Advanced Mathematics

Lab 4: System of differential equations

Date of issue: 12 November 2018 Due date: 19 November 2018, 11:30 a. m.

GEODÄTISCHES INSTITUT
der Universität Stuttgart
anerkannt
82%

Family name: Wang

Given name: Yi

Student ID: 3371561

1a	1b	2a	2b	3	problem
10	16	19	18	19	points

Systems with constant coefficients

1. Given the matrix

$$\underline{\mathbf{B}} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 0 & 1 & 4 \\ 2 & 1 & 4 & 3 \\ 1 & 6 & 3 & 2 \end{pmatrix}$$

a) Calculate the determinant by the method of GRASSMANN-STEINITZ (10 points)

b) Determine the eigen vector of the smallest eigen value $\mu_{\min}$ and 'normalize' in such a way, that the nominators are integer and with the component $x_1 = 1$. (20 points)

2. Given the differential equation

$$y' = \begin{pmatrix} -9 & 23 & 5 \\ -5 & 7 & 5 \\ -1 & 13 & -3 \end{pmatrix} y + b$$

a) Determine the homogeneous solution and sort the columns of the solution $\underline{\mathbf{X}}$ in increasing order of the corresponding eigen values.

(19 points)

b) Solve the inhomogeneous problem for $b = \left(0, \ 0, \ \frac{1}{1+e^{2x}}\right)^{\mathsf{T}}$.

- *A complete matrix inversion is not necessary.*
- *The solution can be given in the form*

$$y = \underline{\mathbf{X}}\left(c + Y_p\right),$$

where the final matrix multiplication is not performed.

(26 points)

System with variable coefficients

3. Determine one vector y, which solves the coupled system with variable coefficients

$$y' = \underline{\mathbf{A}}(x) \cdot y = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \cdot y(x).$$

a) Find first an adequate transformation $\underline{\mathbf{U}}^\top \underline{\mathbf{A}}\, \underline{\mathbf{U}} = \underline{\mathbf{D}}$, to obtain a diagonal matrix $\underline{\mathbf{D}}$.

b) De-couple the system by the matrix $\underline{\mathbf{U}}$ and solve the ordinary differential equations of first order.

(final exam SS16, 25 points)

[V] inverse problem

iv The system $y' = \underline{\mathbf{A}}y + b$ has the homogeneous solution

$$y_h = \underline{\mathbf{X}}c = \begin{pmatrix} 0 & e^{2x} & e^{x} \\ e^{3x} & -2e^{2x} & 4e^{x} \\ -e^{3x} & e^{2x} & -2e^{x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

a) Keep the sorting of eigen values and vectors and find the corresponding matrix $\underline{\mathbf{A}}$.

b) Solve the inhomogeneous system for the vector $b = \left(\frac{1}{\cos^2 e^{-x}},\ 0,\ 0\right)^\top$. The solution can be given in the form

$$y = \underline{\mathbf{X}}\big(c + Y(x)\big),$$

where the final matrix multiplication is not performed.

Systems with constant coefficients

Grassmann Steinitz

1.

a). $\underline{B} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 0 & 1 & 4 \\ 2 & 1 & 4 & 3 \\ 1 & 6 & 3 & 2 \end{pmatrix}$ ✓

$\Rightarrow$ Grassmann-Steinitz Method

	x_1	x_2	x_3	x_4
y_1	4	3	2	1
y_2	3	0	1	4
y_3	2	1	4	3
y_4	1	6	3	2
	-4	-3	-2	*

✓

	x_1	x_2	x_3
y_2	-13	-12	-7
y_3	-10	-8	-2
y_4	-7	0	-1
	-7	0	*

	x_1	x_2
y_2	36	-12
y_3	4	-8
	*	2

✓

	x_2
y_2	6

✓

$$\Rightarrow \det(\underline{B}) = 60 \cdot (-1)^{2+2} \cdot 4 \cdot (-1)^{1+3} \cdot (-1) \cdot (-1)^{3+4} \cdot 1 \cdot (-1)^{1+4} = -240$$ ✓

$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$

$\Downarrow$

$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix}$

b) $|\underline{B} - \mu \underline{I}| = 0 \Rightarrow \begin{vmatrix} 4-\mu & 3 & 2 & 1 \\ 3 & -\mu & 1 & 4 \\ 2 & 1 & 4-\mu & 3 \\ 1 & 6 & 3 & 2-\mu \end{vmatrix} = 0.$ ✓

similar to a) using "G-S" Method

$\Rightarrow (\mu_1 - 10)(\mu_2 + \sqrt{13} + 1)(\mu_3 + 1 - \sqrt{13})(\mu_4 - 2) = 0$ ✓

$\Rightarrow \mu_{min} = -\sqrt{13} - 1$ ✓

Invert into the $(\underline{B} - \mu)\,\underline{x} = \underline{0}$

$$\Longrightarrow \overbrace{\begin{pmatrix} 4+\sqrt{13}+1 & 3 & 2 & 1 \\ 3 & \sqrt{13}+1 & 1 & 4 \\ 2 & 1 & 4+\sqrt{13}+1 & 3 \\ 1 & 6 & 3 & 2+\sqrt{13}+1 \end{pmatrix}}^{A} \cdot \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}^{y}$$ with $x_1 = 1$. ✓

$$\Rightarrow [A|y] \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{17\sqrt{13}+47}{(2\sqrt{13}+3)(5\sqrt{13}+17)} & 0 \\ 0 & 1 & 0 & -\frac{49\sqrt{13}+181}{(2\sqrt{13}+3)(5\sqrt{13}+17)} & 0 \\ 0 & 0 & 1 & \frac{\sqrt{13}+7}{5\sqrt{13}+17} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

with $x_1 = 1$ ✗

$$\Rightarrow V|_{\mu = \mu_{min}} = \begin{bmatrix} 1 \\ -\frac{49\sqrt{13}+181}{17\sqrt{13}+47} \\ -\frac{(2\sqrt{13}+3)(\sqrt{13}+7)}{17\sqrt{13}+47} \\ \frac{(2\sqrt{13}+3)(5\sqrt{13}+17)}{17\sqrt{13}+47} \end{bmatrix}$$ ✗

Step 5?

eigen value

$$\det(A-\mu I) = \begin{vmatrix} -9-\mu & 23 & 5 \\ -5 & 7-\mu & 5 \\ -1 & 13 & -3-\mu \end{vmatrix} = \begin{vmatrix} -9-\mu & 23 & 5 \\ 4+\mu & -16-\mu & 0 \\ -1 & 13 & -3-\mu \end{vmatrix} = \begin{vmatrix} -9-\mu & 18 & 5 \\ 4+\mu & -16-\mu & 0 \\ -1 & 16+\mu & -3-\mu \end{vmatrix}$$

$$= \begin{vmatrix} -9-\mu & 18 & 5 \\ 4+\mu & -16-\mu & 0 \\ 3+\mu & 0 & -3-\mu \end{vmatrix} = \begin{vmatrix} -9-\mu & 18 & -4-\mu \\ 4+\mu & -16-\mu & 4+\mu \\ 3+\mu & 0 & 0 \end{vmatrix} = (3+\mu)(2-\mu)(4+\mu) \checkmark$$

$\Rightarrow$ eigen values $\mu = 2, -3, -4.$ ✓

vert $\mu=-4$, $\Rightarrow A-\mu I = \begin{bmatrix} -5 & 23 & 5 \\ -5 & 11 & 5 \\ -1 & 13 & 1 \end{bmatrix} \sim \begin{bmatrix} -5 & 23 & 5 \\ 0 & -12 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_A(\mu=-4) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ✓

$\mu=-3 \Rightarrow A-\mu I = \begin{bmatrix} -6 & 23 & 5 \\ -5 & 10 & 5 \\ -1 & 13 & 0 \end{bmatrix} \sim \begin{bmatrix} -6 & 23 & 5 \\ 0 & -55 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_A(\mu=-3) = \begin{bmatrix} 13 \\ 1 \\ 11 \end{bmatrix}$ ✓

$\mu=2 \Rightarrow A-\mu I = \begin{bmatrix} -11 & 23 & 5 \\ -5 & 5 & 5 \\ -1 & 13 & -5 \end{bmatrix} \sim \begin{bmatrix} -11 & 23 & 5 \\ 0 & -60 & 30 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_A(\mu=2) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ ✓

the homogeneous solution:

$$y_h = \begin{bmatrix} e^{-4x} & 13e^{-3x} & 3e^{2x} \\ 0 & e^{-3x} & e^{2x} \\ e^{-4x} & 11e^{-3x} & 2e^{2x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \checkmark$$

$$\left(\quad \right) \begin{pmatrix} C_1' \\ C_2' \\ C_3' \end{pmatrix} = b$$

$$[X \mid b] = \left(\begin{array}{ccc|c} 1 & 13 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 11 & 2 & \frac{1}{1+e^{2x}} \end{array}\right) \overset{r_3 = -r_1 + 2r_2 + r_3}{\sim} \begin{pmatrix} 1 & 13 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{1+e^{2x}} \end{pmatrix}$$

+7.

$$\begin{cases} e^{-4x}C_1' + 13C_2'e^{-3x} + 3C_3'\cdot e^{2x} = 0 \\ C_2'\cdot e^{-3x} + C_3'\cdot e^{2x} = 0 \\ C_3'\cdot e^{2x} = \frac{1}{1+e^{2x}} \end{cases} \Rightarrow \begin{cases} C_1' = \frac{10e^{4x}}{1+e^{2x}} \checkmark \\ C_2' = -\frac{e^{3x}}{(1+e^{2x})} \checkmark \\ C_3' = \frac{1}{e^{2x}(1+e^{2x})} \checkmark \end{cases}$$

$C_1 = \int \frac{5\cdot e^{2x}}{1+e^{2x}} de^{2x} \overset{\text{substitution}}{=} 5(e^{2x} - \ln(e^{2x}+1))$ +4

$C_2 = \int -\frac{(e^x)^2}{1+(e^x)^2} de^x \overset{\text{substitution}}{=} \arctan(e^x) - e^x$ ✓ +4

$C_3 = \frac{1}{2}\int \frac{1}{(e^{2x})^2\cdot(1+e^{2x})} de^{2x} = \frac{1}{2}\int \left(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{t+1}\right)dt = \frac{1}{2}\left(\ln(e^{2x}+1) - 2x - \frac{1}{e^{2x}}\right)$ ✓

steps?

general solution:

$$\Rightarrow y = \begin{pmatrix} e^{-4x} & 13e^{-3x} & 3e^{2x} \\ 0 & e^{-3x} & e^{2x} \\ e^{-4x} & 11e^{-3x} & 2e^{2x} \end{pmatrix} \cdot \begin{pmatrix} C_1 + 5(e^{2x} - \ln(e^{2x}+ \\ C_2 + \arctan(e^x) - e^x \\ C_3 + \frac{1}{2}[\ln(e^{2x}+1) - 2x - e \end{pmatrix}$$

+3

Date of issue:

system with variable coefficients

$$y' = \underline{A}(x)\cdot y = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}\cdot y(x)$$

$A = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$

eigen value $\begin{vmatrix} 1-\mu & x \\ x & 1-\mu \end{vmatrix} = 0 \Rightarrow \mu = 1 \pm x$

eigen vector $[1,1]^T$, $[1,-1]^T$ normalize $\Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Since $\underline{A}$ is a 2x2 matrix, we assume $\underline{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$D = \begin{pmatrix} 1+x & \\ 0 & \end{pmatrix}$

$D = \underline{U}^T \underline{A}\,\underline{U} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} a^2+2acx+c^2 & ab+cd+(ad+bc)x \\ ab+cd+(ad+bc)x & b^2+2bdx+d^2 \end{pmatrix}$$

or we can use eigen vectors of A (orthogonal and maybe unitization) to build the orthogonal matrix $\underline{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

in this case $U^T = U^{-1}$, so Question. b) may be more simple to solve.

is diagonal $\Rightarrow \begin{cases} ab+cd = 0 \\ ad+bc = 0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=1 \\ c=-1 \\ d=1 \end{cases}$ is one solution.

$\det(U) = 2 \neq 1$

U should be orthogonal matrix

$\Rightarrow \underline{U} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\underline{D} = \frac{1}{2}\begin{pmatrix} 2-2x & 0 \\ 0 & 2+2x \end{pmatrix}$ ✗

$\underline{U} = \frac{\underline{U}}{\sqrt{\det(\underline{U})}}$

Assume $\underset{\sim}{Y} = \underline{U}\cdot\underset{\sim}{y}$, then $\underset{\sim}{Y}' = \underline{U}\cdot\underset{\sim}{y}' = \underline{U}\cdot\underline{A}\cdot\underset{\sim}{y} = \underline{U}\cdot\underline{A}\cdot\underline{U}^{-1}\cdot\underset{\sim}{Y}$

$$\Rightarrow \underset{\sim}{Y}' = \underline{U}\cdot\underline{A}\cdot\underline{U}^{-1}\cdot\underset{\sim}{Y}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}\cdot\frac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}{2}\cdot\underset{\sim}{Y}$$

$$= \frac{1}{2}\cdot\underline{U}\cdot\underline{A}\cdot\underline{U}^T\cdot\underset{\sim}{Y} = \left[\frac{1}{2}\cdot\underline{D}\right]\underset{\sim}{Y}$$

This is D upside down

Hence $\begin{cases} Y_1' = (1-x)Y_1 \\ Y_2' = (1+x)Y_2 \end{cases} \Rightarrow \begin{cases} \frac{1}{Y_1}dY_1 = (1-x)dx \\ \frac{1}{Y_2}dY_2 = (1+x)dx \end{cases} \Rightarrow \begin{cases} Y_1 = e^{x-\frac{1}{2}x^2+C_1} \\ Y_2 = e^{x+\frac{1}{2}x^2+C_2} \end{cases}$

$\Rightarrow \underset{\sim}{y} = \underline{U}^{-1}\cdot\underset{\sim}{Y} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{e^{x-\frac{1}{2}x^2+C_1} - e^{x-\frac{1}{2}x^2+C_2}}{2} \\ \frac{e^{x-\frac{1}{2}x^2+C_1} + e^{x-\frac{1}{2}x^2+C_2}}{2} \end{bmatrix}$ ✗ ✗

U is not correct

Another way: $\underset{\sim}{Y} = U^T\cdot\underset{\sim}{y} \Rightarrow Y' = D\,\underset{\sim}{Y}$

[V] inverse problem

4. (the opposite process has been shown in lecture).

a) from $\underline{X}$ we can get the eigen values $\{3,2,1\}$ ✓

and the eigen vectors $\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$ $(*)$ of $\underline{A}$.

let $P=(*)$ whose components are eigen vectors of $\underline{A}$.,

then we known $P^{-1}AP = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ whose diagonal elements are eigen values

$$\Rightarrow \quad A = PDP^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$ ✓

$$= \begin{bmatrix} \frac{5}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{4}{3} & -\frac{1}{3} & -\frac{10}{3} \\ \frac{2}{3} & \frac{5}{3} & \frac{14}{3} \end{bmatrix}$$ ✓

+19

b) $$[\underline{X} \mid b] = \left(\begin{array}{ccc|c} 0 & 1 & 1 & b_1 \\ 1 & -2 & 4 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 1 & b_1 \\ 1 & -2 & 4 & b \\ 0 & 0 & 3 & b_1 \end{array}\right)$$ ✓

$$\Rightarrow \begin{cases} 3C_3' \cdot e^x = b_1 \\ C_2' e^{2x} + C_3' e^x = b_1 \\ C_1' e^{3x} - 2\cdot C_2' + 4C_3' = 0 \end{cases} \quad \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = \int \frac{2}{3} \frac{e^{-2x}}{\cos^2 e^{-x}} dx = -\frac{2}{3} e^{-x} \tan e^{-x} - \frac{2}{3} \ln(\cos e^{-x}) \\ C_3 = -\frac{1}{3} \tan e^{-x} \end{cases}$$

-1

$= \int \frac{e^{-x}}{3\cos^2 e^{-x}} dx$

$$\Rightarrow \quad y = \begin{pmatrix} 0 & e^{2x} & e^x \\ e^{3x} & -2e^{2x} & 4e^x \\ -e^{3x} & e^{2x} & -2e^x \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 - \frac{2}{3} e^{-x} \tan e^{-x} - \frac{2}{3} \ln(\cos e^{-x}) \\ C_3 - \frac{1}{3} \tan e^{-x} \end{pmatrix}$$ ✓

+12