Advanced Mathematics

Lab 6: Arc length in curvilinear frames

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anerkannt

1	2	3a	3b	4a	4b	problem
25	5	18	12	10	13	points

eneral remark: In the first use of a particular (new) curvilinear frame, the "frame vectors" and eir normalization should be denoted/derived!

1. Determine the interval so that the curve $\boldsymbol{\Psi}(t) = \left(\cos(8t) + 4\cos(2t) \quad \sin(8t) + 4\sin(2t)\right)^{\mathsf{T}}$ is closed and figure out the arc length for this case. (25 points)

2. A curve is called a loxodrome $\boldsymbol{\ell}_S(t)$ of the surface S, if the intersection angles between the curve and the (orthogonal) parameter lines are constant. Derive a representation of a loxodrome around a unit sphere, which is parametrized w.r.t. longitude and co-latitude, and determine the arc length s as a function of the intersection angle $\omega \in (0, 0.5\pi)$.
Use the co-latitude as curve parameter (midterm WS17/18 22 points)

3. The relationship between Cartesian and a set of curvilinear coordinates (α, β, γ) is given by

$$x = \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\cos\gamma$$
$$y = \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}\sin\gamma$$
$$z = \frac{\alpha^2-\beta^2}{2(\alpha^2+\beta^2)^2}.$$

   a) Derive the normalized frame vectors $\hat{\boldsymbol{h}}_{q_i}$ of thise system.

   b) Determine the tangent vector $\boldsymbol{T}$ and the arc length s of the 'meridian' with α = const. and γ = const. for $\beta = [0, B]$. *The integral should be solved step by step here, e.g. by substitution, partial fraction decomposition or integration by parts!*

(partial midterm WS15/16, 30 points)

4. Given the curve $\Psi = (10 \sin t)\hat{h}_p + 8 \cos t \hat{h}_z$ with $\varphi = \frac{3t}{5}$ in cylindrical coordinates.

   a) Calculate the tangent vector T and the arc length s without using Cartesian expressions.

   b) Determine the coordinates $\{\alpha(t), \beta(t), \gamma(t)\}$ and the (positive) scaling parameter p for this curve in (modified) oblate spheroidal coordinates

$$
\begin{aligned}
x &= p \cosh \alpha \sin \beta \sin \gamma \\
y &= p \cosh \alpha \sin \beta \cos \gamma \\
z &= p \sinh \alpha \cos \beta
\end{aligned}
$$

   and express the tangent vector T in terms of *non-normalized* 'frame vectors' without calculating h_{q_i} explicitly. All points of the curve are on the surface $\alpha = \text{const.}$

(midterm WS16/17 23 points)

V] Closed curve

5. Calculate the arc length of the graph $y = x^{5/4}$ for the interval $x \in [0, 1]$. *In this question, the multiplications can be performed by pocket calculator*

$\psi(t) = \left(\cos 8t + 4\cos 2t \quad \sin 8t + 4\sin 2t \right)^T$

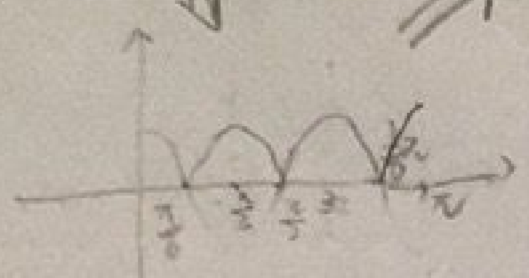
$T_x = \frac{2\pi}{2} = \pi, \quad T_y = \frac{2\pi}{2} = \pi. \Rightarrow$ the interval is $[0, \pi]$ ✓

$$T = \psi' = \frac{\partial \psi(t)}{\partial t} = \begin{bmatrix} \frac{\partial(\cos 8t + 4\cos 2t)}{\partial t} \\ \frac{\partial(\sin 8t + 4\sin 2t)}{\partial t} \end{bmatrix} = \begin{bmatrix} -8\sin 8t - 8\sin 2t \\ 8\cos 8t + 8\cos 2t \end{bmatrix}$$

$= \int_0^{\pi} \sqrt{64\cdot(2 + 2\sin 8t \sin 2t + 2\cos 8t \cos 2t)}\, dt$ ✓

$$= \int_0^{\pi} \sqrt{T^T \cdot T}\, dt = \int_0^{\pi} \sqrt{8^2\cdot(\sin 8t + \sin 2t)^2 + 8^2\cdot(\cos 8t + \cos 2t)^2}\, dt = \int_0^{\pi} \sqrt{64\cdot(2 + 2\cos 6t)}\, dt$$ ✓

$$= \int_0^{\pi} 8\sqrt{2\cdot 2\cos^2 3t}\, dt = \int_0^{\pi} 16|\cos 3t|\, dt = 6\cdot 16 \int_0^{\frac{\pi}{6}} \cos 3t\, dt = 32.$$ ✓



because $\psi_1(t)$ is on the spherical surface

$\Rightarrow$ it can be written as $\psi_1 = \begin{pmatrix} \cos\lambda(t)\sin\Delta(t) \\ \sin\lambda(t)\sin\Delta(t) \\ \cos\Delta(t) \end{pmatrix}$.

$\dot{T}_1 = \dot{\lambda}\cdot|h_\lambda|\cdot\hat{h_\lambda} + \dot{\Delta}\cdot|h_\Delta|\cdot\hat{h_\Delta}$

Loxodrome

spherical system:

$$\begin{cases} x = r\cos\lambda\sin\phi \\ y = r\sin\lambda\sin\phi \\ z = r\cos\phi \end{cases}$$

in this case,

$$\cancel{\begin{cases} x = \cos\lambda\cos\Delta \\ y = \sin\lambda\cos\Delta \\ z = \sin\Delta \end{cases}}$$

$= \dot{\lambda}\cdot\sin\Delta\,\hat{h_\lambda} + \dot{\Delta}\cdot 1\cdot\hat{h_\Delta}\ (t)$

let $\Delta = t$,

$\Rightarrow T_1 = \dot{\lambda}\sin t\,\hat{h_\lambda} + \hat{h_\Delta}$

$+2$

$h_\lambda = \left(\frac{\partial x}{\partial\lambda}, \frac{\partial y}{\partial\lambda}, \frac{\partial z}{\partial\lambda}\right)^T = (-\sin\lambda\sin\Delta, \cos\lambda\sin\Delta, 0)^T$

chose parallels to calculate the intersection

$\psi_2 = \begin{pmatrix} \cos\lambda(t)\sin\Delta_0 \\ \sin\lambda(t)\sin\Delta_0 \\ \cos\Delta_0 \end{pmatrix}, \quad T_2 = \dot{\lambda}\cdot\sin t\,\hat{h_\lambda}$

$\hat{h_\lambda} = (-\sin\lambda, \cos\lambda, 0)^T \quad |h_\lambda| = \sin\Delta.$

$h_\Delta = (\cos\lambda\cos\Delta, \sin\lambda\cos\Delta, -\sin\Delta)^T$

$\hat{h_\Delta} = (\cos\lambda\cos\Delta, \sin\lambda\cos\Delta, -\sin\Delta)^T. \quad |h_\Delta| = 1$

$$\Rightarrow \cos w = \frac{T_1\cdot T_2}{|T_1|\cdot|T_2|} = \frac{\dot{\lambda}^2\sin^2 t}{\sqrt{\dot{\lambda}^2\sin^2 t + 1}\cdot\dot{\lambda}\sin t} = \frac{\dot{\lambda}\sin t}{\sqrt{\dot{\lambda}^2\sin^2 t + 1}} = \text{const}$$

$$\Rightarrow \dot{\lambda} = \frac{c}{\sin t} \Rightarrow \lambda = c\int\frac{1}{\sin t}dt = c\int\frac{1 + \tan^2\frac{t}{2}}{2\tan\frac{t}{2}}dt = c\ln|\tan\frac{t}{2}| + \lambda_0$$

$$\Rightarrow S = \int\sqrt{T^T T}\,dt = \int\sqrt{c^2+1}\,dt = \sqrt{c^2+1}\;t.$$

~~tangent vectors of~~

~~unit sphere & loxodrome~~

obviously $\hat{h_\lambda}\cdot\hat{h_\Delta} = 0.$

assume the unit vector of the loxodrome is

$$\hat{h_w} = \sin w\cdot\hat{h_\lambda} + \cos w\cdot\hat{h_\Delta}$$

$$\Rightarrow \psi_s(t) = \begin{pmatrix} \cos(c\cdot\ln|\tan\frac{t}{2}| + \lambda_0)\sin t \\ \sin(c\cdot\ln|\tan\frac{t}{2}| + \lambda_0)\sin t \\ \cos t \end{pmatrix}$$

in which case $\hat{h_w}\cdot\hat{h_\Delta} = \cos w$, i.e. the angles between $\hat{h_w}$ and $\hat{h_\Delta}$ are w for $\forall\lambda$ and ϕ ✗

since the loxodrome is on the unit sphere

$\Rightarrow \quad dr = r\,\hat{h_w}\cdot ds \quad$ i.e. $\frac{\partial r}{\partial\lambda}d\lambda + \frac{\partial r}{\partial\Delta}d\Delta = (\sin w\,\hat{h_\lambda} + \cos w\,\hat{h_\Delta})\cdot ds$

$\Rightarrow \cos\Delta\,\hat{h_\lambda}\cdot d\lambda + \hat{h_\Delta}d\Delta = \sin w\,\hat{h_\lambda}\cdot ds + \cos w\,\hat{h_\Delta}\cdot ds$

$\Rightarrow \quad ds = \frac{\cos\Delta}{\sin w}d\lambda = \frac{d\Delta}{\cos w}$

difficult to understand

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$\Rightarrow \frac{d\lambda}{d\Delta} = \tan w \cdot \sec\Delta.$ read the question

& $s = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos w} d\Delta = \frac{\pi}{\cos w}$. $\underset{\sim}{\ell}_s(t)$ t is your parameter not w.

$\Rightarrow \lambda = \int \tan w \cdot \sec\Delta d\Delta = \tan w \cdot \int \frac{\cos\Delta}{\cos^2\Delta} d\Delta = \tan w \cdot \int \frac{1}{1-\sin^2\Delta} d\sin\Delta$

$\xrightarrow{x=\sin\Delta} \int \frac{1}{1-x^2} dx = \operatorname{artanh}(\sin\Delta) \cdot \tan w$ (assume $\Delta = [0, \pi]$).

$\Rightarrow$ the loxodrome expression: +3

$$\begin{cases} x = \cos\lambda \cos\Delta = \cos\left(\tan w \cdot \operatorname{artanh}(\sin\Delta)\right) \cos\Delta \\ y = \sin\lambda \cos\Delta = \sin\left(\tan w \cdot \operatorname{artanh}(\sin\Delta)\right) \cos\Delta \\ z = \sin\Delta \end{cases}$$ ✗

a)

$$\underset{\sim}{h_\alpha} = \left(\frac{\partial x}{\partial \alpha}, \frac{\partial y}{\partial \alpha}, \frac{\partial z}{\partial \alpha}\right)^T = \begin{pmatrix} (\beta^3 - 3\alpha^2\beta)\cos r \\ (\beta^3 - 3\alpha^2\beta)\sin r \\ -\alpha^3 + 3\alpha\beta^2 \end{pmatrix} \frac{1}{(\alpha^2+\beta^2)^3}$$ ✓

$$|\underset{\sim}{h_\alpha}| = \sqrt{(\beta^3 - 3\alpha^2\beta)^2 + (-\alpha^3 + 3\alpha\beta^2)^2} \cdot \frac{1}{(\alpha^2+\beta^2)^3} = \frac{\sqrt{(\alpha^2+\beta^2)^3}}{(\alpha^2+\beta^2)^3}$$ ✓

$$\Rightarrow \hat{\underset{\sim}{h_\alpha}} = \frac{1}{\sqrt{(\alpha^2+\beta^2)^3}} \cdot \begin{pmatrix} (\beta^3 - 3\alpha^2\beta)\cos r \\ (\beta^3 - 3\alpha^2\beta)\sin r \\ -\alpha^3 + 3\alpha\beta^2 \end{pmatrix}$$ ✓

$$\underset{\sim}{h_\beta} = \left(\frac{\partial x}{\partial \beta}, \frac{\partial y}{\partial \beta}, \frac{\partial z}{\partial \beta}\right)^T = \begin{pmatrix} (\alpha^3 - 3\beta^2\alpha)\cos r \\ (\alpha^3 - 3\beta^2\alpha)\sin r \\ \beta^3 - 3\beta\alpha^2 \end{pmatrix} \frac{1}{(\alpha^2+\beta^2)^3}$$ ✓

$$|\underset{\sim}{h_\beta}| = \sqrt{(\alpha^3 - 3\beta^2\alpha)^2 + (\beta^3 - 3\beta\alpha^2)^2} \cdot \frac{1}{(\alpha^2+\beta^2)^3} = \frac{\sqrt{(\alpha^2+\beta^2)^3}}{(\alpha^2+\beta^2)^3}$$ ✓

$$\Rightarrow \hat{\underset{\sim}{h_\beta}} = \frac{1}{\sqrt{(\alpha^2+\beta^2)^3}} \begin{pmatrix} (\alpha^3 - 3\alpha\beta^2)\cos r \\ (\alpha^3 - 3\alpha\beta^2)\sin r \\ \beta^3 - 3\alpha^2\beta \end{pmatrix}$$ ✓

$$\underset{\sim}{h_r} = \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}\right)^T = \begin{pmatrix} -\sin r \\ \cos r \\ 0 \end{pmatrix} \cdot \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}$$ ✓

$$|\underset{\sim}{h_r}| = \sqrt{\sin^2 r + \cos^2 r + 0} \cdot \frac{\sqrt{\alpha^2\beta^2}}{(\alpha^2+\beta^2)^2} = \frac{\alpha\beta}{(\alpha^2+\beta^2)^2}$$ ✓

$$\Rightarrow \hat{\underset{\sim}{h_r}} = \begin{pmatrix} -\sin r \\ \cos r \\ 0 \end{pmatrix}$$ ✓

$$\Rightarrow \quad \hat{h}_\alpha = \frac{1}{\sqrt{(\alpha^2+\beta^2)^3}} \begin{pmatrix} (\beta^3-3\alpha^2\beta)\cos\gamma \\ (\beta^3-3\alpha^2\beta)\sin\gamma \\ -\alpha^3+3\alpha\beta^2 \end{pmatrix}$$

$$\hat{h}_\beta = \frac{1}{\sqrt{(\alpha^2+\beta^2)^3}} \cdot \begin{pmatrix} (\alpha^3-3\alpha\beta^2)\cos\gamma \\ (\alpha^3-3\alpha\beta^2)\sin\gamma \\ \beta^3-3\alpha^2\beta \end{pmatrix}$$

$$\hat{h}_\gamma = \begin{pmatrix} -\sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

b) ~~$T = \sum_{i=1}^{3} \dot{q}_i h_i \ldots \frac{\partial h_\gamma}{\ldots} \ldots (\alpha,\beta = \text{const})$~~

$$T = \sum_{i=1}^{3} \dot{q}_i h_i = \frac{\partial\alpha}{\partial\beta} h_\alpha + \frac{\partial\beta}{\partial\beta} h_\beta + \frac{\partial\gamma}{\partial\beta} h_\gamma \quad (\alpha,\beta = \text{const})$$

$$= 0 + h_\beta + 0 = \frac{1}{(\alpha^2+\beta^2)^3} \begin{pmatrix} (\alpha^3-3\beta^2\alpha)\cos\gamma \\ (\alpha^3-3\beta^2\alpha)\sin\gamma \\ \beta^3-3\beta\alpha^2 \end{pmatrix}$$

$$T^T T = \frac{1}{(\alpha^2+\beta^2)^6} \cdot \left((\alpha^3-3\beta^2\alpha)^2 + (\beta^3-3\beta\alpha^2)^2 \right) = \frac{1}{(\alpha^2+\beta^2)^3}$$

$$\Rightarrow \quad L = \int_0^B \sqrt{T^T T}\, d\beta = \int_0^B \sqrt{\frac{1}{(\alpha^2+\beta^2)^3}}\, d\beta$$

$$= \frac{1}{\alpha^3} \int_0^B \frac{1}{1+(\frac{\beta}{\alpha})^2} \cdot \frac{1}{\sqrt{1+(\frac{\beta}{\alpha})^2}}\, d\beta$$

$$\xrightarrow[\substack{t=[0,t_0] \\ t_0 = \arctan\frac{B}{\alpha}}]{\frac{\beta}{\alpha} = \tan t} = \frac{1}{\alpha^3} \cdot \int_0^{t_0} \frac{1}{1+\tan^2 t} \cdot \frac{1}{\sqrt{1+\tan^2 t}}\, d(\alpha \tan t)$$

$$= \frac{1}{\alpha^3} \cdot \int_0^{t_0} \cos^2 t \cdot \cos t \cdot \alpha \cdot \frac{1}{\cos^2 t}\, dt$$

$$= \frac{1}{\alpha^2} \int_0^{t_0} \cos t\, dt = \frac{1}{\alpha^2} \cdot \sin t \Big|_0^{t_0}$$

$$= \frac{1}{\alpha^2} \cdot \sin t_0$$

because ~~t_0~~ $\frac{B}{\alpha} = \tan t_0 = \frac{\sin t_0}{\cos t_0} = \sqrt{\frac{\sin^2 t_0}{1-\cos^2 t_0}}$

solve this equation $\Rightarrow$ $\sin t_0 = \frac{B}{\sqrt{\alpha^2+B^2}}$

$$\Rightarrow L = \frac{1}{\alpha^2} \cdot \frac{B}{\sqrt{\alpha^2+B^2}}$$

Cylindrical system:

$$\begin{cases} \underset{\sim}{h_\rho} = \cos\phi\, i + \sin\phi\, j \\ \underset{\sim}{h_\phi} = (-\sin\phi\, i + \cos\phi\, j)\rho \\ \underset{\sim}{h_z} = \underset{\sim}{k} \end{cases} \Rightarrow \begin{matrix} |h_\rho| = 1 \\ |h_\phi| = \rho \\ |h_z| = 1. \end{matrix}$$

$$\rho = 10\sin t,\quad \phi = \frac{3}{5}t,\quad z = 8\cos t,$$

$$\Rightarrow T = \sum_{i=1}^{3} \dot{q}_i \underset{\sim}{h_i} = \dot{\rho}|h_\rho|\hat{h_\rho} + \dot{\phi}|h_\phi|\hat{h_\phi} + \dot{z}|h_z|\hat{h_z}$$

$$= 10\cos t \cdot 1 \cdot \hat{h_\rho} + \frac{3}{5}\cdot 10\sin t \cdot \hat{h_\phi} + (-8\sin t)\cdot 1 \cdot \hat{h_z}$$

$$= 10\cos t\, \hat{h_\rho} + 6\sin t\, \hat{h_\phi} - 8\sin t\, \hat{h_z}$$

$$T^T T = 10^2\cos^2 t + 6^2\sin^2 t + 8^2\sin^2 t = 100$$

$$S = \int\sqrt{T^T T}\, dt = \int 10\, dt = 10t$$

Cylindrical coordinates:

$$P = \rho\cos\phi\, \underset{\sim}{i} + \rho\sin\phi\, \underset{\sim}{j} + z\underset{\sim}{k}.$$

$$\Rightarrow \psi(t) = 10\sin t\cos\tfrac{3t}{5}\, \underset{\sim}{i} + 10\sin t\sin\tfrac{3t}{5}\, \underset{\sim}{j} + 8\cos t\, \underset{\sim}{k}.$$

$$\begin{cases} \rho\cosh\alpha\sin\beta\sin\gamma = 10\sin t\cos(\frac{3t}{5}) \\ \rho\cosh\alpha\sin\beta\cos\gamma = 10\sin t\sin(\frac{3t}{5}) \\ \rho\sinh\alpha\cos\beta = 8\cos t \end{cases}$$

$$\Rightarrow \begin{cases} \rho\cosh\alpha = 10 \\ \cancel{\rho\cosh\alpha} \\ \rho\sinh\alpha = 8 \end{cases} \ \& \ \begin{cases} \sin\beta = \sin t \\ \cos\beta = \cos t \end{cases} \ \& \ \begin{cases} \sin\gamma = \cos\frac{3t}{5} \\ \cos\gamma = \sin\frac{3t}{5} \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = \operatorname{artanh}\frac{5}{4} \\ \rho = 6 \end{cases} \qquad \beta = t \qquad \gamma = \frac{\pi}{2} - \frac{3t}{5}$$

$$\underset{\sim}{h_\beta} = \begin{pmatrix} \rho\cosh\alpha\sin\gamma\cos\beta \\ \rho\cosh\alpha\cos\gamma\cos\beta \\ -\rho\sinh\alpha\cos\beta \end{pmatrix} \qquad \underset{\sim}{h_\gamma} = \begin{pmatrix} \rho\cosh\alpha\sin\beta\cos\gamma \\ -\rho\cosh\alpha\sin\beta\sin\gamma \\ 0 \end{pmatrix}$$

not necessary

$$T = \sum_{i=1}^{3} \dot{q}_i \underset{\sim}{h_i} = \frac{\partial\beta}{\partial t}\underset{\sim}{h_\beta} + \frac{\partial\gamma}{\partial t}\underset{\sim}{h_\gamma} = \underset{\sim}{h_\beta} - \frac{3}{5}\underset{\sim}{h_\gamma} = \begin{pmatrix} 10\sin\gamma\cos\beta - 6\sin\beta\cos\gamma \\ 10\cos\gamma\cos\beta + 6\sin\beta\sin\gamma \\ -6\cos\beta \end{pmatrix} = \begin{pmatrix} 10\cos\frac{3}{5}t\cos t - 6\sin t\sin\frac{3}{5}t \\ 10\sin\frac{3}{5}t\cos t + 6\sin t\cos\frac{3}{5}t \\ -6\cos t \end{pmatrix}$$

see the graph as a curve $(x, x^{\frac{5}{4}})^T$.

$$T = \sum_{i=1}^{2} \dot{q}_i h_i = \frac{\partial x}{x} \cdot i + \frac{\partial (x^{\frac{5}{4}})}{x} \cdot j$$

$$= i + \frac{5}{4} x^{\frac{1}{4}} \cdot j$$

$$\Rightarrow S = \int_0^1 \sqrt{T^T T}\, dx = \int_0^1 \sqrt{\left(1 + \frac{25}{16} x^{\frac{1}{2}}\right)}\, dx$$

t $t = \sqrt{1 + \frac{25}{16} x^{\frac{1}{2}}} \in [1, \sqrt{\frac{41}{16}}]$

$$x = \left(\frac{16}{25}\right)^2 \cdot (t^2 - 1)^2$$

$$dx = \left(\frac{1024}{625} t^3 - \frac{1024}{625} t\right) dt$$

$$\Rightarrow S = \int_1^{\sqrt{\frac{41}{16}}} t \cdot \left(\frac{1024}{625} t^3 - \frac{1024}{625} t\right) dt$$

$$= \frac{1024}{625} \left(\frac{1}{5} t^5 - \frac{1}{3} t^3\right) \Big|_1^{\sqrt{\frac{41}{16}}}$$

$$\approx 1.423$$