Advanced Mathematics

Lab 5: Numerical methods

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1	2	3a	3b	4	problem
15	19	14	12	24	points

Power series

1. Determine the polynomial solution of the ODE

$$0.5xy'' - (x+5)y' + 5y = 0$$

via power series. $y = \sum_{k=0}^{\infty} a_k (x-\cancel{x_0})^k$ (18 points)

Numerical integration and re-writting

2. Implement the **Runge-Kutta** method of order 4 for numerical integration:

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\big(x_i + 0.5h,\, y_i + 0.5hk_1\big)$$
$$k_3 = f\big(x_i + 0.5h,\, y_i + 0.5hk_2\big)$$
$$k_4 = f\big(x_{i+1},\, y_i + hk_3\big)$$
$$y_{i+1} = y(x_{i+1}) = y(x_i) + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Solve the problem $y'' + xy = 0$ in the interval $x \in [0,6]$ with the initial values $x_0 = 0$ and $y_0 = 0.355\,028\,053\,88$ with the stepwidth $h = 0.01$ and visualize the result.

Write a Runge-Kutta-solver in MALTAB which is called by:
`[X, Y] = RungeKutta4(fxy, x0, h, xmax, y0)`

- The differential equation should be provided by a function handle `fxy`
- The arguments `y0` can be scalar or column vector $u_1 = y$, $u_2 = y'$ $\Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} u_2 \\ -x \cdot u_1 \end{bmatrix}$ $y_0 = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$
- Using the routine without output argument should lead to visualization
- Check all input arguments for type and dimsnion and provide helpfull messages

(26 points)

$y(2), y(1)/x^2 + y(2)/x]$

,h, xmax, y0)

3. a) Rewrite the Euler-Cauchy differential equation

$$16x^2y'' - 9xy' + 8y = 0$$

into a system [S] of the form $y' = \underline{B}y$ and note down the solution which fulfills the initial condition $y(1) = \begin{pmatrix}1, & 1\end{pmatrix}^{\mathsf{T}}$. (14 points)

b) Approximate the solution $u(2)$ of the system [S] by the explicit Euler method with the stepwidth $h = 1$ and note down all steps in exact fractions. (12 points)

$u_1 = u_0 + h\,\underline{B}\,u_0$

Eigen values

4. Symmetric eigen value problems can be solved iterativly by (p,q)-rotation of Jacobi:

$$\underline{A}^{[\ell+1]} = \underline{U}^{\mathsf{T}}\underline{A}^{[\ell]}\underline{U}$$

where the matrix $\underline{A}^{[\ell+1]}$ converges after ℓ iterations to a diagonal matrix. In each step, the matrix products eliminates the off-diagonal element $A(p,q)$ with largest absolute value. The non-zero entries of $\underline{U}$ are defined by

- $U(i,i) = 1$
- $U(p,p) = U(q,q) = \cos\varphi$ with $\cot 2\varphi = \frac{A(q,q)-A(p,p)}{2A(p,q)}$
- $U(p,q) = \sin\varphi$, $U(q,p) = -\sin\varphi$.

Implement a function `[A] = jacobipq(A,iterMax,tol)` which applies the transformation up to `iterMax` iterations and set values smaller than the tolerance `tol` $= 1e-10$ to zero. Check and ensure the symmetry in the procedure!

Test the routine for the matrix

$$\underline{A} = \begin{pmatrix} 9 & -3 & 2 & 1 & 2 \\ -3 & 2 & 1 & 8 & 2 \\ 2 & 1 & -4 & -4 & 1 \\ 1 & 8 & -4 & 4 & 1 \\ 2 & 2 & 1 & 1 & 10 \end{pmatrix}$$

and note down all elements of the first step and the final result with 4 decimals. (30 points)

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(a) $5x y'' - (x+5)y' + 5y = 0$

normalize: $y'' + (-2 - \frac{10}{x})y' + \frac{10}{x} y = 0$

obviously $(-2-\frac{10}{x})$ and $(\frac{10}{x})$ are infinitely often differentiable on the open interval.

so they can be expanded into power series.

Hence the solution of the ODE can be a power series:

$$y = \sum_{k=0}^{\infty} a_k x^k$$

then $y' = \sum_{k=1}^{\infty} a_k k x^{k-1}$, $y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$

Insert into the ODE:

$$\sum_{k=2}^{\infty} a_k k(k-1)x^{k-2} + (-2-\frac{10}{x}) \sum_{k=1}^{\infty} a_k k x^{k-1} + \frac{10}{x} \cdot \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} + (-2) \cdot \sum_{k=1}^{\infty} a_k k x^{k-1} + (-10) \cdot \sum_{k=1}^{\infty} a_k k x^{k-2} + 10 \cdot \sum_{k=0}^{\infty} a_k x^{k-1} = 0$$

$$\sum_{k=1}^{\infty} a_{k+1}(k+1) k x^{k-1} + (-2) \cdot \sum_{k=1}^{\infty} a_k k x^{k-1} + (-10) \cdot \sum_{k=0}^{\infty} a_{k+1}(k+1) x^{k-1} + 10 \cdot \sum_{k=0}^{\infty} a_k x^{k-1} = 0$$

$$\sum_{k=1}^{\infty} \left[a_{k+1}(k+1)k - 2a_k k - 10 a_{k+1}(k+1) + 10 a_k \right] x^{k-1} + \left(-10 \cdot a_1 x^{-1} + 10 \cdot a_0 \cdot x^{-1} \right) = 0.$$

$$\begin{cases} a_0 = a_1 \\ a_{k+1} = \dfrac{(2k-10)a_k}{(k+1)(k-10)}, & k \geq 1. \end{cases}$$

more steps to calculate

$a_3, a_4, a_5 \quad \dots \quad a_6 = a_7 = \dots a$

the polynomial solution:

$\frac{1}{9}a_0 \quad \frac{1}{63}a_0 \quad \frac{1}{945}a_0.$

$$y = a_0 + a_0 x + \frac{4}{9} a_0 x^2 + \frac{1}{4} a_0 x^3 + \frac{1}{28} a_0 x^4 + \frac{1}{420} a_0 x^5$$

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I don't mind if you write down your idea; otherwise I can't see the mistake

3. $$16x^2y'' - 9xy' + 8y = 0.$$

a) let $u_1 = y$, $u_2 = y'$.

then $\begin{cases} u_1' = u_2 \\ u_2' = -\frac{1}{2x^2}u_1 + \frac{9}{16x}u_2 \end{cases}$

$\Rightarrow$ system [S]: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2x^2} & \frac{9}{16x} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Now we solve the ODE with Euler-Cauchy method.

normalize. $x^2y'' - \frac{9}{16}xy' + \frac{1}{2}y = 0$

the characteristic equation:

$$\lambda^2 - \frac{25}{16}\lambda + \frac{1}{2} = 0$$

$\Rightarrow$ roots $\lambda_1 = \frac{\sqrt{113}+25}{32}$, $\lambda_2 = \frac{-\sqrt{113}+25}{32}$

$\Rightarrow$ general solution $y = C_1 \cdot x^{\frac{\sqrt{113}+25}{32}} + C_2 \cdot x^{\frac{-\sqrt{113}+25}{32}}$

Consider the initial condition $\underset{\sim}{y}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e. $u_1(1) = y(1) = 1$. $u_2(1) = y'(1) = 1$.

$\Rightarrow \begin{cases} y(1) = C_1 + C_2 = 1 \\ y'(1) = C_1 \cdot \frac{\sqrt{113}+25}{32} + C_2 \cdot \frac{25-\sqrt{113}}{32} = 1. \end{cases}$

$\Rightarrow C_1 = \frac{7\sqrt{113}+113}{226}$ $C_2 = \frac{113-7\sqrt{113}}{226}$

$\Rightarrow$ the special solution: $y = \frac{7\sqrt{113}+113}{226} \cdot x^{\frac{\sqrt{113}+25}{32}} + \frac{113-7\sqrt{113}}{226} \cdot x^{\frac{25-\sqrt{113}}{32}}$

~~y~~ $\underset{\sim}{u}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\underset{\sim}{u}(2) = u(1) + h \cdot f(x_1, \underset{\sim}{y_1})$

$= \underset{\sim}{u}(1) + h \cdot \underline{B} \cdot \underset{\sim}{u}(1)$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{9}{16} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ \frac{17}{16} \end{bmatrix}$