**Advanced Mathematics**

# Lab 10: PDE of Laplace and ODE of Legendre – ansatz of separation

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anerkannt

90%

| 1a | 1b | 2a | 2b | 3 | 4a | 4b | problem |
|---|---|---|---|---|---|---|---|
| 9 | 16 | 9 | 22 | 15 | 7 | 12 | points |

## Ansatz of separation

1. Solve the partial differential equation

$$x^2u_{xx} + xu_x - (\cos^4 y)u_{yy} + (2\tan y\cos^4 y)u_y = 0$$

by the ansatz of separation.

a) The differential equation in the variable $x$ is of Euler-Cauchy type. Choose the the constant $\pm n^2$ in such a way, that the characteristic equation has imaginary roots for $n \in \mathbb{N}\setminus\{0\}$.

b) Use the substitution $y = \arctan t$ to solve the 2nd equation.

(final exam SS17, 25 points)

## Laplace equation

2. The Laplace operator in two-dimensional curvilinear coordinates $(v, w)$ is given by

$$\Delta_{vw}\Phi = \frac{1}{\cosh v}\left[\frac{1}{\cosh v}\frac{\partial}{\partial v}\left\{\cosh v\frac{\partial\Phi}{\partial v}\right\} + \frac{1}{\cos w}\frac{\partial}{\partial w}\left\{\cos w\frac{\partial\Phi}{\partial w}\right\}\right].$$

a) Apply the ansatz of separation to get *two* ordinary differential equations. The constants should be choosen in such a way, that the function $\varphi(v, w) = \sin w \cdot \sinh v$ is one of the solutions.

b) Consider now the differential equation in $v$ for the constant of $\varphi(v, w)$ and determine a independent solution via reduction of order.

(final exam WS16,33 points)

## Legendre-ODE

3. Given the Legendre-differential equation and its solution of the lecture notes. Determine the non-polynomial solution for the degree $n = 1$. (15 points)

4. Similar to a Fourier-series expansion, the Legendre polynomials $P_n(t)$ can be used for approximation of an arbitrary function $g(t)$ in the interval $[-1, 1]$ via

$$g(t) \approx g^N(t) = \sum_{n=0}^{N} \frac{2n+1}{2} a_n P_n(t)$$

by the synthesis formula:

$$a_n = \int_{-1}^{1} g(t) P_n(t) \mathrm{d}t.$$

a) Derive a recursive formula for the integral $J_k = \int t^k \cosh t dt$ for $k \in \mathbb{N}$ (10 points)

b) and approximate than $g(t) = \cosh t$ by a linear combination of Legendre polynomials up to degree 4 with $P_3 = \frac{1}{2}(5x^3 - 3x)$ and $P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$ (17 points)

## [V] properties of Legendre functions

v The associated Legendre functions can be calculated by the Formula of Rodrigues/Ferrers:

$$\overline{P}_{n,m}(t) = N_{n,m} \cdot \frac{1}{2^n n!}(1-t^2)^{m/2} \frac{\mathrm{d}^{n+m}(t^2-1)^n}{\mathrm{d}t^{n+m}}$$

$$N_{n,m} = \sqrt{(2-\delta_{m,0})(2n+1)\frac{(n-m)!}{(n+m)!}}$$

a) Determine all functions of degree $n = 0, 1, 2$ and $0 \leq m \leq n$ in the variable $t$.

b) Verify for degree $n = 2$ the **addition theorem**

$$P_n(\cos\psi_{QX}) = \frac{1}{2n+1} \sum_{m=0}^{n} \overline{P}_{n,m}(\cos\vartheta_Q)\overline{P}_{n,m}(\cos\vartheta_X)\cos\left[m(\lambda_X - \lambda_Q)\right]$$

for the location $Q = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ and $X = \left(\frac{\sqrt{3}}{\sqrt{8}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{2}}\right)$. The angle $\psi_{QX}$ is the angle between $X$ and $Q$ – also known as spherical distance – and $P_n(..)$ are the Legendre polynomials with order $m = 0$ and without normalization factor $N_{n,m}$.

1.

$$x^2 u_{xx} + x u_x - (\cos^4 y)\cdot u_{yy} + (2\tan y \cos^4 y)\, u_y = 0$$

insert $u = F(x)\cdot G(y)$ into the PDE

$$\Rightarrow x^2\cdot F_{xx} G + x\cdot F_x G - \cos^4 y\cdot F\cdot G_{yy} + 2\tan y\cos^4 y\cdot F\cdot G_y = 0$$

divide by $F\cdot G$,

$$\frac{x^2 F_{xx}}{F} + \frac{x F_x}{F} = \frac{\cos^4 y\, G_{yy}}{G} - \frac{2\tan y\cos^4 y\cdot G_y}{G}$$

a) $$\frac{x^2 F_{xx}}{F} + \frac{x F_x}{F} = \text{const} = C$$

$\Rightarrow x^2 F_{xx} + x F_x - cF = 0$ "Euler-Cauchy ODE"

$\Rightarrow$ solution type $F = x^k$, insert into the ODE

$\Rightarrow$ characteristic equation: $k^2 - c = 0$

imaginary roots $\Big] \Rightarrow c = -n^2$

$\Rightarrow$ solutions $\begin{cases} F_1 = \cos(n\ln x) \\ F_2 = \sin(n\ln x) \end{cases}$

b) $$\frac{\cos^4 y\, G_{yy}}{G} - \frac{2\tan y\cos^4 y\, G_y}{G} = \text{const} = -n^2$$

$\Rightarrow \cos^4 y\, G_{yy} - 2\tan y\cos^4 y\, G_y + n^2 G = 0.$

substitution $y = \arctan t$, $t = \tan y$,

$$G_y = \frac{\partial G}{\partial t}\cdot\frac{\partial t}{\partial y} = \frac{1}{\cos^2 y}\cdot G_t = (1+t^2)G_t$$

$$G_{yy} = \frac{\partial G_y}{\partial t}\cdot\frac{\partial t}{\partial y} = \left(2t G_t + (1+t^2)G_{tt}\right)\cdot\frac{1}{\cos^2 y} = (1+t^2)\left(2tG_t + (1+t^2)G_{tt}\right)$$

$$\Rightarrow \frac{1+t^2}{(1+t^2)^2}\cdot\left(2tG_t + \cancel{(1+t^2)}G_{tt}\right) - 2t\cdot\frac{1}{(1+t^2)^2}\cdot(1+t^2)G_t + n^2 G = 0$$

$\Rightarrow \quad G_{tt} + n^2 G = 0$ . "ODE with constant coefficients."

solutions type $G = e^{ky}$ . Insert into the ODE

$\Rightarrow$ characteristic equation: $k^2 + n^2 = 0$ , $k = \pm ni$

$\Rightarrow$ solutions : $G_1 = \cos nt$
$G_2 = \sin nt$ , $t = \tan y$

$$\Rightarrow \begin{cases} G_1 = \cos(n \tan y) \\ G_2 = \sin(n \tan y) \end{cases}$$

To sum up. the complete solution of the PDE is

$$u(x,y) = \sum_{n=1}^{\infty} \beta_n \begin{Bmatrix} \cos(n \ln x) \\ \sin(n \ln x) \end{Bmatrix} \cdot \begin{Bmatrix} \cos(n \tan y) \\ \sin(n \tan y) \end{Bmatrix} , \quad n \in N$$

2.

$$\Delta_{vw}\phi = \frac{1}{\cosh^2 v}\cdot\left(\sinh v\,\frac{\partial\phi}{\partial v}+\frac{\partial^2\phi}{\partial v^2}\cdot\cosh v\right)+\frac{1}{\cosh v}\cdot\frac{1}{\cos w}\cdot\left(-\sin w\cdot\frac{\partial\phi}{\partial w}+\frac{\partial^2\phi}{\partial w^2}\cdot\cos w\right)=0$$

$$\Rightarrow \left(\tanh v\cdot\phi_v+\phi_{vv}\right)+\left(-\tan w\cdot\phi_w+\phi_{ww}\right)=0$$

a) let $\phi = F(v)G(w)$

$$\Rightarrow \tanh v\cdot G\cdot F_v + G\cdot F_{vv} + F\cdot G_{ww} - \tan w\cdot F\cdot G_w = 0$$

divide by $F\cdot G$, $\Rightarrow$

$$\frac{\tanh v\cdot F_v}{F}+\frac{F_{vv}}{F}=\frac{\tan w\cdot G_w}{G}-\frac{G_{ww}}{G}=\text{const}$$

insert $\phi(v,w)=\sin w\cdot\sinh v$ ( i.e. $F=\sinh v$, $G=\sin w$ ) into the above.

$$= \frac{\tan w\cdot \sin w}{\sin w}-\frac{-\sin w}{\sin w}$$

$$\Rightarrow \text{const}=\frac{\tanh v\cdot\cosh v}{\sinh v}+\frac{\sinh v}{\sinh v}=2$$

check for v -3

$\Rightarrow$ the two ODEs:

$$\begin{cases} F_{vv}+\tanh v\cdot F_v-2F=0 \\ G_{ww}-\tan w\cdot G_w+2G=0\end{cases}$$

b) Consider $F_{vv}+\tanh v\cdot F_v-2F=0$

Obviously $F_1=\sinh v$ is one solution.

Basing on "Reduction of order" method, set $F_2=u\cdot\sinh v$, insert into the ODE

then $u=\int\frac{1}{F_1^2}\cdot e^{-\int\tanh v\,dv}dv=\int\frac{1}{\sinh^2 v}\cdot e^{-\int\tanh v\,dv}dv$

$$\int\tanh v\,dv=\int\frac{\sinh v}{\cosh v}dv=\int\frac{1}{\cosh v}\cdot d(\cosh v)=\ln(\cosh v)$$

$$e^{-\int\tanh v\,dv}=e^{-\ln(\cosh v)}=\frac{1}{\cosh v}$$

$$t=\sinh v$$

$$=\int\frac{1}{\sinh^2 v}\cdot\frac{1}{\cosh v}dv=\int\frac{1\cdot\cosh v}{\sinh^2 v\cosh^2 v}dv=\int\frac{1}{\sinh^2 v(1+\sinh^2 v)}d\sinh v=\int\frac{1}{t^2(1+t^2)}dt=\int\left(\frac{1}{t^2}-\frac{1}{t^2+1}\right)dt$$

$$= \int \frac{1}{t^2}dt - \int \frac{1}{t^2+1}dt = -\frac{1}{t} - \arctan(t) = -\frac{1}{\sinh v} - \arctan(\sinh v)$$ ✓

$$\Rightarrow F_2 = u \cdot F_1 = -1 - \sinh v \cdot \arctan(\sinh v)$$ ✓

$$\Rightarrow F = C_1 \cdot \sinh v + C_2 \cdot \left( -1 - \sinh v \cdot \arctan(\sinh v) \right)$$ ✓

3.

Legendre-differential equation:

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

when $n=1$. $(1-x^2)y''-2xy'+2y=0$

obviously $y_1=x$ is one solution, ✓

using "Reduction of order" method. $y_2=u\cdot x$ ✓

then $u=\int \frac{1}{y_1^2}\cdot e^{-\int p dx}dx$

We normalize the ODE:

$$y''+\frac{-2x}{1-x^2}y'+\frac{2}{1-x^2}y=0$$

$$\Rightarrow u=\int\frac{1}{x^2}\cdot e^{-\int\frac{-2x}{1-x^2}dx}dx=\int\frac{1}{x^2}\cdot e^{\int\frac{2x}{1-x^2}dx}dx$$

good!

$$\int\frac{2x}{1-x^2}dx=\int\frac{1}{1-x^2}dx^2=-\ln(1-x^2)$$

$$e^{\int\frac{2x}{1-x^2}dx}=e^{-\ln(1-x^2)}=\frac{1}{1-x^2}$$ ✓

$$\Rightarrow u=\int\frac{1}{x^2}\cdot\frac{1}{1-x^2}dx=\int\left(\frac{1}{x^2}-\frac{1}{x^2-1}\right)dx=\int\frac{1}{x^2}dx-\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$$

$$=-\frac{1}{x}-\frac{1}{2}\left[\ln(x-1)-\ln(x+1)\right]=-\frac{1}{x}-\frac{1}{2}\ln\frac{x-1}{x+1}$$ ✓

$$\Rightarrow y_2=u\cdot y_1=-1-\frac{1}{2}x\cdot\ln\frac{x-1}{x+1}$$ ✓

$\Rightarrow$ the general non-polynomial solution:

$$y=C_1x+C_2\cdot\left(-1-\frac{1}{2}x\cdot\ln\frac{x-1}{x+1}\right)$$ ✓

4.

a) $J_k = \int t^k \cosh t \, dt$

$= \int t^k \, d\sinh t = t^k \cdot \sinh t - \int \sinh t \, dt^k = t^k \sinh t - \int \sinh t \cdot k t^{k-1} dt$

$\int \sinh t \cdot t^{k-1} dt = \int t^{k-1} d\cosh t = t^{k-1}\cosh t - \int \cosh t \, dt^{k-1} = t^{k-1}\cosh t - \int \cosh t \cdot (k-1) t^{k-2} dt$

$\Rightarrow J_k = t^k \sinh t - k \cdot \left( t^{k-1}\cosh t - (k-1) \cdot \int \cosh t \cdot t^{k-2} dt \right)$

$= t^k \sinh t - k t^{k-1} \cosh t + k(k-1) \cdot J_{k-2}$

Calculate $J_0$, $J_1$

$\Rightarrow$ the recursive formula: $J_k = t^k \sinh t - k \cdot t^{k-1}\cosh t + k(k-1) \cdot J_{k-2}$, $k \geq 2, k \in N$

b)

$a_0 P_0 = 1$ ; $P_1 = t$, $P_2 = \frac{3}{2}t^2 - \frac{1}{2}$, $P_3 = \frac{1}{2}(5t^3 - 3t)$, $P_4 = \frac{1}{8}(35t^4 - 30t^2 + 3)$

$a_0 = \int_{-1}^{1} \cosh t \cdot P_0 \, dt = \int_{-1}^{1} \cosh t \, dt = \sinh t \Big|_{-1}^{1} = e - \frac{1}{e}$

$a_1 = \int_{-1}^{1} \cosh t \cdot P_1 \, dt = \int_{-1}^{1} \cosh t \cdot t \, dt = t \sinh t - \int \sinh t \, dt \Big|_{-1}^{1} = t\sinh t - \cosh t \Big|_{-1}^{1} = 0$

$a_2 = \int_{-1}^{1} \cosh t \cdot P_2 \, dt = \int_{-1}^{1} \cosh t \cdot (\frac{3}{2}t^2 - \frac{1}{2}) dt = \frac{3}{2}\int_{-1}^{1} t^2 \cosh t \, dt - \frac{1}{2}\int_{-1}^{1} \cosh t \, dt$

$= \frac{3}{2} \cdot \left( t^2 \sinh t - 2t\cosh t + 2a_0 \Big|_{-1}^{1} \right) - \frac{1}{2} \cdot a_0 = e - \frac{7}{e}$

$a_3 = \int_{-1}^{1} \cosh t \cdot P_3 \, dt = \int_{-1}^{1} \cosh t \cdot \frac{1}{2}(5t^3 - 3t) dt = \frac{5}{2}\int_{-1}^{1} t^3 \cosh t \, dt - \frac{3}{2}\int_{-1}^{1} t\cosh t \, dt$

$= \frac{5}{2} \cdot \left( t^3 \sinh t - 3 \cdot t^2 \cosh t + 6 \cdot a_1 \right)\Big|_{-1}^{1} - \frac{3}{2} \cdot a_1 = 0$

$a_4 = \int_{-1}^{1} \cosh t \cdot P_4 \, dt = \int_{-1}^{1} \cosh t \cdot \frac{1}{8}(35t^4 - 30t^2 + 3) dt = \frac{35}{8}\int_{-1}^{1} t^4 \cosh t \, dt - \frac{30}{8}\int_{-1}^{1} t^2 \cosh t \, dt + \frac{3}{8}\int_{-1}$

$= \frac{35}{8} \cdot (t^4 \sinh t - 4 \cdot t^3 \cosh t + 12 a_2)\Big|_{-1}^{1} - \frac{30}{8} \cdot (t^2 \sinh t - 2t\cosh t + 2a_0)\Big|_{-1}^{1} + \frac{3}{8} \cdot a_0 = 36e - \frac{266}{e}$

$$\Rightarrow g(t) \simeq g^N(t) = \sum_{n=0}^{N} \frac{2n+1}{2} a_n P_n(t)$$

$$= \sum_{n=0}^{4} \frac{2n+1}{2} a_n P_n(t)$$

Insert $a_n$ then it's correct.

$$= \frac{1}{2} \cdot a_0 \cdot P_0(t) + \frac{3}{2} \cdot a_1 \cdot P_1(t) + \frac{5}{2} \cdot a_2 \cdot P_2(t) + \frac{7}{2} \cdot a_3 \cdot P_3(t) + \frac{9}{2} \cdot a_4 \cdot P_4(t)$$

$$= \frac{1}{2} \cdot \left(e - \frac{1}{e}\right) \cdot 1 + \frac{3}{2} \cdot 0 + \frac{5}{2} \cdot \left(e - \frac{7}{e}\right) \cdot \left(\frac{3}{2}t^2 - \frac{1}{2}\right) + \frac{7}{2} \cdot 0 + \frac{9}{2} \cdot \left(36e - \frac{266}{e}\right) \cdot \frac{1}{8}(35t^4 - 30t^2 + 3)$$

$$= \frac{315}{8}\left(18e - \frac{133}{e}\right) t^4 - \frac{15}{4}\left(161e - 1190e^{-1}\right) t^2 + 60e - \frac{3525}{8} e^{-1}$$

$$\simeq 0.0436\, t^4 + 0.4994\, t^2 + 1.$$

+12

Is $\delta_{m,n}$ a Dirac function? ✓ ~~yes~~ no but the Kronecker delta

a)

$$\delta_{m,0} = \begin{cases} 1, & m=0 \\ 0, & m\neq 0 \end{cases}$$

$n=0 \Rightarrow m=0 \Rightarrow N_{0,0} = \sqrt{(2-\delta_{0,0})\cdot(2\cdot 0+1)\cdot\frac{0!}{0!}} = 1$

$\Rightarrow \overline{P_{0,0}}(t) = N_{0,0}\cdot\frac{1}{2^0\cdot 0!}\cdot(1-t^2)^{\frac{0}{2}}\cdot\frac{d^0(t^2-1)^0}{dt^0} = 1$ ✓

$n=1 \Rightarrow m=0$ or $1$

when $m=0$, $N_{1,0} = \sqrt{(2-\delta_{0,0})\cdot(2\cdot 1+1)\cdot\frac{1!}{1!}} = \sqrt{3}$

$\overline{P_{1,0}}(t) = N_{1,0}\cdot\frac{1}{2^1\cdot 1!}\cdot(1-t^2)^{\frac{0}{2}}\cdot\frac{d^1(t^2-1)^1}{dt} = \sqrt{3}\,t$ ✓

when $m=1$, $N_{1,1} = \sqrt{(2-\delta_{1,0})\cdot(2+1)\cdot\frac{0!}{2!}} = \sqrt{3}$

$\overline{P_{1,1}}(t) = N_{1,1}\cdot\frac{1}{2^1\cdot 1!}\cdot(1-t^2)^{\frac{1}{2}}\cdot\frac{d^2(t^2-1)^1}{dt^2} = \sqrt{3(1-t^2)}$ ✓

~~when m=~~

$n=2 \Rightarrow m=0$ or $1$ or $2$

when $m=0$, $N_{2,0} = \sqrt{(2-\delta_{0,0})\cdot(2\cdot 2+1)\cdot\frac{2!}{2!}} = \sqrt{5}$

$\overline{P_{2,0}}(t) = N_{2,0}\cdot\frac{1}{2^2\cdot 2!}\cdot(1-t^2)^{\frac{0}{2}}\cdot\frac{d^2(t^2-1)^2}{dt^2} = \frac{\sqrt{5}}{2}(3t^2-1)$ ✓

when $m=1$, $N_{2,1} = \sqrt{(2-\delta_{1,0})\cdot(2\cdot 2+1)\cdot\frac{1!}{3!}} = \sqrt{\frac{5}{3}}$

$\overline{P_{2,1}}(t) = N_{2,1}\cdot\frac{1}{2^2\cdot 2!}\cdot(1-t^2)^{\frac{1}{2}}\cdot\frac{d^3(t^2-1)^2}{dt^3} = \sqrt{15}\,t\sqrt{1-t^2}$ ✓

when $m=2$, $N_{2,2} = \sqrt{\frac{5}{12}}$, $\overline{P_{2,2}}(t) = \frac{\sqrt{15}}{2}(1-t^2)$ ✓

+18.

b)

$P_2(\cos\psi_{QX}) = \frac{1}{2}\cdot\left(3\cdot(\cos\psi_{QX})^2-1\right)$ ✓

$\cos\psi_{QX} = \frac{\vec{OQ}\cdot\vec{OX}}{|\vec{OQ}|\cdot|\vec{OX}|} = \frac{(\frac{\sqrt{3}}{2},\frac{1}{2},0)\cdot(\frac{\sqrt{3}}{\sqrt{8}},\frac{1}{\sqrt{8}},\frac{1}{\sqrt{2}})}{\sqrt{(\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2}\cdot\sqrt{\frac{3}{8}+\frac{1}{8}+\frac{1}{2}}} = \frac{\sqrt{2}}{2}$ ✓

$\Rightarrow P_2(\cos\psi_{QX}) = \frac{1}{2}\cdot\left(3\cdot\frac{1}{2}-1\right) = \frac{1}{4}$ ✓

: $\cos\theta_Q = \frac{z_Q}{r} = 0$, $\cos\theta_X = \frac{z_X}{r} = \frac{\sqrt{2}}{2}$, $\lambda_X-\lambda_Q = \arctan\frac{Y_X}{X_X} - \arctan\frac{Y_Q}{X_Q} = 0$

$\overline{P_{2,0}}(t) = \frac{\sqrt{5}}{2}(3t^2-1)$, $\overline{P_{2,1}}(t) = \sqrt{15}\,t\sqrt{1-t^2}$, $\overline{P_{2,2}}(t) = \frac{\sqrt{15}}{2}(1-t^2)$ ✓

$$\Rightarrow RHS = \frac{1}{2\cdot 2+1}\cdot \sum_{m=0}^{2} \overline{P_{2,m}}(0)\cdot \overline{P_{2,m}}\left(\frac{\sqrt{2}}{2}\right)\cdot \cos 0$$

$$= \frac{1}{5}\cdot\left(\overline{P_{2,0}}(0)\cdot\overline{P_{2,0}}\left(\frac{\sqrt{2}}{2}\right) + \overline{P_{2,1}}(0)\cdot\overline{P_{2,1}}\left(\frac{\sqrt{2}}{2}\right) + \overline{P_{2,2}}(0)\cdot\overline{P_{2,2}}\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{1}{5}\cdot\left(-\frac{\sqrt{5}}{2}\cdot\frac{\sqrt{5}}{4} + 0 + \frac{\sqrt{15}}{2}\cdot\frac{\sqrt{15}}{4}\right)$$

$$= \frac{1}{5}\cdot\frac{5}{4}$$

$$= \frac{1}{4}$$

$$\Rightarrow LHS = \frac{1}{4} = RHS$$

$\Rightarrow$ verified