Advanced Mathematics

Lab 7: Gradient, curl, divergence in curvilinear coordinates

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GEODÄTISCHES INSTITUT
der Universität Stuttgart
anerkannt
91%

1	2	3a	3b	4a	4b	exercise
17	18	12	9	25	10	points

Spherical and cylindrical coordinates

1. Determine the *divergence* and the *curl* of the vector field

$$G = \frac{-\cos\lambda}{4\cos\vartheta}\hat{\boldsymbol{h}}_r + \operatorname{artanh}\left(\cot\frac{\vartheta}{2}\right)\cos\lambda\,\hat{\boldsymbol{h}}_\vartheta + r^2\hat{\boldsymbol{h}}_\lambda$$

w.r.t. to spherical coordinates (λ : longitude, ϑ : co-latitude, r : radius). (17 points)

2. Calculate the curl and the divergence of the vector field

$$G_{\mu,\nu} = z\sin(\varphi+\mu)\hat{\boldsymbol{h}}_\rho + z\sin(\lambda+\nu)\hat{\boldsymbol{h}}_\varphi + \rho\sin(\varphi)\hat{\boldsymbol{h}}_z$$

in cylindrical coordinates. For which values of $\{\mu,\nu\}$ is the field curl-free, for which divergence-free? (22 points)

Planar curvilinear coordinate systems

3. A curvilinear coordinate system is given by the relation

$$x = \frac{1}{N}\sinh\alpha$$
$$y = \frac{1}{N}\sin\beta$$

with $N := \cosh\alpha - \cos\beta$, $\alpha \in \mathbb{R}$ and $\beta \in [0, 2\pi]$.

a) Derive the gradient in this system w.r.t. the normalized 'frame vectors'. (14 points)

b) Express the frame vectors $\hat{\boldsymbol{h}}_{q_i}$ of standard polar coordinates by the coordinates $\{\alpha,\beta\}$. *The final answer can be given as matrix-vector product without explicit multiplication. To avoid ambiguities, only the first quadrant can be considered.* (12 points)

(final exam WS 17/18)

Ellipsoidal coordinates

4. The transformation from Cartesian to ellipsoidal (geodetic) coordinates is given by

$$x = (N + H)\cos B \cos L$$
$$y = (N + H)\cos B \sin L$$
$$z = ((1 - E^2)N + H)\sin B$$

with $N = \dfrac{A}{\sqrt{1 - E^2 \sin^2 B}}$, $M = \dfrac{A(1 - E^2)}{(1 - E^2 \sin^2 B)^{3/2}}$, $\dfrac{\partial N}{\partial B} = (N - M)\tan B.$

a) Determine the *gradient* in curvilinear coordinates of an arbitrary scalar field $\Phi(L, B, H)$ described on the ellipsoid. (25 points)

b) Calculate the curl of the vector field $\mathbf{G}(L, B, H) = \cos B \sin L \hat{\boldsymbol{h}}_H + \cos L \hat{\boldsymbol{h}}_L + H \cos B \hat{\boldsymbol{h}}_B$ via the general formula:

$$\operatorname{curl} \mathbf{G}(\alpha, \beta, \gamma) = \frac{1}{h_\alpha h_\beta h_\gamma} \det \begin{pmatrix} h_\alpha \hat{\boldsymbol{h}}_\alpha & h_\beta \hat{\boldsymbol{h}}_\beta & h_\gamma \hat{\boldsymbol{h}}_\gamma \\ \frac{\partial}{\partial \alpha} & \frac{\partial}{\partial \beta} & \frac{\partial}{\partial \gamma} \\ h_\alpha G_\alpha & h_\beta G_\beta & h_\gamma G_\gamma \end{pmatrix}$$

which is valid for all orthogonal coordinates systems $\{\alpha, \beta, \gamma\}$ (10 points)

[V] Ellipsoidal coordinates 2

v Another ellipsoidal frame is given by

$$x = \cosh\alpha \cos\beta \cos\varphi$$
$$y = \cosh\alpha \cos\beta \sin\varphi$$
$$z = \sinh\alpha \sin\beta.$$

Express the gradient of an arbitrary field and the scalar function

$$f = \sqrt{\sinh^2\alpha \cos^2\beta + \cosh^2\alpha \sin^2\beta}$$

in this system.

$$G = \frac{-\cos\lambda}{4\cos\theta}\,\hat{h_r} + \operatorname{artanh}(\cot\tfrac{\theta}{2})\cos\lambda\,\hat{h_\theta} + r^2\,\hat{h_\lambda}$$

$g_1 \qquad g_2 \qquad g_3$

spherical coordinates:

$$\operatorname{div}(G) = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2\cdot G_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot G_2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\lambda}(G_3)$$

$$= \frac{1}{r^2}\cdot\left(2r\cdot\frac{-\cos\lambda}{4\cos\theta}\right) + \frac{1}{r\sin\theta}\cdot\cos\lambda\left(\cos\theta\operatorname{artanh}(\cot\tfrac{\theta}{2}) + \frac{\sin\theta}{2\cos\theta}\right) + 0$$

$$= \frac{-\cos\lambda}{2r\cos\theta} + \frac{\cos\lambda\cos\theta\operatorname{artanh}(\cot\frac{\theta}{2})}{r\sin\theta} + \frac{\cos\lambda}{2r\cos\theta}$$

$$= \frac{\cos\lambda\cot\theta\operatorname{artanh}(\cot\frac{\theta}{2})}{r}$$

$$\operatorname{curl}(G) = \frac{1}{r\sin\theta}\cdot\left(\frac{\partial}{\partial\theta}(\sin\theta\cdot G_3) - \frac{\partial G_2}{\partial\lambda}\right)\hat{h_r}$$

$$+ \left(\frac{1}{r\sin\theta}\cdot\frac{\partial G_1}{\partial\lambda} - \frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot G_3)\right)\hat{h_\theta}$$

$$+ \left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot G_2) - \frac{1}{r}\cdot\frac{\partial G_1}{\partial\theta}\right)\hat{h_\lambda}$$

$$= \frac{1}{r\sin\theta}\cdot\left(r^2\cos\theta + \operatorname{artanh}(\cot\tfrac{\theta}{2})\sin\lambda\right)\hat{h_r}$$

$$+ \left(\frac{1}{r\sin\theta}\cdot\frac{\sin\lambda}{4\cos\theta} - \frac{1}{r}\cdot 3r^2\right)\hat{h_\theta} + \left(\frac{1}{r}\cdot\operatorname{artanh}(\cot\tfrac{\theta}{2})\cos\lambda + \frac{1}{r}\cdot\frac{\cos\lambda\sin\theta}{4\cos^2\theta}\right)\hat{h_\lambda}$$

$$= \left[r\cot\theta + \frac{\operatorname{artanh}(\cot\frac{\theta}{2})\sin\lambda}{r\sin\theta}\right)\hat{h_r} + \left(\frac{\sin\lambda}{2r\sin 2\theta} - 3r\right)\hat{h_\theta} + \frac{\cos\lambda}{r}\left(\operatorname{artanh}(\cot\tfrac{\theta}{2}) + \frac{\sin\theta}{4\cos^2\theta}\right)$$

Should "λ" be "φ"? both are right.

2. $G_{\mu,\nu} = z\sin(\varphi+\mu)\,\hat{h}_\rho + z\sin(\varphi+\nu)\,\hat{h}_\varphi + \rho\sin(\varphi)\,\hat{h}_z$

$$\mathrm{div}(G) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho G_1) + \frac{1}{\rho}\cdot\frac{\partial G_2}{\partial\varphi} + \frac{\partial G_3}{\partial z}$$

$$= \frac{1}{\rho}\cdot z\sin(\varphi+\mu) + \frac{1}{\rho}\cdot z\cos(\varphi+\nu)$$

$$= \frac{1}{\rho}\cdot z\left(\sin(\varphi+\mu)+\cos(\varphi+\nu)\right)$$

$$\mathrm{curl}(G) = \left(\frac{1}{\rho}\cdot\frac{\partial G_3}{\partial\varphi} - \frac{\partial G_2}{\partial z}\right)\hat{h}_\rho + \left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial\rho}\right)\hat{h}_\varphi + \left(\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}(\rho G_2) - \frac{1}{\rho}\frac{\partial G_1}{\partial\varphi}\right)\hat{h}_z$$

$$= \left(\frac{1}{\rho}\cdot\rho\cos\varphi - \sin(\varphi+\nu)\right)\hat{h}_\rho + \left(\sin(\varphi+\mu)-\sin\varphi\right)\hat{h}_\varphi + \left(\frac{1}{\rho}\cdot z\sin(\varphi+\nu) - \frac{1}{\rho}\cdot z\cdot\cos(\varphi+\mu)\right)\hat{h}_z$$

$$= \left[\cos\varphi - \sin(\varphi+\nu)\right]\hat{h}_\rho + \left(\sin(\varphi+\mu)-\sin\varphi\right)\hat{h}_\varphi + \frac{z}{\rho}\left[\sin(\varphi+\nu)-\cos(\varphi+\mu)\right]\hat{h}_z$$

curl-free $\Rightarrow$ $\mathrm{curl}(G)=0$, i.e.

$$\begin{cases}\cos\varphi = \sin(\varphi+\nu)\\ \sin(\varphi+\mu)=\sin\varphi\\ \sin(\varphi+\nu)=\cos(\varphi+\mu)\end{cases} \Rightarrow \begin{cases}\mu = 2k_1\pi\\ \varphi = 2k_2\pi+\frac{\pi}{2}\end{cases}, k_1, k_2 \in \mathbb{Z}$$

divergence-free $\Rightarrow$ $\mathrm{div}(\mathrm{curl}(G))=0$ × div G = 0. −4

$$\mathrm{div}(\mathrm{curl}(G)) = \frac{1}{\rho}\cdot\left[\cos\varphi-\sin(\varphi+\nu)\right] + \frac{1}{\rho}\cdot\left[\cos(\varphi+\mu)-\cos\varphi\right] + \frac{1}{\rho}\left[\sin(\varphi+\nu)-\cos(\varphi+\mu)\right]$$

$$= \frac{1}{\rho}\left[\cos\varphi-\cos\varphi-\sin(\varphi+\nu)+\sin(\varphi+\nu)+\cos(\varphi+\mu)-\cos(\varphi+\mu)\right]$$

$$= 0.$$

$\Rightarrow$ For $\forall \mu,\nu \in \mathbb{R}$, it is divergence-free.

$$x = \frac{1}{N}\sinh\alpha$$
$$y = \frac{1}{N}\sin\beta \quad , \quad N = \cosh\alpha - \cos\beta \quad , \quad \alpha \in \mathbb{R}, \ \beta \in [0, 2\pi].$$

a)

$$h_\alpha = \left(\frac{\partial x}{\partial \alpha}, \frac{\partial y}{\partial \alpha}\right)^T = \begin{bmatrix} \frac{1-\cosh\alpha\cos\beta}{(\cosh\alpha-\cos\beta)^2} \\ -\frac{\sin\beta\sinh\alpha}{(\cosh\alpha-\cos\beta)^2} \end{bmatrix} , \quad |h_\alpha| = \frac{\sqrt{(1-\cosh\alpha\cos\beta)^2 + \sin^2\beta\sinh^2\alpha}}{(\cosh\alpha-\cos\beta)^2} = \frac{1}{N}$$

simplify

$$\hat{h}_\alpha = \left[\frac{1-\cosh\alpha\cos\beta}{\sqrt{(1-\cosh\alpha\cos\beta)^2+\sin^2\beta\sinh^2\alpha}}, -\frac{\sinh\alpha\sin\beta}{\sqrt{(1-\cosh\alpha\cos\beta)^2+\sin^2\beta\sinh^2\alpha}}\right]^T .$$

$$= \frac{1}{N}$$

$$h_\beta = \left(\frac{\partial x}{\partial \beta}, \frac{\partial y}{\partial \beta}\right)^T = \left[-\frac{\sinh\alpha\sin\beta}{N^2}, \frac{\cosh\alpha\cos\beta-1}{N^2}\right]^T , \quad |h_\beta| = \frac{\sqrt{\sinh^2\alpha\sin^2\beta+(\cosh\alpha\cos\beta-1)^2}}{N^2}$$

$$\hat{h}_\beta = \left[\frac{-\sinh\alpha\sin\beta}{\sqrt{\sinh^2\alpha\sin^2\beta+(\cosh\alpha\cos\beta-1)^2}}, \frac{\cosh\alpha\cos\beta-1}{\sqrt{\sinh^2\alpha\sin^2\beta+(\cosh\alpha\cos\beta-1)^2}}\right]^T .$$

$$\Rightarrow \nabla f = \frac{1}{h_\alpha}\frac{\partial f}{\partial \alpha}\hat{h}_\alpha + \frac{1}{h_\beta}\frac{\partial f}{\partial \beta}\hat{h}_\beta$$

$$= \frac{(\cosh\alpha-\cos\beta)^2}{\sqrt{(1-\cosh\alpha\cos\beta)^2+\sin^2\beta\sinh^2\alpha}}\cdot\frac{\partial f}{\partial \alpha}\hat{h}_\alpha + \frac{(\cosh\alpha-\cos\beta)^2}{\sqrt{\sinh^2\alpha\sin^2\beta+(\cosh\alpha\cos\beta-1)^2}}\cdot\frac{\partial f}{\partial \beta}\cdot\hat{h}_\beta$$

−2

$$= |N|\frac{\partial f}{\partial \alpha}\hat{h}_\alpha + |N|\frac{\partial f}{\partial \beta}\cdot\hat{h}_\beta$$

b) standard polar coordinates: $\rho = \sqrt{x^2+y^2}$, $\varphi = \arctan\frac{y}{x}$.

$$\Rightarrow \rho = \frac{1}{N}\sqrt{\sinh^2\alpha+\sin^2\beta} \quad , \quad \varphi = \arctan\left(\frac{\sin\beta}{\sinh\alpha}\right) \quad , \quad \begin{cases} \cos\varphi = \frac{x}{\sqrt{x^2+y^2}} \\ \sin\varphi = \frac{y}{\sqrt{x^2+y^2}} \end{cases}$$

in polar coordinates: $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$

$$\Rightarrow h_\rho = \left[\frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}\right]^T = [\cos\varphi, \sin\varphi]^T , \quad \hat{h}_\rho = [\cos\varphi, \sin\varphi]^T$$

$$h_\varphi = \left[\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}\right]^T = [-\rho\sin\varphi, \rho\cos\varphi]^T , \quad \hat{h}_\varphi = [-\sin\varphi, \cos\varphi]^T$$

$$\Rightarrow \begin{cases} \hat{h}_\rho = \left[\cos\left(\arctan\frac{\sin\beta}{\sinh\alpha}\right), \sin\left(\arctan\frac{\sin\beta}{\sinh\alpha}\right)\right]^T \\ \hat{h}_\varphi = \left[-\sin\left(\arctan\frac{\sin\beta}{\sinh\alpha}\right), \cos\left(\arctan\frac{\sin\beta}{\sinh\alpha}\right)\right]^T \end{cases} \text{or} \begin{cases} \hat{h}_\rho = \left[\frac{\sinh\alpha}{\sqrt{\sinh^2\alpha+\sin^2\beta}}, \frac{\sin\beta}{\sqrt{\sinh^2\alpha+\sin^2\beta}}\right]^T \\ \hat{h}_\varphi = \left[-\frac{\sin\beta}{\sqrt{\sinh^2\alpha+\sin^2\beta}}, \frac{\sinh\alpha}{\sqrt{\sinh^2\alpha+\sin^2\beta}}\right]^T \end{cases}$$

$$\begin{bmatrix} \hat{h}_\rho \\ \hat{h}_\varphi \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} \hat{h}_\alpha \\ \hat{h}_\beta \end{bmatrix} \ ? \qquad -3$$

in polar coordinates:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \Rightarrow \begin{aligned} h_\rho &= \left[\frac{\partial x}{\partial \rho}, \frac{\partial y}{\partial \rho}\right]^T = [\cos\varphi, \sin\varphi]^T \\ h_\varphi &= \left[\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}\right]^T = [-\rho\sin\varphi, \rho\cos\varphi]^T \end{aligned}$$

$$\Rightarrow \begin{aligned} \hat{h}_\rho &= [\cos\varphi, \sin\varphi]^T \\ \hat{h}_\varphi &= [-\cancel{\rho}\sin\varphi, \cos\varphi]^T \end{aligned} \qquad \text{i.e.} \begin{bmatrix} \hat{h}_\rho \\ \hat{h}_\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

ccording to a), $$\begin{bmatrix} \hat{h}_\alpha \\ \hat{h}_\beta \end{bmatrix} = \begin{bmatrix} \frac{1-\cosh\alpha\cos\beta}{N} & \frac{-\sinh\alpha\sin\beta}{N} \\ \frac{-\sinh\alpha\sin\beta}{N} & \frac{\cosh\alpha\cos\beta-1}{N} \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{h}_\rho \\ \hat{h}_\varphi \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \frac{1-\cosh\alpha\cos\beta}{N} & \frac{-\sinh\alpha\sin\beta}{N} \\ \frac{-\sinh\alpha\sin\beta}{N} & \frac{\cosh\alpha\cos\beta-1}{N} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \hat{h}_\alpha \\ \hat{h}_\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \frac{\cosh\alpha\cos\beta-1}{N} & \frac{\sinh\alpha\sin\beta}{N} \\ \frac{\sinh\alpha\sin\beta}{N} & \frac{1-\cosh\alpha\cos\beta}{N} \end{bmatrix} \begin{bmatrix} \hat{h}_\alpha \\ \hat{h}_\beta \end{bmatrix}$$

$$\begin{cases} x = (N+H)\cos B\cos L \\ y = (N+H)\cos B\sin L \\ z = \left((1-E^2)N+H\right)\sin B \end{cases} \qquad N = \frac{A}{\sqrt{1-E^2\sin^2 B}}. \qquad \frac{\partial N}{\partial B} = (N-M)\tan B$$

$$M = \frac{A(1-E^2)}{(1-E^2\sin^2 B)^{\frac{3}{2}}}$$

a)

$$h_L = \left[\frac{\partial x}{\partial L}, \frac{\partial y}{\partial L}, \frac{\partial z}{\partial L}\right]^T = \begin{bmatrix} -(N+H)\cos B\sin L \\ (N+H)\cos B\cos L \\ 0 \end{bmatrix}$$

$$|h_L| = \sqrt{[(N+H)\cos B\sin L]^2 + [(N+H)\cos B\cos L]^2} = (N+H)\cos B$$

$$\hat{h_L} = [-\sin L, \cos L, 0]^T.$$

$$h_B = \left[\frac{\partial x}{\partial B}, \frac{\partial y}{\partial B}, \frac{\partial z}{\partial B}\right]^T = \begin{bmatrix} -(N-M)\tan B\cos B\cos L + (N+H)(-\sin B)\cos L \\ (N-M)\tan B\cos B\sin L + (N+H)(-\sin B)\sin L \\ (1-E^2)(N-M)\tan B\sin B + [(1-E^2)N+H]\cdot\cos B \end{bmatrix}$$

$$= \begin{bmatrix} -(M+H)\sin B\cos L \\ -(M+H)\sin B\sin L \\ (1-E^2)\frac{AE^2\cos^2 B}{(1-E^2\sin^2 B)^{\frac{3}{2}}}\cdot\frac{\sin^2 B}{\cos B} + (1-E^2)\cdot\frac{A(\cancel{1}-E^2\sin^2 B)\cos B}{(1-E^2\sin^2 B)^{\frac{3}{2}}} + H\cos B \end{bmatrix}$$

$$= \begin{bmatrix} -(M+H)\sin B\cos L \\ -(M+H)\sin B\sin L \\ (M+H)\cos B \end{bmatrix}$$

$$|h_B| = \sqrt{[(M+H)\sin B\cos L]^2 + [(M+H)\sin B\sin L]^2 + [(M+H)\cos B]^2} = M+H.$$

$$|\hat{h_B}| = \begin{bmatrix} -\sin B\cos L \\ -\sin B\sin L \\ \cos B \end{bmatrix}$$

$$h_H = \left[\frac{\partial x}{\partial H}, \frac{\partial y}{\partial H}, \frac{\partial z}{\partial H}\right]^T = \begin{bmatrix} \cos B\cos L \\ \cos B\sin L \\ \sin B \end{bmatrix}$$

$$|h_H| = \sqrt{(\cos B\cos L)^2 + (\cos B\sin L)^2 + \sin^2 B} = 1$$

$$\hat{h_H} = [-\sin B\cos L, -\sin B\sin L, \cos B]^T.$$

$$\Rightarrow \text{gradient } \nabla\phi = \sum_{i=1}^{3}\frac{1}{|h_i|}\cdot\frac{\partial\phi}{\partial q_i}\hat{h_i} = \frac{1}{(N+H)\cos B}\frac{\partial\phi}{\partial L}\hat{h_L} + \frac{1}{M+H}\frac{\partial\phi}{\partial B}\hat{h_B} + \frac{\partial\phi}{\partial H}\hat{h_H}$$

b)

$$\text{curl } G(L,B,H) = \frac{1}{h_L\cdot h_B\cdot h_H}\det\begin{pmatrix} h_L\hat{h_L} & h_B\hat{h_B} & h_H\hat{h_H} \\ \frac{\partial}{\partial L} & \frac{\partial}{\partial B} & \frac{\partial}{\partial H} \\ h_L G_L & h_B G_B & h_H G_H \end{pmatrix}.$$

$$\frac{1}{(N+H)\cos B\cdot(M+H)\cdot 1}\cdot \det\begin{pmatrix} |h_L|\cdot\hat{h}_L & |h_B|\cdot\hat{h}_B & |h_H|\cdot\hat{h}_H \\ \frac{\partial}{\partial L} & \frac{\partial}{\partial B} & \frac{\partial}{\partial H}\\ |h_L|\cdot G_L & |h_B|\cdot G_B & |h_H|\cdot G_H\end{pmatrix}$$

$$\frac{1}{(N+H)\cos B\cdot(M+H)}\cdot\Big[|h_L|\cdot\hat{h}_L\cdot\Big(\frac{\partial}{\partial B}(|h_H|\cdot G_H)-\frac{\partial}{\partial H}(|h_B|\cdot G_B)\Big)-|h_B|\cdot\hat{h}_B\cdot\Big(\frac{\partial}{\partial L}(|h_H| G_H)-\frac{\partial}{\partial H}(|h_L|\cdot G_L)\Big)$$

$$+|h_H|\cdot\hat{h}_H\Big(\frac{\partial}{\partial L}(|h_B| G_B)-\frac{\partial}{\partial B}(|h_L|\cdot G_L)\Big)\Big]$$

$$\frac{1}{(N+H)\cdot\cos B\cdot(M+H)}\cdot\Big[(N+H)\cos B\cdot\hat{h}_L\cdot\big(-\sin B\sin L-(M+2H)\cos B\big)-(M+H)\cdot\hat{h}_B\cdot\big(\cos B\cos L-\cos B\cos L\big)$$

$$+1\cdot\hat{h}_H\big(0-(N+H)(-\sin B)\cdot\cos L\big)\Big]$$

$$\frac{\sin B\sin L-(M+2H)\cos B}{M+H}\hat{h}_L+\frac{\sin B\cos L}{\cos B(M+H)}\hat{h}_H$$

$$\begin{cases} x = \cosh\alpha\cos\beta\cos\varphi \\ y = \cosh\alpha\cos\beta\sin\varphi \\ z = \sinh\alpha\sin\beta \end{cases} \qquad f = \sqrt{\sinh^2\alpha\cos^2\beta + \cosh^2\alpha\sin^2\beta}$$

$$\Rightarrow \quad h_\alpha = \left[\frac{\partial x}{\partial\alpha}, \frac{\partial y}{\partial\alpha}, \frac{\partial z}{\partial\alpha}\right]^T = [\sinh\alpha\cos\beta\cos\varphi, \sinh\alpha\cos\beta\sin\varphi, \cosh\alpha\sin\beta]^T \quad ✓$$

$$|h_\alpha| = \sqrt{(\sinh\alpha\cos\beta\cos\varphi)^2 + (\sinh\alpha\cos\beta\sin\varphi)^2 + (\cosh\alpha\sin\beta)^2} = \sqrt{\sinh^2\alpha + \sin^2\beta} \quad \times$$

$$\hat{h}_\alpha = \frac{1}{\sqrt{\sinh^2\alpha + \sin^2\beta}}[\sinh\alpha\cos\beta\cos\varphi, \sinh\alpha\cos\beta\sin\varphi, \cosh\alpha\sin\beta]^T$$

$$h_\beta = \left[\frac{\partial x}{\partial\beta}, \frac{\partial y}{\partial\beta}, \frac{\partial z}{\partial\beta}\right]^T = [-\cosh\alpha\sin\beta\cos\varphi, -\cosh\alpha\sin\beta\sin\varphi, \sinh\alpha\cos\beta]^T \quad ✓$$

$$|h_\beta| = \sqrt{(\cosh\alpha\sin\beta\cos\varphi)^2 + (-\cosh\alpha\sin\beta\sin\varphi)^2 + (\sinh\alpha\cos\beta)^2} = \sqrt{(\sinh\alpha)^2 + \sin^2\beta} \quad \times$$

$$\hat{h}_\beta = \frac{1}{\sqrt{\sinh^2\alpha + \sin^2\beta}}[-\cosh\alpha\sin\beta\cos\varphi, -\cosh\alpha\sin\beta\sin\varphi, \sinh\alpha\cos\beta]^T$$

$$h_\varphi = \left[\frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}\right]^T = [-\cosh\alpha\cos\beta\sin\varphi, \cosh\alpha\cos\beta\cos\varphi, 0]^T \quad ✓$$

$$|h_\varphi| = \sqrt{(\cosh\alpha\cos\beta\sin\varphi)^2 + (\cosh\alpha\cos\beta\cos\varphi)^2} = \cosh\alpha\cos\beta \quad ✓$$

$$\hat{h}_\varphi = [-\sin\varphi, \cos\varphi, 0]^T \quad ✓$$

$$\Rightarrow \text{gradient}, \quad \nabla f = \frac{1}{h_\alpha}\frac{\partial f}{\partial\alpha}\hat{h}_\alpha + \frac{1}{h_\beta}\frac{\partial f}{\partial\beta}\hat{h}_\beta + \frac{1}{h_\varphi}\frac{\partial f}{\partial\varphi}\hat{h}_\varphi$$

$$\text{while } \frac{\partial f}{\partial\alpha} = \frac{2\sinh\alpha\cosh\alpha\cos^2\beta + 2\cosh\alpha\sinh\alpha\sin^2\beta}{2\sqrt{\sinh^2\alpha\cos^2\beta + \cosh^2\alpha\sin^2\beta}} = \frac{\sinh\alpha\cosh\alpha}{\sqrt{\sinh^2\alpha(1-\sin^2\beta) + \cosh^2\alpha\sin^2\beta}} = \frac{\sinh 2\alpha}{2\sqrt{\sinh^2\alpha + \sin^2\beta}}$$

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$$\text{arly}, \quad \frac{\partial f}{\partial\beta} = \frac{\sin 2\beta}{2\sqrt{\sinh^2\alpha + \sin^2\beta}}$$

$$\frac{\partial f}{\partial\varphi} = 0$$

$$\Rightarrow \nabla f = \frac{1}{\sqrt{\sinh^2\alpha + \sin^2\beta}}\cdot\frac{\sin 2\alpha}{2\sqrt{\sinh^2\alpha + \sin^2\beta}}\hat{h}_\alpha + \frac{1}{\sqrt{\sinh^2\alpha + \sin^2\beta}}\cdot\frac{\sin 2\beta}{2\sqrt{\sinh^2\alpha + \sin^2\beta}}\hat{h}_\beta + 0$$

×

$$= \frac{\sin 2\alpha}{\sinh^2\alpha + \sin^2\beta}\hat{h}_\alpha + \frac{\sin 2\beta}{\sinh^2\alpha + \sin^2\beta}\hat{h}_\beta$$