Zoom-in on Modulator: An Evolutionary Algorithm Enhanced Tree-width Dynamic Programming [*]

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Abstract. *Tree-width*(TW), obtained via *tree decomposition*, measures the similarity between a given graph and an exact tree. It is extensively utilized alongside with dynamic programming to design *fixed-parameter tractable* algorithms with complexity $\mathcal{O}(f(TW) \cdot n^{O(1)})$ when solving Np-hard graph combinatorial optimization problems. With the increase on TW, an exponential growth on f is inevitable, which directly restricts scalability only on graphs with bounded tree-widths. However, in practice, it is unrealistic to expect tree-width remains within a desired threshold.

In this work, we propose a general framework to improve the standard tree decomposition based dynamic programming by harnessing the power of elementary evolutionary algorithms such as (1+1)EA. To accelerate the efficiency, we put efforts on reducing the tree-width and reusing the constructed tree decomposition. On reducing tree-width, we adopt *tree-width modulator* with necessary adjustments, propose a more practical version of modulator. On reusing tree decomposition, we first fixate a *partial solution* on the vertices of modulator through a zoom-in EA. With the known partial solution, the dynamic programming can proceed and scale to much larger TWs.

To demonstrate the applicability of the model, we select three graph combinatorial problems to execute three sets of experiments analyzing the performance of the model in term of *solution quality*, *the number of fitness evaluations*, *wall-clock time* and *the first hitting time* of the obtained best solutions. As the *first study* to combine structural parameters and evolutionary computations, we focus on demonstrating the extent to which basic (1+1)EA and tree-width based dynamic programming can mutually benefit. Our experimental results indicate that the proposed framework significantly improves the scalability of the dynamic programming with non-optimal tree decompositions. It also substantially reduces the number of fitness evaluations, time cost and quickly yields a near-optimal solution. The improvements will be stronger by tuning the target tree-width.

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1   Introduction

Evolutionary computation community has studied NP-HARD combinatorial optimization problems through the lens of parameterized complexity. Theoretical results on selected parameters make substantial progress for a wide spectrum of problems including vertex cover [15], maximum leaf spanning trees [14], Euclidean TSP [20], makespan scheduling [23], closest string [22], and so on [13]. Commonly studied parameters include the solution size [15] and hand-crafted problem-specific graph structures [14, 20]. Other recent developments include jump-and-repair operators [8] and multivariate analysis [3].

This paper focuses on a specific fixed parameter algorithm called *tree decomposition based dynamic programming* (TD-DP) [5], which is one of the most well-known fixed parameter algorithms and has been widely applied on graph combinatorial optimization. We start with a high-level description and defer the more technical descriptions to Section 3. Fixed parameter analysis studies how to identify and exactly solve easy instances of NP-hard problems. Many NP-hard combinatorial optimization problems that are hard on general graphs become tractable when the actual graph instances are *similar* to trees. Focusing on graph structural invariants, *tree-width* (TW) measures the similarity between a given graph and an exact tree. We present its formal definition in Definition 1. TW is calculated via a complicated *tree decomposition* process [10], which maps any given general graph to a tree structure, where each tree node, often called a *bag*, corresponds to a subset of vertices from the original graph. Graphs closer to trees tend to lead to smaller bags. The tree-width is defined as the maximum bag size minus one. A significant result on tree-width is *Courcelle's theorem* [9], which states that **every** graph property expressible by *Monadic Second Order Logic* (MSO_2) is decidable in linear time on graphs with bounded tree-width — there is often a *straightforward* way to construct TD-DP with time complexity $O(f(TW) \cdot n^{O(1)})$. For NP-hard problems, f is exponential, therefore, TD-DP only scales on graphs with bounded tree-widths.

Despite the general applicability of TD-DP, in practice, direct usage of TD-DP seems out of reach because the graph instances do not necessarily have small enough tree-widths. In this paper, we apply evolutionary computations, specifically (1+1)EA, to scale up TD-DP. We focus on vertex selection problems, where the solution is a subset of vertices, for example, *independent set*, *vertex cover* and *max cut*. For presentation purposes, for the rest of this paragraph, we utilize *maximum independent set* (MIS) as our context. For MIS, there exists a TD-DP with complexity $\mathcal{O}^*(2^{TW})$[1]. When the tree-width is below a threshold (i.e., 10), TD-DP is certainly scalable. For graphs with larger tree-widths, we resort to a concept called *Tree-width Modulator* (TM), which concerns how to

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[1] For simplicity, we use $\mathcal{O}^*(\cdot)$ to highlight the dominating terms and omit polynomial terms. The notation is widely used.

optimally delete a small set of vertices in order to lower the tree-width of the remaining graph below a threshold. A naive way to utilize TM is to simply ignore all vertices in TM (i.e., the final MIS solution does not contain any vertex from TM), which forces TD-DP to be able to scale on the remaining graph. However, naively ignoring all vertices in TM generally alters the solution space. For example, the optimal MIS solution may indeed need to include some vertices from TM. Additionally, directly ignoring TM may also break the feasibility. For example, for *minimum vertex cover* (MVC), simple TM deletion may lead to uncovered edges. Therefore, instead of naively ignoring all vertices in TM, we use evolutionary algorithm(EA) to generate partial solutions involving vertices only in TM, named ZOOM-IN EA. For MIS, assume EA is built by *bit strings*, the partial solution settles the states of $v \in TM$ from *uncertain bit* (?) to *selected bit* (1) or *discarded bit* (0) as a part of the final MIS solution. With the partial solution fixated, we run TD-DP to complete the full solution on the remaining graph, which is scalable as the remaining graph's tree-width is below our desired threshold. The fitness of ZOOM-IN EA is evaluated based on the completed solution after running TD-DP. When mutating a partial solution within TM, we need to respect *local constraints*. Specifically for MIS, edge is not allowed. There are two scenarios, first, the conflicts may appear inside TM due to EA's randomness. This can be resolved by introducing a *penalty* term during evaluation. Secondly, the conflicts may appear outside of TM. This can be completely avoided by applying *marking mechanism*. For instance, if $v \in TM$ is *selected* by EA, v marks $\forall v' \in N(v)^2 \setminus TM$ as *discarded*. After fixating a partial solution for TM, states of $N(v) \setminus TM$ are known, and can be directly filled to the tree decomposition to proceed the TD-DP. Specifically, for each bag, the searching space is reduced by revealing some of the **?** bits to **0 or 1** based on the partial solution EA made on TM. Overall, our approach is problem-agnostic, although we do need problem-specific local constraints, which are rather trivial to apply.

A key challenge we have yet mentioned is the computation of the tree-width modulator. Unfortunately, the computation of tree-width modulator is known NP-HARD, and efficient heuristics also seem beyond reach [11, 18].

On the challenge of application aspect, vertex deletion would trigger the recalculation of tree decomposition for the remaining graph[3], which is made even worse by the fact that optimal tree decomposition is also NP-HARD and commonly used tree decomposition heuristics are expensive for large graphs. This motivates us to propose a more scalable version of tree width modulator, named as *Tree-width Modulator on a Given Tree Decomposition (TMGTD)*. The idea is to run an expensive tree decomposition heuristic *only once*. We then focus on optimally deleting vertices from the existing tree decomposition without ever regenerating the tree decomposition (i.e., never recreate the bags nor modify the tree structure). We show that our more scalable version is still NP-hard. However,

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[2] neighbor function, defined later.

[3] The scope of our discussions focus on calculating tree-widths by constructing tree decompositions. Other exact or approximation methods directly calculating tree-widths are out of scope.

in practice, it can often be solved efficiently via mixed integer programming or via cheap heuristics.

In summary, we apply evolutionary algorithm, specifically (1+1)EA, to scale up TD-DP, leading to a practically scalable and more importantly a *problem-agnostic general* approach that can be applied to a variety of graph combinatorial optimization problems. In our experiments, we demonstrate the effectiveness of our approach on *maximum independent set*, *minimum vertex cover* and *max cut*. We run exhaustive experiments on a variety of graphs and provide comprehensive experimental results in terms of *solution qualities*, *the number of fitness evaluations*, *wall-clock time* and the *first hitting time* of the best solutions obtained. By integrating (1+1)EA, TD-DP can scale to much larger tree-widths, and the number of evaluations of EA is significantly shortened compared to (1+1)EA alone. Meanwhile, near-optimal results will appear at much earlier stages. Notably, previous works on combining parameterized complexity and EA focused on specific optimization problems or problem-specific parameters, while our work is the *first study* that combines problem-agnostic graph structural parameters and EA.

2   Preliminaries

Given a graph $G = (V, E)$, we note $|V| = n, |E| = m$. Given $V' \subseteq V$, we define the induced subgraphs as $G(V')$, and we use $N[\cdot]$ and $N(\cdot)$ to represent the inclusion-wise and exclusion-wise neighborhood, respectively. We define *boundary* of V' as $\partial_G(V') = \{v \in V' : \exists u \in G \setminus V', (u, v) \in E\}$.

The formal definitions of tree decomposition and tree-width are given as:

Definition 1 (Tree Decomposition & Tree-width [10]). *A tree decomposition is a mapping from G to a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree and every tree node $t \in V(T)$ corresponds to a subset of vertices from the original graph, denoted as $X_t \subseteq V(G)$. X_t is often referred to as a **bag**. For a tree decomposition to be valid, we must have*

- *The union of all bags contain all the vertices from the original graph:*
  $\bigcup_{t \in V(T)} X_t = V(G)$
- *For any edge (u, v) in the original graph, there must exist at least one bag that contains both u and v:*
  $\forall (u, v) \in E(G), \exists t \in T$ *with* $\{u, v\} \subseteq X_t$
- *For any vertex u from the original graph, all bags containing u must form a subtree of T:*
  $\forall u \in V(G), T_u = \{t \in V(T) | u \in X_t\}$ *is a connected component of T*

*The **width** of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is defined by*

$$\max_{t \in V(T)} |X_t| - 1$$

The **tree-width** *of a graph G is denoted by TW^*, which is the minimum width of all valid tree decompositions of G. We denote the tree decomposition producing the minimum tree-width as $\mathcal{T}^*$.*

Using tree decomposition, we can construct *fixed-parameter tractable* dynamic programming for many graph combinatorial optimization problems [5].

The time complexity of these NP-hard problems is typically $O(f(TW){\cdot}n^{O(1)})$, where f is inevitably exponential in TW. For practical scalability, we need small tree-widths. That is, it is desirable to work with the optimal tree-width TW^* as it is the smallest possible. However, the decision version of optimal tree-width is NP-COMPLETE [2], and it remains NP-COMPLETE for many special graphs [6]. Furthermore, for many graphs, even the optimal tree-width can still be too large. To handle graphs with large tree-widths, we resort to the following concept called *tree-width modulator*, which concerns how to optimally delete a small set of vertices in order to reduce the remaining graph's tree-width to below a threshold.

Definition 2 (Tree-width Modulator [11](TM)). *Let $\eta \geq 0$ be an integer and G be a graph. A set $X \subset V(G)$ is called an η-tree-width modulator in G if $TW^*(G \setminus X) \leq \eta$. We use $TM_\eta^*(G)$ to denote the optimal tree-width modulator with the minimum size (i.e., when $|X|$ is minimized).*

As a special case of *vertex deletion problem* with hereditary property [11,18], the decision version of tree-width modulator is NP-complete. Deriving the optimal tree-width modulator $TM_\eta^*(G)$ involves two optimization subtasks. We need to optimally decide X (which vertices to delete) and also need to calculate the optimal tree-width of the remaining graph $TW^*(G \setminus X)$. As mentioned above, the decision version of optimal tree-width is NP-complete. Furthermore, even if we assume that we have access to an oracle that returns the optimal tree-width, the optimal vertex deletion part is still NP-complete — We use $MVC(G)$ and $MFVS(G)$ to denote the *minimum vertex cover* (MVC) and the *minimum feedback vertex set* (MFVS), respectively. It is easy to see that $TM_0^*(G)$ is MVC and $TM_1^*(G)$ is MFVS, and these two problems only involve trivial tree decomposition, so these problems are hard even with an optimal tree-width oracle.

Considering the above hardness results, a natural idea is to settle for heuristics. That is, we could heuristically select the vertices for deletion and also heuristically calculate the tree-width of the remaining graph. For example, the vertex deletion subtask can perhaps be carried out via evolutionary computation or reinforcement learning, and the tree decomposition subtask can perhaps be carried out using standard heuristics such as *min degree* or *min fill-in* [7]. Unfortunately, in practice, tree decomposition is expensive on large graphs even if we settle for heuristics. If we use evolutionary computation to heuristically select the vertices for deletion, then we cannot afford too many fitness evaluations. Similarly, reinforcement learning for vertex deletion is also not scalable as it also requires frequent regeneration of tree decompositions.

Therefore, for practical scalability, we propose a more scalable version of tree-width modulator, which we call *Tree-width Modulator on a Given Tree De-*

composition (TMGTD). The edge-centric version of this problem has previously been studied in the field of bioinformatics [19], named as Tree-diet, and a more general form of the problem is recognized by the term $\mathcal{F}$-M-DELETION [4]. Our more scalable version only generates the tree decomposition once, which is affordable even with an expensive heuristic. We only focus on deleting vertices from the bags of this constructed tree decomposition. Thus, the bags are neither regenerated nor the tree structure is modified. We provide the formal definition of TMGTD, together with an example in Fig. 1.
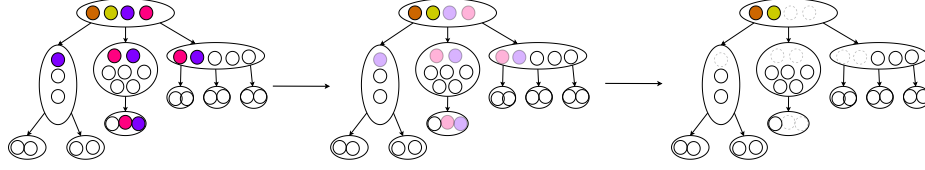

Fig. 1: Recall that the tree-width is the largest bag size minus one. By removing the purple and the red vertex from the existing bags without regenerating the tree structure, we reduced the tree-width from 6 to 4.

Definition 3 (Tree-width Modulator on a Given Tree Decomposition (TMGTD)). *Given a graph G and a tree decomposition of G denoted as $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a parameter k. We simply delete vertices from the existing bags (the X_t) and do not regenerate the tree structure T. That is, after deleting $Y \subseteq V(G)$, the resulting tree decomposition is $\mathcal{T}' = (T, \{X_t \setminus Y\}_{t \in V(T)})$ and the resulting tree-width is $\max_{t \in V(T)} |X_t \setminus Y| - 1$.*

We want to decide whether there exists a vertex set $Y \subseteq V(G), |Y| \leq k$, whose deletion will reduce the width of $\mathcal{T}'$ to at most η.

First of all, we show that deleting vertices from the bags without regenerating the tree structure guarantees the validity of the resulting tree decomposition.

Proposition 1. *Given a graph G and a tree decomposition of G denoted as $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree and every tree node $t \in V(T)$ corresponds to a subset of vertices from the original graph, denoted as $X_t \subseteq V(G)$.*

For any set of vertices $Y \subseteq V(G)$, after deleting Y from $\mathcal{T}$ without regenerating the tree decomposition, the result $\mathcal{T}' = (T, \{X_t \setminus Y\}_{t \in V(T)})$ is still a valid tree decomposition.

Proof. Assume the set of deleting vertices is Y, and we execute the deletion on a valid tree decomposition. By removing Y from each X_t, we eliminate the existence of all vertices in Y from T, so as all related incident edges. Clearly, this operation satisfies the first two properties in Def. 1. Additionally, the third property in Def. 1 will never be violated since we remove Y completely from every bag.

Next, we claim that even though TMGTD avoid regenerating tree decomposition, it is still NP-COMPLETE. For space limit, the proof is deferred to **Appendix A.1**.

Theorem 1 (Hardness of Def. 3). *Tree-width modulator on a given tree decomposition is* NP-COMPLETE.

To wrap up the complexity discussion, we provide two trivial bounds to roughly illustrate the size of TMGTD. Proof is in **Appendix A.2**.

Lemma 1 (bounds). *We use $*$ to denote the optimality. And $\mathcal{C}$ represents the target tree-width. $TW_\mathcal{T}$ means the width of a given tree decomposition $\mathcal{T}$.*

1. *Given an optimal tree-width TW^*, $|TM^*_\eta| \geq |TM^*_{\eta'}|$ when $\eta \leq \eta'$*
2. *$|TMGTD^*(TW_\mathcal{T}, \mathcal{C})| \geq |TM^*(TW^*, \mathcal{C})|$*

In practice, despite the NP-completeness, optimally solving TMGTD on practical graphs is a scalable task due to the limited number of large bags. The goal is to reduce every bag's size to at most $\eta + 1$. Initially, bags of size at most $\eta + 1$ are discarded. For a bag with size x, at least $x - \eta - 1$ vertices must be removed so the resulting TW is desired. The task of minimizing the number of vertices removed can be easily formulated as a mixed integer program. With mixed integer programming solvers such as Gurobi, deriving the optimal TMGTD is scalable. For particularly difficult or large instances, incumbent results can serve as heuristic solutions.

Lastly, given that TMGTD only generates a tree decomposition once, intuitively speaking, the quality of the initial tree decomposition could significantly influence the quality of the modulator. It raises the following question: whether better initial tree decompositions (i.e., start with smaller tree-widths) always lead to better modulators (i.e., end up having to delete fewer vertices)? The answer is no. We provide the following observation, and defer the examples and detailed discussions to **Appendix. A.3** due to the space limit.

Observation 1 *Optimally solving $TMGTD$ via optimal tree decompositions $\mathcal{T}^*$ is not a sufficient condition to access the optimal TM^*. On the other hand, optimally solving TMGTD via non-optimal tree decompositions is possible to access the optimal TM^*.*

3   Framework

Our proposed approach utilizes evolutionary computation to scale up TD-DP. To simplify the discussion, assume a tree modulator TM for a given tree decomposition TD is known, which is obtained via solving the scalable TMGTD problem. Based on the definition of TM, the removal of all vertices in TM from TD reduce the width of TD below a threshold, thereby allowing TD-DP to proceed. The high-level idea of our approach is to apply evolutionary computation to mutate the partial solution, focusing exclusively on the vertices within the TM. In this paper, we use the elementary (1+1)EA. Due to its simplicity, we defer the pseudo-code of (1+1)EA to Alg. 5 in **Appendix C**. With the partial solution fixated, the solution for the remaining graph can then be completed via TD-DP. The fitness evaluation of the partial solution provided by (1+1)EA

focuses on the quality of the completed solution. We name our approach ZOOM-IN EA-enhanced dynamic programming, reflecting the fact that the ZOOM-IN $(1+1)$EA only cares about the tree modulator vertices.

When evolving the partial solution, adhering problem-specific *local constraints* is essential. Specifically, a partial solution is feasible if and only if it can be completed into a feasible full solution. For example, for MIS, the partial solution cannot contain two connected vertices. We also need a *marking mechanism*, in the sense that the decisions within the tree modulator may affect the decisions on the rest of the graph. For example, for MIS, for selected vertices in TM, their neighbors outside of TM must not be selected. Integrating the marking mechanism into TD-DP is trivial, because TD-DP is exhaustive in nature and the marking mechanism merely filters out some of the irrelevant states based on the partial solution given by EA.

3.1   Technical Details of Our Approach

We first outline the fundamental principles of *tree decomposition based dynamic programming* (TD-DP) and the details of *local constraints* and *marking mechanism* used in our ZOOM-IN *EA-enhanced dynamic programming*.
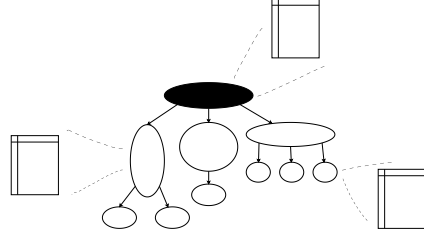

Fig. 2: Tree decomposition based dynamic programming, where every bag is associated with a DP table and the final result is assembled at the root bag.
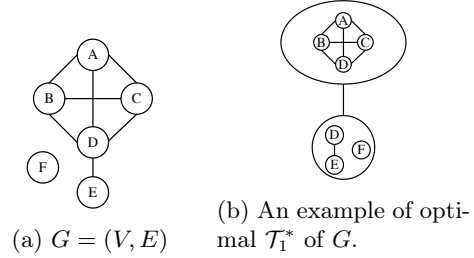
(a) $G = (V, E)$

(b) An example of optimal $\mathcal{T}_1^*$ of G.

Fig. 3: Simple example of optimal tree decomposition.

Tree Decomposition based Dynamic Programming (TD-DP)
We defer the pseudo-code of TD-DP to **Appendix C** due to space constraint. Tree decomposition allows the execution of dynamic programming in a bottom-up fashion. The primary difference to conventional tabular DPs has been highlighted in Fig. 2, TD-DP disassembles the global table into segments, with each *bag* possessing its own local table. After exactly filling each local table, global solutions are assembled at the root *bag*. The general TD-DP procedures are shown in Alg. 6. In the pseudo-code, both X and Y represent the dynamic programming tabulation. We traverse the tree decomposition via breath-first search from leaf bags to the root bag. For each bag $\mathcal{B}_t$, there is at most one parent bag and multiple child bags. The valid state collection $\mathcal{S}_t$ of each bag $\mathcal{B}_t$ is calculated by problem-specific constraints. To elaborate the process, we use MIS and Fig. 3 as a concrete example. For the leaf bag $\{D, E, F\}$, the valid states

$\mathcal{S} = \{\{\}, \{D\}, \{E\}, \{F\}, \{D, F\}, \{E, F\}\}$. To calculate the valid state collection, we apply MIS's *no-edge* constraint to the *power set* of $\{D, E, F\}$ to filter out all invalid states. Then, we use table Y to decide which state is able to pass to the parent bag $\{A, B, C, D\}$. Every *locally valid* state needs to be passed via the *intersection* between child bag and parent bag. Therefore, in our example, all valid states $s \in \mathcal{S}$, able to pass from child bag to the parent bag must satisfy: $s \subseteq \mathcal{B}_{parent} \bigcap \mathcal{B}_{child} = \{D\}$. As outlined by the seventh line in Alg. 6, we use table Y to record the **best** state from the subtree rooted at $\mathcal{B}_t$. After that, we utilize table X to assemble the partial solutions from the children bags. In our example, we use the mentioned method to enumerate all valid states of $\{A, B, C, D\}$. Next, in the sixth line, we use the *intersection* $(s \bigcap \mathcal{B}_{child})$ as the **key** to extract the corresponding entry from the Y table filled by the child bag. By using the two-stage strategy, we finish computing one bag.

The pseudo-code assumes the objective is maximization. If minimization is the goal, the 'max' function in the code can simply be replaced with 'min'.

Local Constraint Property and Marking Mechanism

Given a tree decomposition $\mathcal{T}$ and a modulator of it $\mathrm{TM}_{\mathcal{T}}$ [4], derived from the problem TMGTD, we apply (1+1)EA on the vertices in $\mathrm{TM}_{\mathcal{T}}$. (1+1)EA will occasionally make mistakes due to its inherent randomness. If the EA decisions on vertices in $\mathrm{TM}_{\mathcal{T}}$ already lead to conflicts, then we consider this partial solution infeasible. Fig. 4 shows an example for MIS. In this figure, the circle represents the area of $\mathrm{TM}_{\mathcal{T}}$. There are two nodes in the circle which are connected. An example local constraint violation is when EA selects both vertices in the circle. When this happens, this partial solution can never be completed into a full solution that is feasible. In terms of marking mechanism, in Fig 4, the black vertex represents a vertex selected by (1+1)EA. For MIS, if a vertex is included in the solution set, all its neighbours must be excluded. That is, the three vertices to the left of the circle must be excluded. We can summarise similar local constraints and marking mechanism for other combinatorial optimization models.
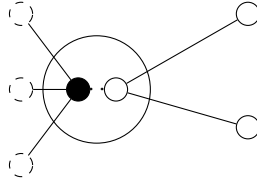

Fig. 4: If the black vertex is picked by (1+1)EA, dashed vertices are ensured out of the current MIS solution. And we can safely remove them. If the white vertex (inside the circle that represents the tree modulator) is not picked by (1+1)EA, then we can not make decisions for the two vertices to the right of the circle. We keep them.

It is obvious that if (1+1)EA makes the correct decisions on the tree modulator vertices, then the completed solution after TD-DP is **globally optimal**.

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[4] TM and $TM_{\mathcal{T}}$ share the same meaning. We use the later when we emphasize $\mathcal{T}$.

For MIS, MVC and MC, we list the problem-specific **local constraints** and **marking mechanism**. We assume the local decisions produced by (1+1)EA is $TM = TM^1 \bigcup TM^0$, where TM^1 represents vertices *selected* as a part of the final solution, and TM^0 refers to vertices *discarded* out of the final solution.

- **(LC-MIS)** TM can be *safely* removed from $\mathcal{T}$ with marks on $N[TM^1]\backslash TM$
- **(LC-MVC)** TM can be *safely* removed from $\mathcal{T}$ with marks on $N[TM^0]\backslash TM$
- **(LC-MC)** $TM\backslash\partial_G(TM)$ can be *safely* removed from $\mathcal{T}$ with marks on $\partial_G(TM)$

Algorithm 1: Zoom-in Mutation

Input : A random bit string
$x : \{0,1\}^n$, n=|V|
Modulator of a given tree decomposition $TM_\mathcal{T}$
$z = (z_1, z_2, ...z_n)$

1 $z_i = \begin{cases} 1 & i \in TM_\mathcal{T}, \\ 0 & i \in V \backslash TM_\mathcal{T} \end{cases}$

2 $y \leftarrow \text{BitwiseFlipping}(x, \frac{1}{|TM_\mathcal{T}|})$;

3 $y \leftarrow y \wedge z$;

4 **return** y;

After removing the *safe* vertices, we need to inform the dynamic programming what (1+1)EA has done. As mentioned earlier, TM can be safely removed from G since the vertices inside TM has no connection to the rest of the graph. However, for MIS and MVC, decisions on $N[TM^1]\backslash TM$ and $N[TM^0]\backslash TM$ will be influenced by decisions made by (1+1)EA on TM. Same principle applies to the $\partial_G(TM)$ in the case of MC. We use Fig.3 and MIS as an example. Suppose $\mathbf{A} \in TM$, (1+1)EA decides to select the vertex $\mathbf{A}$ based on TM, we know that $\mathbf{B, C, D}$ must not be in the solution. In the context of (1+1)EA, the **bit** at the position of $\mathbf{A}$ is assigned with **1**, and the **bits** at the positions of $\mathbf{B, C, D}$ must be **0**. We then take the tree decomposition in Fig.3b to continue this example. For the root bag, we do not need to compute valid states anymore, because the valid state has been confirmed. We have $< 1,0,0,0 >$ lexicographically corresponds to $\{A,B,C,D\}$ based on (1+1)EA's decision. Comparing with plain TD-DP, when we firstly encounter the bag $\{A,B,C,D\}$, the bit string status is $<?,?,?,?>$. Thus, we need to enumerate all $\mathbf{2^4}$ to construct the valid state collection. Reducing from 16 state validity checks to zero, it is a big improvement. On the child bag $\{D,E,F\}$, as we already know D is assigned by **0**, we have $< 0,?,? >$ corresponds to $\{D,E,F\}$. Therefore, the number of states for the child bag $\{D,E,F\}$ is already halved from 2^3 to 2^2 than using TD-DP only. Considering we use TMGTD, we successfully reduce the TW to the target TW′. Thus, in the worst case, we need to compute $O(2^{TW'+1})$ options for each bag for MIS. If the target TW′ is small enough, the TD-DP has been reduced to the time $O(2^{TW'} \cdot poly(TW') \cdot n)$, where $poly(TW')$ is the time spent on state validity checking. In summary, local constraints and marking mechanism allows us to proceed TD-DP on the remaining graph in a scalable fashion, without modifying the solution space. There is the additional computational cost for validity checking, but that is only polynomial in the tree-width.

Having established all preconditions, we now present the complete zoom-in EA enhanced dynamic programming algorithm.

Zoom-in **EA Enhanced Dynamic Programming**

We use Alg. 1 to explain the ZOOM-IN EA. We represent the graph with a bit string $x : \{0,1\}^n$, where each bit corresponds to a $v \in V$. Firstly, we build a mutation mask z based on the calculated $\text{TM}_{\mathcal{T}}$ via the indicator function specified in the first line. Next, we apply the standard bit flipping mutation to x with $\frac{1}{|\text{TM}_{\mathcal{T}}|}$ as the mutation ratio. Finally, we apply the mutation mask z to eliminate the mutations on all irrelevant bits. This approach confines the mutations within the $\text{TM}_{\mathcal{T}}$ area. Following this, we apply the local constraint property to $\mathcal{T}$ and at the same time, use the marking mechanism to inform the TD-DP which bits are confirmed at $(1+1)$EA stage. For the fitness function, we define it by **summing up** the fitness value from $(1+1)$EA and the exact solution from TD-DP. TD-DP will **never produce conflicts** due to the correctness of the framework. That is to say, the penalties will always come from $(1+1)$EA by violating the local constraint properties within the area of $\text{TM}_{\mathcal{T}}$. We complete the model description by showing Alg. 2.

Algorithm 2: ZOOM-IN EA enhanced TD-DP

Input : Graph $G = (V, E)$, a tree decomposition $\mathcal{T}$,
Modulator of the given tree decomposition $\text{TM}_{\mathcal{T}}$,
Fitness function $f : \{0,1\}^n \to \mathbb{R}$,
A bit string: $x : \{0,1\}^n$,
LC: problem-specific local constraint property,
DP tabulations X and Y

1 **while not** *terminate criteria* **do**
2     $y \leftarrow$ ZOOM-IN mutation$(x, TM_{\mathcal{T}})$;
3     $r^{\text{IN}} \leftarrow f(y')$;
4   Calculate $\mathcal{S}$ by applying LC(G, y);
5   **for** $X_t \in \mathcal{T}$ **do**
6        $X_t \leftarrow X_t \setminus \mathcal{S}$
7   Mark the table X and Y;
8     $r^{\text{OUT}} \leftarrow$ TD-DP(G, $\mathcal{T}$);
9   **if** $r^{\text{IN}} + r^{\text{OUT}} \geq r^{\text{BEST}}$ **then**
10        $x \leftarrow y$;
11        $r^{\text{BEST}} \leftarrow r^{\text{IN}} + r^{\text{OUT}}$
12   **for** $X_t \in \mathcal{T}$ **do**
13        $X_t \leftarrow X_t \bigcup \mathcal{S}$

---

4 Experiments

In our experiments, we aim to demonstrate the following:

– $(1+1)$EA significantly improves the scalability of TD-DP to handle larger tree-width.
– TD-DP efficiently reduces the number of fitness evaluations typically required by $(1+1)$EA. Good solutions will appear at an early stage.

- Allocated the same time budget, our model shows a distinct advantage over (1+1)EA with various initialization strategies.
- Our framework can adapt to various graph combinatorial problems.

It is important to note that our objective is to utilize (1+1) EA to strengthen the scalability of TD-DP. Thus, our experiments focus on graphs whose tree-width is moderately larger than the upper limit manageable by standard TD-DP. We note that the magnitude of tree-width in our experiments is sufficient for many practical graphs, such as road networks, bio-information graphs, and etc. In general, these graphs do not possess extremely large cliques.

For the sake of reproducibility, we use **Appendix.B.1** and **Appendix.B.2** to explain **experiment configurations** and **EA implementation details**, respectively.

4.1   Standalone TD-DP in Practice

In our initial set of experiments, we demonstrate the practical efficiency of plain TD-DP. We utilize graphs generated by random regular model and *Erdős–Rényi* model. We manually adjust the parameters of the models to ensure the width of the tree decompositions falling within a range that allows us to observe the appearance of the exponential runtime increase. For each instance, TD-DP is given with only **1 shot** and timeout after **30 minutes**. As indicated in Table. 4 in **Appendix. D**, with the calculated tree-width increases, the runtime escalates exponentially. The results align with the theoretical time complexity of TD-DP. Therefore, it will be prohibitive to proceed the TD-DP beyond a certain threshold of tree-width, since the single exponential term always leads to impractically long runtime.

4.2   Target TW

In this set of experiment, we use small-size instances to assess the impact of the various size of TMs on our model. For each instance, we set three different target **TW=5,7,9**. For both our model and (1+1)EA, we gave **3,000 and 1,000,000** steps of fitness evaluations, respectively. We let our model and (1+1)EA start with the exact same bit string, which is initialized by basic feasible solutions. Specifically, for MVC, we use $\mathbf{1}^n$; for MIS, we use $\mathbf{0}^n$ and for MC, $\mathbf{0}^n$ and $\mathbf{1}^n$ will have the same impact, we choose to use $\mathbf{0}^n$ in our experiments. In the visualization in Fig. 5, we follow the rules: *1) Once the curves converge to the same value, we choose not to draw the rest of the steps because they are not relevant to our emphasis, and no benefits helping readers understand., 2) We drop the results at the* **very first step***, because the starting point are the same for all curves based on our initialization.*

The specific initial tree-width, target tree-width and the corresponding modulator sizes are given in Fig. 5. The results indicates that, mostly, smaller TMs will bring good solutions at an earlier evolution stage in terms of the number of
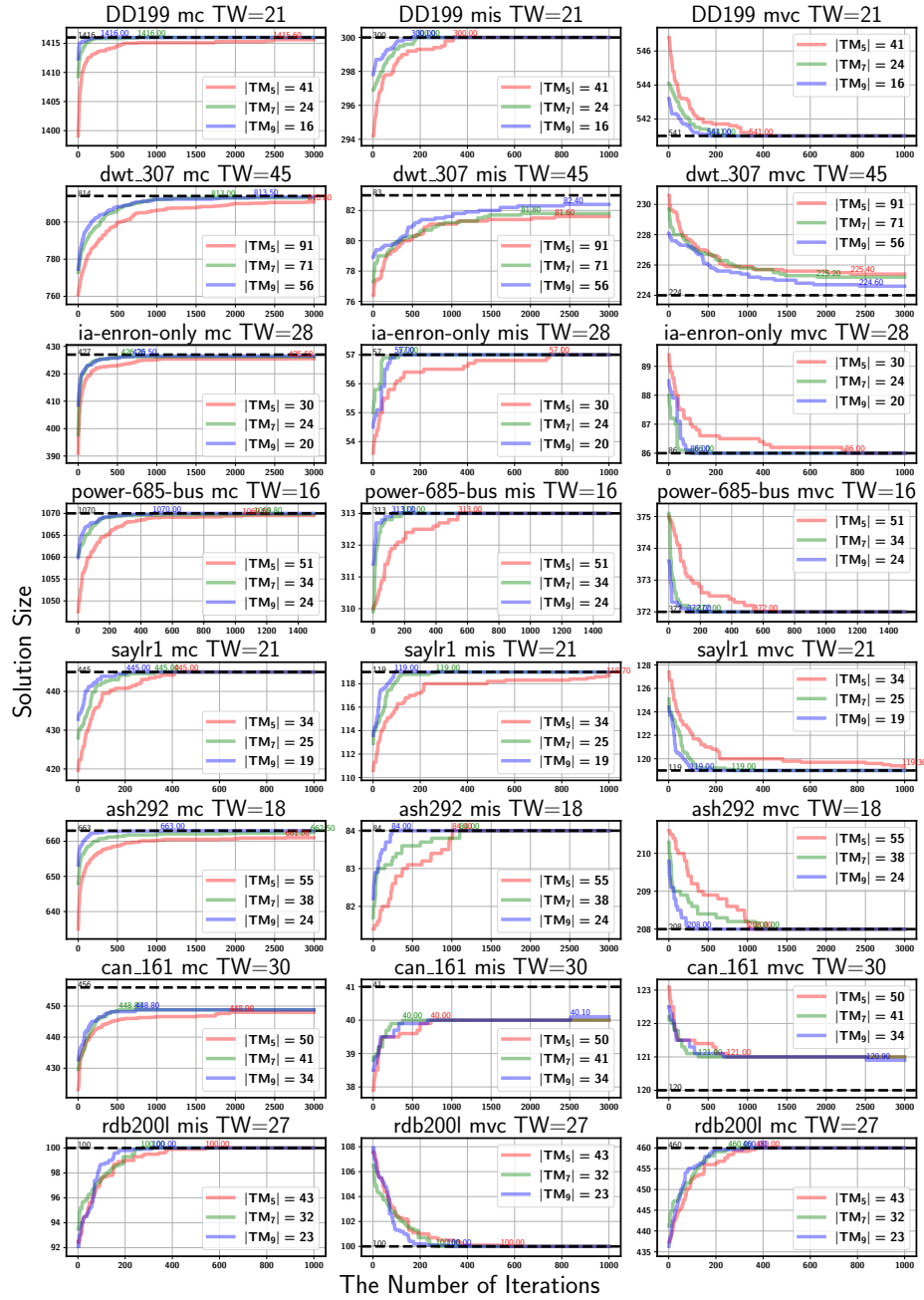
Fig. 5: The colorful numbers represents *the average first hitting time of best solutions for each line.* More details refer to Table. 1. The ----- indicates the Gurobi solution.

Table 1: For each data pair, the FIRST VALUE is *the average best solution size*, the SECOND VALUE (in parentheses) is *the average first hitting time of the best solution*. (5), (7) and (9) are different *target tree-width* values.

Graph	Problem	(1+1)EA	ZOOM-IN(5)	ZOOM-IN(7)	ZOOM-IN(9)	GUROBI
DD199	MC	1393.6 (553.52)	1415.6 (230.77)	1416.0 (80.89)	**1416.0 (68.32)**	**1416**
	MIS	273.0 (604.30)	300.0 (23.66)	**300.0 (16.55)**	300.0 (16.70)	**300**
	MVC	568.0 (622.84)	541.0 (23.36)	**541.0 (16.03)**	541.0 (16.05)	**541**
dwt_307	MC	802.3 (82.94)	810.8 (152.70)	813.0 (291.82)	**813.5** (763.10)	**814**
	MIS	77.4 (255.50)	81.6 (24.59)	81.8 (52.01)	**82.4** (100.02)	**83**
	MVC	229.6 (258.66)	225.4 (24.69)	225.2 (43.19)	**224.6** (71.79)	**224**
ia-enron-only	MC	422.9 (23.03)	425.5 (36.28)	426.2 (31.29)	**426.5** (170.50)	**427**
	MIS	56.9 (65.66)	57.0 (2.00)	**57.0 (0.39)**	57.0 (0.88)	**57**
	MVC	86.1 (66.11)	86.0 (2.11)	**86.0 (0.34)**	86.0 (0.66)	**86**
power-685-bus	MC	1039.9 (280.53)	1069.6 (98.78)	1069.8 (96.26)	**1070.0** (173.46)	**1070**
	MIS	297.1 (436.09)	313.0 (25.24)	313.0 (7.51)	**313.0 (4.38)**	**313**
	MVC	387.9 (444.00)	372.0 (25.86)	372.0 (7.31)	**372.0 (4.09)**	**372**
saylr1	MC	440.5 (5.91)	**445.0 (13.16)**	445.0 (16.60)	445.0 (33.21)	**445**
	MIS	110.9 (139.41)	119.0 (11.19)	**119.0 (8.21)**	119.0 (11.23)	**119**
	MVC	127.1 (142.74)	119.0 (11.30)	**119.0 (7.81)**	119.0 (10.45)	**119**
ash292	MC	655.6 (162.55)	661.0 (104.27)	662.5 (304.59)	**663.0** (245.65)	**663**
	MIS	81.2 (230.45)	84.0 (16.69)	84.0 (20.92)	**84.0 (11.34)**	**84**
	MVC	210.8 (234.57)	208.0 (16.33)	208.0 (17.59)	**208.0 (8.22)**	**208**
can_161	MC	**450.4 (2.33)**	448.0 (29.48)	448.8 (39.56)	448.8 (109.05)	**456**
	MIS	**40.2** (60.69)	40.0 (2.68)	**40.0 (1.88)**	40.1 (7.12)	**41**
	MVC	**120.8** (62.19)	121.0 (2.88)	**121.0 (1.79)**	120.9 (6.01)	**120**
rdb200l	MC	441.4 (109.44)	**460.0 (11.66)**	460.0 (22.35)	460.0 (77.16)	**460**
	MIS	94.2 (108.77)	**100.0 (7.65)**	100.0 (12.01)	100.0 (28.57)	**100**
	MVC	105.8 (108.43)	**100.0 (7.73)**	100.0 (11.49)	100.0 (26.34)	**100**

fitness evaluations. However, smaller TMs will sometimes undermine the run-time for TD-DP due to the fact that a larger target TW will make TD-DP harder to proceed (imagine for each bag, 2^5 and 2^9 will pose a clear difference). Additionally, our model also shows a strong performance in terms of the first hitting time of the best solutions. As shown by Table. 1, our framework will quickly yield the good results in terms of wall-clock timing. And the first hitting time of the obtained best solution is superior, especially when we set the target TW to 7 and 9.

4.3   Large Instances

In this set of experiments, we extensively test the scalability of our model on medium and large instances. We give (1+1)EA and our framework **1 hour** limit. We give Gurobi **3-hour** limit, and use the best solution of the 10 trials as our truth. We also use 4 different initialization strategies, including basic feasible initialization, random feasible initialization, 30% random greedy initialization and 60% random greedy initialization, to generate different initial bit strings. We fix the **target tree-width** in our model to **5**. Our model shows a stable performance on instances with tree-width around one hundred. The solutions are mostly near-optimal. Due to the space limit, we **direct the readers to the complete Table. 5 and Table. 6 in Appendix D.3**.

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