Advanced Mathematics

Lab 1: Homogenous differential equations – constant coefficients and reduction of order

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GEODÄTISCHES INSTITUT der Universität Stuttgart
anerkannt
65%

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1.1	1.2	1.3	2	3	4	exercise points
12	11	10	14	11	10.	

Constant coefficients

1. Solve the homogeneous differential equations with constant coefficients

$$y'' - 4y' + 13y = 0 \qquad y(\pi/6) = -8 \text{ and } y'(\pi/6) = 2 \tag{1.1}$$
$$y'' - 26y' + 169y = 0 \qquad y(2) = 2 \text{ and } y'(0) = 4 \tag{1.2}$$
$$4y'' + 16y' + 18y = 0 \qquad y(2) = 4 + 2\imath \text{ and } y'(2) = -1 - 4\imath \tag{1.3}$$

and consider the initial values. (12+11+14 points)

Matlab

2. In case of constant coefficients, the procedure can be extended to higher order differential equations, which requires the roots of a polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with degree n. Implement the *Horner scheme*

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ & b_n x_0 & b_{n-1}x_0 & \cdots & b_2 x_0 & b_1 x_0 \\ \hline b_n & b_{n-1} & b_{n-2} & \cdots & b_1 & b_0 = P(x_0) \end{array} \qquad \text{with } b_i = \begin{cases} a_n & i = n \\ a_i + b_{i+1}x_0 & \text{else} \end{cases}$$

and note down the solution of the differential equation

$$2y'''' + 4y''' - 34y'' - 36y' + 144y = 0$$

- function call: horner(an,x0)
  with the coefficient vector an= $[a_n, a_{n-1}, \ldots, a_1, a_0]$ and the guess x0 for the root
- Reasonable help text!
- Check for dimension and variable type (numerical, scalar, vector)
- The horner scheme should be presented on the screen.

(16 points)

Reduction of the order

3. Solve the following differential equation by **reduction of order**:

$$-xy'' + (x-2)y' + y = 0$$

$y_1 = e^x$, $y_2 =$

and consider the initial values $y(1) = 1$ and $y'(1) = 1$. (14 points)

4. Apply the reduction of order onto the differential equation

$$\cos^2 x y'' - 6y = 0,$$

where the first solution is of the form $y_1 = (\beta \tan^2 x + 1)$ with $\beta \in \mathbb{R}$.
(midterm exam WS17/18 33 points)

[V]: Reduction of order

v Solve the following differential equation by **reduction of order**:

$$x^2(x-2)y'' - 2x(2x-3)y' + 6(x-1)y = 0$$

[V] Refresh: integration of hyperbolic functions

vi Solve the integrals by integration by parts, substitution and/or partial fraction decomposition

$$\int \sinh x \sin x \mathrm{d}x \quad \text{(iv.1)}$$

$$\int \frac{1}{\sinh^2 x \cosh x} \mathrm{d}x \quad \text{(iv.2)}$$

$$\int \sqrt{\coth x} \mathrm{d}x \quad \text{(iv.3)}$$

Hint for problem (iv.3): 'Divide' by the unit, express the nominator via hyperbolic functions and use afterwards the substitution $v = \sqrt{\coth x}$.

Constant coefficients

1.

1.1 $y''-4y'+13y=0$ with $y(\frac{\pi}{6})=-8$ and $y'(\frac{\pi}{6})=2$

$\Rightarrow$ Consider the characteristic equation of this ODE:

$$\lambda^2-4\lambda+13=0$$

We get two conjugated complex roots:

$$\lambda_1=2+3i,\ \lambda_2=2-3i$$

Hence we get two solutions $y_1=e^{(2+3i)x}$, $y_2=e^{(2-3i)x}$

According to the superposition principle,

$$Y_1=\frac{y_1+y_2}{2}=\frac{e^{2x}(e^{3ix}+e^{-3ix})}{2}=e^{2x}\cdot\cos 3x$$

$$Y_2=\frac{y_1-y_2}{2i}=\frac{e^{2x}(e^{3ix}-e^{-3ix})}{2i}=e^{2x}\cdot\sin 3x$$

are both the solutions of the ODE.

Hence we get the general solution:

$$\underline{y=e^{2x}(C_1\cos 3x+C_2\sin 3x)}$$

Then take into consideration the initial values:

$$y(\tfrac{\pi}{6})=e^{\frac{\pi}{3}}(C_1\cos\tfrac{\pi}{2}+C_2\sin\tfrac{\pi}{2})=e^{\frac{\pi}{3}}\cdot C_2=-8$$

$$y'(\tfrac{\pi}{6})=e^{\frac{\pi}{3}}(-3C_1\sin\tfrac{\pi}{2}+3C_2\cos\tfrac{\pi}{2})+2\cdot e^{\frac{\pi}{3}}(C_1\cos\tfrac{\pi}{2}+C_2\sin\tfrac{\pi}{2})=e^{\frac{\pi}{3}}(-3C_1+2C_2)=2$$

Hence $C_1=\frac{-6}{e^{\frac{\pi}{3}}}$, $C_2=\frac{-8}{e^{\frac{\pi}{3}}}$

And the final solution is:

$$\underline{y=e^{2x-\frac{\pi}{3}}(-6\cos 3x-8\sin 3x)}$$

1.2 $y'' - 26y' + 169y = 0 \qquad y(2) = 2$ and $y'(0) = 4$.

$\Rightarrow$ Consider the characteristic equation:

$$\lambda^2 - 26\lambda + 169 = 0$$

we get real double root $\lambda = 13$. and one solution $y_1 = e^{13x}$.

We assume another solution $y_2 = u(x)\, y_1$.

insert y_2' and y_2'' into the ODE:

$$e^{13x}(u'' + 26u' + 169u) - 26e^{13x}(u' + 13u) + 169\, e^{13x} \cdot u = 0$$

We get $u'' = 0$, hence one u with $u = x$ fits the condition.

Hence we get $y_2 = x \cdot e^{13x}$.

Hence we get the general solution:

$$y = e^{13x}(C_1 + C_2 x)$$

Now we take into consideration the initial values:

$$y(2) = e^{26}(C_1 + 2C_2) = 2$$

$$y'(0) = C_2 + 13C_1 = 4$$

$$\Rightarrow \begin{cases} C_1 = \frac{1}{25} \cdot \left(8 - \frac{2}{e^{26}}\right) \\ C_2 = \frac{1}{25} \cdot \left(\frac{26}{e^{26}} - 4\right) \end{cases}$$

Hence the final solution is:

$$y = \frac{e^{13x}}{25}\left[8 - \frac{2}{e^{26}} + \left(\frac{26}{e^{26}} - 4\right) \cdot x\right]$$

1.3 $4y'' + 16y' + 18y = 0$. $y(2) = 4+2i$ and $y'(2) = -1-4i$

$\Rightarrow$ Consider the characteristic equation:

$$4\lambda^2 + 16\lambda + 18 = 0$$ ✓

We get two conjugated complex roots: ✓

$$\lambda_1 = -2 + \frac{\sqrt{2}}{2} i, \quad \lambda_2 = -2 - \frac{\sqrt{2}}{2} i$$

Hence we get two solutions $y_1 = e^{(-2+\frac{\sqrt{2}}{2}i)x}$, $y_2 = e^{(-2-\frac{\sqrt{2}}{2}i)x}$ ✓

Then the general solution:

$$y = C_1 \cdot \frac{y_1 + y_2}{2} + C_2 \cdot \frac{y_1 - y_2}{2i}$$

$$= e^{-2x}\left(C_1 \cos\frac{\sqrt{2}}{2}x + C_2 \sin\frac{\sqrt{2}}{2}x\right)$$ ✓

Now take into consideration the initial values:

$$y(2) = e^{-4}\left(C_1 \cos\sqrt{2} + C_2 \sin\sqrt{2}\right) = 4+2i$$ ✓

$$y'(2) = e^{-4}\left(-\frac{\sqrt{2}}{2} C_1 \sin\sqrt{2} + \frac{\sqrt{2}}{2} C_2 \cos\sqrt{2}\right) + (-2)\cdot e^{-4}\left(C_1 \cos\sqrt{2} + C_2 \sin\sqrt{2}\right) = -1-4i$$ ✓

$\Rightarrow$ $C_1 = 4\cos\sqrt{2} + \cancel{5\sqrt{2}\sin\sqrt{2} + (2\cos\sqrt{2} + 6\sqrt{2}}\sin\sqrt{2})i$ × $C_1 = \left((4+2i)\cos\sqrt{2} - 7\sqrt{2}\sin\sqrt{2}\right) e^{4}$

$C_2 = 4\sin\sqrt{2} + \cancel{5\sqrt{2}\cos\sqrt{2} + (2\sin\sqrt{2} + }6\sqrt{2}\cos\sqrt{2})i$ × $C_2 = \left((4+2i)\sin\sqrt{2} + 7\sqrt{2}\cos\sqrt{2}\right) \cdot e^{4}$ -4

Hence the final solution is:

$$y = e^{4-2x}\left(\left[4\cos\sqrt{2} + \cancel{5\sqrt{2}\sin\sqrt{2} + (2\cos\sqrt{2} + 6\sqrt{2}\sin\sqrt{2})}i\right]\cdot\cos\frac{\sqrt{2}}{2}x\right.$$ ×

$$\left. + \left[4\sin\sqrt{2} \cancel{+ 5\sqrt{2}\cos\sqrt{2} + (2\sin\sqrt{2} + 6\sqrt{2}\cos\sqrt{2})i}\right]\cdot\sin\frac{\sqrt{2}}{2}x\right)$$ ×

$$y = e^{4-2x}\left(\left[(4+2i)\cos\sqrt{2} - 7\sqrt{2}\sin\sqrt{2}\right]\cos\frac{\sqrt{2}}{2}x + \left[(4+2i)\sin\sqrt{2} + 7\sqrt{2}\cos\sqrt{2}\right]\sin\frac{\sqrt{2}}{2}x\right)$$

Reduction of the order:

3. $-xy''+(x-2)y'+y=0.$ $y(1)=1$ and $y'(1)=1$

$\Rightarrow$ We assume that $y_1=x^k$ is the of the solutions.

insert into the ODE:

$$y_1'=kx^{k-1},\quad y_1''=k(k-1)x^{k-2}$$

Hence $-x\cdot k(k-1)x^{k-2}+(x-2)kx^{k-1}+x^k=0$

$\Rightarrow (k-x)(k+1)x^{k-1}=0$

$\Rightarrow k=-1.$ ✓

Hence $y_1=\frac{1}{x}$ is a solution. ✓

According to the "Reduction of order" theory.

we get $y_2=\int\frac{1}{y_1^2}\cdot e^{-\int(\frac{x-2}{x})dx}dx\cdot y_1$

step for $\int x^2 e^{x-2\ln x}dx=e^x$

$=\int x^2\cdot e^{x-2\ln x}dx\cdot y_1$

-3

$=\int x^2\cdot\frac{e^x}{e^{\ln x^2}}dx\cdot y_1=\int\frac{x^2}{x^2}\cdot e^x dx\cdot y_1=e^x\cdot y_1=\frac{e^x}{x}$

Hence the general solution:

$$y=\frac{C_1+C_2\cdot e^x}{x}$$

Now consider the initial values:

$y(1)=\frac{C_1+C_2\cdot e}{1}=1$ ✓

$y'(1)=\frac{C_2e\cdot 1-C_1-C_2e}{1^2}=-C_1=1$ ✓

$\Rightarrow C_1=-1,\ C_2=\frac{2}{e}$

Hence the final solution

$$y=\frac{-1+\frac{2}{e}e^x}{x}$$

or $y=\frac{-1+2\cdot e^{x-1}}{x}$ ✓

$\cos^2 x\, y'' - 6y = 0$ where the first solution is of the form $y_1 = \beta\tan^2 x + 1$

$\Rightarrow$ insert $y_1 = \beta\tan^2 x + 1$ into the ODE:

$y_1' = \beta \cdot 2\tan x \cdot (1+\tan^2 x)$ ✓

$y_1'' = 2\beta \cdot \frac{1}{\cos^2 x}(1+3\tan^2 x)$ ✓

Hence $\cos^2 x \cdot 2\beta \cdot \frac{1}{\cos^2 x}(1+3\tan^2 x) - 6\cdot(\beta\tan^2 x+1) = 0.$

$\Rightarrow$ $\beta = 3$ ✓ +5.

Hence we get one solution $\underline{y_1 = 3\tan^2 x + 1}$. ✓

According to the "reduction of order" theory, we can get y_2.

$y_2 = \int \frac{1}{y_1^2} \cdot e^{-\int 0 dx} dx \cdot y_1$ — steps for normalization +2.

$= \int \frac{1}{(3\tan^2 x+1)^2} dx \cdot (3\tan^2 x + 1)$

$u=\tan x$ $= \int \frac{1}{(3u^2+1)^2} \cdot \frac{1}{1+u^2} du \cdot (3\tan^2 x + 1)$ $\int \frac{4u^2+1}{(3u^2+1)^2(u^2+1)} du$ × =

$u = \sqrt{3}\tan \cancel{x}$ $= \left[-\frac{1}{2}\int \frac{1}{(3u^2+1)^2} du + \frac{3}{2}\int \frac{1}{(3u^2+1)(u^2+1)} du\right] \cdot (3\tan^2 x+1)$ ×

$= \left[-\frac{1}{2}\int \frac{1}{((\sqrt{3}u)^2+1)^2} d(\sqrt{3}u) \cdot \frac{1}{\sqrt{3}} + \frac{3}{4}\cdot\left(\int \frac{3}{3u^2+1} du - \int \frac{1}{u^2+1} du\right)\right] \cdot (3\tan^2 x+1)$

$\xrightarrow{\tan v}$ $\int \frac{1}{(\tan^2 v+1)^2} d(\tan v) = \int \frac{1}{\tan^2 v + 1} dv = \int \cos^2 v\, dv = \int \frac{1+\cos 2v}{2} dv = \frac{1}{2}v + \frac{1}{4}\sin 2v$ +1

$= \frac{1}{2}v + \frac{1}{4}\cdot\frac{2\tan v}{1+\tan^2 v} = \frac{1}{2}\cdot\arctan(\sqrt{3}u) + \frac{1}{2}\cdot\frac{\sqrt{3}u}{1+3u^2}$

$\rightarrow \int \frac{3}{3u^2+1} du = \int \frac{\sqrt{3}}{t^2+1} dt = \sqrt{3}\arctan(t) = \sqrt{3}\arctan(\sqrt{3}u)$ +1

$\int \frac{1}{u^2+1} du = \arctan(u)$

$y_2 = \left[-\frac{1}{2\sqrt{3}} \cdot \frac{1}{2}\left(\arctan(\sqrt{3}u) + \frac{\sqrt{3}u}{1+3u^2}\right) + \frac{3}{4}\cdot\sqrt{3}\arctan(\sqrt{3}u) - \frac{3}{4}\cdot\arctan u\right] \cdot (3\tan^2 x+1)$

$\xrightarrow{\tan x}$ $= \left[\frac{2}{3}\sqrt{3}\arctan(\sqrt{3}\tan x) - \frac{1}{4}\frac{\tan x}{1+3\tan^2 x} - \frac{3}{4}x\right] \cdot (3\tan^2 x+1)$ (reverse side)

+1

Hence the general solution:

$$y = C_1(3\tan^2 x+1) + C_2\cdot\left[\frac{2}{3\sqrt{3}}\arctan(\sqrt{3}\tan x) - \frac{1}{4}\cdot\frac{\tan x}{1+3\tan^2 x} - \frac{3}{4}x\right]\cdot(3\tan$$

$$y_2 = \int \frac{1}{y_1^2} e^{-\int p\,dx}\,dx \cdot y_1 = \int \frac{1}{(3\tan^2 x+1)^2}\cdot e^{-\int 0\,dx}\,dx\cdot(3\tan^2 x+1)$$

$$= \int \frac{1}{(3\tan^2 x+1)^2}\,dx \cdot (3\tan^2 x+1)$$

$$\int \frac{1}{(3\tan^2 x+1)^2}\,dx \xlongequal[x=\arctan\frac{u}{\sqrt{3}}]{u=\sqrt{3}\tan x} \int \frac{1}{(u^2+1)^2}\cdot\frac{1}{1+\frac{u^2}{3}}\,du = \int \frac{3}{(u^2+1)^2(u^2+3)}\,du$$

$$= \frac{3}{2}\int\left[\frac{1}{(u^2+1)^2} - \frac{1}{(u^2+3)(u^2+1)}\right]du \quad (*)$$

① $$\int \frac{1}{(u^2+1)^2}\,du \xlongequal[v=\arctan u]{u=\tan v} \int \frac{1}{(\tan^2 v+1)^2}\cdot(1+\tan^2 v)\,dv = \int \frac{1}{\tan^2 v+1}\,dv = \int \cos^2 v\,dv$$

$$= \int \frac{1+\cos 2v}{2}\,dv = \frac{1}{2}v + \frac{1}{2}\int \cos 2v\,dv = \frac{1}{2}v + \frac{1}{4}\sin 2v = \frac{1}{2}\arctan u + \frac{1}{2}\cdot\frac{u}{u^2+1}$$

② $$\int \frac{1}{(u^2+3)(u^2+1)}\,du = \frac{1}{2}\int\left(\frac{1}{u^2+1} - \frac{1}{u^2+3}\right)du = \frac{1}{2}\left(\arctan u - \frac{1}{\sqrt{3}}\cdot\arctan\frac{u}{\sqrt{3}}\right).$$

$$\Rightarrow (*) = \frac{3}{2}\cdot\left(\cancel{\frac{1}{2}\arctan u} + \frac{1}{2}\frac{u}{u^2+1} - \cancel{\frac{1}{2}\arctan u} + \frac{1}{2\sqrt{3}}\arctan\frac{u}{\sqrt{3}}\right)$$

$$= \frac{3}{4}\left(\frac{u}{u^2+1} + \frac{1}{\sqrt{3}}\arctan\frac{u}{\sqrt{3}}\right)$$

$$\xlongequal{u=\sqrt{3}\tan x} \frac{3}{4}\left(\frac{\sqrt{3}\tan x}{1+3\tan^2 x} + \frac{1}{\sqrt{3}}\cdot x\right) = \frac{\sqrt{3}}{4}x + \frac{3\sqrt{3}\tan x}{4(1+3\tan^2 x)}$$

$$\Rightarrow y_2 = \frac{\sqrt{3}}{4}x(3\tan^2 x+1) + \frac{3\sqrt{3}\tan x}{4}$$

$= 4.$

$+ 9 \cdot t - t.$

$y_0 + (9-1)t = x$

$\cancel{3t + t^2 + y_0 - x_0} =$

[V] Reduction of order

5. $x^2(x-2)y'' - 2x(2x-3)y' + 6(x-1)y = 0.$

Assume $y_1 = x^k$, insert into the ODE:

$$y_1' = kx^{k-1}, \quad y_1'' = k(k-1)x^{k-2}$$

$$\Rightarrow x^2(x-2)\cdot k(k-1)x^{k-2} - 2x(2x-3)\cdot kx^{k-1} + 6(x-1)\cdot x^k = 0$$

simplify this equation and we get $k=3$.

$\Rightarrow \quad y_1 = x^3$ is one solution

Then we assume $y_2 = u(x)\cdot y_1$, by "reduction of order" we get

$$y_2 = \int \frac{1}{y_1^2} e^{-\int p\,dx}\,dx \cdot y_1 = \int \frac{1}{x^6} e^{2\cdot\int \frac{2x-3}{x(x-2)}dx} = \int \frac{1}{x^6}\cdot x^3(x-2)\,dx$$

$$= -\frac{x-1}{x^2}\cdot x^3$$

Hence the general solution

$$y = C_1\cdot x^3 - C_2\cdot x(x-1)$$

[V] Refresh: integration of hyperbolic functions

6.1 $\int \sinh x \sin x\,dx$

② you can just use $\sinh x$ and $\cosh x$

$$\begin{cases} d\sinh x = \cosh x \\ d\sin x = -\cos x \end{cases}$$

① $= \int \frac{e^x - e^{-x}}{2}\sin x\,dx = \frac{1}{2}\left(\int e^x \sin x\,dx - \int e^{-x}\sin x\,dx\right)$

while $\int e^x \sin x\,dx = \int \sin x\,d(e^x) = e^x\sin x - \int e^x d(\sin x) + C_1 = e^x\sin x - \int e^x\cos x\,dx + C_1$

$$= e^x\sin x - \left(e^x\cos x - \int e^x d(\cos x) + C_2\right) + C_1$$

$$= e^x\sin x - e^x\cos x - \int e^x\sin x\,dx - C_2 + C_1$$

$$\begin{cases} \int \sinh x \sin x\,dx = -\sinh x\cos x + \int \\ \int \cos x\cosh x\,dx = \cosh x\sin x - \int \sinh \end{cases}$$

$$\Rightarrow \frac{\cosh x\sin x - \sinh x\cos x}{2} + C$$

$$\Rightarrow 2\int e^x\sin x\,dx = e^x\sin x - e^x\cos x + C_1 - C_2$$

$$\Rightarrow \int e^x\sin x\,dx = \frac{e^x\sin x - e^x\cos x}{2} + C_3$$

Similarly, $\int e^{-x}\sin x\,dx = -\frac{e^{-x}\sin x + e^{-x}\cos x}{2} + C_4$

Hence $\int \sinh x\sin x\,dx = \frac{1}{4}\left[e^x(\sin x - \cos x) + e^{-x}(\sin x + \cos x)\right] + C$

reverse

.2

$$\int \frac{1}{\sinh^2 x \cosh x}\,dx$$

$$= \int \frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x \cosh x}\,dx = \int \frac{\cosh x}{\sinh^2 x}\,dx - \int \frac{1}{\cosh x}\,dx$$

$$= \int \frac{1}{\sinh^2 x}\,d(\sinh x) - \int \frac{\cosh x}{\cosh^2 x}\,dx$$

$$= \cancel{\ln(\sinh x)}\ -\frac{1}{\sinh x} + C_1 - \int \frac{1}{1+\sinh^2 x}\cdot d(\sinh x)$$

$$= \cancel{\ln(\sinh x)}\ -\frac{1}{\sinh x} + C_1 - \arctan(\sinh x) + C_2$$

$$= \cancel{\ln(\sinh x)}\ -\frac{1}{\sinh x} - \arctan(\sinh x) + C$$

3

$$\int \sqrt{\coth x}\,dx$$

$$= \int \frac{\sqrt{\coth x}}{\cosh^2 x - \sinh^2 x}\,dx = \int \frac{\sqrt{\coth x}\cdot(-2\sqrt{\coth x}\cdot \sinh^2 x)}{\cosh^2 x - \sinh^2 x}\,d(\sqrt{\coth x}) \quad = \int \frac{-2\cosh x \sinh x}{\cosh^2 x - \sinh^2 x}\,d\sqrt{\coth x}$$

$$\times \frac{1/\sinh^2 x}{1/\sinh^2 x} = \int \frac{-2\coth x}{\coth^2 x - 1}\,d\sqrt{\coth x}$$

$$= -\int \frac{2\coth x}{\coth^2 x - 1}\,d(\sqrt{\coth x})$$

$$= 2\int \frac{v^2}{1-v^4}\,dv = \cancel{\tfrac{1}{2}}\int \left(\frac{1}{1-v^2} - \frac{1}{1+v^2}\right)dv$$

$$= \cancel{\tfrac{1}{2}}\left(\int \frac{1}{1-v^2}\,dv - \int \frac{1}{1+v^2}\,dv\right) \qquad = \int \frac{-2v}{v^2-1}\cdot dv$$

$$= \cancel{\tfrac{1}{2}}(\operatorname{arccoth} v - \arctan v) + C$$

$$= \cancel{\tfrac{1}{2}}\left[\operatorname{arccoth}(\sqrt{\coth x}) - \arctan(\sqrt{\coth x})\right] + C \quad ✗$$

...mine the Wronskian of y_1 and y_2.