Advanced Mathematics

Lab 9: Integral theorems: Flux and circulation

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anerkannt

53%

1	2	3a	3b	4	problem
15	9	10	8	11	points

1. A incomplete rotational surface $\mathcal{S}$ is defined by the expression $(\rho-1)^2+z^2=2$ with $\rho \geq 0$ and the rotational angle $\varphi \in [0,\pi]$ in cylindrical coordinates. Determine the flux of the vector field $G(r,\varphi,z) = 5z^2\rho\hat{h}_\rho + \cos\varphi\hat{h}_z$ through the surface $\mathcal{S}$.

$S:\ \rho = \sqrt{2}\cos u + 1,\ \varphi = v,\ z = \sqrt{2}\sin u \Rightarrow \left.\begin{matrix} r_p = \\ r_u = \end{matrix}\right\} \Rightarrow N = \ \Rightarrow \iint G^T N\, du\, dv$

(30 points)

2. The solid sphere $\Sigma = \{x \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 4\}$ is divided into two spherical caps C_1 and C_2 by the intersection with the plane $E: y+z=\frac{1}{2}$. Calculate the flux $\mathcal{F}$ of the vector feld

$$F = \left(-xy^2,\ \ -yz^2, -z^2y\right)$$

through the volume of the smaller cap C_1. *Introduce a rotated spherical coordinate system where the new equator is parallel to the plane.* (19 points)

3. Given the asymmetric cone, which is defined by the apex (='the singular point') $A = \begin{pmatrix}0, & 0, & 4\end{pmatrix}^T$ and the planar figure $\mathcal{B} = \left\{x \in \mathbb{R}^3 : 4x^2+3xy+4y^2+y \leq x, z=0\right\}$.

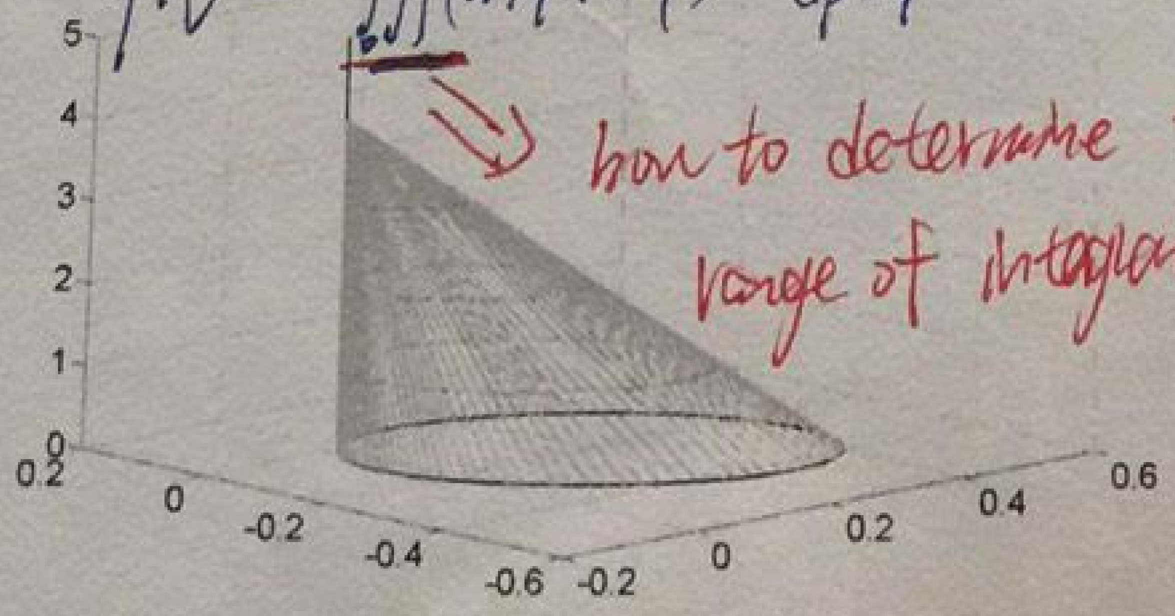
a) The boundary of $\mathcal{B}$ in the plane $z = 0$ is a shifted and rotated ellipse. Determine its normal form to figure out the geometry. (15 points)

$(x,y)\cdot A\cdot\begin{pmatrix}x\\y\end{pmatrix} = (\bar{x},\bar{y})\ \underline{UAU^T}\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

b) Calculate the flux of the vector field $\mathbf{G} = 4\rho\hat{h}_\rho + \cos\varphi\hat{h}_\varphi + (z - z\frac{1}{\rho}\cos\varphi)\hat{h}_z$ in cylindrical coordinates through the volume of the cone via the integral theorem of Gauß. (21 points)

$\text{flux} = \iiint \operatorname{div} G \cdot dV = \iiint \left(9 - \frac{\sin\varphi + \cos\varphi}{\rho}\right) dV$

$= 9\cdot V - \int\!\!\int\!\!\int^{4} (\sin\varphi + \cos\varphi)\cdot d\rho\, d\varphi\, dz$



how to determine the range of integration?

Hints:

- *The volume of a cone – with a planar boundary curve – is given by $V = \frac{1}{3}B\cdot h$ with the height h and the base area B.*
- *Split the volume integral into two parts. One part can determined by using the results of (3a) without explicit integration*

(final exam WS 18/19)

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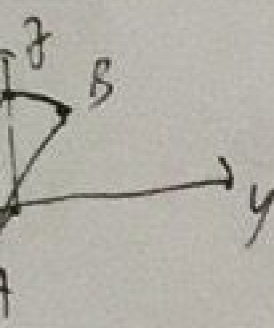
Circulation

4. Evaluate the circulation of the vector field $\mathbf{G}(r, \lambda, \vartheta) = r\cos\lambda\,\hat{h}_\lambda$ through the spherical triangle with the corner points $\mathbf{A} = \begin{pmatrix}1, & 0, & 0\end{pmatrix}^\mathsf{T}$, $\mathbf{B} = \begin{pmatrix}0, & \frac{3}{5}, & \frac{4}{5}\end{pmatrix}^\mathsf{T}$ and $\mathbf{C} = \begin{pmatrix}0, & 0, & 1\end{pmatrix}^\mathsf{T}$. *Consider that the boundaries of a spherical triangle consist in great circles* (15 points)



$$\iint_S (\text{curl}\,\underline{G})^T n\,dA = \oint_c \underline{F}\,dr = \oint_c \underline{F}^T \cdot \underline{J} \cdot dt$$

AB: $\quad r = 1, \quad \lambda=\lambda,\ \theta=\theta \quad (\underline{OA} \times \underline{OB}) \cdot \underline{e}_r = 0$

$$(1,0,0) \times (0, \tfrac{3}{5}, \tfrac{4}{5}) \cdot (\sin\theta\cos\lambda, \sin\theta\sin\lambda, \cos\theta) = 0$$

$$\Rightarrow \sin\lambda = \tfrac{3}{4}\cot\theta, \quad \lambda = \arcsin(\tfrac{3}{4}\cot\theta), \quad \theta \in [\qquad]$$

$$\underline{T}_{AB} = 0 + \dot{\lambda} \cdot r\sin\theta \cdot \hat{h}_\lambda + 1 \cdot r \cdot \hat{h}_\theta$$

BC: $\quad r = 1, \ \lambda = \frac{\pi}{2}, \ \theta = \theta \in [\qquad]$

$$\underline{T}_{BC} = 0 + 0 + 1 \cdot r \cdot \hat{h}_\theta$$

CA: $\quad r = 1, \ \lambda = 0, \ \theta = \theta \in [\qquad]$

$$\underline{T}_{CA} = 0 + 0 + 1 \cdot r \cdot \hat{h}_\theta$$

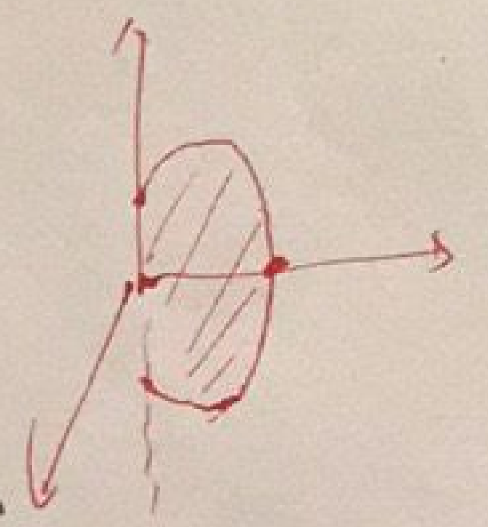
(directly)

the surface S: $(\rho-1)^2+z^2=2$, $\rho\ge 0$ and $\varphi\in[0,\pi]$

integral
How to determine the boundary?
↓
Draw the surface?

let $\rho=\rho$, $\varphi=\varphi$, $z=\pm\sqrt{2-(\rho-1)^2}$, $\rho\ge 0$, $\varphi\in[0,\pi]$

① $z_1=\sqrt{2-(\rho-1)^2}$



the parameter expression of S:

$$\underline{S}=\rho\,\hat{h}_\rho+\varphi\,\hat{h}_\varphi+\sqrt{2-(\rho-1)^2}\,\hat{h}_z \quad ✓$$

tangent vector:

$$r_\rho=1\cdot 1\cdot\hat{h}_\rho+0+\frac{-2(\rho-1)}{2\sqrt{2-(\rho-1)^2}}\cdot 1\cdot\hat{h}_z=\hat{h}_\rho+\frac{-(\rho-1)}{\sqrt{2-(\rho-1)^2}}\hat{h}_z \quad ✓$$

$$r_\varphi=0+1\cdot\rho\cdot\hat{h}_\varphi+0=\rho\,\hat{h}_\varphi \quad ✓$$

normal vector:

$$\underline{N}=r_\rho\times r_\varphi=\det\begin{vmatrix}\hat{h}_\rho & \hat{h}_\varphi & \hat{h}_z\\ 1 & 0 & \frac{1-\rho}{\sqrt{2-(\rho-1)^2}}\\ 0 & \rho & 0\end{vmatrix}=-\rho\cdot\frac{(\rho-1)}{\sqrt{2-(\rho-1)^2}}\hat{h}_\rho+\rho\,\hat{h}_z$$

sign (direction) of N
Should we chang[e]

the vector field along the surface:

$$\underline{G}_1=5z^2\rho\,\hat{h}_\rho+\cos\varphi\,\hat{h}_z=5\cdot(2-(\rho-1)^2)\cdot\rho\,\hat{h}_\rho+\cos\varphi\,\hat{h}_z$$

$$\Rightarrow \text{flux}\iint_{S_1}\underline{G}^T\underline{n}\,dA=\iint\underline{G}^T\underline{N}\,d\rho\,d\varphi=\iint 5\rho^2(\rho-1)\sqrt{2-(\rho-1)^2}+\rho\cos\varphi\,d\rho\,d\varphi$$

$$=\int_0^{\sqrt{2}+1}\left(5\rho^2(\rho-1)\sqrt{2-(\rho-1)^2}\,\varphi+\rho\sin\varphi\right)\Big|_0^\pi d\rho=\int_0^{\sqrt{2}+1}5\rho^2(\rho-1)\sqrt{2-(\rho-1)^2}\cdot\pi\,d\rho \;✓ =\left(4+\frac{15}{4}\pi\right)\pi$$

② $z=-\sqrt{2-(\rho-1)^2}$

[Similar]ly, $\underline{S}=\rho\,\hat{h}_\rho+\varphi\,\hat{h}_\varphi+\sqrt{2-(\rho-1)^2}$, $\underline{N}=-\frac{\rho(\rho-1)}{\sqrt{2-(\rho-1)^2}}\hat{h}_\rho+\rho\,\hat{h}_z$, $\underline{G}_1=5(2-(\rho-1)^2)\rho\,\hat{h}_\rho+\cos\varphi\,\hat{h}_z$

$$\Rightarrow\iint_{S_2}\underline{G}^T\underline{n}\,dA=\iint -5\rho^2(\rho-1)\sqrt{2-(\rho-1)^2}+\rho\cos\varphi\,d\rho\,d\varphi=\int_0^{\sqrt{2}+1}-5\rho^2(\rho-1)\sqrt{2-(\rho-1)^2}\,\pi\,d\rho$$

$$=\left(4+\frac{15}{4}\pi\right)\pi$$

~~[wit]h ① and ②, we get~~ $\cancel{\iint_{S_1}\underline{G}^T\underline{n}\,dA+\iint_{S_2}\underline{G}^T\underline{n}\,dA=0.}$ ✗

①, ② $\Rightarrow$ flux $=8\pi+\frac{15}{2}\pi^2$

(via triangle substitution)

the surface S: $(\rho-1)^2+z^2=2$, $\rho\geq 0$ and $\varphi\in[0,\pi]$.

we let $\rho=\sqrt{2}\cos u+1$, $\varphi=v$, $z=\sqrt{2}\sin u$, $u\in[-\frac{3}{4}\pi,\frac{3}{4}\pi]$, $v\in[0,\pi]$

thus the parameter expression of the surface S:

$$\underset{\sim}{S}=(\sqrt{2}\cos u+1)\,\hat{h}_\rho+v\cdot\hat{h}_\varphi+\sqrt{2}\sin u\cdot\hat{h}_z$$

the tangent vector of this surface:

$$\begin{cases}|h_\rho|=1\\|h_\varphi|=\rho\\|h_z|=1\end{cases}\quad\text{in cylindrical coordinates}$$

$$r_u=-\sqrt{2}\sin u\,\hat{h}_\rho+0+\sqrt{2}\cos u\,\hat{h}_z$$

$$r_v=0+1\cdot(\sqrt{2}\cos u+1)\,\hat{h}_\varphi+0$$

the normal vector of this surface:

$$\underset{\sim}{N}=r_u\times r_v=\det\begin{vmatrix}\hat{h}_\rho & \hat{h}_\varphi & \hat{h}_z\\ -\sqrt{2}\sin u & 0 & \sqrt{2}\cos u\\ 0 & \sqrt{2}\cos u+1 & 0\end{vmatrix}=-(\sqrt{2}\cos u+1)\left(\sqrt{2}\cos u\,\hat{h}_\rho+\sqrt{2}\sin u\,\hat{h}_z\right)$$

Consider the vector field along the surface:

$$\underset{\sim}{G}=5\cdot(\sqrt{2}\sin u)^2\cdot(\sqrt{2}\cos u+1)\,\hat{h}_\rho+\cos v\,\hat{h}_z$$

-4

the flux

$$\iint_S \underset{\sim}{G}^T\underset{\sim}{n}\,dA=\iint \underset{\sim}{G}^T\underset{\sim}{N}\,du\,dv=\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}\int_0^{\pi}\left(-10\sqrt{2}\sin^2u\cos u(\sqrt{2}\cos u+1)^2-(\sqrt{2}\cos u+1)\sqrt{2}\sin u\cos v\right)dv\,du$$

$$=\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}-10\sqrt{2}\sin^2u\cos u(\sqrt{2}\cos u+1)^2\cdot v-(\sqrt{2}\cos u+1)\sqrt{2}\sin u\sin v\Big|_0^{\pi}du=\int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}-10\sqrt{2}\sin^2u\cos u(\sqrt{2}\cos u+1)^2\pi\,du$$

$$=\left(\frac{\sin x}{2}-\frac{\sin 5x}{40}-\frac{\sin 3x}{8}-\frac{\sqrt{2}\sin 4x}{16}+\frac{\sqrt{2}x}{4}\right)\Big|_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}=\frac{\sqrt{2}}{40}\cancel{(15\pi+16)}\qquad \frac{15}{2}\pi^2+8\pi$$

Different from "Direct Method" (last page). Why?

Both methods are correct, but there are some mistake

Introduce the rotated spherical coordinate system where the new equator is parallel to the plane:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} r\sin\Delta\cos\lambda \\ r\sin\Delta\sin\lambda \\ r\cos\Delta \end{bmatrix}$$

this coordinate system is rotated form the standard one along the x axes with $(-\frac{\pi}{4})$ angle,

so the coordinates of the original normal cartesian system is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ 0 & \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

① $\int_0^{\arccos\frac{\sqrt2}{8}} d\Delta \int_{\frac{\sqrt2}{4\cos\Delta}}^{2} dr \int_0^{2\pi} -\frac{\pi}{2} r^4 \cos^2\Delta\sin\Delta \, d\lambda$

② $\int_{\frac{\sqrt2}{4}}^{2} dz' \int_{-\sqrt{\frac{31}{8}}}^{\sqrt{\frac{31}{8}}} dy' \int_{-\sqrt{\frac{31}{8}}}^{\sqrt{\frac{31}{8}}} 2z'^2 \, dx'$

+3

hence
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\Delta\cos\lambda \\ \frac{\sqrt2}{2} r\sin\Delta\sin\lambda + \frac{\sqrt2}{2} r\cos\Delta \\ -\frac{\sqrt2}{2} r\sin\Delta\sin\lambda + \frac{\sqrt2}{2} r\cos\Delta \end{bmatrix}$$

In this case ($r=2$, rotation angle $=\frac{\pi}{4}$)

the intersection plane: $y+z=\frac{1}{2}$

$\Rightarrow \sqrt2 r\cos\Delta = \frac12$, $\lambda\in[0, \ldots]$, $\Delta\in[0, \ldots \frac{\sqrt2}{8}]$

Hence $\Omega = r\hat{h}_r + \Delta\hat{h}_\Delta + \lambda\hat{h}_\lambda$, $\lambda\in[0,2\pi]$, $\Delta\in[0,\arccos\frac{\sqrt2}{8}]$, $r\in[\frac{\sqrt2}{4\cos\Delta}, 2]$

+6

in spherical coordinates, $dV = r^2\sin\Delta \, dr\, d\lambda\, d\Delta$

vector field F: $(-xy^2, -yz^2, -z^2y)$ ~~should it be $-zy$~~

$\operatorname{div} F = (-y^2 - z^2 - 2yz) =$

$= -(y+z)^2 = -\frac{1}{4} \quad -2r^2\cos^2\Delta$ validate only for plane E $y+z=\frac12$

$\iiint \operatorname{div} F \, dV = \int_{\frac{\sqrt2}{4\cos\Delta}}^{2}\int_0^{\arccos\frac{\sqrt2}{8}}\int_0^{2\pi} -2r^2\cos^2\Delta\cdot r^2\sin\Delta \; d\lambda\, d\Delta\, dr =$

$\int_{\frac{\sqrt2}{4\cos\Delta}}^{2}\int_0^{\arccos\frac{\sqrt2}{8}} -\frac{\pi}{2} r^2 \sin\Delta \, d\Delta\, dr = \int_0^{\arccos\frac{\sqrt2}{8}}$

... function $\varphi(v,w) = \sin w\cdot\sinh v$ is one of the ... The constants

b) Consider now the differential equation in v for the constant of $\varphi(v,w)$ ... independent solution via reduction of ord...

a) the most simple form of an ellipse is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

with shiftness. it turns to $\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$ ✓

with rotation, it turns to $\frac{[(x-x_0)\cos\theta+(y-y_0)\sin\theta]^2}{a^2}+\frac{[(x-x_0)\sin\theta-(y-y_0)\cos\theta]}{b^2}$ ✓

introduce this into $4x^2+3xy+4y^2+y-x=0$.

$$\Rightarrow\ 4(x-\tfrac{1}{5})^2+3(x-\tfrac{1}{5})(y+\tfrac{1}{5})+4(y+\tfrac{1}{5})^2-\tfrac{1}{5}=0 \quad\text{, i.e.}\quad x_0=\tfrac{1}{5},\ y_0=-\tfrac{1}{5}$$ ✓

$$\Rightarrow\ 4x'^2+3x'y'+4y'^2-\tfrac{1}{5}=0 \quad ,\ [x',y']=[x-\tfrac{1}{5},\ y+\tfrac{1}{5}]$$

$$\Rightarrow\ \frac{(x'\cos(-\frac{\pi}{4})+y'\sin(-\frac{\pi}{4}))^2}{\left(\frac{\sqrt{8+\frac{2}{\sqrt5}}+\sqrt{8-\frac{2}{\sqrt5}}}{2}\right)^2}+\frac{(x'\sin(-\frac{\pi}{4})+y'\cos(-\frac{\pi}{4}))^2}{\left(\frac{\sqrt{8+\frac{2}{\sqrt5}}-\sqrt{8-\frac{2}{\sqrt5}}}{2}\right)^2}=1$$

+10.

i.e. $a=\frac{\sqrt{8+\frac{2}{\sqrt5}}+\sqrt{8-\frac{2}{\sqrt5}}}{2}$ ×, $b=\frac{\sqrt{8+\frac{2}{\sqrt5}}-\sqrt{8-\frac{2}{\sqrt5}}}{2}$ ×

$\theta=-\frac{\pi}{4}$. ✓

⇒ the normal form of the ellipse:

$$\frac{[(x-\frac15)\cos(-\frac{\pi}{4})+(y+\frac15)\sin(-\frac{\pi}{4})]^2}{\left(\frac{\sqrt{8+\frac{2}{\sqrt5}}+\sqrt{8-\frac{2}{\sqrt5}}}{2}\right)^2}+\frac{[(x-\frac15)\sin(-\frac{\pi}{4})-(y+\frac15)\cos(-\frac{\pi}{4})]^2}{\left(\frac{\sqrt{8+\frac{2}{\sqrt5}}-\sqrt{8-\frac{2}{\sqrt5}}}{2}\right)^2}=1$$

b) $G=4\rho\hat{h}_\rho+\cos\varphi\,\hat{h}_\varphi+(z-z\frac{1}{\rho}\cos\varphi)\hat{h}_z$ ✓

$$\operatorname{div} G=\frac{1}{\rho}\frac{\partial(\rho v_1)}{\partial\rho}+\frac{1}{\rho}\frac{\partial(v_2)}{\partial\varphi}+\frac{\partial v_3}{\partial z}=\frac{1}{\rho}\cdot 8\rho+\frac{1}{\rho}\cdot(-\sin\varphi)+(1-\frac{1}{\rho}\cos\varphi)$$ +

$$=9-\frac{\sin\varphi+\cos\varphi}{\rho}$$ ✓

$$\Rightarrow \text{flux}=\iiint_V \operatorname{div} G\,dV=\iiint_V\left(9-\frac{\sin\varphi+\cos\varphi}{\rho}\right)dV=\iiint_V 9\,dV-\iiint_V\frac{\sin\varphi+\cos\varphi}{\rho}\,dV$$

$=9\cdot\frac{1}{3}(\pi ab)\cdot 4-\iiint_V\frac{\sin\varphi+\cos\varphi}{\rho}\,dV$ what is dV? draw a fig may help

$12\pi ab-$

a) the ellipse can be written in a matrix form:

$$(x,y)\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + (x,y)\cdot\begin{pmatrix} e \\ f \end{pmatrix} + G = 0.$$

insert into the equation $\Rightarrow$

$$(x,y)\underbrace{\begin{pmatrix} 4 & \frac{3}{2} \\ \frac{3}{2} & 4 \end{pmatrix}}_{A}\cdot\begin{pmatrix} x \\ y \end{pmatrix} + (x,y)\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$

In order to determine the normal form of the ellipse, terms with "xy" should

i.e. find new $\bar{x}, \bar{y}$ which make A diagonal.

or $\quad U A U^T = D$ is diagonal

Thus we should calculate the eigen values & vectors of A:

$$\det\begin{vmatrix} 4-\mu & \frac{3}{2} \\ \frac{3}{2} & 4-\mu \end{vmatrix} = 0 \Rightarrow \mu = \frac{11}{2} \text{ or } \frac{5}{2}$$

$$E(\mu=\tfrac{11}{2}) = \left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right)^T, \quad E(\mu=\tfrac{5}{2}) = \left(\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\right)^T$$

$$\Rightarrow U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad D = UAU^T = \begin{pmatrix} \frac{11}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix} \cancel{= U(x,y)\cdot A\cdot U^T \begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\text{And } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

$$\Rightarrow (\bar{x},\bar{y})\cdot\begin{pmatrix} \frac{11}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix}\cdot\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} + (\bar{x},\bar{y})\cdot\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad \frac{11}{2}\bar{x}^2 + \frac{5}{2}\bar{y}^2 - \sqrt{2}\,\bar{y} = 0$$

$$\Rightarrow \quad \frac{\bar{x}^2}{\frac{2}{11}} + \frac{(\bar{y}-\frac{\sqrt{2}}{5})^2}{\frac{2}{5}} = \frac{1}{5}$$

$$\text{i.e.} \quad \frac{\bar{x}^2}{\frac{2}{55}} + \frac{(\bar{y}-\frac{\sqrt{2}}{5})^2}{\frac{2}{25}} = 1$$

Circulation: $\iint_S \text{curl}\, \underset{\sim}{F}^T \cdot \underset{\sim}{n}\, dA = \oint_C \underset{\sim}{F}\, d\underset{\sim}{r}$ (Stokes theorem)

the boundary of the spherical triangle (let A on the prime meridian):

$\underset{\sim}{C_1} = 1 \cdot \hat{h_r} + 0 \cdot \hat{h_\lambda} + \theta \cdot \hat{h_\theta}$ ✓ , $\theta \in [0, \frac{\pi}{2}]$

$\begin{pmatrix} |h_r| = 1 \\ |h_\lambda| = r\sin\theta \\ |h_\theta| = r \end{pmatrix}$

$\underset{\sim}{C_2} = 1 \cdot \hat{h_r} + \lambda \cdot \hat{h_\lambda} + \theta \cdot \hat{h_\theta}$ ✓ ~~where~~

where $(r\sin\theta\cos\lambda, r\sin\theta\sin\lambda, r\cos\theta)^T \cdot \left((1,0,0) \times (0, \frac{3}{5}, \frac{4}{5})\right) = 0.$

$\Rightarrow \sin\lambda = \frac{3}{4}\cot\theta \Rightarrow \lambda = \arcsin(\frac{3}{4}\cot\theta)$, $\theta \in [\arcsin\frac{3}{5}, \frac{\pi}{2}]$

$C_3 = 1 \cdot \hat{h_r} + \frac{\pi}{2} \cdot \hat{h_\lambda} + \theta \cdot \hat{h_\theta}$ ✓ $\theta \in [0, \arcsin\frac{3}{5}]$

ent vector:

+11

$\underset{\sim}{T_1} = 0 + 0 + 1 \cdot 1 \cdot \hat{h_\theta} = \hat{h_\theta}$ ✓

$\underset{\sim}{T_2} = 0 + 1 \frac{-\frac{3}{4} \cdot \frac{1}{\sin^2\theta}}{\sqrt{1-(\frac{3}{4}\cot\theta)^2}} \cdot 1\sin\theta \cdot \hat{h_\lambda} + 1 \cdot 1 \cdot \hat{h_\theta} = \frac{-3}{4} \cdot \frac{1}{\sin\theta\sqrt{1-(\frac{3}{4}\cot\theta)^2}} \hat{h_\lambda} + \hat{h_\theta}$ ✗

use Cartesian coordinate for $\underset{\sim}{T_2}$

$\underset{\sim}{T_3} = 0 + 0 + 1 \cdot 1 \cdot \hat{h_\theta} = \hat{h_\theta}$

$= r\cos\lambda\, \hat{h_\lambda} = 1 \cdot \sqrt{1-(\frac{3}{4}\cot\theta)^2}\, \hat{h_\lambda}$

$\oint_{C_1} \underset{\sim}{F}\, d\underset{\sim}{r} = \int \underset{\sim}{F}^T \underset{\sim}{T_1}\, d\theta = 0$ ✓

$\oint_{C_2} \underset{\sim}{F}\, d\underset{\sim}{r} = \int \underset{\sim}{F}^T \underset{\sim}{T_2}\, d\theta = \int_{\arcsin\frac{3}{5}}^{\frac{\pi}{2}} -\frac{3}{4} \frac{1}{\sin\theta}\, d\theta = -\frac{3}{4}\ln|\csc\theta - \cot\theta| \Big|_{\arcsin\frac{3}{5}}^{\frac{\pi}{2}} = -\frac{3}{4}\ln 3$ ✗

$\oint_{C_3} \underset{\sim}{F}\, d\underset{\sim}{r} = \int \underset{\sim}{F}^T \underset{\sim}{T_3}\, d\theta = \int 0$ ✓

$\oint_C \underset{\sim}{F}\, d\underset{\sim}{r} = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} = -\frac{3}{4}\ln 3$ ✗, ? incorrect answer?

should be choosen in such a way, that the function $\varphi(v, w) = \sin w \cdot \sinh v$ is one of the solutions