[Appendix 1]. Trigonometric function & hyperbolic function.

1. fundamental trigonometric formulas.

① $\tan\alpha\cot\alpha = 1$, $\sin\alpha\csc\alpha = 1$, $\cos\alpha\sec\alpha = 1$.

② $\sin^2\alpha + \cos^2\alpha = 1$, $1+\tan^2\alpha = \sec^2\alpha = \frac{1}{\cos^2\alpha}$, $1+\cot^2\alpha = \csc^2\alpha = \frac{1}{\sin^2\alpha}$

③ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$, $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

④ $\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, $\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, $\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\tan\alpha+\tan\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$

⑤ $\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$ $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$\cos 2\alpha = 2\cos^2\alpha - 1 = 1-2\sin^2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$,

⑥ $\sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$, $\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$, $\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

⑦ $a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\sin(\alpha+\beta)$, $\beta = \arctan\frac{b}{a}$.

2. hyperbolic function

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$, $\tanh x = \frac{\sinh x}{\cosh x}$, $\coth x = \frac{\cosh x}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$.

hyperbolic $\Leftrightarrow$ trigonometric : items with "$\sinh\alpha\cdot\sinh\beta$" time "$-1$".

3. derivative

① $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \frac{1}{\cos^2 x} = 1+\tan^2 x$, $(\cot x)' = -\frac{1}{\sin^2 x}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $(\arctan x)' = \frac{1}{1+x^2}$, $(\text{arccot}\, x)' = -\frac{1}{1+x^2}$

② $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$, $(\tanh x)' = \text{sech}^2 x = 1-\tanh^2 x = \frac{1}{\cosh^2 x}$, $(\coth x)' = -\frac{1}{\sinh^2 x}$

$(\text{arsinh}\, x)' = \frac{1}{\sqrt{x^2+1}}$, $(\text{arcosh}\, x)' = \frac{1}{\sqrt{x^2-1}}$, $(\text{artanh}\, x)' = \frac{1}{1-x^2}$, $(\text{arcoth}\, x)' = \frac{1}{1-x^2}$

I. Differential equations

1. Reduction of order

Consider the linear homogeneous ODE $y''+py'+qy=0$.

if one solution y_1 is known or can be guessed, $\Rightarrow$ usually $y_1 = x^t$, we can assume $y_2 = u(x)\, y_1$

insert into the ODE and get: $y_2 = uy_1 = \int \frac{1}{y_1^2} e^{-\int p dx} dx \cdot y_1$.

2. Linear homogeneous ODE with constant coefficients

Consider the linear homo-ODE with constant coefficients $y''+ay'+by=0$.

it has solution with the form $y = e^{\lambda x}$. (insert into the ODE)

characteristic equation: $\lambda^2 + a\lambda + b = 0$.

$\Rightarrow$ two roots $\lambda_1 = \frac{-a+\sqrt{a^2-4b}}{2}$, $\lambda_2 = \frac{-a-\sqrt{a^2-4b}}{2}$.

① "$\lambda_1 \neq \lambda_2$" & real.

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

② "$\lambda_1 = \lambda_2$" & real ($\lambda_1 = \lambda_2 = -\frac{a}{2}$).

use "reduction of order" to get $y_2 \Rightarrow y = C_1 e^{-\frac{a}{2}x} + C_2 x e^{-\frac{a}{2}x}$

③ $\lambda_1 . \lambda_2$ conjugated complex ($\lambda_1 = -\frac{a}{2} + wi$, $\lambda_2 = -\frac{a}{2} - wi$)

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

or using 'euler formular': $y = C_1 \cdot (y_1 + y_2) + C_2 \cdot (y_1 - y_2)$

$$= e^{-\frac{a}{2}x}(C_1 \cos wx + C_2 \sin wx)$$

3. Euler-Cauchy ODE

Consider the Euler ODE with normalized form: $x^2 y'' + axy' + by = 0$.

it has solution with the form $y = x^\lambda$. (insert into the ODE)

characteristic equation: $\lambda^2 + (a-1)\lambda + b = 0$.

$\Rightarrow$ two roots: $\lambda_1 = \frac{1-a+\sqrt{(a-1)^2-4b}}{2}$, $\lambda_2 = \frac{1-a-\sqrt{(a-1)^2-4b}}{2}$

again there are 3 cases.

$\begin{matrix} 0 & e^{2x} & e & \\ 3x & -2e^{2x} & 4e^{x} & b_2 \\ & & 2e^{x} & b_3 \end{matrix}$

ar ODE

① two real roots

$$y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$$

② real double roots

again use 'reduction of order' to get $y_2 \Rightarrow y = C_1 x^{\frac{1-a}{2}} + C_2 \ln x \cdot x^{\frac{1-a}{2}}$

③ conjugated complex roots ($\lambda = \frac{1-a}{2} \pm wi$)

$$y = C_1 y_1 + C_2 y_2 = C_1 \cdot \frac{y_1 + y_2}{2} + C_2 \cdot \frac{y_1 - y_2}{2}$$
$$= x^{\frac{1-a}{2}} \left(C_1 \cos(w \ln x) + C_2 \sin(w \ln x) \right)$$

emark]

1°. Euler ODE can be transformed to constant-coefficients-ODE with $x = e^t$.

$\Rightarrow \ddot{y} + (a-1)\dot{y} + by = 0$ ($t = \ln x$).

2°. offline Euler ODE: $(Ax+B)^2 y'' + a \cdot (Ax+B) y' + by = 0$.

4. Wronskian, Cramer's rule and substitution of argument.

1). Wronskian of two functions y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

2) set up ODE with two solutions y_1, y_2:

Assume $A(x) y'' + B(x) y' + C(x) y = 0$, divided by "$-C(x)$"

$\Rightarrow \bar{A} y'' + \bar{B} y' = y$

$\Rightarrow \begin{cases} \bar{A} y_1'' + \bar{B} y_1' = y_1 \\ \bar{A} y_2'' + \bar{B} y_2' = y_2 \end{cases}$ use Cramer's rule $\Rightarrow$ $\bar{A} = \dfrac{\det\begin{pmatrix} y_1 & y_1' \\ y_2 & y_2' \end{pmatrix}}{\det\begin{pmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{pmatrix}}$, $\bar{B} = \dfrac{\det\begin{pmatrix} y_1'' & y_1 \\ y_2'' & y_2 \end{pmatrix}}{\det\begin{pmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{pmatrix}}$

3) substitution of argument

In some cases, to solve an ODE we can substitute variables like $x = t^k$, $x = e^t$. ...

4) if the Wronskian of two functions y_1 and y_2 $W(y_1, y_2)$ is equal to 0.

then these two functions are linear dependent.

· Nonhomogeneous linear ODE

Consider a non-homogeneous linear ODE: $y''+p(x)y'+q(x)y=r(x)$, $r(x)\neq 0$.

Its general solution is $y=y_h+y_p$

where y_h is the general solution of " $y''+py'+qy=0$ ".

y_p is a particular solution of " $y''+py'+qy=r$ ".

emark] if y_1, y_2 are solutions of the non-homo ODE, (y_1-y_2) is solution of homo-ODE.

① Method of variation of (constants) parameters

first normalize the ODE: $y''+p(x)y'+q(x)y=r(x)$.

then get the general solution of the homo ODE. $y_h=C_1y_1(x)+C_2y_2(x)$

to get y_p, assume $y_p=C_1(x)\,y_1(x)+C_2(x)\,y_2(x)$.

insert into the ODE, together with another condition $C_1'y_1+C_2'y_2=0$

$$\Rightarrow \begin{cases} C_1'y_1+C_2'y_2=0 \\ C_1'y_1'+C_2'y_2'=r \end{cases}$$

solve this by Cramer's rule:

$$C_1'=\frac{\det\begin{pmatrix}0 & y_2\\ r & y_2'\end{pmatrix}}{\det\begin{pmatrix}y_1 & y_2\\ y_1' & y_2'\end{pmatrix}} \qquad C_2'=\frac{\det\begin{pmatrix}y_1 & 0\\ y_1' & r\end{pmatrix}}{\det\begin{pmatrix}y_1 & y_2\\ y_1' & y_2'\end{pmatrix}}$$

$$C_1'=\frac{-ry_2}{y_1y_2'-y_1'y_2} \qquad C_2'=\frac{ry_1}{y_1y_2'-y_1'y_2}$$

$$\Rightarrow y_p=\int\frac{-ry_2}{y_1y_2'-y_1'y_2}dx\cdot y_1+\int\frac{ry_1}{y_1y_2'-y_1'y_2}dx\cdot y_2$$

Hence we get the final solution: $y=y_h+y_p$

② Method of undetermined coefficients

Consider some special cases of ODE, they have special choices of y_p:

$$r(x)=\begin{cases} e^{rx} \\ x^n \\ \cos(\omega x),\ \sin(\omega x) \\ e^{\alpha x}\cos(\omega x),\ e^{\alpha x}\sin(\omega x) \end{cases} \Rightarrow \quad y_p(x)=\begin{cases} Ce^{rx} \\ k_n x^n + k_{n-1}x^{n-1}+\cdots+k_0 \\ k\cos(\omega x)+m\sin(\omega x) \\ e^{\alpha x}(k\cos\omega x + m\sin\omega x). \end{cases}$$

Modification rule: If y_p happens to be a solution of homo-ODE, try $x\cdot y_p$, $x^2\cdot y_p$,

Grassmann-Steinitz method

For determinant of matrix:

e.g. $\quad \underline{B}=\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 0 & 1 & 4 \\ 2 & 1 & 4 & 3 \\ 1 & 6 & 3 & 2 \end{pmatrix}$

$\Rightarrow$ ①

	x_1	x_2	x_3	x_4
y_1	4	3	2	1
y_2	3	0	1	4
y_3	2	1	4	3
y_4	1	6	3	2
	-3	0	*	-4

②

	x_1	x_2	x_4
y_1	-2	3	-7
y_3	-10	1	-13
y_4	-8	6	-10
	10	*	13

③

	x_1	x_4
y_1	28	32
y_4	52	68
	*	$-\frac{8}{7}$

④

	x_4
y_4	$\frac{60}{7}$

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$\Rightarrow \det(\underline{B}) = \frac{60}{7}\cdot(-1)^{4+4}\cdot 28\cdot(-1)^{1+1}\cdot 1\cdot(-1)^{2+3}\cdot 1\cdot(-1)^{2+3} = -240.$$

② For eigen vectors:

first insert μ into $A-\mu I$, e.g. $\begin{pmatrix} -4 & -1 & -1 \\ -4 & -10 & -10 \\ 2 & 5 & 5 \end{pmatrix}$

$\Rightarrow$

	x_1	x_2	x_3
y_1	-4	-1	-1
y_2	-4	-10	-10
y_3	2	5	5
	-4	-1	*

	x_1	x_2
x_3	-4	-1
y_2	36	0
y_3	18	0
	*	0

	x_2	*
x_3	-1	
y_2	0	
x_1	0	

$\Rightarrow \underset{\sim}{E} = [0, 1, -1]^T$

7. ODE systems

1) Consider a non-homogeneous ODE system with first order:

$$\underline{y}' = \underline{A}(x)\,\underline{y} + \underline{b}(x)$$

① Solution of the homogeneous ODE: $\underline{y}' = \underline{A}(x)\cdot\underline{y}$

the solution has a form of $\underline{y} = \underline{v}\,e^{\mu x}$, insert into the ODE

and get $(A-\mu I)\,\underline{v} = \underline{0}$.

$\Rightarrow$ calculate the eigen values and eigen vectors of A

and get the solution $\underline{y}_h = \sum_{i=1}^{n} C_i \underline{v}_i e^{\mu_i x}$

② Variation of constants for the non-homogeneous ODE system.

After finding the solution of the homo-ODE system

$$\underline{y}_h = C_1\underline{y}_1 + C_2\underline{y}_2 + \dots + C_n\underline{y}_n.$$

Assume the non-homo solution to be of the form

$$\underline{y}_p = C_1(x)\underline{y}_1 + C_2(x)\underline{y}_2 + \dots + C_n(x)\underline{y}_n.$$ insert into the ODE

$\Rightarrow$ $\underline{y}_p' = [\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n]\,\underline{C}'(x) + \underline{A}\,\underline{y}_p$

$\Rightarrow$ $\underline{y}_p' - \underline{A}\,\underline{y}_p = [\underline{y}_1, \underline{y}_2, \dots, \underline{y}_n]\,\underline{C}'(x) = \underline{b}(x)$

solve the equation system $\underline{X}\cdot\underline{C}' = \underline{b}$, $\underline{C}' = \underline{X}^{-1}\cdot\underline{b}$.

$\Rightarrow$ $\underline{y}_p = \int \underline{C}'(x)\,dx$

$\Longrightarrow$ general solution for the non-homo ODE system:

$$\underline{y} = \underline{y}_h + \underline{y}_p$$

2) If $\underline{b}(x)$ consist only sine/cosine, polynomial and/or exponential, the concept "undetermined parameters" can be used again.

e.g. $\underline{b}(x) = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$, then solution $y = \begin{bmatrix} A\cos x + B\sin x \\ C\cos x + D\sin x \end{bmatrix}$.

) ODE systems with non-constant coefficients

① Every linear ODE of order n can be re-written into a 1st order system

In case of 2nd order ODE: $y'' + p(x) y' + q(x) y = r(x)$.

re-write $\Rightarrow \begin{cases} u = y \\ v = y' \end{cases}$

$\Rightarrow \begin{cases} u' = v \\ v' = -p(x) v - q(x) u + r(x) \end{cases} \Rightarrow \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -q(x) & -p(x) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ r(x) \end{pmatrix}$

② system with variable coefficients

Consider an ODE system $\underset{\sim}{y}' = \underline{A}\, \underset{\sim}{y}$, where $\underline{A}$ is symmetric and non-constan

we can transform this ODE system into a system with constant coefficients

$\Rightarrow$ calculate the eigen values and eigen vectors of $\underline{A}$.

let $\underline{U}$ be the normalized eigen vector matrix.

let $\underline{D}$ be the eigen value matrix.

$\Rightarrow \quad \underline{U}^T \underline{A}\, \underline{U} = \underline{D}$, where $\underline{D}$ is a diagonal matrix $\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$.

$\Rightarrow$ Assume $Y = \underline{U}^T \underset{\sim}{y}$, then $Y' = \underline{U}^T \cdot \underset{\sim}{y}' = \underline{U}^T \cdot \underline{A} \cdot \underset{\sim}{y} = \underline{U}^T \underline{A} \cdot \underline{U}^{T-1} \cdot \underset{\sim}{Y} = \underline{U}^T \underline{A} \underline{U}^{*} \underset{\sim}{Y} = \underline{D}\, \underset{\sim}{Y}$

$\Rightarrow \begin{cases} Y_1' = \mu_1 Y_1 \\ Y_2' = \mu_2 Y_2 \end{cases} \Rightarrow$ get Y_1 and Y_2.

$\Rightarrow$ final solution $\underset{\sim}{y} = \underline{U}^{*} \cdot \underset{\sim}{Y}$

k] generally, if matrix $\underline{P}$ is a col matrix of eigen vectors of $\underline{A}$,
and $\underline{D}$ is a diagonal matrix of eigen values of $\underline{A}$

then $\underline{P}^{-1} \underline{A}\, \underline{P} = \underline{D}$

8. Numerical Solution

1) Power series method

Consider the normalized ODE $y''+py'+qy=0$.

if p, q are continuous and have a convergent power series

$$p=\sum_{k=0}\frac{p^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$q=\sum_{k=0}\frac{q^{(k)}(x_0)}{k!}(x-x_0)^k$$

Then the solution of the ODE might have a power series expansion

$$y(x)=\sum_{k=0}^{\infty}a_k(x-x_0)^k$$

insert $y''=\sum_{k=2}^{\infty}a_k k(k-1)(x-x_0)^{k-2}$, $y'=\sum_{k=1}^{\infty}a_k k(x-x_0)^{k-1}$ and y into the ODE

and re-order by power of $(x-x_0)^k$, compare the coefficients to get the recursion formula and some initial values

$\Rightarrow$ finally get the solution y.

2) Numerical integration

① explicit Euler method $u_{i-1}'=f(x_{i-1}, y_{i-1})$.

$$u_i=u_{i-1}+h\cdot f(x_{i-1}, y_{i-1})$$

② implicit Euler method $u_i'=f(x_i, y_i)$

$$u_i=u_{i-1}+h\cdot f(x_i, y_i).$$

II. Vector analysis in curvilinear coordinates.

1. Cartesian coordinate system: $f = f(x,y,z)$. $\underline{v} = v_1(x,y,z)\underline{i} + v_2(x,y,z)\underline{j} + v_3(x,y,z)\underline{k}$.
$= X\underline{i} + Y\underline{j} + Z\underline{k}$.

Curvilinear coordinate system: $f = f(q_1,q_2,q_3)$ $\underline{v} = v_1(q_1,q_2,q_3)\hat{\underline{h_1}} + v_2(q_1,q_2,q_3)\hat{\underline{h_2}} + v_3(q_1,q_2,q_3)\hat{\underline{h_3}}$

Relationship between Cartesian & Curvilinear:

$$\begin{cases} x = x(q_1,q_2,q_3) \\ y = y(q_1,q_2,q_3) \\ z = z(q_1,q_2,q_3) \end{cases}, \quad \begin{cases} \underline{h_1} = \frac{\partial P}{\partial q_1} = \left[\frac{\partial x}{\partial q_1}, \frac{\partial y}{\partial q_1}, \frac{\partial z}{\partial q_1}\right]^T \\ \underline{h_2} = \frac{\partial P}{\partial q_2} = \left[\frac{\partial x}{\partial q_2}, \frac{\partial y}{\partial q_2}, \frac{\partial z}{\partial q_2}\right]^T \\ \underline{h_3} = \frac{\partial P}{\partial q_3} = \left[\frac{\partial x}{\partial q_3}, \frac{\partial y}{\partial q_3}, \frac{\partial z}{\partial q_3}\right]^T \end{cases}, \quad \begin{cases} \hat{\underline{h_1}} = \frac{\underline{h_1}}{|\underline{h_1}|} \\ \hat{\underline{h_2}} = \frac{\underline{h_2}}{|\underline{h_2}|} \\ \hat{\underline{h_3}} = \frac{\underline{h_3}}{|\underline{h_3}|} \end{cases}$$

2. Common curvilinear coordinates.

1°. spherical coordinates (r, λ, Δ). $(\Delta = 90° - \phi)$.

$$\begin{cases} x = r\sin\Delta\cos\lambda \\ y = r\sin\Delta\sin\lambda \\ z = r\cos\Delta \end{cases}, \quad \begin{cases} \underline{h_r} = [\cos\lambda\sin\Delta, \sin\lambda\sin\Delta, \cos\Delta]^T \\ \underline{h_\lambda} = [-r\sin\Delta\sin\lambda, r\sin\Delta\cos\lambda, 0]^T \\ \underline{h_\Delta} = [r\cos\lambda\cos\Delta, r\sin\lambda\cos\Delta, -r\sin\Delta]^T \end{cases}, \quad \begin{cases} |\underline{h_r}| = 1 \\ |\underline{h_\lambda}| = r\sin\Delta \\ |\underline{h_\Delta}| = r \end{cases}, \quad \begin{cases} \hat{\underline{h_r}} = [\cos\lambda\sin\Delta, \sin\lambda\sin\Delta, \cos\Delta]^T \\ \hat{\underline{h_\lambda}} = [-\sin\lambda, \cos\lambda, 0]^T \\ \hat{\underline{h_\Delta}} = [\cos\lambda\cos\Delta, \sin\lambda\cos\Delta, -\sin\Delta]^T \end{cases}$$

$dV = r^2\sin\Delta\, dr\, d\lambda\, d\Delta$

2°. cylindrical coordinates (ρ, ϕ, z).

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases}, \quad \begin{cases} \underline{h_\rho} = [\cos\phi, \sin\phi, 0]^T \\ \underline{h_\phi} = [-\rho\sin\phi, \rho\cos\phi, 0]^T \\ \underline{h_z} = [0, 0, 1]^T \end{cases}, \quad \begin{cases} |\underline{h_\rho}| = 1 \\ |\underline{h_\phi}| = \rho \\ |\underline{h_z}| = 1 \end{cases}, \quad \begin{cases} \hat{\underline{h_\rho}} = [\cos\phi, \sin\phi, 0]^T \\ \hat{\underline{h_\phi}} = [-\sin\phi, \cos\phi, 0]^T \\ \hat{\underline{h_z}} = [0, 0, 1]^T \end{cases}$$

$dV = \rho\, d\rho\, d\phi\, dz$

[Remark] transformation between Cartesian & curvilinear.

scalar: $f(x,y,z) =$ $\qquad f(q_1,q_2,q_3) = f(x(q_1,q_2,q_3), y(q_1,q_2,q_3), z(q_1,q_2,q_3))$.

vector: $\begin{bmatrix} \underline{h_1} \\ \underline{h_2} \\ \underline{h_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2} \\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3} \end{bmatrix} \cdot \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix}$ $\qquad \begin{bmatrix} \underline{i} \\ \underline{j} \\ \underline{k} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2} \\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3} \end{bmatrix}^{-1} \cdot \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} \cdot \begin{bmatrix} \hat{\underline{h_1}} \\ \hat{\underline{h_2}} \\ \hat{\underline{h_3}} \end{bmatrix}$

3. Tangential vector and arc length.

1°. Tanget vector : $\underset{\sim}{T} = \lim_{\Delta t \to 0} \frac{\psi(t+\Delta t) - \psi(t)}{\Delta t}$

Cartesian coordinates , $\underset{\sim}{T} = x'(t)\,\underset{\sim}{i} + y'(t)\,\underset{\sim}{j} + z'(t)\,\underset{\sim}{k}$

curvilinear coordinates , $\underset{\sim}{T} = \sum_{i=1}^{3} \dot{q}_i \cdot |h_i| \cdot \hat{h_i}$

2°. arc length : $S = \int_a^b \sqrt{T^T T}\, dt$.

Cartesian coordinates , $s = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$

curvilinear coordinates , $s = \int_a^b \sqrt{\dot{q}_1^2 h_1^2 + \dot{q}_2^2 h_2^2 + \dot{q}_3^2 h_3^2}\, dt$

[Remark]
① $\frac{d\psi}{ds} = \frac{d\psi}{dt} \cdot \frac{dt}{ds} = \frac{\psi'}{\sqrt{T^T T}} = \frac{T}{\sqrt{T^T T}} \Rightarrow \left|\frac{d\psi}{ds}\right| = 1$

② if $t = s$, then the tangent vector is normalized.

4. Gradient of a scalar field

the gradient of a scalar field $f(x, y, z)$

in Cartesian Coordinates : $\nabla f = \text{grad}\, f = \frac{\partial f}{\partial x}\underset{\sim}{i} + \frac{\partial f}{\partial y}\underset{\sim}{j} + \frac{\partial f}{\partial z}\underset{\sim}{k} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^T$

in Curvilinear Coordinates : $\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{h_1} + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{h_2} + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{h_3}$

if $\underset{\sim}{G} = \nabla f$, G is the vector field . then f is called the potential of $\underset{\sim}{G}$, and it is

[Remark] Laplace equation : $\nabla^2 V = \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$.

5. Divergence and curl of a vector field

$\underset{\sim}{V} = V_1(x,y,z)\,\underset{\sim}{i} + V_2(x,y,z)\,\underset{\sim}{j} + V_3(x,y,z)\,\underset{\sim}{k}$

1°. divergence of $\underset{\sim}{V}$: $\text{div}\, \underset{\sim}{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

in cylindrical coordinates : $\text{div}\, \underset{\sim}{V} = \frac{1}{\rho} \cdot \frac{\partial}{\partial \rho}(\rho \cdot V_1) + \frac{1}{\rho} \cdot \frac{\partial V_2}{\partial \varphi} + \frac{\partial V_3}{\partial z}$ (ρ, φ, z)

in spherical coordinates : $\text{div}\, \underset{\sim}{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, V_2) + \frac{1}{r\sin\theta}\frac{\partial V_3}{\partial \lambda}$ (r, θ, λ)

[Remark] $\text{div}(\text{grad}\, V) = \nabla^2 V = \Delta V$

2°. curl of $\underset{\sim}{V}$: $\text{curl } \underset{\sim}{V} = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{pmatrix}$

in cylindrical coordinates: (ρ, φ, z)

$$\text{curl } \underset{\sim}{V} = \left(\frac{1}{\rho}\frac{\partial V_3}{\partial \varphi} - \frac{\partial V_2}{\partial z}\right)\hat{h_\rho} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial \rho}\right)\hat{h_\varphi} + \left(\frac{1}{\rho}\frac{\partial(\rho V_2)}{\partial \rho} - \frac{1}{\rho}\frac{\partial V_1}{\partial \varphi}\right)\hat{h_z}$$

in spherical coordinates: (r, θ, λ)

$$\text{curl } \underset{\sim}{V} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta V_3) - \frac{\partial V_2}{\partial \lambda}\right)\hat{h_r} + \left[\frac{1}{r\sin\theta}\frac{\partial V_1}{\partial \lambda} - \frac{1}{r}\frac{\partial}{\partial r}(rV_3)\right]\hat{h_\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rV_2) - \frac{1}{r}\frac{\partial V_1}{\partial \theta}\right)\hat{h}$$

[Theorem] conservative fields are curl-free: $\text{curl}(\text{grad } v) = 0$

curl of a vector field is divergence-free: $\text{div}(\text{curl } v) = 0$.

3°. Line integrals

a line integral of a vector field along a curve C:

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_a^b \underset{\sim}{F}(\underset{\sim}{r}(t))^T \cdot \underset{\sim}{r}'(t)\, dt$$, $r'(t)$ is the tangent vector of the curve

[Remark] in a conservative field, the line integral for a curve is path-independent.

that is to say, the Work done by a constant force $\underset{\sim}{F}$ along a curve C.

1st tangent vectors of the curve $r'(t)$.

2nd field along the curve $F(r(t))$

3rd integration.

or more complicatedly, we should first calculate the curve (e.g. intersection of two su

using parameterization

III. Integral Theorems.

1. Green's theorem   area

let functions $F_1(x,y)$, $F_2(x,y)$ (or $\underset{\sim}{F}$),

then $\iint_R (\partial_x F_2 - \partial_y F_1)\,dx\,dy = \oint F_1 dx + F_2 dy = \oint \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int \underset{\sim}{F}(\underset{\sim}{r}(t))\cdot \underset{\sim}{r}'(t)\,dt$

specially, the area of a region R: $A = \iint_R dx\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx$.

[Remark] • If the origin is a part of the curve, then we can assume $x = y\cdot t$ or $y = x\cdot t$

• If $y = x\cdot t$, then $A = \int x^2\,dt$

• in case of polar coordinates, $A = \frac{1}{2}\int \rho^2\,d\varphi$.

2. Surface integrals   flux through a surface

Representation of surfaces: $\underset{\sim}{r}(u,v) = x(u,v)\,\underset{\sim}{i} + y(u,v)\,\underset{\sim}{j} + z(u,v)\,\underset{\sim}{k}$

Tangent plane: $T = \underset{\sim}{r}(u_0,v_0) + s\cdot \underset{\sim}{r_u}(u_0,v_0) + t\cdot \underset{\sim}{r_v}(u_0,v_0)$

$$\begin{cases} \underset{\sim}{r_u} = \frac{\partial}{\partial u}\underset{\sim}{r}(u,v) = G_1 \\ \underset{\sim}{r_v} = \frac{\partial}{\partial v}\underset{\sim}{r}(u,v) = G_2 \end{cases}$$

Normal vector: $\underset{\sim}{N} = \underset{\sim}{r_u} \times \underset{\sim}{r_v}$, $\underset{\sim}{n} = \frac{\underset{\sim}{r_u}\times\underset{\sim}{r_v}}{\|\underset{\sim}{r_u}\times\underset{\sim}{r_v}\|} = G_3 \rightarrow$ unit normal vector

Surface integral of a vector field $\underset{\sim}{F}$ over the surface S:

$$\iint_S \underset{\sim}{F}^T\cdot \underset{\sim}{n}\,dA := \iint_R \underset{\sim}{F}(\underset{\sim}{r}(u,v))^T\cdot \underset{\sim}{N}(u,v)\,du\,dv$$

3. Integral theorem of Gauß   flux through a volume $\Leftrightarrow$ all the surfaces

$$\iiint_T \operatorname{div}\underset{\sim}{F}\,dV = \iint_S \underset{\sim}{F}^T\cdot\underset{\sim}{n}\,dA$$

where $dV = dx\,dy\,dz = |J|\,dq_1\,dq_2\,dq_3 = \det\left(\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \frac{\partial V}{\partial q_3}\right) dq_1\,dq_2\,dq_3$

• $\iiint_T (f\cdot\Delta g + \operatorname{grad} f^T\cdot\operatorname{grad} g)\,dV = \iint_S f\cdot\frac{\partial g}{\partial n}\,dA$

• $\iiint_T (f\cdot\Delta g - g\cdot\Delta f)\,dV = \iint_S \left(f\frac{\partial g}{\partial n} - f\frac{\partial f}{\partial n}\right)dA$

4. stokes theorem

$$\iint_S (\text{curl } F)^T \cdot \underset{\sim}{n}\, dA = \oint_C \underset{\sim}{F}\, d\underset{\sim}{r} = \oint \underset{\sim}{F}^T \cdot \underset{\sim}{T}\, dt$$

[Remark] Coordinates' Rotation

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Right-hand rule).

point along the axis

$\updownarrow$ converse

coordinates along the axis

IV. Partial differential equations

1. Classification of PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \phi(x, y, u, u_x, u_y) = 0$$

$AC - B^2 < 0$: hyperbolic type

$AC - B^2 = 0$: parabolic type

$AC - B^2 > 0$: elliptic type

The solutions $y(x)$ of the linear ODE $A(y')^2 - 2By' + C = 0$ are called the characteristics of the l

$\phi(x, y) = const$, $\psi(x, y) = const$. where $\frac{dy}{dx} = -\frac{\psi_x}{\psi_y} = -\frac{\phi_x}{\phi_y}$

$\Rightarrow$ new variables v, w :

① hyperbolic $v = \phi$, $w = \psi$

$$u_x = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = u_v \cdot \phi_x + u_w \cdot \psi_x$$

similarly, $u_y = \cdots$, and u_{xx}, u_{xy}, u_{yy}

insert into the PDE

$$\Rightarrow u_{vw} = F_1(u, u_v, u_w)$$

② parabolic $v = x$, $w = \phi = \psi$

$$\Rightarrow u_{ww} = f_2$$

③ elliptic $v = \frac{\phi + \psi}{2}$, $w = \frac{\phi - \psi}{2}$

$$\Rightarrow u_{vv} + u_{ww} = F_3$$

2. Ansatz of separation

Sometimes (usually homogeneous & constant coefficients) set $u(x, y) = F(x) \cdot G(y)$

insert into the PDE and divide by $F \cdot G$, we can get:

$$H_1(F_{xx}, F_x, F) = H_2(G_{yy}, G_y, G) = const$$

Thus there are two ODEs and the solutions F_1, F_2, G_1, G_2

The final solution of the PDE is the linear combination of all possible products of F, G:

$$u(x, y) = \Sigma \beta \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \cdot \begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix}$$

- Laplace equation. $\Delta u = div(grad\, u) = 0$