GIS Institute of Geodesy University of Stuttgart

Advanced Mathematics – Final Exam

120 minutes *28 February 2019* open book

Linear algebra

1. Given the symmetric matrix

$$\underline{\mathbf{A}} = \begin{pmatrix} 9 & 6 & 3 & 3 & 3 \\ & 4 & e & 2 & 2 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & \text{sym.} & & & 1 \end{pmatrix}.$$

Apply **one** step of the (p, q)-rotation of Jacobi to find the eigen values. Note down the non-zero elements of the transformation matrix $\underline{\mathbf{U}}$ and the upper triangle of the resulting matrix $\underline{\mathbf{A}}^{[1]}$. Non-integer values should be given with 4 digits after the decimal separator. (15 points)

Laplace Operator

2. Use the 'frame vectors' $\hat{h}_{q_i}$ to derive (step-by-step) the Laplace operator in the system:

$$x = \arctan\frac{\alpha}{\beta}$$

$$y = \frac{1}{2}\ln(\alpha^2 + \beta^2).$$

For which functions $g(\beta)$ is the field $\Phi(\alpha, \beta) = g(\beta)\sinh\alpha - 3\alpha^2\beta + \beta^3$ harmonic in this coordinate system? (28 points)

ntegral theorems

$(3x^2, -7y^2, 0)$

3. Calculate the flux of the vector field $F = \left(x^3, \cancel{-y^3, z}\right)^{\mathsf{T}}$ through the body $\mathcal{B}$, which is defined in the following way:

- The curved 'bottom surface' is given by

$$\mathcal{H}_0 = \left\{x \in \mathbb{R}^3 : x^2 + y^2 - 8z^2 = 4, x \geq 0, y \geq 0, 1 \leq z \leq 2\right\}.$$

- Move each point of the surface $\mathcal{H}_0$ (without rotation) parallel to the vector $v = \begin{pmatrix}1, & 1, & 2\end{pmatrix}^{\mathsf{T}}$ until the plane $\mathcal{P} = \left\{x \in \mathbb{R}^3 : z + 2x + 2y = 6\right\}$.
- Every location in space which is passed by one of the points is part of the volume $\mathcal{B}$.

In case of an adequate parametrization, the trigonometric relations

$$\cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right), \qquad \cos x - \sin x = \sqrt{2}\sin\left(\frac{\pi}{4} - x\right)$$

might be helpful and can be used without proof. (30 points)

Differential equation

4. Given the differential equation

$$x^2 y'' + 3xy' = \frac{1}{y^3 x^4}.$$

a) Verify, that the ODE is *scale invariant.* Hence, there is a value $p \in \mathbb{R}\setminus\{0\}$ in such a way, that the substitution $x := a\bar{x}$ and $y := a^p \bar{y}$ leads to the same ODE in the expressions $\{\bar{x}, \bar{y}\}$. (10 points)

b) Introduce a new dependent function $u(x) = x^p y(x)$ and eliminate the variable x from the ODE by the substitution $x = \mathrm{e}^t$. The resulting ODE of first order can be solved by using $v = \dot{u}$ as new variable. *The determination of u is not required!* (17 points)

Good luck!