Advanced Mathematics

Lab 8: Work, potential and theorem of Green

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1	2a	2b	3a	3b	4a	4b	exercise points
10	13	16	10	X	4	26	

Work

1. Evaluate the work for moving a unit mass in the vector field $\mathbf{G} = \rho\hat{\boldsymbol{h}}_\rho + \rho\hat{\boldsymbol{h}}_\varphi + \sqrt{1-\rho^2}\hat{\boldsymbol{h}}_z$ along the curve $\boldsymbol{\Psi} : \left\{x \in \mathbb{R}^3 : \|x\| = 1,\ \sqrt{x^2+y^2} = (1+\cos\lambda), z \geq 0\right\}$. (16 points)

$G^T \cdot \dot{\Psi} dt$

2. Given the curvilinear coordinate system by the relationship

$$x = \alpha\beta$$
$$y = \frac{1}{2}(\alpha^2 - \beta^2)$$
$$z = \gamma$$

a) Calculate the 'frame vectors' $\hat{\boldsymbol{h}}_{q_i}$ and note down the gradient. (13 points)

b) For an adequate choice of the parameter μ, the vector field

$$G_\mu = \frac{1}{\alpha^2+\beta^2-\gamma^2}\left(\frac{\alpha\gamma}{\alpha^2+\beta^2}\hat{\boldsymbol{h}}_\alpha + \frac{\beta\gamma}{\alpha^2+\beta^2}\hat{\boldsymbol{h}}_\beta + \mu\sqrt{\alpha^2+\beta^2}\hat{\boldsymbol{h}}_\gamma\right)$$

is conservative. Determine μ and the corresponding potential $\Phi_\mu(\alpha, \beta, \gamma)$. (16 points)

$\nabla\phi = \frac{1}{|h_\alpha|}\cdot\frac{\partial\phi}{\partial\alpha}\hat{h}_\alpha + \frac{1}{|h_\beta|}\cdot\frac{\partial\phi}{\partial\beta}\cdot\hat{h}_\beta + \frac{1}{|h_r|}\frac{\partial\phi}{\partial r}\hat{h}_r$

Theorem of Green

pass the origin → $x = y \cdot t$ or $y = x t$

3. Given the domain $\mathcal{D} = \left\{(x, y) \in \mathbb{R}^2 : y^2(1-y^2) - x^2 \leq 0, y \geq 0\right\}$

a) Determine the enclosed area via line integrals (10 points)

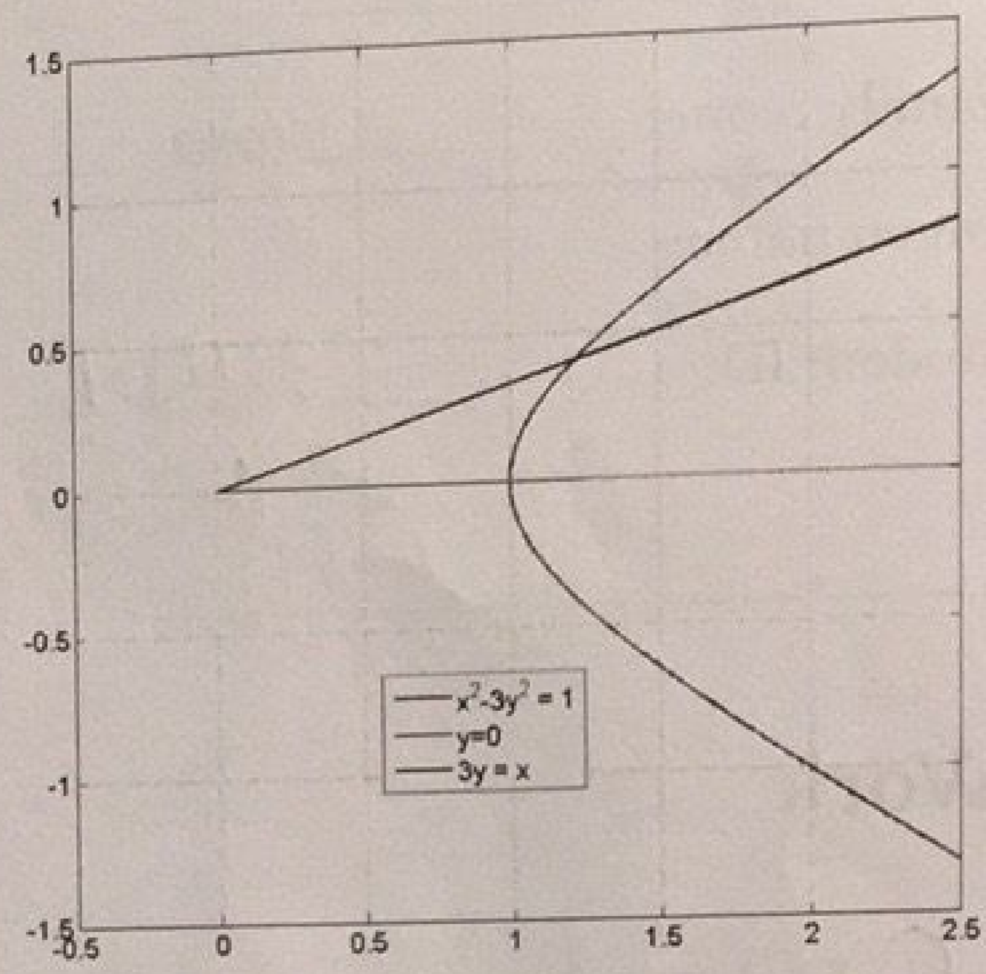
b) Calculate the effect of the Laplacian of the vector field $w = \left(x^3y,\ y^3\right)^\mathsf{T}$ within the enclosed area via line integral and surface integral. (15 points)

4. A generalized sector is defined as the 'triangular area' A, where 2 boundaries are given by straight lines g_1, g_2 through the origin, while the third boundary is given by a smooth curve $F(x, y) = 0$ without singularities.

   a) Verify in general, that straight lines g_i through the origin will never contribute to the area(!) in the theorem of Green (**sector formula of Leibniz**).

   b) Consider now hyperbola sector A with the boundaries $h : x^2 - 3y^2 = 1$, $f : y = 0$ and the straight lines $g : 3y = x$. Determine the x-coordinate of geometric center of A by choosing $F_1 = 0$ and $F_2 = \frac{x^2}{2A}$ in the 'line integral' of Green.
   For the parametrization and limits, the hyperbolic functions and their inverse might be helpful
   $\Longrightarrow$ for checking purposes: $X = \frac{\sqrt{2}}{3\text{arsinh}\frac{1}{\sqrt{2}}}$

   (30 points)



x-coordinate of Geometric Center:

$$\frac{\iint x\,ds}{\iint ds} = \iint \frac{x}{A}\,dxdy$$

$$\underline{F} = (0, \frac{x^2}{2A})$$

$$\partial_x F_2 - \partial_y F_1 = \frac{x}{A} - 0 = \frac{x}{A})$$

$$\iint \frac{x}{A}\,dxdy = \iint (\partial_x F_2 - \partial_y F_1)\,dxdy = \oint F_1 dx + \oint F_2 dy = \oint \underline{F} \cdot d\underline{r} = \oint \underline{F}^T \cdot \underline{\dot{r}} \cdot dt$$

h:

g:

f:

curve ψ: $\{ \|\underset{\sim}{x}\| = 1,\ \sqrt{x^2+y^2} = (1+\cos\lambda),\ z \geq 0 \}$.

Consider in cylindrical coordinates. $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$ ✓

$\Rightarrow \begin{cases} (\rho\cos\varphi)^2 + (\rho\sin\varphi)^2 + z^2 = 1 \\ \sqrt{(\rho\cos\varphi)^2 + (\rho\sin\varphi)^2} = 1+\cos\lambda \end{cases}$ ✓ ✓ $z \geq 0$.

t, and λ are both variables

$\Rightarrow \quad \rho = 1+\cos\lambda,\ \varphi = t,\ z = \sqrt{-\cos^2\lambda - 2\cos\lambda}$, $(\cos\lambda \leq 0)$ this is a circle $(0 \leq t \leq 2\pi)$

Tangent vector

$$\underset{\sim}{T} = \frac{\partial\rho}{\partial t}|h_\rho|\cdot\hat{h_\rho} + \frac{\partial\varphi}{\partial t}|h_\varphi|\cdot\hat{h_\varphi} + \frac{\partial z}{\partial t}|h_z|\cdot\hat{h_z}$$

$$= 0 + 1\cdot(1+\cos\lambda)\cdot\hat{h_\varphi} + 0$$

$$= (1+\cos\lambda)\,\hat{h_\varphi}$$ ✗

vector field along the curve

$$\underset{\sim}{G}(\underset{\sim}{\psi}) = (1+\cos\lambda)\,\hat{h_\rho} + (1+\cos\lambda)\,\hat{h_\varphi} + \sqrt{-\cos^2\lambda - 2\cos\lambda}\,\hat{h_z}$$

$\Rightarrow$ Work, $W = \int_0^{2\pi} \underset{\sim}{G}^T \cdot \underset{\sim}{T}\, dt = \int_0^{2\pi} (1+\cos\lambda)^2 \cdot dt = (1+\cos\lambda)^2 \cdot 2\pi$ ✗

+10

2.

$$\begin{cases} x = \alpha\beta \\ y = \frac{1}{2}(\alpha^2 - \beta^2) \\ z = r \end{cases}$$

a) $h_\alpha = (\beta, \alpha, 0)^T$, $|h_\alpha| = \sqrt{\alpha^2+\beta^2}$, $\hat{h_\alpha} = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\beta, \alpha, 0)^T$

$h_\beta = (\alpha, -\beta, 0)^T$, $|h_\beta| = \sqrt{\alpha^2+\beta^2}$, $\hat{h_\beta} = \frac{1}{\sqrt{\alpha^2+\beta^2}}(\alpha, -\beta, 0)^T$

$h_r = (0, 0, 1)^T$, $|h_r| = 1$, $\hat{h_r} = (0, 0, 1)^T$

gradient:

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial q_i}\hat{h_i} = \frac{1}{\sqrt{\alpha^2+\beta^2}}\frac{\partial f}{\partial \alpha}\hat{h_\alpha} + \frac{1}{\sqrt{\alpha^2+\beta^2}}\frac{\partial f}{\partial \beta}\hat{h_\beta} + \frac{\partial f}{\partial r}\hat{h_r}$$

b)

Because G_μ is conservative, $G = \nabla\phi$, ϕ is the potential

thus $\frac{1}{\alpha^2+\beta^2-r^2}\cdot\frac{\alpha r}{\alpha^2+\beta^2} = \frac{1}{\sqrt{\alpha^2+\beta^2}}\frac{\partial\phi}{\partial\alpha}$ ①

$\frac{1}{\alpha^2+\beta^2-r^2}\cdot\frac{\beta r}{\alpha^2+\beta^2} = \frac{1}{\sqrt{\alpha^2+\beta^2}}\frac{\partial\phi}{\partial\beta}$ ②

$\frac{1}{\alpha^2+\beta^2-r^2}\cdot\mu\sqrt{\alpha^2+\beta^2} = \frac{\partial\phi}{\partial r}$ ③

$$① \Rightarrow \phi = \int \frac{r}{\alpha^2+\beta^2-r^2}\cdot\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}d\alpha = \int\frac{r}{\alpha^2+\beta^2-r^2}\cdot d\sqrt{\alpha^2+\beta^2} = \int\frac{r}{t^2-r^2}dt = \frac{1}{2}\int\left(\frac{1}{t-r}-\frac{1}{t+r}\right)dt =$$

$$= \frac{1}{2}\cdot\ln\frac{t-r}{t+r}+C = \frac{1}{2}\cdot\ln\frac{\sqrt{\alpha^2+\beta^2}-r}{\sqrt{\alpha^2+\beta^2}+r}+C$$

this can also be acquired from ②.

$$\frac{d\phi}{dr} = \left(\frac{1}{2}\left[\ln(\sqrt{\alpha^2+\beta^2}-r) - \ln(\sqrt{\alpha^2+\beta^2}+r)\right]\right)' = \frac{1}{2}\left(\frac{-1}{\sqrt{\alpha^2+\beta^2}-r} - \frac{1}{\sqrt{\alpha^2+\beta^2}+r}\right) = \frac{-\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2-r^2}$$

combine this and ③ $\Rightarrow \mu = -1$.

and $\phi = \frac{1}{2}\ln\frac{\sqrt{\alpha^2+\beta^2}-r}{\sqrt{\alpha^2+\beta^2}+r}+C$ (C: const).

a) the origin is part of the curve, so we can assume $x = y \cdot t$.

then $y^2(1-y^2) - y^2t^2 = 0$.

$\Rightarrow \quad y = \sqrt{1-t^2} \qquad (-1 \le t \le 1)$

and $x = t \cdot \sqrt{1-t^2}$

$\Rightarrow$ enclosed area $A = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\oint x\dot{y} - y\dot{x}\,dt$

$= \left|\frac{1}{2}\int_{-1}^{1} yt\dot{y} - y\cdot(\dot{y}t + y)\,dt\right|$

$= \left|\frac{1}{2}\int_{-1}^{1} -y^2\,dt\right| = \left|\frac{1}{2}\int_{-1}^{1}(t^2-1)\,dt\right| = \left|\frac{1}{2}\left(\frac{t^3}{3} - t\right)\Big|_{-1}^{1}\right| = \frac{4}{3}$

b)

$\Delta = \nabla w = 3x^2y + 3y^2$

$\iint \Delta\,dx\,dy = \iint (3x^2y + 3y^2)\,dx\,dy =$

a)

$$A = \frac{1}{2}\oint x\,dy - y\,dx$$

if $y = kx$, then $dy = k\,dx$

$\Rightarrow A = \frac{1}{2}\oint x \cdot k\,dx - kx\,dx = 0.$

$\Rightarrow$ verified

b)

first calculate the Area A, based on a) we know ~~A~~ has only ~~relationship~~ relationship with $h: x^2 - 3y^2 = 1$. set $x = \cosh t$, $y = \frac{1}{\sqrt{3}}\sinh t$. $(0 \le t \le \operatorname{arsinh}\frac{\sqrt{2}}{2})$

$\rightarrow$ this from the interse… points of 3 li…

$$A = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\oint \cosh t \cdot \frac{1}{\sqrt{3}}\cosh t - \frac{1}{\sqrt{3}}\sinh t \cdot \sinh t\,dt$$

$$= \frac{1}{2}\int_0^{\operatorname{arsinh}\frac{\sqrt{2}}{2}} \frac{\sqrt{3}}{3}\,dt = \frac{\sqrt{3}}{6}\operatorname{arsinh}\frac{\sqrt{2}}{2}$$

the x-coordinate of geometric centre of A

$$X = \frac{\iint x\,ds}{\iint ds} = \frac{\iint x\,dxdy}{\iint dxdy} = \iint \frac{x}{A}\,dxdy$$

set $F_1 = 0$, $F_2 = \frac{x^2}{2A}$, so $X = \iint (\partial_x F_2 - \partial_y F_1)\,dxdy = \oint \underline{F}\,d\underline{r}$

Consider line $h: x^2 - 3y^2 = 1$ or $(\cosh t, \frac{1}{\sqrt{3}}\sinh t)^T$,

tangent vector $\underline{T}_h = (\sinh t, \frac{1}{\sqrt{3}}\cosh t)^T$.

$$\Rightarrow \int_h \underline{F}\,d\underline{r} = \int_0^{\operatorname{arsinh}\frac{\sqrt{2}}{2}} \underline{F}^T \cdot \underline{T}\,dt = \int_0^{\operatorname{arsinh}\frac{\sqrt{2}}{2}} \left(\frac{\cosh^2 t}{2A} \cdot \frac{1}{\sqrt{3}}\cosh t\right)dt = \int_0^{\operatorname{arsinh}\frac{\sqrt{2}}{2}} \frac{\sqrt{3}}{6A}\cosh^3 t\,dt$$

$$= \int_0^{\operatorname{arsinh}\frac{\sqrt{2}}{2}} \frac{\sqrt{3}}{6A}(\sinh^2 t + 1)\,d\sinh t = \frac{\sqrt{3}}{6A}\left(\frac{\sinh^3 t}{3} + \sinh t\right)\Big|_0^{\operatorname{arsinh}\frac{\sqrt{2}}{2}} = \frac{7}{72A}\sqrt{6}$$

range of integration?

… the height h and the base area B.

- Split the volume integral into two parts consider line f: $y=0$. or $(x,0)^T$.

tangent vector $\underline{T}_f = (1,0)^T$

$$\int_f \underline{E}\, d\underline{r} = \int_f F^T \cdot \underline{T}\, dx = 0 .$$

consider line g: $3y = x$. or $(x, \frac{1}{3}x)^T$. $\quad 0 \le x \le \frac{\sqrt{6}}{2}$

tangent vector $\underline{T}_g = (1 . \frac{1}{3})^T$

$$\int_g \underline{E}\, d\underline{r} = \int_g \underline{E}^T \cdot \underline{T}_g\, dx = \int_{\frac{\sqrt{6}}{2}}^{0} \frac{x^2}{6A}\, dx = -\frac{x^3}{18A}\Big|_0^{\frac{\sqrt{6}}{2}} = -\frac{\sqrt{6}}{24A}$$

$$\Rightarrow X = \int_h \underline{E}\, d\underline{r} + \int_f \underline{E}\, d\underline{r} + \int_g \underline{E}\, d\underline{r}$$

$$= \frac{7}{72A}\sqrt{6} + 0 + \left(-\frac{\sqrt{6}}{24A}\right) = \frac{\sqrt{6}}{18A} = \frac{\sqrt{6}}{18 \cdot \frac{\sqrt{3}}{6} \operatorname{arsinh} \frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{3 \operatorname{arsinh} \frac{\sqrt{2}}{2}}$$ check