Advanced Mathematics

Lab 2: Euler differential equations – substitution of argument

Date of issue: 24 October 2018 Due date: 5 November 2018, 11:30 a. m.

GEODÄTISCHES INSTITUT
der Universität Stuttgart
anerkannt
85%

Family name: Wang

Given name: Yi

Student ID: 3371561

1.1	1.2	1.3	2	3a	3b	3c	4	5	exercise points
4	4	7	17	15	5	3	~~18~~ 30	~~13~~	

Euler differential equations

1. Solve the Euler-(Cauchy) differential equations

$$x^2y'' - 15xy' + 66y = 0 \tag{1.1}$$
$$x^2y'' - 4xy' + 6y = 0 \tag{1.2}$$
$$(9x^2 - 12x + 4)y'' + (9x - 6)y' + y = 0 \tag{1.3}$$

(4+4+7 points)

2. Transform the problem

$$x^3y''' + x^2y'' - 4xy' + 6y = 0$$

via substitution $x = e^t$ into a differential equation with constant coefficients and determine its solution in the variables t and x.

(17 points)

ubstitution of argument

3. a) Find **via substitution** a differential equation with the two solutions $y_1 = \sqrt{\frac{x-1}{x+1}}$ and $y_2 = \sqrt{\frac{x+1}{x-1}}$ based on a Euler-Cauchy equation with adequate coefficients. The answer should be given in the form: $a(x)y'' + b(x)y' - 1 \cdot y = 0$.

(21 points)

b) Determine the Wronskian of y_1 and y_2.

(5 points)

c) Calculate the particular solution which fulfills $y(2) = 10$ and $y'(2) = -5$.

(6 points)

can assume
$pdx\ dx \cdot y_1$.

4. Solve the differential equation

$$y'' + \left(\frac{4x^3}{(x^4-1)} - \frac{1}{x}\right)y' - \left(\frac{4x}{1-x^4}\right)^2 y = 0 \qquad x > 1$$

via the substitution $x = \sqrt{\coth t}$ and evaluate also the Wronskian determinant in the variable t and x (31 points)

[V] Setup an ode

v Given a linear differential equation of second order in the form

$$Ay'' + By' + y = 0.$$

Verify that the choice of 2 linear independent solutions $y_1 = x^a$ and $y_2 = x^b$ for $a, b \in \mathbb{C}$ leads necessarily to an Euler-Cauchy differential equation and express the coefficients depending on (a, b, x). *Hint: Cramer's rule might lead to an elegant and compact solution*

Euler differential equations

1. Solve the Euler-Cauchy differential equations.

1.1 $x^2y''-15xy'+66y=0$.

$\Rightarrow$ Assume $y_1 = x^k$ is one solution and insert into the ODE:

$$y_1' = kx^{k-1}, \quad y_1'' = k(k-1)x^{k-2}$$

and the characteristic equation:

$$k^2-16k+66=0.$$

we get conjugated complex roots $k = 8\pm\sqrt{2}i$

Hence we get two solutions $y_1 = x^{8+\sqrt{2}i}$, $y_2 = x^{8-\sqrt{2}i}$

According to the superposition theory, we get

$\frac{y_1+y_2}{2} = x^8\cos(\sqrt{2}\ln x)$ and $\frac{y_1-y_2}{2} = x^8\sin(\sqrt{2}\ln x)$ are also solutions.

Hence we get the general solution:

$$y = x^8\left[C_1\cos(\sqrt{2}\ln x) + C_2\sin(\sqrt{2}\ln x)\right]$$

1.2 $x^2y''-4xy'+6y=0$

$\Rightarrow$ Assume $y_1 = x^k$ is one solution and insert into the ODE,

We get the characteristic equation:

$$k^2-5k+6=0$$

Hence we get two real roots $k_1=2$ and $k_2=3$,

and the two solutions $y_1 = x^2$, $y_2 = x^3$

$\Rightarrow$ we get the general solution:

$$y = C_1x^2+C_2x^3$$

(reverse side)

1.3 $(9x^2-12x+4)y''+(9x-6)y'+y=0$

$\Rightarrow$ We can transform this equation into simple Euler ODE:

$$(3x-2)^2 y''+3\cdot(3x-2)y'+y=0.$$

We assume $y_1=(3x-2)^k$ is one solution and insert into the ODE,

and get the characteristic equation:

$$k^2+\frac{1}{9}=0.$$

From it we get two conjugated complex roots $k=\pm\frac{1}{3}i$ and two solutions

$$y_1=(3x-2)^{\frac{1}{3}i}, \quad y_2=(3x-2)^{-\frac{1}{3}i}$$

Hence we get the general solution:

$$y=C_1\cdot(3x-2)^{\frac{1}{3}i}+C_2\cdot(3x-2)^{-\frac{1}{3}i}$$

or $y=C_1\cos\left[\frac{1}{3}\ln(3x-2)\right]+C_2\sin\left[\frac{1}{3}\ln(3x-2)\right]$

2. $x^3y''' + x^2y'' - 4xy' + 6y = 0.$

$\Rightarrow$ We use substitution $x = e^t$ and insert into the ODE: $(t = \ln x)$.

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \dot{y}\cdot\frac{1}{x} = \dot{y}\cdot\frac{1}{e^t}.$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d(\dot{y}\cdot\frac{1}{x})}{dx} = \frac{d(\dot{y}\cdot\frac{1}{x})}{dt}\cdot\frac{dt}{dx} = \ddot{y}\cdot\frac{1}{x^2} - \frac{1}{x^2}\cdot\dot{y}$$

$$y''' = \frac{d^3y}{dx^3} = \frac{d(\ddot{y}\cdot\frac{1}{x^2} - \dot{y}\cdot\frac{1}{x^2})}{dt}\cdot\frac{dt}{dx} = \frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y})$$

$$x^3\cdot\frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y}) + x^2\cdot\frac{1}{x^2}(\ddot{y} - \cancel{\dot{y}}) - 4x\cdot\frac{1}{x}\cdot\dot{y} + 6y = 0$$

$$\Rightarrow \quad \dddot{y} - 2\ddot{y} - 3\dot{y} + 6y = 0$$

Hence we get the characteristic equation (assume $y = e^{kt}$)

$$k^3 - 2k^2 - 3k + 6 = 0$$

Then we get 3 real roots: $k_1 = 2$, $k_2 = \sqrt{3}$, $k_3 = -\sqrt{3}$

And the general solution

$$y = C_1e^{2t} + C_2e^{\sqrt{3}t} + C_3e^{-\sqrt{3}t}$$

insert $t = \ln x$ into this,

We get the final solution:

$$y = C_1x^2 + C_2x^{\sqrt{3}} + C_3x^{-\sqrt{3}}$$

3. Substitution of argument

a) set $t=\frac{x-1}{x+1}$, then $y_1=t^{\frac{1}{2}}$, $y_2=t^{-\frac{1}{2}}$

Consider an ODE with solution of the form $y=t^k$.

We can find an Euler equation with this solution:

$t^2y''+Aty'+By=0$ $\cancel{Ax^2y''+Bxy'+Cy=0}$ $y'\neq\mathring{y}$ $\mathring{y}=\frac{dy}{dt}$.

Consider the characteristic equation: $A\ddot{\mathring{y}}+B\mathring{y}+C=0$.

$k^2+(A-1)k+B=0$ $\cancel{Ak^2+Bk+C=0}$

from y_1, y_2 we know it has 2 real roots $k_1=\frac{1}{2}, k_2=-\frac{1}{2}$

thus we can get a characteristic equation:

$k^2-\frac{1}{4}=0$ ansatz: $y=t^k\Rightarrow A-1=0, B=-\frac{1}{4}$

Hence we get ~~the Euler equation~~. $Ak(k-1)+Bk+C=0$.

The idea is correct but the characterstic equation is wrong

$t^2\ddot{y}+t\mathring{y}-\frac{1}{4}y=0$. $\cancel{t^2\ddot{y}-\frac{1}{4}y=0}$.

Since $t=\frac{x-1}{x+1}$. $\mathring{y}=\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=y'\cdot\frac{1}{\left(\frac{x-1}{x+1}\right)'}=\frac{(x+1)^2}{2}\cdot y'$ ✓

$\ddot{y}=\frac{d(\mathring{y})}{dt}=\frac{d\left(\frac{(x+1)^2}{2}\cdot y'\right)}{dx}\cdot\frac{1}{\frac{dt}{dx}}=\frac{(x+1)^3}{4}\left[2y'+(x+1)y''\right]$ ✓

$\Rightarrow$ the ODE can be transformed with the unknown x: $+\left(\frac{x-1}{x+1}\right)\cdot\frac{(x+1)^2}{2}\cdot y'$ −6

$$\left(\frac{x-1}{x+1}\right)^2\cdot\frac{(x+1)^3}{4}\left[2y'+(x+1)y''\right]-\frac{1}{4}y=0$$

simplify $\Rightarrow$ $(x+1)^2(x-1)^2y''+\cancel{2(x+1)(x-1)^2}y'-y=0$

$2x(x^2-1)$

b) the Wronskian of y_1 and y_2

$$W(y_1,y_2)=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix}\sqrt{\frac{x-1}{x+1}} & \sqrt{\frac{x+1}{x-1}}\\ \frac{1}{(x+1)^{\frac{3}{2}}(x-1)^{\frac{1}{2}}} & \frac{-1}{(x-1)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\end{vmatrix}=-\frac{2}{x^2-1}$$ ✓

we can assume $y_2 = u(x)\,y_1$

=)

invert y_1, y_2 into the general solution

$$y = c_1 \cdot y_1 + c_2 \cdot y_2$$

$$\Rightarrow \begin{cases} c_1 y_1(2) + c_2 y_2(2) = 10 \\ c_1 y_1'(2) + c_2 \cdot y_2'(2) = -5 \end{cases}$$

$\Longrightarrow$ insert $\begin{cases} y_1(2) = \sqrt{\frac{1}{3}} \\ y_2(2) = \sqrt{3} \end{cases}$ $\begin{cases} y_1'(2) = \frac{1}{3\sqrt{3}} \\ y_2'(2) = -\frac{1}{\sqrt{3}} \end{cases}$

Based on Cramer's rule, we get

$$c_1 = \frac{\det\begin{bmatrix} 10 & y_2(2) \\ -5 & y_2'(2) \end{bmatrix}}{\det\begin{bmatrix} y_1(2) & y_2(2) \\ y_1'(2) & y_2'(2) \end{bmatrix}} = \frac{\begin{vmatrix} 10 & \sqrt{3} \\ -5 & -\frac{1}{\sqrt{3}} \end{vmatrix}}{-\frac{2}{3}} = \cancel{\frac{1}{\sqrt{3}}} \quad -\frac{5}{2}\sqrt{3}$$

$$c_2 = \frac{\det\begin{bmatrix} y_1(2) & 10 \\ y_1'(2) & -5 \end{bmatrix}}{\det\begin{bmatrix} y_1(2) & y_2(2) \\ y_1'(2) & y_2'(2) \end{bmatrix}} = \frac{\begin{vmatrix} \frac{\sqrt{3}}{3} & 10 \\ \frac{1}{3\sqrt{3}} & -5 \end{vmatrix}}{-\frac{2}{3}} = \cancel{-\frac{25}{9}\sqrt{3}} \quad \frac{25}{6}\sqrt{3}.$$

=) the particular solution:

+3

$$\cancel{y = \frac{5}{3}\sqrt{3} \cdot \sqrt{\frac{x-1}{x+1}} - \frac{25}{9}\sqrt{3} \cdot \sqrt{\frac{x+1}{x-1}}}$$

$$y = -\frac{5}{2}\sqrt{3} \cdot \sqrt{\frac{x-1}{x+1}} + \frac{25}{6}\sqrt{3}\sqrt{\frac{x+1}{x-1}}.$$

$$y''+\left(\frac{4x^3}{x^4-1}-\frac{1}{x}\right)y'+\left(\frac{4x}{1-x^4}\right)^2 y=0 \qquad x>1.$$

⇒ substitution $x=\sqrt{\coth t}$

we get: $\frac{dx}{dt}=\frac{-\frac{1}{\sinh^2 t}}{2\sqrt{\coth t}}=\frac{1-\coth^2 t}{2\sqrt{\coth t}}$ ✓

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{1}{\frac{dx}{dt}}=\dot y\cdot\frac{2\sqrt{\coth t}}{1-\coth^2 t}$$ ✓

$$y''=\frac{d\left(\dot y\cdot\frac{2\sqrt{\coth t}}{1-\coth^2 t}\right)}{dt}\cdot\frac{1}{\frac{dx}{dt}}=\underbrace{\left(\ddot y\cdot\frac{2\sqrt{\coth t}}{1-\coth^2 t}+\dot y\cdot\frac{1-\coth^2 t+4\coth^2 t}{\sqrt{\coth t}\cdot(1-\coth^2 t)}\right)}_{\text{simplify}}\cdot\frac{2\sqrt{\coth t}}{1-\coth^2 t}$$

−1

⇒ we can transform the ODE into:

$$\frac{2\sqrt{\coth t}}{1-\coth^2 t}\cdot\left(\frac{2\sqrt{\coth t}}{1-\coth^2 t}\cdot\ddot y+\frac{1-\coth^2 t+4\coth^2 t}{\sqrt{\coth t}\cdot(1-\coth^2 t)}\cdot\dot y\right)+\left(\frac{4\cdot\coth t\sqrt{\coth t}}{\coth^2 t-1}-\frac{1}{\sqrt{\coth t}}\right)\cdot\frac{2\sqrt{\coth t}}{1-\coth^2 t}\cdot\dot y$$
$$-\frac{(4\sqrt{\coth t})^2}{(1-\coth^2 t)^2}\cdot y=0.$$

simplify ⇒ $$\frac{4\coth t}{(1-\coth^2 t)^2}\ddot y-\frac{16\coth t}{(1-\coth^2 t)^2}y=0$$

⇒ $$\ddot y-4y=0,$$ ✓

characteristic equation of this ODE: $\lambda^2-4=0$.

two real roots $\lambda=\pm 2$. ✓

⇒ general solution: $y=C_1e^{2t}+C_2e^{-2t}$ ✓

insert $t=\operatorname{arccoth}(x^2)$

⇒ general solution:

$$y=C_1e^{2\operatorname{arccoth}(x^2)}+C_2\cdot e^{-2\operatorname{arccoth}(x^2)}$$ ✓

$e^{2\operatorname{arccoth}x^2-2\operatorname{arccoth}x^2}=1$

the Wronskian determinant

$$W(t)=\begin{vmatrix}e^{2t}&e^{-2t}\\2e^{2t}&-2e^{-2t}\end{vmatrix}=-4$$ ✓

$$W(x)=\begin{vmatrix}e^{2\operatorname{arccoth}(x^2)}&e^{-2\operatorname{arccoth}(x^2)}\\2\operatorname{arccoth}(x^2)\cdot e^{2\operatorname{arccoth}x^2}\cdot\frac{2x}{1-x^4}&-2\operatorname{arccoth}x^2\cdot e^{-2\operatorname{arccoth}x^2}\cdot\frac{2x}{1-x^4}\end{vmatrix}=\frac{8x\operatorname{arccoth}(x^2)}{x^4-1}$$ ✗

Room 5.322 — markus.antoni@

5.

Since $y_1 = x^a$, $y_2 = x^b$ are solutions,

We get $\begin{cases} A y_1'' + B y_1' + y_1 = 0 \\ A y_2'' + B y_2' + y_2 = 0 \end{cases}$

$\Rightarrow$ Also $\begin{cases} A y_1'' + B y_1' = -y_1 \\ A y_2'' + B y_2' = -y_2 \end{cases}$

$$\Rightarrow \quad A = \frac{\begin{vmatrix} -y_1 & y_1' \\ -y_2 & y_2' \end{vmatrix}}{\begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix}}, \quad B = \frac{\begin{vmatrix} y_1'' & -y_1 \\ y_2'' & -y_2 \end{vmatrix}}{\begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix}}$$

$$\begin{vmatrix} -y_1 & y_1' \\ -y_2 & y_2' \end{vmatrix} = \begin{vmatrix} -x^a & a x^{a-1} \\ -x^b & b x^{b-1} \end{vmatrix} = (a-b)\, x^{a+b-1}$$

$$\begin{vmatrix} y_1'' & -y_1 \\ y_2'' & -y_2 \end{vmatrix} = \begin{vmatrix} a(a-1) x^{a-2} & -x^a \\ b(b-1) x^{b-2} & -x^b \end{vmatrix} = [b(b-1) - a(a-1)]\, x^{a+b-2}$$

$$\begin{vmatrix} y_1'' & y_1' \\ y_2'' & y_2' \end{vmatrix} = \begin{vmatrix} a(a-1) x^{a-2} & a x^{a-1} \\ b(b-1) x^{b-2} & b x^{b-1} \end{vmatrix} = ab(a-b)\, x^{a+b-3}$$

$$\Rightarrow \quad A = \frac{(a-b)\, x^{a+b-1}}{ab(a-b)\, x^{a+b-3}} = \frac{1}{ab} \cdot x^{2}$$

$$B = \frac{(b+a-1)(b-a)\, x^{a+b-2}}{ab(a-b)\, x^{a+b-3}} = \frac{1-a-b}{ab} x$$

$\Rightarrow$ Euler ODE: $\frac{1}{ab} x^2 y'' + \frac{1-a-b}{ab} x y' + y = 0.$