

THEORY OF ELASTICITY

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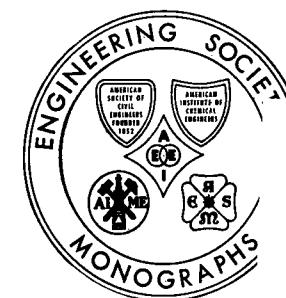
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THEORY OF ELASTICITY

By S. TIMOSHENKO

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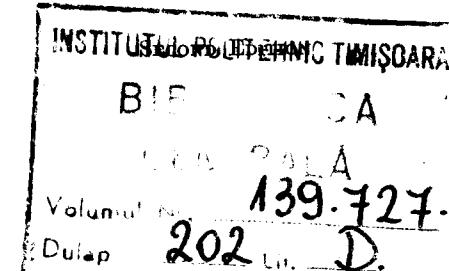
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THEORY OF ELASTICITY

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PREFACE TO THE SECOND EDITION

The many developments and clarifications in the theory of elasticity and its applications which have occurred since the first edition was written are reflected in numerous additions and emendations in the present edition. The arrangement of the book remains the same for the most part.

The treatments of the photoelastic method, two-dimensional problems in curvilinear coordinates, and thermal stress have been rewritten and enlarged into separate new chapters which present many methods and solutions not given in the former edition. An appendix on the method of finite differences and its applications, including the relaxation method, has been added. New articles and paragraphs incorporated in the other chapters deal with the theory of the strain gauge rosette, gravity stresses, Saint-Venant's principle, the components of rotation, the reciprocal theorem, general solutions, the approximate character of the plane stress solutions, center of twist and center of shear, torsional stress concentration at fillets, the approximate treatment of slender (*e.g.*, solid airfoil) sections in torsion and bending, and the circular cylinder with a band of pressure.

Problems for the student have been added covering the text as far as the end of the chapter on torsion.

It is a pleasure to make grateful acknowledgment of the many helpful suggestions which have been contributed by readers of the book.

S. TIMOSHENKO
J. N. GOODIER

PALO ALTO, CALIF.
February, 1951

PREFACE TO THE FIRST EDITION

During recent years the theory of elasticity has found considerable application in the solution of engineering problems. There are many cases in which the elementary methods of strength of materials are inadequate to furnish satisfactory information regarding stress distribution in engineering structures, and recourse must be made to the more powerful methods of the theory of elasticity. The elementary theory is insufficient to give information regarding local stresses near the loads and near the supports of beams. It fails also in the cases when the stress distribution in bodies, all the dimensions of which are of the same order, has to be investigated. The stresses in rollers and in balls of bearings can be found only by using the methods of the theory of elasticity. The elementary theory gives no means of investigating stresses in regions of sharp variation in cross section of beams or shafts. It is known that at reentrant corners a high stress concentration occurs and as a result of this cracks are likely to start at such corners, especially if the structure is submitted to a reversal of stresses. The majority of fractures of machine parts in service can be attributed to such cracks.

During recent years considerable progress has been made in solving such practically important problems. In cases where a rigorous solution cannot be readily obtained, approximate methods have been developed. In some cases solutions have been obtained by using experimental methods. As an example of this the photoelastic method of solving two-dimensional problems of elasticity may be mentioned. The photoelastic equipment may be found now at universities and also in many industrial research laboratories. The results of photoelastic experiments have proved especially useful in studying various cases of stress concentration at points of sharp variation of cross-sectional dimensions and at sharp fillets of reentrant corners. Without any doubt these results have considerably influenced the modern design of machine parts and helped in many cases to improve the construction by eliminating weak spots from which cracks may start.

Another example of the successful application of experiments in the solution of elasticity problems is the soap-film method for determining stresses in torsion and bending of prismatical bars. The

difficult problems of the solution of partial differential equations with given boundary conditions are replaced in this case by measurements of slopes and deflections of a properly stretched and loaded soap film. The experiments show that in this way not only a visual picture of the stress distribution but also the necessary information regarding magnitude of stresses can be obtained with an accuracy sufficient for practical application.

Again, the electrical analogy which gives a means of investigating torsional stresses in shafts of variable diameter at the fillets and grooves is interesting. The analogy between the problem of bending of plates and the two-dimensional problem of elasticity has also been successfully applied in the solution of important engineering problems.

In the preparation of this book the intention was to give to engineers, in a simple form, the necessary fundamental knowledge of the theory of elasticity. It was also intended to bring together solutions of special problems which may be of practical importance and to describe approximate and experimental methods of the solution of elasticity problems.

Having in mind practical applications of the theory of elasticity, matters of more theoretical interest and those which have not at present any direct applications in engineering have been omitted in favor of the discussion of specific cases. Only by studying such cases with all the details and by comparing the results of exact investigations with the approximate solutions usually given in the elementary books on strength of materials can a designer acquire a thorough understanding of stress distribution in engineering structures, and learn to use, to his advantage, the more rigorous methods of stress analysis.

In the discussion of special problems in most cases the method of direct determination of stresses and the use of the compatibility equations in terms of stress components has been applied. This method is more familiar to engineers who are usually interested in the magnitude of stresses. By a suitable introduction of stress functions this method is also often simpler than that in which equations of equilibrium in terms of displacements are used.

In many cases the energy method of solution of elasticity problems has been used. In this way the integration of differential equations is replaced by the investigation of minimum conditions of certain integrals. Using Ritz's method this problem of variational calculus is reduced to a simple problem of finding a minimum of a function. In this manner useful approximate solutions can be obtained in many practically important cases.

To simplify the presentation, the book begins with the discussion of two-dimensional problems and only later, when the reader has familiarized himself with the various methods used in the solution of problems of the theory of elasticity, are three-dimensional problems discussed. The portions of the book that, although of practical importance, are such that they can be omitted during the first reading are put in small type. The reader may return to the study of such problems after finishing with the most essential portions of the book.

The mathematical derivations are put in an elementary form and usually do not require more mathematical knowledge than is given in engineering schools. In the cases of more complicated problems all necessary explanations and intermediate calculations are given so that the reader can follow without difficulty through all the derivations. Only in a few cases are final results given without complete derivations. Then the necessary references to the papers in which the derivations can be found are always given.

In numerous footnotes references to papers and books on the theory of elasticity which may be of practical importance are given. These references may be of interest to engineers who wish to study some special problems in more detail. They give also a picture of the modern development of the theory of elasticity and may be of some use to graduate students who are planning to take their work in this field.

In the preparation of the book the contents of a previous book ("Theory of Elasticity," vol. I, St. Petersburg, Russia, 1914) on the same subject, which represented a course of lectures on the theory of elasticity given in several Russian engineering schools, were used to a large extent.

The author was assisted in his work by Dr. L. H. Donnell and Dr. J. N. Goodier, who read over the complete manuscript and to whom he is indebted for many corrections and suggestions. The author takes this opportunity to thank also Prof. G. H. MacCullough, Dr. E. E. Weibel, Prof. M. Sadowsky, and Mr. D. H. Young, who assisted in the final preparation of the book by reading some portions of the manuscript. He is indebted also to Mr. L. S. Veenstra for the preparation of drawings and to Mrs. E. D. Webster for the typing of the manuscript.

S. TIMOSHENKO

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NOTATION

x, y, z	Rectangular coordinates.
r, θ	Polar coordinates.
ξ, η	Orthogonal curvilinear coordinates; sometimes rectangular coordinates.
R, ψ, θ	Spherical coordinates.
N	Outward normal to the surface of a body.
l, m, n	Direction cosines of the outward normal.
A	Cross-sectional area.
I_x, I_y	Moments of inertia of a cross section with respect to x - and y -axes.
I_p	Polar moment of inertia of a cross section.
g	Gravitational acceleration.
ρ	Density.
q	Intensity of a continuously distributed load.
p	Pressure.
X, Y, Z	Components of a body force per unit volume.
$\bar{X}, \bar{Y}, \bar{Z}$	Components of a distributed surface force per unit area.
M	Bending moment.
M_t	Torque.
$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to x -, y -, and z -axes.
σ_n	Normal component of stress parallel to n .
σ_r, σ_θ	Radial and tangential normal stresses in polar coordinates.
σ_ξ, σ_η	Normal stress components in curvilinear coordinates.
$\sigma_r, \sigma_\theta, \sigma_z$	Normal stress components in cylindrical coordinates.
$\Theta = \sigma_x + \sigma_y + \sigma_z = \sigma_r + \sigma_\theta + \sigma_z$	
τ	Shearing stress.
$\tau_{xy}, \tau_{xz}, \tau_{yz}$	Shearing-stress components in rectangular coordinates.
$\tau_{r\theta}$	Shearing stress in polar coordinates.
$\tau_{\xi\eta}$	Shearing stress in curvilinear coordinates.
$\tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$	Shearing-stress components in cylindrical coordinates.
S	Total stress on a plane.
u, v, w	Components of displacements.
ϵ	Unit elongation.
$\epsilon_x, \epsilon_y, \epsilon_z$	Unit elongations in x -, y -, and z -directions.

NOTATION

$\epsilon_r, \epsilon_\theta$	Radial and tangential unit elongations in polar coordinates.
$e = \epsilon_x + \epsilon_y + \epsilon_z$	Volume expansion.
γ	Unit shear.
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	Shearing-strain components in rectangular coordinates.
$\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{rz}$	Shearing-strain components in cylindrical coordinates.
E	Modulus of elasticity in tension and compression.
G	Modulus of elasticity in shear. Modulus of rigidity.
ν	Poisson's ratio.
$\mu = G, \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$	Lamé's constants.
ϕ	Stress function.
$\psi(z), x(z)$	Complex potentials; functions of the complex variable $z = x + iy$.
\bar{z}	The conjugate complex variable $x - iy$.
C	Torsional rigidity.
θ	Angle of twist per unit length.
$F = 2G\theta$	Used in torsional problems.
V	Strain energy.
V_0	Strain energy per unit volume.
t	Time.
T	Certain interval of time. Temperature.
α	Coefficient of thermal expansion.

CHAPTER 1

INTRODUCTION

1. Elasticity. All structural materials possess to a certain extent the property of *elasticity*, *i.e.*, if external forces, producing *deformation* of a structure, do not exceed a certain limit, the deformation disappears with the removal of the forces. Throughout this book it will be assumed that the bodies undergoing the action of external forces are *perfectly elastic*, *i.e.*, that they resume their initial form completely after removal of forces.

The molecular structure of elastic bodies will not be considered here. It will be assumed that the matter of an elastic body is *homogeneous* and continuously distributed over its volume so that the smallest element cut from the body possesses the same specific physical properties as the body. To simplify the discussion it will also be assumed that the body is *isotropic*, *i.e.*, that the elastic properties are the same in all directions.

Structural materials usually do not satisfy the above assumptions. Such an important material as steel, for instance, when studied with a microscope, is seen to consist of crystals of various kinds and various orientations. The material is very far from being homogeneous; but experience shows that solutions of the theory of elasticity based on the assumptions of homogeneity and isotropy can be applied to steel structures with very great accuracy. The explanation of this is that the crystals are very small; usually there are millions of them in one cubic inch of steel. While the elastic properties of a single crystal may be very different in different directions, the crystals are ordinarily distributed at random and the elastic properties of larger pieces of metal represent averages of properties of the crystals. So long as the geometrical dimensions defining the form of a body are large in comparison with the dimensions of a single crystal the assumption of homogeneity can be used with great accuracy, and if the crystals are orientated at random the material can be treated as isotropic.

When, due to certain technological processes such as rolling, a certain orientation of the crystals in a metal prevails, the elastic properties of the metal become different in different directions and the condition of *anisotropy* must be considered. We have such a condition, for instance, in the case of cold-rolled copper.

2. Stress. Let Fig. 1 represent a body in equilibrium. Under the action of external forces P_1, \dots, P_7 , internal forces will be produced between the parts of the body. To study the magnitude of these forces at any point O , let us imagine the body divided into two parts A and B by a cross section mm through this point. Considering one of these parts, for instance, A , it can be stated that it is in equilibrium under the action of external forces P_1, \dots, P_7 and the inner forces distributed over the cross section mm and representing the actions of the material of the part B on the material of the part A . It will be assumed that these forces are continuously distributed over the area mm in the same way that hydrostatic pressure or wind pressure is continuously distributed over the surface on which it acts. The magnitudes of such forces are usually defined by their *intensity*, i.e., by the amount of force per unit area of the surface on which they act. In discussing internal forces this intensity is called *stress*.

In the simplest case of a prismatical bar submitted to tension by forces uniformly distributed over the ends (Fig. 2), the internal forces are also uniformly distributed over any cross section mm . Hence the intensity of this distribution, i.e., the stress, can be obtained by dividing the total tensile force P by the cross-sectional area A .

In the case just considered the stress was uniformly distributed over the cross section. In the general case of Fig. 1 the stress is not uniformly distributed over mm . To obtain the magnitude of stress acting on a small area δA , cut out from the cross section mm at any point O , we assume that the forces acting across this elemental area, due to the action of material of the part B on the material of the part A , can be reduced to a resultant δP . If we now continuously contract the elemental area δA , the limiting value of the ratio $\delta P/\delta A$ gives us the magnitude of the stress acting on the cross section mm at the point O . The limiting direction of the resultant δP is the direction of the stress. In the general case the direction of

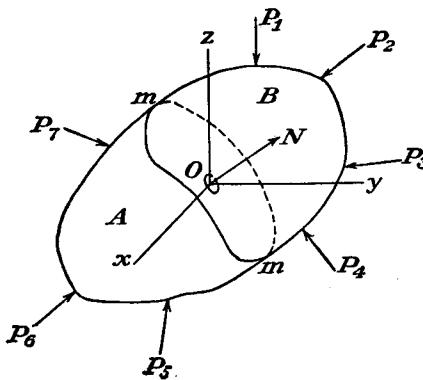


FIG. 1.

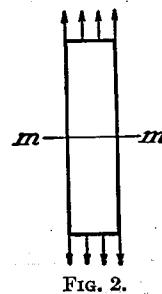


FIG. 2.

stress is inclined to the area δA on which it acts and we usually resolve it into two components: a *normal stress* perpendicular to the area, and a *shearing stress* acting in the plane of the area δA .

3. Notation for Forces and Stresses. There are two kinds of external forces which may act on bodies. Forces distributed over the surface of the body, such as the pressure of one body on another, or hydrostatic pressure, are called *surface forces*. Forces distributed over the volume of a body, such as gravitational forces, magnetic forces, or in the case of a body in motion, inertia forces, are called *body forces*. The surface force per unit area we shall usually resolve into three components parallel to the coordinate axes and use for these components the notation $\bar{X}, \bar{Y}, \bar{Z}$. We shall also resolve the body force per unit volume into three components and denote these components by X, Y, Z .

We shall use the letter σ for denoting normal stress and the letter τ for shearing stress. To indicate the direction of the plane on which the stress is acting, subscripts to these letters are used. If we take a very small cubic element at a point O , Fig. 3, with sides parallel to the coordinate axes, the notations for the components of stress acting on the sides of this element and the directions taken as positive are as indicated in Fig. 3. For the sides of the element perpendicular to the y -axis, for instance, the normal components of stress acting on these sides are denoted by σ_y . The subscript y indicates that the stress is acting on a plane normal to the y -axis. The normal stress is taken positive when it produces tension and negative when it produces compression.

The shearing stress is resolved into two components parallel to the coordinate axes. Two subscript letters are used in this case, the first indicating the direction of the normal to the plane under consideration and the second indicating the direction of the component of the stress. For instance, if we again consider the sides perpendicular to the y -axis, the component in the x -direction is denoted by τ_{yx} and that in the z -direction by τ_{yz} . The positive directions of the components of shearing stress on any side of the cubic element are taken as the positive directions of the coordinate axes if a tensile stress on the same side would have the positive direction of the corresponding axis. If the

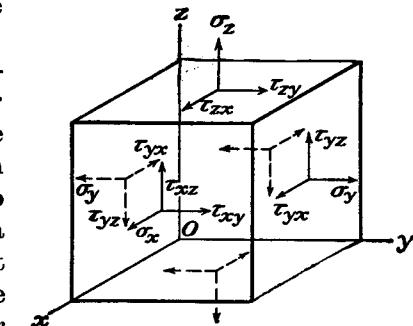


FIG. 3.

tensile stress has a direction opposite to the positive axis, the positive direction of the shearing-stress components should be reversed. Following this rule the positive directions of all the components of stress acting on the right side of the cubic element (Fig. 3) coincide with the positive directions of the coordinate axes. The positive directions are all reversed if we are considering the left side of this element.

4. Components of Stress. From the discussion of the previous article we see that, for each pair of parallel sides of a cubic element, such as in Fig. 3, one symbol is needed to denote the normal component of stress and two more symbols to denote the two components of shearing stress. To describe the stresses acting on the six sides of a cubic element three symbols, $\sigma_x, \sigma_y, \sigma_z$, are necessary for normal stresses; and six symbols, $\tau_{xy}, \tau_{yz}, \tau_{zx}, \tau_{xz}, \tau_{yz}, \tau_{zy}$, for shearing stresses. By a simple consideration of the equilibrium of the element the number of symbols for shearing stresses can be reduced to three.

If we take the moments of the forces acting on the element about the x -axis, for instance, only the surface stresses shown in Fig. 4 need be considered. Body forces, such as the weight of the element, can be neglected in this instance, which follows from the fact that in reducing the

dimensions of the element the body forces acting on it diminish as the cube of the linear dimensions while the surface forces diminish as the square of the linear dimensions. Hence, for a very small element, body forces are small quantities of higher order than surface forces and can be neglected in calculating the surface forces. Similarly, moments due to nonuniformity of distribution of normal forces are of higher order than those due to the shearing forces and vanish in the limit. Also the forces on each side can be considered to be the area of the side times the stress at the middle. Then denoting the dimensions of the small element in Fig. 4 by dx, dy, dz , the equation of equilibrium of this element, taking moments of forces about the x -axis, is

$$\tau_{zy} dx dy dz = \tau_{yz} dx dy dz$$

The two other equations can be obtained in the same manner. From these equations we find

$$\tau_{xy} = \tau_{yx}, \quad \tau_{zx} = \tau_{xz}, \quad \tau_{zy} = \tau_{yz} \quad (1)$$

Hence for two perpendicular sides of a cubic element the components of

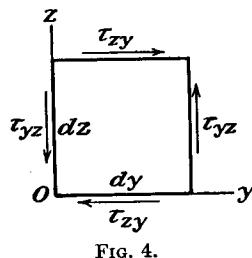


FIG. 4.

shearing stress perpendicular to the line of intersection of these sides are equal.

The six quantities $\sigma_x, \sigma_y, \sigma_z, \tau_{xy} = \tau_{yx}, \tau_{zx} = \tau_{xz}, \tau_{yz} = \tau_{zy}$ are therefore sufficient to describe the stresses acting on the coordinate planes through a point; these will be called the *components of stress* at the point.

It will be shown later (Art. 67) that with these six components the stress on any inclined plane through the same point can be determined.

5. Components of Strain. In discussing the deformation of an elastic body it will be assumed that there are enough constraints to prevent the body from moving as a rigid body, so that no displacements of particles of the body are possible without a deformation of it.

In this book, only small deformations such as occur in engineering structures will be considered. The small displacements of particles of a deformed body will usually be resolved into components u, v, w parallel to the coordinate axes x, y, z , respectively. It will be assumed that these components are very small quantities varying continuously over the volume of the body. Consider a small element $dx dy dz$ of an elastic body (Fig. 5). If the body undergoes a deformation and u, v, w are the components of the displacement of the point O , the displacement in the x -direction of an adjacent point A on the x -axis is

$$u + \frac{\partial u}{\partial x} dx$$

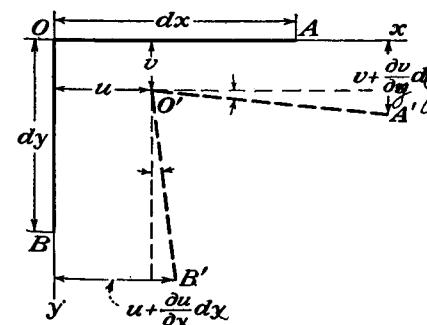


FIG. 5.

due to the increase $(\partial u / \partial x) dx$ of the function u with increase of the coordinate x . The increase in length of the element OA due to deformation is therefore $(\partial u / \partial x) dx$. Hence the *unit elongation* at point O in the

x -direction is $\partial u / \partial x$. In the same manner it can be shown that the unit elongations in the y - and z -directions are given by the derivatives $\partial v / \partial y$ and $\partial w / \partial z$.

Let us consider now the distortion of the angle between the elements OA and OB , Fig. 6. If u and v are the displacements of the point O in the x - and y -directions, the displacement of the point A in the y -direc-

tion and of the point B in the x -direction are $v + (\partial v / \partial x) dx$ and $u + (\partial u / \partial y) dy$, respectively. Due to these displacements the new direction $O'A'$ of the element OA is inclined to the initial direction by the small angle indicated in the figure, equal to $\partial v / \partial x$. In the same manner the direction $O'B'$ is inclined to OB by the small angle $\partial u / \partial y$. From this it will be seen that the initially right angle AOB between the two elements OA and OB is diminished by the angle $\partial v / \partial x + \partial u / \partial y$. This is the *shearing strain* between the planes xz and yz . The shearing strains between the planes xy and xz and the planes yx and yz can be obtained in the same manner.

We shall use the letter ϵ for unit elongation and the letter γ for unit shearing strain. To indicate the directions of strain we shall use the same subscripts to these letters as for the stress components. Then from the above discussion

$$\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x}, & \epsilon_y &= \frac{\partial v}{\partial y}, & \epsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\end{aligned}\quad (2)$$

It will be shown later that, having the three unit elongations in three perpendicular directions and three unit shear strains related to the same directions, the elongation in *any* direction and the distortion of the angle between *any* two directions can be calculated (see Art. 73). The six quantities $\epsilon_x, \dots, \gamma_{yz}$ are called the *components of strain*.

6. Hooke's Law. The relations between the components of stress and the components of strain have been established experimentally and are known as *Hooke's law*. Imagine an elemental rectangular parallelopiped with the sides parallel to the coordinate axes and submitted to the action of normal stress σ_x uniformly distributed over two opposite sides. Experiments show that in the case of an isotropic material these normal stresses do not produce any distortion of angles of the element. The magnitude of the unit elongation of the element is given by the equation

$$\epsilon_x = \frac{\sigma_x}{E} \quad (a)$$

in which E is the *modulus of elasticity in tension*. Materials used in engineering structures have moduli which are very large in comparison with allowable stresses, and the unit elongation (a) is a very small quantity. In the case of structural steel, for instance, it is usually smaller than 0.001.

Extension of the element in the x -direction is accompanied by lateral contractions,

$$\epsilon_y = -\nu \frac{\sigma_x}{E}, \quad \epsilon_z = -\nu \frac{\sigma_x}{E} \quad (b)$$

in which ν is a constant called *Poisson's ratio*. For many materials Poisson's ratio can be taken equal to 0.25. For structural steel it is usually taken equal to 0.3.

Equations (a) and (b) can be used also for simple compression. Within the elastic limit the modulus of elasticity and Poisson's ratio in compression are the same as in tension.

If the above element is submitted to the action of normal stresses $\sigma_x, \sigma_y, \sigma_z$, uniformly distributed over the sides, the resultant components of strain can be obtained by using Eqs. (a) and (b). Experiments show that to get these components we have to superpose the strain components produced by each of the three stresses. By this method of superposition we obtain the equations

$$\begin{aligned}\epsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)]\end{aligned}\quad (3)$$

In our further discussion we shall often use this *method of superposition* in calculating total deformations and stresses produced by several forces. This method is legitimate as long as the deformations are small and the corresponding small displacements do not affect substantially the action of the external forces. In such cases we neglect small changes in dimensions of deformed bodies and also small displacements of the points of application of external forces and base our calculations on initial dimensions and initial shape of the body. The resultant displacements will then be obtained by superposition in the form of linear functions of external forces, as in deriving Eqs. (3).

There are, however, exceptional cases in which small deformations cannot be neglected but must be taken into consideration. As an example of this kind the case of the simultaneous action on a thin bar of axial and lateral forces may be mentioned. Axial forces alone produce simple tension or compression, but they may have a substantial effect on the bending of the bar if they are acting simultaneously with lateral forces. In calculating the deformation of bars under such con-

ditions, the effect of the deflection on the moment of the external forces must be considered, even though the deflections are very small.¹ Then the total deflection is no longer a linear function of the forces and cannot be obtained by simple superposition.

Equations (3) show that the relations between elongations and stresses are completely defined by two physical constants E and ν . The same constants can also be used to define the relation between shearing strain and shearing stress.

Let us consider the particular case of deformation of the rectangular parallelopiped in which $\sigma_y = -\sigma_z$ and $\sigma_x = 0$. Cutting out an element $abcd$ by planes parallel to the x -axis and at 45 deg. to the y - and z -axes (Fig. 7), it may be seen from Fig. 7b, by summing up the forces along and perpendicular to bc , that the normal stress on the sides of this element is zero and the shearing stress on the sides is

$$\tau = \frac{1}{2}(\sigma_z - \sigma_y) = \sigma_z \quad (c)$$

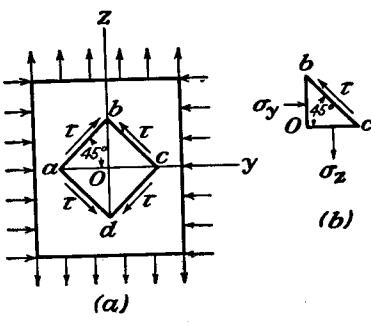


FIG. 7.

Such a condition of stress is called *pure shear*. The elongation of the vertical element Ob is equal to the shortening of the horizontal elements Oa and Oc , and neglecting a small quantity of the second order we conclude that the lengths ab and bc of the element do not change during deformation. The angle between the sides ab and bc changes, and the corresponding magnitude of shearing strain γ may be found from the triangle Obc . After deformation, we have

$$\frac{Oc}{Ob} = \tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{1 + \epsilon_y}{1 + \epsilon_z}$$

Substituting, from Eqs. (3),

$$\begin{aligned} \epsilon_z &= \frac{1}{E}(\sigma_z - \nu\sigma_y) = \frac{(1 + \nu)\sigma_z}{E} \\ \epsilon_y &= -\frac{(1 + \nu)\sigma_z}{E} \end{aligned}$$

and noting that for small γ

¹ Several examples of this kind can be found in S. Timoshenko, "Strength of Materials," vol. II, pp. 25-49.

$$\tan\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\gamma}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{\gamma}{2}} = \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$$

we find

$$\gamma = \frac{2(1 + \nu)\sigma_z}{E} = \frac{2(1 + \nu)\tau}{E} \quad (4)$$

Thus the relation between shearing strain and shearing stress is defined by the constants E and ν . Often the notation

$$G = \frac{E}{2(1 + \nu)} \quad (5)$$

is used. Then Eq. (4) becomes

$$\gamma = \frac{\tau}{G}$$

The constant G , defined by (5), is called the *modulus of elasticity in shear* or the *modulus of rigidity*.

If shearing stresses act on the sides of an element, as shown in Fig. 3, the distortion of the angle between any two coordinate axes depends only on shearing-stress components parallel to these axes and we obtain

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \quad \gamma_{zx} = \frac{1}{G}\tau_{zx} \quad (6)$$

The elongations (3) and the distortions (6) are independent of each other. Hence the general case of strain, produced by three normal and three shearing components of stress, can be obtained by superposition: on the three elongations given by Eqs. (3) are superposed three shearing strains given by Eqs. (6).

Equations (3) and (6) give the components of strain as functions of the components of stress. Sometimes the components of stress expressed as functions of the components of strain are needed. These can be obtained as follows. Adding equations (3) together and using the notations

$$\begin{aligned} e &= \epsilon_x + \epsilon_y + \epsilon_z \\ \Theta &= \sigma_x + \sigma_y + \sigma_z \end{aligned} \quad (7)$$

we obtain the following relation between the volume expansion e and the sum of normal stresses:

$$e = \frac{1 - 2\nu}{E} \Theta \quad (8)$$

In the case of a uniform hydrostatic pressure of the amount p we have

$$\sigma_x = \sigma_y = \sigma_z = -p$$

and Eq. (8) gives

$$e = -\frac{3(1-2\nu)p}{E}$$

which represents the relation between unit volume expansion e and hydrostatic pressure p .

The quantity $E/3(1-2\nu)$ is called the *modulus of volume expansion*.

Using notations (7) and solving Eqs. (3) for $\sigma_x, \sigma_y, \sigma_z$, we find

$$\begin{aligned}\sigma_x &= \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{1+\nu} \epsilon_x \\ \sigma_y &= \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{1+\nu} \epsilon_y \\ \sigma_z &= \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{1+\nu} \epsilon_z\end{aligned}\quad (9)$$

or using the notation

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (10)$$

and Eq. (5), these become

$$\begin{aligned}\sigma_x &= \lambda e + 2G\epsilon_x \\ \sigma_y &= \lambda e + 2G\epsilon_y \\ \sigma_z &= \lambda e + 2G\epsilon_z\end{aligned}\quad (11)$$

Problems

- Show that Eqs. (1) continue to hold if the element of Fig. 4 is in motion and has an angular acceleration like a rigid body.
- Suppose an elastic material contains a large number of evenly distributed small magnetized particles, so that a magnetic field exerts on any element $dx dy dz$ a moment $\mu dx dy dz$ about an axis parallel to the x -axis. What modification will be needed in Eqs. (1)?
- Give some reasons why the formulas (2) will be valid for *small* strains only.
- An elastic layer is sandwiched between two perfectly rigid plates, to which it is bonded. The layer is compressed between the plates, the compressive stress being σ_z . Supposing that the attachment to the plates prevents lateral strain being ϵ_x, ϵ_y completely, find the apparent Young's modulus (*i.e.*, σ_z/ϵ_z) in terms of E and ν . Show that it is many times E if the material of the layer is nearly incompressible by hydrostatic pressure.
- Prove that Eq. (8) follows from Eqs. (11), (10), and (5).

CHAPTER 2

PLANE STRESS AND PLANE STRAIN

7. Plane Stress. If a thin plate is loaded by forces applied at the boundary, parallel to the plane of the plate and distributed uniformly over the thickness (Fig. 8), the stress components $\sigma_z, \tau_{xz}, \tau_{yz}$ are zero on both faces of the plate, and it may be assumed, tentatively, that they are zero also within the plate. The state of stress is then specified by $\sigma_x, \sigma_y, \tau_{xy}$ only, and is called *plane stress*. It may also be assumed that

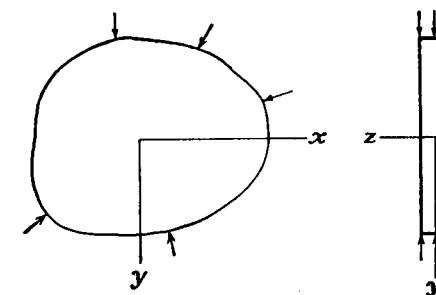


FIG. 8.

these three components are independent of z , *i.e.*, they do not vary through the thickness. They are then functions of x and y only.

8. Plane Strain. A similar simplification is possible at the other extreme when the dimension of the body in the z -direction is very large. If a long cylindrical or prismatical body is loaded by forces which are perpendicular to the longitudinal elements and do not vary along the length, it may be assumed that all cross sections are in the same condition. It is simplest to suppose at first that the end sections are confined between fixed smooth rigid planes, so that displacement in the axial direction is prevented. The effect of removing these will be examined later. Since there is no axial displacement at the ends, and, by symmetry, at the mid-section, it may be assumed that the same holds at every cross section.

There are many important problems of this kind—a retaining wall with lateral pressure (Fig. 9), a culvert or tunnel (Fig. 10), a cylindrical tube with internal pressure, a cylindrical roller compressed by forces in

a diametral plane as in a roller bearing (Fig. 11). In each case of course the loading must not vary along the length. Since conditions are the same at all cross sections, it is sufficient to consider only a slice between two sections unit distance apart. The components u and v of the displacement are functions of x and y but are independent of the

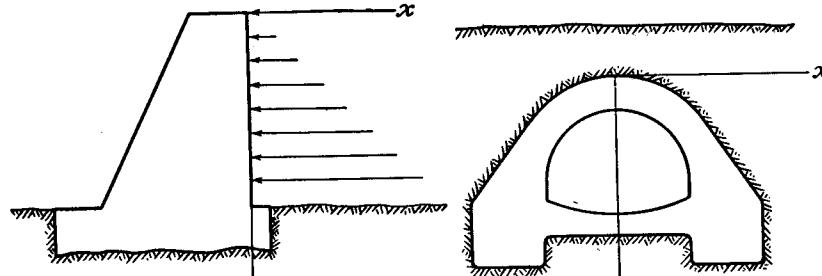


FIG. 9.

FIG. 10.

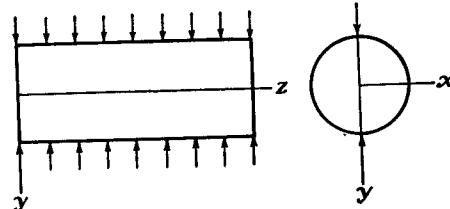


FIG. 11.

longitudinal coordinate z . Since the longitudinal displacement w is zero, Eqs. (2) give

$$\begin{aligned}\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \\ \epsilon_z &= \frac{\partial w}{\partial z} = 0\end{aligned}\quad (a)$$

The longitudinal normal stress σ_z can be found in terms of σ_x and σ_y by means of Hooke's law, Eqs. (3). Since $\epsilon_z = 0$ we find

$$\begin{aligned}\sigma_z - \nu(\sigma_x + \sigma_y) &= 0 \\ \text{or} \quad \sigma_z &= \nu(\sigma_x + \sigma_y)\end{aligned}\quad (b)$$

These normal stresses act over the cross sections, including the ends, where they represent forces required to maintain the plane strain, and provided by the fixed smooth rigid planes.

By Eqs. (a) and (6), the stress components τ_{xz} and τ_{yz} are zero, and, by Eq. (b), σ_z can be found from σ_x and σ_y . Thus the plane strain problem, like the plane stress problem, reduces to the determination of σ_x , σ_y , and τ_{xy} as functions of x and y only.

9. Stress at a Point. Knowing the stress components σ_x , σ_y , τ_{xy} at any point of a plate in a condition of plane stress or plane strain, the stress acting on any plane through this point perpendicular to the plate and inclined to the x - and y -axes can be calculated from the equations of statics. Let O be a point of the stressed plate and suppose the stress components σ_x , σ_y , τ_{xy} are known (Fig. 12). To find the stress for any plane through the z -axis and inclined to the x - and y -axes, we take a plane BC parallel to it, at a small distance from O , so that this latter plane together with the coordinate planes cuts out from the plate a very small triangular prism OBC . Since the stresses vary continuously over the volume of the body the stress acting on the plane BC will approach the stress on the parallel plane through O as the element is made smaller.

In discussing the conditions of equilibrium of the small triangular prism, the body force can be neglected as a small quantity of a higher order (page 4). Likewise, if the element is very small, we can neglect the variation of the stresses over the sides and assume that the stresses are uniformly distributed. The forces acting on the triangular prism can therefore be determined by multiplying the stress components by the areas of the sides. Let N be the direction of the normal to the plane BC , and denote the cosines of the angles between the normal N and the axes x and y by

$$\cos Nx = l, \quad \cos Ny = m$$

Then, if A denotes the area of the side BC of the element, the areas of the other two sides are Al and Am .

If we denote by \bar{X} and \bar{Y} the components of stress acting on the side BC , the equations of equilibrium of the prismatical element give

$$\begin{aligned}\bar{X} &= l\sigma_x + m\tau_{xy} \\ \bar{Y} &= m\sigma_y + l\tau_{xy}\end{aligned}\quad (12)$$

Thus the components of stress on any plane defined by the direction

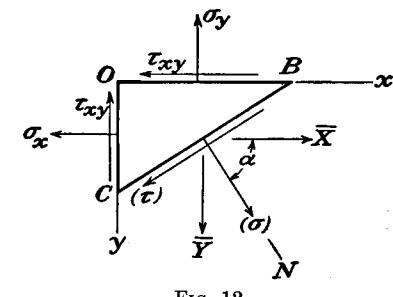


FIG. 12.

cosines l and m can easily be calculated from Eqs. (12), provided the three components of stress σ_x , σ_y , τ_{xy} at the point O are known.

Letting α be the angle between the normal N and the x -axis, so that $l = \cos \alpha$ and $m = \sin \alpha$, the normal and shearing components of stress on the plane BC are (from Eqs. 12)

$$\begin{aligned}\sigma &= \bar{X} \cos \alpha + \bar{Y} \sin \alpha = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \\ &\quad + 2\tau_{xy} \sin \alpha \cos \alpha \quad (13) \\ \tau &= \bar{Y} \cos \alpha - \bar{X} \sin \alpha = \tau_{xy}(\cos^2 \alpha - \sin^2 \alpha) \\ &\quad + (\sigma_y - \sigma_x) \sin \alpha \cos \alpha\end{aligned}$$

It may be seen that the angle α can be chosen in such a manner that the shearing stress τ becomes equal to zero. For this case we have

$$\tau_{xy}(\cos^2 \alpha - \sin^2 \alpha) + (\sigma_y - \sigma_x) \sin \alpha \cos \alpha = 0$$

or

$$\frac{\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\sin \alpha \cos \alpha}{\cos^2 \alpha - \sin^2 \alpha} = \frac{1}{2} \tan 2\alpha \quad (14)$$

From this equation two perpendicular directions can be found for which the shearing stress is zero. These directions are called *principal directions* and the corresponding normal stresses *principal stresses*.

If the principal directions are taken as the x - and y -axes, τ_{xy} is zero and Eqs. (13) are simplified to

$$\begin{aligned}\sigma &= \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha \quad (13') \\ \tau &= \frac{1}{2} \sin 2\alpha(\sigma_y - \sigma_x)\end{aligned}$$

The variation of the stress components σ and τ , as we vary the angle α , can be easily represented graphically by making a diagram in which σ and τ are taken as coordinates.¹ For each plane there will correspond a point on this diagram, the coordinates of which represent the values of σ and τ for this plane. Figure 13 represents such a diagram. For the planes perpendicular to the principal directions we obtain points A and B with abscissas σ_x and σ_y , respectively. Now it can be proved that the stress components for any plane BC with an angle α (Fig. 12) that the stress components for any plane BC with an angle α (Fig. 12) will be represented by coordinates of a point on the circle having AB as a diameter. To find this point it is only necessary to measure from the point A in the same direction as α is measured in Fig. 12 an arc subtending an angle equal to 2α . If D is the point obtained in this manner, then, from the figure,

$$OF = OC + CF = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha$$

$$DF = CD \sin 2\alpha = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\alpha$$

¹ This graphical method is due to O. Mohr, *Zivilingenieur*, 1882, p. 113. See also his "Technische Mechanik," 2d ed., 1914.

Comparing with Eqs. (13') it is seen that the coordinates of point D give the numerical values of stress components on the plane BC at the angle α . To bring into coincidence the sign of the shearing component we take τ positive in the upward direction (Fig. 13) and consider shearing stresses as positive when they give a couple in the clockwise direction, as on the sides bc and ad of the element $abcd$ (Fig. 13b). Shearing stresses of opposite direction, as on the sides ab and dc of the element, are considered as negative.¹

As the plane BC rotates about an axis perpendicular to the xy -plane (Fig. 12) in the clockwise direction, and α varies from 0 to $\pi/2$, the

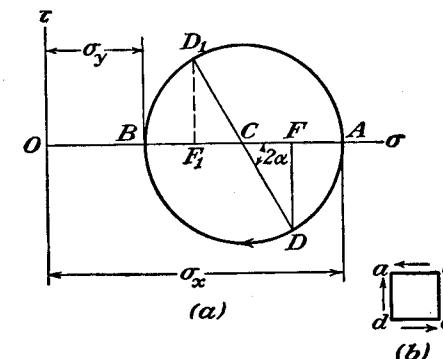


FIG. 13.

point D in Fig. 13 moves from A to B , so that the lower half circle determines the stress variation for all values of α within these limits. The upper half of the circle gives stresses for $\pi/2 \leq \alpha \leq \pi$.

Prolonging the radius CD to the point D_1 (Fig. 13), i.e., taking the angle $\pi + 2\alpha$, instead of 2α , the stresses on the plane perpendicular to BC (Fig. 12) are obtained. This shows that the shearing stresses on two perpendicular planes are numerically equal as previously proved. As for normal stresses, we see from the figure that $OF_1 + OF = 2OC$, i.e., the sum of the normal stresses over two perpendicular cross sections remains constant when the angle α changes.

The maximum shearing stress is given in the diagram (Fig. 13) by the maximum ordinate of the circle, i.e., is equal to the radius of the circle. Hence

$$\tau_{\max.} = \frac{\sigma_x - \sigma_y}{2} \quad (15)$$

It acts on the plane for which $\alpha = \pi/4$, i.e., on the plane bisecting the angle between the two principal stresses.

¹ This rule is used only in the construction of Mohr's circle. Otherwise the rule given on p. 3 holds.

The diagram can be used also in the case when one or both principal stresses are negative (compression). It is only necessary to change the sign of the abscissa for compressive stress. In this manner Fig. 14a represents the case when both principal stresses are negative and Fig. 14b the case of pure shear.

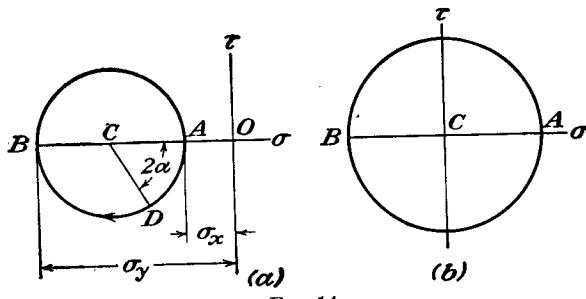


FIG. 14.

From Figs. 13 and 14 it is seen that the stress at a point can be resolved into two parts: One, uniform tension or compression, the magnitude of which is given by the abscissa of the center of the circle; and the other, pure shear, the magnitude of which is given by the radius of the circle. When several plane stress distributions are superposed, the uniform tensions or compressions can be added together

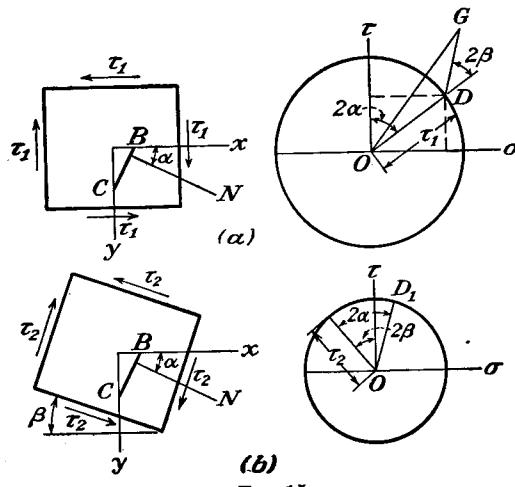


FIG. 15.

algebraically. The pure shears must be added together by taking into account the directions of the planes on which they are acting. It can be shown that, if we superpose two systems of pure shear whose planes of maximum shear make an angle of β with each other, the resulting system will be another case of pure shear. For example, Fig. 15 represents the determination of stress on any plane defined by α , produced by two pure shears of magnitude τ_1 and τ_2 acting one on the planes

xz and yz (Fig. 15a) and the other on the planes inclined to xz and yz by the angle β (Fig. 15b). In Fig. 15a the coordinates of point D represent the shear and normal stress on plane CB produced by the first system, while the coordinate of D_1 (Fig. 15b) gives the stresses on this plane for the second system. Adding OD and OD_1 geometrically we obtain OG , the resultant stress on the plane due to both systems, the coordinates of G giving us the shear and normal stress. Note that the magnitude of OG does not depend upon α . Hence, as the result of the superposition of two shears, we obtain a Mohr circle for pure shear, the magnitude of which is given by OG , the planes of maximum shear being inclined to the xz and yz planes by an angle equal to half the angle GOD .

A diagram, such as shown in Fig. 13, can be used also for determining principal stresses if the stress components σ_x , σ_y , τ_{xy} for any two perpendicular planes (Fig. 12) are known. We begin in such a case with the plotting of the two points D and D_1 , representing stress conditions on the two coordinate planes (Fig. 16). In this manner the diameter DD_1 of the circle is obtained. Constructing the circle, the principal stresses σ_1 and σ_2 are obtained from the intersection of the circle with the abscissa axis. From the figure we find

$$\begin{aligned}\sigma_1 &= OC + CD = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_2 &= OC - CD = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}\end{aligned}\quad (16)$$

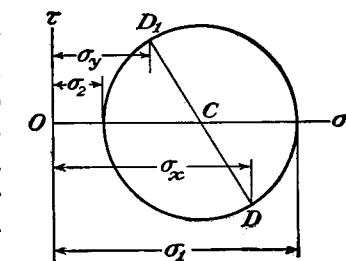


FIG. 16.

The maximum shearing stress is given by the radius of the circle, i.e.,

$$\tau_{\max.} = \frac{1}{2} (\sigma_1 - \sigma_2) = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (17)$$

In this manner all necessary features of the stress distribution at a point can be obtained if only the three stress components σ_x , σ_y , τ_{xy} are known.

10. Strain at a Point. When the strain components ϵ_x , ϵ_y , γ_{xy} at a point are known, the unit elongation for any direction, and the decrease of a right angle—the shearing strain—of any orientation at the point can be found. A line element PQ (Fig. 17a) between the points (x, y) , $(x + dx, y + dy)$ is translated, stretched (or contracted) and rotated into the line element $P'Q'$ when the deformation occurs. The dis-

placement components of P are u, v , and those of Q are

$$u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad v + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

If $P'Q'$ in Fig. 17a is now translated so that P' is brought back to P , it is in the position PQ'' of Fig. 17b, and QR, RQ'' represent the components of the displacement of Q relative to P . Thus

$$QR = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \quad RQ'' = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad (a)$$

The components of this relative displacement QS, SQ'' , normal to PQ'' and along PQ'' , can be found from these as

$$QS = -QR \sin \theta + RQ'' \cos \theta, \quad SQ'' = QR \cos \theta + RQ'' \sin \theta \quad (b)$$

ignoring the small angle QPS in comparison with θ . Since the short line QS may be identified with an arc of a circle with center P , SQ''

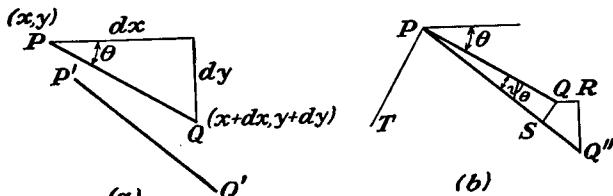


FIG. 17.

gives the stretch of PQ . The unit elongation of $P'Q'$, denoted by ϵ_θ , is SQ''/PQ . Using (b) and (a) we have

$$\begin{aligned} \epsilon_\theta &= \cos \theta \left(\frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} \right) + \sin \theta \left(\frac{\partial v}{\partial x} \frac{dx}{ds} + \frac{\partial v}{\partial y} \frac{dy}{ds} \right) \\ &= \frac{\partial u}{\partial x} \cos^2 \theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \sin \theta \cos \theta + \frac{\partial v}{\partial y} \sin^2 \theta \end{aligned} \quad (c)$$

or

$$\epsilon_\theta = \epsilon_x \cos^2 \theta + \gamma_{xy} \sin \theta \cos \theta + \epsilon_y \sin^2 \theta \quad (c)$$

which gives the unit elongation for any direction θ .

The angle ψ_θ through which PQ is rotated is QS/PQ . Thus from (b) and (a),

$$\psi_\theta = -\sin \theta \left(\frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds} \right) + \cos \theta \left(\frac{\partial v}{\partial x} \frac{dx}{ds} + \frac{\partial v}{\partial y} \frac{dy}{ds} \right)$$

or

$$\psi_\theta = \frac{\partial v}{\partial x} \cos^2 \theta + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \sin \theta \cos \theta - \frac{\partial u}{\partial y} \sin^2 \theta \quad (d)$$

The line element PT at right angles to PQ makes an angle $\theta + (\pi/2)$ with the x -direction, and its rotation $\psi_{\theta+\frac{\pi}{2}}$ is therefore given by (d) when $\theta + (\pi/2)$ is substituted for θ . Since $\cos [\theta + (\pi/2)] = -\sin \theta$, $\sin [\theta + (\pi/2)] = \cos \theta$, we find

$$\psi_{\theta+\frac{\pi}{2}} = \frac{\partial v}{\partial x} \sin^2 \theta - \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \sin \theta \cos \theta - \frac{\partial u}{\partial y} \cos^2 \theta \quad (e)$$

The shear strain γ_θ for the directions PQ, PT is $\psi_\theta - \psi_{\theta+\frac{\pi}{2}}$, so

$$\gamma_\theta = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) (\cos^2 \theta - \sin^2 \theta) + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) 2 \sin \theta \cos \theta$$

or

$$\frac{1}{2} \gamma_\theta = \frac{1}{2} \gamma_{xy} (\cos^2 \theta - \sin^2 \theta) + (\epsilon_y - \epsilon_x) \sin \theta \cos \theta \quad (f)$$

Comparing (c) and (f) with (13), we observe that they may be obtained from (13) by replacing σ by ϵ_θ , τ by $\gamma_\theta/2$, σ_x by ϵ_x , σ_y by ϵ_y , τ_{xy} by $\gamma_{xy}/2$, and α by θ . Consequently for each deduction made from (13) as to σ and τ , there is a corresponding deduction from (c) and (f) as to ϵ_θ and $\gamma_\theta/2$. Thus there are two values of θ , differing by 90 deg., for which γ_θ is zero. They are given by

$$\frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \tan 2\theta$$

The corresponding strains ϵ_θ are *principal strains*. A Mohr circle diagram analogous to Fig. 13 or Fig. 16 may be drawn, the ordinates representing $\gamma_\theta/2$ and the abscissas ϵ_θ . The principal strains ϵ_1, ϵ_2 will be the algebraically greatest and least values of ϵ_θ as a function of θ . The greatest value of $\gamma_\theta/2$ will be represented by the radius of the circle. Thus the greatest shearing strain $\gamma_{\theta \max}$ is given by

$$\gamma_{\theta \max} = \epsilon_1 - \epsilon_2$$

11. Measurement of Surface Strains. The strains, or unit elongations, on a surface are usually most conveniently measured by means of electric-resistance strain gauges.¹ The simplest form of such a gauge is a short length of wire insulated from and glued to the surface. When stretching occurs the resistance of the wire is increased, and the strain can thus be measured electrically. The effect is usually magnified by looping the wires backward and forward several times, to form several gauge lengths connected in series. The wire is glued between two tabs of paper, and the assembly glued to the surface.

The use of these gauges is simple when the principal directions are

¹ A detailed account of this method is given in the "Handbook of Experimental Stress Analysis," Chaps. 5 and 9.

known. One gauge is placed along each principal direction and direct measurements of ϵ_1, ϵ_2 obtained. The principal stresses σ_1, σ_2 may then be calculated from Hooke's law, Eqs. (3), with $\sigma_x = \sigma_1, \sigma_y = \sigma_2, \sigma_z = 0$, the last holding on the assumption that there is no stress acting on the surface to which the gauges are attached. Then

$$(1 - \nu^2)\sigma_1 = E(\epsilon_1 + \nu\epsilon_2), \quad (1 - \nu^2)\sigma_2 = E(\epsilon_2 + \nu\epsilon_1)$$

When the principal directions are not known in advance, three measurements are needed. Thus the state of strain is completely determined if $\epsilon_x, \epsilon_y, \gamma_{xy}$ can be measured. But since the strain gauges mea-

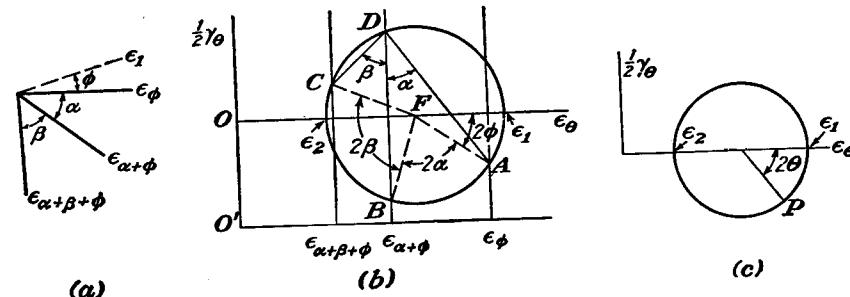


FIG. 18.

ure extensions, and not shearing strain directly, it is convenient to measure the unit elongations in three directions at the point. Such a set of gauges is called a "strain rosette." The Mohr circle can be drawn by the simple construction¹ given in Art. 12, and the principal drawn by the simple construction¹ given in Art. 12, and the principal strains can then be read off. The three gauges are represented by the three full lines in Fig. 18a. The broken line represents the (unknown) direction of the larger principal strain ϵ_1 , from which the direction of the first gauge is obtained by a clockwise rotation ϕ .

If the x - and y -directions for Eqs. (c) and (f) of Art. 10 had been taken as the principal directions, ϵ_x would be ϵ_1 , ϵ_y would be ϵ_2 , and γ_{xy} would be zero. The equations would then be

$$\epsilon_\theta = \epsilon_1 \cos^2 \theta + \epsilon_2 \sin^2 \theta, \quad \frac{1}{2}\gamma_\theta = -(\epsilon_1 - \epsilon_2) \sin \theta \cos \theta$$

where θ is the angle measured from the direction of ϵ_1 . These may be written

$$\epsilon_\theta = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2) \cos 2\theta, \quad \frac{1}{2}\gamma_\theta = -\frac{1}{2}(\epsilon_1 - \epsilon_2) \sin 2\theta$$

and these values are represented by the point P on the circle in Fig. 18c. If θ takes the value ϕ , P corresponds to the point A on the circle in Fig.

¹ Glenn Murphy, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 12, p. A-209, 1945; N. J. Hoff, *ibid.*

18b, the angular displacement from the ϵ_θ -axis being 2ϕ . The abscissa of this point is ϵ_ϕ , which is known. If θ takes the value $\phi + \alpha$, P moves to B , through a further angle $AFB = 2\alpha$, and the abscissa is the known value $\epsilon_{\alpha+\phi}$. If θ takes the value $\phi + \alpha + \beta$, P moves on to C , through a further angle $BFC = 2\beta$, and the abscissa is $\epsilon_{\alpha+\beta+\phi}$.

The problem is to draw the circle when these three abscissas and the two angles α, β are known.

12. Construction of Mohr Strain Circle for Strain Rosette. A temporary horizontal ϵ -axis is drawn horizontally from any origin O' , Fig. 18b, and the three measured strains $\epsilon_\phi, \epsilon_{\alpha+\phi}, \epsilon_{\alpha+\beta+\phi}$ laid off along it. Verticals are drawn through these points. Selecting any point D on the vertical through $\epsilon_{\alpha+\phi}$, lines DA, DC are drawn at angles α and β to the vertical at D as shown, to meet the other two verticals at A and C . The circle drawn through D, A , and C is the required circle. Its center F is determined by the intersection of the perpendicular bisectors of CD, DA . The points representing the three gauge directions are A, B , and C . The angle AFB , being twice the angle ADB at the circumference, is 2α , and BFC is 2β . Thus A, B, C are at the required angular intervals round the circle, and have the required abscissas. The ϵ_θ axis can now be drawn as OF , and the distances from O to the intersections with the circle give ϵ_1, ϵ_2 . The angle 2ϕ is the angle of FA below this axis.

13. Differential Equations of Equilibrium. We now consider the equilibrium of a small rectangular block of edges h, k , and unity (Fig. 19). The stresses acting on the faces 1, 2, 3, 4, and their positive directions are indicated in the figure. On account of the variation of stress

throughout the material, the value of, for instance, σ_x is not quite the same for face 1 as for face 3. The symbols $\sigma_x, \sigma_y, \tau_{xy}$ refer to the point x, y , the mid-point of the rectangle in Fig. 19. The values at the mid-points of the faces are denoted by $(\sigma_x)_1, (\sigma_x)_3$, etc. Since the faces are very small, the corresponding forces are obtained by multiplying these values by the areas of the faces on which they act.¹

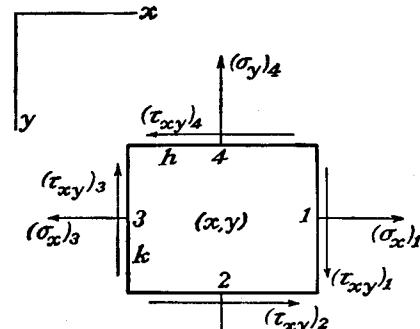


FIG. 19.

¹ More precise considerations would introduce terms of higher order which vanish in the final limiting process.

The body force on the block, which was neglected as a small quantity of higher order in considering the equilibrium of the triangular prism of Fig. 12, must be taken into consideration, because it is of the same order of magnitude as the terms due to the variations of the stress components which are now under consideration. If X, Y denote the components of body force per unit volume, the equation of equilibrium for forces in the x -direction is

$$(\sigma_x)_1 k - (\sigma_x)_3 k + (\tau_{xy})_2 h - (\tau_{xy})_4 h + Xhk = 0$$

or, dividing by hk ,

$$\frac{(\sigma_x)_1 - (\sigma_x)_3}{h} + \frac{(\tau_{xy})_2 - (\tau_{xy})_4}{k} + X = 0$$

If now the block is taken smaller and smaller, i.e., $h \rightarrow 0, k \rightarrow 0$, the limit of $[(\sigma_x)_1 - (\sigma_x)_3]/h$ is $\partial\sigma_x/\partial x$ by the definition of such a derivative. Similarly $[(\tau_{xy})_2 - (\tau_{xy})_4]/k$ becomes $\partial\tau_{xy}/\partial y$. The equation of equilibrium for forces in the y -direction is obtained in the same manner. Thus

$$\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + Y &= 0 \end{aligned} \quad (18)$$

In practical applications the weight of the body is usually the only body force. Then, taking the y -axis downward and denoting by ρ the mass per unit volume of the body, Eqs. (18) become

$$\begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} + \rho g &= 0 \end{aligned} \quad (19)$$

These are the differential equations of equilibrium for two-dimensional problems.

14. Boundary Conditions. Equations (18) or (19) must be satisfied at all points throughout the volume of the body. The stress components vary over the volume of the plate, and when we arrive at the boundary they must be such as to be in equilibrium with the external forces on the boundary of the plate, so that external forces may be regarded as a continuation of the internal stress distribution. These conditions of equilibrium at the boundary can be obtained from Eqs. (12). Taking the small triangular prism OBC (Fig. 12), so that the side BC coincides with the boundary of the plate, as shown in Fig. 20,

and denoting by \bar{X} and \bar{Y} the components of the surface forces per unit area at this point of the boundary, we have

$$\begin{aligned} \bar{X} &= l\sigma_x + m\tau_{xy} \\ \bar{Y} &= m\sigma_y + l\tau_{xy} \end{aligned} \quad (20)$$

in which l and m are the direction cosines of the normal N to the boundary.

In the particular case of a rectangular plate the coordinate axes are usually taken parallel to the sides of the plate and the boundary conditions (20) can be simplified. Taking, for instance, a side of the plate parallel to the x -axis we have for this part of the boundary the normal N parallel to the y -axis; hence $l = 0$ and $m = \pm 1$. Equations (20) then become

$$\bar{X} = \pm\tau_{xy}, \quad \bar{Y} = \pm\sigma_y$$

Here the positive sign should be taken if the normal N has the positive direction of the y -axis and the negative sign for the opposite direction of N . It is seen from this that at the boundary the stress components become equal to the components of the surface forces per unit area of the boundary.

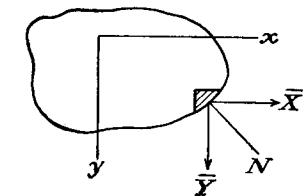


FIG. 20.

15. Compatibility Equations. The problem of the theory of elasticity usually is to determine the state of stress in a body submitted to the action of given forces. In the case of a two-dimensional problem it is necessary to solve the differential equations of equilibrium (18), and the solution must be such as to satisfy the boundary conditions (20). These equations, derived by application of the equations of statics for absolutely rigid bodies, and containing three stress components $\sigma_x, \sigma_y, \tau_{xy}$, are not sufficient for the determination of these components. The problem is a statically indeterminate one, and in order to obtain the solution the elastic deformation of the body must also be considered.

The mathematical formulation of the condition for compatibility of stress distribution with the existence of continuous functions u, v, w defining the deformation will be obtained from Eqs. (2). In the case of two-dimensional problems only three strain components need be considered, namely,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (a)$$

These three strain components are expressed by two functions u and v ; hence they cannot be taken arbitrarily, and there exists a certain rela-

tion between the strain components which can easily be obtained from (a). Differentiating the first of the Eqs. (a) twice with respect to y , the second twice with respect to x , and the third once with respect to x and once with respect to y , we find

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (21)$$

This differential relation, called the *condition of compatibility*, must be satisfied by the strain components to secure the existence of functions u and v connected with the strain components by Eqs. (a). By using Hooke's law, [Eqs. (3)], the condition (21) can be transformed into a relation between the components of stress.

In the case of plane stress distribution (Art. 7), Eqs. (3) reduce to

$$\epsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y), \quad \epsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad (22)$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} \quad (23)$$

Substituting in Eq. (21), we find

$$\frac{\partial^2}{\partial y^2} (\sigma_x - \nu \sigma_y) + \frac{\partial^2}{\partial x^2} (\sigma_y - \nu \sigma_x) = 2(1+\nu) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} \quad (b)$$

This equation can be written in a different form by using the equations of equilibrium. For the case when the weight of the body is the only body force, differentiating the first of Eqs. (19) with respect to x and the second with respect to y and adding them, we find

$$2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = - \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

Substituting in Eq. (b), the compatibility equation in terms of stress components becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0 \quad (24)$$

Proceeding in the same manner with the general equations of equilibrium (18) we find

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -(1+\nu) \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \quad (25)$$

In the case of plane strain (Art. 8), we have

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

and from Hooke's law (Eqs. 3), we find

$$\epsilon_x = \frac{1}{E} [(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y] \quad (26)$$

$$\epsilon_y = \frac{1}{E} [(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x]$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} \quad (27)$$

Substituting in Eq. (21), and using, as before, the equations of equilibrium (19), we find that the compatibility equation (24) holds also for plane strain. For the general case of body forces we obtain from Eqs. (21) and (18) the compatibility equation in the following form:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = - \frac{1}{1-\nu} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \quad (28)$$

The equations of equilibrium (18) or (19) together with the boundary conditions (20) and one of the above compatibility equations give us a system of equations which is usually sufficient for the complete determination of the stress distribution in a two-dimensional problem.¹ The particular cases in which certain additional considerations are necessary will be discussed later (page 117). It is interesting to note that in the case of constant body forces the equations determining stress distribution do not contain the elastic constants of the material. Hence the stress distribution is the same for all isotropic materials, provided the equations are sufficient for the complete determination of the stresses. The conclusion is of practical importance: we shall see later that in the case of transparent materials, such as glass or xylonite, it is possible to determine stresses by an optical method using polarized light (page 131). From the above discussion it is evident that experimental results obtained with a transparent material in most cases can be applied immediately to any other material, such as steel.

It should be noted also that in the case of constant body forces the compatibility equation (24) holds both for the case of plane stress and for the case of plane strain. Hence the stress distribution is the same in these two cases, provided the shape of the boundary and the external forces are the same.²

¹ In plane stress there are compatibility conditions other than (21) which are in fact violated by our assumptions. It is shown in Art. 84 that in spite of this the method of the present chapter gives good approximations for thin plates.

² This statement may require modification when the plate or cylinder has holes, for then the problem can be correctly solved only by considering the displacements as well as the stresses. See Art. 39.

16. Stress Function. It has been shown that a solution of two-dimensional problems reduces to the integration of the differential equations of equilibrium together with the compatibility equation and the boundary conditions. If we begin with the case when the weight of the body is the only body force, the equations to be satisfied are (see Eqs. 19 and 24)

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g &= 0\end{aligned}\quad (a)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0 \quad (b)$$

To these equations the boundary conditions (20) should be added. The usual method of solving these equations is by introducing a new function, called the *stress function*.¹ As is easily checked, Eqs. (a) are satisfied by taking any function ϕ of x and y and putting the following expressions for the stress components:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} - \rho gy, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} - \rho gy, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (29)$$

In this manner we can get a variety of solutions of the equations of equilibrium (a). The true solution of the problem is that which satisfies also the compatibility equation (b). Substituting expressions (29) for the stress components into Eq. (b) we find that the stress function ϕ must satisfy the equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (30)$$

Thus the solution of a two-dimensional problem, when the weight of the body is the only body force, reduces to finding a solution of Eq. (30) which satisfies the boundary conditions (20) of the problem. In the following chapters this method of solution will be applied to several examples of practical interest.

Let us now consider a more general case of body forces and assume that these forces have a potential. Then the components X and Y in Eqs. (18) are given by the equations

¹ This function was introduced in the solution of two-dimensional problems by G. B. Airy, *Brit. Assoc. Advancement Sci. Rept.*, 1862, and is sometimes called the *Airy stress function*.

$$\begin{aligned}X &= -\frac{\partial V}{\partial x} \\ Y &= -\frac{\partial V}{\partial y}\end{aligned}\quad (c)$$

in which V is the potential function. Equations (18) become

$$\begin{aligned}\frac{\partial}{\partial x} (\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial}{\partial y} (\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} &= 0\end{aligned}$$

These equations are of the same form as Eqs. (a) and can be satisfied by taking

$$\sigma_x - V = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y - V = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (31)$$

in which ϕ is the stress function. Substituting expressions (31) in the compatibility equation (25) for plane stress distribution, we find

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1-\nu) \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \quad (32)$$

An analogous equation can be obtained for the case of plane strain.

When the body force is simply the weight, the potential V is $-\rho gy$. In this case the right-hand side of Eq. (32) reduces to zero. By taking the solution $\phi = 0$ of (32), or of (30), we find the stress distribution from (31), or (29),

$$\sigma_x = -\rho gy, \quad \sigma_y = -\rho gy, \quad \tau_{xy} = 0 \quad (d)$$

as a possible state of stress due to gravity. This is a state of hydrostatic pressure ρgy in two dimensions, with zero stress at $y = 0$. It can exist in a plate or cylinder of any shape provided the corresponding boundary forces are applied. Considering a boundary element as in Fig. 12, Eqs. (13) show that there must be a normal pressure ρgy on the boundary, and zero shear stress. If the plate or cylinder is to be supported in some other manner we have to superpose a boundary normal tension ρgy and the new supporting forces. The two together will be in equilibrium, and the determination of their effects is a problem of boundary forces only, without body forces.¹

Problems

1. Show that Eqs. (12) remain valid when the element of Fig. 12 has acceleration.
2. Find graphically the principal strains and their directions from rosette measurements

$$\epsilon_\phi = 2 \times 10^{-3}, \quad \epsilon_{\alpha+\phi} = 1.35 \times 10^{-3}, \quad \epsilon_{\alpha+\beta+\phi} = 0.95 \times 10^{-3} \text{ in. per inch}$$

where $\alpha = \beta = 45^\circ$.

¹ This problem, and the general case of a potential V such that the right-hand side of Eq. (32) vanishes, have been discussed by M. Biot, *J. Applied Mechanics (Trans. A.S.M.E.)*, 1935, p. A-41.

3. Show that the line elements at the point x, y which have the maximum and minimum rotation are those in the two perpendicular directions θ determined by

$$\tan 2\theta = \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) / \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

4. The stresses in a rotating disk (of unit thickness) can be regarded as due to centrifugal force as body force in a stationary disk. Show that this body force is derivable from the potential $V = -\frac{1}{2}\rho\omega^2(x^2 + y^2)$, where ρ is the density, and ω the angular velocity of rotation (about the origin).

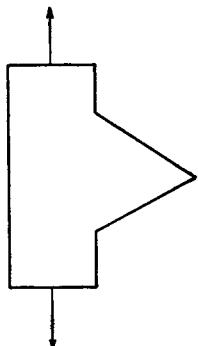
5. A disk with its axis horizontal has the gravity stress represented by Eqs. (d) of Art. 16. Make a sketch showing the boundary forces which support its weight. Show by another sketch the auxiliary problem of boundary forces which must be solved when the weight is entirely supported by the reaction of a horizontal surface on which the disk stands.

6. A cylinder with its axis horizontal has the gravity stress represented by Eqs. (d) of Art. 16. Its ends are confined between smooth fixed rigid planes which maintain the condition of *plane strain*. Sketch the forces acting on its surface, including the ends.

7. Using the stress-strain relations, and Eqs. (a) of Art. 15 in the equations of equilibrium (18), show that in the absence of body forces the displacements in problems of plane stress must satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{1-\nu} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

and a companion equation.



8. The figure represents a "tooth" on a plate in a state of plane stress in the plane of the paper. The faces of the tooth (the two straight lines) are free from force. Prove that there is no stress at all at the apex of the tooth. (N.B.: The same conclusion *cannot* be drawn for a reentrant, *i.e.*, internal, corner.)

CHAPTER 3

TWO-DIMENSIONAL PROBLEMS IN RECTANGULAR COORDINATES

17. Solution by Polynomials. It has been shown that the solution of two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (a)$$

having regard to boundary conditions (20). In the case of long rectangular strips, solutions of Eq. (a) in the form of polynomials are of interest. By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved.¹

Beginning with a polynomial of the second degree

$$\phi_2 = \frac{a_2}{2} x^2 + b_2 xy + \frac{c_2}{2} y^2 \quad (b)$$

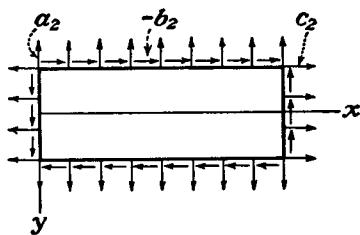


FIG. 21.

which evidently satisfies Eq. (a), we find from Eqs. (29), putting $\rho g = 0$,

$$\sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} = c_2, \quad \sigma_y = \frac{\partial^2 \phi_2}{\partial x^2} = a_2, \quad \tau_{xy} = -\frac{\partial^2 \phi_2}{\partial x \partial y} = -b_2$$

All three stress components are constant throughout the body, *i.e.*, the stress function (b) represents a combination of uniform tensions or compressions² in two perpendicular directions and a uniform shear. The forces on the boundaries must equal the stresses at these points as discussed on page 23; in the case of a rectangular plate with sides parallel to the coordinate axes these forces are shown in Fig. 21.

¹ A. Mesnager, *Compt. rend.*, vol. 132, p. 1475, 1901. See also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

² This depends on the sign of coefficients a_2 and b_2 . The directions of stresses indicated in Fig. 21 are those corresponding to positive values of a_2, b_2, c_2 .

Let us consider now a stress function in the form of a polynomial of the third degree:

$$\phi_3 = \frac{a_3}{3 \cdot 2} x^3 + \frac{b_3}{2} x^2 y + \frac{c_3}{2} x y^2 + \frac{d_3}{3 \cdot 2} y^3 \quad (c)$$

This also satisfies Eq. (a). Using Eqs. (29) and putting $\rho g = 0$, we find

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \partial y} = -b_3 x - c_3 y$$

For a rectangular plate, taken as in Fig. 22, assuming all coefficients except d_3 equal to zero, we obtain pure bending. If only coefficient a_3 is different from zero, we obtain pure bending by normal stresses applied to the sides $y = \pm c$ of the plate. If coefficient b_3 or c_3 is taken

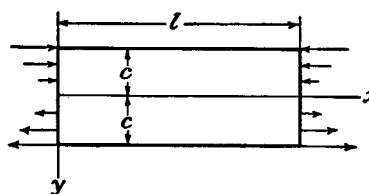


FIG. 22.

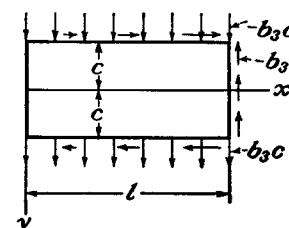


FIG. 23.

different from zero, we obtain not only normal but also shearing stresses acting on the sides of the plate. Figure 23 represents, for instance, the case in which all coefficients, except b_3 in function (c), are equal to zero. The directions of stresses indicated are for b_3 positive. Along the sides $y = \pm c$ we have uniformly distributed tensile and compressive stresses, respectively, and shearing stresses proportional to x . On the side $x = l$ we have only the constant shearing stress $-b_3 l$, and there are no stresses acting on the side $x = 0$. An analogous stress distribution is obtained if coefficient c_3 is taken different from zero.

In taking the stress function in the form of polynomials of the second and third degrees we are completely free in choosing the magnitudes of the coefficients, since Eq. (a) is satisfied whatever values they may have. In the case of polynomials of higher degrees Eq. (a) is satisfied only if certain relations between the coefficients are satisfied. Taking,

for instance, the stress function in the form of a polynomial of the fourth degree,

$$\phi_4 = \frac{a_4}{4 \cdot 3} x^4 + \frac{b_4}{3 \cdot 2} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3 \cdot 2} x y^3 + \frac{e_4}{4 \cdot 3} y^4 \quad (d)$$

and substituting it into Eq. (a), we find that the equation is satisfied only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\sigma_x = \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 x y - (2c_4 + a_4) y^2$$

$$\sigma_y = \frac{\partial^2 \phi_4}{\partial x^2} = a_4 x^2 + b_4 x y + c_4 y^2$$

$$\tau_{xy} = \frac{\partial^2 \phi_4}{\partial x \partial y} = -\frac{b_4}{2} x^2 - 2c_4 x y - \frac{d_4}{2} y^2$$

Coefficients a_4, \dots, d_4 in these expressions are arbitrary, and by suitably adjusting them we obtain various conditions of loading of a rectangular plate. For instance, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4 x y, \quad \sigma_y = 0, \quad \tau_{xy} = -\frac{d_4}{2} y^2 \quad (e)$$

Assuming d_4 positive, the forces acting on the rectangular plate shown in Fig. 24 and producing the stresses (e) are given. On the longitudinal sides $y = \pm c$ are uniformly distributed shearing forces; on the ends shearing forces are distributed according to a parabolic law. The shearing forces acting on the boundary of the plate reduce to the couple¹

$$M = \frac{d_4 c^2 l}{2} \cdot 2c - \frac{1}{3} \frac{d_4 c^2}{2} \cdot 2c \cdot l = \frac{2}{3} d_4 c^3 l$$

This couple balances the couple produced by the normal forces along the side $x = l$ of the plate.

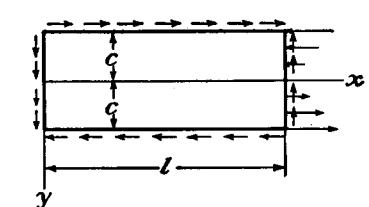


FIG. 24.

Let us consider a stress function in the form of a polynomial of the fifth degree.

$$\phi_5 = \frac{a_5}{5 \cdot 4} x^5 + \frac{b_5}{4 \cdot 3} x^4 y + \frac{c_5}{3 \cdot 2} x^3 y^2 + \frac{d_5}{3 \cdot 2} x^2 y^3 + \frac{e_5}{4 \cdot 3} x y^4 + \frac{f_5}{5 \cdot 4} y^5 \quad (f)$$

¹ The thickness of the plate is taken equal to unity.

Substituting in Eq. (a) we find that this equation is satisfied if

$$\begin{aligned} e_5 &= -(2c_5 + 3a_5) \\ f_5 &= -\frac{1}{3}(b_5 + 2d_5) \end{aligned}$$

The corresponding stress components are:

$$\begin{aligned} \sigma_x &= \frac{\partial^2 \phi_5}{\partial y^2} = \frac{c_5}{3}x^3 + d_5x^2y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3 \\ \sigma_y &= \frac{\partial^2 \phi_5}{\partial x^2} = a_5x^3 + b_5x^2y + c_5xy^2 + \frac{d_5}{3}y^3 \\ \tau_{xy} &= -\frac{\partial^2 \phi_5}{\partial x \partial y} = -\frac{1}{3}b_5x^3 - c_5x^2y - d_5xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3 \end{aligned}$$

Again coefficients a_5, \dots, d_5 are arbitrary, and in adjusting them we obtain solutions for various loading conditions of a plate. Taking,

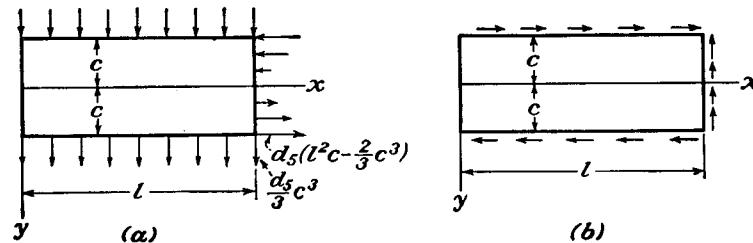


FIG. 25.

for instance, all coefficients, except d_5 , equal to zero we find

$$\begin{aligned} \sigma_x &= d_5(x^2y - \frac{2}{3}y^3) \\ \sigma_y &= \frac{1}{3}d_5y^3 \\ \tau_{xy} &= -d_5xy^2 \end{aligned} \quad (g)$$

The normal forces are uniformly distributed along the longitudinal sides of the plate (Fig. 25a). Along the side $x = l$, the normal forces consist of two parts, one following a linear law and the other following the law of a cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the plate and follow a parabolic law along the side $x = l$. The distribution of these stresses is shown in Fig. 25b.

Since Eq. (a) is a linear differential equation, it may be concluded that a sum of several solutions of this equation is also a solution. We can superpose the elementary solutions considered in this article and in this manner arrive at new solutions of practical interest. Several examples of the application of this method of superposition will be considered.

18. Saint-Venant's Principle.¹ In the previous article several cases were discussed in which exact solutions for rectangular plates were obtained by taking very simple forms for the stress function ϕ . In each case all the equations of elasticity are satisfied, but the solutions are exact only if the surface forces are distributed in the manner given. In the case of pure bending, for instance (Fig. 22), the bending moment must be produced by tensions and compressions on the ends, these tensions and compressions being proportional to the distance from the neutral axis. The fastening of the end, if any, must be such as not to interfere with distortion of the plane of the end. If the above conditions are not fulfilled, i.e., the bending moment is applied in some different manner or the constraint is such that it imposes other forces on the end section, the solution given in Art. 17 is no longer an exact solution of the problem. The practical utility of the solution however is not limited to such a specialized case. It can be applied with sufficient accuracy to cases of bending in which the conditions at the ends are not rigorously satisfied. Such an extension in the application of the solution is usually based on the so-called principle of Saint-Venant. This principle states that if the forces acting on a small portion of the surface of an elastic body are replaced by another statically equivalent system of forces acting on the same portion of the surface, this redistribution of loading produces substantial changes in the stresses locally but has a negligible effect on the stresses at distances which are large in comparison with the linear dimensions of the surface on which the forces are changed. For instance, in the case of pure bending of a rectangular strip (Fig. 22) the cross-sectional dimensions of which are small in comparison with its length, the manner of application of the external bending moment affects the stress distribution only in the vicinity of the ends and is of no consequence for distant cross sections, at which the stress distribution will be practically as given by the solution to which Fig. 22 refers.

The same is true in the case of axial tension. Only near the loaded end does the stress distribution depend on the manner of applying the tensile force, and in cross sections at a distance from the end the stresses are practically uniformly distributed. Some examples illustrating this statement and showing how rapidly the stress distribution becomes practically uniform will be discussed later (see page 52).

¹ This principle was stated in the famous memoir on torsion in *Mém. savants étrangers*, vol. 14, 1855. Its relation to the principle of conservation of energy is discussed later (see p. 150).

19. Determination of Displacements. When the components of stress are found from the previous equations, the components of strain can be obtained by using Hooke's law, Eqs. (3) and (6). Then the displacements u and v can be obtained from the equations

$$\frac{\partial u}{\partial x} = \epsilon_x, \quad \frac{\partial v}{\partial y} = \epsilon_y, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} \quad (a)$$

The integration of these equations in each particular case does not present any difficulty, and we shall have several examples of their application. It may be seen at once that the strain components (a) remain unchanged if we add to u and v the linear functions

$$u_1 = a + by, \quad v_1 = c - bx \quad (b)$$

in which a , b , and c are constants. This means that the displacements are not entirely determined by the stresses and strains. On the displacements due to the internal strains a displacement like that of a rigid body can be superposed. The constants a and c in Eqs. (b) represent a translatory motion of the body and the constant b is a small angle of rotation of the rigid body about the z -axis.

It has been shown (see page 25) that in the case of constant body forces the stress distribution is the same for plane stress distribution or plane strain. The displacements however are different for these two problems, since in the case of plane stress distribution the components of strain, entering into Eqs. (a), are given by equations

$$\epsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y), \quad \epsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x), \quad \gamma_{xy} = \frac{1}{G} \tau_{xy}$$

and in the case of plane strain the strain components are:

$$\begin{aligned} \epsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{1}{E} [(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y] \\ \epsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] = \frac{1}{E} [(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x] \\ \gamma_{xy} &= \frac{1}{G} \tau_{xy} \end{aligned}$$

It is easily verified that these equations can be obtained from the preceding set for plane stress by replacing E in the latter by $E/(1 - \nu^2)$, and ν by $\nu/(1 - \nu)$. These substitutions leave G , which is $E/2(1 + \nu)$, unchanged. The integration of Eqs. (a) will be shown later in discussing particular problems.

20. Bending of a Cantilever Loaded at the End. Consider a cantilever having a narrow rectangular cross section of unit width bent by a force P applied at the end (Fig. 26). The upper and lower edges are free from load, and shearing forces, having a resultant P , are distributed along the end $x = 0$. These conditions can be satisfied by a proper combination of pure shear, with the stresses (e) of Art. 17 represented in Fig. 24. Superposing the pure shear $\tau_{xy} = -b_2$ on the stresses (e), we find

$$\begin{aligned} \sigma_x &= d_4 xy, & \sigma_y &= 0 \\ \tau_{xy} &= -b_2 - \frac{d_4}{2} y^2 & (a) \end{aligned}$$

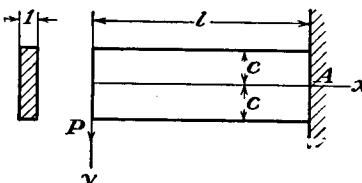


FIG. 26.

To have the longitudinal sides $y = \pm c$ free from forces we must have

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2} c^2 = 0$$

from which

$$d_4 = -\frac{2b_2}{c^2}$$

To satisfy the condition on the loaded end the sum of the shearing forces distributed over this end must be equal to P . Hence¹

$$-\int_{-c}^c \tau_{xy} \cdot dy = \int_{-c}^c \left(b_2 - \frac{b_2}{c^2} y^2 \right) dy = P$$

from which

$$b_2 = \frac{3P}{4c}$$

Substituting these values of d_4 and b_2 in Eqs. (a) we find

$$\begin{aligned} \sigma_x &= -\frac{3}{2} \frac{P}{c^3} xy, & \sigma_y &= 0 \\ \tau_{xy} &= -\frac{3P}{4c} \left(1 - \frac{y^2}{c^2} \right) \end{aligned}$$

Noting that $\frac{1}{3}c^3$ is the moment of inertia I of the cross section of the cantilever, we have

$$\begin{aligned} \sigma_x &= -\frac{Pxy}{I}, & \sigma_y &= 0 \\ \tau_{xy} &= -\frac{P}{I} \frac{1}{2} (c^2 - y^2) \end{aligned} \quad (b)$$

¹ The minus sign before the integral follows from the rule for the sign of shearing stresses. Stress τ_{xy} on the end $x = 0$ is positive if it is upward (see p. 3).

This coincides completely with the elementary solution as given in books on the strength of materials. It should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to y . If the forces at the ends are distributed in any other manner, the stress distribution (b) is not a correct solution for the ends of the cantilever, but, by virtue of Saint-Venant's principle, it can be considered satisfactory for cross sections at a considerable distance from the ends.

Let us consider now the displacement corresponding to the stresses (b). Applying Hooke's law we find

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI}, \quad \epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI} \quad (c)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2) \quad (d)$$

The procedure for obtaining the components u and v of the displacement consists in integrating Eqs. (c) and (d). By integration of Eqs. (c) we find

$$u = -\frac{Px^2y}{2EI} + f(y), \quad v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

in which $f(y)$ and $f_1(x)$ are as yet unknown functions of y only and x only. Substituting these values of u and v in Eq. (d) we find

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2IG}(c^2 - y^2)$$

In this equation some terms are functions of x only, some are functions of y only, and one is independent of both x and y . Denoting these groups by $F(x)$, $G(y)$, K , we have

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_1(x)}{dx}, \quad G(y) = \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

and the equation may be written

$$F(x) + G(y) = K$$

Such an equation means that $F(x)$ must be some constant d and $G(y)$ some constant e . Otherwise $F(x)$ and $G(y)$ would vary with x and y , respectively, and by varying x alone, or y alone, the equality would be

violated. Thus

$$e + d = -\frac{Pc^2}{2IG} \quad (e)$$

and

$$\frac{df_1(x)}{dx} = \frac{Px^2}{2EI} + d, \quad \frac{df(y)}{dy} = -\frac{Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

Functions $f(y)$ and $f_1(x)$ are then

$$f(y) = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$f_1(x) = \frac{Px^3}{6EI} + dx + h$$

Substituting in the expressions for u and v we find

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g \quad (g)$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + dx + h$$

The constants d , e , g , h may now be determined from Eq. (e) and from the three conditions of constraint which are necessary to prevent the beam from moving as a rigid body in the xy -plane. Assume that the point A , the centroid of the end cross section, is fixed. Then u and v are zero for $x = l$, $y = 0$, and we find from Eqs. (g),

$$g = 0, \quad h = -\frac{Pl^3}{6EI} - dl$$

The deflection curve is obtained by substituting $y = 0$ into the second of Eqs. (g). Then

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x) \quad (h)$$

For determining the constant d in this equation we must use the third condition of constraint, eliminating the possibility of rotation of the beam in the xy -plane about the fixed point A . This constraint can be realized in various ways. Let us consider two cases: (1) When an element of the axis of the beam is fixed at the end A . Then the condition of constraint is

$$\left(\frac{\partial v}{\partial x} \right)_{y=0} = 0 \quad (k)$$

(2) When a vertical element of the cross section at the point A is fixed,

Then the condition of constraint is

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \quad (l)$$

In the first case we obtain from Eq. (h)

$$d = -\frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting all the constants in Eqs. (g), we find

$$\begin{aligned} u &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y \\ v &= \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \end{aligned} \quad (m)$$

The equation of the deflection curve is

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \quad (n)$$

which gives for the deflection at the loaded end ($x = 0$) the value $Pl^3/3EI$. This coincides with the value usually derived in elementary books on the strength of materials.

To illustrate the distortion of cross sections produced by shearing stresses let us consider the displacement u at the fixed end ($x = l$). For this end we have from Eqs. (m),

$$\begin{aligned} (u)_{x=l} &= -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pc^2y}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{x=l} &= -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} - \frac{Pc^2}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{y=0} &= -\frac{Pc^2}{2IG} = -\frac{3P}{4cG} \end{aligned} \quad (o)$$

The shape of the cross section after distortion is as shown in Fig. 27a. Due to the shearing stress $\tau_{xy} = -3P/4c$ at the point A, an element of the cross section at A rotates in the xy -plane about the point A through an angle $3P/4cG$ in the clockwise direction.

If a vertical element of the cross section is fixed at A (Fig. 27b).

instead of a horizontal element of the axis, we find from condition (l) and the first of Eqs. (g)

$$e = \frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$d = -\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting in the second of Eqs. (g) we find

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pc^2}{2IG}(l-x) \quad (r)$$

Comparing this with Eq. (n) it can be concluded that, due to rotation

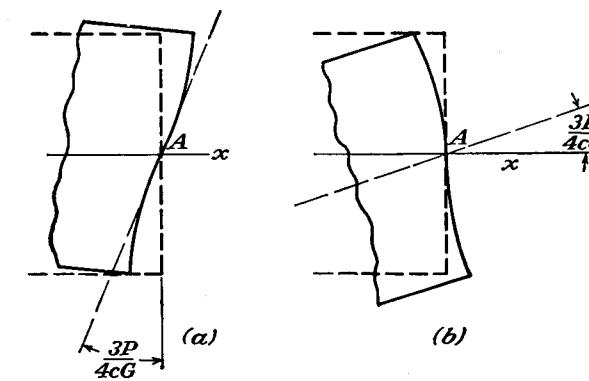


FIG. 27.

of the end of the axis at A (Fig. 27b), the deflections of the axis of the cantilever are increased by the quantity

$$\frac{Pc^2}{2IG}(l-x) = \frac{3P}{4cG}(l-x)$$

This is the so-called *effect of shearing force* on the deflection of the beam. In practice, at the built-in end we have conditions different from those shown in Fig. 27. The fixed section is usually not free to distort and the distribution of forces at this end is different from that given by Eqs. (b). Solution (b) is, however, satisfactory for comparatively long cantilevers at considerable distances from the terminals.

21. Bending of a Beam by Uniform Load. Let a beam of narrow rectangular cross section of unit width, supported at the ends, be bent

by a uniformly distributed load of intensity q , as shown in Fig. 28. The conditions at the upper and lower edges of the beam are:

$$(\tau_{xy})_{y=\pm c} = 0, \quad (\sigma_y)_{y=+c} = 0, \quad (\sigma_y)_{y=-c} = -q \quad (a)$$

The conditions at the ends $x = \pm l$ are

$$\int_{-c}^c \tau_{xy} dy = \mp ql, \quad \int_{-c}^c \sigma_x dy = 0, \quad \int_{-c}^c \sigma_{xy} dy = 0 \quad (b)$$

The last two of Eqs. (b) state that there is no longitudinal force and no bending couple applied at the ends of the beam. All the conditions (a) and (b) can be satisfied by combining certain solutions in the form

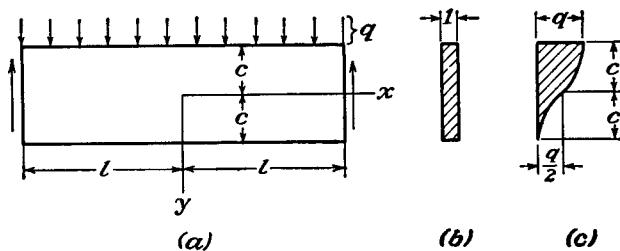


FIG. 28.

of polynomials as obtained in Art. 17. We begin with solution (g) illustrated by Fig. 25. To remove the tensile stresses along the side $y = c$ and the shearing stresses along the sides $y = \pm c$ we superpose a simple compression $\sigma_y = a_2$ from solution (b), Art. 17, and the stresses $\sigma_y = b_3 y$ and $\tau_{xy} = -b_3 x$ in Fig. 23. In this manner we find

$$\begin{aligned} \sigma_x &= d_5(x^2y - \frac{2}{3}y^3) \\ \sigma_y &= \frac{1}{3}d_5y^3 + b_3y + a_2 \\ \tau_{xy} &= -d_5xy^2 - b_3x \end{aligned} \quad (c)$$

From the conditions (a) we find

$$\begin{aligned} -d_5c^2 - b_3 &= 0 \\ \frac{1}{3}d_5c^3 + b_3c + a_2 &= 0 \\ -\frac{1}{3}d_5c^3 - b_3c + a_2 &= -q \end{aligned}$$

from which

$$a_2 = -\frac{q}{2}, \quad b_3 = \frac{3}{4}\frac{q}{c}, \quad d_5 = -\frac{3}{4}\frac{q}{c^3}$$

Substituting in Eqs. (c) and noting that $2c^3/3$ is equal to the moment of inertia I of the rectangular cross-sectional area of unit width, we find

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) = -\frac{q}{2I}\left(x^2y - \frac{2}{3}y^3\right) \\ \sigma_y &= -\frac{3q}{4c^3}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) \quad (d) \\ \tau_{xy} &= -\frac{3q}{4c^3}(c^2 - y^2)x = -\frac{q}{2I}(c^2 - y^2)x \end{aligned}$$

It can easily be checked that these stress components satisfy not only conditions (a) on the longitudinal sides but also the first two conditions (b) at the ends. To make the couples at the ends of the beam vanish we superpose on solution (d) a pure bending, $\sigma_x = d_3 y$, $\sigma_y = \tau_{xy} = 0$, shown in Fig. 22, and determine the constant d_3 from the condition at $x = \pm l$

$$\int_{-c}^c \sigma_{xy} dy = \int_{-c}^c \left[-\frac{3}{4}\frac{q}{c^3}\left(l^2y - \frac{2}{3}y^3\right) + d_3y \right] y dy = 0$$

from which

$$d_3 = \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)$$

Hence, finally,

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)y \\ &= \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) \end{aligned} \quad (33)$$

The first term in this expression represents the stresses given by the usual elementary theory of bending, and the second term gives the necessary correction. This correction does not depend on x and is small in comparison with the maximum bending stress, provided the span of the beam is large in comparison with its depth. For such beams the elementary theory of bending gives a sufficiently accurate value for the stresses σ_x . It should be noted that expression (33) is an exact solution only if at the ends $x = \pm l$ the normal forces are distributed according to the law

$$\bar{X} = \frac{3}{4}\frac{q}{c^3}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right)$$

i.e., if the normal forces at the ends are the same as σ_x for $x = \pm l$ from Eq. (33). These forces have a resultant force and a resultant couple equal to zero. Hence, from Saint-Venant's principle we can conclude

that their effects on the stresses at considerable distances from the ends, say at distances larger than the depth of the beam, can be neglected. Solution (33) at such points is therefore accurate enough for the case when there are no forces \bar{X} .

The discrepancy between the exact solution (33) and the approximate solution, given by the first term of (33), is due to the fact that in deriving the approximate solution it is assumed that the longitudinal fibers of the beam are in a condition of simple tension. From solution (d) it can be seen that there are compressive stresses σ_y between the fibers. These stresses are responsible for the correction represented by the second term of solution (33). The distribution of the compressive stresses σ_y over the depth of the beam is shown in Fig. 28c. The distribution of shearing stress τ_{xy} , given by the third of Eqs. (d), over a cross section of the beam coincides with that given by the usual elementary theory.

When the beam is loaded by its own weight instead of the distributed load q , the solution must be modified by putting $q = 2\rho gc$ in (33) and the last two of Eqs. (d), and adding the stresses

$$\sigma_z = 0, \quad \sigma_y = \rho g(c - y), \quad \tau_{xy} = 0 \quad (e)$$

For the stress distribution (e) can be obtained from Eqs. (29) by taking

$$\phi = \frac{1}{2}\rho g(cx^2 + y^3/3)$$

and therefore represents a possible state of stress due to weight and boundary forces. On the upper edge $y = -c$ we have $\sigma_y = 2\rho gc$, and on the lower edge $y = c$, $\sigma_y = 0$. Thus when the stresses (e) are added to the previous solution, with $q = 2\rho gc$, the stress on both horizontal edges is zero, and the load on the beam consists only of its own weight.

The displacements u and v can be calculated by the method indicated in the previous article. Assuming that at the centroid of the middle cross section ($x = 0, y = 0$) the horizontal displacement is zero and the vertical displacement is equal to the deflection δ , we find, using solutions (d) and (33),

$$\begin{aligned} u &= \frac{q}{2EI} \left[\left(l^2x - \frac{x^3}{3} \right) y + x \left(\frac{2}{3}y^3 - \frac{2}{5}c^2y \right) + \nu x \left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3 \right) \right] \\ v &= -\frac{q}{2EI} \left\{ \frac{y^4}{12} - \frac{c^2y^2}{2} + \frac{2}{3}c^3y + \nu \left[(l^2 - x^2) \frac{y^2}{2} + \frac{y^4}{6} - \frac{1}{5}c^2y^2 \right] \right\} \\ &\quad - \frac{q}{2EI} \left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu \right) c^2x^2 \right] + \delta \end{aligned}$$

It can be seen from the expression for u that the neutral surface of the

beam is not at the center line. Due to the compressive stress

$$(\sigma_y)_{y=0} = -\frac{q}{2}$$

the center line has a tensile strain $\nu q/2E$, and we find

$$(u)_{y=0} = \frac{\nu q x}{2E}$$

From the expression for v we find the equation of the deflection curve,

$$(v)_{y=0} = \delta - \frac{q}{2EI} \left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu \right) c^2x^2 \right] \quad (f)$$

Assuming that the deflection is zero at the ends ($x = \pm l$) of the center line, we find

$$\delta = \frac{5}{24} \frac{ql^4}{EI} \left[1 + \frac{12}{5} \frac{c^2}{l^2} \left(\frac{4}{5} + \frac{\nu}{2} \right) \right] \quad (34)$$

The factor before the brackets is the deflection which is derived by the elementary analysis, assuming that cross sections of the beam remain plane during bending. The second term in the brackets represents the correction usually called the *effect of shearing force*.

By differentiating Eq. (f) for the deflection curve twice with respect to x , we find the following expression for the curvature:

$$\left(\frac{d^2v}{dx^2} \right)_{y=0} = \frac{q}{EI} \left[\frac{l^2 - x^2}{2} + c^2 \left(\frac{4}{5} + \frac{\nu}{2} \right) \right] \quad (35)$$

It will be seen that the curvature is not exactly proportional to the bending moment¹ $q(l^2 - x^2)/2$. The additional term in the brackets represents the necessary correction to the usual elementary formula. A more general investigation of the curvature of beams shows² that the correction term given in expression (35) can also be used for any case of continuously varying intensity of load. The effect of shearing force on the deflection in the case of a concentrated load will be discussed later (page 107).

An elementary derivation of the effect of the shearing force on the curvature of the deflection curve of beams has been made by Rankine³ in England and by Grashof⁴ in Germany. Taking the maximum shearing strain at the neutral

¹This was pointed out first by K. Pearson, *Quart. J. Math.*, vol. 24, p. 63, 1889.

²See paper by T. v. Kármán, *Abhandl. aerodynam. Inst., Tech. Hochschule, Aachen*, vol. 7, p. 3, 1927.

³Rankine, "Applied Mechanics," 14th ed., p. 344, 1895.

⁴Grashof, "Elastizität und Festigkeit," 2d ed., 1878.

axis of a rectangular beam of unit width as $\frac{1}{2}(Q/2cG)$, where Q is the shearing force, the corresponding increase in curvature is given by the derivative of the above shearing strain with respect to x , which gives $\frac{1}{2}(q/2cG)$. The corrected expression for the curvature by elementary analysis then becomes

$$\frac{q}{EI} \cdot \frac{l^2 - x^2}{2} + \frac{3}{2} \frac{q}{2cG} = \frac{q}{EI} \left[\frac{l^2 - x^2}{2} + c^2(1 + \nu) \right]$$

Comparing this with expression (35), it is seen that the elementary solution gives an exaggerated value¹ for the correction.

The correction term in expression (35) for the curvature cannot be attributed to the shearing force alone. It is produced partially by the compressive stresses σ_y . These stresses are not uniformly distributed over the depth of the beam. The lateral expansion in the x -direction produced by these stresses diminishes from the top to the bottom of the beam, and in this way a reversed curvature (convex upwards) is produced. This curvature together with the effect of shearing force accounts for the correction term in Eq. (35).

22. Other Cases of Continuously Loaded Beams. By increasing the degree of polynomials representing solutions of the two-dimensional

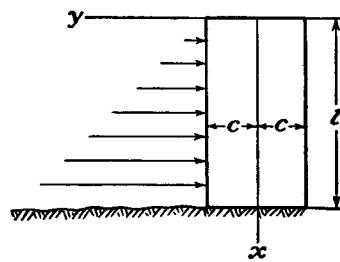


FIG. 29.

problem (Art. 17), we may obtain solutions of bending problems with various types of continuously varying load.² By taking, for instance, a solution in the form of a polynomial of the sixth degree and combining it with the previous solutions of Art. 17, we may obtain the stresses in a vertical cantilever loaded by hydrostatic pressure, as shown in

Fig. 29. In this manner it can be shown that all conditions on the longitudinal sides of the cantilever are satisfied by the following system of stresses:

$$\begin{aligned} \sigma_x &= \frac{qx^3y}{4c^3} + \frac{q}{4c^3} \left(-2xy^3 + \frac{6}{5}c^2xy \right) \\ \sigma_y &= -\frac{qx}{2} + qx \left(\frac{y^3}{4c^3} - \frac{3y}{4c} \right) \\ \tau_{xy} &= \frac{3qx^2}{8c^3} (c^2 - y^2) - \frac{q}{8c^3} (c^4 - y^4) + \frac{q}{4c^3} \frac{3}{5} c^2(c^2 - y^2) \end{aligned} \quad (a)$$

Here q is the weight of unit volume of the fluid, so that the intensity of the load at a depth x is qx . The shearing force and the bending moment at the same depth are $qx^2/2$ and $qx^3/6$, respectively. It is

¹ A better approximation is given by elementary strain-energy considerations. See S. Timoshenko, "Strength of Materials," 2d ed., vol. 1, p. 299.

² See papers by Timpe, *loc. cit.*; W. R. Osgood, *J. Research Natl. Bur. Standards*, vol. 28, p. 159, 1942.

evident that the first terms in the expressions for σ_x and τ_{xy} are the values of the stresses calculated by the usual elementary formulas.

On the top end of the beam ($x = 0$) the normal stress is zero. The shearing stress is

$$\tau_{xy} = -\frac{q}{8c^3} (c^4 - y^4) + \frac{q}{4c^3} \frac{3}{5} c^2(c^2 - y^2)$$

Although these stresses are different from zero, they are very small all over the cross section and their resultant is zero, so that the condition approaches that of an end free from external forces.

By adding to σ_x in Eqs. (a) the term $-q_1x$, in which q_1 is the weight of unit volume of the material of the cantilever, the effect of the weight of the beam on the stress distribution is taken into account. It has been proposed¹ to use the solution obtained in this way for calculating the stresses in masonry dams of rectangular cross section. It should be noted that this solution does not satisfy the conditions at the bottom of the dam. Solution (a) is exact if, at the bottom, forces are acting which are distributed in the same manner as σ_x and τ_{xy} in solution (a). In an actual case the bottom of the dam is connected with the foundation, and the conditions are different from those represented by this solution. From Saint-Venant's principle it can be stated that the effect of the constraint at the bottom is negligible at large distances from the bottom, but in the case of a masonry dam the cross-sectional dimension $2c$ is usually not small in comparison with the height l and this effect cannot be neglected.²

By taking for the stress function a polynomial of the seventh degree the stresses in a beam loaded by a parabolically distributed load may be obtained.

In the general case of a continuous distribution of load q , Fig. 30, the stresses at any cross section at a considerable distance from the ends, say at a distance larger than the depth of the beam, can be approximately calculated from the following equations:³

¹ M. Levy, *Compt. rend.*, vol. 126, p. 1235, 1898.

² The problem of stresses in masonry dams is of great practical interest and has been discussed by various authors. See K. Pearson, On Some Disregarded Points in the Stability of Masonry Dams, *Drapers' Co. Research Mem.*, 1904; K. Pearson and C. Pollard, An Experimental Study of the Stresses in Masonry Dams, *Drapers' Co. Research Mem.*, 1907. See also papers by L. F. Richardson, *Trans. Roy. Soc. (London)* series A, vol. 210, p. 307, 1910; and S. D. Carothers, *Proc. Roy. Soc. Edinburgh*, vol. 33, p. 292, 1913. I. Muller, *Publications du laboratoire de photo-élasticité*, Zürich, 1930. Fillunger, *Oesterr. Wochschr. öffentl. Baudienst*, 1913, Heft. 45. K. Wolf, *Sitzber. Akad. Wiss. Wien*, vol. 123, 1914.

³ F. Seewald, *Abhandl. aerodynam. Inst., Tech. Hochschule, Aachen*, vol. 7, p. 11, 1927.

$$\begin{aligned}\sigma_x &= \frac{My}{I} + q \left(\frac{y^3}{2c^3} - \frac{3}{10} \frac{y}{c} \right) \\ \sigma_y &= -\frac{q}{2} + q \left(\frac{3y}{4c} - \frac{y^3}{4c^3} \right) \\ \tau_{xy} &= \frac{Q}{2I} (c^2 - y^2)\end{aligned}\quad (36)$$

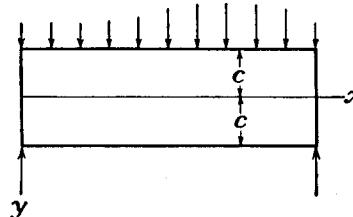


FIG. 30.

in which M and Q are the bending moment and shearing forces calculated in the usual way and q is the intensity of load at the cross section under consideration. These equations agree with those previously obtained for a uniformly loaded beam (see Art. 21).

If the load of intensity q , in the downward direction, is distributed along the lower edge ($y = +c$) of the beam, the expressions

for the stresses are obtained from Eqs. (36) by superposing a uniform tensile stress, $\sigma_y = q$, and

$$\begin{aligned}\sigma_x &= \frac{My}{I} + q \left(\frac{y^3}{2c^3} - \frac{3}{10} \frac{y}{c} \right) \\ \sigma_y &= \frac{q}{2} + q \left(\frac{3y}{4c} - \frac{y^3}{4c^3} \right) \\ \tau_{xy} &= \frac{Q}{2I} (c^2 - y^2)\end{aligned}\quad (36')$$

23. Solution of the Two-dimensional Problem in the Form of a Fourier Series. It has been shown that if the load is continuously distributed along the length of a rectangular beam of narrow cross section a stress function in the form of a polynomial may be used in certain simple cases. If the load is discontinuous, a stress function in the form of a trigonometric series should be used.¹ The equation for the stress function,

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (a)$$

may be satisfied by taking the function ϕ in the form

$$\phi = \sin \frac{m\pi x}{l} f(y) \quad (b)$$

in which m is an integer and $f(y)$ a function of y only. Substituting (b) into Eq. (a) and using the notation $m\pi/l = \alpha$, we find the following equation for determining $f(y)$:

$$\alpha^4 f(y) - 2\alpha^2 f''(y) + f''''(y) = 0 \quad (c)$$

¹ The first application of trigonometric series in the solution of beam problems was given by M. C. Ribière in a thesis, *Sur divers cas de la flexion des prismes rectangles*, Bordeaux, 1889. See also his paper in *Compt. rend.*, vol. 126, pp. 402-404 and 1190-1192. Further progress in the application of this solution was made by L. N. G. Filon, *Phil. Trans.*, series A, vol. 201, p. 63, 1903. Several particular examples were worked out by F. Bleich, *Bauingenieur*, vol. 4, p. 255, 1923.

The general integral of this linear differential equation with constant coefficients is

$$f(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y$$

The stress function then is

$$\phi = \sin \alpha x (C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y) \quad (d)$$

and the corresponding stress components are

$$\begin{aligned}\sigma_x &= \frac{\partial^2 \phi}{\partial y^2} = \sin \alpha x [C_1 \alpha^2 \cosh \alpha y + C_2 \alpha^2 \sinh \alpha y + C_3 \alpha (2 \sinh \alpha y + \alpha y \cosh \alpha y) + C_4 \alpha (2 \cosh \alpha y + \alpha y \sinh \alpha y)] \\ \sigma_y &= -\frac{\partial^2 \phi}{\partial x^2} = -\alpha^2 \sin \alpha x (C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y) \quad (e) \\ \tau_{xy} &= -\frac{\partial^2 \phi}{\partial x \partial y} = -\alpha \cos \alpha x [C_1 \alpha \sinh \alpha y + C_2 \alpha \cosh \alpha y + C_3 (\cosh \alpha y + \alpha y \sinh \alpha y) + C_4 (\sinh \alpha y + \alpha y \cosh \alpha y)]\end{aligned}$$

Let us consider a particular case of a rectangular beam supported at the ends and subjected along the upper and lower edges to the action of continuously

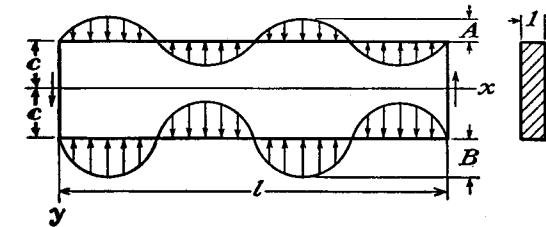


FIG. 31.

distributed vertical forces of the intensity $A \sin \alpha x$ and $B \sin \alpha x$, respectively. Figure 31 shows the case when $\alpha = 4\pi/l$ and indicates also the positive values of A and B . The stress distribution for this case can be obtained from solution (e). The constants of integration C_1, \dots, C_4 may be determined from the conditions on the upper and lower edges of the beam, $y = \pm c$. These conditions are:

For $y = +c$,

$$\tau_{xy} = 0, \quad \sigma_y = -B \sin \alpha x \quad (f)$$

For $y = -c$,

$$\tau_{xy} = 0, \quad \sigma_y = -A \sin \alpha x$$

Substituting these values in the third of Eqs. (e), we find

$$\begin{aligned}C_1 \alpha \sinh \alpha c + C_2 \alpha \cosh \alpha c + C_3 (\cosh \alpha c + \alpha c \sinh \alpha c) &+ C_4 (\sinh \alpha c + \alpha c \cosh \alpha c) = 0 \\ -C_1 \alpha \sinh \alpha c + C_2 \alpha \cosh \alpha c + C_3 (\cosh \alpha c + \alpha c \sinh \alpha c) &- C_4 (\sinh \alpha c + \alpha c \cosh \alpha c) = 0\end{aligned}$$

from which

$$\begin{aligned} C_3 &= -C_2 \frac{\alpha \cosh \alpha c}{\cosh \alpha c + \alpha c \sinh \alpha c} \\ C_4 &= -C_1 \frac{\alpha \sinh \alpha c}{\sinh \alpha c + \alpha c \cosh \alpha c} \end{aligned} \quad (g)$$

Using the conditions on the sides $y = \pm c$ in the second of Eqs. (e), we find

$$\begin{aligned} \alpha^2(C_1 \cosh \alpha c + C_2 \sinh \alpha c + C_3 \cosh \alpha c + C_4 \sinh \alpha c) &= B \\ \alpha^2(C_1 \cosh \alpha c - C_2 \sinh \alpha c - C_3 \cosh \alpha c + C_4 \sinh \alpha c) &= A \end{aligned}$$

By adding and subtracting these equations and using Eqs. (g), we find

$$\begin{aligned} C_1 &= \frac{A+B}{\alpha^2} \cdot \frac{\sinh \alpha c + \alpha c \cosh \alpha c}{\sinh 2\alpha c + 2\alpha c} \\ C_2 &= -\frac{A-B}{\alpha^2} \cdot \frac{\cosh \alpha c + \alpha c \sinh \alpha c}{\sinh 2\alpha c - 2\alpha c} \\ C_3 &= \frac{A-B}{\alpha^2} \cdot \frac{\alpha \cosh \alpha c}{\sinh 2\alpha c - 2\alpha c} \\ C_4 &= -\frac{A+B}{\alpha^2} \cdot \frac{\alpha \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \end{aligned} \quad (h)$$

Substituting in Eqs. (e), we find the following expressions for the stress components:

$$\begin{aligned} \sigma_x &= (A+B) \frac{(\alpha c \cosh \alpha c - \sinh \alpha c) \cosh \alpha y - \alpha y \sinh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \sin \alpha x \\ &\quad - (A-B) \frac{(\alpha c \sinh \alpha c - \cosh \alpha c) \sinh \alpha y - \alpha y \cosh \alpha y \cosh \alpha c}{\sinh 2\alpha c - 2\alpha c} \cdot \sin \alpha x \\ \sigma_y &= -(A+B) \frac{(\alpha c \cosh \alpha c + \sinh \alpha c) \cosh \alpha y - \alpha y \sinh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \sin \alpha x \\ &\quad + (A-B) \frac{(\alpha c \sinh \alpha c + \cosh \alpha c) \sinh \alpha y - \alpha y \cosh \alpha y \cosh \alpha c}{\sinh 2\alpha c - 2\alpha c} \cdot \sin \alpha x \\ \tau_{xy} &= -(A+B) \frac{\alpha c \cosh \alpha c \sinh \alpha y - \alpha y \cosh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \cos \alpha x \\ &\quad + (A-B) \frac{\alpha c \sinh \alpha c \cosh \alpha y - \alpha y \sinh \alpha y \cosh \alpha c}{\sinh 2\alpha c - 2\alpha c} \cdot \cos \alpha x \end{aligned} \quad (k)$$

These stresses satisfy the conditions shown in Fig. 31 along the sides $y = \pm c$. At the ends of the beam $x = 0$ and $x = l$, the stresses σ_x are zero and only shearing stress τ_{xy} is present. This stress is represented by two terms [see Eqs. (k)]. The first term, proportional to $A+B$, represents stresses which, for the upper and lower halves of the end cross section, are of the same magnitude but of opposite sign. The resultant of these stresses over the end is zero. The second term, proportional to $A-B$, has resultants at the ends of the beam which maintain equilibrium with the loads applied to the longitudinal sides ($y = \pm c$).

If these loads are the same for both sides, coefficient A is equal to B , and the reactive forces at the ends vanish. Let us consider this particular case more in detail, assuming that the length of the beam is large in comparison with its depth. From the second of Eqs. (k) the normal stresses σ_y over the middle plane $y = 0$ of the beam are

$$\sigma_y = -2A \frac{\alpha c \cosh \alpha c + \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \sin \alpha x \quad (l)$$

For long beams αc , equal to $m\pi c/l$, is small, provided the number of waves m is not large. Then, substituting in (l),

$$\sinh \alpha c = \alpha c + \frac{(\alpha c)^3}{6} + \frac{(\alpha c)^5}{120} + \dots ; \quad \cosh \alpha c = 1 + \frac{(\alpha c)^2}{2} + \frac{(\alpha c)^4}{24} + \dots$$

and neglecting small quantities of higher order than $(\alpha c)^4$, we find

$$\sigma_y = -A \sin \alpha x \left(1 - \frac{(\alpha c)^4}{24} \right)$$

Hence for small values of αc the distribution of stresses over the middle plane is practically the same as on both horizontal edges ($y = \pm c$) of the beam. It can be concluded that pressures are transmitted through a beam or plate without any substantial change, provided the variation of these pressures along the sides is not rapid.

The shearing stresses τ_{xy} for this case are very small. On the upper and lower halves of the end cross sections they add up to the small resultants necessary to balance the small difference between the pressures on the horizontal edges ($y = \pm c$) and the middle plane ($y = 0$).

In the most general case the distribution of vertical loading along the upper and lower edges of a beam (Fig. 32) can be represented by the following series:¹

For the upper edge,

$$q_u = A_0 + \sum_{m=1}^{\infty} A_m \sin \frac{m\pi x}{l} + \sum_{m=1}^{\infty} A'_m \cos \frac{m\pi x}{l} \quad (m)$$

For the lower edge,

$$q_l = B_0 + \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{l} + \sum_{m=1}^{\infty} B'_m \cos \frac{m\pi x}{l}$$

The constant terms A_0 and B_0 represent a uniform loading of the beam, which was discussed in Art. 21. Stresses produced by terms containing $\sin(m\pi x/l)$ are obtained by summing up solutions (k). The stresses produced by terms containing $\cos(m\pi x/l)$ are easily obtained from (k) by exchanging $\sin \alpha x$ for $\cos \alpha x$ and vice versa, and by changing the sign of τ_{xy} .

To illustrate the application of this general method of stress calculation in rectangular plates, let us consider the case shown in Fig. 33. For this case of symmetrical loading the terms with $\sin(m\pi x/l)$ vanish from expressions (m) and the coefficients A_0 and A'_m are obtained in the usual manner;

¹ For Fourier series see Osgood, "Advanced Calculus," 1928; or Byerly, "Fourier Series and Spherical Harmonics," 1902; or Churchill "Fourier Series and Boundary Value Problems," 1941.

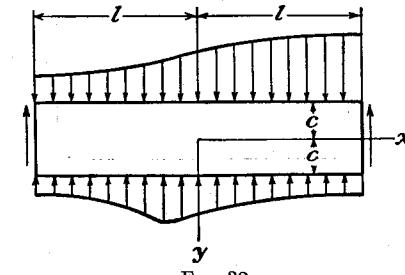


FIG. 32.

$$A_0 = B_0 = \frac{qa}{l}, \quad A_m' = B_m' = \frac{1}{l} \int_{-a}^a q \cos \frac{m\pi x}{l} dx = \frac{2q \sin \frac{m\pi a}{l}}{m\pi} \quad (n)$$

The terms A_0 and B_0 represent a uniform compression in the y -direction equal to qa/l . The stresses produced by the trigonometric terms are obtained by using solutions (k), exchanging $\sin \alpha x$ for $\cos \alpha x$ in this solution and changing the sign of τ_{xy} .

Let us consider the middle plane $y = 0$, on

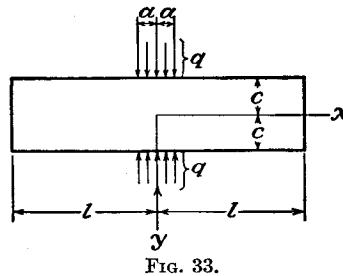


FIG. 33.

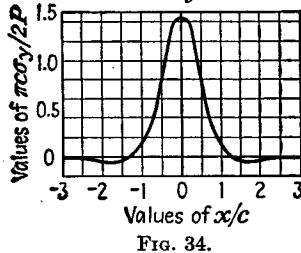


FIG. 34.

which there is only the normal stress σ_y . By using the second of Eqs. (k) we find

$$\sigma_y = -\frac{qa}{l} - \frac{4q}{\pi} \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi a}{l}}{m} \cdot \frac{\frac{m\pi c}{l} \cosh \frac{m\pi c}{l} + \sinh \frac{m\pi c}{l}}{\sinh \frac{2m\pi c}{l} + 2 \frac{m\pi c}{l}} \cdot \cos \frac{m\pi x}{l}$$

This stress was evaluated by Filon¹ for an infinitely long strip when the dimension a is very small (*i.e.*, concentrated force $P = 2qa$). The results of this calculation

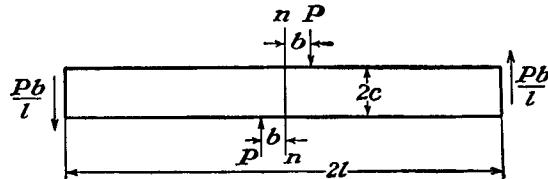


FIG. 35.

are shown in Fig. 34. It will be seen that σ_y diminishes very rapidly with x . At a value $x/c = 1.35$, it becomes zero and is then replaced by tension. Filon discusses

¹ L. N. G. Filon, *Trans. Roy. Soc. (London)*, series A, vol. 201, p. 67, 1903. The same problem was discussed also by A. Timpe, *Z. Math. Physik*, vol. 55, p. 149, 1907; G. Mesmer, *Vergleichende spannungsoptische Untersuchungen . . .*, Dissertation, Göttingen, 1929; F. Seewald, *Abhandl. aerodynam. Inst., Tech. Hochschule, Aachen*, vol. 7, p. 11, 1927; and H. Bay, *Ingenieur-Archiv*, vol. 3, p. 435, 1932. An approximate solution of the same problem was given by M. Pigeaud, *Compt. rend.*, vol. 161, p. 673, 1915. The investigation of the problem in the case of a rectangular plate of finite length was made by J. N. Goodier, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 54, no. 18, p. 173, 1932.

also the case shown in Fig. 35 when the forces P are displaced one with respect to the other. The distribution of shearing stresses over the cross section nn in this case is of practical interest and is shown in Fig. 36. It may be seen that for small values of the ratio b/c this distribution does not resemble the parabolic distribution given by the elementary theory, and that there are very large stresses at the top and bottom of the beam while the middle portion of the beam is practically free from shearing stresses.

In the problem of Fig. 34 there will by symmetry be no shear stress and no vertical displacement at the middle line $y = 0$. The upper half therefore corresponds to an elastic layer resting on a rigid smooth base.¹

Let us consider now another extreme case when the depth of the plate $2c$ is

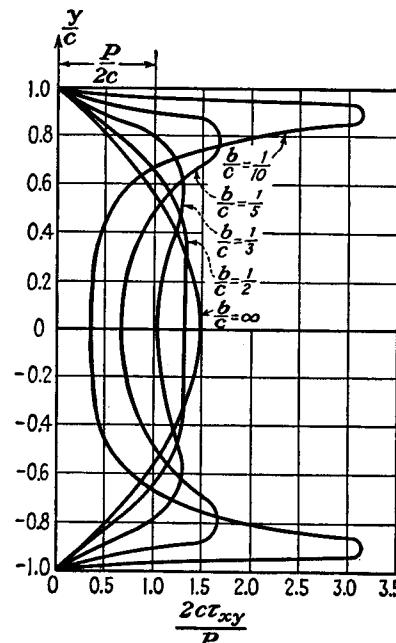


FIG. 36.

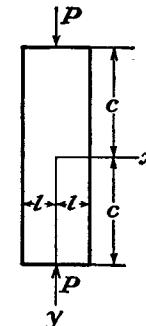


FIG. 37.

large in comparison with the length $2l$ (Fig. 37). We shall use this case to show that the distribution of stresses over cross sections rapidly approaches uniformity as the distance from the point of application of the forces P increases. By using the second of Eqs. (k) with $\cos \alpha x$ instead of $\sin \alpha x$ and expressions (n) for coefficients A_m' , equal to B_m' , we find

$$\sigma_y = -\frac{qa}{l} - \frac{4q}{\pi} \sum_{m=1}^{\infty} \frac{\sin \alpha a}{m} \cdot \frac{(\alpha c \cosh \alpha c + \sinh \alpha c) \cosh \alpha y - \alpha y \sinh \alpha c \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \cos \alpha x \quad (p)$$

in which $qa = P/2$. If l is small in comparison with c , αc is a large number and it can be neglected in comparison with $\sinh \alpha c$. We can also put

$$\sinh \alpha c = \cosh \alpha c = \frac{1}{2} e^{\alpha c}$$

For cross sections at a large distance from the middle of the plate we can write

¹ The rough base is considered by K. Marguerre, *Ingenieur-Archiv*, vol. 2, p. 108, 1931, and a flexible but inextensible layer embedded in the elastic material, a case of interest in soil mechanics, by M. A. Biot, *Physics*, vol. 6, p. 367, 1935.

$\sinh \alpha y = \cosh \alpha y = \frac{1}{2}e^{\alpha y}$. Substituting these in Eq. (p), we find

$$\begin{aligned}\sigma_y &= -\frac{qa}{l} - \frac{4q}{\pi} \sum_{m=1}^{\infty} \frac{\sin \alpha a}{2m} [(\alpha c + 1)e^{\alpha(y-c)} - \alpha y e^{\alpha(y-c)}] \cos \alpha x \\ &= -\frac{qa}{l} - \frac{4q}{\pi} \sum_{m=1}^{\infty} \frac{\sin \frac{m\pi a}{l}}{2m} \left[\frac{m\pi}{l}(c-y) + 1 \right] e^{\frac{m\pi}{l}(y-c)} \cos \frac{m\pi x}{l}\end{aligned}$$

If $c - y$ is not very small, say $c - y > l/2$, this series converges very rapidly and it is only necessary to take a few terms in calculating σ_y . Then we can take

$$\sin \frac{m\pi a}{l} = \frac{m\pi a}{l}$$

and putting $2aq = P$, we find

$$\sigma_y = -\frac{P}{2l} - \frac{P}{l} \sum_{m=1}^{\infty} \left[\frac{m\pi}{l}(c-y) + 1 \right] e^{\frac{m\pi}{l}(y-c)} \cos \frac{m\pi x}{l}$$

For $y = c - l$, for instance,

$$\sigma_y = -\frac{P}{2l} - \frac{P}{l} \left(\frac{\pi+1}{e^\pi} \cos \frac{\pi x}{l} + \frac{2\pi+1}{e^{2\pi}} \cos \frac{2\pi x}{l} + \frac{3\pi+1}{e^{3\pi}} \cos \frac{3\pi x}{l} + \dots \right)$$

The first three terms of the series are sufficient to give good accuracy and the stress distribution is as shown in Fig. 38b. In the same figure the stress distributions for $c - y = l/2$ and $c - y = 2l$ are also given.¹ It is evident that at a distance from the end equal to the width of the strip the stress distribution is practically uniform, which confirms the conclusion usually made on the basis of Saint-Venant's principle.

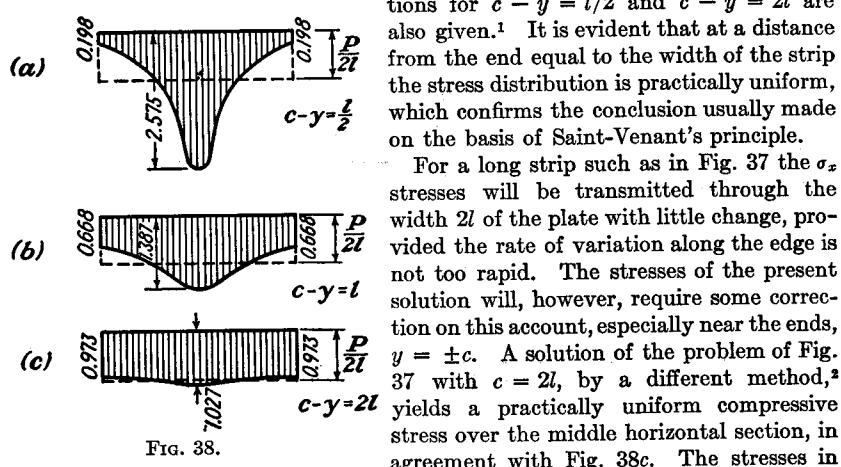


FIG. 38. The stress distributions σ_y versus position x for different distances $c - y$ from the end of a strip.

the vicinity of the points of application of the loads P will be discussed later (see page 85).

¹ See paper by F. Bleich, *loc. cit.*

² J. N. Goodier, *Trans. A.S.M.E.*, vol. 54, p. 173, 1932.

24. Other Applications of Fourier Series. Gravity Loading. The problems considered in Art. 23 concerned a single "span" l or $2l$. The solutions, however, can equally well be regarded as representing periodic states of stress in long strips parallel to the x -axis, since a Fourier series represents a periodic function. A continuous beam consisting of a sequence of equal spans similarly loaded will have such a periodic stress distribution if the end conditions are appropriate. If, as in certain reinforced-concrete bunker constructions, the beam is essentially a wall supported at points whose distance apart is comparable with the depth (Fig. 39), useful results can be obtained by the present method.¹ The elementary beam

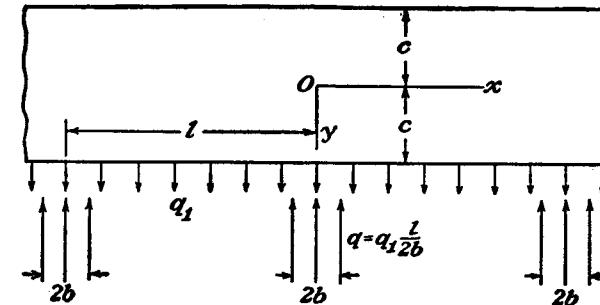


FIG. 39.

theory is not adequate. A uniformly distributed load q_1 on the lower edge, supported by upward reactions uniformly distributed in widths $2b$ at intervals l , presents a special case covered by Eqs. (m) of Art. 23. If the load q_1 is applied on the upper edge it is merely necessary to add the stress distribution due to equal and opposite uniformly distributed pressures q_1 on both upper and lower edges.

If the load is the weight of the beam itself the resulting body-force problem may at once be reduced to an edge-load problem. The simple stress distribution

$$\sigma_x = 0, \quad \sigma_y = -\rho g(y+c), \quad \tau_{xy} = 0$$

satisfies the equations of equilibrium and compatibility (19) and (24). It clearly represents support by uniformly distributed pressure $2\rho gc$ on the lower edge in Fig. 39. The condition that σ_y is zero at the lower edge, except at the supports (of width $2b$), is satisfied by adding this stress distribution to that represented by Fig. 39 when q_1 is replaced by $2\rho gc$, and the stress is due to q and q_1 without body force.

Problems

1. Investigate what problem of plane stress is solved by the stress function

$$\phi = \frac{3F}{4c} \left(xy - \frac{xy^3}{3c^2} \right) + \frac{P}{2} y^2$$

2. Investigate what problem is solved by

$$\phi = -\frac{F}{d^3} xy^2(3d-2y)$$

applied to the region included in $y = 0, y = d, x = 0$, on the side x positive.

¹ Problems of this kind are discussed, with references, in the book "Die Statik im Eisenbetonbau," by K. Beyer, 2d ed., p. 723, 1934; see also H. Craemer, *Ingenieur-Archiv*, vol. 7, p. 325, 1936.

3. Show that

$$\phi = \frac{q}{8c^3} \left[x^2 (y^4 - 3c^2y + 2c^4) - \frac{1}{5} y^4 (y^2 - 2c^2) \right]$$

is a stress function, and find what problem it solves when applied to the region included in $y = \pm c$, $x = 0$, on the side x positive.

4. The stress function

$$\phi = s \left(\frac{1}{4} xy - \frac{xy^2}{4c} - \frac{xy^3}{4c^2} + \frac{ly^2}{4c} + \frac{ly^3}{4c^2} \right)$$

is proposed as giving the solution for a cantilever ($y = \pm c$, $0 < x < l$) loaded by uniform shear along the lower edge, the upper edge and the end $x = l$ being free from load. In what respects is this solution imperfect? Compare the expressions for the stresses with those obtainable from elementary tension and bending formulas.

5. The beam of Fig. 28 is loaded by its own weight instead of the load q on the upper edge. Find expressions for the displacement components u and v . Find also an expression for the change of the (originally unit) thickness.

6. The cantilever of Fig. 26, instead of having a narrow rectangular cross section, has a wide rectangular cross section, and is maintained in plane strain by suitable forces along the vertical sides. The load is P per unit width on the end.

Justify the statement that the stresses σ_x , σ_y , τ_{xy} are the same as those found in Art. 20. Find an expression for the stress σ_z , and sketch its distribution along the sides of the cantilever. Write down expressions for the displacement components u and v when a horizontal element of the axis is fixed at $x = l$.

7. Show that if V is a plane harmonic function, i.e., it satisfies the Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

then the functions xV , yV , $(x^2 + y^2)V$ will each satisfy Eq. (a) of Art. 17, and so can be used as stress functions.

8. Show that

$$(Ae^{ax} + Be^{-ax} + Cye^{ay} + Dye^{-ay}) \sin ax$$

is a stress function.

Derive series expressions for the stresses in a semi-infinite plate, $y > 0$, due to normal pressure on the straight edge ($y = 0$) having the distribution

$$\sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{l}$$

Show that the stress σ_x at a point on the edge is a compression equal to the applied pressure at that point. Assume that the stress tends to disappear as y becomes large.

9. Show that (a) the stresses given by Eqs. (e) of Art. 23 and (b) the stresses in Prob. 8 satisfy Eq. (b) of Art. 16.

CHAPTER 4

TWO-DIMENSIONAL PROBLEMS IN POLAR COORDINATES

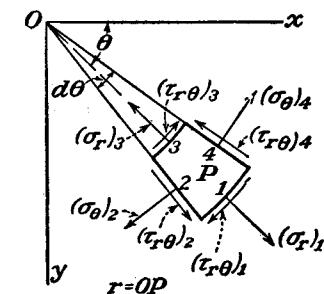
25. General Equations in Polar Coordinates. In discussing stresses in circular rings and disks, curved bars of narrow rectangular cross section with a circular axis, etc., it is advantageous to use polar coordinates. The position of a point in the middle plane of a plate is then defined by the distance from the origin O (Fig. 40) and by the angle θ between r and a certain axis Ox fixed in the plane.

Let us now consider the equilibrium of a small element 1234 cut out from the plate by the radial sections 04, 02, normal to the plate, and by two cylindrical surfaces 3, 1, normal to the plate. The normal stress component in the radial direction is denoted by σ_r , the normal component in the circumferential direction by σ_θ , and the shearing-stress component by $\tau_{r\theta}$, each symbol representing stress at the point r , θ , which is the mid-point P of the element. On account of the variation of stress the values at the mid-points of the sides 1, 2, 3, 4 are not quite the same as the values σ_r , σ_θ , $\tau_{r\theta}$, and are denoted by $(\sigma_r)_1$, etc., in Fig. 40. The radii of the sides 3, 1 are denoted by r_3 , r_1 . The radial force on the side 1 is $\sigma_{r1} r_1 d\theta$ which may be written $(\sigma_{rr})_1 d\theta$, and similarly the radial force on side 3 is $-(\sigma_{rr})_3 d\theta$. The normal force on side 2 has a component along the radius through P of $-(\sigma_\theta)_2 (r_1 - r_3) \sin(d\theta/2)$, which may be replaced by $-(\sigma_\theta)_2 dr (d\theta/2)$. The corresponding component from side 4 is $-(\sigma_\theta)_4 dr (d\theta/2)$. The shearing forces on sides 2 and 4 give $[(\tau_{r\theta})_2 - (\tau_{r\theta})_4]dr$.

Summing up forces in the radial direction, including body force R per unit volume in the radial direction, we obtain the equation of equilibrium

$$\begin{aligned} (\sigma_{rr})_1 d\theta - (\sigma_{rr})_3 d\theta - (\sigma_\theta)_2 dr \frac{d\theta}{2} - (\sigma_\theta)_4 dr \frac{d\theta}{2} \\ + [(\tau_{r\theta})_2 - (\tau_{r\theta})_4] dr + Rr d\theta dr = 0 \end{aligned}$$

FIG. 40.



Dividing by $dr d\theta$ this becomes

$$\frac{(\sigma_r r)_1 - (\sigma_r r)_3}{dr} - \frac{1}{2} [(\sigma_\theta)_2 + (\sigma_\theta)_4] + \frac{(\tau_{r\theta})_2 - (\tau_{r\theta})_4}{d\theta} + Rr = 0$$

If the dimensions of the element are now taken smaller and smaller, to the limit zero, the first term of this equation is in the limit $\partial(\sigma_r)/\partial r$. The second becomes σ_θ , and the third $\partial\tau_{r\theta}/\partial\theta$. The equation of equilibrium in the tangential direction may be derived in the same manner. The two equations take the final form

$$\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r} \frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + R &= 0 \\ \frac{1}{r} \frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned} \quad (37)$$

These equations take the place of Eqs. (18) when we solve two-dimensional problems by means of polar coordinates. When the body force R is zero they are satisfied by putting

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} \\ \tau_{r\theta} &= \frac{1}{r^2} \frac{\partial\phi}{\partial\theta} - \frac{1}{r} \frac{\partial^2\phi}{\partial r \partial\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\phi}{\partial\theta} \right) \end{aligned} \quad (38)$$

where ϕ is the stress function as a function of r and θ . This of course may be verified by direct substitution. A derivation of (38) is included in what follows.

To yield a possible stress distribution, this function must ensure that the condition of compatibility is satisfied. In Cartesian coordinates (see page 26) this condition is

$$\frac{\partial^4\phi}{\partial x^4} + 2 \frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \quad (a)$$

For the present purpose we need this equation transformed to polar coordinates. The relation between polar and Cartesian coordinates is given by

Since $x^2 + y^2 = r^2$, $\theta = \arctan \frac{y}{x}$, from which

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta, & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta \\ \frac{\partial\theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin\theta}{r}, & \frac{\partial\theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos\theta}{r} \end{aligned}$$

Using these, and considering ϕ as a function of r and θ , we find

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta} \frac{\partial\theta}{\partial x} = \frac{\partial\phi}{\partial r} \cos\theta - \frac{1}{r} \frac{\partial\phi}{\partial\theta} \sin\theta$$

To get the second derivative with respect to x , it is only necessary to repeat the above operation; hence

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} &= \left(\frac{\partial}{\partial r} \cos\theta - \frac{1}{r} \sin\theta \frac{\partial}{\partial\theta} \right) \left(\frac{\partial\phi}{\partial r} \cos\theta - \frac{1}{r} \frac{\partial\phi}{\partial\theta} \sin\theta \right) \\ &= \frac{\partial^2\phi}{\partial r^2} \cos^2\theta - 2 \frac{\partial^2\phi}{\partial\theta \partial r} \frac{\sin\theta \cos\theta}{r} + \frac{\partial\phi}{\partial r} \frac{\sin^2\theta}{r} + 2 \frac{\partial\phi}{\partial\theta} \frac{\sin\theta \cos\theta}{r^2} \\ &\quad + \frac{\partial^2\phi}{\partial\theta^2} \frac{\sin^2\theta}{r^2} \end{aligned} \quad (b)$$

In the same manner we find

$$\begin{aligned} \frac{\partial^2\phi}{\partial y^2} &= \frac{\partial^2\phi}{\partial r^2} \sin^2\theta + 2 \frac{\partial^2\phi}{\partial\theta \partial r} \frac{\sin\theta \cos\theta}{r} + \frac{\partial\phi}{\partial r} \frac{\cos^2\theta}{r} \\ &\quad - 2 \frac{\partial\phi}{\partial\theta} \frac{\sin\theta \cos\theta}{r^2} + \frac{\partial^2\phi}{\partial\theta^2} \frac{\cos^2\theta}{r^2} \end{aligned} \quad (c)$$

Adding together (b) and (c), we obtain

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} \quad (d)$$

Using the identity

$$\frac{\partial^4\phi}{\partial x^4} + 2 \frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \right)$$

and Eq. (d), the compatibility equation (a) in polar coordinates becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial\theta^2} \right) \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2} \right) = 0 \quad (39)$$

From various solutions of this partial differential equation we obtain solutions of two-dimensional problems in polar coordinates for various boundary conditions. Several examples of such problems will be discussed in this chapter.

The first and second of the expressions (38) follow from Eqs. (b) and (c). If we choose any point in the plate, and let the x -axis pass through it, we have $\theta = 0$, and σ_x , σ_y are the same, for this particular point, as σ_r , σ_θ . Thus from (c), putting $\theta = 0$,

$$\sigma_r = \sigma_x = \left(\frac{\partial^2\phi}{\partial y^2} \right)_{\theta=0} = \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2\phi}{\partial\theta^2}$$

This expression continues to represent σ_r , whatever the orientation of the x -axis. We find similarly from (b), putting $\theta = 0$,

$$\sigma_\theta = \sigma_u = \left(\frac{\partial^2 \phi}{\partial x^2} \right)_{\theta=0} = \frac{\partial^2 \phi}{\partial r^2}$$

and the third expression of (38) can be obtained likewise by finding the expression for $-\partial^2 \phi / \partial x \partial y$ analogous to (b) and (c).

26. Stress Distribution Symmetrical about an Axis. If the stress distribution is symmetrical with respect to the axis through O perpendicular to the xy -plane (Fig. 40), the stress components do not depend on θ and are functions of r only. From symmetry it follows also that the shearing stress $\tau_{r\theta}$ must vanish. Then only the first of the two equations of equilibrium (37) remains, and we have

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \quad (40)$$

If the body force R is zero, we may use the stress function ϕ . When this function depends only on r , the equation of compatibility (39) becomes

$$\begin{aligned} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left(\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d \phi}{dr} \right) \\ = \frac{d^4 \phi}{dr^4} + \frac{2}{r} \frac{d^3 \phi}{dr^3} - \frac{1}{r^2} \frac{d^2 \phi}{dr^2} + \frac{1}{r^3} \frac{d \phi}{dr} = 0 \end{aligned} \quad (41)$$

This is an ordinary differential equation, which can be reduced to a linear differential equation with constant coefficients by introducing a new variable t such that $r = e^t$. In this manner the general solution of Eq. (41) can easily be obtained. This solution has four constants of integration, which must be determined from the boundary conditions. By substitution it can be checked that

$$\phi = A \log r + Br^2 \log r + Cr^2 + D \quad (42)$$

is the general solution. The solutions of all problems of symmetrical stress distribution and no body forces can be obtained from this. The corresponding stress components from Eqs. 38 are

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{A}{r^2} + B(1 + 2 \log r) + 2C \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2 \log r) + 2C \\ \tau_{r\theta} &= 0 \end{aligned} \quad (43)$$

If there is no hole at the origin of coordinates, constants A and B vanish, since otherwise the stress components (43) become infinite when

$r = 0$. Hence, for a plate without a hole at the origin and with no body forces, only one case of stress distribution symmetrical with respect to the axis may exist, namely that when $\sigma_r = \sigma_\theta = \text{constant}$ and the plate is in a condition of uniform tension or uniform compression in all directions in its plane.

If there is a hole at the origin, other solutions than uniform tension or compression can be derived from expressions (43). Taking B as zero,¹ for instance, Eqs. 43 become

$$\begin{aligned} \sigma_r &= \frac{A}{r^2} + 2C \\ \sigma_\theta &= -\frac{A}{r^2} + 2C \end{aligned} \quad (44)$$

This solution may be adapted to represent the stress distribution in a hollow cylinder submitted to uniform pressure on the inner and outer surfaces² (Fig. 41). Let a and b denote the inner and outer radii of the cylinder, and p_i and p_o the uniform internal and external pressures. Then the boundary conditions are:

$$(\sigma_r)_{r=a} = -p_i, \quad (\sigma_r)_{r=b} = -p_o \quad (a)$$

Substituting in the first of Eqs. (44), we obtain the following equations to determine A and C :

$$\begin{aligned} \frac{A}{a^2} + 2C &= -p_i \\ \frac{A}{b^2} + 2C &= -p_o \end{aligned}$$

from which

$$\begin{aligned} A &= \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \\ 2C &= \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \end{aligned}$$

Substituting these in Eqs. (44) the following expressions for the stress components are obtained:

$$\begin{aligned} \sigma_r &= \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \\ \sigma_\theta &= -\frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \end{aligned} \quad (45)$$

¹ Proof that B must be zero requires consideration of displacements. See p. 68.

² The solution of this problem is due to Lamé, "Leçons sur la théorie . . . de l'élasticité," Paris, 1852.

It is interesting to note that the sum $\sigma_r + \sigma_\theta$ is constant through the thickness of the wall of the cylinder. Hence the stresses σ_r and σ_θ produce a uniform extension or contraction in the direction of the axis of the cylinder, and cross sections perpendicular to this axis remain plane. Hence the deformation produced by the stresses (45) in an element of the cylinder cut out by two adjacent cross sections does not interfere with the deformation of the neighboring elements, and it is justifiable to consider the element in the condition of plane stress as we did in the above discussion.

In the particular case when $p_o = 0$ and the cylinder is submitted to internal pressure only, Eqs. 45 give

$$\begin{aligned}\sigma_r &= \frac{a^2 p_i}{b^2 - a^2} \left(1 - \frac{b^2}{r^2}\right) \\ \sigma_\theta &= \frac{a^2 p_i}{b^2 - a^2} \left(1 + \frac{b^2}{r^2}\right)\end{aligned}\quad (46)$$

These equations show that σ_r is always a compressive stress and σ_θ a tensile stress. The latter is greatest at the inner surface of the cylinder, where

$$(\sigma_\theta)_{\max.} = \frac{p_i(a^2 + b^2)}{b^2 - a^2} \quad (47)$$

$(\sigma_\theta)_{\max.}$ is always numerically greater than the internal pressure and approaches this quantity as b increases, so that it can never be reduced below p_i , however much material is added on the outside. Various applications of Eqs. (46) and (47) in machine design are usually discussed in elementary books on the strength of materials.¹

The corresponding problem for a cylinder with an eccentric bore is considered in Art. 66. It was solved by G. B. Jeffery.² If the radius of the bore is a and that of the external surface b , and if the distance between their centers is e , the maximum stress, when the cylinder is under an internal pressure p_i , is the tangential stress at the internal surface at the thinnest part, if $e < \frac{1}{2}a$, and is of the magnitude

$$\sigma = p_i \left[\frac{2b^2(b^2 + a^2 - 2ae - e^2)}{(a^2 + b^2)(b^2 - a^2 - 2ae - e^2)} - 1 \right]$$

If $e = 0$, this coincides with Eq. (47).

¹ See, for instance, S. Timoshenko, "Strength of Materials," vol. 2, p. 236, 1941.

² Trans. Roy. Soc. (London), series A, vol. 221, p. 265, 1921. See also Brit. Assoc. Advancement Sci. Repts., 1921.

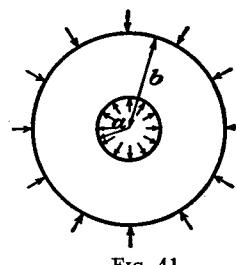


FIG. 41.

27. Pure Bending of Curved Bars. Let us consider a curved bar with a constant narrow rectangular cross section¹ and a circular axis bent in the plane of curvature by couples M applied at the ends (Fig. 42). The bending moment in this case is constant along the length of the bar and it is natural to expect that the stress distribution is the same in all radial cross sections, and that the solution of the problem

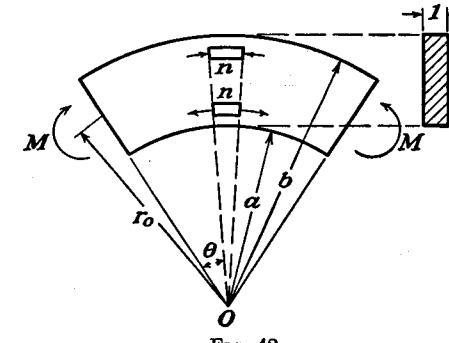


FIG. 42.

can therefore be obtained by using expression (42). Denoting by a and b the inner and the outer radii of the boundary and taking the width of the rectangular cross section as unity, the boundary conditions are

$$(1) \quad \sigma_r = 0 \text{ for } r = a \text{ and } r = b$$

$$(2) \quad \int_a^b \sigma_\theta dr = 0, \quad \int_a^b \sigma_r r dr = -M \quad (a)$$

$$(3) \quad \tau_{r\theta} = 0 \text{ at the boundary}$$

Condition (1) means that the convex and concave boundaries of the bar are free from normal forces; condition (2) indicates that the normal stresses at the ends give rise to the couple M only, and condition (3) indicates that there are no tangential forces applied at the boundary. Using the first of Eqs. (43) with (1) of the boundary conditions (a) we obtain

$$\frac{A}{a^2} + B(1 + 2 \log a) + 2C = 0 \quad (b)$$

$$\frac{A}{b^2} + B(1 + 2 \log b) + 2C = 0$$

¹ From the general discussion of the two-dimensional problem, Art. 15, it follows that the solution obtained below holds also for another extreme case when the dimension of the cross section perpendicular to the plane of curvature is very large, as, for instance, in the case of a tunnel vault (see Fig. 10), if the load is the same along the length of the tunnel.

From (2) of conditions (a) we find

$$\int_a^b \sigma_\theta dr = \int_a^b \frac{\partial^2 \phi}{\partial r^2} dr = \left| \frac{\partial \phi}{\partial r} \right|_a^b = 0$$

or substituting for ϕ its expression (42), we find

$$\begin{aligned} \left[\frac{A}{b} + B(b + 2b \log b) + 2Cb \right] \\ - \left[\frac{A}{a} + B(a + 2a \log a) + 2Ca \right] = 0 \quad (c) \end{aligned}$$

Comparing this with (b), it is easy to see that (c) is satisfied, and the forces at the ends are reducible to a couple, provided conditions (b) are satisfied. To have the bending couple equal to M , the condition

$$\int_a^b \sigma_\theta r dr = \int_a^b \frac{\partial^2 \phi}{\partial r^2} r dr = -M \quad (d)$$

must be fulfilled. We have

$$\int_a^b \frac{\partial^2 \phi}{\partial r^2} r dr = \left| \frac{\partial \phi}{\partial r} r \right|_a^b - \int_a^b \frac{\partial \phi}{\partial r} dr = \left| \frac{\partial \phi}{\partial r} r \right|_a^b - \left| \phi \right|_a^b$$

and noting that on account of (b)

$$\left| \frac{\partial \phi}{\partial r} r \right|_a^b = 0$$

we find from (d),

$$\left| \phi \right|_a^b = M$$

or substituting expression (42) for ϕ ,

$$A \log \frac{b}{a} + B(b^2 \log b - a^2 \log a) + C(b^2 - a^2) = M \quad (e)$$

This equation, together with the two Eqs. (b), completely determines the constants A , B , C , and we find

$$\begin{aligned} A &= -\frac{4M}{N} a^2 b^2 \log \frac{b}{a}, & B &= -\frac{2M}{N} (b^2 - a^2) \\ C &= \frac{M}{N} [b^2 - a^2 + 2(b^2 \log b - a^2 \log a)] \end{aligned} \quad (f)$$

where for simplicity we have put

$$N = (b^2 - a^2)^2 - 4a^2 b^2 \left(\log \frac{b}{a} \right)^2 \quad (g)$$

Substituting the values (f) of the constants into the expressions (43) for the stress components, we find

$$\begin{aligned} \sigma_r &= -\frac{4M}{N} \left(\frac{a^2 b^2}{r^2} \log \frac{b}{a} + b^2 \log \frac{r}{b} + a^2 \log \frac{a}{r} \right) \\ \sigma_\theta &= -\frac{4M}{N} \left(-\frac{a^2 b^2}{r^2} \log \frac{b}{a} + b^2 \log \frac{r}{b} + a^2 \log \frac{a}{r} + b^2 - a^2 \right) \\ \tau_{r\theta} &= 0 \end{aligned} \quad (48)$$

This gives the stress distribution satisfying all the boundary conditions¹ (a) for pure bending and represents the exact solution of the problem, provided the distribution of the normal forces at the ends is that given by the second of Eqs. (48). If the forces giving the bending couple M are distributed over the ends of the bar in some other manner, the stress distribution at the ends will be different from that of the solution (48). But on the basis of Saint-Venant's principle it can be concluded that the deviations from solution (48) are very small and may be neglected at large distances from the ends, say at distances greater than the depth of the bar.

It is of practical interest to compare solution (48) with the elementary solutions usually given in books on the strength of materials. If the depth of the bar, $b - a$, is small in comparison with the radius of the central axis, $(b + a)/2$, the same stress distribution as for straight bars is usually assumed. If this depth is not small it is usual in practice to assume that cross sections of the bar remain plane during the bending, from which it can be shown that the distribution of the normal stresses σ_θ over any cross sections follows a hyperbolic law.² In all cases the maximum and minimum values of the stress σ_θ can be presented in the form

$$\sigma_\theta = m \frac{M}{a^2} \quad (h)$$

¹ This solution is due to H. Golovin, *Trans. Inst. Tech.*, St. Petersburg, 1881. The paper, published in Russian, remained unknown in other countries, and the same problem was solved later by M. C. Ribiére (*Compt. rend.*, vol. 108, 1889, and vol. 132, 1901) and by L. Prandtl. See A. Föppl, "Vorlesungen über technische Mechanik," vol. 5, p. 72, 1907; also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

² This approximate theory was developed by H. RéSal, *Ann. mines*, p. 617, 1862, and by E. Winkler, *Zivilingenieur*, vol. 4, p. 232, 1858; see also his book "Die Lehre von der Elastizität und Festigkeit," Chap. 15, Prag, 1867. Further development of the theory was made by F. Grashof, "Elastizität und Festigkeit," p. 251, 1878, and by K. Pearson, "History of the Theory of Elasticity," vol. 2, pt. 1, p. 422, 1893.

The following table gives the values of the numerical factor m calculated by the two elementary methods, referred to above, and by the

COEFFICIENT m OF Eq. (48)			
$\frac{b}{a}$	Linear stress distribution	Hyperbolic stress distribution	Exact solution
1.3	± 66.67	$+72.98, -61.27$	$+73.05, -61.35$
2	± 6.000	$+7.725, -4.863$	$+7.755, -4.917$
3	± 1.500	$+2.285, -1.095$	$+2.292, -1.130$

exact formula (48).¹ It can be seen from this table that the elementary solution based on the hypothesis of plane cross sections gives very accurate results.

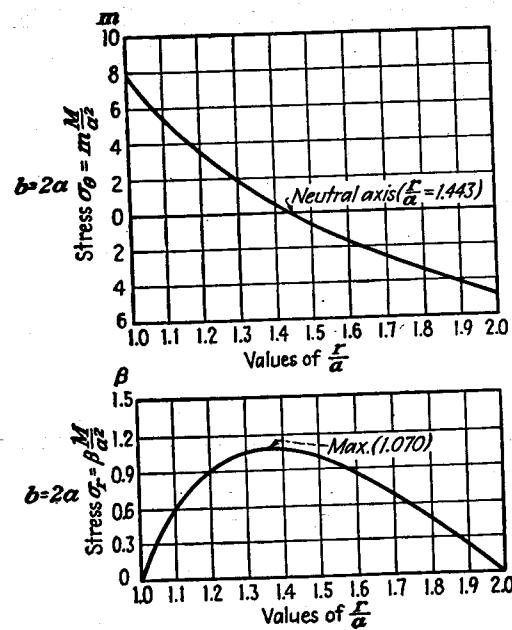


FIG. 43.

It will be shown later that, in the case of pure bending, the cross sections actually do remain plane, and the discrepancy between the elementary and the exact solutions comes from the fact that in the elementary solution the stress component σ_r is neglected and it is

¹ The results are taken from the doctorate thesis, Univ. Michigan, 1931, of V. Billevic.

assumed that longitudinal fibers of the bent bar are in simple tension or compression.

From the first of Eqs. (48) it can be shown that the stress σ_r is always positive for the direction of bending shown in Fig. 42. The same can be concluded at once from the direction of stresses σ_θ acting on the elements $n - n$ in Fig. 42. The corresponding tangential forces give resultants in the radial direction tending to separate longitudinal fibers and producing tensile stress in the radial direction. This stress increases toward the neutral surface and becomes a maximum near this surface. This maximum is always much smaller than $(\sigma_\theta)_{\max}$. For instance, for $b/a = 1.3$, $(\sigma_r)_{\max} = 0.060(\sigma_\theta)_{\max}$; for $b/a = 2$, $(\sigma_r)_{\max} = 0.138(\sigma_\theta)_{\max}$; for $b/a = 3$, $(\sigma_r)_{\max} = 0.193(\sigma_\theta)_{\max}$. In Fig. 43 the distribution of σ_θ and σ_r for $b/a = 2$ is given. From this figure we see that the point of maximum stress σ_r is somewhat displaced from the neutral axis in the direction of the center of curvature.

28. Strain Components in Polar Coordinates. In considering the displacement in polar coordinates let us denote by u and v the compo-

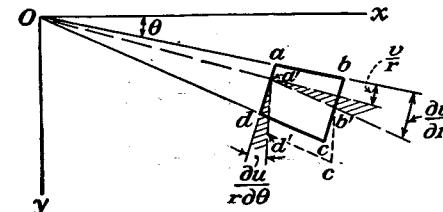


FIG. 44.

nents of the displacement in the radial and tangential directions, respectively. If u is the radial displacement of the side ad of the element $abcd$ (Fig. 44), the radial displacement of the side bc is $u + (\partial u / \partial r) dr$. Then the unit elongation of the element $abcd$ in the radial direction is

$$\epsilon_r = \frac{\partial u}{\partial r} \quad (49)$$

As for the strain in the tangential direction it should be observed that it depends not only on the displacement v but also on the radial displacement u . Assuming, for instance, that the points a and d of the element $abcd$ (Fig. 44) have only the radial displacement u , the new length of the arc ad is $(r + u) d\theta$ and the tangential strain is therefore

$$\frac{(r + u) d\theta - r d\theta}{r d\theta} = \frac{u}{r}$$

The difference in the tangential displacement of the sides ab and cd of

the element $abcd$ is $(\partial v / \partial \theta) d\theta$, and the tangential strain due to the displacement v is accordingly $\partial v / r \partial \theta$. The total tangential strain is thus¹

$$\epsilon_\theta = \frac{u}{r} + \frac{\partial v}{r \partial \theta} \quad (50)$$

Considering now the shearing strain, let $a'b'c'd'$ be the position of the element $abcd$ after deformation (Fig. 44). The angle between the direction ad and $a'd'$ is due to the radial displacement u and is equal to $\partial u / r \partial \theta$. In the same manner the angle between $a'b'$ and ab is equal to $\partial v / \partial r$. It should be noted that only part of this angle (shaded in the figure) contributes to the shearing strain and the other part, equal to v/r , represents the angular displacement due to rotation of the element $abcd$ as a rigid body about the axis through O . Hence the total change in the angle dab , representing the shearing strain, is

$$\gamma_{r\theta} = \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \quad (51)$$

Substituting now the expressions for the strain components (49), (50), (51) into the equations of Hooke's law,²

$$\begin{aligned} \epsilon_r &= \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \\ \epsilon_\theta &= \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \\ \gamma_{r\theta} &= \frac{1}{G} \tau_{r\theta} \end{aligned} \quad (52)$$

we can obtain sufficient equations for determining u and v .

29. Displacements for Symmetrical Stress Distributions. Substituting in the first of Eqs. (52) the stress components from Eqs. 43, we find

$$\frac{\partial u}{\partial r} = \frac{1}{E} \left[\frac{(1+\nu)A}{r^2} + 2(1-\nu)B \log r + (1-3\nu)B + 2(1-\nu)C \right]$$

By integration we obtain

$$u = \frac{1}{E} \left[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br \log r - B(1+\nu)r + 2C(1-\nu)r \right] + f(\theta) \quad (a)$$

¹ The symbol ϵ_θ was used with a different meaning in Art. 10.

² It is assumed here that we have to do with plane stress, i.e., that there is no stress σ_z perpendicular to the plane of the plate (see p. 11).

in which $f(\theta)$ is a function of θ only. From the second of Eqs. (52), we find, by using Eq. 50,

$$\frac{\partial v}{\partial \theta} = \frac{4Br}{E} - f(\theta)$$

from which, by integration,

$$v = \frac{4Br\theta}{E} - \int f(\theta) d\theta + f_1(r) \quad (b)$$

where $f_1(r)$ is a function of r only. Substituting (a) and (b) in Eq. (51) and noting that $\gamma_{r\theta}$ is zero since $\tau_{r\theta}$ is zero, we find

$$\frac{1}{r} \frac{\partial f(\theta)}{\partial \theta} + \frac{\partial f_1(r)}{\partial r} + \frac{1}{r} \int f(\theta) d\theta - \frac{1}{r} f_1(r) = 0 \quad (c)$$

from which

$$f_1(r) = Fr, \quad f(\theta) = H \sin \theta + K \cos \theta \quad (d)$$

where F , H , and K are constants to be determined from the conditions of constraint of the curved bar or ring. Substituting expressions (d) into Eqs. (a) and (b), we find the following expressions for the displacements.¹

$$u = \frac{1}{E} \left[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br \log r - B(1+\nu)r + 2C(1-\nu)r \right] + H \sin \theta + K \cos \theta \quad (53)$$

$$v = \frac{4Br\theta}{E} + Fr + H \cos \theta - K \sin \theta$$

in which the values of constants A , B , and C for each particular case should be substituted. Consider, for instance, pure bending. Taking the centroid of the cross section from which θ is measured (Fig. 42) and also an element of the radius at this point, as rigidly fixed, the conditions of constraint are

$$u = 0, \quad v = 0, \quad \frac{\partial v}{\partial r} = 0 \text{ for } \theta = 0 \text{ and } r = r_0 = \frac{a+b}{2}$$

Applying these to expressions (53), we obtain the following equations for calculating the constants of integration F , H , and K :

¹ Equation (c) is satisfied only when $\int f(\theta) d\theta$ is taken from (d) without an additive constant.

$$\frac{1}{E} \left[-\frac{(1+\nu)A}{r_0} + 2(1-\nu)Br_0 \log r_0 - B(1+\nu)r_0 + 2C(1-\nu)r_0 \right] + K = 0$$

$$Fr_0 + H = 0$$

$$F = 0$$

From this it follows that $F = H = 0$, and for the displacement v we obtain

$$v = \frac{4Br\theta}{E} - K \sin \theta \quad (54)$$

This means that the displacement of any cross section consists of a translatory displacement $-K \sin \theta$, the same for all points in the cross section, and of a rotation of the cross section by the angle $4B\theta/E$ about the center of curvature O (Fig. 42). We see that cross sections remain plane in pure bending as is usually assumed in the elementary theory of the bending of curved bars.

In discussing the symmetrical stress distribution in a full ring (page 59) the constant B in the general solution (43) was taken as zero, and in this manner we arrived at a solution of Lamé's problem. Now, after obtaining expressions (53) for displacements, we see what is implied by taking B as zero. B contributes to the displacement v the term $4Br\theta/E$. This term is not *single valued*, as it changes when we increase θ by 2π , i.e., if we arrive at a given point after making a complete circle round the ring. Such a *many-valued* expression for a displacement is physically impossible in a full ring, and so, for this case, we must take $B = 0$ in the general solution (43).

A full ring is an example of a multiply-connected body, i.e., a body such that some sections can be cut clear across without dividing the body into two parts. In determining the stresses in such bodies we usually arrive at the conclusion that the boundary conditions referring to the stresses are not sufficient to determine completely the stress distribution, and additional equations, representing the conditions that the displacements should be single valued, must be considered (see page 118).

The physical meaning of many-valued solutions can be explained by considering the *initial stresses* possible in a multiply-connected body. If a portion of the ring between two adjacent cross sections is cut out (Fig. 45), and the ends of the ring are joined again by welding or other

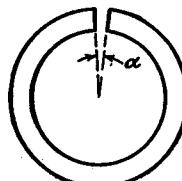


FIG. 45.

means, a ring with initial stresses is obtained, i.e., there are stresses in the ring when external forces are absent. If α is the small angle measuring the portion of the ring which was cut out, the tangential displacement necessary to bring the ends of the ring together is

$$v = \alpha r \quad (e)$$

The same displacement, obtained from Eq. (54) by putting $\theta = 2\pi$, is

$$v = 2\pi \frac{4Br}{E} \quad (f)$$

From (e) and (f) we find

$$B = \frac{\alpha E}{8\pi} \quad (g)$$

The constant B , entering into the many-valued term for the displacement (54) has now a definite value depending on the way in which the initial stresses were produced in the ring. Substituting (g) into Eqs. (f) of Art. 27 (see page 62), we find that the bending moment necessary to bring the ends of the ring together (Fig. 45) is

$$M = -\frac{\alpha E}{8\pi} \frac{(b^2 - a^2)^2 - 4a^2b^2 \left(\log \frac{b}{a}\right)^2}{2(b^2 - a^2)} \quad (h)$$

From this the initial stresses in the ring can easily be calculated by using the solution (48) for pure bending.

30. Rotating Disks. The stress distribution in rotating circular disks is of great practical importance.¹ If the thickness of the disk is small in comparison with its radius, the variation of radial and tangential stresses over the thickness can be neglected² and the problem can be easily solved.³ If the thickness of the disk is constant Eq. (40) can be applied, and it is only necessary to put the body force equal to the inertia force.⁴ Then

$$R = \rho \omega^2 r \quad (a)$$

¹ A complete discussion of this problem and the bibliography of the subject can be found in the well-known book by A. Stodola, "Dampf- und Gas-Turbinen," 6th ed., pp. 312 and 889, 1924.

² An exact solution of the problem for a disk having the shape of a flat ellipsoid of revolution was obtained by C. Chree, see *Proc. Roy. Soc. (London)*, vol. 58, p. 39, 1895. It shows that the difference between the maximum and the minimum stress at the axis of revolution is only 5 per cent of the maximum stress in a uniform disk with thickness one-eighth of its diameter.

³ A more detailed discussion of the problem will be given later (see Art. 119).

⁴ The weight of the disk is neglected.

where ρ is the mass per unit volume of the material of the disk and ω the angular velocity of the disk.

Equation (40) can then be written in the form

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2r^2 = 0 \quad (b)$$

This equation is satisfied if we derive the stress components from a stress function F in the following manner:

$$r\sigma_r = F, \quad \sigma_\theta = \frac{dF}{dr} + \rho\omega^2r^2 \quad (c)$$

The strain components in the case of symmetry are, from Eqs. (49) and (50),

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}$$

Eliminating u between these equations, we find

$$\epsilon_\theta - \epsilon_r + r \frac{d\epsilon_\theta}{dr} = 0 \quad (d)$$

Substituting for the strain components their expressions in terms of the stress components, (52), and using Eqs. (c), we find that the stress function F should satisfy the following equation:

$$r^2 \frac{d^2F}{dr^2} + r \frac{dF}{dr} - F + (3 + \nu)\rho\omega^2r^3 = 0 \quad (e)$$

It can be verified by substitution that the general solution of this equation is

$$F = Cr + C_1 \frac{1}{r} - \frac{3 + \nu}{8} \rho\omega^2r^3 \quad (f)$$

and from Eqs. (c) we find

$$\begin{aligned} \sigma_r &= C + C_1 \frac{1}{r^2} - \frac{3 + \nu}{8} \rho\omega^2r^2 \\ \sigma_\theta &= C - C_1 \frac{1}{r^2} - \frac{1 + 3\nu}{8} \rho\omega^2r^2 \end{aligned} \quad (g)$$

The integration constants C and C_1 are determined from the boundary conditions.

For a solid disk we must take $C_1 = 0$ since otherwise the stresses (g) become infinite at the center. The constant C is determined from the condition at the periphery ($r = b$) of the disk. If there are no forces

applied there, we have

$$(\sigma_r)_{r=b} = C - \frac{3 + \nu}{8} \rho\omega^2b^2 = 0$$

from which

$$C = \frac{3 + \nu}{8} \rho\omega^2b^2$$

and the stress components, from Eqs. (g), are

$$\begin{aligned} \sigma_r &= \frac{3 + \nu}{8} \rho\omega^2(b^2 - r^2) \\ \sigma_\theta &= \frac{3 + \nu}{8} \rho\omega^2b^2 - \frac{1 + 3\nu}{8} \rho\omega^2r^2 \end{aligned} \quad (55)$$

These stresses are greatest at the center of the disk, where

$$\sigma_r = \sigma_\theta = \frac{3 + \nu}{8} \rho\omega^2b^2 \quad (56)$$

In the case of a disk with a circular hole of radius a at the center, the constants of integration in Eqs. (g) are obtained from the conditions at the inner and outer boundaries. If there are no forces acting on these boundaries, we have

$$(\sigma_r)_{r=a} = 0, \quad (\sigma_r)_{r=b} = 0 \quad (h)$$

from which we find that

$$C = \frac{3 + \nu}{8} \rho\omega^2(b^2 + a^2); \quad C_1 = -\frac{3 + \nu}{8} \rho\omega^2a^2b^2$$

Substituting in Eqs. (g),

$$\begin{aligned} \sigma_r &= \frac{3 + \nu}{8} \rho\omega^2 \left(b^2 + a^2 - \frac{a^2b^2}{r^2} - r^2 \right) \\ \sigma_\theta &= \frac{3 + \nu}{8} \rho\omega^2 \left(b^2 + a^2 + \frac{a^2b^2}{r^2} - \frac{1 + 3\nu}{3 + \nu} r^2 \right) \end{aligned} \quad (57)$$

We find the maximum radial stress at $r = \sqrt{ab}$, where

$$(\sigma_r)_{\max.} = \frac{3 + \nu}{8} \cdot \rho\omega^2(b - a)^2 \quad (58)$$

The maximum tangential stress is at the inner boundary, where

$$(\sigma_\theta)_{\max.} = \frac{3 + \nu}{4} \rho\omega^2 \left(b^2 + \frac{1 - \nu}{3 + \nu} a^2 \right) \quad (59)$$

It will be seen that this stress is larger than $(\sigma_r)_{\max.}$.

When the radius a of the hole approaches zero, the maximum tangential stress approaches a value twice as great as that for a solid disk (56); i.e., by making a small circular hole at the center of a solid rotating disk we double the maximum stress. This phenomenon of *stress concentration at a hole* will be discussed later (see page 78).

Assuming that the stresses do not vary over the thickness of the disk, the method of analysis developed above for disks of constant thickness can be extended also to *disks of variable thickness*. If h is the thickness of the disk, varying with radius r , the equation of equilibrium of such an element as shown in Fig. 40 is

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta + h\rho\omega^2r^2 = 0 \quad (k)$$

This equation is satisfied by putting

$$hr\sigma_r = F, \quad h\sigma_\theta = \frac{dF}{dr} + h\rho\omega^2r^2$$

where F is again a stress function.

Substituting these expressions for the stress components into the compatibility equation (d) we arrive at the following equation for the stress function F :

$$r^2 \frac{d^2F}{dr^2} + r \frac{dF}{dr} - F + (3 + \nu)\rho\omega^2hr^3 - \frac{r}{h} \frac{dh}{dr} \left(r \frac{dF}{dr} - \nu F \right) = 0 \quad (l)$$

In this manner the problem of finding the stress distribution in a disk of variable thickness is reduced to the solution of Eq. (l). In the particular case where the thickness varies according to the equation

$$h = Cr^n \quad (m)$$

in which C is a constant and n any number, Eq. (l) can easily be integrated.¹ The general solution has the form

$$F = mr^{n+3} + Ar^\alpha + Br^\beta$$

in which

$$m = -\frac{(3 + \nu)\rho\omega^2C}{(\nu n + 3n + 8)}$$

while α and β are the roots of the quadratic equation

$$x^2 - nx + \nu n - 1 = 0$$

and A and B are integration constants which are determined from the boundary conditions.

¹ This case was investigated by Stodola, *loc. cit.* See also H. Holzer, *Z. ges. Turbinenwesen*, 1915.

A good approximation to the actual shapes of rotating disks can be obtained by dividing the disk into parts and fitting approximately to each part a curve of the type (m).¹ The case of a conical disk has been discussed by several authors.² Very often the calculations are made by dividing the disk into parts and considering each part as a disk of constant thickness.³

31. Bending of a Curved Bar by a Force at the End.⁴ We begin with the simple case shown in Fig. 46. A bar of a narrow rectangular cross section and with a circular axis is constrained at the lower end and bent by a force P applied at the upper end in the radial direction. The bending moment at any cross section mn is proportional to $\sin \theta$, and the normal stress σ_θ , according to elementary theory of the bending of curved bars, is proportional to the bending moment. Assuming that this holds also for the exact solution, an assumption which the results will justify, we find from the second of Eqs. (38) that the stress function ϕ , satisfying the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) = 0 \quad (a)$$

should be proportional to $\sin \theta$. Taking

$$\phi = f(r) \sin \theta \quad (b)$$

and substituting in Eq. (a), we find that $f(r)$ must satisfy the following ordinary differential equation:

¹ See M. Grüber, *V.D.I.*, vol. 50, p. 535, 1906.

² See A. Fischer, *Z. oesterr. Ing. Arch. Vereins*, vol. 74, p. 46, 1922; H. M. Martin, *Engineering*, vol. 115, p. 1, 1923; B. Hodkinson, *Engineering*, vol. 116, p. 274, 1923; K. E. Bisschopp, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 11, p. A-1, 1944.

³ This method was developed by M. Donath; see his book, "Die Berechnung rotierender Scheiben und Ringe," Berlin, 1912. It is described in English by H. Hearle in *Engineering*, vol. 106, p. 131, 1918. A further development of the method was given by R. Grammel, *Dinglers Polytech. J.*, vol. 338, p. 217, 1923. The case when material does not follow Hooke's law was investigated by M. Grüber, *V.D.I.*, vol. 41, p. 860, 1897, and vol. 44, p. 1157, 1900. See also H. Schlechtweg, *Z. angew. Math. Mech.*, vol. 11, p. 17, 1931, and *Ingenieur-Archiv*, vol. 2, p. 212, 1931.

⁴ H. Golovin, *loc. cit.*

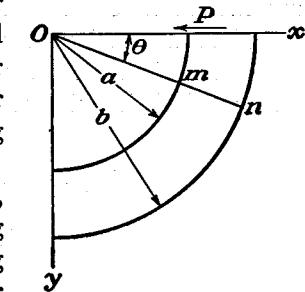


FIG. 46.

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) \left(\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{f}{r^2} \right) = 0 \quad (c)$$

This equation can be transformed into a linear differential equation with constant coefficients (see page 58), and its general solution is

$$f(r) = Ar^3 + B\frac{1}{r} + Cr + Dr \log r \quad (d)$$

in which A , B , C , and D are constants of integration, which are determined from the boundary conditions. Substituting solution (d) in expression (b) for the stress function, and using the general formulas (38), we find the following expressions for the stress components:

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \left(2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} = \left(6Ar + \frac{2B}{r^3} + \frac{D}{r} \right) \sin \theta \\ \tau_{r\theta} &= - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = - \left(2Ar - \frac{2B}{r^3} + \frac{D}{r} \right) \cos \theta \end{aligned} \quad (60)$$

From the conditions that the outer and inner boundaries of the curved bar (Fig. 46) are free from external forces, we require that

$$\sigma_r = \tau_{r\theta} = 0 \text{ for } r = a \text{ and } r = b$$

or, from Eqs. (60),

$$\begin{aligned} 2Aa - \frac{2B}{a^3} + \frac{D}{a} &= 0 \\ 2Ab - \frac{2B}{b^3} + \frac{D}{b} &= 0 \end{aligned} \quad (e)$$

The last condition is that the sum of the shearing forces distributed over the upper end of the bar should equal the force P . Taking the width of the cross section as unity—or P as the load per unit thickness of the plate—we obtain for $\theta = 0$,

$$\begin{aligned} \int_a^b \tau_{r\theta} dr &= - \int_a^b \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) dr = \left[\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right]_b^a \\ &= \left[Ar^2 + \frac{B}{r^2} + C + D \log r \right]_b^a = P \end{aligned}$$

or

$$-A(b^2 - a^2) + B \frac{(b^2 - a^2)}{a^2 b^2} - D \log \frac{b}{a} = P \quad (f)$$

From Eqs. (e) and (f) we find

$$A = \frac{P}{2N}, \quad B = -\frac{Pa^2 b^2}{2N}, \quad D = -\frac{P}{N} (a^2 + b^2) \quad (g)$$

in which

$$N = a^2 - b^2 + (a^2 + b^2) \log \frac{b}{a}$$

Substituting the values (g) of the constants of integration in Eqs. (60), we obtain the expressions for the stress components. For the upper end of the bar, $\theta = 0$, we find

$$\begin{aligned} \sigma_\theta &= 0 \\ \tau_{r\theta} &= - \frac{P}{N} \left[r + \frac{a^2 b^2}{r^3} - \frac{1}{r} (a^2 + b^2) \right] \end{aligned} \quad (h)$$

For the lower end, $\theta = \pi/2$,

$$\begin{aligned} \tau_{r\theta} &= 0 \\ \sigma_\theta &= \frac{P}{N} \left[3r - \frac{a^2 b^2}{r^3} - (a^2 + b^2) \frac{1}{r} \right] \end{aligned} \quad (k)$$

The expressions (60) constitute an exact solution of the problem only when the forces at the ends of the curved bar are distributed in the

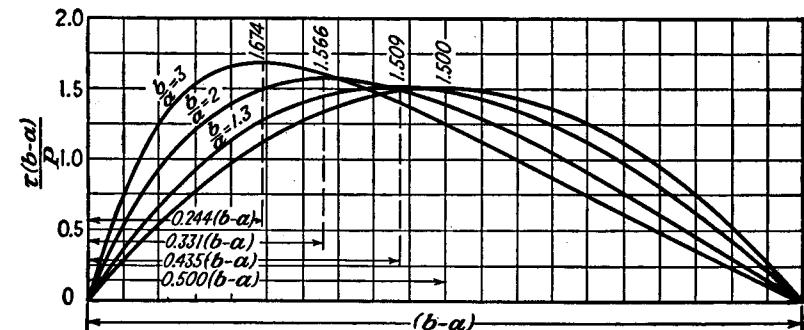


FIG. 47.

manner given by Eqs. (h) and (k). For any other distribution of forces the stress distribution near the ends will be different from that given by solution (60), but at larger distances this solution will be valid by Saint-Venant's principle. Calculations show that the simple theory, based on the assumption that cross sections remain plane during bending, again gives very satisfactory results.

In Fig. 47 the distribution of the shearing stress $\tau_{r\theta}$ over the cross section $\theta = 0$ (for the cases $b = 3a$, $2a$, and $1.3a$) is shown. The abscissas are the radial distances from the inner boundary ($r = a$). The ordinates represent numerical factors with which we multiply the average shearing stress $P/(b - a)$ to get the shearing stress at the

point in question. A value 1.5 for this factor gives the maximum shearing stress as calculated from the parabolic distribution for rectangular straight beams. From the figures it may be seen that the distribution of shearing stresses approaches the parabolic distribution when the depth of the cross section is small. For such proportions as are usual in arches and vaults the parabolic distribution of shearing stress, as in straight rectangular bars, can be assumed with sufficient accuracy.

Let us consider now the displacements produced by the force P (Fig. 46). By using Eqs. (49) to (52), and substituting for the stress components the expressions (60), we find

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\sin \theta}{E} \left[2Ar(1 - 3\nu) - \frac{2B}{r^3}(1 + \nu) + \frac{D}{r}(1 - \nu) \right] \\ \frac{\partial v}{\partial \theta} &= r\epsilon_\theta - u \\ \gamma_{r\theta} &= \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\end{aligned}\quad (l)$$

From the first of these equations we obtain by integration

$$u = \frac{\sin \theta}{E} \left[Ar^2(1 - 3\nu) + \frac{B}{r^2}(1 + \nu) + D(1 - \nu) \log r \right] + f(\theta) \quad (m)$$

where $f(\theta)$ is a function of θ only. Substituting (m) in the second of Eqs. (l) together with the expression for ϵ_θ and integrating, we find

$$\begin{aligned}v &= -\frac{\cos \theta}{E} \left[Ar^2(5 + \nu) + \frac{B}{r^2}(1 + \nu) - D \log r(1 - \nu) \right. \\ &\quad \left. + D(1 - \nu) \right] - \int f(\theta) d\theta + F(r) \quad (n)\end{aligned}$$

in which $F(r)$ is a function of r only. Substituting now (m) and (n) in the third of Eqs. (l) we arrive at the equation

$$\int f(\theta) d\theta + f'(\theta) + rF'(r) - F(r) = -\frac{4D \cos \theta}{E}$$

This equation is satisfied by putting

$$F(r) = Hr, \quad f(\theta) = -\frac{2D}{E} \theta \cos \theta + K \sin \theta + L \cos \theta \quad (p)$$

in which H , K , and L are arbitrary constants, to be determined from the conditions of constraint. The components of displacements, from

(m) and (n), are then

$$\begin{aligned}u &= -\frac{2D}{E} \theta \cos \theta + \frac{\sin \theta}{E} \left[D(1 - \nu) \log r + A(1 - 3\nu)r^2 \right. \\ &\quad \left. + \frac{B(1 + \nu)}{r^2} \right] + K \sin \theta + L \cos \theta \\ v &= \frac{2D}{E} \theta \sin \theta - \frac{\cos \theta}{E} \left[A(5 + \nu)r^2 + \frac{B(1 + \nu)}{r^2} \right. \\ &\quad \left. - D(1 - \nu) \log r \right] + \frac{D(1 + \nu)}{E} \cos \theta + K \cos \theta - L \sin \theta + Hr\end{aligned}\quad (q)$$

The radial deflection of the upper end of the bar is obtained by putting $\theta = 0$ in the expression for u , which gives

$$(u)_{\theta=0} = L \quad (r)$$

The constant L is obtained from the condition at the built-in end (Fig. 46). For $\theta = \pi/2$ we have $v = 0$; $\partial v / \partial r = 0$, hence, from the second of Eqs. (q),

$$H = 0, \quad L = \frac{D\pi}{E} \quad (s)$$

The deflection of the upper end is, therefore, using (g),

$$(u)_{\theta=0} = \frac{D\pi}{E} = -\frac{P\pi(a^2 + b^2)}{E \left[(a^2 - b^2) + (a^2 + b^2) \log \frac{b}{a} \right]} \quad (61)$$

The application of this formula will be given later. When b approaches a , and the depth of the curved bar, $h = b - a$, is small in comparison with a , we can use the expression

$$\log \frac{b}{a} = \log \left(1 + \frac{h}{a} \right) = \frac{h}{a} - \frac{1}{2} \frac{h^2}{a^2} + \frac{1}{3} \frac{h^3}{a^3} - \dots$$

Substituting in (61) and neglecting small terms of higher order, we obtain

$$(u)_{\theta=0} = -\frac{3\pi a^3 P}{E h^3}$$

which coincides with the elementary formula for this case.¹

By taking the stress function in the form

$$\phi = f(r) \cos \theta$$

¹ See S. Timoshenko, "Strength of Materials," vol. 2, Art. 13, 1941.

and proceeding as above, we get a solution for the case when a vertical force and a couple are applied to the upper end of the bar (Fig. 46). Subtracting from this solution the stresses produced by the couple (see Art. 27), the stresses due to a vertical force applied at the upper end of the bar remain. Having the solutions for a horizontal and for a vertical load, the solution for any inclined force can be obtained by superposition.

In the above discussion it was always assumed that Eqs. (e) are satisfied and that the circular boundaries of the bar are free from forces. By taking the expressions in (e) different from zero, we obtain the case when normal and tangential forces proportional to $\sin \theta$ and $\cos \theta$ are distributed over circular boundaries of the bar. Combining such solutions with the solutions previously obtained for pure bending and for bending by a force applied at the end we can approach the condition of loading of a vault covered with sand or soil.¹

32. The Effect of Circular Holes on Stress Distributions in Plates.

Figure 48 represents a plate submitted to a uniform tension of magnitude S in the x -direction. If a small circular hole is made in the middle of the plate, the stress distribution in the neighborhood of the hole will be changed, but we can conclude from Saint-Venant's principle that the change is negligible at distances which are large compared with a , the radius of the hole.

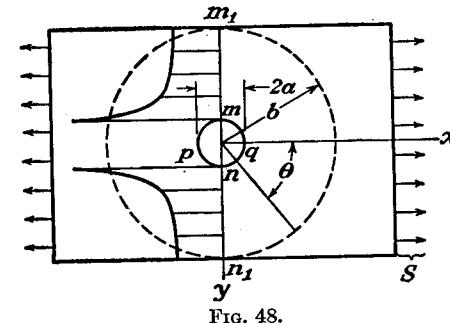


FIG. 48.

tude S in the x -direction. If a small circular hole is made in the middle of the plate, the stress distribution in the neighborhood of the hole will be changed, but we can conclude from Saint-Venant's principle that the change is negligible at distances which are large compared with a , the radius of the hole.

Consider the portion of the plate within a concentric circle of radius b , large in comparison with a . The stresses at the radius b are effectively the same as in the plate without the hole and are therefore given by

$$\begin{aligned} (\sigma_r)_{r=b} &= S \cos^2 \theta = \frac{1}{2}S(1 + \cos 2\theta) \\ (\tau_{r\theta})_{r=b} &= -\frac{1}{2}S \sin 2\theta \end{aligned} \quad (a)$$

¹ Several examples of this kind were discussed by Golovin, *loc. cit.*, and Ribière, *loc. cit.*, and *Compt. rend.*, vol. 132, p. 315, 1901.

These forces, acting around the outside of the ring having the inner and outer radii $r = a$ and $r = b$, give a stress distribution within the ring which we may regard as consisting of two parts. The first is due to the constant component $\frac{1}{2}S$ of the normal forces. The stresses it produces can be calculated by means of Eqs. (45) (page 59). The remaining part, consisting of the normal forces $\frac{1}{2}S \cos 2\theta$, together with the shearing forces $-\frac{1}{2}S \sin 2\theta$, produces stresses which may be derived from a stress function of the form

$$\phi = f(r) \cos 2\theta \quad (b)$$

Substituting this into the compatibility equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) = 0$$

we find the following ordinary differential equation to determine $f(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{4}{r^2} \right) \left(\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{4f}{r^2} \right) = 0$$

The general solution is

$$f(r) = Ar^2 + Br^4 + C \frac{1}{r^2} + D$$

The stress function is therefore

$$\phi = \left(Ar^2 + Br^4 + C \frac{1}{r^2} + D \right) \cos 2\theta \quad (c)$$

and the corresponding stress components, from Eqs. (38), are

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = - \left(2A + \frac{6C}{r^4} + \frac{4D}{r^2} \right) \cos 2\theta \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4} \right) \cos 2\theta \\ \tau_{r\theta} &= - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2} \right) \sin 2\theta \end{aligned} \quad (d)$$

The constants of integration are now to be determined from conditions (a) for the outer boundary and from the condition that the edge of the hole is free from external forces. These conditions give

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{1}{2}S$$

$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$

$$2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = -\frac{1}{2}S$$

$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

Solving these equations and putting $a/b = 0$, i.e., assuming an infinitely large plate, we obtain

$$A = -\frac{S}{4}, \quad B = 0, \quad C = -\frac{a^4}{4}S, \quad D = \frac{a^2}{2}S$$

Substituting these values of constants into Eqs. (d) and adding the stresses produced by the uniform tension $\frac{1}{2}S$ on the outer boundary calculated from Eqs. (45) we find¹

$$\begin{aligned}\sigma_r &= \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta \\ \sigma_\theta &= \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} &= -\frac{S}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta\end{aligned}\quad (62)$$

If r is very large, σ_r and $\tau_{r\theta}$ approach the values given in Eqs. (a). At the edge of the hole, $r = a$ and we find

$$\sigma_r = \tau_{r\theta} = 0, \quad \sigma_\theta = S - 2S \cos 2\theta$$

It can be seen that σ_θ is greatest when $\theta = \pi/2$ or $\theta = 3\pi/2$, i.e., at the ends m and n of the diameter perpendicular to the direction of the tension (Fig. 48). At these points $(\sigma_\theta)_{\max.} = 3S$. This is the maximum tensile stress and is three times the uniform stress S , applied at the ends of the plate.

At the points p and q , θ is equal to π and 0 and we find

$$\sigma_\theta = -S$$

so that there is a compression stress in the tangential direction at these points.

¹ This solution was obtained by Prof. G. Kirsch; see *V.D.I.*, vol. 42, 1898. It has been well confirmed many times by strain measurements and by the photo-elastic method (see Chap. 5 and the books cited on p. 131).

For the cross section of the plate through the center of the hole and perpendicular to the x -axis, $\theta = \pi/2$, and, from Eqs. (62),

$$\tau_{r\theta} = 0, \quad \sigma_\theta = \frac{S}{2}\left(2 + \frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)$$

It is evident that the effect of the hole is of a very *localized character*, and, as r increases, the stress σ_θ approaches the value S very rapidly. The distribution of this stress is shown in the figure by the shaded area. The localized character of the stresses around the hole justifies the application of the solution (e), derived for an infinitely large plate, to a plate of finite width. If the width of the plate is not less than four diameters of the hole the error of the solution (62) in calculating $(\sigma_\theta)_{\max.}$ does not exceed 6 per cent.¹

Having the solution (d) for tension or compression in one direction, the solution for tension or compression in two perpendicular directions can be easily obtained by superposition. By taking, for instance, tensile stresses in two perpendicular directions equal to S , we find at the boundary of the hole a tensile stress $\sigma_\theta = 2S$ (see page 72). By taking a tensile stress S in the x -direction (Fig. 49) and a compressive stress $-S$ in the y -direction, we obtain the case of pure shear. The tangential stresses at the boundary of the hole are, from Eqs. (62),

$$\sigma_\theta = S - 2S \cos 2\theta - [S - 2S \cos (2\theta - \pi)]$$

For $\theta = \pi/2$ or $\theta = 3\pi/2$, i.e., at the points n and m , we find $\sigma_\theta = 4S$. For $\theta = 0$ or $\theta = \pi$, i.e., at n_1 and m_1 , $\sigma_\theta = -4S$. Hence, for a large plate under pure shear, the maximum tangential stress at the boundary of the hole is four times larger than the applied pure shear.

The high *stress concentration* found at the edge of a hole is of great practical importance. As an example, holes in ships' decks may be mentioned. When the hull of a ship is bent, tension or compression is produced in the decks and there is a high stress concentration at the holes. Under the cycles of stress produced by waves, fatigue of the metal at the overstressed portions may result finally in fatigue cracks.²

¹ See S. Timoshenko, *Bull. Polytech. Inst., Kiew*, 1907. We must take S equal to the load divided by the gross area of the plate.

² See paper by T. L. Wilson, *The S.S. Leviathan, Damage, Repairs and Strength Analysis*, presented at a meeting of the American Society of Naval Architects and Marine Engineers, November, 1930.

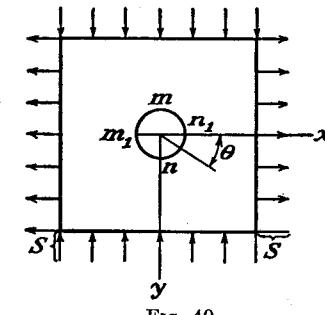


FIG. 49.

It is often necessary to reduce the stress concentration at holes, such as access holes in airplane wings and fuselages. This can be done by adding a bead¹ or reinforcing ring.² The analytical problem has been solved by extending the method employed for the hole, and the results have been compared with strain-gauge measurements.²

The case of a circular hole near the straight boundary of a semi-infinite plate under tension parallel to this boundary (Fig. 50) was analyzed by G. B. Jeffery.³ A corrected result and a comparison with photoelastic tests (see Chap. 5) were given later by R. D. Mindlin.⁴

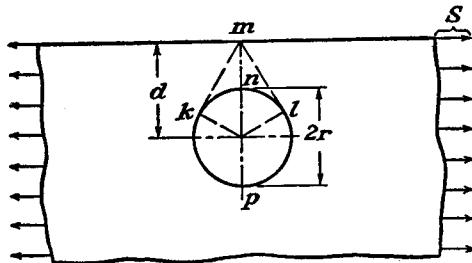


FIG. 50.

The stress at the hole, at the point n nearest the edge, becomes a very large multiple of the undisturbed tensile stress when mn is small compared with np .

G. B. Jeffery also investigated the case of a uniform normal pressure p_i acting on the boundary of the hole. This is a problem of practical interest. It gives the stresses near a rivet hole while the hot plastic rivet is being forced home under pressure. If the hole is very far from the straight edge the stresses at the boundary of the hole, from Eqs. (46) (page 60), are

$$\sigma_\theta = p_i, \quad \sigma_r = -p_i$$

If the hole is near the straight edge, the tangential stresses are no longer constant along the boundary of the hole. The maximum tangential stress is at the points k and l and is given by the formula

$$(\sigma_\theta)_{\max.} = p_i \frac{d^2 + r^2}{d^2 - r^2} \quad (63)$$

¹ See S. Timoshenko, *J. Franklin Inst.*, vol. 197, p. 505, 1924; also S. Timoshenko, "Strength of Materials," 2d ed., vol. 2, p. 317.

² S. Levy, A. E. McPherson, and F. C. Smith, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 15, p. 160, 1948. References to prior work may be found in this paper.

³ *Trans. Roy. Soc. (London)*, series A, vol. 221, p. 265, 1921.

⁴ *Proc. Soc. Expl. Stress Analysis*, vol. 5, p. 56, 1948.

This stress should be compared with the tensile stress at the point m on the straight edge of the plate, given by the formula

$$\sigma_\theta = \frac{4p_i r^2}{d^2 - r^2} \quad (64)$$

For $d = r\sqrt{3}$, the two expressions have the same magnitude. If d is greater than this the maximum stress is at the circular boundary, and if it is less, the maximum stress is at the point m .

The case of a plate of finite width with a circular hole on the axis of symmetry (Fig. 51) was discussed by R. C. J. Howland.¹ He found, for instance, that when $2r = \frac{1}{2}d$, $\sigma_\theta = 4.3S$ at the point n and

$$\sigma_\theta = 0.75S$$

at the point m .

The method used in this article for analyzing stresses round a small circular hole can be applied when the plate is subjected to pure bending.² The cases of a row of circular holes in an infinite plate,^{3,4,5} a row of holes in a strip,^{5,6} and in a semi-infinite plate,⁵ and a ring of holes in a plate⁷ (under all-round tension) have also been investigated. A method devised by Hengst has been applied to the case of a hole in a square plate⁸ under equal tension in both directions, and under shear⁹ when the hole is plain or reinforced.

Solutions have been obtained for the infinite plate with a circular hole when forces are applied to the boundary of the hole,¹⁰ for the corre-

¹ R. C. J. Howland, *Trans. Roy. Soc. (London)*, Series A, vol. 229, p. 49, 1930.

² Z. Tuzi, *Phil. Mag.*, February, 1930, p. 210; also *Sci. Papers Inst. Phys. Chem. Research (Tokyo)*, vol. 9, p. 65, 1928. The corresponding problem for an elliptical hole was solved earlier by K. Wolf, *Z. tech. Physik*, 1922; p. 160. The circular hole in a strip is discussed by R. C. J. Howland and A. C. Stevenson, *Trans. Roy. Soc. (London)*, series A, vol. 232, p. 155, 1933. A proof of convergence of the series solutions is given by R. C. Knight, *Quart. J. Math., Oxford series*, vol. 5, p. 255, 1934.

³ M. Sadowsky, *Z. angew. Math. Mech.*, vol. 8, p. 107, 1928.

⁴ R. C. J. Howland, *Proc. Roy. Soc. (London)*, series A, vol. 148, p. 471, 1935.

⁵ K. J. Schulz, *Proc. Nederl. Akad. van Wetenschappen*, vol. 45, pp. 233, 341, 457, and 524, 1942, vol. 48, pp. 282 and 292, 1945.

⁶ C. B. Ling, P. S. Wang, and K. S. Jing, *Tech. Rept. No. 9, Bur. Aeronaut. Research*, Chengtu, China, Jan. 1944.

⁷ C. B. Ling and P. S. Wang, *Tech. Rept. No. 6, ibid.*, June, 1943.

⁸ H. Hengst, *Z. angew. Math. Mech.*, vol. 18, p. 44, 1938.

⁹ C. K. Wang, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 13, p. A-77, 1946.

¹⁰ W. G. Bickley, *Trans. Roy. Soc. (London)*, series A, vol. 227, p. 383, 1928.

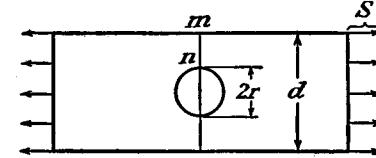


FIG. 51.

sponding problem of the strip,¹ and for a row of holes parallel and near to the straight edge of a semi-infinite plate² (row of rivet holes).

If an elliptical hole is made in an infinite plate under tension S , with one of the principal axes parallel to the tension, the stresses at the ends of the axis of the hole perpendicular to the direction of the tension are

$$\sigma = S \left(1 + 2 \frac{a}{b} \right) \quad (65)$$

where $2a$ is the axis of the ellipse perpendicular to the tension, and $2b$ is the other axis. This and other problems concerning ellipses, hyperbolas, and two circles are discussed in Chap. 7, where references will be found.

A very slender hole (a/b large) perpendicular to the direction of the tension causes a very high stress concentration.³ This explains why cracks transverse to applied forces tend to spread. The spreading can be stopped by drilling holes at the ends of the crack to eliminate the sharp curvature responsible for the high stress concentration.

When a hole is filled with material which is rigid or has elastic constants different from those of the plate (plane stress) or body (plane strain) itself, we have the problem of the rigid or elastic *inclusion*. This has been solved for circular⁴ and elliptic inclusions.⁵ The results for the rigid circular inclusion have been confirmed by the photoelastic method⁶ (see Chap. 5).

The stresses given by Eqs. (62) for the problem indicated by Fig. 48 are the same for plane strain as for plane stress. In plane strain, however, the axial stress

$$\sigma_z = v(\sigma_r + \sigma_\theta)$$

must act on the plane ends, which are parallel to the xy -plane, in order to make ϵ_z zero. Removal of these stresses from the ends, to arrive at the condition of free ends, will produce further stress which will not be of a two-dimensional (plane stress or plane strain) character. If

¹ R. C. Knight, *Phil. Mag.*, series 7, vol. 19, p. 517, 1935.

² C. B. Ling and M. C. Hsu, *Tech. Rept. No. 16, Bur. Aeronaut. Research*, Chengtu, China, February, 1945.

³ The problem of a narrow slot was discussed by M. Sadowsky, *Z. angew. Math. Mech.*, vol. 10, p. 77, 1930.

⁴ K. Sezawa and G. Nishimura, *Rept. Aeronaut. Research Inst., Tokyo Imp. Univ.*, vol. 6, no. 25, 1931; J. N. Goodier, *Trans. A.S.M.E.*, vol. 55, p. 39 (1933).

⁵ L. H. Donnell, "Theodore von Kármán Anniversary Volume," p. 293, Pasadena, 1941.

⁶ W. E. Thibodeau and L. A. Wood, *J. Research Natl. Bur. Standards*, vol. 20, p. 393, 1938.

the hole is small in diameter compared with the thickness between the ends, the disturbance will be confined to the neighborhood of the ends. But if the diameter and the thickness are of the same order of magnitude, the problem must be treated as essentially three-dimensional throughout. Investigations of this kind¹ have shown that σ_θ remains the largest stress component and its value is very close to that given by the two-dimensional theory.

33. Concentrated Force at a Point of a Straight Boundary. Let us consider now a concentrated vertical force P acting on a horizontal

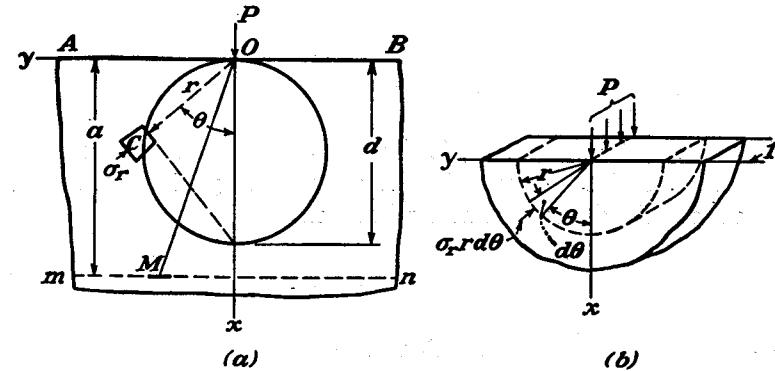


FIG. 52.

straight boundary AB of an infinitely large plate (Fig. 52a). The distribution of the load along the thickness of the plate is uniform, as indicated in Fig. 52b. The thickness of the plate is taken as unity so that P is the load per unit thickness.

The distribution of stress in this case is a very simple one² and is called a *simple radial distribution*. Any element C at a distance r from the point of application of the load is subjected to a simple compression in the radial direction, the radial stress being

$$\sigma_r = -\frac{2P \cos \theta}{\pi r} \quad (66)$$

¹ A. E. Green, *Trans. Roy. Soc. (London)*, series A, vol. 193, p. 229, 1948; E. Sternberg and M. Sadowsky, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 16, p. 27, 1949.

² The solution of this problem was obtained by way of the three-dimensional solution of J. Boussinesq (p. 362) by Flamant, *Compt. rend.*, vol. 114, p. 1465, 1892, Paris. The extension of the solution to the case of an inclined force is due to Boussinesq, *Compt. rend.*, vol. 114, p. 1510, 1892. See also the paper by J. H. Michell, *Proc. London Math. Soc.*, vol. 32, p. 35, 1900. The experimental investigation of stress distribution, which suggested the above theoretical work, was done by Carus Wilson, *Phil. Mag.*, vol. 32, p. 481, 1891.

The tangential stress σ_θ and the shearing stress $\tau_{r\theta}$ are zero. It is easy to see that these values of the stress components satisfy the equations of equilibrium (37) (page 56).

The boundary conditions are also satisfied because σ_r and $\tau_{r\theta}$ are zero along the straight edge of the plate, which is free from external forces except at the point of application of the load ($r = 0$). Here σ_r becomes infinite. The resultant of the forces acting on a cylindrical surface of radius r (Fig. 52b) must balance P . It is obtained by summing the vertical components $\sigma_r r d\theta \cos \theta$ acting on each element $r d\theta$ of the surface. In this manner we find

$$2 \int_0^{\frac{\pi}{2}} \sigma_r \cos \theta \cdot r d\theta = - \frac{4P}{\pi} \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = -P$$

To prove that solution (66) is the exact solution of the problem we must consider also the equation of compatibility (39). The above solution is derived from the stress function

$$\phi = - \frac{P}{\pi} r \theta \sin \theta \quad (a)$$

We can verify this by using Eqs. (38) as follows:

$$\begin{aligned} \sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = - \frac{2P}{\pi} \frac{\cos \theta}{r} \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} = 0 \\ \tau_{r\theta} &= - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = 0 \end{aligned} \quad (66')$$

which coincides with solution (66). Substituting the function (a) into Eq. (39), we can easily show that this equation is satisfied. Hence, (a) represents the true stress function and Eqs. (66') give the true stress distribution.

Taking a circle of any diameter d with center on the x -axis and tangent to the y -axis at O (Fig. 52a), we have, for any point C of the circle, $d \cos \theta = r$. Hence, from Eq. (66),

$$\sigma_r = - \frac{2P}{\pi d}$$

i.e., the stress is the same at all points on the circle, except the point O , the point of application of the load.

Taking a horizontal plane mn at a distance a from the straight edge of the plate, the normal and shearing components of the stress on this

plane at any point M (Fig. 52a) are calculated from the simple compression in the radial direction,

$$\begin{aligned} \sigma_x &= \sigma_r \cos^2 \theta = - \frac{2P}{\pi} \frac{\cos^2 \theta}{r} = - \frac{2P}{\pi a} \cos^4 \theta \\ \sigma_y &= \sigma_r \sin^2 \theta = - \frac{2P}{\pi a} \sin^2 \theta \cos^2 \theta \\ \tau_{xy} &= \sigma_r \sin \theta \cos \theta = - \frac{2P}{\pi} \frac{\sin \theta \cos^2 \theta}{r} = - \frac{2P}{\pi a} \sin \theta \cos^3 \theta \end{aligned} \quad (67)$$

In Fig. 53 the distribution of stresses σ_x and τ_{xy} along the horizontal plane mn is represented graphically.

At the point of application of the load the stress is theoretically infinitely large because a finite force is acting on an infinitely small

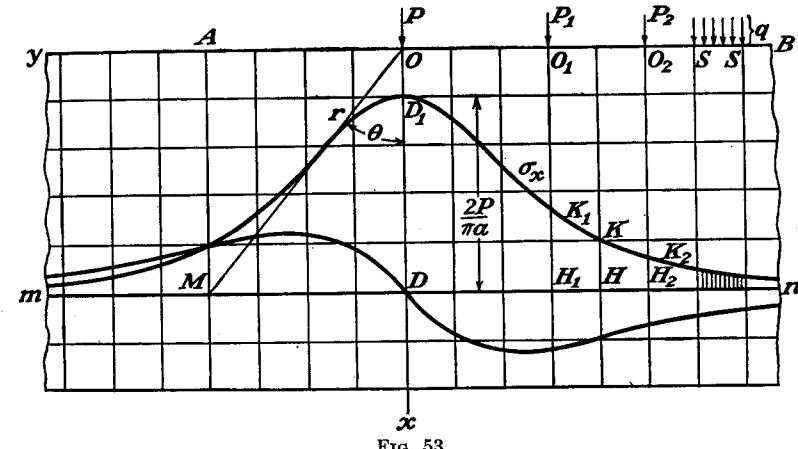


FIG. 53.

area. In practice, at the point of application there is always a certain yielding of material and as a result of this the load will become distributed over a finite area. Imagine that the portion of material which suffered a plastic flow is cut out from the plate by a circular cylindrical surface of small radius as shown in Fig. 52b. Then the equations of elasticity can be applied to the remaining portion of the plate.

An analogous solution can be obtained for a horizontal force P applied to the straight boundary of the semi-infinite plate (Fig. 54). The stress components for this case are obtained from the same Eqs. (66'); it is only necessary to measure the angle θ from the direction of the force, as shown in the figure. By calculating the resultant of the forces acting on a cylindrical surface, shown in Fig. 54 by the dotted

line, we find

$$-\frac{2P}{\pi} \int_0^\pi \cos^2 \theta d\theta = -P$$

This resultant balances the external force P , and, as the stress components $\tau_{r\theta}$ and σ_θ at the straight edge are zero, solution (66') satisfies the boundary conditions.

Having the solutions for vertical and horizontal concentrated forces, solutions for inclined forces are obtained by superposition. Resolving the inclined force P into two components, $P \cos \alpha$ vertically and $P \sin \alpha$

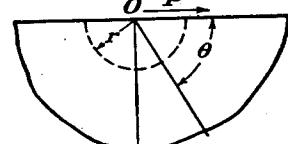


FIG. 54.

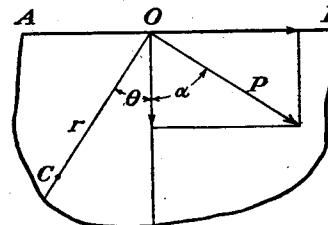


FIG. 55.

α horizontally (Fig. 55), the radial stress at any point C is, from Eqs. (66'),

$$\sigma_r = -\frac{2}{\pi r} \left[P \cos \alpha \cos \theta + P \sin \alpha \cos \left(\frac{\pi}{2} + \theta \right) \right] = -\frac{2P}{\pi r} \cos(\alpha + \theta) \quad (68)$$

Hence Eqs. (66') can be used for any direction of the force, provided in each case we measure the angle θ from the direction of the force.

The stress function (a) may be used also in the case when a couple is acting on the straight boundary of an infinite plate (Fig. 56a). It is easy to see that the stress function for the case when the tensile force P is at the point O_1 , at a distance a from the origin, is obtained from ϕ , Eq. (a), regarded for the moment as a function of x and y instead of r and θ , by writing $y + a$ instead of y and also $-P$ instead of P . This and the original stress function ϕ can be combined, and we then obtain the stress function for the two equal and opposite forces applied at O and O_1 , in the form

$$-\phi(x, y + a) + \phi(x, y)$$

When a is very small, this approaches the value

$$\phi_1 = -a \frac{\partial \phi}{\partial y} \quad (b)$$

Substituting (a) in Eq. (b), and noting (see page 57) that

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \sin \theta + \frac{\partial \phi}{\partial \theta} \frac{\cos \theta}{r}$$

we find

$$\phi_1 = \frac{Pa}{\pi} (\theta + \sin \theta \cos \theta) = \frac{M}{\pi} (\theta + \sin \theta \cos \theta) \quad (69)$$

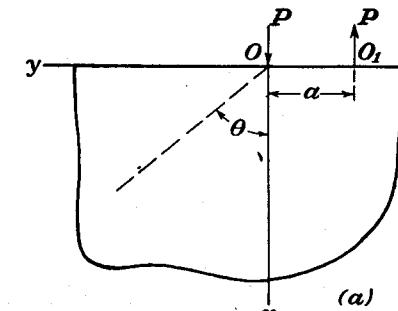
in which M is the moment of the applied couple.

Reasoning in the same manner, we find that by differentiation of ϕ_1 , we obtain the stress function ϕ_2 for the case when two equal and opposite couples M are acting at two points O and O_1 a very small distance apart (Fig. 56b). We thus find that

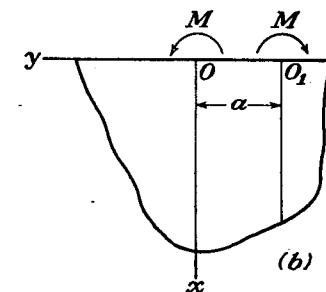
$$\begin{aligned} \phi_2 &= \phi_1 - \left(\phi_1 + \frac{\partial \phi_1}{\partial y} a \right) = -a \frac{\partial \phi_1}{\partial y} \\ &= -\frac{2Ma}{\pi r} \cos^3 \theta \end{aligned} \quad (70)$$

If the directions of the couples are changed it is only necessary to change the sign of the function (70).

A series of stress functions obtained by successive differentiation has been employed to solve the problem of stress concentration due to a semicircular notch in a semi-infinite plate in tension parallel to the edge.¹ The maximum tensile stress is slightly greater than three times the undisturbed tensile stress away from the notch. The strip with a semicircular notch in each edge has also been investigated.² The stress-concentration factor (ratio of maximum to mean stress at minimum section) falls below three, approaching unity as the notches are made larger.



(a)



(b)

Having the distribution of stresses, the corresponding displacements can be obtained in the usual way by applying Eqs. (49) to (51). For a force normal to the straight boundary (Fig. 52) we have

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r} = -\frac{2P}{\pi E} \frac{\cos \theta}{r} \\ \epsilon_\theta &= \frac{u}{r} + \frac{\partial v}{r \partial \theta} = \nu \frac{2P}{\pi E} \frac{\cos \theta}{r} \\ \gamma_{r\theta} &= r \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0 \end{aligned} \quad (c)$$

¹ F. G. Maunsell, *Phil. Mag.*, vol. 21, p. 765, 1936.

² C. B. Ling, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 14, p. A-275, 1947; H. Poritsky, H. D. Snively, and C. R. Wylie, *ibid.*, vol. 6, p. A-63, 1939.

Integrating the first of these equations, we find

$$u = -\frac{2P}{\pi E} \cos \theta \log r + f(\theta) \quad (d)$$

where $f(\theta)$ is a function of θ only. Substituting in the second of Eqs. (c) and integrating it, we obtain

$$v = \frac{2\nu P}{\pi E} \sin \theta + \frac{2P}{\pi E} \log r \sin \theta - \int f(\theta) d\theta + F(r) \quad (e)$$

in which $F(r)$ is a function of r only. Substituting (d) and (e) in the third of Eqs. (c), we conclude that

$$f(\theta) = -\frac{(1-\nu)P}{\pi E} \theta \sin \theta + A \sin \theta + B \cos \theta, \quad F(r) = Cr \quad (f)$$

where A , B , and C are constants of integration which are to be determined from the conditions of constraint. The expressions for the displacements, from Eqs. (d) and (e), are

$$\begin{aligned} u &= -\frac{2P}{\pi E} \cos \theta \log r - \frac{(1-\nu)P}{\pi E} \theta \sin \theta + A \sin \theta + B \cos \theta \\ v &= \frac{2\nu P}{\pi E} \sin \theta + \frac{2P}{\pi E} \log r \sin \theta - \frac{(1-\nu)P}{\pi E} \theta \cos \theta \\ &\quad + \frac{(1-\nu)P}{\pi E} \sin \theta + A \cos \theta - B \sin \theta + Cr \end{aligned} \quad (g)$$

Assume that the constraint of the semi-infinite plate (Fig. 52) is such that the points on the x -axis have no lateral displacement. Then $v = 0$, for $\theta = 0$, and we find from the second of Eqs. (g) that $A = 0$, $C = 0$. With these values of the constants of integration the vertical displacements of points on the x -axis are

$$(u)_{\theta=0} = -\frac{2P}{\pi E} \log r + B \quad (h)$$

To find the constant B let us assume that a point of the x -axis at a distance d from the origin does not move vertically. Then from Eq. (h) we find

$$B = \frac{2P}{\pi E} \log d$$

Having the values of all the constants of integration, the displacements of any point of the semi-infinite plate can be calculated from Eqs. (g).

Let us consider, for instance, the displacements of points on the

straight boundary of the plate. The horizontal displacements are obtained by putting $\theta = \pm\pi/2$ in the first of Eqs. (g). We find

$$(u)_{\theta=\frac{\pi}{2}} = -\frac{(1-\nu)P}{2E}, \quad (u)_{\theta=-\frac{\pi}{2}} = -\frac{(1-\nu)P}{2E} \quad (71)$$

The straight boundary on each side of the origin thus has a constant displacement (71), at all points, directed toward the origin. We may regard such a displacement as a physical possibility, if we remember that around the point of application of the load P we removed the portion of material bounded by a cylindrical surface of a small radius (Fig. 52b) within which portion the equations of elasticity do not hold. Actually of course this material is plastically deformed and permits displacement (71) along the straight boundary. The vertical displacements on the straight boundary are obtained from the second of Eqs. (g). Remembering that ν is positive if the displacement is in the direction of increasing θ , and that the deformation is symmetrical with respect to the x -axis, we find for the vertical displacements in the downward direction at a distance r from the origin

$$(v)_{\theta=-\frac{\pi}{2}} = -(v)_{\theta=\frac{\pi}{2}} = \frac{2P}{\pi E} \log \frac{d}{r} - \frac{(1+\nu)P}{\pi E} \quad (72)$$

At the origin this equation gives an infinitely large displacement. To remove this difficulty we must assume as before that a portion of material around the point of application of the load is cut out by a cylindrical surface of small radius. For other points of the boundary, Eq. (72) gives finite displacements.

34. Any Vertical Loading of a Straight Boundary. The curves for σ_x and τ_{xy} of the preceding article (Fig. 53) can be used as *influence lines*. We assume that these curves represent the stresses for P equal to a unit force, say 1 lb. Then for any other value of the force P the stress σ_x at any point H of the plane mn is obtained by multiplying the ordinate \overline{HK} by P .

If several vertical forces P , P_1 , P_2 , . . . , act on the horizontal straight boundary AB of the semi-infinite plate, the stresses on the horizontal plane mn are obtained by superposing the stresses produced by each of these forces. For each of them, the σ_x and τ_{xy} curves are obtained by shifting the σ_x and τ_{xy} curves, constructed for P , to the new origins O_1 , O_2 , From this it follows that the stress σ_x produced, for instance, by the force P_1 on the plane mn at the point D is obtained by multiplying the ordinate $\overline{H_1K_1}$ by P_1 . In the same manner the σ_x stress at D produced by P_2 is $\overline{H_2K_2} \cdot P_2$, and so on. The total normal

stress at D on the plane mn produced by P, P_1, P_2, \dots is

$$\sigma_x = \overline{DD_1} \cdot P + \overline{H_1K_1} \cdot P_1 + \overline{H_2K_2} \cdot P_2 + \dots$$

Hence the σ_x curve shown in Fig. 53 is the *influence line* for the normal stress σ_x at the point D . In the same manner we conclude that the τ_{xy} curve is the influence line for the shearing stress on the plane mn at the point D .

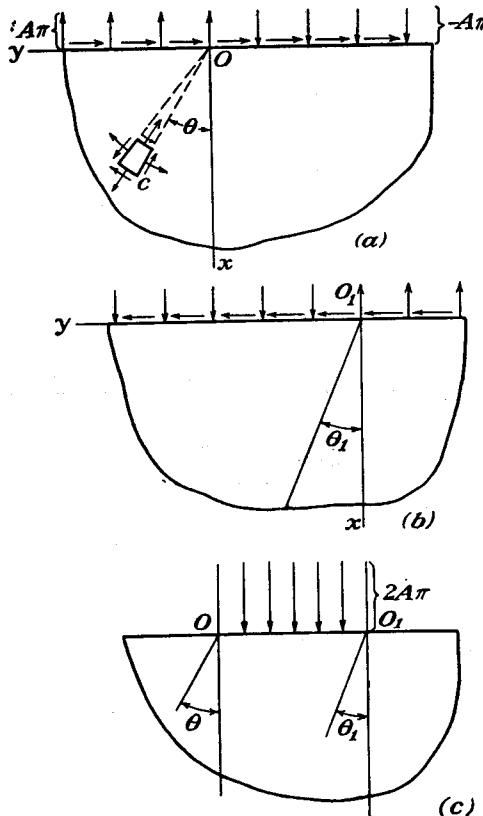


FIG. 57.

Having these curves, the stress components at D for any kind of vertical loading of the edge AB of the plate can easily be obtained.

If, instead of concentrated forces, we have a uniform load of intensity q , distributed over a portion ss of the straight boundary (Fig. 53), the normal stress σ_x produced by this load at the point D is obtained by multiplying by q the corresponding *influence area* shaded in the figure.

The problem of the uniformly distributed load can be solved in

another manner by means of a stress function in the form

$$\phi = Ar^2\theta \quad (a)$$

in which A is a constant. The corresponding stress components are

$$\begin{aligned}\sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 2A\theta \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} = 2A\theta \\ \tau_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = -A\end{aligned} \quad (b)$$

Applying this to the semi-infinite plate we arrive at the load distribution shown in Fig. 57a. On the straight edge of the plate there acts a uniformly distributed shearing force of intensity $-A$ and a uniformly distributed normal load of the intensity $A\pi$, abruptly changing sign at the origin O . The directions of the forces follow from the positive directions of the stress components acting on an element C .

By shifting the origin to O_1 and changing the sign of stress function ϕ , we arrive at the load distribution shown in Fig. 57b. Superposing the two cases of load distribution (Figs. 57a and 57b), we obtain the case of uniform loading of a portion of the straight boundary of the semi-infinite plate shown in Fig. 57c. To obtain the given intensity q of uniform load, we take

$$2A\pi = q, \quad A = \frac{1}{2\pi} q$$

The stress at any point of the plate is then given by the stress function¹

$$\phi = A(r^2\theta - r_1^2\theta_1) = \frac{q}{2\pi} (r^2\theta - r_1^2\theta_1) \quad (c)$$

From Eqs. (b) we see that the first term of the stress function (c) gives, at any point M of the plate (Fig. 58a), a uniform tension in all directions in the plane of the plate equal to $2A\theta$ and a pure shear $-A$. In the same manner the second term of the stress function gives a uniform compression $-2A\theta_1$ and a pure shear A . The uniform tension and compression can be simply added together and we find a uniform compressive stress

$$p = 2A\theta - 2A\theta_1 = 2A(\theta - \theta_1) = -2A\alpha \quad (d)$$

in which α is the angle between the radii r and r_1 .

In superposing the two kinds of pure shear, one corresponding to the direction r and the other to the direction r_1 , we shall use Mohr's circle (Fig. 58b), which in this case has a radius equal to the numerical value of the pure shears A . By taking

¹ This solution of the problem is due to J. H. Michell, *Proc. London Math. Soc.*, vol. 34, p. 134, 1902.

the two diameters, DD_1 parallel to r and FF_1 perpendicular to r , as the r - and σ -axes, we have a representation of the pure shear corresponding to the r -direction. The radii CF and CF_1 represent the principal stresses A and $-A$ making angles $\pi/4$ with r at the point M , corresponding to this pure shear, and the radius CD represents the shearing stress $-A$ on the plane mn perpendicular to r . For any plane inclined at an angle β to mn (Fig. 58a), the stress components are given by the coordinates σ and τ of the point G of the circle, with the angle GCD equal to 2β .

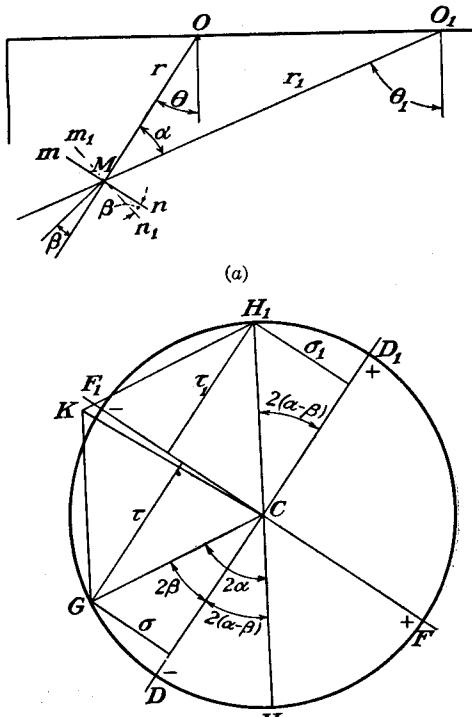


FIG. 58.

The same circle can be used also to get the stress components due to pure shear in the direction r_1 (see page 16). Considering again the plane m_1n_1 , and noting that the normal to this plane makes an angle $\alpha - \beta$ with the direction r_1 (Fig. 58a), it appears that the stress components are given by the coordinates of the point H of the circle. To take care of the sign of the pure shear corresponding to the r_1 -direction, we must change the signs of the stress components, and we obtain in this manner the point H_1 on the circle. The total stress acting on the plane m_1n_1 is given by the vector CK , the components of which give the normal stress $-(\sigma + \sigma_1)$ and the shearing stress $r_1 - \tau$. The vector CK has the same magnitude for all values of β since the lengths of its components CH_1 and CG , and the angle between them, $\pi - 2\alpha$, are independent of β . Hence, by combining two pure shears we obtain again a pure shear (see page 17).

When $\tau_1 - \tau = 0$, the angle β determines the direction of one of the principal stresses at M . From the figure we see that τ and τ_1 are numerically equal if $2\beta = 2(\alpha - \beta)$, from which $\beta = \alpha/2$. The direction of the principal stress therefore bisects the angle between the radii r and r_1 . The magnitudes of the principal stresses are therefore

$$\pm 2\sigma = \pm 2A \sin 2\beta = \pm 2A \sin \alpha \quad (e)$$

Combining this with the uniform compression (d) we find, for the total values of the principal stresses at any point M ,

$$-2A(\alpha + \sin \alpha), \quad -2A(\alpha - \sin \alpha) \quad (f)$$

Along any circle through O and O_1 the angle α remains constant, and so the principal stresses (f) are also constant. At the boundary, between the points O and O_1 ,

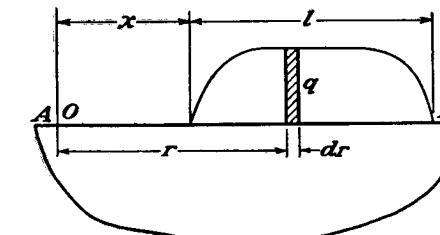


FIG. 59.

(Fig. 58a), the angle α is equal to π , and we find, from (f), that both principal stresses are equal to $-2\pi A = -q$. For the remaining portions of the boundary $\alpha = 0$, and both principal stresses are zero.

Hence if an arbitrary load distribution (Fig. 59) is regarded as composed of a large number of loads of varying intensities on short elements of the boundary, the horizontal stress σ_x under one such load element (as indicated in Fig. 59) is entirely due to that element, and

$$\sigma_x = \sigma_y = -q \quad (g)$$

all along the straight boundary.

Several other cases of distributed load on a straight boundary of the semi-infinite plate were discussed by S. D. Carothers,¹ and by M. Sadowsky.² Another manner of solving this problem will be discussed later (see page 125).

The deflections of the straight boundary of the plate can be found for any load distribution by using Eq. (72) obtained for the case of a concentrated force. If q is the intensity of vertical load distribution (Fig. 59), the deflection produced at any point O at a distance r from the shaded element $q dr$ of the load, from Eq. (72), is

$$\frac{2q}{\pi E} \log \frac{d}{r} dr - \frac{(1 + \nu)q}{\pi E} dr$$

¹ Proc. Roy. Soc. (London), series A, vol. 97, p. 110, 1920.

² Z. angew. Math. Mech., vol. 8, p. 107, 1928.

and the total deflection at O is

$$v_0 = \frac{2}{\pi E} \int_x^{l+x} q \log \frac{d}{r} dr - \frac{1+\nu}{\pi E} \int_x^{l+x} q dr \quad (h)$$

In the case of a uniformly distributed load, q is constant and we find

$$v_0 = \frac{2q}{\pi E} \left[(l+x) \log \frac{d}{l+x} - x \log \frac{d}{x} \right] + \frac{1-\nu}{\pi E} ql \quad (i)$$

In the same manner, for a point under the load (Fig. 60), we find

$$v_0 = \frac{2q}{\pi E} \left[(l-x) \log \frac{d}{l-x} + x \log \frac{d}{x} \right] + \frac{1-\nu}{\pi E} ql \quad (j)$$

Equation (h) can be used also for finding the intensity q of load distribution, which produces a given deflection at the straight boundary.

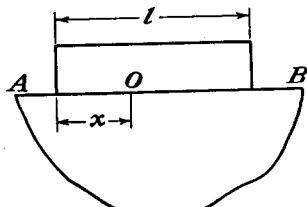


FIG. 60.

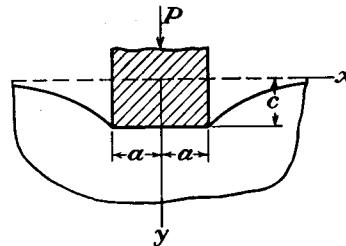


FIG. 61.

Assuming, for instance, that the deflection is constant along the loaded portion of the straight boundary (Fig. 61), it can be shown that the distribution of pressure along this portion is given by the equation¹

$$q = \frac{P}{\pi \sqrt{a^2 - x^2}}$$

35. Force Acting on the End of a Wedge. The simple radial stress distribution discussed in Art. 33 can be used also in investigating the stresses in a wedge due to a concentrated force at its apex. Let us consider a symmetrical case, as shown in Fig. 62. The thickness of the wedge in the direction perpendicular to the xy -plane is taken as unity. The conditions along the faces, $\theta = \pm \alpha$, of the wedge are satisfied by taking for the stress components the values

$$\sigma_r = -\frac{kP \cos \theta}{r}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0 \quad (a)$$

¹ Sadowsky, loc. cit.

The constant k will now be adjusted so as to satisfy the condition of equilibrium at the point O . Making the resultant of the pressures on the cylindrical surface (shown by the dotted line) equal to $-P$ we find

$$-2 \int_0^\alpha \frac{kP \cos^2 \theta}{r} r d\theta = -kP \left(\alpha + \frac{1}{2} \sin 2\alpha \right) = -P$$

from which

$$k = \frac{1}{\alpha + \frac{1}{2} \sin 2\alpha}$$

Then, from Eqs. (a),¹

$$\sigma_r = -\frac{P \cos \theta}{r(\alpha + \frac{1}{2} \sin 2\alpha)} \quad (73)$$

By making $\alpha = \pi/2$ we arrive at solution (66) for a semi-infinite plate, which has already been discussed. It may be seen that the distribu-

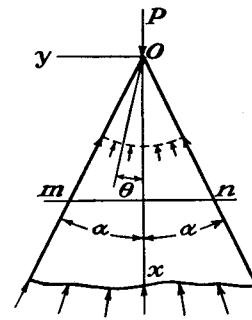


FIG. 62.

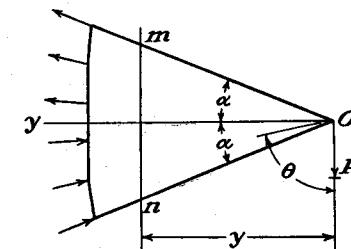


FIG. 63.

tion of normal stresses over any cross section mn is not uniform, and the ratio of the normal stress at the points m or n to the maximum stress at the center of the cross section is found to be equal to $\cos^4 \alpha$.

If the force is perpendicular to the axis of the wedge (Fig. 63), the same solution (a) can be used if θ is measured from the direction of the force. The constant factor k is found from the equation of equilibrium

$$\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} \sigma_r \cos \theta \cdot r d\theta = -P$$

from which

$$k = \frac{1}{\alpha - \frac{1}{2} \sin 2\alpha}$$

and the radial stress is

$$\sigma_r = -\frac{P \cos \theta}{r(\alpha - \frac{1}{2} \sin 2\alpha)} \quad (74)$$

¹ This solution is due to Michell, loc. cit. See also A. Mesnager, Ann. ponts et chaussées, 1901,

The normal and shearing stresses over any cross section mn are

$$\begin{aligned}\sigma_y &= -\frac{Pyx \sin^4 \theta}{y^3(\alpha - \frac{1}{2} \sin 2\alpha)} \\ \tau_{xy} &= -\frac{Px^2 \sin^4 \theta}{y^3(\alpha - \frac{1}{2} \sin 2\alpha)}\end{aligned}\quad (b)$$

In the case of a small angle α , we can put

$$2\alpha - \sin 2\alpha = \frac{(2\alpha)^3}{6}$$

Then writing I for the moment of inertia of the cross section mn we find from (b) that

$$\begin{aligned}\sigma_y &= -\frac{Pyx}{I} \cdot \left(\frac{\tan \alpha}{\alpha}\right)^3 \sin^4 \theta \\ \tau_{xy} &= -\frac{Px^2}{I} \cdot \left(\frac{\tan \alpha}{\alpha}\right)^3 \sin^4 \theta\end{aligned}\quad (c)$$

For small values of α , the factor $(\tan \alpha / \alpha)^3 \sin^4 \theta$ can be taken as nearly unity. Then the expression for σ_y coincides with that given by the elementary beam formula. The maximum shearing stress occurs at the points m and n and is twice as great as that given by the elementary theory for the centroid of a rectangular cross section of a beam.

Since we have solutions for the two cases represented in Figs. 62 and 63, we can deal with any direction of the force P in the xy -plane by resolving the force into two components and using the method of superposition.¹ It should be noted that solutions (73) and (74) represent an exact solution only in the case when, at the supported end, the wedge is held by radially directed forces distributed in the manner given by the solutions. Otherwise the solutions are accurate only at points at large distances from the supported end.

The problem of the wedge loaded by a bending couple M , in the plane of the wedge, and concentrated at the tip, is solved by the stress function.²

$$\phi = M \frac{\sin 2\theta - 2\theta \cos 2\alpha}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)} \quad (d)$$

where θ is as indicated in Fig. 62 and the applied couple M is counter-

¹ Several examples of stress distribution in wedges are discussed by Akira Miura, "Spannungskriven in rechteckigen und keilförmigen Trägern," Berlin, 1928. Forces not at the vertex, on a wedge, or a plate from which a wedge has been cut, are considered by J. H. A. Braatz, *Physics*, vol. 4, p. 56, 1933, and by W. M. Shepherd, *Proc. Roy. Soc. (London)*, series A, vol. 148, p. 284, 1935.

² C. E. Inglis, *Trans. Inst. Nav. Arch. (London)*, 1922, vol. 64.

clockwise. The stresses are

$$\begin{aligned}\sigma_r &= -\frac{M}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)} \cdot \frac{4}{r^2} \sin 2\theta, \quad \sigma_\theta = 0 \\ \tau_{r\theta} &= \frac{M}{2(\sin 2\alpha - 2\alpha \cos 2\alpha)} \cdot \frac{2}{r^2} (\cos 2\theta - \cos 2\alpha)\end{aligned}\quad (e)$$

36. Concentrated Force Acting on a Beam. The problem of stress distribution in a beam subjected to the action of a concentrated force is of great practical interest. It was shown before (Art. 22) that in continuously loaded beams of narrow rectangular cross section the stress distribution is obtained with satisfactory accuracy by the usual elementary theory of bending. Near the point of application of a concentrated force, however, a serious local perturbation in stress distribution should be expected and a further investigation of the problem

is necessary. The first study of these local stresses was made experimentally by Carus Wilson.¹ Experimenting with a rectangular beam of glass on two supports (Fig. 64) loaded at the middle, and using polarized light (see page 132), he showed that at the point A , where the load is applied, the stress distribution approaches that produced in a semi-infinite plate by a normal concentrated force. Along the cross section AD the normal stress σ_x does not follow a linear law, and at the point D , opposite to A , the tensile stress is smaller than would be expected from the elementary beam theory. These results were explained on the basis of certain empirical assumptions by G. G. Stokes.² The system represented in Fig. 64 can be obtained by superposing two systems shown in Fig. 65. The radial compressive stresses acting on the sections mn , np , and pq of a semi-infinite plate (Fig. 65a) are removed by equal radial tensile stresses acting on the sides of the rectangular beam supported at n and p (Fig. 65b). The stresses in this beam should be superposed on the stresses in the semi-infinite plate in order to get the case discussed by Stokes.

In calculating the stresses in the beam, the elementary beam formula will be applied. The bending moment at the middle cross section AD of the beam is obtained by taking the moment of the reac-

¹ Loc. cit.

² Wilson, loc. cit.; also G. G. Stokes, "Mathematical and Physical Papers," vol. 5, p. 238.

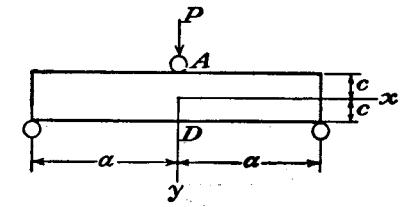


FIG. 64.

tion $P/2$ and subtracting the moment of all the radially directed tensile forces applied to one-half of the beam. This latter moment is easily calculated if we observe that the radially distributed tensile forces are statically equivalent to the pressure distribution over the quadrant ab of the cylindrical surface abc at the point A (Fig. 65c) or, using Eq. (66), are equivalent to a horizontal force P/π and a vertical force $P/2$,

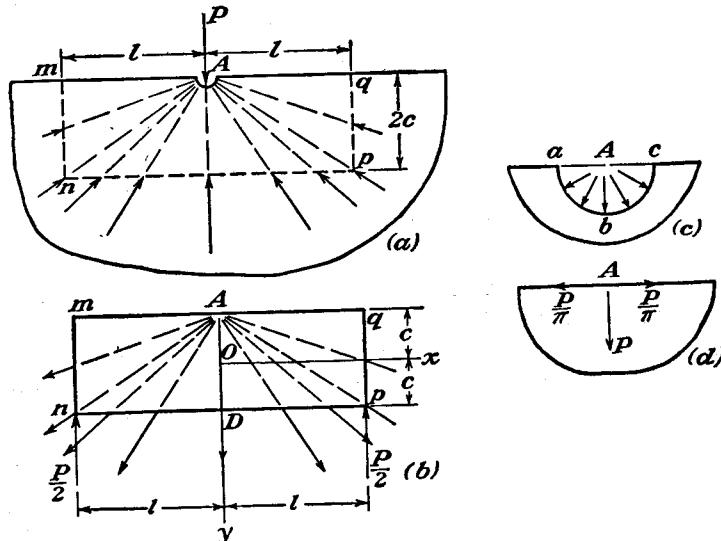


FIG. 65.

applied at A (Fig. 65d). Then the bending moment, *i.e.*, the moment about the point O , is

$$\frac{P}{2}l - \frac{P}{\pi}c$$

and the corresponding bending stresses are¹

$$\sigma_x' = \frac{P}{I} \left(\frac{l}{2} - \frac{c}{\pi} \right) y = \frac{3P}{2c^3} \left(\frac{l}{2} - \frac{c}{\pi} \right) y$$

To these bending stresses the uniformly distributed tensile stress $P/2\pi c$ produced by the tensile force P/π should be added. The normal $P/2\pi c$ over the cross section AD , as obtained in this elementary way, are therefore

$$\sigma_x = \frac{3P}{2c^3} \left(\frac{l}{2} - \frac{c}{\pi} \right) y + \frac{P}{2\pi c}$$

This coincides with the formula given by Stokes.

¹ As before we take P as the force per unit thickness of the plate.

A better approximation is obtained if we observe that a continuously distributed load is applied to the bottom of the beam (Fig. 65b) and use Eqs. (36'). The intensity of this load at the point D , from Eq. (66), is $P/\pi c$. Substituting this in (36') and combining with the value of σ_x above, we obtain, as a second approximation,

$$\begin{aligned}\sigma_x &= \frac{3P}{2c^3} \left(\frac{l}{2} - \frac{c}{\pi} \right) y + \frac{P}{2\pi c} + \frac{P}{\pi c} \left(\frac{y^3}{2c^3} - \frac{3}{10} \frac{y}{c} \right) \\ \sigma_y &= \frac{P}{2\pi c} + \frac{P}{\pi c} \left(\frac{3y}{4c} - \frac{y^3}{4c^3} \right)\end{aligned}\quad (a)$$

These stresses should be superposed on the stresses

$$\sigma_x = 0, \quad \sigma_y = -\frac{2P}{\pi(c+y)} \quad (b)$$

as for a semi-infinite plate, in order to obtain the total stresses along the section AD .

A comparison with a more accurate solution, given below (see table, page 106), shows that Eqs. (a) and (b) give the stresses with very good accuracy at all points except the point D at the bottom of the beam, at which the correction to the simple beam formula is given as

$$-\frac{3P}{2\pi c} + \frac{P}{2\pi c} + \frac{1}{5} \frac{P}{\pi c} = -0.254 \frac{P}{c}$$

while the more accurate solution gives only $-0.133(P/c)$.

The first attempt to get a more accurate solution of the problem was made by J. Boussinesq.¹ He used Flamant's solution (see Art. 33) for the semi-infinite plate. To annul the stresses over the boundary np (Fig. 65a), he superimposes an equal and opposite system of stresses and uses again the Flamant solution, *i.e.*, considers the beam as a semi-infinite plate extending above the line np . This corrective system introduces extra stresses over the top of the beam, which again can be removed by repeated application of Flamant's solution, and so on. This process is too slowly convergent and did not lead to a satisfactory result.

A solution of the problem by means of trigonometric series was obtained by L. N. G. Filon.² He applied this solution to the case of concentrated loads and made calculations for several particular cases (see Art. 23), which are in good agreement with more recent investigations.

¹ Compt. rend., vol. 114, p. 1510, 1892.

² L. N. G. Filon, Trans. Roy. Soc. (London), series A, vol. 201, p. 63, 1903.

Further progress in the solution of the problem was made by H. Lamb.¹ Considering an infinite beam loaded at equal intervals by equal concentrated forces acting in the upward and downward directions alternately, he simplified the solution of the two-dimensional problem and obtained for several cases expressions for the deflection curves. It was shown in this manner that the elementary Bernoulli-Euler theory of bending is very accurate if the depth of the beam is small in comparison with its length. It was shown also that the correction for shearing force as given by Rankine's and Grashof's elementary theory (see page 43) is somewhat exaggerated and should be diminished to about 0.75 of its value.²

A more detailed study of the stress distribution and of the curvature near the point of application of a concentrated load was made by T. v. Kármán³ and F. Seewald.⁴ Kármán considers an infinitely long beam and makes use of the solution for a semi-infinite plate with two equal and opposite couples acting on two neighboring points of its straight boundary (Fig. 56b). The stresses along the bottom of the beam which are introduced by this procedure can be removed by using a solution in the form of a trigonometric series (Art. 23) which, for an infinitely long beam, will be represented by a Fourier integral. In this manner Kármán arrives at the stress function

$$\phi = \frac{Ma}{\pi} \int_0^{\infty} \frac{(\alpha c \cosh \alpha c + \sinh \alpha c) \cosh \alpha y - \sinh \alpha c \sinh \alpha y \cdot \alpha y}{\sinh 2\alpha c + 2\alpha c} \cos \alpha x \, d\alpha$$

$$- \frac{Ma}{\pi} \int_0^{\infty} \frac{(\alpha \sinh \alpha c + \cosh \alpha c) \sinh \alpha y - \cosh \alpha c \cosh \alpha y \cdot \alpha y}{\sinh 2\alpha c - 2\alpha c} \cos \alpha x \, d\alpha \quad (c)$$

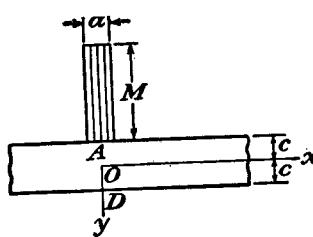


FIG. 66.

This function gives the stress distribution in the beam when the bending moment diagram consists of a very narrow rectangle, as shown in Fig. 66. For the most general loading of the beam by vertical forces applied at the top of the beam⁴ the corresponding bending-moment diagram can be divided into elementary rectangles such as the one shown in Fig. 66, and the corresponding stress function will be obtained by integrating expression (c) along the length of the beam.

This method of solution was applied by Seewald to the case of a beam loaded by a concentrated force P (Fig. 64). He shows that the stress σ_x can be split into two

¹ H. Lamb, *Atti IV congr. intern. matemat.*, vol. 3, p. 12, Rome, 1909.

² Filon came to the same conclusion in his paper (*loc. cit.*).

³ *Abhandl. aerodynam. Inst., Tech. Hochschule, Aachen*, vol. 7, 1927.

⁴ The case of a concentrated load applied half way between the top and bottom of the beam was discussed by R. C. J. Howland, *Proc. Roy. Soc. (London)*, vol. 124, p. 89, 1929 (see p. 115 below), and pairs of forces within the beam by K. Girkmann, *Ingenieur-Archiv*, vol. 13, p. 273, 1943. Concentrated longitudinal forces in the web of an I-beam are considered by Girkmann in *Oesterr. Ingenieur-Archiv*, vol. 1, p. 420, 1946.

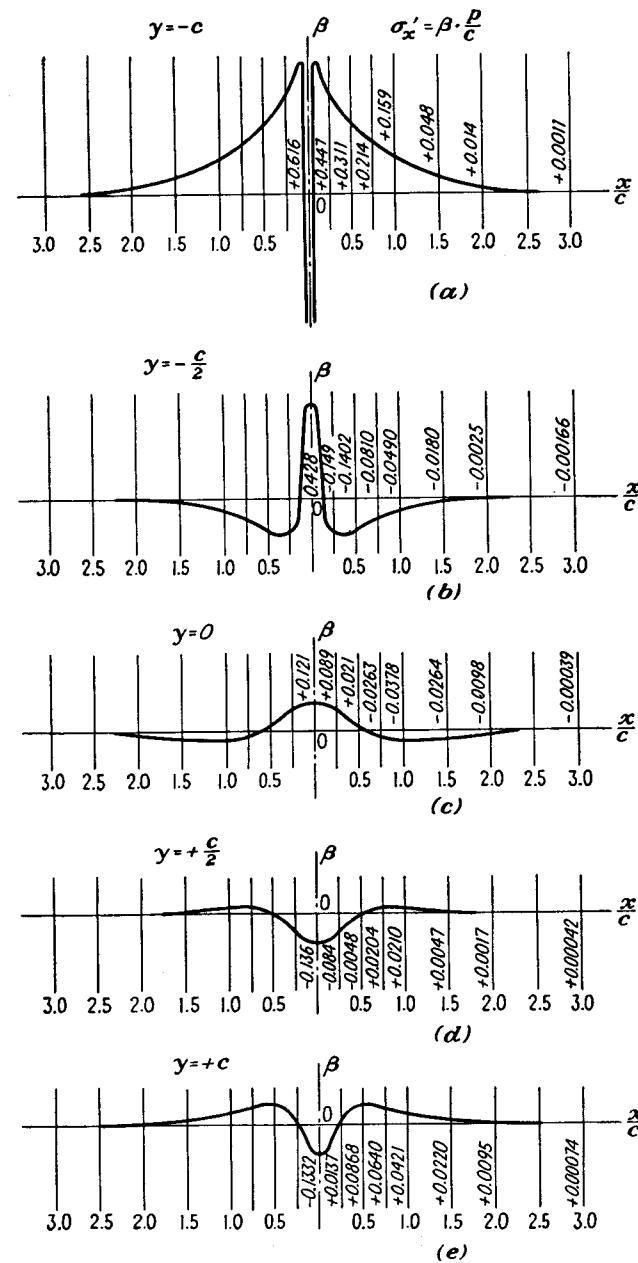


FIG. 67.

parts: one, which can be calculated by the usual elementary beam formula; and another, which represents the local effect near the point of application of the load. This latter part, called σ_x' , can be represented in the form $\beta(P/c)$, in which β is a numerical factor depending on the position of the point for which the local stresses are calculated. The values of this factor are given in Fig. 67. The two other stress components σ_y and τ_{xy} can also be represented in the form $\beta(P/c)$. The corresponding values of β are given in Figs. 68 and 69. It can be seen from the figures that the local stresses decrease very rapidly with increase of distance from

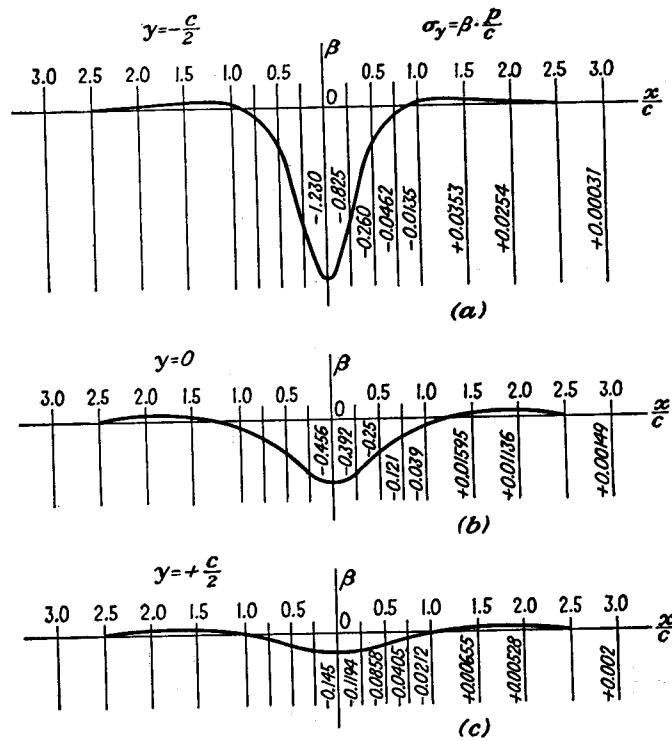


FIG. 68.

the point of application of the load and at a distance equal to the depth of the beam are usually negligible. Using the values of the factor β for $x = 0$, the local stresses at five points of the cross section AD under the load (Fig. 64) are tabulated below. For comparison the local stresses,¹ as obtained from Eqs. (a) and (b) (page 101), are also given. It is seen that these equations give the local stresses with sufficient accuracy.

Knowing the stresses, the curvature and the deflection of the beam can be calculated without any difficulty. These calculations show that the curvature of the deflection curve can also be split into two parts—one as given by the

¹ That is, stresses which must be superposed on those obtained from the ordinary beam formula.

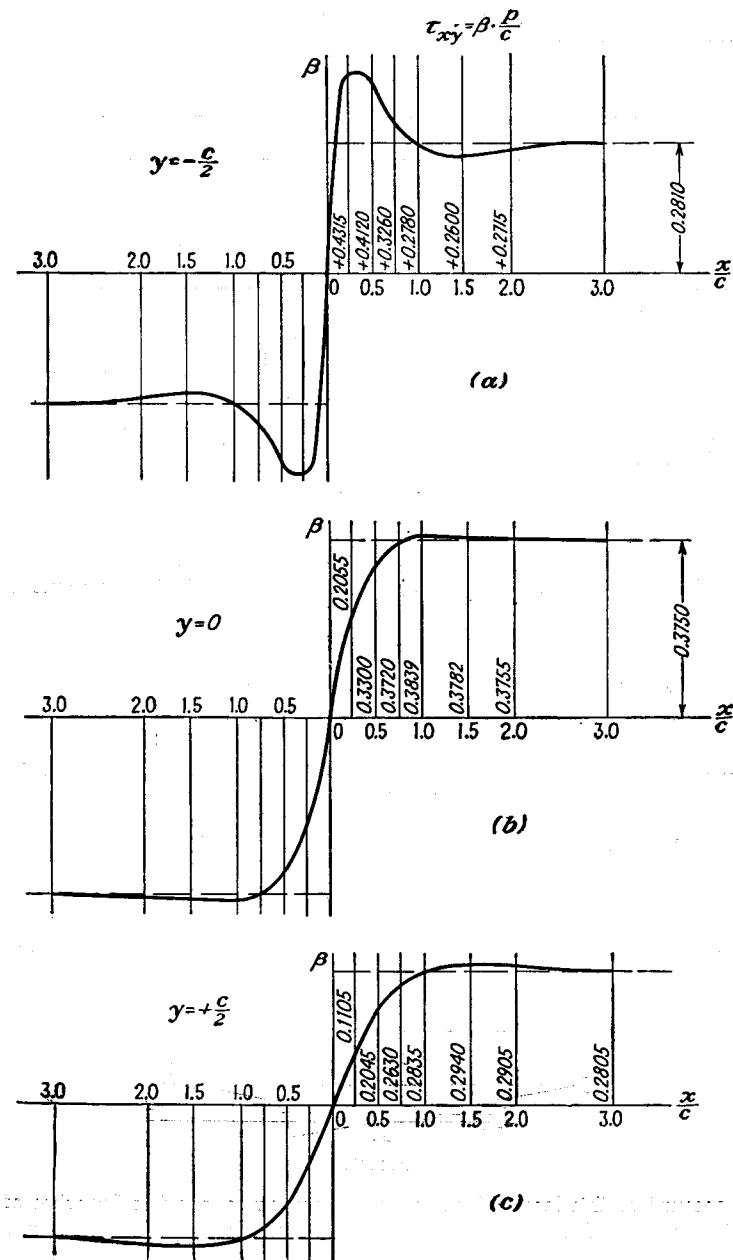


FIG. 69.

TABLE OF FACTORS β FOR THE MIDDLE

$y =$	$-c$	$-\frac{c}{2}$	0	$\frac{c}{2}$	c
Exact solution					
σ_x'	0.428	0.121	-0.136	-0.133
σ_y	∞	-1.23	-0.456	-0.145	0
Approximate solution					
σ_x'	0.573	0.426	0.159	-0.108	-0.254
σ_y	∞	-1.22	-0.477	-0.155	0

elementary beam theory and the other representing the local effect of the concentrated load P . This additional curvature of the center line can be represented by the formula

$$\frac{1}{r} = \alpha \frac{P}{Ec^2} \quad (d)$$

in which α is a numerical factor varying along the length of the beam. Several values of this factor are given in Fig. 70. It is seen that at cross sections at a

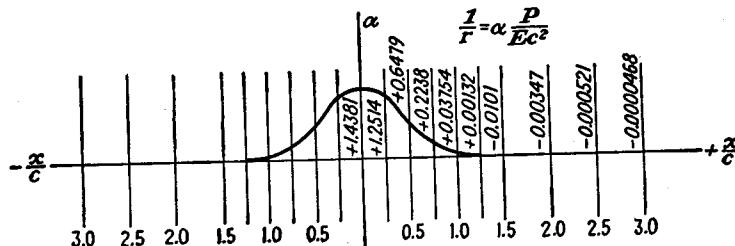


FIG. 70.

distance greater than half of the depth of the beam the additional curvature is negligible.

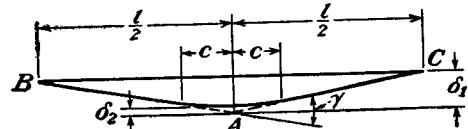


FIG. 71.

On account of this localized effect on the curvature, the two branches of the deflection curve AB and AC (Fig. 71) may be considered to meet at an angle equal to

$$\gamma = \frac{P}{c} \left(\frac{3}{4G} - \frac{3}{10E} - \frac{3\nu}{4E} \right) \quad (e)$$

The corresponding deflection at the middle is

$$\delta_1 = \frac{\gamma l}{4} = \frac{Pl}{4c} \left(\frac{3}{4G} - \frac{3}{10E} - \frac{3\nu}{4E} \right) \quad (f)$$

From this deflection a small further correction δ_2 , removing the sharp change of slope at A , should be subtracted. This quantity was also calculated by Seewald and is equal to

$$\delta_2 = 0.21 \frac{P}{E}$$

Denoting now by δ_0 the deflection as calculated by using the elementary theory, the total deflection under the load is

$$\delta = \delta_0 + \delta_1 - \delta_2 = \frac{Pl^3}{48EI} + \frac{Pl}{4c} \left(\frac{3}{4G} - \frac{3}{10E} - \frac{3\nu}{4E} \right) - 0.21 \frac{P}{E} \quad (75)$$

Taking $\nu = 0.3$, this gives

$$\delta = \frac{Pl^3}{48EI} \left[1 + 2.85 \left(\frac{2c}{l} \right)^2 - 0.84 \left(\frac{2c}{l} \right)^3 \right] \quad (75')$$

The elementary Rankine-Gashof theory (see page 43) gives for this case

$$\delta = \frac{Pl^3}{48EI} \left[1 + 3.90 \left(\frac{2c}{l} \right)^2 \right] \quad (g)$$

It appears that Eq. (g) gives an exaggerated value for the correction due to shear. In these formulas the deflection due to local deformation at the supports is not taken into account.

37. Stresses in a Circular Disk.

Let us begin with the simple case of two equal and opposite forces P acting along a diameter AB (Fig. 72). Assuming that each of the forces produces a simple radial stress distribution [Eq. (66)], we can find what forces should be applied at the circumference of the disk in order to maintain such a stress distribution. At any point M of the circumference we have compressions in the directions of r and r_1 equal to $\frac{2P}{\pi r} \cos \theta$ and $\frac{2P}{\pi r_1} \cos \theta_1$, respectively.¹ Since r and r_1 are perpendicular to each other and

$$\frac{\cos \theta}{r} = \frac{\cos \theta_1}{r_1} = \frac{1}{d} \quad (a)$$

¹ It is assumed that P is the force per unit thickness of the disk.

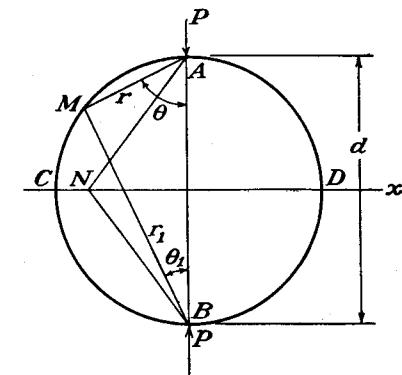


FIG. 72.

where d is the diameter of the disk, we conclude that the two principal stresses at M are two equal compressive stresses of magnitude $2P/\pi d$. Hence the same compressive stress is acting on any plane through M perpendicular to the plane of the disk, and normal compressive forces of the constant intensity $2P/\pi d$ should be applied to the circumference of the disk in order to maintain the assumed pair of simple radial stress distributions.

If the boundary of the disk is free from external forces, the stress at any point is therefore obtained by superposing a uniform tension in the plane of the disk of the magnitude $2P/\pi d$ on the above two simple radial stress distributions. Let us consider the stress on the horizontal diametral section of the disk at N . From symmetry it can be concluded that there will be no shearing stress on this plane. The normal stress produced by the two equal radial compressions is

$$-2 \cdot \frac{2P \cos \theta}{\pi} \cdot \cos^2 \theta$$

in which r is the distance AN , and θ the angle between AN and the vertical diameter. Superposing on this the uniform tension $2P/\pi d$, the total normal stress on the horizontal plane at N is

$$\sigma_y = -\frac{4P}{\pi} \frac{\cos^3 \theta}{r} + \frac{2P}{\pi d}$$

or, using the fact that

$$\cos \theta = \frac{d}{\sqrt{d^2 + 4x^2}}$$

we find

$$\sigma_y = \frac{2P}{\pi d} \left[1 - \frac{4d^4}{(d^2 + 4x^2)^2} \right] \quad (b)$$

The maximum compressive stress along the diameter CD is at the center of the disk, where

$$\sigma_y = -\frac{6P}{\pi d}$$

At the ends of the diameter the compressive stress σ_y vanishes.

Consider now the case of two equal and opposite forces acting along a chord AB (Fig. 73). Assuming again two simple radial distributions radiating from A and B , the stress on a plane tangential to the circumference at M is obtained by superposing the two radial compressions $\frac{2P \cos \theta}{\pi r}$ and $\frac{2P \cos \theta_1}{\pi r_1}$ acting in the directions r and r_1 , respectively.

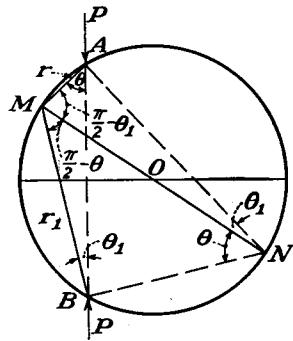


FIG. 73.

The normal MN to the tangent at M is the diameter of the disk; hence MAN and MBN are right-angled triangles and the angles which the normal MO makes with r and r_1 are $\pi/2 - \theta_1$ and $\pi/2 - \theta$, respectively. The normal and shearing stresses on an element of the boundary at M are then

$$\begin{aligned} \sigma &= -\frac{2P}{\pi} \frac{\cos \theta}{r} \cos^2 \left(\frac{\pi}{2} - \theta_1 \right) - \frac{2P}{\pi} \frac{\cos \theta_1}{r_1} \cos^2 \left(\frac{\pi}{2} - \theta \right) \\ &= -\frac{2P}{\pi} \left(\frac{\cos \theta \sin^2 \theta_1}{r} + \frac{\cos \theta_1 \sin^2 \theta}{r_1} \right) \\ \tau &= -\frac{2P}{\pi} \left(\frac{\cos \theta}{r} \sin \theta_1 \cos \theta_1 - \frac{\cos \theta_1}{r_1} \sin \theta \cos \theta \right) \end{aligned} \quad (c)$$

These equations can be simplified if we observe that, from the triangles MAN and MBN ,

$$r = d \sin \theta_1, \quad r_1 = d \sin \theta$$

Substituting in Eqs. (c), we find

$$\sigma = -\frac{2P}{\pi d} \sin(\theta + \theta_1), \quad \tau = 0 \quad (d)$$

From Fig. 73 it may be seen that $\sin(\theta + \theta_1)$ remains constant around the boundary. Hence uniformly distributed compressive forces of the intensity $2P/\pi d \sin(\theta + \theta_1)$ should be applied to the boundary in order to maintain the assumed radial stress distributions. To obtain the solution for a disk with its boundary free from uniform compression it is only necessary to superpose on the above two simple radial distributions a uniform tension of the intensity $2P/\pi d \sin(\theta + \theta_1)$.

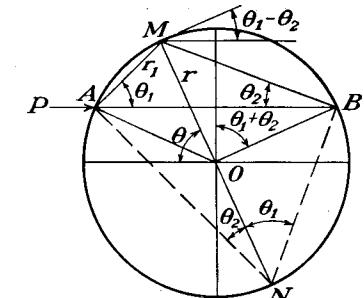


FIG. 74.

The problem of the stress distribution in a disk can be solved for the more general case when any system of forces in equilibrium is acting on the boundary of the disk.¹ Let us take one of these forces, acting at A in the direction of the chord AB (Fig. 74). Assuming again a simple radial stress distribution we have at point M a simple radial compression of the magnitude $(2P/\pi) \cos \theta_1/r_1$ acting in the direction of AM .

¹ The problems discussed in this article were solved by H. Hertz, *Z. Math. Physik*, vol. 28, 1883, or "Gesammelte Werke," vol. 1, p. 283; and J. H. Michell, *Proc. London Math. Soc.*, vol. 32, p. 44, 1900, and vol. 34, p. 134, 1901. The problem corresponding to Fig. 72 when the disk is replaced by a rectangle is considered by J. N. Goodier, *Trans. A.S.M.E.*, vol. 54, p. 173, 1932, including the effects of distribution of the load over small segments of the boundary.

Let us take as origin of polar coordinates the center O of the disk, and measure θ as shown in the figure. Then the normal and the shearing components of the stress acting on an element tangential to the boundary at M can easily be calculated if we observe that the angle between the normal MO to the element and the direction r_1 of the compression is equal to $\pi/2 - \theta_2$. Then

$$\begin{aligned}\sigma_r &= -\frac{2P \cos \theta_1}{\pi r_1} \sin^2 \theta_2 \\ \tau_{r\theta} &= -\frac{2P \cos \theta_1}{\pi r_1} \sin \theta_2 \cos \theta_2\end{aligned}\quad (e)$$

Since, from the triangle AMN , $r_1 = d \sin \theta_2$, Eqs. (e) can be written in the form

$$\begin{aligned}\sigma_r &= -\frac{P}{\pi d} \sin (\theta_1 + \theta_2) - \frac{P}{\pi d} \sin (\theta_2 - \theta_1) \\ \tau_{r\theta} &= -\frac{P}{\pi d} \cos (\theta_1 + \theta_2) - \frac{P}{\pi d} \cos (\theta_2 - \theta_1)\end{aligned}\quad (f)$$

This stress acting on the element tangential to the boundary at point M can be obtained by superposing the following three stresses on the element.

(1) A normal stress uniformly distributed along the boundary:

$$-\frac{P}{\pi d} \sin (\theta_1 + \theta_2) \quad (g)$$

(2) A shearing stress uniformly distributed along the boundary:

$$-\frac{P}{\pi d} \cos (\theta_1 + \theta_2) \quad (h)$$

(3) A stress of which the normal and shearing components are

$$-\frac{P}{\pi d} \sin (\theta_2 - \theta_1) \quad \text{and} \quad -\frac{P}{\pi d} \cos (\theta_2 - \theta_1) \quad (k)$$

Observing that the angle between the force P and the tangent at M is $\theta_1 - \theta_2$, it can be concluded that the stress (k) is of magnitude $P/\pi d$ and acts in the direction opposite to the direction of the force P .

Assume now that there are several forces acting on the disk and each of them produces a simple radial stress distribution. Then the forces to be applied at the boundary in order to maintain such a stress distribution are:

(1) A normal force uniformly distributed along the boundary, of intensity

$$-\sum \frac{P}{\pi d} \sin (\theta_1 + \theta_2) \quad (l)$$

(2) Shearing forces of intensity

$$-\sum \frac{P}{\pi d} \cos (\theta_1 + \theta_2) \quad (m)$$

(3) A force, the intensity and direction of which are obtained by vectorial summation of expressions (k). The summation must extend over all forces acting on the boundary.

The moment of all the external forces with respect to O , from Fig. 74, is

$$\sum \frac{P \cos (\theta_1 + \theta_2) d}{2}$$

and, as this moment must be zero for a system in equilibrium, we conclude that the shearing forces (m) are zero. The force obtained by summation of the stresses (k), proportional to the vectorial sum of the external forces, is also zero for a system in equilibrium. Hence it is only necessary to apply at the boundary of the disk a uniform compression (l) in order to maintain the simple radial distributions. If the boundary is free from uniform compression, the stress at any point of the disk is obtained by superposing a uniform tension of magnitude

$$\sum \frac{P}{\pi d} \sin (\theta_1 + \theta_2)$$

on the simple radial distributions.

By using this general method, various other cases of stress distribution in disks can easily be solved.¹ We may select, for instance, the case of a couple acting on the disk (Fig. 75), balanced by a couple applied at the center of the disk. Assuming two equal radial stress distributions at A and B , we see that, in this case, (l) and the summation of (k) are zero and only shearing forces (m) need be applied at the boundary in order to maintain the simple radial stress distributions. The intensity of these forces, from (m), is

$$-\frac{2P}{\pi d} \cos (\theta_1 + \theta_2) = -\frac{2M_t}{\pi d^2} \quad (n)$$

where M_t is the moment of the couple. To free the boundary of the disk from shearing forces and transfer the couple balancing the pair of forces P from the circumference of the disk to its center, it is necessary to superpose on the simple radial distributions the stresses of the case shown in Fig. 75b. These latter stresses, produced by pure circumferential shear, can easily be calculated if we observe that for each concentric circle of radius r the shearing stresses must give a couple M_t . Hence,

$$\tau_{r\theta} 2\pi r^2 = M_t, \quad \tau_{r\theta} = \frac{M_t}{2\pi r^2} \quad (p)$$

These stresses may also be derived from the general equations (38) by taking as the stress function

$$\phi = \frac{M_t \theta}{2\pi} \quad (q)$$

from which

$$\sigma_r = \sigma_\theta = 0, \quad \tau_{r\theta} = \frac{M_t}{2\pi r^2}$$

¹ Several interesting examples are discussed by J. H. Michell, *loc. cit.*

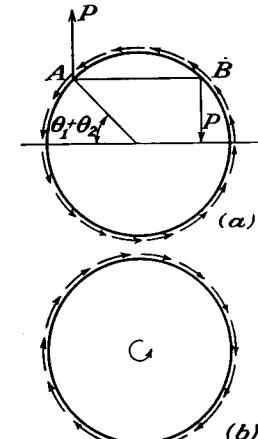


FIG. 75.

38. Force at a Point of an Infinite Plate. If a force P acts in the middle plane of an infinite plate (Fig. 76a), the stress distribution can easily be obtained by superposition of systems which we have already discussed. We cannot, however, construct a solution by simple superposition of two solutions for a semi-infinite plate as shown in Figs. 76b and 76c. Although the vertical displacements are the same in both these cases, the horizontal displacements along the straight boundaries are different. While in the case 76b this displacement is away from

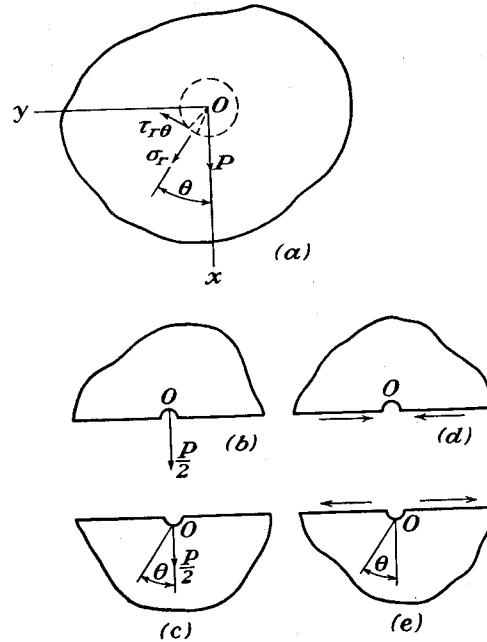


FIG. 76.

the point O , in the case 76c it is toward the point O . The magnitudes of these displacements in both cases, from Eq. (71), is

$$\frac{1 - \nu}{4E} \cdot P \quad (a)$$

This difference in the horizontal displacements may be eliminated by combining the cases 76b and 76c with the cases 76d and 76e in which shearing forces act along the straight boundaries. The displacements for these latter cases can be obtained from the problem of bending of a curved bar, shown in Fig. 46. Making the inner radius of this bar approach zero and the outer radius increase indefinitely, we arrive at

the case of a semi-infinite plate. The displacement along the straight boundary of this plate in the direction of the shearing force acting on the boundary is, from Eq. (61),

$$\frac{D\pi}{E} \quad (b)$$

The constant of integration D must now be adjusted so as to make the displacement resulting from (a) and (b) vanish. Then

$$\frac{D\pi}{E} = \frac{1 - \nu}{4E} P, \quad D = \frac{1 - \nu}{4\pi} P \quad (c)$$

With this adjustment the result of superposing cases 76b, 76c, 76d, and 76e is an infinite plate loaded at a point, Fig. 76a.

The stress distribution in the plate is now easily obtained by superposing the stresses in a semi-infinite plate produced by a normal load $P/2$ at the boundary (see Art. 33) on the stresses in the curved bar containing the constant of integration D . Observing the difference in measuring the angle θ in Figs. 46 and 76 and using Eqs. (60), the stresses in the curved bar are, for θ as in Fig. 76,

$$\begin{aligned}\sigma_r &= \frac{D \cos \theta}{r} = \frac{1 - \nu}{4\pi} \frac{P \cos \theta}{r} \\ \sigma_\theta &= \frac{D \cos \theta}{r} = \frac{1 - \nu}{4\pi} \frac{P \cos \theta}{r} \\ \tau_{r\theta} &= \frac{D \sin \theta}{r} = \frac{1 - \nu}{4\pi} \frac{P \sin \theta}{r}\end{aligned}$$

Combining this with stresses (66) calculated for the load $P/2$, we obtain the following stress distribution in the infinite plate:

$$\begin{aligned}\sigma_r &= \frac{1 - \nu}{4\pi} \frac{P \cos \theta}{r} - \frac{P \cos \theta}{\pi r} = -\frac{(3 + \nu)}{4\pi} \frac{P \cos \theta}{r} \\ \sigma_\theta &= \frac{1 - \nu}{4\pi} \frac{P \cos \theta}{r} \\ \tau_{r\theta} &= \frac{1 - \nu}{4\pi} \frac{P \sin \theta}{r}\end{aligned} \quad (76)$$

By cutting out from the plate at the point O (Fig. 76a) a small element bounded by a cylindrical surface of radius r , and projecting the forces acting on the cylindrical boundary of the element on the x - and y -axes, we find

$$X = 2 \int_0^\pi (\sigma_r \cos \theta - \tau_{r\theta} \sin \theta) r d\theta = P$$

$$Y = 2 \int_0^\pi (\sigma_r \sin \theta + \tau_{r\theta} \cos \theta) r d\theta = 0$$

i.e., the forces acting on the boundary of the cylindrical element represent the load P applied at the point O . By using Eqs. (13) the stress components, in Cartesian coordinates, are found from Eqs. (76):

$$\begin{aligned}\sigma_x &= \frac{P \cos \theta}{4\pi r} [-(3 + \nu) + 2(1 + \nu) \sin^2 \theta] \\ \sigma_y &= \frac{P \cos \theta}{4\pi r} [1 - \nu - 2(1 + \nu) \sin^2 \theta] \\ \tau_{xy} &= -\frac{P \sin \theta}{4\pi r} [1 - \nu + 2(1 + \nu) \cos^2 \theta]\end{aligned}\quad (77)$$

From solution (77), for one concentrated force, solutions for other kinds of loading can be obtained by superposition. Take, for instance, the case shown in Fig. 77, in which two equal and opposite forces acting on an infinite plate are applied at two points O and O_1 a very small distance d apart. The stress at any point M is obtained by superposing on the stress produced by the force at O the stress produced by the other force at O_1 . Considering, for instance, an element at M perpendicular to the x -axis and denoting by σ_x the normal stress produced on the element by the force at O , the normal stress σ'_x produced by the two forces shown in the figure is

$$\sigma'_x = \sigma_x - \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} d \right) = -d \frac{\partial \sigma_x}{\partial x} = -d \left(\frac{\partial \sigma_x}{\partial r} \cos \theta - \frac{\partial \sigma_x}{\partial \theta} \frac{\sin \theta}{r} \right)$$

Thus the stress components for the case of Fig. 77 are obtained from Eqs. (77) by differentiation. In this manner we find

$$\begin{aligned}\sigma_x &= \frac{dP}{4\pi r^2} [-(3 + \nu) \cos^2 \theta + (1 - \nu) \sin^2 \theta + 8(1 + \nu) \sin^2 \theta \cos^2 \theta] \\ \sigma_y &= \frac{dP}{4\pi r^2} [(1 - \nu) \cos^2 \theta + (1 + 3\nu) \sin^2 \theta \\ &\quad - 8(1 + \nu) \sin^2 \theta \cos^2 \theta]\end{aligned}\quad (78)$$

$$\tau_{xy} = \frac{dP}{4\pi r^2} [-(6 + 2\nu) + 8(1 + \nu) \sin^2 \theta] \sin \theta \cos \theta$$

It can be seen that the stress components decrease rapidly, as r increases, and are negligible when r is large in comparison with d . Such a result is to be expected in accordance with Saint-Venant's principle if we have two forces in equilibrium applied very near to each other.

By superposing two stress distributions such as given by Eqs. (78),

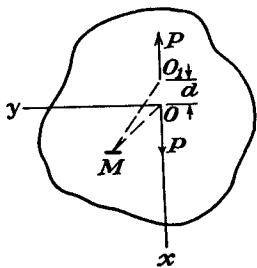


FIG. 77.

we can obtain the solution of the problem shown in Fig. 78. The stress components for this case are

$$\begin{aligned}\sigma_x &= -2(1 - \nu) \frac{dP}{4\pi r^2} (1 - 2 \sin^2 \theta) \\ \sigma_y &= 2(1 - \nu) \frac{dP}{4\pi r^2} (1 - 2 \sin^2 \theta) \\ \tau_{xy} &= -2(1 - \nu) \frac{dP}{4\pi r^2} \sin 2\theta\end{aligned}$$

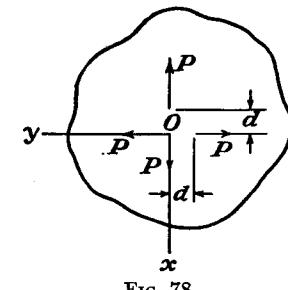


FIG. 78.

The same stress distribution expressed in polar coordinates is

$$\sigma_r = -2(1 - \nu) \frac{dP}{4\pi r^2}, \quad \sigma_\theta = 2(1 - \nu) \frac{dP}{4\pi r^2}, \quad \tau_{r\theta} = 0 \quad (79)$$

This solution can be made to agree with solution (46) for a thick cylinder submitted to the action of internal pressure if the outer diameter of the cylinder is taken as infinitely great.

In the same manner we can get a solution for the case shown in Fig. 79a. The stress components are¹

$$\sigma_r = \sigma_\theta = 0, \quad \tau_{r\theta} = -\frac{M}{2\pi r^2} \quad (80)$$

They represent the stresses produced by a couple M applied at the origin (Fig. 79b).

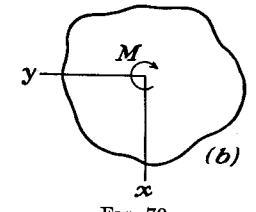
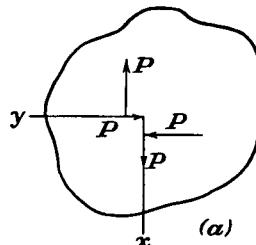


FIG. 79.

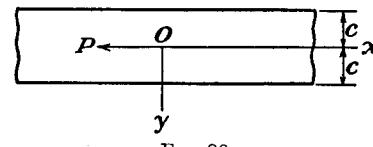


FIG. 80.

If instead of an infinite plate we have to deal with an infinitely long strip subjected to the action of a longitudinal force P (Fig. 80), we may begin with solution (77) as if the plate were infinite in all directions. The stresses along the edges of the strip resulting from this procedure can be annulled by superposing an equal and opposite system. The stresses produced by this corrective system can be determined by using the general method described in Art. 23. Calculations made by R. C. J. Howland² show that the local stresses produced by the concentrated force P diminish rapidly as the distance from the point of application of the load increases, and at distances greater than the width of the strip the distribution of stresses over the cross section is practically uniform. In the table below several

¹ A. E. H. Love, "Theory of Elasticity," p. 214, Cambridge, 1927.

² Loc. cit. See also a paper by E. Melan, *Z. angew. Math. Mech.*, vol. 5, p. 314, 1925.

values of the stresses σ_x and σ_y are given, calculated on the assumption that the strip is fixed at the end $x = +\infty$ and Poisson's ratio is $\frac{1}{4}$.

	$\frac{x}{c} =$	$-\frac{\pi}{3}$	$-\frac{\pi}{9}$	$-\frac{\pi}{18}$	$-\frac{\pi}{30}$	0
$y = 0$	$\frac{\sigma_x 2c}{P} =$	-0.118	-0.992	∞
$y = c$	$\frac{\sigma_x 2c}{P} =$	+0.159	+0.511	0.532	0.521	0.500
$y = 0$	$\frac{\sigma_y 2c}{P} =$	0.110	0.364
	$\frac{x}{c} =$	$\frac{\pi}{30}$	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = 0$	$\frac{\sigma_x 2c}{P} =$	1.992	1.118	1.002
$y = c$	$\frac{\sigma_x 2c}{P} =$	0.479	0.468	0.489	0.841	0.973
$y = 0$	$\frac{\sigma_y 2c}{P} =$	-0.364	-0.110	-0.049

Stresses produced in a semi-infinite plate by a force applied at some distance from the edge have been discussed by E. Melan.¹

39. General Solution of the Two-dimensional Problem in Polar Coordinates. Having discussed various particular cases of the two dimensional problem in polar coordinates we are now in a position to write down the general solution of the problem. The general expression for the stress function ϕ , satisfying the compatibility equation (39) is²

$$\begin{aligned}\phi = & a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2 \theta + a'_0 \theta \\ & + \frac{a_1}{2} r \theta \sin \theta + (b_1 r^3 + a'_1 r^{-1} + b'_1 r \log r) \cos \theta \\ & - \frac{c_1}{2} r \theta \cos \theta + (d_1 r^3 + c'_1 r^{-1} + d'_1 r \log r) \sin \theta \\ & + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a'_n r^{-n} + b'_n r^{-n+2}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c'_n r^{-n} + d'_n r^{-n+2}) \sin n\theta \quad (81)\end{aligned}$$

¹ Z. angew. Math. Mech., vol. 12, p. 343, 1932.

² This solution was given by J. H. Michell, Proc. London Math. Soc., vol. 31, p. 100, 1899. See also A. Timpe, Z. Math. Physik, vol. 52, p. 348, 1905. An analogous solution for the case of an elliptical ring was given by A. Timpe, Math. Z., vol. 17, p. 189, 1923.

The first three terms in the first line of this expression represent the solution for the stress distribution symmetrical with respect to the origin of coordinates (see Art. 26). The fourth term gives the stress distribution for the case shown in Fig. 57. The fifth term gives the solution for pure shear (Fig. 75b). The first term in the second line is the simple radial distribution for a load in the direction $\theta = 0$. The remaining terms of the second line represent the solution for a portion of a circular ring bent by a radial force (Fig. 46). By a combination of all the terms of the second line the solution for a force acting on an infinite plate was obtained (Art. 38). Analogous solutions are obtained also from the third line of expression (81), the only difference being that the direction of the force is changed by $\pi/2$. The further terms of (81) represent solutions for shearing and normal forces, proportional to $\sin n\theta$ and $\cos n\theta$, acting on the inner and outer boundaries of a circular ring. We had an example of this kind in discussing the stress distribution around a small circular hole (Art. 32).

In the case of a portion of a circular ring the constants of integration in expression (81) can be calculated without any difficulty from the boundary conditions. If we have a complete ring, certain additional investigations of the displacements are sometimes necessary in determining these constants. We shall consider the general case of a complete ring and assume that the intensities of the normal and shearing forces at the boundaries $r = a$ and $r = b$ are given by the following trigonometrical series:

$$\begin{aligned}(\sigma_r)_{r=a} &= A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \\ (\sigma_r)_{r=b} &= A'_0 + \sum_{n=1}^{\infty} A'_n \cos n\theta + \sum_{n=1}^{\infty} B'_n \sin n\theta \\ (\tau_{r\theta})_{r=a} &= C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} D_n \sin n\theta \\ (\tau_{r\theta})_{r=b} &= C'_0 + \sum_{n=1}^{\infty} C'_n \cos n\theta + \sum_{n=1}^{\infty} D'_n \sin n\theta \quad (a)\end{aligned}$$

in which the constants A_0, A_n, B_n, \dots , are to be calculated in the usual manner from the given distribution of forces at the boundaries (see page 49). Calculating the stress components from expression (81) by using Eqs. (38), and comparing the values of these components for $r = a$ and $r = b$ with those given by Eqs. (a), we obtain a sufficient number of equations to determine the constants of integration in all cases with $n \geq 2$. For $n = 0$, i.e., for the terms in the first line of expression (81), and for $n = 1$, i.e., for the terms in the second and third lines, further investigations are necessary.

Taking the first line of expression (81) as a stress function, the constant a_0 is determined by the magnitude of the shearing forces uniformly distributed along the boundaries (see page 111). The stress distribution given by the term with the factor d_0 is many valued (see page 93) and, in a complete ring, we must assume $d_0 = 0$. For the determination of the remaining three constants a_0, b_0 , and c_0 we have only two equations,

$$(\sigma_r)_{r=a} = A_0 \quad \text{and} \quad (\sigma_r)_{r=b} = A'_0$$

The additional equation for determining these constants is obtained from the consideration of displacements. The displacements in a complete ring should be single-valued functions of θ . Our previous investigation shows (see Art. 26) that this condition is fulfilled if we put $c_0 = 0$. Then the remaining two constants a_0 and b_0 are determined from the two boundary conditions stated above.

Let us consider now, in more detail, the terms for which $n = 1$. For determining the eight constants a_1, b_1, \dots, d_1' entering into the second and the third lines of expression (81), we calculate the stress components σ_r and $\tau_{r\theta}$ using this portion of ϕ . Then using conditions (a) and equating corresponding coefficients of $\sin n\theta$ and $\cos n\theta$ at the inner and outer boundaries, we obtain the following eight equations:

$$\begin{aligned} (a_1 + b_1')a^{-1} + 2b_1a - 2a_1'a^{-3} &= A_1 \\ (a_1 + b_1')b^{-1} + 2b_1b - 2a_1'b^{-3} &= A_1' \\ (c_1 + d_1')a^{-1} + 2d_1a - 2c_1'a^{-3} &= B_1 \\ (c_1 + d_1')b^{-1} + 2d_1b - 2c_1'b^{-3} &= B_1' \end{aligned} \quad (b)$$

$$\begin{aligned} 2d_1a - 2c_1'a^{-3} + d_1'a^{-1} &= -C_1 \\ 2d_1b - 2c_1'b^{-3} + d_1'b^{-1} &= -C_1' \\ 2b_1a - 2a_1'a^{-3} + b_1'a^{-1} &= D_1 \\ 2b_1b - 2a_1'b^{-3} + b_1'b^{-1} &= D_1' \end{aligned} \quad (c)$$

Comparing Eqs. (b) with (c) it can be seen that they are compatible only if

$$\begin{aligned} a_1a^{-1} &= A_1 - D_1 \\ a_1b^{-1} &= A_1' - D_1' \\ c_1a^{-1} &= B_1 + C_1 \\ c_1b^{-1} &= B_1' + C_1' \end{aligned} \quad (d)$$

from which it follows that

$$a(A_1 - D_1) = b(A_1' - D_1'), \quad a(B_1 + C_1) = b(B_1' + C_1') \quad (e)$$

It can be shown that Eqs. (e) are always fulfilled if the forces acting on the ring are in equilibrium. Taking, for instance, the sum of the components of all the forces in the direction of the x -axis as zero, we find

$$\int_0^{2\pi} \{[b(\sigma_r)_{r=a} - a(\sigma_r)_{r=b}] \cos \theta - [b(\tau_{r\theta})_{r=a} - a(\tau_{r\theta})_{r=b}] \sin \theta\} d\theta = 0$$

Substituting for σ_r and $\tau_{r\theta}$ from (a), we arrive at the first of Eqs. (e). In the same manner, by resolving all the forces along the y -axis, we obtain the second of Eqs. (e).

When a_1 and c_1 are determined from Eqs. (d) the two systems of Eqs. (b) and (c) become identical, and we have only four equations for determining the remaining six constants. The necessary two additional equations are obtained by considering the displacements. The terms in the second line in expression (81) represent the stress function for a combination of a simple radial distribution and the bending stresses in a curved bar (Fig. 46). By superposing¹ the general expressions for the displacements in these two cases, namely Eqs. (g) (page 90) and Eqs. (q) (page 77), and, substituting $a_1/2$ for $-P/\pi$ in Eqs. (g) and b_1' for D in Eqs. (q) (page 77),

¹ It should be noted that $\theta + (\pi/2)$ must be substituted for θ if the angle is measured from the vertical axis, as in Fig. 52; instead of from the horizontal axis, as in Fig. 46.

Eqs. (q), we find the following many-valued terms in the expressions for the displacements u and v , respectively:

$$\begin{aligned} \frac{a_1}{2} \frac{1-\nu}{E} \theta \sin \theta + \frac{2b_1'}{E} \theta \sin \theta \\ \frac{a_1}{2} \frac{1-\nu}{E} \theta \cos \theta + \frac{2b_1'}{E} \theta \cos \theta \end{aligned}$$

These terms must vanish in the case of a complete ring, hence

$$\frac{a_1}{2} \frac{1-\nu}{E} + \frac{2b_1'}{E} = 0$$

or

$$b_1' = -\frac{a_1(1-\nu)}{4} \quad (f)$$

Considering the third line of expression (81) in the same manner, we find

$$d_1' = -\frac{c_1(1-\nu)}{4} \quad (g)$$

Equations (f) and (g), together with Eqs. (b) and (c), are now sufficient for determining all the constants in the stress function represented by the second and the third lines of expression (81).

We conclude that in the case of a complete ring the boundary conditions (a) are not sufficient for the determination of the stress distribution, and it is necessary to consider the displacements. The displacements in a complete ring must be single valued and to satisfy this condition we must have

$$c_0 = 0, \quad b_1' = -\frac{a_1(1-\nu)}{4}, \quad d_1' = -\frac{c_1(1-\nu)}{4} \quad (82)$$

We see that the constants b_1' and d_1' depend on Poisson's ratio. Accordingly the stress distribution in a complete ring will usually depend on the elastic properties of the material. It becomes independent of the elastic constants only when a_1 and c_1 vanish so that, from Eq. (82), $b_1' = d_1' = 0$. This particular case occurs if [see Eqs. (d)]

$$A_1 = D_1 \quad \text{and} \quad B_1 = -C_1$$

We have such a condition when the resultant of the forces applied to each boundary of the ring vanishes. Take, for instance, the resultant component in the x -direction of forces applied to the boundary $r = a$. This component, from (a), is

$$\int_0^{2\pi} (\sigma_r \cos \theta - \tau_{r\theta} \sin \theta) a d\theta = a\pi(A_1 - D_1)$$

If it vanishes we find $A_1 = D_1$. In the same manner, by resolving the forces in the y -direction, we obtain $B_1 = -C_1$ when the y -component is zero. From this we may conclude that the stress distribution in a complete ring is independent of the elastic constants of the material if the resultant of the forces applied to each boundary is zero. The moment of these forces need not be zero.

These conclusions for the case of a circular ring hold also in the most general case of the two-dimensional problem for a *multiply-connected* body. From general investigations made by J. H. Michell,¹ it follows that, for multiply-connected bodies (Fig. 81), equations analogous to Eqs. (82) and expressing the condition that the displacements are single valued should be derived for each *independent circuit* such as the circuits *A* and *B* in the figure. The stress distributions in such bodies generally depend on the elastic constants of the material. They are independent of these constants only if the resultant force on each boundary vanishes.² Quantitatively the effect of the moduli on the maximum stress is usually very small, and in practice it can be neglected.³ This conclusion is of practical importance. We shall see later that in the case of transparent materials, such as glass or bakelite, it is possible to determine the stresses by an optical method, using polarized light (see page 131) and this conclusion means that the experimental results obtained with a transparent material can be applied immediately to any other material such as steel if the external forces are the same.

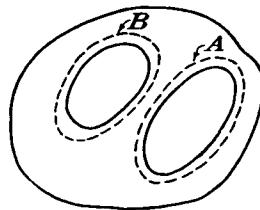


FIG. 81.

It is possible to determine the stresses by an optical method, using polarized light (see page 131) and this conclusion means that the experimental results obtained with a transparent material can be applied immediately to any other material such as steel if the external forces are the same.

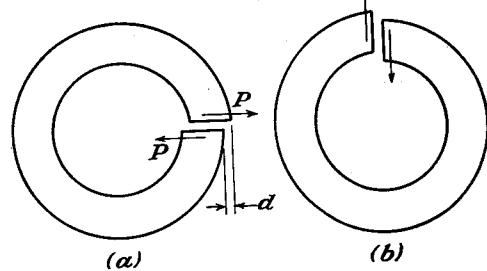


FIG. 82.

It was mentioned before (see page 68) that the physical meaning of many-valued solutions can be demonstrated by considering *initial stresses* in a multiply-connected body. Suppose, for instance, that Eq. (f) above is not satisfied. The corresponding displacement is shown in Fig. 82a. Such a displacement can be produced by cutting the ring and applying forces P . If now the ends of the ring are joined again by welding or other means, a ring with initial stresses is obtained. The magnitudes of these stresses depend on the initial displacement d .⁴ A similar

¹ Loc. cit.² It must be remembered that the body forces were taken as zero.³ An investigation of this subject is given by L. N. G. Filon, *Brit. Assoc. Advancement Sci. Rept.*, 1921. See E. G. Coker and L. N. G. Filon, "Photo-elasticity," Arts. 6.07 and 6.16.⁴ A discussion of such stresses is given by A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905. A general theory is given by V. Volterra, *Ann. école norm.*, Paris, series 3, vol. 24, pp. 401-517, 1907. See also A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 221, 1927; J. N. Goodier, *Proc. Fifth Intern. Congr. Applied Mechanics*, 1938, p. 129.

effect is obtained by making a cut along a vertical radius and imposing an initial displacement of one end of the ring with respect to the other in the vertical direction (Fig. 82b). The initial stresses produced in the cases shown in Figs. 82a and 82b correspond to the many-valued terms of the general solution when Eqs. (f) and (g) are not satisfied.

The complete solution of these problems can be obtained by applying the results of Art. 31. The displacements given by Eqs. (q) of Art. 31 will be found to have the required type of discontinuity when applied to a ring (see Prob. 4, page 126).

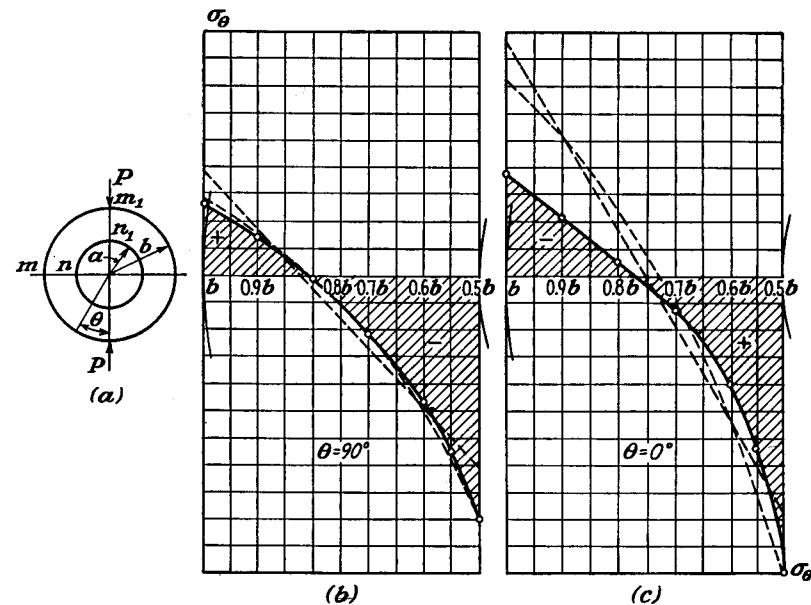


FIG. 83.

40. Applications of the General Solution in Polar Coordinates. As a first application of the general solution of the two-dimensional problem in polar coordinates let us consider a circular ring compressed by two equal and opposite forces acting along a diameter¹ (Fig. 83a). We begin with the solution for a solid disk (Art. 37). By cutting out a concentric hole of radius a in this disk, we are left with normal and shearing forces distributed round the edge of the hole. These forces can be annulled by superposing an equal and opposite system of forces. This latter system can be represented with sufficient accuracy by using the first few terms of a Fourier series. Then the corresponding stresses in the ring are obtained by using the general solution of the previous article. These stresses together with the stresses calculated as for a solid disk constitute the total stresses

¹ See S. Timoshenko, *Bull. Polytech. Inst. Kiew*, 1910, and *Phil. Mag.*, vol. 44, p. 1014, 1922. See also K. Wieghardt, *Sitzber. Akad. Wiss., Wien*, vol. 124, Abt. II, p. 1119, 1915.

in the ring. The ratios $\sigma_\theta:2P/\pi b$, calculated in this manner for various points of the cross sections mn and m_1n_1 for the case $b = 2a$, are given in the table below.¹

$r =$	b	0.9b	0.8b	0.7b	0.6b	0.5b
Exact theory						
mn	2.610	1.477	-0.113	-2.012	-4.610	-8.942
m_1n_1	-3.788	-2.185	-0.594	1.240	4.002	10.147
Hyperbolic stress distribution						
mn	2.885	1.602	0.001	-2.060	-4.806	-8.653
m_1n_1	-7.036	-5.010	-2.482	0.772	5.108	11.18
Linear stress distribution						
mn	3.90	1.71	-0.48	-2.67	-4.86	-7.04
m_1n_1	-8.67	-5.20	-1.73	1.73	5.20	8.67

For comparison we give the values of the same stresses calculated from the two elementary theories based on the following assumptions: (1) that cross sections remain plane; in which case the normal stresses over the cross section follow a hyperbolic law; (2) that the stresses are distributed according to a linear law. The table shows that for the cross section mn , which is at a comparatively large distance from the points of application of the loads P , the hyperbolic stress distribution gives results which are very nearly exact. The error in the maximum stress is only about 3 per cent. For the cross section m_1n_1 the errors of the approximate solution are much larger. It is interesting to note that the resultant of the normal stresses over the cross section m_1n_1 is P/π .

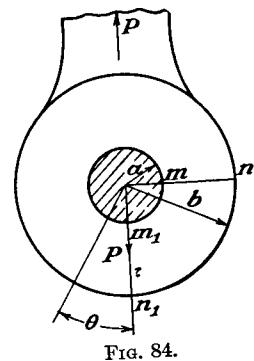


FIG. 84.

above to the case of two equal and opposite forces can be used for the general case of loading of a circular ring by concentrated forces.²

As a second example we consider the end of an eyebar³ (Fig. 84). The distribution of pressures along the edge of the hole depends on the amount of clearance between the bolt and the hole. The following results are obtained on the assumption

¹ The thickness of the plate is taken as unity.

² L. N. G. Filon, The Stresses in a Circular Ring, *Selected Engineering Papers*,

No. 12, London, 1924, published by the Institution of Civil Engineers.

³ H. Reissner, *Jahrb. wiss. Gesellsch. Luftfahrt*, p. 126, 1928; H. Reissner, and

F. Strauch, *Ingenieur-Archiv*, vol. 4, p. 481, 1933.

tion that there are only normal pressures acting on the inner and outer boundaries having the magnitudes:¹

$$(\sigma_r)_{r=a} = -\frac{2P}{\pi} \frac{\cos \theta}{a} \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$(\sigma_r)_{r=b} = -\frac{2P}{\pi} \frac{\cos \theta}{b} \text{ for } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$$

i.e., the pressures are distributed along the lower half of the inner edge and the upper half of the outer edge of the eye-shaped end of the bar. After expanding

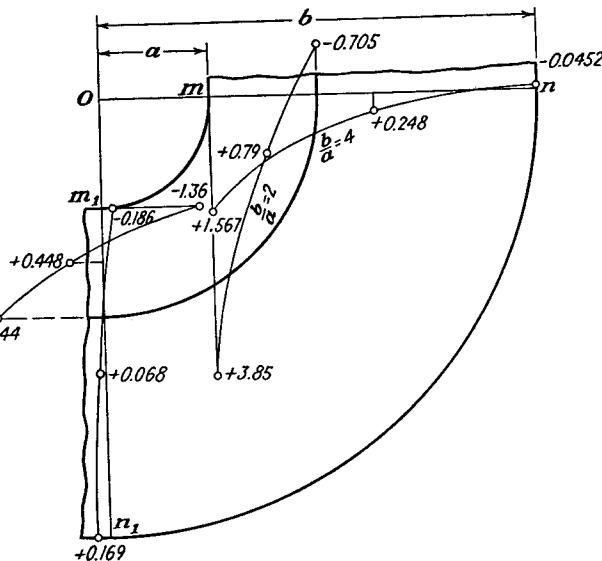


FIG. 85.

these distributions into trigonometric series, the stresses can be calculated by using the general solution (81) of the previous article. Figure 85 shows the values of the ratio $\sigma_\theta:P/2a$, calculated for the cross sections mn and m_1n_1 , for $b/a = 4$ and $b/a = 2$.² It should be noted that in this case the resultant of the forces acting on each boundary does not vanish, hence the stress distribution depends on elastic constants of the material. The above calculations are for Poisson's ratio $\nu = 0.3$.

41. A Wedge Loaded along the Faces. The general solution (81) can be used also for polynomial distributions of load on the faces of a wedge.³ By calculating

¹ P is the force per unit thickness of the plate.

² For experimental determinations of the stress distribution in eyebars by the photoelastic method see E. G. Coker and L. N. G. Filon, "Photo-elasticity," Art. 6.18, and K. Takemura and Y. Hosokawa, *Rept. 12*, 1926, *Aeronaut. Research Inst., Tōkyō Imp. Univ.* The stress distribution in steel eyebars was investigated by J. Mathar, *Forschungsarbeiten*, No. 306, 1928.

³ See S. Timoshenko, "Theory of Elasticity," Russian edition, p. 119, St. Petersburg, 1914.

the stress components from Eq. (81) in the usual way, and taking only the terms containing r^n with $n \geq 0$, we find the following expressions for the stress components in ascending powers of r :

$$\begin{aligned}\sigma_\theta &= 2b_0 + 2d_0\theta + 2a_2 \cos 2\theta + 2c_2 \sin 2\theta \\ &\quad + 6r(b_1 \cos \theta + d_1 \sin \theta + a_3 \cos 3\theta + c_3 \sin 3\theta) \\ &\quad + 12r^2(b_2 \cos 2\theta + d_2 \sin 2\theta + a_4 \cos 4\theta + c_4 \sin 4\theta) \\ &\quad \dots \\ (n+2)(n+1)r^n[b_n \cos n\theta + d_n \sin n\theta + a_{n+2} \cos(n+2)\theta \\ &\quad + c_{n+2} \sin(n+2)\theta]\end{aligned}\quad (83)$$

$$\begin{aligned}\tau_{r\theta} &= -d_0 + 2a_2 \sin 2\theta - 2c_2 \cos 2\theta \\ &\quad + r(2b_1 \sin \theta - 2d_1 \cos \theta + 6a_3 \sin 3\theta - 6c_3 \cos 3\theta) \\ &\quad + r^2(6b_2 \sin 2\theta - 6d_2 \cos 2\theta + 12a_4 \sin 4\theta - 12c_4 \cos 4\theta) \\ &\quad \dots \\ &\quad + r^n[n(n+1)b_n \sin n\theta - n(n+1)d_n \cos n\theta + (n+1)(n+2) \\ &\quad a_{n+2} \sin(n+2)\theta - (n+1)(n+2)c_{n+2} \cos(n+2)\theta]\end{aligned}$$

Thus each power of r is associated with four arbitrary parameters so that, if the applied stresses on the boundaries, $\theta = \alpha$ and $\theta = \beta$, are given as polynomials in r , the stresses in the wedge included between these boundaries are determined.

If, for instance, the boundary conditions are¹

$$\begin{aligned}(\sigma_\theta)_{\theta=\alpha} &= N_0 + N_1 r + N_2 r^2 + \dots \\ (\sigma_\theta)_{\theta=\beta} &= N'_0 + N'_1 r + N'_2 r^2 + \dots \\ (\tau_{r\theta})_{\theta=\alpha} &= S_0 + S_1 r + S_2 r^2 + \dots \\ (\tau_{r\theta})_{\theta=\beta} &= S'_0 + S'_1 r + S'_2 r^2 + \dots\end{aligned}\quad (a)$$

we have, by equating coefficients of powers of r ,

$$\begin{aligned}2(b_0 + d_0\alpha + a_2 \cos 2\alpha + c_2 \sin 2\alpha) &= N_0 \\ 6(b_1 \cos \alpha + d_1 \sin \alpha + a_3 \cos 3\alpha + c_3 \sin 3\alpha) &= N_1\end{aligned}\quad (b)$$

and generally

$$(n+2)(n+1)[b_n \cos n\alpha + d_n \sin n\alpha + a_{n+2} \cos(n+2)\alpha \\ + c_{n+2} \sin(n+2)\alpha] = N_n$$

with three other groups of equations for σ_θ at $\theta = \beta$ and $\tau_{r\theta}$ at $\theta = \alpha$ and $\theta = \beta$. These equations are sufficient for calculating the constants entering into the solution (83).

Let us consider, as an example, the case shown in Fig. 86. A uniform normal pressure q is acting on the face $\theta = 0$ of the wedge and the other face $\theta = \beta$ is free from forces. Using only the first lines in the expressions (83) for σ_θ and $\tau_{r\theta}$ the equations for determining constants b_0 , d_0 , a_2 , and c_2 are

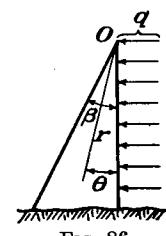


FIG. 86.

$$2b_0 + 2a_2 = -q$$

$$2b_0 + 2d_0\beta + 2a_2 \cos 2\beta + 2c_2 \sin 2\beta = 0$$

$$-d_0 - 2c_2 = 0$$

$$-d_0 + 2a_2 \sin 2\beta - 2c_2 \cos 2\beta = 0$$

¹ The terms N_0 , N'_0 , S_0 , S'_0 are not independent. They represent stress at the corner $r = 0$ and only three can be assigned.

from which (writing $k = \tan \beta - \beta$) we find

$$c_2 = \frac{q}{4k}, \quad a_2 = -\frac{q \tan \beta}{4k}, \quad d_0 = -\frac{q}{2k}, \quad 2b_0 = -q + \frac{q \tan \beta}{2k}$$

Substituting in Eqs. (83), we obtain¹

$$\begin{aligned}\sigma_\theta &= \frac{q}{k} \left(-k + \frac{1}{2} \tan \beta - \theta - \frac{1}{2} \tan \beta \cos 2\theta + \frac{1}{2} \sin 2\theta \right) \\ \tau_{r\theta} &= \frac{q}{k} \left(\frac{1}{2} - \frac{1}{2} \tan \beta \sin 2\theta - \frac{1}{2} \cos 2\theta \right) \\ \sigma_r &= \frac{q}{k} \left(-k + \frac{1}{2} \tan \beta - \theta - \frac{1}{2} \sin 2\theta + \frac{1}{2} \tan \beta \cos 2\theta \right)\end{aligned}\quad (c)$$

The stress components for any other term in the polynomial load distribution (a) may be obtained in a similar manner.

The method developed above for calculating stresses in a wedge is applicable to a semi-infinite plate by making the angle β of the wedge equal to π . The stresses for the case shown in Fig. 87, for instance, are obtained from Eqs. (c) by substituting $\beta = \pi$ in these equations. Then

$$\begin{aligned}\sigma_\theta &= -\frac{q}{\pi} \left(\pi - \theta + \frac{1}{2} \sin 2\theta \right) \\ \tau_{r\theta} &= -\frac{q}{2\pi} (1 - \cos 2\theta) \\ \sigma_r &= -\frac{q}{\pi} \left(\pi - \theta - \frac{1}{2} \sin 2\theta \right)\end{aligned}\quad (d)$$

Problems

1. Verify Eq. (d) of Art. 25 in the case

$$\phi = x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = r^4 \cos 2\theta$$

2. Examine the significance of the stress function $C\theta$ where C is a constant. Apply it to a ring $a < r < b$, and to an infinite plate.

A ring is fixed at $r = a$ and subjected to a uniform circumferential shear at $r = b$ forming a couple M . Using Eqs. (49), (50), (51), find an expression for the circumferential displacement v at $r = b$.

3. Show that in the problem of Fig. 45, if the inner radius a is small compared with the outer radius b , the value of σ_θ at the inside is given by

$$\frac{\alpha E}{4\pi} \left(1 - 2 \log \frac{b}{a} \right)$$

and so is large, and negative when α is positive (the gap is being closed).

¹ This solution was obtained by another method by M. Levy, *Compt. rend.*, vol. 126, p. 1235, 1898. See also P. Fillunger, *Z. Math. Physik*, vol. 60, 1912. An application of stress functions of this type to tapered box beams is given by E. Reissner, *J. Aeronaut. Sci.*, vol. 7, p. 353, 1940. Other loads on wedges are considered by C. J. Tranter, *Quart. J. Mech. and Appl. Math.*, vol. 1, p. 125, 1948.

What is the largest gap (value of α) which can be closed without exceeding the elastic limit, if $b/a = 10$, $E = 3 \times 10^7$ p.s.i., elastic limit = 4×10^4 p.s.i.?

4. Use the results of Art. 31 to obtain formulas for the stresses due to closing the parallel gap δ in Fig. 88a, and due to the sliding of amount δ in Fig. 88b, in terms of δ .

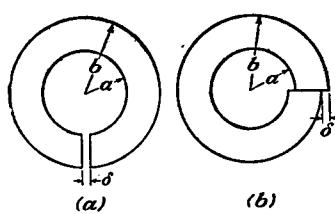


FIG. 88.

5. Find by superposition from Eqs. (62) the stresses in the infinite plate with a hole when the undisturbed stress at infinity is uniform tension S in both the x - and y -directions.

The results should correspond with Eqs. (45) for the special case $b/a \rightarrow \infty$, $p_i = 0$, $p_o = -S$. Use this as a check.

6. Find expressions for the displacements

corresponding to the stresses (62), and verify that they are single-valued.

7. Convert the stress function (a) of Art. 33 to Cartesian coordinates and hence derive the values of σ_x , σ_y , τ_{xy} which are equivalent to the stress distribution of Eqs. (66'). Show that these values approach zero as the distance from the force increases in any direction.

8. Verify that in the special case of $\alpha = \pi/2$ the stress function (d), page 98, agrees with Eq. (69), and investigate whether the stress distribution (e), page 99, tends to agree with elementary bending theory for small α .

9. Show by evaluating the force resultants that the stress distribution (e), page 99, does in fact correspond to loading by a pure couple M at the tip of the wedge.

10. A force P per unit thickness is applied by a knife-edge to the bottom of a 90-deg. notch in a large plate as indicated in Fig. 89. Evaluate the stresses, and the horizontal force transmitted across an arc AB .

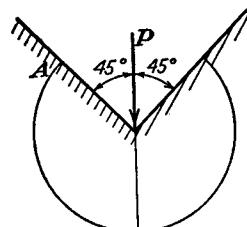


FIG. 89.

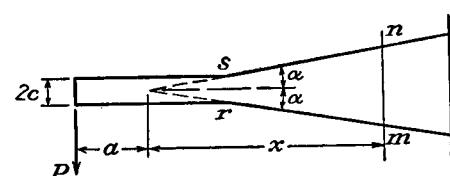


FIG. 90.

11. Find an expression for the stress σ_x on the section mn indicated in Fig. 90. The wedge theory of the present chapter and the cantilever theory of Chap. 3 give different stress distributions for the junction rs . Comment on this.

12. Determine the value of the constant C in the stress function

$$\phi = C[r^2(\alpha - \theta) + r^2 \sin \theta \cos \theta - r^2 \cos^2 \theta \tan \alpha]$$

required to satisfy the conditions on the upper and lower edges of the triangular plate shown in Fig. 91. Evaluate the stress components σ_x , τ_{xy} for a vertical

section mn . Draw curves for the case $\alpha = 20$ deg. and draw also for comparison the curves given by elementary beam theory.

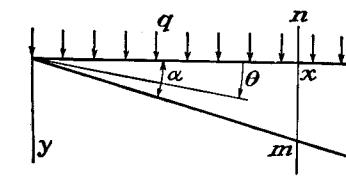


FIG. 91.

13. Determine the value of the constant C in the stress function

$$\phi = Cr^2(\cos 2\theta - \cos 2\alpha)$$

required to satisfy the conditions

$$\begin{aligned} \sigma_\theta &= 0, & \tau_{r\theta} &= s \text{ on } \theta = \alpha \\ \sigma_\theta &= 0, & \tau_{r\theta} &= -s \text{ on } \theta = -\alpha \end{aligned}$$

corresponding to uniform shear loading on each edge of a wedge, directed away from the vertex. Verify that no concentrated force or couple acts on the vertex.

14. Find the stress function of the type

$$a_3r^3 \cos 3\theta + b_1r^3 \cos \theta$$

which satisfies the conditions

$$\begin{aligned} \sigma_\theta &= 0, & \tau_{r\theta} &= sr \text{ on } \theta = \alpha \\ \sigma_\theta &= 0, & \tau_{r\theta} &= -sr \text{ on } \theta = -\alpha \end{aligned}$$

s being a constant. Sketch the loading for positive s .

15. Find the stress function of the type

$$a_4r^4 \cos 4\theta + b_2r^4 \cos 2\theta$$

which satisfies the conditions

$$\begin{aligned} \sigma_\theta &= 0, & \tau_{r\theta} &= sr^2 \text{ on } \theta = \alpha \\ \sigma_\theta &= 0, & \tau_{r\theta} &= -sr^2 \text{ on } \theta = -\alpha \end{aligned}$$

s being a constant. Sketch the loading.

16. Derive the stress distribution

$$\begin{aligned} \sigma_x &= -\frac{p}{\pi} \left(\arctan \frac{y}{x} + \frac{xy}{x^2 + y^2} \right), & \tau_{xy} &= -\frac{p}{\pi} \frac{y^2}{x^2 + y^2} \\ \sigma_y &= -\frac{p}{\pi} \left(\arctan \frac{y}{x} - \frac{xy}{x^2 + y^2} \right) \end{aligned}$$

from the stress function [see Eq. (a), Art. 34]

$$\phi = -\frac{p}{2\pi} \left((x^2 + y^2) \arctan \frac{y}{x} - xy \right)$$

and show that it solves the problem of the semi-infinite plate indicated in Fig. 92, with axes as shown. The load extends indefinitely to the left.

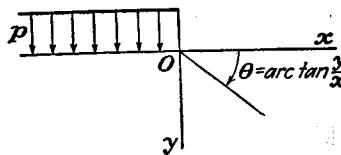


FIG. 92.

Examine the value of τ_{xy} , (a) approaching O along the boundary Ox , (b) approaching O along the y -axis (the discrepancy is due to the discontinuity of loading at O).

17. Show that the stress function

$$\phi = \frac{s}{\pi} \left[\frac{1}{2} y^2 \log(x^2 + y^2) + xy \arctan \frac{y}{x} - y^2 \right]$$

solves the problem of the semi-infinite plate indicated in Fig. 93, the uniform shear loading s extending from O indefinitely to the left. Show that σ_x grows without limit as O is approached from any direction. (This is due to the discontinuity of load at O . A finite value is obtained when this is smoothed out, depending on the loading curve in the neighborhood of O .)

18. By superposition, using the results of Prob. 16, obtain σ_x , σ_y , τ_{xy} , for pressure p on a segment $-a < x < a$ of the straight edge of the semi-infinite plate. Show that the shear stress is

$$\tau_{xy} = -\frac{p}{\pi} \frac{4axy^2}{[(x-a)^2 + y^2][(x+a)^2 + y^2]}$$

and examine the behavior of this stress as the point $x = a$, $y = 0$ is approached (a) along the boundary, (b) along the line $x = a$.

19. Using the results of Prob. 17, sketch the variation of σ_x along the edge $y = 0$, for a uniform shear load s applied to the segment $-a < x < a$ of the edge.

20. Show that the stress function

$$\phi = -\frac{p}{2\pi a} \left[\left(\frac{1}{3} x^3 + xy^2 \right) \arctan \frac{y}{x} + \frac{1}{3} y^3 \log(x^2 + y^2) - \frac{1}{3} x^2 y \right]$$

solves the problem of the semi-infinite plate indicated in Fig. 94, the linearly increasing pressure load extending indefinitely to the left.

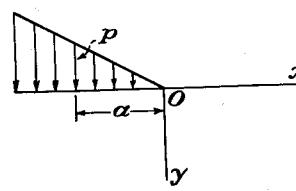


FIG. 94.

21. Show that if the pressure loading of Prob. 20 is replaced by shear loading, s , replacing p , the appropriate stress function is

$$\phi = \frac{s}{2\pi a} \left[xy^2 \log(x^2 + y^2) + (x^2 y - y^3) \arctan \frac{y}{x} - 3xy^2 \right]$$

22. Show how the distributions of load indicated in Fig. 95 may be obtained by superposition from loading of the type indicated in Fig. 94.

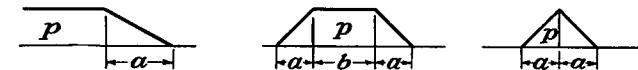


FIG. 95.

23. Show that the parabolic loading indicated in Fig. 96 is given by the stress function

$$-\frac{p}{\pi} \left\{ -\frac{xy^3}{3a^2} \log \frac{r_2^2}{r_1^2} - \left[\frac{a^2}{4} + \frac{1}{2} (x^2 + y^2) \left(1 - \frac{x^2}{6a^2} + \frac{y^2}{2a^2} \right) \right] \alpha + \frac{2}{3} ax\beta + \frac{1}{2} ay \left(1 - \frac{x^2}{3a^2} + \frac{y^2}{a^2} \right) \right\}$$

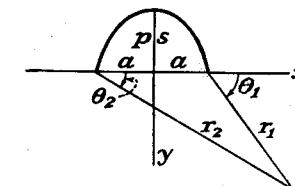


FIG. 96.

for pressure, and

$$\frac{s}{\pi} \left\{ \frac{y^2}{6a^2} (3a^2 - 3x^2 + y^2) \log \frac{r_2^2}{r_1^2} + \frac{2}{3} ay\beta + \frac{xy}{3a^2} (x^2 - 3y^2 - 3a^2)\alpha + \frac{4xy^2}{3a} \right\}$$

for shear, where

$$\begin{aligned} r_1^2 &= (x - a)^2 + y^2, & r_2^2 &= (x + a)^2 + y^2 \\ \alpha &= \theta_1 - \theta_2 = \arctan \frac{2ay}{x^2 + y^2 - a^2}, & \beta &= \theta_1 + \theta_2 = \arctan \frac{2xy}{x^2 - y^2 - a^2} \end{aligned}$$

24. Show that in the problem of Fig. 72 there is a tensile stress $\sigma_x = 2P/\pi d$ along the vertical diameter, except at A and B . Account for the equilibrium of the semicircular part ADB by considering small semicircles about A and B in the manner of Figs. 65c and d.

25. Verify that the stress function

$$\begin{aligned} \phi = -\frac{P}{\pi} \left\{ \nu r \cos \theta - \frac{1}{4} (1 - \nu)r \log r \cos \theta - \frac{1}{2} r\theta \sin \theta \right. \\ \left. + \frac{d}{4} \log r - \frac{d^2}{32} (3 - \nu) \frac{1}{r} \cos \theta \right\} \end{aligned}$$

satisfies the boundary conditions for a force P acting in a hole in an infinite plate with zero stress at infinity, and that the circumferential stress round the hole is

$$\frac{P}{\pi d} [2 + (3 - \nu) \cos \theta]$$

except at A (Fig. 97).

Show that it also corresponds to single-valued displacements.

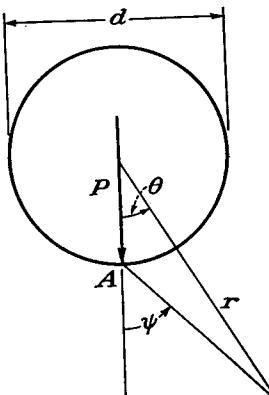


FIG. 97.

26. Deduce from Prob. 25 by integration the circumferential stress round the hole due to uniform pressure p in the hole, and check the result by means of Eqs. (46).

27. Find the general form of $f(r)$ in the stress function $\theta f(r)$, and find the expressions for the stress components σ_r , σ_θ , $\tau_{r\theta}$. Could such a stress function apply to a closed ring?

CHAPTER 5

THE PHOTOELASTIC METHOD

42. Photoelastic Stress Measurement. The boundaries of the plates so far considered have been of simple geometrical form. For more complex shapes the difficulties of obtaining analytical solutions become formidable, but these difficulties can be avoided by resorting to numerical methods (which are discussed in the Appendix) or to experimental methods, such as the measurement of surface strains by extensometers and strain gauges, or the *photoelastic* method. This method is based on the discovery of David Brewster¹ that when a piece of glass is stressed and viewed by *polarized* light transmitted through it, a brilliant color pattern due to the stress is seen. He suggested that these color patterns might serve for the measurement of stresses in engineering structures such as masonry bridges, a glass model being examined in polarized light under various loading conditions. This suggestion went unheeded by engineers at the time. Comparisons of photoelastic color patterns with analytical solutions were made by the physicist Maxwell.² The suggestion was adopted much later by C. Wilson in a study of the stresses in a beam with a concentrated load,³ and by A. Mesnager in an investigation of arch bridges.⁴ The method was developed and extensively applied by E. G. Coker⁵ who introduced celluloid as the model material. Later investigators have used bakelite, and more recently, fosterite.⁶

In the following we consider only the simplest form of photoelastic apparatus.⁷ Ordinary light is regarded as consisting of vibrations in

¹ D. Brewster, *Trans. Roy. Soc. (London)*, 1816, p. 156.

² J. Clerk Maxwell, *Sci. Papers*, vol. 1, p. 30.

³ C. Wilson, *Phil. Mag.*, vol. 32, p. 481, 1891.

⁴ A. Mesnager, *Ann. ponts et chaussées*, 4^e Trimestre, p. 129, 1901, and 9^e Series, vol. 16, p. 135, 1913.

⁵ The numerous publications of Prof. Coker are compiled in his papers: *Gen. Elec. Rev.*, vol. 23, p. 870, 1920, and *J. Franklin Inst.*, vol. 199, p. 289, 1925. See also the book by E. G. Coker and L. N. G. Filon, "Photo-elasticity," Cambridge University Press, 1931.

⁶ M. M. Leven, *Proc. Soc. Expl. Stress Analysis*, vol. 6, no. 1, p. 19, 1948.

⁷ More complete treatments may be found in the following books: "Handbook of Experimental Stress Analysis," 1950; M. M. Frocht, "Photoelasticity," 2 vols., 1941 and 1948; and the book cited in footnote 5.

all directions transverse to the direction of the ray. By reflection from a piece of plate glass covered on one side with black paint, or by transmission through a polarizer—a *Nicol prism*, or *Polaroid plate*—we obtain a more or less polarized beam of light in which transverse vibrations in a definite direction prevail. The plane containing this direction and a ray is the plane of polarization. This is the kind of light used in the photoelastic investigation of stress. We shall consider only monochromatic light.

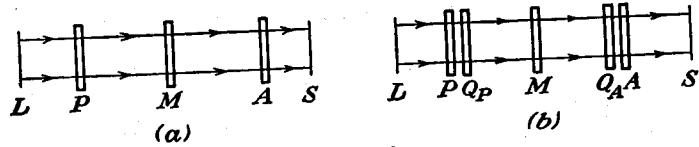


FIG. 98.

Figure 98a represents diagrammatically a *plane polariscope*. A beam of light originating at L passes through a polarizer P , then through the transparent model M which modifies the light according to the stress, then through an *analyzer*—another polarizer A —to a screen S , on which a pattern of interference fringes (Figs. 100 to 104) is formed.

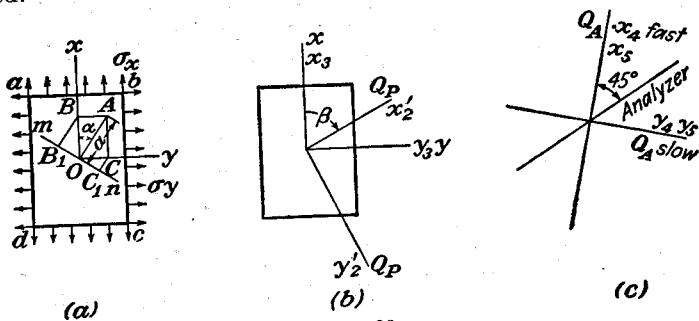


FIG. 99.

In Fig. 99a, $abcd$ represents a small element of the left-hand face of the model M , the directions of the principal stresses σ_x , σ_y being drawn vertically and horizontally for convenience. A ray of light polarized in the vertical plane OA (Fig. 99) arrives from P , the direction of the ray in Fig. 99 being through the paper. The vibration is simple harmonic and may be represented by the transverse "displacement"

$$s = a \cos pt \quad (a)$$

in the direction OA , where p is 2π times the frequency, depending on the color of the light, and t is the time.

The displacement (a) in the plane OA is resolved into components with amplitudes $OB = a \cos \alpha$ and $OC = a \sin \alpha$ in the planes Ox , Oy respectively. The corresponding displacement components are

$$x = a \cos \alpha \cos pt, \quad y = a \sin \alpha \cos pt \quad (b)$$

The effect of the principal stresses σ_x and σ_y , acting at the point O of the plate, is to change the velocities with which these components are propagated through the plate. Let v_x and v_y denote the velocities in the planes Ox and Oy . If h is the thickness of the plate, the times required for the two components to traverse the thickness are

$$t_1 = \frac{h}{v_x}, \quad t_2 = \frac{h}{v_y} \quad (c)$$

Since the light waves are transmitted without change of form, the x -displacement, x_1 , of the light leaving the plate at time t corresponds to the x -displacement of the light entering the plate at a time t_1 earlier. Thus

$$x_1 = a \cos \alpha \cos p(t - t_1), \quad y_1 = a \sin \alpha \cos p(t - t_2) \quad (d)$$

On leaving the plate, therefore, these components have a *phase difference* $\Delta = p(t_2 - t_1)$. It was established experimentally that for a given material at a given temperature, and for light of a given wave length, this phase difference is proportional to the difference in the principal stresses. It is also proportional to the thickness of the plate. The relationship is usually expressed in the form

$$\Delta = \frac{2\pi h}{\lambda} \cdot C(\sigma_x - \sigma_y) \quad (e)$$

where λ is the wave length (*in vacuo*), and C the experimentally determined *stress-optical coefficient*. C depends on the wave length and temperature as well as the material.

The analyzer A transmits only vibrations or components in its own plane of polarization. If this is at right angles to the plane of polarization of the polarizer,¹ and if the model is removed, no light is transmitted by A and the screen is dark. We now consider what occurs when the model is present. The components (d) on arrival at the analyzer may be represented as

$$x_2 = a \cos \alpha \cos \psi, \quad y_2 = a \sin \alpha \cos (\psi - \Delta) \quad (f)$$

since they retain the phase difference Δ in traveling from M to A . Here ψ denotes $pt + \text{constant}$.

¹ The polarizer and analyzer are then said to be "crossed."

The plane of polarization of A is represented by mn in Fig. 99a, for convenience. It is set perpendicular to OA . The components of the vibrations (f) which are transmitted by A are the components along Om , which are, using Eqs. (f),

$$x_2 \sin \alpha = \frac{1}{2}a \sin 2\alpha \cos \psi, \quad -y_2 \cos \alpha = -\frac{1}{2}a \sin 2\alpha \cos (\psi - \Delta)$$

The resultant vibration along mn is therefore

$$\frac{1}{2}a \sin 2\alpha [\cos \psi - \cos (\psi - \Delta)] = -a \sin 2\alpha \sin \frac{\Delta}{2} \sin \left(\psi - \frac{\Delta}{2}\right)$$

The factor $\sin \left(\psi - \frac{\Delta}{2}\right)$ represents the simple harmonic variation with time. The amplitude is

$$a \sin 2\alpha \sin \frac{\Delta}{2} \quad (g)$$

It follows that some light will reach the screen unless either $\sin 2\alpha = 0$ or $\sin \Delta/2 = 0$. If $\sin 2\alpha = 0$ the directions of the principal stresses are parallel to the (perpendicular) directions of polarization of P and A . Thus rays which pass through such points of M will be extinguished and the corresponding points on the screen S will be dark. These points usually lie on one or more curves, indicated by a dark band on S . Such a curve is called an "isoclinic." Very short lines parallel to the axes of P and A may be drawn at numerous points on it to record the (parallel) directions of the principal stresses at these points. By setting P and A in different (perpendicular) orientations, different isoclinics are obtained. The short lines then cover the field like a pattern of iron filings over a magnet, and it is possible to draw curves which are tangential at each point to the principal axes of stress. The latter lines are trajectories of the principal stresses.

If $\sin \Delta/2 = 0$, then $\Delta = 2n\pi$ where $n = 0, 1, 2, \dots$. When $\Delta = 0$, the principal stresses are equal. Points where this occurs are called isotropic points, and will of course be dark. Points at which $n = 1$ form a dark band, or fringe, of the first order, points for which $n = 2$ a fringe of the second order, and so on. These fringes are called *isochromatics* (because, when white light is used, they correspond to extinction of a certain wave length and therefore to a color band). It follows from Eq. (e) that $\sigma_x - \sigma_y$ on a fringe $n = 2$ has twice the value of $\sigma_x - \sigma_y$ on a fringe $n = 1$, and so on. To evaluate the principal stress differences it is therefore necessary to know the order of the fringes, and the stress difference represented by the fringe of the first order, or *fringe value*.

The fringe value may be determined by loading a strip in simple tension. Since the stress is uniform there are no fringes, the whole piece appearing uniformly bright or dark on the screen. At zero load it will be dark. As the stress is increased, it will brighten, then darken as the stress difference (here simply the tensile stress) approaches the fringe value. On further increase of load it brightens once more, then darkens again when the stress is twice the fringe value, and so on.

Similar cycles of brightness and darkness will clearly occur at any point of a nonuniform stress field as the load is increased—provided the stress difference at the point reaches a multiple of the fringe value. These cycles at individual points correspond, in the view of the whole field, to gradual movement of the fringes, including the entrance of new

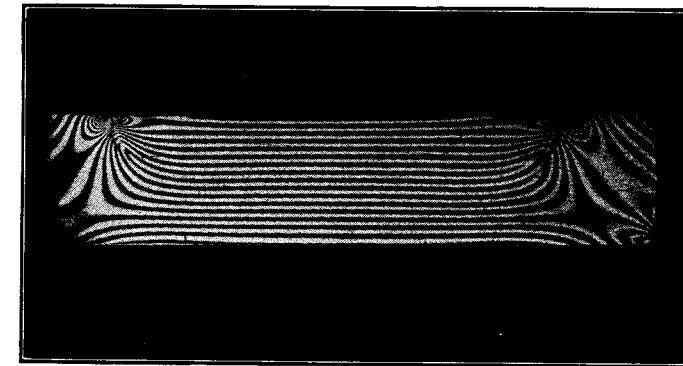


FIG. 100.

fringes, as the load is increased. The orders of the fringes may therefore be determined by observing this movement and counting the fringes.

For instance a strip in pure bending gives a fringe pattern as shown in Fig. 100. The parallel fringes accord with the fact that in the portion of the strip away from the points of application of the loads, the stress distribution is the same in all vertical cross sections. By watching the screen as the load is gradually increased we should observe that new fringes appear at the top and bottom of the strip and move toward the middle, the fringes as a whole becoming more and more closely packed. There will be one fringe at the neutral axis which remains dark throughout. This will clearly be the fringe of zero order ($n = 0$).

43. Circular Polariscopic. We have seen that the plane polariscope just discussed provides, for a chosen value of α , the corresponding isoclinic as well as the isochromatics or fringes. Figure 100 should

therefore show darkness wherever the orientations of the principal stresses coincide with the orientations of the polarizer and analyzer. Figure 100 was actually obtained in a *circular polariscope*, which is a modification of the *plane polariscope* designed to eliminate the isoclinics.¹ It is indicated diagrammatically in Fig. 98b, which corresponds to Fig. 98a with the addition of two *quarter-wave plates* Q_P, Q_A . A quarter-wave plate is a crystal plate having two polarizing axes, which affect the light like a uniformly stressed model, introducing a phase difference Δ as in Eq. (f) — but the thickness of the quarter-wave plate is chosen so that $\Delta = \pi/2$. Using Eq. (f) with this value of Δ for the light leaving Q_P , we observe that a simple result is obtained by choosing α , now denoting the angle between the plane of polarization of P and one of the axes of Q_P , as 45 deg. Then we may write

$$x_2' = \frac{a}{\sqrt{2}} \cos \psi, \quad y_2' = \frac{a}{\sqrt{2}} \cos \left(\psi - \frac{\pi}{2} \right) = \frac{a}{\sqrt{2}} \sin \psi \quad (h)$$

Here x_2' corresponds to the "fast" axis of the quarter-wave plate. A point moving with these displacement components (ψ always having the form $pt + \text{constant}$ for a given point along the ray) moves in a circle. Such light is therefore described as *circular polarized*.

The components (h) are along the axes of polarization of Q_P . Using β for the angle between x_2' and the direction of σ_x in the model (Fig. 99b), and Δ once more for the phase difference caused by the stressed element, we have, for the light leaving the model, due to x_2' only

$$x_3 = \frac{a}{\sqrt{2}} \cos \beta \cos \psi, \quad y_3 = \frac{a}{\sqrt{2}} \sin \beta \cos (\psi - \Delta) \quad (i)$$

and for the light due to y_2' only

$$x_3 = -\frac{a}{\sqrt{2}} \sin \beta \sin \psi, \quad y_3 = \frac{a}{\sqrt{2}} \cos \beta \sin (\psi - \Delta) \quad (j)$$

Adding the components in Eqs. (i) and (j) we find for the light leaving the model

$$x_3 = \frac{a}{\sqrt{2}} \cos \psi', \quad y_3 = \frac{a}{\sqrt{2}} \sin (\psi' - \Delta) \quad (k)$$

where $\psi' = \psi + \beta$.

¹ If the polarizer and analyzer rotate, their axes remaining perpendicular, the fringes remain stationary and the isoclinics move. If the rotation is rapid the isoclinics are no longer visible. The circular polariscope achieves the same effect by purely optical means.

Before examining the effects of Q_A and A upon the light it is convenient to represent the motion (k) as a superposition of two circular motions. This may be done as follows. Denoting $\psi' - (\Delta/2)$ by ψ'' , and $a/\sqrt{2}$ by b , Eqs. (k) give

$$x_3 = b \cos \left(\psi'' + \frac{\Delta}{2} \right) = b \left(\cos \frac{\Delta}{2} \cos \psi'' - \sin \frac{\Delta}{2} \sin \psi'' \right) \quad (l)$$

$$y_3 = b \sin \left(\psi'' - \frac{\Delta}{2} \right) = b \left(\cos \frac{\Delta}{2} \sin \psi'' - \sin \frac{\Delta}{2} \cos \psi'' \right) \quad (m)$$

which represent the superposition of a circular motion of radius $b \cos (\Delta/2)$, clockwise in Fig. 99b (where the ray passes downward through the paper), and a circular motion of radius $b \sin (\Delta/2)$, counterclockwise.

We may now show that if the polarizing axis of A is set at 45 deg. to the polarizing axes of Q_A , one of these circular motions is transmitted to the screen S , the other extinguished, and the desired result—isochromatics without isoclinics—is obtained.

The components x_3, y_3 in Eqs. (l) and (m) are along the directions of principal stress in a model. A change of axes for a circular motion will merely result in a change of the phase angle ψ'' by a constant. Thus the clockwise circular motion can be represented by components of the form

$$x_4 = c \cos \psi, \quad y_4 = c \sin \psi \quad (n)$$

along the axes of Q_A , where ψ is again of the form $pt + \text{constant}$. Identifying x_4 with the *fast* axis of Q_A , we shall have on emergence from Q_A

$$x_5 = c \cos \psi, \quad y_5 = c \sin \left(\psi - \frac{\pi}{2} \right) = -c \cos \psi \quad (o)$$

ψ having again changed by a constant.

If we now set the analyzer (A) axis at 45 deg. to Ox_4 and Oy_4 (Fig. 99c), the components of the displacements (o) along it give

$$c \cos 45^\circ \cos \psi - c \cos 45^\circ \cos \psi$$

or zero. Thus the clockwise circular motion is extinguished.

Considering in the same way the counterclockwise part of the motion of Eqs. (l) and (m), i.e.,

$$x_4' = -c \sin \psi, \quad y_4' = -c \cos \psi \quad (n')$$

we find that the transmitted displacement along the analyzer axis is

$$-c \cos 45^\circ \sin \psi - c \cos 45^\circ \sin \psi$$

and the amplitude is thus

$$\sqrt{2} c \quad \text{or} \quad \sqrt{2} b \sin \frac{\Delta}{2} \quad \text{or} \quad a \sin \frac{\Delta}{2} \quad (p)$$

remembering that b denotes $a/\sqrt{2}$ and that a is the amplitude leaving the polarizer. No account has been taken of course of loss of light in the apparatus. Comparing this result with the result (g) for the plane polariscope, we observe that the factor $\sin 2\alpha$ is now absent, and therefore the isochromatics appear on the screen, but no isoclinics.

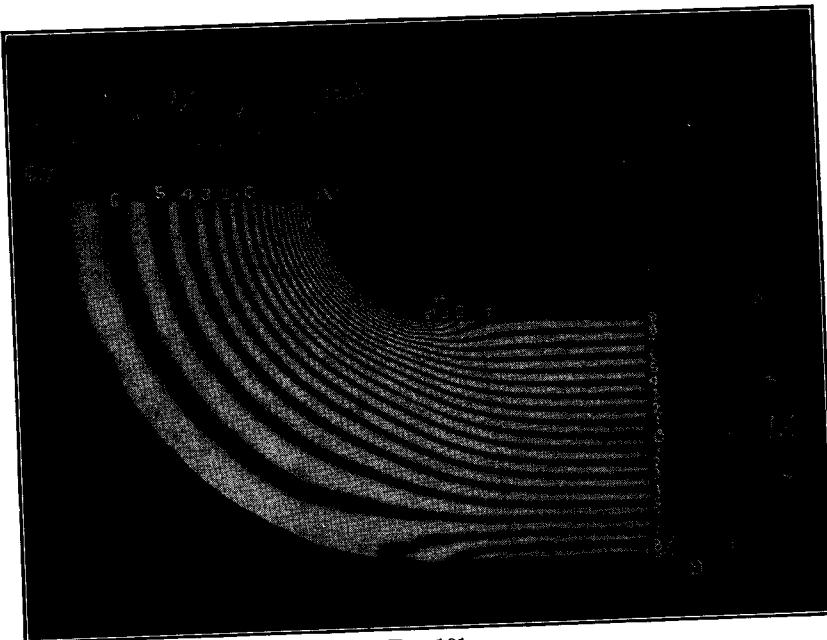


FIG. 101.

If Δ is zero, the amplitude (p) is also zero. Thus if there is no model, or if the model is unloaded, the screen is dark. We have a *dark field* setting. If the analyzer axis is turned through 90 deg. with respect to the Q_A we should have a *light field* and light fringes taking the place of the former dark fringes. The same effect is brought about in the plane polariscope by having the polarizer and analyzer axes parallel instead of at right angles.

44. Examples of Photoelastic Stress Determination. The photoelastic method has yielded especially important results in the study of stress concentration at the boundaries of holes and reentrant corners.

In such cases the maximum stress is at the boundary, and it can be obtained directly by the optical method because one of the principal stresses vanishes at the free boundary.

Figure 101 shows the fringe pattern of a curved bar¹ bent by couples M . The outer radius is three times the inner. The order numbers of the fringes marked on the right-hand end show a maximum of 9 at both

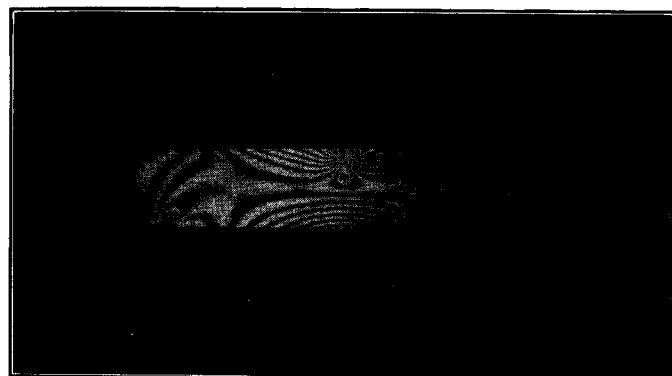


FIG. 102.

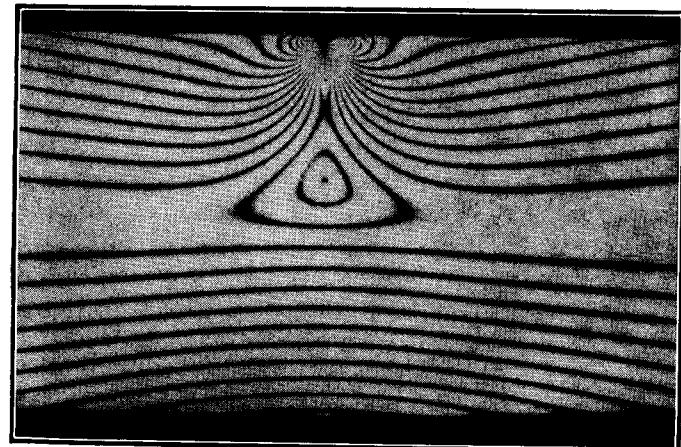


FIG. 103.

top and bottom. The regular spacing corresponds to the linear distribution of bending stress in the straight shank. The fringe orders marked along the top edge show the stress distribution in the curved

¹ E. E. Weibel, *Trans. A.S.M.E.*, vol. 56, p. 637, 1934.

part (the complete model continues above this top edge, which is its axis of symmetry), indicating a compressive stress at the inside represented by 13.5, and a tensile stress at the outside represented by 6.7. These values are in very close agreement, proportionally, with the theoretical "exact-solution" values in the last line of the table on page 64.

Figures 102, 103 represent¹ the case of bending of a beam by a force applied at the middle. The density of distribution of dark fringes indicates high stresses near the point of application of the load. The number of fringes crossing a cross section diminishes as the distance of

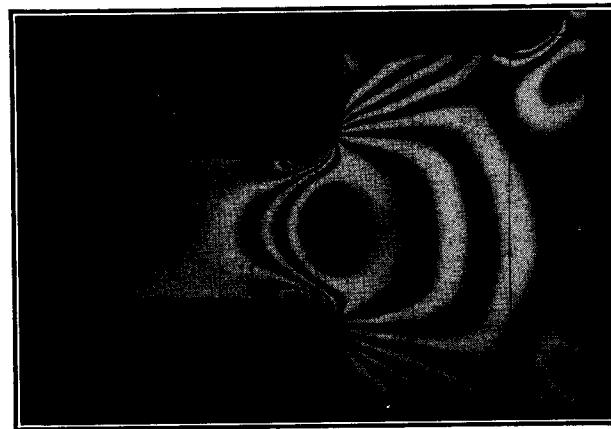


FIG. 104.

the cross section from the middle of the beam increases. This is due to decrease in bending moment.

Figure 104 represents the stress distribution in a plate of two different widths submitted to centrally applied tension. It is seen that the maximum stress occurs at the ends of the fillets. The ratio of this maximum stress to the average stress in the narrower portion of the plate is called the *stress-concentration factor*. It depends on the ratio of the radius R of the fillet to the width d of the plate. Several values of the stress-concentration factor obtained experimentally² are given in Fig. 105. It is seen that the maximum stress is rapidly increasing as the ratio R/d is decreasing, and when $R/d = 0.1$ the maximum stress is more than twice the average tensile stress. Figure 106 repre-

¹ M. M. Frocht, *Trans. A.S.M.E.*, vol. 53, 1931.

² See paper by Weibel, *loc. cit.*

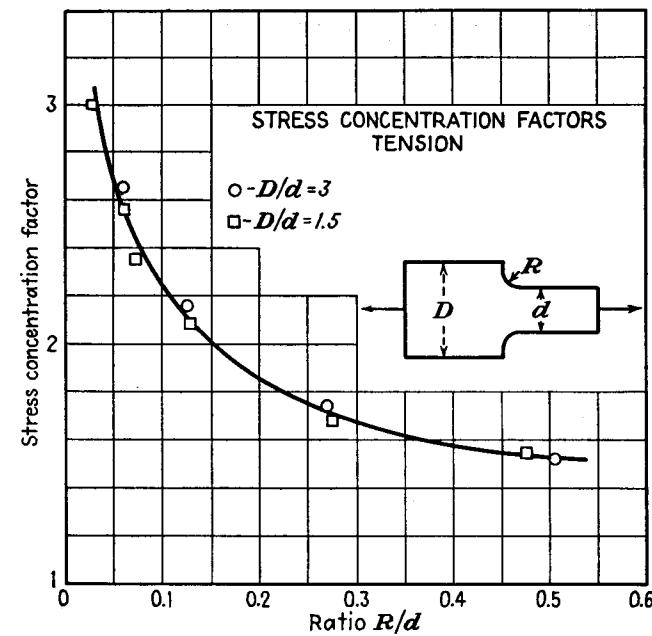


FIG. 105.

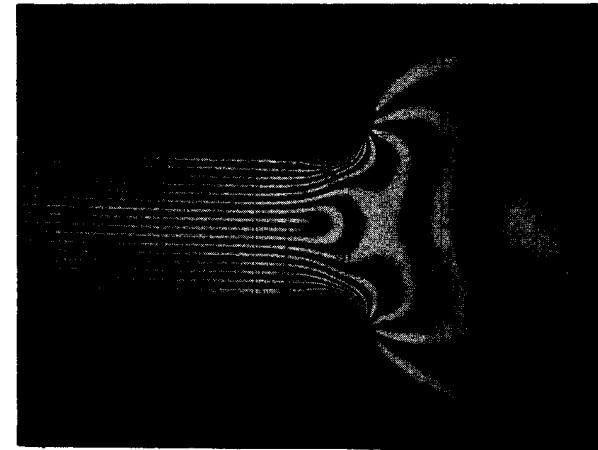


FIG. 106.

sents the same plate submitted to pure bending by a couple applied at the end and acting in the middle plane of the plate. Figure 107 gives the stress-concentration factors for this case.

45. Determination of the Principal Stresses. The ordinary polariscope, as we have seen, determines only the difference of the principal stresses, and their directions. When it is required to determine the principal stresses throughout the model, or at a boundary where there is unknown loading, further measurement, or calculation, is required.

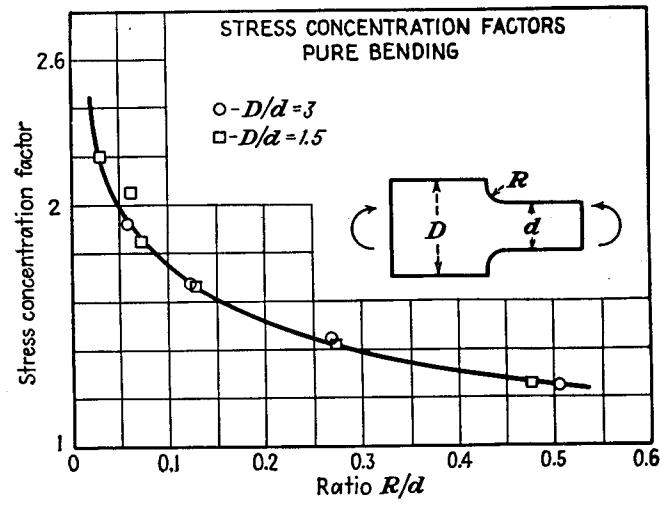


FIG. 107.

Many methods have been used, or proposed. Only a brief description of some of these will be given here.¹

The sum of the principal stresses can be found by measuring the changes in the thickness of the plate.² The decrease in thickness due to the stress is

$$\Delta h = \frac{h\nu}{E} (\sigma_x + \sigma_y) \quad (a)$$

whence $\sigma_x + \sigma_y$ may be calculated if Δh is measured at each point where the stresses are to be evaluated. Several special forms of extensometer have been designed for this purpose.³ The pattern of interference fringes formed when a model is placed against an optical flat, so as to

¹ For further information see the references cited in footnote 7 on p. 131.

² This method was suggested by Mesnager, *loc. cit.*

³ See M. M. Frocht, "Photoelasticity," vol. 2.

form an air film with thickness variations determined by the thickness variations in the plate, yields the required information in a single photograph.

The differential equation satisfied by the sum of the principal stresses, Eq. (b) on page 26, is also satisfied by the deflection of a membrane of constant tension, such as a soap film, and if the boundary values are made to correspond, the deflection represents $\sigma_x + \sigma_y$ to a certain scale.¹ In many cases the boundary values of $\sigma_x + \sigma_y$ required for the construction of the membrane can be found from the photoelastic fringe pattern. The latter gives $\sigma_x - \sigma_y$. At a free boundary one principal stress, say σ_y , is zero, and $\sigma_x + \sigma_y$ becomes the same as $\sigma_x - \sigma_y$. Also at a boundary point where the loading is purely normal to the boundary and of known magnitude, it constitutes one principal stress itself, and the photoelastic measurement of the difference suffices to determine the sum. The same differential equation is satisfied by the electric potential in flow of current through a plate, and this can be made the basis of an electrical method.² Effective numerical methods have been developed as alternatives to these experimental procedures. These are discussed in the Appendix. The principal stresses can also be determined by purely photoelastic observations, more elaborate than those considered in Arts. 42 and 43.²

46. Three-dimensional Photoelasticity. The models used in the ordinary photoelastic test are loaded at room temperature, are elastic, and the fringe pattern disappears when the load is removed. Since the light must pass through the whole thickness, interpretation of the fringe pattern is feasible only when the model is in a state of plane stress—the stress components then being very nearly uniform through the thickness. When this is not the case, as in a three-dimensional stress distribution, the optical effect is an integral involving the stress at all points along the ray.³

This difficulty has been surmounted by a method based on observations made by Brewster and by Clerk Maxwell,⁴ that gelatinous materials, such as isinglass, allowed to dry under load, then unloaded, retain a permanent fringe pattern in the polariscope as though still loaded and still elastic. Resins such as bakelite and fosterite loaded while hot, then cooled, have been found by later investigators to possess the

¹ J. P. Den Hartog, *Z. angew. Math. Mech.*, vol. 11, p. 156, 1931.

² See R. D. Mindlin, *J. Applied Phys.*, vol. 10, p. 282, 1939.

³ See the article by D. C. Drucker in "Handbook of Experimental Stress Analysis," which gives a comprehensive account of three-dimensional photoelasticity.

⁴ J. Clerk Maxwell, *Sci. Papers*, vol. 1, p. 30.

same property. The explanation¹ is that these materials have the structure of a strong elastic skeleton, or molecular network, which is unaffected by heat, the spaces being filled by a mass of loosely bonded molecules which softens on heating. When the hot specimen is loaded the elastic skeleton bears the load and is elastically deformed without hindrance. On cooling, the softened mass in which this skeleton is embedded becomes "frozen," and holds the skeleton almost to the

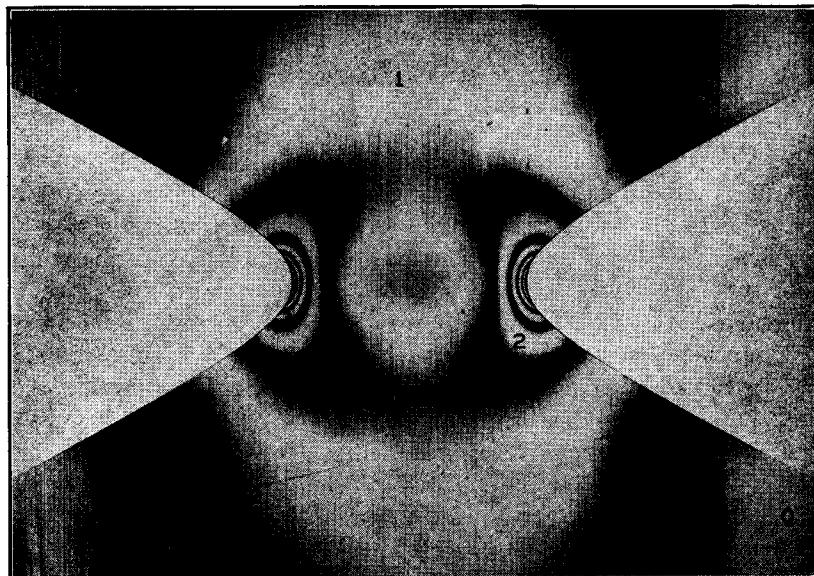
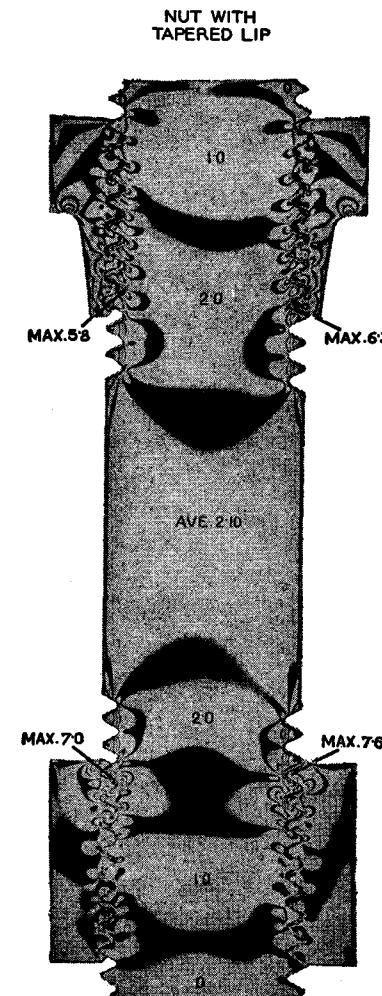


FIG. 108.

same deformation even when the load is removed. The optical effect is likewise substantially retained, and is not disturbed by cutting the specimen into pieces. A three-dimensional specimen may therefore be cut into thin slices, and each slice examined in the polariscope. The state of stress which produced the optical effect in the slice is not plane stress, but the other components τ_{xz} , τ_{yz} , σ_z are known to have no effect on a ray along the z -direction, *i.e.*, normal to the slice. The fringe pattern shown in Fig. 108 was obtained from such a slice cut centrally from a round shaft (of fosterite) with a hyperbolic groove.² The maximum stress obtained from this pattern is within two or three percent of the theoretical value. Figure 109 shows another fringe pattern

¹ M. Hetényi, *J. Applied Phys.*, vol. 10, p. 295, 1939.

² Leven, *loc. cit.*



CONVENTIONAL NUT

FIG. 109.

of the same type, obtained from a (bakelite) model of a bolt and nut fastening.¹ The lower nut was a conventional type. The upper one has a tapered lip and shows a lower stress concentration than the conventional nut.

¹ M. Hetényi, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 10, p. A-93, 1943. Results for several other forms of nut are given in this paper.

CHAPTER 6

STRAIN-ENERGY METHODS

47. Strain Energy. When a uniform bar is loaded in simple tension the forces on the ends do a certain amount of work as the bar stretches.

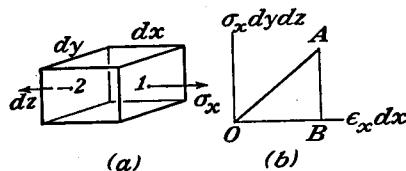


FIG. 110.

Thus if the element shown in Fig. 110 is subject to normal stresses σ_x only, we have a force $\sigma_x dy dz$ which does work on an extension $\epsilon_x dx$. The relation between these two quantities during loading is represented by a straight line such

as OA in Fig. 110b, and the work done during deformation is given by the area $\frac{1}{2}(\sigma_x dy dz)(\epsilon_x dx)$ of the triangle OAB . Writing dV for this work we have

$$dV = \frac{1}{2}\sigma_x \epsilon_x dx dy dz \quad (a)$$

It is evident that the same amount of work is done on all such elements, if their volumes are the same. We now inquire what becomes of this work—what kind or kinds of energy is it converted into?

In the case of a gas, adiabatic compression causes a rise of temperature. When an ordinary steel bar is adiabatically compressed there is an analogous, but quite small, rise of temperature. The corresponding amount of heat is, however, only a very small fraction of the work done by the compressive forces.¹ For our purposes it is sufficiently accurate to disregard this small fraction. Then none of the work done is accounted for by heat, and we may say that it is all stored within the element as *strain energy*. It is assumed that the element remains elastic and that no kinetic energy is developed.

The same considerations apply when the element has all six components of stress, $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$ acting on it (Fig. 3). Conservation of energy requires that the work done do not depend on the order in which the forces are applied, but only on the final magnitudes. Otherwise we could load in one order, and unload in another order corre-

¹ If this were not so there would be a substantial difference between adiabatic and isothermal moduli of elasticity. The actual differences are very slight. See G. F. C. Searle, "Experimental Elasticity," Chap. 1.

sponding to a larger amount of work. Thus a net amount of work would have been gained from the element in a complete cycle.

The calculation of the work done is simplest if the forces, or stresses, all increase simultaneously in the same ratio. Then the relation between each force and the corresponding displacement is still linear, as in Fig. 110b, and the work done by all the forces is

$$dV = V_0 dx dy dz \quad (b)$$

where

$$V_0 = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{xz} \gamma_{xz}) \quad (c)$$

Thus V_0 is the amount of work per unit volume, or strain energy per unit volume.

In the preceding discussion the stresses were regarded as the same on opposite faces of the element, and there was no body force. Let us now reconsider the work done on the element when the stress varies through the body and body force is included. Considering first the force $\sigma_x dy dz$ on the face 1 of the element in Fig. 110a, it does work on the displacement u of this face, of amount $\frac{1}{2}(\sigma_x u)_1 dy dz$, where the subscript 1 indicates that the functions σ_x, u must be evaluated at the point 1. The force $\sigma_x dy dz$ on the face 2 does work $-\frac{1}{2}(\sigma_x u)_2 dy dz$. The total for the two faces

$$\frac{1}{2}[(\sigma_x u)_1 - (\sigma_x u)_2] dy dz$$

is the same, in the limit, as

$$\frac{1}{2} \frac{\partial}{\partial x} (\sigma_x u) dx dy dz \quad (d)$$

Computing the work done by the shear stresses τ_{xy}, τ_{xz} on the faces 1 and 2, and adding to (d), we have the work done on the two faces by all three components of stress as

$$\frac{1}{2} \frac{\partial}{\partial x} (\sigma_x u + \tau_{xy} v + \tau_{xz} w) dx dy dz$$

where v and w are the components of displacement in the y - and z -directions. The work done on the other two pairs of faces can be similarly expressed. We find, for the total work done by the stresses on the faces,

$$\begin{aligned} \frac{1}{2} \left[\frac{\partial}{\partial x} (\sigma_x u + \tau_{xy} v + \tau_{xz} w) + \frac{\partial}{\partial y} (\sigma_y v + \tau_{yz} w + \tau_{xy} u) \right. \\ \left. + \frac{\partial}{\partial z} (\sigma_z w + \tau_{xz} u + \tau_{yz} v) \right] dx dy dz \quad (e) \end{aligned}$$

As the body is loaded the body forces $X dx dy dz$ etc. do work

$$\frac{1}{2}(Xu + Yv + Zw) dx dy dz \quad (f)$$

The total work done on the element is the sum of (e) and (f). On carrying out the differentiations in (e) we find that the total work becomes

$$\frac{1}{2} \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial w}{\partial z} + \tau_{xy} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \tau_{yz} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) + \tau_{zx} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + u \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X \right) + v \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial x} + Y \right) + w \left(\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z \right) \right] dx dy dz$$

But on account of the equations of equilibrium (127) derived in Art. 76 the brackets multiplying u , v , w are zero. The quantities multiplying the stress components are, from Eqs. 2, $\epsilon_x, \dots, \gamma_{xy}, \dots$. Thus the total work done on the element reduces to the value given by (b) and (c). These formulas therefore continue to give the work done on the element, or strain energy stored in it, when the stress is not uniform and body forces are included.

By means of Hooke's law, Eqs. (3) and (6), we can express V_o , given by Eq. (c), as a function of the stress components only. Then

$$V_o = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E} (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + \frac{1}{2G} (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) \quad (84)$$

Alternatively we may use Eqs. (11) and express V_o as a function of the strain components only. Then

$$V_o = \frac{1}{2} \lambda e^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2} G(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2) \quad (85)$$

in which

$$e = \epsilon_x + \epsilon_y + \epsilon_z, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

This form shows at once that V_o is always positive.

It is easy to show that the derivative of V_o , as given by (85), with respect to any strain component gives the corresponding stress component. Thus taking the derivative with respect to ϵ_x and using Eq. (11), we find

$$\frac{\partial V_o}{\partial \epsilon_x} = \lambda e + 2G\epsilon_x = \sigma_x \quad (g)$$

For the case of plane stress, in which $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, we have from (84)

$$V_o = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2) - \frac{\nu}{E} \sigma_x \sigma_y + \frac{1}{2G} \tau_{xy}^2 \quad (86)$$

The total strain energy of a deformed elastic body is obtained from the strain energy per unit volume V_o by integration:

$$V = \iiint V_o dx dy dz \quad (87)$$

It represents the total work done against internal forces during loading. If we think of the body as consisting of a very large number of particles interconnected by springs, it would represent the work done in stretching or contracting the springs.

By using Eq. (84) or (85) it can be represented either as a function of stress components or as a function of strain components. The application of both these forms will be illustrated in the following discussion.

The quantity of strain energy stored per unit volume of the material is sometimes used as a basis for determining the limiting stress at which failure occurs.¹ In order to bring this theory into agreement with the fact that isotropic materials can sustain very large hydrostatic pressures without yielding, it has been proposed to split the strain energy into two parts, one due to the change in volume and the other due to the distortion, and consider only the second part in determining the strength.²

We know that the volume change is proportional to the sum of the three normal stress components [Eq. (8)], so if this sum is zero the deformation consists of distortion only. We may resolve each stress component into two parts,

$$\sigma_x = \sigma'_x + p, \quad \sigma_y = \sigma'_y + p, \quad \sigma_z = \sigma'_z + p$$

where

$$p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}\Theta \quad (h)$$

Since, from this,

$$\sigma'_x + \sigma'_y + \sigma'_z = 0$$

the stress condition $\sigma'_x, \sigma'_y, \sigma'_z$ produces only distortion, and the change in volume depends entirely³ on the magnitude of the uniform tension p . The part of the total energy due to this change in volume is, from Eq. (8),

$$\frac{ep}{2} = \frac{3(1-2\nu)}{2E} p^2 = \frac{1-2\nu}{6E} (\sigma_x + \sigma_y + \sigma_z)^2 \quad (i)$$

Subtracting this from (84), and using the identity

$$\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x = -\frac{1}{2}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + (\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$$

we can present the part of the total energy due to distortion in the form

$$V_o - \frac{1-2\nu}{6E} (\sigma_x + \sigma_y + \sigma_z)^2 = \frac{1+\nu}{6E} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + \frac{1}{2G} (\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) \quad (88)$$

¹ The various strength theories are discussed in S. Timoshenko, "Strength of Materials," vol. 2.

² M. T. Huber, *Czasopismo techniczne*, Lwów, 1904. See also R. v. Mises, *Göttingen Nachrichten, Math.-phys. Klasse*, 1913, p. 582, and F. Schleicher, *Z. angew. Math. Mech.*, vol. 5, p. 199, 1925.

³ The shearing components τ_{xy} , τ_{yz} , τ_{xz} produce shearing strains which do not involve any change of volume.

In the case of simple tension in the x -direction, σ_x alone is different from zero, and the strain energy of distortion (88) is $(1 + \nu)/3E \cdot \sigma_x^2$. In the case of pure shear, say between the xz , and yz -planes, τ_{xy} alone is different from zero and the energy of distortion is $(1/2G)\tau_{xy}^2$. If it is true that, whatever the stress system, failure occurs when the strain energy of distortion reaches a certain limit (characteristic of the material), the ratio between the critical value for tensile stress alone and for shearing stress alone is found from the equation

$$\frac{1}{2G}\tau_{xy}^2 = \frac{1 + \nu}{3E}\sigma_x^2$$

from which

$$\tau_{xy} = \frac{1}{\sqrt{3}}\sigma_x = 0.557\sigma_x \quad (j)$$

Experiments with steel show¹ that the ratio between the yield point in tension and the yield point in shear is in very good agreement with that given by (j).

Saint-Venant's principle (see p. 33) can be shown by consideration of strain energy to be a consequence of the conservation of energy.² According to the principle, two different distributions of force having the same resultant, acting on a small part of an elastic body, will produce the same stress except in the immediate neighborhood of the loaded part. If one of these distributions is reversed, and combined with the other, there will be zero stress except in this neighborhood. The combined loads are self-equilibrating, and the principle is in fact equivalent to the statement that a self-equilibrating distribution of force on a small part of an elastic solid produces only local stress.

Such a distribution of force does work during its application only because there is deformation of the loaded region. Let one surface element of this region be fixed in position and orientation. If p denotes the order of magnitude (e.g., average) of the force per unit area, and a a representative linear dimension (e.g., diameter) of the loaded part, the strain components are of order (p/E) and the displacements within the loaded part are of order pa/E . The work done is of order $pa^2 \cdot pa/E$ or p^2a^3/E .

On the other hand, stress components of order p imply strain energy of order p^2/E per unit volume. The work done is therefore sufficient only for a volume of order a^3 , in accordance with the statement of the principle.

It has been supposed here that the body obeys Hooke's law and is of solid form. The former restriction may be dispensed with, E in the above argument then denoting merely the order of magnitude of the slopes of the stress-strain curves of the material. If the body is not a solid form, as for instance a beam with a very thin web, or a thin cylindrical shell, a self-equilibrating distribution of force on one end may make itself felt at distances many times the depth or diameter.³

¹ See the papers by W. Lode, *Z. Physik*, vol. 36, p. 913, 1926, and *Forschungssarbeiten*, No. 303, Berlin, 1928.

² J. N. Goodier, *Phil. Mag.*, series 7, vol. 24, p. 325, 1937; *J. Applied Phys.*, vol. 13, p. 167, 1942.

³ V. Z. Vlasov, "Thin Walled Elastic Bars," Moscow, 1940; J. N. Goodier and M. V. Barton, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 11, p. A-35, 1944; N. J. Hoff, *J. Aeronaut. Sci.*, vol. 12, p. 455, 1945.

The above argument can be repeated without change for a load with non-zero resultant, so long as there is a fixed surface element within or near to the loaded part. Thus if a deformable material is bonded to a rigid one, pressure applied to a small part of the former near to the attachment will produce only local stress.¹

48. Principle of Virtual Work. In the solution of problems of elasticity it is sometimes advantageous to use the *principle of virtual work*. In the case of a particle, this principle states that if a particle is in equilibrium the total work of all forces acting on the particle in any *virtual displacement* vanishes. As a virtual displacement of a particle, free to move in any direction, any small displacement can be taken. If δu , δv , δw are components of a virtual displacement in the x -, y -, and z -directions and ΣX , ΣY , ΣZ are the sums of projections on the same directions of forces, acting on the particle, the principle of virtual work gives

$$\delta u \Sigma X = 0, \quad \delta v \Sigma Y = 0, \quad \delta w \Sigma Z = 0$$

These equations are satisfied for any virtual displacement if

$$\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0$$

Thus we arrive at the known equations of equilibrium of a particle. In applying the principle of virtual work the acting forces are considered as constant during a virtual displacement. If some of the forces acting on a point are elastic reactions, as reactions of bars in the case of a hinge of a truss, we assume that virtual displacements are so small that the change in magnitudes or directions of reactions can be neglected.

An elastic body at rest, with its surface and body forces, constitutes a system of particles on each of which acts a set of forces in equilibrium. In any virtual displacement the total work done by the forces on any particle vanishes, and therefore the total work done by all the forces of the system vanishes.

A *virtual displacement* in the case of an elastic body is any small displacement compatible with the condition of continuity of the material and with the conditions for the displacements at the surface of the body, if such conditions are prescribed. If it is given, for instance, that a certain portion of the surface of the body, say a built-in end of a beam, is immovable or has a given displacement, the virtual displacement for this portion must be zero.

Let us consider, as an example, the case of a plane stress distribution in a plate. Denote by u and v the components of the actual displacements due to the loads and by δu and δv the components of a virtual displacement from the loaded position of equilibrium. These latter

¹ Goodier, *J. Applied Phys.*, loc. cit.

components are arbitrary small quantities satisfying the conditions of continuity of an elastic deformation, i.e., they are continuous functions of x and y .

For any system of displacements the work done against the mutual actions between the particles is equal to the strain energy stored, i.e., the strain energy corresponding to the displacements. If we change u and v by δu and δv , therefore, the work done against the mutual actions between the particles is the difference between the strain energy corresponding to $u + \delta u, v + \delta v$, and the strain energy corresponding to u and v . The virtual displacements $\delta u, \delta v$ produce the change in strain components

$$\delta\epsilon_x = \frac{\partial \delta u}{\partial x}, \quad \delta\epsilon_y = \frac{\partial \delta v}{\partial y}, \quad \delta\gamma_{xy} = \frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}$$

The corresponding change in strain energy per unit volume, from expression (85), is

$$\delta V_0 = \frac{\partial V_0}{\partial \epsilon_x} \delta\epsilon_x + \frac{\partial V_0}{\partial \epsilon_y} \delta\epsilon_y + \frac{\partial V_0}{\partial \gamma_{xy}} \delta\gamma_{xy} = \sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \tau_{xy} \delta\gamma_{xy} \quad (a)$$

The change of the total strain energy of the body is then $\iint \delta V_0 dx dy$, in which the integration is taken over the whole area of the plate of unit thickness.

As already stated, this change in strain energy measures the work done against the mutual actions between the particles. In order to get the work done by the mutual actions the sign must be reversed. Hence the work done by these forces during the virtual displacement is

$$-\iint \delta V_0 dx dy \quad (b)$$

In calculating the work done by external forces during a virtual displacement, consideration must be given the forces applied at the boundary of the plate and the body forces. Denoting by \bar{X} and \bar{Y} the components of the boundary forces per unit area, the work done by these forces on the virtual displacements having components δu and δv may be written down at once as

$$\int (\bar{X} \delta u + \bar{Y} \delta v) ds \quad (c)$$

in which the integration is taken along the boundary s of the plate.

Similarly the work done by the body forces is

$$\iint (X \delta u + Y \delta v) dx dy \quad (d)$$

in which X and Y are the components of the body force per unit volume of the plate, and the integration is taken over the whole area of the

plate. The condition that the total work done during the virtual displacement vanishes, now takes the form, from (b), (c), and (d),

$$\int (\bar{X} \delta u + \bar{Y} \delta v) ds + \iint (X \delta u + Y \delta v) dx dy - \iint \delta V_0 dx dy = 0 \quad (89)$$

Since, in applying the principle of virtual work, the given forces and the actual stress components corresponding to the position of equilibrium are considered as constant during a virtual displacement, the variation sign δ can be put before the integral signs in Eq. (89), and, changing signs throughout, we have

$$\delta [\iint V_0 dx dy - \iint (Xu + Yu) dx dy - \int (\bar{X}u + \bar{Y}v) ds] = 0 \quad (89')$$

The first term in the bracket is the potential energy of deformation. The second and the third terms together represent the potential energy of forces acting on the body if the potential energy of these forces for the unstressed condition ($u = v = 0$) is taken as zero. The complete expression in brackets represents the *total potential energy of the system*.

Hence in comparing various values of the displacements u and v , it can be stated that the displacements which actually occur in an elastic system under the action of given external forces are those which lead to zero variation of the total potential energy of the system for any virtual displacement from the position of equilibrium, i.e., the total potential energy of the system at the position of equilibrium is a maximum or a minimum. To decide whether the energy is a maximum or a minimum, the small quantities of higher order, which were neglected¹ in our previous discussion, should be considered. If in this way it can be shown that for any virtual displacement the change in the total potential energy of the system is positive we have the case of a minimum. If this change is always negative we have the case of a maximum. For stable equilibrium it is always necessary to demand a positive work for any virtual displacement of the system from this position, hence in this case the total potential energy of the system at this position is a *minimum*.

An equation analogous to (89) can easily be written down for a three-dimensional stress distribution.

The principle of virtual work is especially useful for finding the deformation of an elastic body produced by given forces. To illustrate the application of the method let us consider here a few simple examples, the solutions of which are already well known.

The first is the deflection curve of a perfectly flexible elastic string AB stretched by forces S between fixed points A and B (Fig. 111) and loaded by a distributed vertical load of intensity q . We assume that

¹ We neglected them when we assumed that stress components and forces remain constant during any virtual displacement.

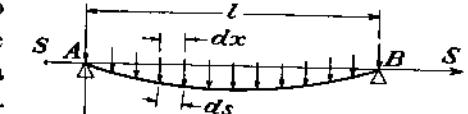


FIG. 111.

the initial tension of the string is so large that the increase in tensile force due to additional stretching during the deflection can be neglected. Then the increase in strain energy due to the deflection is obtained by multiplying the initial tensile forces S by the stretch of the string due to the deflection. Taking coordinates as shown in Fig. 111, we find

$$ds - dx = dx \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} - dx = \frac{1}{2} \left(\frac{dy}{dx} \right)^2 dx$$

The stretching of the string is

$$\int_0^l (ds - dx) = \frac{1}{2} \int_0^l \left(\frac{dy}{dx} \right)^2 dx$$

and the corresponding increase in strain energy of the string is

$$S \int_0^l \left(\frac{dy}{dx} \right)^2 dx \quad (e)$$

To get the total strain energy of the string the constant strain energy due to initial stretching would have to be added to expression (e). The principle of virtual work in this case gives the following equation, analogous to Eq. (89):

$$\int_0^l q \delta y dx - \frac{S}{2} \int_0^l \delta \left(\frac{dy}{dx} \right)^2 dx = 0 \quad (f)$$

Calculating the variation of the second term, we find

$$\int_0^l \delta \left(\frac{dy}{dx} \right)^2 dx = 2 \int_0^l \frac{dy}{dx} \delta \frac{dy}{dx} dx = 2 \int_0^l \frac{dy}{dx} \frac{d}{dx} \delta y dx$$

Integrating by parts and taking into account that at the ends of the string $\delta y = 0$, we find

$$2 \int_0^l \left(\frac{dy}{dx} \right) \frac{d}{dx} \delta y dx = 2 \left(\left. \frac{dy}{dx} \delta y \right|_0^l - \int_0^l \frac{d^2 y}{dx^2} \delta y dx \right) = -2 \int_0^l \frac{d^2 y}{dx^2} \delta y dx$$

Substituting into Eq. (f), we obtain

$$S \int_0^l \frac{d^2 y}{dx^2} \delta y dx + \int_0^l q \delta y dx = 0$$

or

$$\int_0^l \left(S \frac{d^2 y}{dx^2} + q \right) \delta y dx = 0$$

This equation will be satisfied for any virtual displacement δy , only if

$$S \frac{d^2 y}{dx^2} + q = 0 \quad (g)$$

Thus we obtain the known differential equation of a vertically loaded string.

The principle of virtual work can be used not only for establishing a differential equation for the deflection curve, as in the example given above, but also for the direct determination of deflections.¹ Take, for example, a prismatical bar supported at the ends and loaded by a force P (Fig. 112). In the most general case the deflection curve of such a bar can be represented in the form of a trigonometric series,

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{2\pi x}{l} + a_3 \sin \frac{3\pi x}{l} + \dots \quad (h)$$

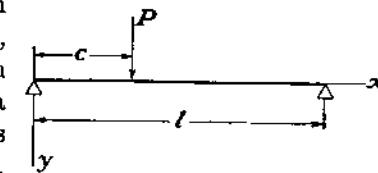


FIG. 112.

Substituting this in the well-known formula for the strain energy of bending of a prismatical bar,² we find

$$V = \frac{EI}{2} \int_0^l \left(\frac{dy}{dx} \right)^2 dx = \frac{EI\pi^4}{4l^3} \sum_{n=1}^{\infty} n^4 a_n^2 \quad (k)$$

Let us consider a virtual displacement from the actual deflection curve obtained by giving to any coefficient a_n in the series (h) a variation δa_n . Then

$$\delta y = \delta a_n \sin \frac{n\pi x}{l} \quad (l)$$

The corresponding change of strain energy, from Eq. (k), is

$$\frac{\partial V}{\partial a_n} \delta a_n = \frac{EI\pi^4}{2l^3} n^4 a_n \delta a_n \quad (m)$$

and the work done by the external force P during the virtual displacement (l) is

$$P \delta a_n \sin \frac{n\pi c}{l} \quad (n)$$

¹ See S. Timoshenko, *Bull. Polytech. Inst.*, Kiew, 1909. See also S. Timoshenko, "Strength of Materials," vol. 2, p. 44, 1941.

² See S. Timoshenko, "Strength of Materials," vol. 1, p. 297, 1941.

By using (m) and (n) the equation of virtual work becomes

$$\frac{EI\pi^4}{2l^3} n^4 a_n \delta a_n - P \delta a_n \sin \frac{n\pi c}{l} = 0$$

from which

$$a_n = \frac{2Pl^3 \sin \frac{n\pi c}{l}}{EI\pi^4 n^4}$$

Substituting in the series (h), we find the deflection curve

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l}}{n^4} \quad (o)$$

This series rapidly converges, and a few terms give a satisfactory approximation. Taking, for instance, the load P at the middle of the span ($c = l/2$), the deflection under the load is

$$(y)_{x=\frac{l}{2}} = \frac{2Pl^3}{EI\pi^4} \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right)$$

By taking only the first term of this series, we obtain

$$(y)_{x=\frac{l}{2}} = \frac{Pl^3}{48.7EI}$$

We have a factor 48.7 while the exact value is 48, so that the error made in using only the first term of the series is about $1\frac{1}{2}$ per cent.

In the preceding discussion we had to consider displacement in only one direction and we represented it by a sine series (h). A similar method can be applied in more complicated cases. Let us consider a rectangular plate with fixed edges, Fig. 113, and acted upon by body forces parallel to its plane. General expressions for the displacements u and v can be taken in the form of series,

$$u = \sum \sum A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (p)$$

$$v = \sum \sum B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Each term of these series vanishes at the boundary, so the boundary conditions are satisfied. To calculate the coefficients A_{mn}, B_{mn} we pro-

ceed as in the case of the beam and take virtual displacements in the form

$$\delta u = \delta A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$\delta v = \delta B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

The equation (89) of virtual displacements then gives

$$\begin{aligned} \delta A_{mn} \iint X \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy &= \delta A_{mn} \frac{\partial}{\partial A_{mn}} \iint V_o dx dy \\ \delta B_{mn} \iint Y \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy &= \delta B_{mn} \frac{\partial}{\partial B_{mn}} \iint V_o dx dy \end{aligned} \quad (q)$$

For the calculation of strain energy in the case of plane stress we use the formula

$$V = \iint V_o dx dy = \iint \left[\frac{E}{2(1-\nu^2)} (\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y) + \frac{E}{4(1+\nu)} \gamma_{xy}^2 \right] dx dy \quad (r)$$

Substituting in it¹

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \sum \sum \frac{m\pi}{a} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \epsilon_y &= \frac{\partial v}{\partial y} = \sum \sum \frac{n\pi}{b} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum \sum \frac{n\pi}{b} A_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} + \sum \sum \frac{m\pi}{a} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned}$$

and performing the integrations we find

$$V = \frac{\pi^2 Eab}{4} \left[\sum \sum A_{mn}^2 \left(\frac{1}{2(1-\nu^2)} \frac{m^2}{a^2} + \frac{1}{4(1+\nu)} \frac{n^2}{b^2} \right) + \sum \sum B_{mn}^2 \left(\frac{1}{2(1-\nu^2)} \frac{n^2}{b^2} + \frac{1}{4(1+\nu)} \frac{m^2}{a^2} \right) \right]$$

¹ It is not always legitimate to differentiate a Fourier series term by term. Sufficient conditions may be found in the book "Modern Analysis" by E. T. Whittaker and G. N. Watson, p. 169. These conditions are fulfilled in the present problem.

Substituting this expression for the strain energy into Eqs. (q) we obtain

$$\begin{aligned} A_{mn} \frac{Eab\pi^2}{4} \left(\frac{m^2}{a^2(1-\nu^2)} + \frac{n^2}{2b^2(1+\nu)} \right) &= \int_0^a \int_0^b X \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ B_{mn} \frac{Eab\pi^2}{4} \left(\frac{m^2}{2a^2(1+\nu)} + \frac{n^2}{b^2(1-\nu^2)} \right) &= \int_0^a \int_0^b Y \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \end{aligned}$$

We see that for any kind of volume forces the coefficients in expressions (p) can be readily calculated and the complete solution of the problem can be obtained.

The method of virtual displacements can be used for finding approximate solutions of two-dimensional problems when displacements at the boundary are given. Assume that the displacements u and v can be represented with sufficient accuracy by series

$$\begin{aligned} u &= \phi_o(x,y) + \sum_m a_m \phi_m(x,y) \\ v &= \psi_o(x,y) + \sum_m b_m \psi_m(x,y) \end{aligned} \quad (s)$$

which satisfy the prescribed boundary conditions. For example, we can select ϕ_o and ψ_o so that they will give at the boundary the required displacements, and the rest of the functions ϕ and ψ can then vanish at the boundary. For the calculation of the coefficients $a_1, \dots, a_m, b_1, \dots, b_m$ we use the principle of virtual displacements (89). Taking virtual displacements in the form

$$\delta u_m = \delta a_m \phi_m(x,y), \quad \delta v_m = \delta b_m \psi_m(x,y) \quad (t)$$

we can write as many equations of equilibrium, similar to equations (q) in the preceding example, as the number of the coefficients in the series (s). These equations will be linear with respect to $a_1, \dots, a_m, b_1, \dots, b_m$, and solving them we will find the values of the coefficients in the series (s), representing the approximate solution of the problem.¹

In using the principle of virtual displacements (89) it is assumed that the strain energy per unit volume V_o is represented as a function of the strain components [Eq. (r)] and these are calculated by using expres-

¹ This method of solving problems in elasticity was proposed by W. Ritz and was successfully used by him in an investigation of bending of rectangular plates. See *J. reine u. angew. Math.*, vol. 135, pp. 1-61. See also "Gesammelte Werke," p. 192, Paris, 1911.

sions (s). The calculation of the variation of the strain energy can be simplified if we observe that

$$\begin{aligned} \delta V &= \int \int (\sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \tau_{xy} \delta \gamma_{xy}) dx dy \\ &= \int \int \left[\sigma_x \frac{\partial}{\partial x} \delta u + \sigma_y \frac{\partial}{\partial y} \delta v + \tau_{xy} \left(\frac{\partial}{\partial y} \delta u + \frac{\partial}{\partial x} \delta v \right) \right] dx dy \end{aligned}$$

Integrating by parts and observing that δu and δv vanish at the boundary we obtain

$$\delta V = - \int \int \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) \delta u + \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \right) \delta v \right] dx dy$$

Taking for the virtual displacements expressions (t) we then obtain the necessary equations for calculating the coefficients $a_1, \dots, a_m, b_1, \dots, b_m$ in the following form:

$$\begin{aligned} \delta a_m \int \int X \phi_m(x,y) dx dy &= - \delta a_m \int \int \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) \phi_m(x,y) dx dy \\ \delta b_m \int \int Y \psi_m(x,y) dx dy &= - \delta b_m \int \int \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} \right) \psi_m(x,y) dx dy \end{aligned}$$

or

$$\begin{aligned} \int \int \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X \right) \phi_m(x,y) dx dy &= 0 \\ \int \int \left(\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y \right) \psi_m(x,y) dx dy &= 0 \end{aligned} \quad (90)^1$$

As an example of the application of these equations let us again consider a rectangular plate (Fig. 113) and assume that three sides of it are fixed and along the fourth side ($y = b$) the displacements are given by the equations

$$u = 0, \quad v = Cb \sin \frac{\pi x}{a}$$

The boundary conditions will be satisfied by taking

$$\begin{aligned} u &= \sum_m \sum_n A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ v &= Cy \sin \frac{\pi x}{a} + \sum_m \sum_n B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \quad (u)$$

¹ Equations of virtual displacements in this form are sometimes called Galerkin's equations. However both forms of equations represented by Eqs. (q) and Eqs. (90) are indicated by W. Ritz in the above-mentioned paper. See "Gesammelte Werke," p. 228.

The corresponding stress components will be

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2} \left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} \right) = \frac{E}{1-\nu^2} \left(\sum \sum A_{mn} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\ &\quad \left. + \nu \sum \sum B_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} + \nu C \sin \frac{\pi x}{a} \right) \\ \sigma_y &= \frac{E}{1-\nu^2} \left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \right) = \frac{E}{1-\nu^2} \left(\sum \sum B_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right. \\ &\quad \left. + C \sin \frac{\pi x}{a} + \nu \sum \sum A_{mn} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right) \\ \tau_{xy} &= \frac{E}{2(1+\nu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{E}{2(1+\nu)} \left(\sum \sum A_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right. \\ &\quad \left. + \sum \sum B_{mn} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + Cy \frac{\pi}{a} \cos \frac{\pi x}{a} \right)\end{aligned}$$

Substituting into Eqs. (90) and assuming

$$\delta u = \delta A_m \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad \delta v = \delta B_m \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

we obtain, after integration,

$$\begin{aligned}-\frac{E\pi^2 ab}{4} \left(\frac{m^2}{a^2(1-\nu^2)} + \frac{n^2}{2b^2(1+\nu)} \right) A_{mn} \\ + \int_0^a \int_0^b X \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy = 0 \\ -\frac{E\pi^2 ab}{4} \left(\frac{n^2}{b^2(1-\nu^2)} + \frac{m^2}{2a^2(1+\nu)} \right) B_{mn} \\ -\frac{C\pi^2}{a^2} \int_0^a \int_0^b y \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ + \int \int Y \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy = 0\end{aligned}$$

If the body forces vanish we find that all coefficients A_{mn} vanish also. The coefficients B_{mn} are different from zero only when $m = 1$. Then

$$\begin{aligned}\frac{E\pi^2 ab}{4} \left(\frac{n^2}{b^2(1-\nu^2)} + \frac{1}{2a^2(1+\nu)} \right) B_{1n} \\ = -\frac{C\pi^2}{2a} \int_0^b y \sin \frac{n\pi y}{b} dy = \frac{C\pi b^2 \cos n\pi}{2a} \frac{1}{n}\end{aligned}$$

Determining B_{1n} from this equation and substituting into the formulas (u) we obtain the displacements produced by the assumed displacements at the boundary.

General considerations of the total energy of a system were applied by A. A. Griffith in developing his theory of rupture of brittle materials.¹ It is known that materials always show a strength much smaller than might be expected from the molecular forces. For a certain glass Griffith found a theoretical strength in tension of the order of 1.6×10^6 p.s.i., while tensile tests with glass rods gave only 26×10^8 p.s.i. He showed that this discrepancy between theory and experiments can be explained if we assume that in such materials as glass there exist microscopic cracks or flaws producing high stress concentrations and consequent spreading of the cracks. For purposes of calculation Griffith takes a crack in the form of a very narrow elliptical hole, the major axis of which is perpendicular to the direction of the tensile force. Consider a plate fixed along the sides ab and cd , and stretched by uniformly distributed tensile stress S , acting along the same sides (Fig. 114). If a microscopic elliptical hole AB of length l is made in the plate, ab and cd remaining fixed, the initial strain energy due to the tensile stresses S will be reduced. This reduction can be calculated by using the solution for an elliptical hole,² and for a plate of unit thickness it is equal to

$$V = \frac{\pi l^2 S^2}{4E} \quad (v)$$

If the crack lengthens, there is a further reduction of strain energy stored in the plate. However, the lengthening of the crack means an increase of surface energy, since the surfaces of solids possess a surface tension just as liquids do. Griffith found, for instance, that for the kind of glass used in his experiments the surface energy T per unit surface area was of the order 3.12×10^{-8} in. lb. per square inch. Now if the lengthening of the crack requires an increase of surface energy which can be supplied by the reduction of the strain energy, the lengthening can occur without increase of the total energy. The condition that the crack extends spontaneously is that these two quantities of energy are equal, or, using (v),

$$\frac{dV}{dl} dl = \frac{\pi l S_{cr}^2}{2E} dl = 2 dl T$$

¹ A. A. Griffith, *Trans. Roy. Soc. (London)*, series A, vol. 221, p. 163, 1921. See also his paper in *Proc. Intern. Congr. Appl. Mech.*, Delft, 1924. A bibliography of the subject can be found in "Handbuch der physikalischen und technischen Mechanik," vol. 4, part 2, 1931, article by Adolf Smekal.

² See p. 201.

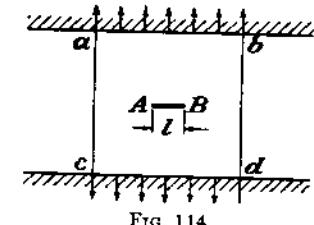


FIG. 114.

from which

$$S_{cr} = \sqrt{\frac{4ET}{\pi l}} \quad (w)$$

Experiments in which cracks of known length were formed with a glass-cutter's diamond showed a very satisfactory agreement with Eq. (w). It was also shown experimentally that, if precautions are taken to eliminate microscopic cracks, a much higher strength than usual can be obtained. Some glass rods tested by Griffith showed an ultimate strength of the order of 900,000 p.s.i., which is more than half of the theoretical strength mentioned above.

49. Castigliano's Theorem. In the previous article the equilibrium configuration of an elastic body submitted to given body forces and given boundary conditions was compared with neighboring configurations arrived at by virtual displacements δu , δv from the position of equilibrium. It was established that the true displacements corresponding to the position of stable equilibrium are those which make the total potential energy of the system a minimum.

Let us consider now, instead of displacements, the stresses corresponding to the position of equilibrium. We take again, as an example, the case of a plane stress distribution. We know that the differential equations of equilibrium (18), together with the boundary conditions (20), are not sufficient for determining the stress components σ_x , σ_y , τ_{xy} . By taking various expressions for the stress function ϕ in Eqs. (29) we may find many different stress distributions satisfying the equations of equilibrium and the boundary conditions, and the question arises: What distinguishes the true stress distribution from all the other statically possible stress distributions?

Let σ_x , σ_y , τ_{xy} be the true stress components corresponding to the position of equilibrium and $\delta\sigma_x$, $\delta\sigma_y$, $\delta\tau_{xy}$ small variations of these components such that the new stress components $\sigma_x + \delta\sigma_x$, $\sigma_y + \delta\sigma_y$, $\tau_{xy} + \delta\tau_{xy}$ satisfy the same equations of equilibrium (18). Then, by subtracting the equations for one set from those of the other, we find that the changes in the stress components satisfy the following equations of equilibrium:

$$\begin{aligned} \frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \delta\sigma_y}{\partial y} + \frac{\partial \delta\tau_{xy}}{\partial x} &= 0 \end{aligned} \quad (a)$$

Corresponding to this variation of stress components there will be some variation in the surface forces. Let $\delta\bar{X}$ and $\delta\bar{Y}$ be these small changes

in boundary forces; then, from boundary conditions (20), we find

$$\begin{aligned} \delta\sigma_x l + \delta\tau_{xy} m &= \delta\bar{X} \\ \delta\sigma_y m + \delta\tau_{xy} l &= \delta\bar{Y} \end{aligned} \quad (b)$$

Consider now the change in strain energy of the body due to the above changes in stress components. Taking the strain energy per unit volume as a function of the stress components (86), the change of this energy is

$$\delta V_0 = \frac{\partial V_0}{\partial \sigma_x} \delta\sigma_x + \frac{\partial V_0}{\partial \sigma_y} \delta\sigma_y + \frac{\partial V_0}{\partial \tau_{xy}} \delta\tau_{xy} \quad (c)$$

in which

$$\begin{aligned} \frac{\partial V_0}{\partial \sigma_x} &= \frac{1}{E} (\sigma_x - \nu\sigma_y) = \epsilon_x \\ \frac{\partial V_0}{\partial \sigma_y} &= \frac{1}{E} (\sigma_y - \nu\sigma_x) = \epsilon_y \\ \frac{\partial V_0}{\partial \tau_{xy}} &= \frac{1}{G} \tau_{xy} = \gamma_{xy} \end{aligned}$$

giving us

$$\delta V_0 = \epsilon_x \delta\sigma_x + \epsilon_y \delta\sigma_y + \gamma_{xy} \delta\tau_{xy}$$

and the total change in the strain energy due to changes of stress components is

$$\delta V = \iint \delta V_0 dx dy = \iint (\epsilon_x \delta\sigma_x + \epsilon_y \delta\sigma_y + \gamma_{xy} \delta\tau_{xy}) dx dy \quad (d)$$

Let us calculate this change in energy, taking into consideration the boundary conditions (b). The first term in (d) gives, integrating by parts,

$$\begin{aligned} \iint \epsilon_x \delta\sigma_x dx dy &= \int dy \int \frac{\partial u}{\partial x} \delta\sigma_x dx = \int dy |u \delta\sigma_x| \\ &\quad - \iint u \frac{\partial \delta\sigma_x}{\partial x} dx dy \end{aligned} \quad (e)$$

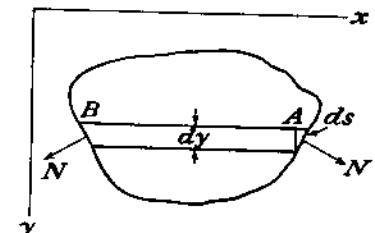


FIG. 115.

in which the expression $|u \delta\sigma_x|$ represents the difference of the values of the function $u \delta\sigma_x$ at two opposite points of the boundary, such as the points A and B in Fig. 115. We then have

$$\begin{aligned} dy |u \delta\sigma_x| &= dy(u \delta\sigma_x)_A - dy(u \delta\sigma_x)_B = ds(u \delta\sigma_x \cos Nx)_A \\ &\quad + ds(u \delta\sigma_x \cos Nx)_B \end{aligned} \quad (f)$$

where $\cos Nx = l$ is the cosine of the angle between the external normal N and the x -axis, and ds is an element of the boundary. Summing up such expressions as given in (f) we find

$$\int dy |u \delta\sigma_x| = \int u \delta\sigma_x l ds$$

and Eq. (e) becomes

$$\iint \epsilon_x \delta\sigma_x dx dy = \int u \delta\sigma_x l ds - \iint u \frac{\partial \delta\sigma_x}{\partial x} dx dy \quad (g)$$

in which the first integral is extended along the boundary and the second over the area of the plate.

In the same manner the second and the third terms on the right side of Eq. (d) may be transformed and we find

$$\begin{aligned} \iint \epsilon_y \delta\sigma_y dx dy &= \int dx \int \frac{\partial v}{\partial y} \delta\sigma_y dy = \int dx |v \delta\sigma_y| \\ &- \iint v \frac{\partial \delta\sigma_y}{\partial y} dx dy = \int v \delta\sigma_y m ds - \iint v \frac{\partial \delta\sigma_y}{\partial y} dx dy \\ \iint \gamma_{xy} \delta\tau_{xy} dx dy &= \int dy \int \frac{\partial v}{\partial x} \delta\tau_{xy} dx + \int dx \int \frac{\partial u}{\partial y} \delta\tau_{xy} dy \\ &= \int dy |v \delta\tau_{xy}| - \iint v \frac{\partial \delta\tau_{xy}}{\partial x} dx dy + \int dx |u \delta\tau_{xy}| \\ &- \iint u \frac{\partial \delta\tau_{xy}}{\partial y} dx dy = \int v \delta\tau_{xy} l ds + \int u \delta\tau_{xy} m ds \\ &- \iint v \frac{\partial \delta\tau_{xy}}{\partial x} dx dy - \iint u \frac{\partial \delta\tau_{xy}}{\partial y} dx dy \end{aligned} \quad (h)$$

Substituting (g) and (h) in Eq. (d) we obtain

$$\begin{aligned} \delta V &= \int [u(\delta\sigma_x l + \delta\tau_{xy} m) + v(\delta\sigma_y m + \delta\tau_{xy} l)] ds \\ &- \iint \left[u \left(\frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} \right) + v \left(\frac{\partial \delta\sigma_y}{\partial y} + \frac{\partial \delta\tau_{xy}}{\partial x} \right) \right] dx dy \end{aligned}$$

in which the first integral is extended along the boundary and the second over the area of the plate. Making use of Eqs. (a) and (b) we finally obtain the following expression for the change of strain energy due to variation of stress components:

$$\delta V = \int (u \delta\bar{X} + v \delta\bar{Y}) ds \quad (91)$$

The right side of this equation represents the work produced by the changes of external forces on the actual displacements.

The true stresses are those which satisfy this equation. An analogous equation can be obtained for the three-dimensional case, with a third term $w \delta\bar{Z}$ added in the bracket in Eq. (91), and the integral extending over the boundary surface instead of the boundary curve.

If we have concentrated loads instead of a continuous distribution of surface forces, the integration in Eq. (91) should be replaced by a summation. Letting P_1, P_2, \dots , be independent concentrated loads and d_1, d_2, \dots , the actual displacements of the points of applications of the loads in the directions of these loads, Eq. (91) becomes

$$\delta V = d_1 \delta P_1 + d_2 \delta P_2 + \dots \quad (92)$$

In this discussion we have taken the most general variations of the stress components fulfilling the equations of equilibrium (a).

Let us consider now a special case when the variations of the stress components are such that they can be actually produced in an elastic body by proper changes in the external forces. We assume that the stress components are expressed as functions of the external loads P_1, P_2, \dots , and we take those changes of stress components which are due to the changes $\delta P_1, \delta P_2, \dots$, of these forces. Considering only cases when the stress components are linear functions of the external loads¹ P_1, P_2, \dots , and substituting these functions in Eq. (84), we obtain the expression for the strain energy as a homogeneous quadratic function of the external forces.

It should be noted that the reactions at the supports, which can be determined from the equations of equilibrium of a rigid body, will be expressed as functions of the given loads P_1, P_2, \dots and will not enter into the expression for the strain energy. If there are redundant constraints, the corresponding reactions should be considered together with the loads P_1, P_2, \dots as statically independent forces.

Having an expression for the strain energy in terms of the external forces, the change of this energy due to changes in the forces is

$$\delta V = \frac{\partial V}{\partial P_1} \delta P_1 + \frac{\partial V}{\partial P_2} \delta P_2 + \dots$$

¹ We exclude for instance such cases as the bending of thin bars by lateral forces with simultaneous axial tension or compression. In these cases the stresses produced by the axial force depend on the deflections due to the lateral forces and are not linear functions of the external loads.

Substituting this in Eq. (92), we find

$$\left(\frac{\partial V}{\partial P_1} - d_1\right) \delta P_1 + \left(\frac{\partial V}{\partial P_2} - d_2\right) \delta P_2 + \dots = 0 \quad (i)$$

The forces P_1, P_2, \dots are, as explained above, statically independent and their changes $\delta P_1, \delta P_2, \dots$ are completely arbitrary. We can take all but one of them equal to zero; hence Eq. (i) requires

$$\frac{\partial V}{\partial P_1} = d_1, \quad \frac{\partial V}{\partial P_2} = d_2 \dots \quad (93)$$

We see that if the strain energy V of an elastic system is represented as a function of statically independent external forces P_1, P_2, \dots , the partial derivatives of this function with respect to any of these forces give the actual displacement of the point of application of the force in the direction of the force. This is the well-known *Castigliano's theorem*.

50. Principle of Least Work. In deriving Eq. (91) we assume any changes in stress components satisfying the equations of equilibrium. If we assume now that the changes are such that the surface forces remain unchanged, then, instead of Eqs. (b) of the previous article, we obtain

$$\begin{aligned}\delta\sigma_x l + \delta\tau_{xy} m &= 0 \\ \delta\sigma_y m + \delta\tau_{xy} l &= 0\end{aligned}$$

and Eq. (91) becomes

$$\delta V = 0 \quad (94)$$

This means that if we have a body with given forces acting on the boundary, and if we consider such changes of stress components as do not affect the equations of equilibrium and the boundary conditions, the true stress components are those making the variation of strain energy vanish. It can be shown that these correct values of the stress components make the strain energy a minimum. Then Eq. (94) expresses the so-called *principle of least work*.

This equation holds also if a portion of the boundary is held rigidly fixed by the constraints and the changes of stress components are such that there are variations of surface forces along this constrained portion of the boundary. Since the displacement along the constrained boundary is zero, the right side of Eq. (91) vanishes, and we arrive again at Eq. (94).

The principle of least work is used very often in elementary treatments of statically indeterminate systems.¹ If X, Y, Z, \dots are

¹ See, for instance, S. Timoshenko, "Strength of Materials," vol. 1, 1940, or Timoshenko and Young, "Theory of Structures."

forces or couples acting in redundant elements or at redundant constraints of an elastic system, the magnitudes of these statically indeterminate quantities can be calculated from the condition that the strain energy of the system, represented as a function of X, Y, Z, \dots must be a minimum, i.e., we have the equations

$$\frac{\partial V}{\partial X} = 0, \quad \frac{\partial V}{\partial Y} = 0, \quad \frac{\partial V}{\partial Z} = 0, \dots \quad (95)$$

Several applications of the principle of least work in the solution of two-dimensional problems will be discussed in the following articles.

51. Applications of the Principle of Least Work—Rectangular Plates. As an example let us consider a rectangular plate. Previously (page 46) it has been shown that by using trigonometric series the conditions on two sides of a rectangular plate can be satisfied. Solutions obtained in this way may be of practical interest when applied to a plate whose width is small in comparison with its length. If both dimensions of a plate are of the same order, the conditions on all four sides must be considered. In the solution of problems of this kind the principle of minimum energy can sometimes be successfully applied.

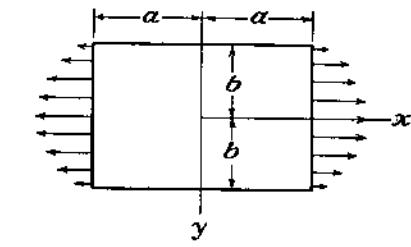


FIG. 116.

Let us consider the case of a rectangular plate in tension, when the tensile forces at the ends are distributed according to a parabolic law¹ (Fig. 116). The boundary conditions in this case are:

For $x = \pm a$,

$$\begin{aligned}\tau_{xy} &= 0, \quad \sigma_x = S \left(1 - \frac{y^2}{b^2}\right) \\ \text{For } y &= \pm b, \\ \tau_{xy} &= 0, \quad \sigma_y = 0\end{aligned} \quad (a)$$

The strain energy for a plate of unit thickness is, from Eq. (86),

$$V = \frac{1}{2E} \int \int [\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1 + \nu)\tau_{xy}^2] dx dy \quad (b)$$

It should be noted that for a simply connected boundary, such as we have in the present case, the stress distribution does not depend on the

¹ See S. Timoshenko, *Phil. Mag.*, vol. 47, p. 1095, 1924.

elastic constants of the material (see page 25) and further calculations can therefore be simplified by taking Poisson's ratio ν as zero. Then, introducing the stress function ϕ , and substituting in (b)

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}, \quad \nu = 0$$

we find

$$V = \frac{1}{2E} \int \int \left[\left(\frac{\partial^2 \phi}{\partial y^2} \right)^2 + \left(\frac{\partial^2 \phi}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 \right] dx dy \quad (c)$$

The correct expression for the stress function is that satisfying conditions (a) and making the strain energy (c) a minimum.

If we apply variational calculus to determine the minimum of (c), we shall arrive at Eq. (30) for the stress function ϕ . Instead of this we shall use the following procedure for an approximate solution of the problem. We take the stress function in the form of a series,

$$\phi = \phi_0 + \alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \dots \quad (d)$$

such that the boundary conditions (a) are satisfied, $\alpha_1, \alpha_2, \alpha_3, \dots$ being constants to be determined later. Substituting this series in expression (c) we find V as a function of the second degree in $\alpha_1, \alpha_2, \alpha_3, \dots$. The magnitude of the constants can then be calculated from the minimum conditions

$$\frac{\partial V}{\partial \alpha_1} = 0, \quad \frac{\partial V}{\partial \alpha_2} = 0, \quad \frac{\partial V}{\partial \alpha_3} = 0, \dots \quad (e)$$

which will be linear equations in $\alpha_1, \alpha_2, \alpha_3, \dots$.

By a suitable choice of the functions ϕ_1, ϕ_2, \dots , we can usually get a satisfactory approximate solution by using only a few terms in the series (d). In our case the boundary conditions (a) are satisfied by taking

$$\phi_0 = \frac{1}{2} S y^2 \left(1 - \frac{y^2}{b^2} \right)$$

since this gives

$$\sigma_y = \frac{\partial^2 \phi_0}{\partial x^2} = 0, \quad \tau_{xy} = -\frac{\partial^2 \phi_0}{\partial x \partial y} = 0, \quad \sigma_x = \frac{\partial^2 \phi_0}{\partial y^2} = S \left(1 - \frac{y^2}{b^2} \right)$$

The remaining functions ϕ_1, ϕ_2, \dots , must be chosen so that the stresses corresponding to them vanish at the boundary. To ensure this we take the expression $(x^2 - a^2)^2(y^2 - b^2)^2$ as a factor in all these

functions; the second derivative of this expression with respect to x vanishes at the sides $y = \pm b$, and the second derivative with respect to y vanishes at the sides $x = \pm a$; the second derivative $\partial^2/\partial x \partial y$ vanishes on all four sides of the plate. The stress function can then be taken as

$$\phi = \frac{1}{2} S y^2 \left(1 - \frac{y^2}{b^2} \right) + (x^2 - a^2)^2(y^2 - b^2)^2(\alpha_1 + \alpha_2 x^2 + \alpha_3 y^2 + \dots) \quad (f)$$

Only even powers of x and y are taken in the series because the stress distribution is symmetrical with respect to the x - and y -axes. Limiting ourselves to the first term α_1 in series (f), we have

$$\phi = \frac{1}{2} S y^2 \left(1 - \frac{y^2}{b^2} \right) + \alpha_1 (x^2 - a^2)^2(y^2 - b^2)^2$$

The first of Eqs. (e) then becomes

$$\alpha_1 \left(\frac{64}{7} + \frac{256}{49} \frac{b^2}{a^2} + \frac{64}{7} \frac{b^4}{a^4} \right) = \frac{S}{a^4 b^2}$$

For a square plate ($a = b$) we find

$$\alpha_1 = 0.04253 \frac{S}{a^6}$$

and the stress components are

$$\begin{aligned} \sigma_x &= S \left(1 - \frac{y^2}{a^2} \right) - 0.1702 S \left(1 - \frac{3y^2}{a^2} \right) \left(1 - \frac{x^2}{a^2} \right)^2 \\ \sigma_y &= -0.1702 S \left(1 - \frac{3x^2}{a^2} \right) \left(1 - \frac{y^2}{a^2} \right)^2 \\ \tau_{xy} &= -0.6805 S \frac{xy}{a^2} \left(1 - \frac{x^2}{a^2} \right) \left(1 - \frac{y^2}{a^2} \right) \end{aligned}$$

The distribution of σ_x on the cross section $x = 0$ is represented by curve II¹ (Fig. 117).

To obtain a closer approximation, we now take three terms in the series (f). Then Eqs. (e), for calculating the constants $\alpha_1, \alpha_2, \alpha_3$, are

¹ Curve I represents the parabolic stress distribution at the ends of the plate.

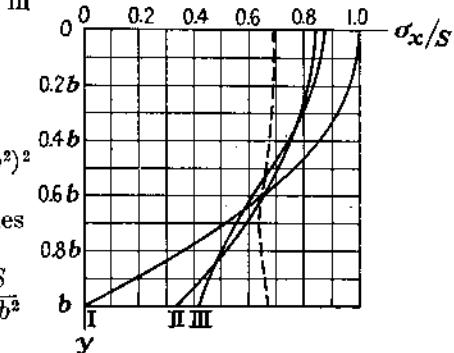


FIG. 117.

$$\begin{aligned} \alpha_1 \left(\frac{64}{7} + \frac{256}{49} \frac{b^2}{a^2} + \frac{64}{7} \frac{b^4}{a^4} \right) + \alpha_2 a^2 \left(\frac{64}{77} + \frac{64}{49} \frac{b^4}{a^4} \right) \\ + \alpha_3 a^2 \left(\frac{64}{49} \frac{b^2}{a^2} + \frac{64}{77} \frac{b^6}{a^6} \right) = \frac{S}{a^4 b^2} \\ \alpha_1 \left(\frac{64}{11} + \frac{64}{7} \frac{b^4}{a^4} \right) + \alpha_2 a^2 \left(\frac{192}{143} + \frac{256}{77} \frac{b^2}{a^2} + \frac{192}{7} \frac{b^4}{a^4} \right) \\ + \alpha_3 a^2 \left(\frac{64}{77} \frac{b^2}{a^2} + \frac{64}{77} \frac{b^6}{a^6} \right) = \frac{S}{a^4 b^2} \quad (g) \end{aligned}$$

$$\begin{aligned} \alpha_1 \left(\frac{64}{7} + \frac{64}{11} \frac{b^4}{a^4} \right) + \alpha_2 a^2 \left(\frac{64}{77} + \frac{64}{77} \frac{b^4}{a^4} \right) \\ + \alpha_3 a^2 \left(\frac{192}{7} \frac{b^2}{a^2} + \frac{256}{77} \frac{b^4}{a^4} + \frac{192}{143} \frac{b^6}{a^6} \right) = \frac{S}{a^4 b^2} \end{aligned}$$

For a square plate these give

$$\alpha_1 = 0.04040 \frac{S}{a^6}, \quad \alpha_2 = \alpha_3 = 0.01174 \frac{S}{a^8}$$

The distribution of σ_x on the cross section $x = 0$ is given by

$$\begin{aligned} (\sigma_x)_{x=0} = S \left(1 - \frac{y^2}{a^2} \right) - 0.16168 \left(1 - 3 \frac{y^2}{a^2} \right) \\ + 0.0235 \left(1 - 12 \frac{y^2}{a^2} + 15 \frac{y^4}{a^4} \right) \end{aligned}$$

In Fig. 117 this stress distribution is shown by the curve III.¹

As the length of the plate increases, the stress distribution over the cross section $x = 0$ becomes more and more uniform. If we take for instance $a = 2b$, we find, from Eqs. (g),

$$\alpha_1 = 0.07983 \frac{S}{a^4 b^2}, \quad \alpha_2 = 0.1250 \frac{S}{a^6 b^2}, \quad \alpha_3 = 0.01826 \frac{S}{a^8 b^2}$$

The corresponding values of σ_x over the cross section $x = 0$ are given below:

$\frac{y}{b} = 0$	0.2	0.4	0.6	0.8	1.0
$\sigma_x = 0.690S$	0.684S	0.669S	0.653S	0.649S	0.675S

This distribution is represented in Fig. 117 by the dotted line. We see that in this case the deviation from the average stress, $\frac{2}{3}S$, is very small.

¹ Similar results were obtained by C. E. Inglis, *Proc. Roy. Soc. (London)*, series A, vol. 103, 1923, and by G. Pickett, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 11, p. 176, 1944.

To deal with other symmetrical distributions of forces over the edges $x = \pm a$ we have only to change the form of the function ϕ_0 in expression (f). Only the right-hand expressions in Eqs. (g) have to be changed.

As an example of stress distribution nonsymmetrical with respect to the x -axis, let us consider the case of bending shown in Fig. 118¹ in which the forces applied at the ends are $(\sigma_x)_{x=\pm a} = Ay^3$ (curve b in Fig. 118b). Clearly, the stress system will be odd with respect to the

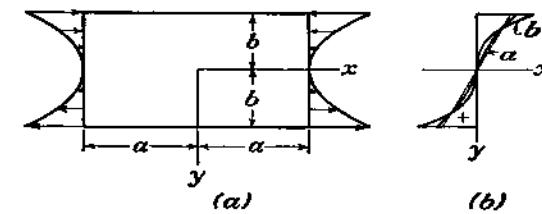


FIG. 118.

x -axis and even with respect to the y -axis. These conditions are satisfied by taking a stress function in the form

$$\phi = \frac{1}{2} \frac{A}{a} y^5 + (x^2 - a^2)^2 (y^2 - b^2)^2 \quad (\alpha_1 y + \alpha_2 yx^2 + \alpha_3 y^3 + \alpha_4 x^2 y^3 + \dots) \quad (h)$$

The first term, as before, satisfies the boundary conditions for ϕ . Using Eq. (h) with four coefficients $\alpha_1, \dots, \alpha_4$ in Eqs. (e), we find for a square plate ($a = b$)

$$\begin{aligned} \sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2Aa^3 & \left\{ \frac{1}{2} \eta^3 - (1 - \xi^2)^2 [0.08392(5\eta^3 - 3\eta) \right. \\ & + 0.004108(21\eta^5 - 20\eta^3 + 3\eta)] \\ & - \xi^2(1 - \xi^2)^2 [0.07308(5\eta^3 - 3\eta) + 0.04179(21\eta^5 - 20\eta^3 + 3\eta)] \} \quad (k) \end{aligned}$$

where $\xi = x/a$ and $\eta = y/b$. The distribution at the middle cross section $x = 0$ is not far from being linear. It is shown in Fig. 118b by curve a.

52. Effective Width of Wide Beam Flanges. As another example of the application of the minimum-energy principle to two-dimensional problems of rectangles, let us consider a beam with very wide flanges (Fig. 119). Such beams are encountered very often in reinforced concrete structures and in the structures of hulls of ships. The elementary theory of bending assumes that the bending stresses are

¹ These calculations are taken from J. N. Goodier's doctor's thesis, Michigan Univ., 1931. See also *Trans. A.S.M.E.*, vol. 54, p. 173, 1932.

proportional to the distance from the neutral axis, i.e., that the stresses do not change along the width of the flange. But if this width is very large it is known that parts of the flanges at a distance from the web do not take their full share in resisting bending moment, and the beam is weaker than the elementary theory of bending indicates. It is the usual practice in calculating stresses in such beams to replace the actual width of the flanges by a certain reduced width, such that the elementary theory of bending applied to such a transformed beam cross section gives the correct value of maximum bending stress. This reduced width of flange is called the *effective width*. In the following discussion a theoretical basis for determining the effective width is given.¹

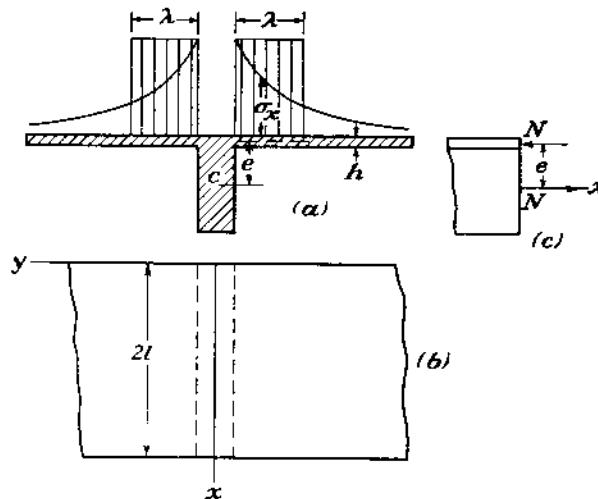


FIG. 119.

To make the problem as simple as possible it is assumed that we have an infinitely long continuous beam on equidistant supports. All spans are equally loaded by loads symmetrical with respect to the middle of the spans. One of the supports of the span shown in Fig. 119 is taken as the origin of coordinates, with the x -axis in the direction of the axis of the beam. Due to symmetry, only one span and one half of the flange, say that corresponding to positive y , need be considered. The width of the flange is assumed infinitely large and its thickness h very small in comparison with the depth of the beam. Bending of the flange as a thin plate can then be neglected, and it can be assumed that during bending of the beam the forces are transmitted to the flange in its middle plane so that the stress distribution in the flange presents a two-dimensional problem. The corresponding stress function ϕ , satisfying the differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (a)$$

¹ The subject was investigated by T. v. Kármán; see "Festschrift August Föppl," p. 114, 1923. Also G. Schnadel, *Werft und Reederei*, vol. 9, p. 92, 1928; E. Reissner, *Der Stahlbau*, 1934, p. 206; E. Chwalla, *Der Stahlbau*, 1936; L. Beschke, *Publs. Intern. Assoc. Bridge and Structural Engineering*, vol. 5, p. 65, 1938.

can be taken for our symmetrical case in the form of the series

$$\phi = \sum_{n=1}^{\infty} f_n(y) \cos \frac{n\pi x}{l} \quad (b)$$

in which $f_n(y)$ are functions of y only. Substituting in Eq. (a), we find the following expression for $f_n(y)$:

$$f_n(y) = A_n e^{-\frac{n\pi y}{l}} + B_n \left(1 + \frac{n\pi y}{l}\right) e^{-\frac{n\pi y}{l}} + C_n e^{\frac{n\pi y}{l}} + D_n \left(1 + \frac{n\pi y}{l}\right) e^{\frac{n\pi y}{l}} \dots \quad (c)$$

To satisfy the condition that stresses must vanish for an infinite value of y , we take $C_n = D_n = 0$. The expression for the stress function is then

$$\phi = \sum_{n=1}^{\infty} \left[A_n e^{-\frac{n\pi y}{l}} + B_n \left(1 + \frac{n\pi y}{l}\right) e^{-\frac{n\pi y}{l}} \right] \cos \frac{n\pi x}{l} \quad (d)$$

The coefficients A_n and B_n will now be determined from the condition that the true stress distribution is that making the strain energy of the flange together with that of the web a minimum. Substituting

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

in the expression for strain energy

$$V_1 = 2 \cdot \frac{h}{2E} \int_0^{\infty} \int_0^{2l} [\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2] dx dy$$

and using Eq. (d) for the stress function, the strain energy of the flange is¹

$$V_1 = 2h \sum_{n=1}^{\infty} \frac{n^3 \pi^3}{l^2} \left(\frac{B_n^2}{E} + \frac{A_n B_n}{2G} + \frac{A_n^2}{2G} \right) \quad (e)$$

In considering the strain energy of the web alone, let A be its cross-sectional area, I its moment of inertia about the horizontal axis through the centroid C , and e the distance from the centroid of the web to the middle plane of the flange (Fig. 119). The total bending moment transmitted at any cross section by the web together with the flange can be represented for our symmetrical case by the series

$$M = M_0 + M_1 \cos \frac{\pi x}{l} + M_2 \cos \frac{2\pi x}{l} + \dots \quad (f)$$

In this series M_0 is a statically indeterminate quantity depending on the magnitude of the bending moment at the supports, and the other coefficients M_1, M_2, \dots , are to be calculated from the conditions of loading. Letting N denote the compressive force in the flange (Fig. 119c), the bending moment M can be divided into

¹ The integrals entering into the expression for strain energy are calculated in the paper by Kármán, *loc. cit.*

two parts: a part M' taken by the web and a part M'' , equal to Ne , due to the longitudinal forces N in the web and flange. From statics the normal stresses over any cross section of the complete beam give a couple M , hence

$$\begin{aligned} N + 2h \int_0^{\infty} \sigma_x dy &= 0 \\ M' - 2he \int_0^{\infty} \sigma_x dy &= M \end{aligned} \quad (g)$$

where $-2he \int_0^{\infty} \sigma_x dy = M''$ is the part of the bending moment taken by the flange. The strain energy of the web is

$$V_1 = \int_0^{2l} \frac{N^2}{2AE} dx + \int_0^{2l} \frac{M'^2}{2EI} dx \quad (h)$$

From the first of Eqs. (g) we find

$$N = -2h \int_0^{\infty} \sigma_x dy = -2h \int_0^{\infty} \frac{\partial^2 \phi}{\partial y^2} dy = 2h \left[\frac{\partial \phi}{\partial y} \right]_0^{\infty}$$

From expression (d) for the stress function it may be seen that

$$\left(\frac{\partial \phi}{\partial y} \right)_{y=\infty} = 0, \quad \left(\frac{\partial \phi}{\partial y} \right)_{y=0} = \sum_{n=1}^{\infty} \frac{n\pi}{l} A_n \cos \frac{n\pi x}{l}$$

Hence

$$N = 2h \sum_{n=1}^{\infty} \frac{n\pi}{l} A_n \cos \frac{n\pi x}{l}$$

$$M' = M + 2he \int_0^{\infty} \sigma_x dy = M + Ne = M + 2he \sum_{n=1}^{\infty} \frac{n\pi}{l} A_n \cos \frac{n\pi x}{l}$$

or, using the notation

$$2h \frac{n\pi}{l} A_n = X_n$$

we may write

$$\begin{aligned} N &= \sum_{n=1}^{\infty} X_n \cos \frac{n\pi x}{l} \\ M' &= M + e \sum_{n=1}^{\infty} X_n \cos \frac{n\pi x}{l} = M_0 + \sum_{n=1}^{\infty} (M_n + eX_n) \cos \frac{n\pi x}{l} \end{aligned} \quad (k)$$

Substituting in (h) and noting that

$$\int_0^{2l} \cos^2 \frac{n\pi x}{l} dx = l, \quad \int_0^{2l} \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx = 0 \text{ (when } m \neq n)$$

we obtain

$$V_1 = \frac{l}{2AE} \sum_{n=1}^{\infty} X_n^2 + \frac{M_0^2 l}{EI} + \frac{l}{2EI} \sum_{n=1}^{\infty} (M_n + eX_n)^2$$

Adding this to the strain energy (e) of the flange, and introducing in this latter the notations

$$2h \frac{n\pi}{l} A_n = X_n, \quad 2h \frac{n\pi}{l} B_n = Y_n$$

we find the following expression for the total strain energy:

$$\begin{aligned} V &= \frac{\pi}{2hE} \sum_{n=1}^{\infty} n[Y_n^2 + (1+\nu)X_n Y_n + (1+\nu)X_n^2] + \frac{l}{2AE} \sum_{n=1}^{\infty} X_n^2 \\ &\quad + \frac{M_0^2 l}{EI} + \frac{l}{2EI} \sum_{n=1}^{\infty} (M_n + eX_n)^2 \end{aligned} \quad (l)$$

The quantities M_0, X_n, Y_n are to be determined from the minimum condition of the strain energy (l). It can be seen that M_0 appears only in the term $M_0^2 l/EI$, and from the minimum requirement for (l) it follows that $M_0 = 0$.

From the condition

$$\frac{\partial V}{\partial Y_n} = 0$$

it follows that

$$\begin{aligned} 2Y_n + (1+\nu)X_n &= 0 \\ Y_n &= -\frac{1+\nu}{2} X_n \end{aligned}$$

Substituting this and $M_0 = 0$ in Eq. (l) we get the following expression for the strain energy:

$$\begin{aligned} V &= \frac{\pi}{2hE} \cdot \frac{3+2\nu-\nu^2}{4} \sum_{n=1}^{\infty} n \cdot X_n^2 + \frac{l}{2AE} \sum_{n=1}^{\infty} X_n^2 \\ &\quad + \frac{l}{2EI} \sum_{n=1}^{\infty} (M_n + eX_n)^2 \end{aligned} \quad (m)$$

From the condition that X_n should make V a minimum it follows that

$$\frac{\partial V}{\partial X_n} = 0$$

from which we find

$$X_n = -\frac{M_n}{e} \frac{1}{1 + \frac{I}{Ae^2} + \frac{n\pi l}{h e^2} \cdot \frac{3+2\nu-\nu^2}{4}} \quad (n)$$

Let us consider a particular case when the bending-moment diagram is a simple cosine line, say $M = M_1 \cos(\pi x/l)$. Then, from Eq. (n),

$$X_1 = -\frac{M_1}{e} \frac{1}{1 + \frac{I}{Ae^2} + \frac{\pi I}{h e^2} \cdot \frac{3+2\nu-\nu^2}{4}}$$

and, from Eq. (k), the moment due to the force N of the flange is

$$M'' = -eN = -eX_1 \cos \frac{\pi x}{l} = \frac{M}{1 + \frac{f}{Ae^2} + \frac{\pi I}{he^2 l} \frac{3 + 2\nu - \nu^2}{4}} \quad (p)$$

The distribution of the stress σ_z along the width of the flange can now be calculated from (d) by taking all coefficients A_n and B_n , except A_1 and B_1 , equal to zero, and by putting (following our notations)

$$A_1 = \frac{lX_1}{2\pi h}, \quad B_1 = -\frac{1 + \nu}{2} A_1 = -\frac{(1 + \nu)lX_1}{4\pi h}$$

This distribution of σ_z is shown by the curves in Fig. 119a. The stress σ_z diminishes as the distance from the web increases.

Let us now determine a width 2λ of the flange (Fig. 119a), of a T-beam, such that a uniform stress distribution over the cross section of the flange, shown by the shaded area, gives the moment M'' calculated above, Eq. (p). This will then be the effective width of the flange.

Denoting, as before, by M' and M'' the portions of the bending moment taken by the web and by the flange, by σ_c the stress at the centroid C of the web, and by σ_e the stress at the middle plane of the flange, we find, from the elementary theory of bending,

$$\sigma_e = \sigma_c + \frac{M'e}{I} \quad (q)$$

and, from the equations of statics,

$$\begin{aligned} 2\lambda h\sigma_e + \sigma_c A &= 0 \\ 2\lambda h\sigma_e e &= M'' \end{aligned} \quad (r)$$

The expressions for the two portions of the bending moment, from Eqs. (q) and (r), are

$$\begin{aligned} M' &= \frac{I}{e} (\sigma_e - \sigma_c) = \frac{I}{e} \left(1 + \frac{2\lambda h}{A}\right) \sigma_e \\ M'' &= 2\lambda h\sigma_e \end{aligned}$$

The ratio of M'' to the total bending moment is

$$\frac{M''}{M' + M''} = \frac{2\lambda h\sigma_e}{2\lambda h\sigma_e + \frac{I}{e} \left(1 + \frac{2\lambda h}{A}\right) \sigma_e} = \frac{1}{1 + \frac{I}{Ae^2} + \frac{I}{2\lambda h e^2}} \quad (s)$$

To make this ratio equal to the ratio M''/M obtained from the exact solution (v), we must take

$$\frac{I}{2\lambda h e^2} = \frac{\pi I}{he^2 l} \frac{3 + 2\nu - \nu^2}{4}$$

From this we obtain the following expression for the effective width 2λ :

$$2\lambda = \frac{4l}{\pi(3 + 2\nu - \nu^2)}$$

Taking, for instance, $\nu = 0.3$, we find

$$2\lambda = 0.181(2l)$$

i.e., for the assumed bending-moment diagram the effective width of the flange is approximately 18 per cent of the span.

In the case of a continuous beam with equal concentrated forces at the middle of the spans, the bending-moment diagram will be as shown in Fig. 120. Representing this bending-moment diagram by a Fourier series and using the general method developed above, we find that the effective width at the supports is

$$2\lambda = 0.85 \cdot \frac{4l}{\pi(3 + 2\nu - \nu^2)}$$

i.e., somewhat less than it is for the case of a moment diagram in the form of a cosine line.

53. Shear Lag. A problem of the same general nature as that discussed in Art. 52 occurs in aircraft structures. Consider a *box beam*, Fig. 121, formed from two channels *ABFE* and *DCHG* to which are attached thin sheets *ABCD* and *EFGH*, by riveting or welding along the edges. If the whole beam is built in at the left-hand end, and loaded as a cantilever by two forces P applied to the channels at the other end, the elementary bending theory will give a tensile bending stress in the sheet *ABCD* uniform across any section parallel to *BC*.

Actually, however, the sheet acquires its tensile stress from shear stresses on its edges communicated to it by the channels, as indicated in Fig. 121, and the distribution of tensile stress across the width will not be uniform, but, as in Fig. 121, higher at the edges than at the middle. This departure from the uniformity assumed by the elementary theory is known as "shear lag," since it involves a shear deformation in the sheets. The problem has been analyzed by strain-energy and other methods, with the help of simplifying assumptions.¹

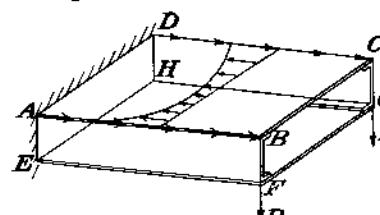


FIG. 121.

deformation in the sheets. The problem has been analyzed by strain-energy and other methods, with the help of simplifying assumptions.¹

Problems

- Find an expression in terms of σ_x , σ_y , τ_{xy} for the strain energy V per unit thickness of a cylinder or prism in plane strain ($\epsilon_z = 0$).

- Write down the integral for the strain energy V in terms of polar coordinates and polar stress components for the case of plane stress [cf. Eq. (b), Art. 51].

The stress distribution given by Eqs. (80) solves the problem indicated in Fig. 122, a couple M being applied by uniform shear to the inside of a ring, and a

¹ E. Reissner, *Quart. Applied Math.*, vol. 4, p. 268, 1946; J. Hadji-Arkyris, *(Brit.) Aeronaut. Research Council, Reports and Memoranda*, No. 2038, 1944; J. Hadji-Arkyris and H. L. Cox, *ibid.*, No. 1969, 1944. References to earlier investigations are given in these papers.

balancing couple to the outside. Evaluate the strain energy in the ring, and by equating this to the work done during loading deduce the rotation of the outside circle when the ring is fixed at the inside (cf. Prob. 2, page 125).

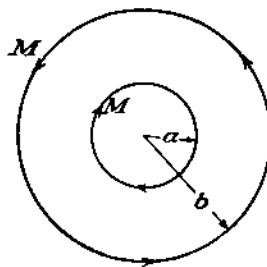


FIG. 122.

3. Evaluate the strain energy per unit length of a cylinder $a < r < b$ subjected to internal pressure p_i [see Eqs. (46)]. Deduce the radial displacement of the inner surface.

Obtain the same result by use of Eq. (50) (taking $v = 0$) and the stress-strain relations of plane stress.

4. Interpret the equation

$$\iint V_0 \, dx \, dy = \frac{1}{2} \iint (Xu + Yv) \, dx \, dy + \frac{1}{2} \iint (\bar{X}u + \bar{Y}v) \, ds$$

and give the justification of the factors $\frac{1}{2}$ on the right.

5. Show from Eq. (84) that if we have a case of plane stress and a corresponding case of plane strain ($\epsilon_z = 0$) in which the stresses $\sigma_x, \sigma_y, \tau_{xy}$ are the same, the strain energy is greater (per unit thickness) for the plane stress.

6. In Fig. 123, (a) represents a strip under compression, in which the stress therefore extends throughout. In (b) the deformable strip is bonded to rigid plates

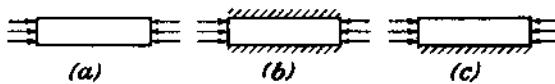


FIG. 123.

on its top and bottom edges. Will there be stress throughout the strip or only locally at the ends? In (c) the upper edge is free, as in (a), but the lower edge is fixed, as in (b). Will the stress be local or not?

7. From the principle that a system in stable equilibrium has less potential energy than that corresponding to any neighboring configuration, show without calculation that the strain energy of the plate in Fig. 114 must either decrease or remain the same when a fine cut AB is made.

8. State the Castigliano theorem expressed by Eq. (91) in a form suitable for use in polar coordinates, the boundary forces \bar{X} and \bar{Y} being replaced by radial and tangential components \bar{R} and \bar{T} , and the displacement components by the polar components u and v of Chap. 4.

9. "Equation (91) is valid when $\delta V, \delta \bar{X}, \delta \bar{Y}$ result from any small changes in the stress components which satisfy the conditions of equilibrium (a) Art. 49, whether these changes violate the conditions of compatibility (Art. 15) or not. In the latter case the changes in the stress are those which actually occur when the boundary forces are changed by $\delta \bar{X}, \delta \bar{Y}$." Is this statement correct?

Assuming that it is, show that the radial displacement of Prob. 3 can be calculated from the formula

$$(u)_{r=a} = \frac{1}{2\pi a} \cdot \frac{\partial V}{\partial p_i}$$

CHAPTER 7

TWO-DIMENSIONAL PROBLEMS IN CURVILINEAR COORDINATES

54. Functions of a Complex Variable. For the problems solved so far, rectangular and polar coordinates have proved adequate. For other boundaries—ellipses, hyperbolae, nonconcentric circles, and less simple curves—it is usually preferable to employ different coordinates. In the consideration of these, and also in the construction of suitable stress functions, it is advantageous to use complex variables.

Two real numbers x, y form the complex number $z = x + iy$, with i representing $\sqrt{-1}$. Since i does not belong to the real-number system, the meaning of equality, addition, subtraction, multiplication, and division must be defined.¹ Thus, by definition, $z + iy = x' + iy'$ means $x = x', y = y'$, and i^2 means -1 . Otherwise the operations are defined just as for real numbers. For instance

$$(x + iy)^2 = x^2 + 2xy + (iy)^2 = x^2 - y^2 + i2xy \quad \text{since } i^2 = -1$$

Converting to polar coordinates, as in Fig. 124,

$$z = x + iy = r(\cos \theta + i \sin \theta) \quad (a)$$

Since

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \dots$$

and

$$i^2 = -1, i^3 = -i, i^4 = 1, \text{ etc.,}$$

we have

$$\begin{aligned} e^{i\theta} &= 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots + i(\theta - \frac{1}{3!}\theta^3 + \dots \\ &= \cos \theta + i \sin \theta \end{aligned}$$

From Eq. (a) therefore

$$z = x + iy = re^{i\theta} \quad (b)$$

¹ The definitions represent operations on pairs of real numbers, the use of i being merely a convenience. See for instance E. T. Whittaker and G. N. Watson, "Modern Analysis," 3d ed., pp. 6-8.

Algebraic, trigonometric, exponential, logarithmic, and other functions can be formed from z as well as from a real variable, provided an analytical rather than a geometrical definition is adopted. Thus $\sin z$, $\cos z$, and e^z may be defined by their power series. Any such function can be separated into "real" and "imaginary" parts, that is, put in the form $\alpha(x,y) + i\beta(x,y)$ where $\alpha(x,y)$, the real part, and $\beta(x,y)$, the imaginary part,¹ are ordinary real functions of x and y (they do not contain i). For instance if the function of z , $f(z)$, is $1/z$, we have

$$f(z) = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} + i \frac{(-y)}{x^2+y^2}$$

Similarly, observing that $\cosh iy = \frac{1}{2}(e^{iy} + e^{-iy})$, $\sinh iy = \frac{1}{2}(e^{iy} - e^{-iy})$, and $e^{\pm iy} = \cos y \pm i \sin y$, we find

$$\begin{aligned}\sinh z &= \sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy \\ &= \sinh x \cos y + i \cosh x \sin y \\ \cosh z &= \cosh(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy \\ &= \cosh x \cos y + i \sinh x \sin y\end{aligned}$$

As an illustration of the general method for converting a complex denominator to a real one, consider the function $\coth z$. We have

$$\begin{aligned}\coth z &= \frac{\cosh z}{\sinh z} = \frac{\cosh(x+iy)}{\sinh(x+iy)} \cdot \frac{\sinh(x-iy)}{\sinh(x-iy)} \\ &= \frac{(\cosh x \cos y + i \sinh x \sin y)(\sinh x \cos y - i \cosh x \sin y)}{(\sinh x \cos y + i \cosh x \sin y)(\sinh x \cos y - i \cosh x \sin y)}\end{aligned}$$

The denominator is the same as the real quantity $(\sinh x \cos y)^2 + (\cosh x \sin y)^2$. When the numerator is multiplied out, and i^2 replaced by -1 , the separation into real and imaginary parts is completed. The result can be simplified to

$$\coth z = \frac{\sinh 2x - i \sin 2y}{\cosh 2x - \cos 2y} \quad (c)$$

An alternative procedure is indicated by Eq. (p) of Art. 62.

The derivative of $f(z)$ with respect to z is by definition

$$\frac{df(z)}{dz} = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad (d)$$

where $\Delta z = \Delta x + i \Delta y$ and $\Delta z \rightarrow 0$ means, of course, both $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$. We can always think of x , y as the Cartesian coordinates of a point in a plane. Then Δx , Δy represent a shift to a neighboring point. It might be expected at first that (d) could be different for different directions of the shift. Nevertheless, the limit in (d) is calculable directly in terms of z and Δz just as if these were real numbers, and the corresponding results, such as

$$\frac{d}{dz}(z^2) = 2z, \quad \frac{d}{dz} \sin z = \cos z$$

must appear, independent of the choice of Δz , and of Δx and Δy . We may say, therefore, that all the functions we may form from z in the usual way will have

¹ It should be observed that this is real in spite of its name.

derivatives which depend on z only, being the same for all directions (of dz) at the point z . Such functions are called *analytic*.

The quantity $x - iy$ may be regarded as a function of z , in the sense that if z is given, x and y are given, and so $x - iy$ is determined. However, $x - iy$ cannot be formed from z as for instance z^n , e^z are formed. Its derivative with respect to z is the limit of $(\Delta x - i \Delta y)/(\Delta x + i \Delta y)$ as $\Delta x, \Delta y \rightarrow 0$. This is not independent of the direction of the shift $\Delta x, \Delta y$. If we take this shift in the x -direction, so that $\Delta y = 0$, we obtain 1 as the value of the limit. If we take the shift in the y -direction, $\Delta x = 0$ and the limit is -1 . Thus $x - iy$ is not an *analytic* function of $x + iy$. Analytic functions together with $x - iy$ will be used later in the construction of stress functions. Any function involving i will be referred to as a "complex function."

An analytic function $f(z)$ will have an indefinite integral, defined as the function having $f(z)$ as its derivative with respect to z , and written $\int f(z) dz$. For instance if $f(z) = 1/z$ we have

$$\int \frac{1}{z} dz = \log z + C$$

the additive constant C being now a complex number $A + iB$, containing two real arbitrary constants A and B .

55. Analytic Functions and Laplace's Equation. An analytic function $f(z)$ can be regarded as a function of x and y , having partial derivatives. Thus

$$\frac{\partial}{\partial x} f(z) = \frac{d}{dz} f(z) \frac{\partial z}{\partial x} = f'(z) \frac{\partial z}{\partial x} = f'(z) \quad (a)$$

since $\partial z/\partial x = 1$. Similarly

$$\frac{\partial}{\partial y} f(z) = f'(z) \frac{\partial z}{\partial y} = if'(z) \quad (b)$$

since $\partial z/\partial y = i$.

But if $f(z)$ is put in the form $\alpha(x,y) + i\beta(x,y)$, or for brevity $\alpha + i\beta$, we have

$$\frac{\partial}{\partial x} f(z) = \frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial y} f(z) = \frac{\partial \alpha}{\partial y} + i \frac{\partial \beta}{\partial y} \quad (c)$$

Comparing Eqs. (c) with Eq. (a) and Eq. (b) yields

$$i \left(\frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x} \right) = \frac{\partial \alpha}{\partial y} + i \frac{\partial \beta}{\partial y} \quad (d)$$

Remembering that α, β are real, $i^2 = -1$, and that the equality implies that real and imaginary parts are separately equal, we find

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}, \quad \frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x} \quad (e)$$

These are called the *Cauchy-Riemann equations*. Eliminating β by differentiating the first with respect to x , the second with respect to y , and adding, we obtain

$$\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} = 0 \quad (f)$$

An equation of this form is called *Laplace's equation* and any solution is called a *harmonic function*. In the same way elimination of α from Eqs. (e) yields

$$\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} = 0 \quad (g)$$

Thus if two functions α and β of x and y are derived as the real and imaginary parts of an analytic function $f(z)$, each will be a solution of Laplace's equation. Laplace's equation is encountered in many physical problems, including those of elasticity [see for instance Eq. (b), Art. 16].

The functions α and β are called *conjugate harmonic functions*. It is evident that if we are given any harmonic function α , Eqs. (e) will, but for a constant, determine another function β , which will be the conjugate to α .

As examples of the derivation of harmonic functions from analytic functions of z , consider e^{inx} , z^n , $\log z$, n being a real constant. We have

$$e^{inx} = e^{inx} e^{-ny} = e^{-ny} \cos nx + i e^{-ny} \sin nx$$

showing that $e^{-ny} \cos nx$, $e^{-ny} \sin nx$ are harmonic functions. Changing n to $-n$ we find that $e^{ny} \cos nx$, $e^{ny} \sin nx$ are also harmonic, and it follows that

$$\sinh ny \sin nx, \quad \cosh ny \sin nx, \quad \sinh ny \cos nx, \quad \cosh ny \cos nx \quad (h)$$

are harmonic since they can be formed by addition and subtraction of the foregoing functions with factors $\frac{1}{2}$. From

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n \cos n\theta + i r^n \sin n\theta$$

we find the harmonic functions

$$r^n \cos n\theta, \quad r^n \sin n\theta, \quad r^{-n} \cos n\theta, \quad r^{-n} \sin n\theta \quad (i)$$

From

$$\log z = \log re^{i\theta} = \log r + i\theta$$

we find the harmonic functions

$$\log r, \quad \theta \quad (j)$$

It is easily verified that the functions (i) and (j) satisfy Laplace's equation in polar coordinates [see Eq. (d), page 57], i.e.,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0 \quad (k)$$

Problems

1. Determine the *real* functions of x and y which are the *real* and *imaginary* parts of the complex functions z^2 , z^3 , $\tanh z$.

[$x^2 - y^2$, $2xy$; $x^3 - 3xy^2$, $3x^2y - y^3$;
 $\sinh 2x(\cosh 2x + \cos 2y)^{-1}$, $\sin 2y(\cosh 2x + \cos 2y)^{-1}$]

2. Determine the *real* functions of r and θ which are the *real* and *imaginary* parts of the complex functions z^{-2} , $z \log z$.

[$r^{-2} \cos 2\theta$, $r^{-2} \sin \theta$; $r \log r \cos \theta - r\theta \sin \theta$, $r \log r \sin \theta + r\theta \cos \theta$]

3. If ξ is a complex variable, and $z = c \cosh \xi$, find

$$\frac{d}{dz} \sinh n\xi$$

Writing $\xi = \xi + i\eta$ find the real and imaginary parts of this derivative when c and n are real.

4. If $z = x + iy$, $\xi = \xi + i\eta$, and $z = ia \coth \frac{1}{2}\xi$ where a is real, show that

$$x = \frac{a \sin \eta}{\cosh \xi - \cos \eta}, \quad y = \frac{a \sinh \xi}{\cosh \xi - \cos \eta}$$

56. Stress Functions in Terms of Harmonic and Complex Functions. If ψ is any function of x and y , we have by differentiation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (x\psi) = x \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + 2 \frac{\partial \psi}{\partial x} \quad (a)$$

If ψ is harmonic, the parenthesis on the right is zero. Also $\partial\psi/\partial x$ is a harmonic function, since $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial \psi}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = 0$. Thus another application of the Laplacian operation to (a) yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (x\psi) = 0 \quad (b)$$

which is the same as

$$\left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) (x\psi) = 0$$

Comparison with Eq. (a), page 29, shows that $x\psi$ may be used as a stress function, ψ being harmonic. The same is true of $y\psi$, and also, of course, of the function ψ itself.

It can easily be shown by differentiation that $(x^2 + y^2)\psi$, that is $r^2\psi$, also satisfies the same differential equation and may therefore be taken as a stress function, ψ being harmonic.

For instance, taking the two harmonic functions

$$\sinh ny \sin nx, \quad \cosh ny \sin nx$$

from the functions (h), page 182, and multiplying them by y , we arrive by superposition at the stress function (d), page 47. Taking the harmonic functions (i) and (j), page 182, as they stand or multiplied by x , y , or r^2 , we can reconstruct all the terms of the stress function in polar coordinates given by Eq. (81), page 116.

The question of whether any stress function at all can be arrived at in this fashion remains open, and will be answered immediately, in the

process of expressing the general stress function in terms of two arbitrary analytic functions.

Denoting the Laplacian operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

by ∇^2 , Eq. (a) on page 29 can be written $\nabla^2(\nabla^2\phi) = 0$ or $\nabla^4\phi = 0$. Writing P for $\nabla^2\phi$, which represents $\sigma_x + \sigma_y$, we observe that P is a harmonic function, and so will have a conjugate harmonic function Q . Consequently $P + iQ$ is an analytic function of z , and we may write

$$f(z) = P + iQ \quad (c)$$

The integral of this function with respect to z is another analytic function, $4\psi(z)$ say. Then, writing p and q for the real and imaginary parts of $\psi(z)$, we have

$$\psi(z) = p + iq = \frac{1}{4}\int f(z) dz \quad (d)$$

so that $\psi'(z) = \frac{1}{4}f(z)$. We have also

$$\frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}\psi(z) = \psi'(z)\frac{\partial z}{\partial x} = \frac{1}{4}f(z) = \frac{1}{4}(P + iQ)$$

Equating real parts of the first and last members we find

$$\frac{\partial p}{\partial x} = \frac{1}{4}P \quad (e)$$

Since p and q are conjugate functions, they satisfy Eqs. (e) of Art. 55, and so

$$\frac{\partial q}{\partial y} = \frac{1}{4}P \quad (f)$$

Recalling that $P = \nabla^2\phi$, Eqs. (e) and (f) enable us to show that $\phi - xp - yq$ is a harmonic function. For

$$\nabla^2(\phi - xp - yq) = \nabla^2\phi - 2\frac{\partial p}{\partial x} - 2\frac{\partial q}{\partial y} = 0 \quad (g)$$

Thus for any stress function ϕ we have

$$\phi - xp - yq = p_1$$

where p_1 is some harmonic function. Consequently

$$\phi = xp + yq + p_1 \quad (96)$$

which shows that any stress function can be formed from suitably chosen conjugate harmonic functions p, q and a harmonic function p_1 .

Equation (96) will prove useful later, but it may be observed that the use of both the functions p and q is not necessary. Instead of Eq. (g) we can write

$$\nabla^2(\phi - 2xp) = \nabla^2\phi - 4\frac{\partial p}{\partial x} = 0$$

showing that $\phi - 2xp$ is harmonic, say equal to p_2 , so that any stress function must be expressible in the form

$$\phi = 2xp + p_2 \quad (h)$$

where p and p_2 are suitably chosen harmonic functions. Similarly, considering $\phi - 2yq$, we may show that any stress function must also be expressible in the form

$$\phi = -2yq + p_3$$

where q and p_3 are suitably chosen harmonic functions.

Returning to the form (96), let us introduce the function q_1 which is the conjugate harmonic to p_1 , and write

$$\chi(z) = p_1 + iq_1$$

Then it is easily verified that the real part of

$$(x - iy)(p + iq) + p_1 + iq_1$$

is identical with the right-hand side of Eq. (96). Thus any stress function is expressible in the form¹

$$\phi = \operatorname{Re}[\bar{z}\psi(z) + \chi(z)] \quad (97)$$

where Re means "real part of," \bar{z} denotes $x - iy$, and $\psi(z)$ and $\chi(z)$ are suitably chosen analytic functions. Conversely (97) yields a stress function, that is a solution of Eq. (a), p. 29, for any choice of $\psi(z)$ and $\chi(z)$. It is applied later to the solution of several problems of practical interest.

Writing the "complex stress function" in brackets in (97) as

$$\bar{z}z \frac{\psi(z)}{z} + \chi(z)$$

and observing that $\bar{z}z = r^2$, and $\psi(z)/z$ is still a function of z , we find that any stress function can also be expressed as

$$r^2p_4 + p_5$$

where p_4, p_5 are harmonic.

¹ E. Goursat, *Bull. soc. math. France*, vol. 26, p. 206, 1898. N. I. Muschelišvili, *Math. Ann.*, vol. 107, pp. 282-312, 1932.

57. Displacement Corresponding to a Given Stress Function. It was shown in Art. 39 that the determination of the stress in a multiply-connected region requires the evaluation of displacement to ensure that it is not discontinuous, that is, to ensure that the stress is not partly due to dislocations. For this reason, as well as for cases where the displacements are of interest in themselves, we require a method of finding the displacement functions u and v when a stress function is given.

The stress-strain relations for plane stress, Eqs. (22), (23), may be written

$$E \frac{\partial u}{\partial x} = \sigma_x - \nu \sigma_y, \quad E \frac{\partial v}{\partial y} = \sigma_y - \nu \sigma_x \quad (a)$$

$$G \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = \tau_{xy} \quad (b)$$

Inserting the stress function into the first, and recalling that $P = \nabla^2 \phi$, we have

$$\begin{aligned} E \frac{\partial u}{\partial x} &= \frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2} = \left(P - \frac{\partial^2 \phi}{\partial x^2} \right) - \nu \frac{\partial^2 \phi}{\partial x^2} \\ &= -(1 + \nu) \frac{\partial^2 \phi}{\partial x^2} + P \end{aligned} \quad (c)$$

and similarly

$$E \frac{\partial v}{\partial y} = -(1 + \nu) \frac{\partial^2 \phi}{\partial y^2} + P \quad (d)$$

But from Eqs. (f) and (g) of Art. 56, we can replace P in Eq. (c) above by $4 \partial p / \partial x$, and in Eq. (d) by $4 \partial q / \partial y$. Then, after division by $1 + \nu$,

$$2G \frac{\partial u}{\partial x} = -\frac{\partial^2 \phi}{\partial x^2} + \frac{4}{1 + \nu} \frac{\partial p}{\partial x}, \quad 2G \frac{\partial v}{\partial y} = -\frac{\partial^2 \phi}{\partial y^2} + \frac{4}{1 + \nu} \frac{\partial q}{\partial y} \quad (e)$$

and these imply, by integration,

$$2Gu = -\frac{\partial \phi}{\partial x} + \frac{4}{1 + \nu} p + f(y), \quad 2Gv = -\frac{\partial \phi}{\partial y} + \frac{4}{1 + \nu} q + f_1(x) \quad (f)$$

where $f(y)$ and $f_1(x)$ are arbitrary functions. If these are substituted in the left of Eq. (b), we obtain

$$-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{2}{1 + \nu} \left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} \right) + \frac{1}{2} \frac{df}{dy} + \frac{1}{2} \frac{df_1}{dx} = \tau_{xy} \quad (g)$$

But the first term on the left is equal to τ_{xy} , and the parenthesis vanishes because p and q are conjugate harmonic functions satisfying

the Cauchy-Riemann equations (Art. 56). Hence

$$\frac{df}{dy} + \frac{df_1}{dx} = 0$$

which implies

$$\frac{df}{dy} = A, \quad \frac{df_1}{dx} = -A$$

where A is a constant. It follows that the terms $f(y)$ and $f_1(x)$ in Eq. (f) represent a rigid body displacement. Discarding these terms we may write Eqs. (f) as¹

$$2Gu = -\frac{\partial \phi}{\partial x} + \frac{4}{1 + \nu} p, \quad 2Gv = -\frac{\partial \phi}{\partial y} + \frac{4}{1 + \nu} q \quad (h)$$

on the understanding that a rigid-body displacement can be added. These equations enable us to find u and v when ϕ is known. We have first to find P as $\nabla^2 \phi$, determine the conjugate function Q by means of the Cauchy-Riemann equations

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x},$$

form the function $f(z) = P + iQ$, and obtain p and q by integration of $f(z)$ as in Eq. (d), Art. 56. The terms of Eqs. (h) can then be evaluated.

The usefulness of Eqs. (h) will appear in later applications, for which the method of determining displacements used in Chaps. 3 and 4 is not suitable.

58. Stress and Displacement in Terms of Complex Potentials. So far the stress and displacement components have been expressed in terms of the stress function ϕ . But since Eq. (97) expresses ϕ in terms of two functions $\psi(z)$, $\chi(z)$, it is possible to express the stress and displacement in terms of these two "complex potentials."

Any complex function $f(z)$ can be put into the form $\alpha + i\beta$ where α and β are real. To this there corresponds the conjugate,² $\alpha - i\beta$, the value taken by $f(z)$ when i is replaced, wherever it occurs in $f(z)$, by $-i$. This change is indicated by the notation

$$\tilde{f}(\bar{z}) = \alpha - i\beta \quad (a)$$

Thus if $f(z) = e^{inz}$ we have

$$\tilde{f}(\bar{z}) = e^{-in\bar{z}} = e^{-in(x-iy)} = e^{-inx} \cdot e^{-iny} \quad (b)$$

¹ A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., Arts. 144, 146, 1926.

² The word is used here with a significance quite distinct from that in the term "conjugate harmonic functions."

This may be contrasted with

$$f(\bar{z}) = e^{iz^2}$$

to illustrate the significance of the bar over the f in Eq. (a).

Evidently

$$f(z) + \bar{f}(\bar{z}) = 2\alpha = 2 \operatorname{Re} f(z)$$

In the same way if we add to the function in brackets in Eq. (97) its conjugate, the sum will be twice the real part of this function. Thus Eq. (97) may be replaced by

$$2\phi = \bar{z}\psi(z) + \chi(z) + z\bar{\psi}(\bar{z}) + \bar{\chi}(\bar{z}) \quad (98)$$

and by differentiation

$$\begin{aligned} 2 \frac{\partial \phi}{\partial x} &= \bar{z}\psi'(z) + \psi(z) + \chi'(z) + z\bar{\psi}'(\bar{z}) + \bar{\psi}(\bar{z}) + \bar{\chi}'(\bar{z}) \\ 2 \frac{\partial \phi}{\partial y} &= i[\bar{z}\psi'(z) - \psi(z) + \chi'(z) - z\bar{\psi}'(\bar{z}) + \bar{\psi}(\bar{z}) - \bar{\chi}'(\bar{z})] \end{aligned}$$

These two equations may be combined into one by multiplying the second by i and adding. Then

$$\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} = \psi(z) + z\bar{\psi}(\bar{z}) + \bar{\chi}'(\bar{z}) \quad (c)$$

Combining Eqs. (h) of Art. 57 in the same way we find

$$2G(u + iv) = - \left(\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right) + \frac{4}{1 + v} (p + iq)$$

or, using Eq. (d) of Art. 56 and Eq. (c) above,

$$2G(u + iv) = \frac{3 - v}{1 + v} \psi(z) - z\bar{\psi}'(\bar{z}) - \bar{\chi}'(\bar{z}) \quad (99)$$

This equation determines u and v for plane stress when the complex potentials $\psi(z)$, $\chi(z)$ are given. For plane strain $v/(1 - v)$ is substituted for v on the right of Eq. (99) in accordance with Art. 19.

The stress components σ_x , σ_y , τ_{xy} can be obtained directly from the second derivatives of Eq. (98). But, in view of later application to curvilinear coordinates, it is preferable to proceed otherwise. Differentiating Eq. (c) with respect to x we have

$$\frac{\partial^2 \phi}{\partial x^2} + i \frac{\partial^2 \phi}{\partial x \partial y} = \psi'(z) + z\bar{\psi}''(\bar{z}) + \bar{\psi}'(\bar{z}) + \bar{\chi}''(\bar{z}) \quad (d)$$

Differentiating it with respect to y and multiplying by i we have

$$i \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} = -\psi'(z) + z\bar{\psi}''(\bar{z}) - \bar{\psi}'(\bar{z}) + \bar{\chi}''(\bar{z}) \quad (e)$$

Simpler forms are obtained by subtracting and adding Eqs. (d) and (e). Then¹

$$\sigma_x + \sigma_y = 2\psi'(z) + 2\bar{\psi}'(\bar{z}) = 4 \operatorname{Re} \psi'(z) \quad (100)$$

$$\sigma_y - \sigma_x - 2i\tau_{xy} = 2[z\bar{\psi}''(\bar{z}) + \bar{\chi}''(\bar{z})] \quad (101)$$

Changing i to $-i$ on both sides of Eq. (101) yields the alternative form

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)] \quad (102)$$

On separation of real and imaginary parts the right side of Eq. (102), or (101), gives $\sigma_x - \sigma_y$ and $2\tau_{xy}$. The two equations (100) and (102) determine the stress components in terms of the complex potentials $\psi(z)$ and $\chi(z)$. Thus by choosing definite functions for $\psi(z)$ and $\chi(z)$ we find a possible state of stress from Eqs. (100) and (102), and the displacements corresponding to this state of stress are easily obtained from Eq. (99).

As a simple illustration of this method, consider the polynomial stress system discussed on page 32. A stress function in the form of a polynomial of the fifth degree will evidently be obtained from Eq. (98) by taking

$$\psi(z) = (a_5 + ib_5)z^5, \quad \chi(z) = (c_5 + id_5)z^5$$

where a_5, b_5, c_5, d_5 are real arbitrary coefficients. Then

$$\begin{aligned} \psi'(z) &= 4(a_5 + ib_5)z^4, & \chi'(z) &= 5(c_5 + id_5)z^4 \\ \psi''(z) &= 12(a_5 + ib_5)z^3, & \chi''(z) &= 20(c_5 + id_5)z^3 \end{aligned}$$

and Eqs. (100) and (102) yield

$$\begin{aligned} \sigma_x + \sigma_y &= 4 \operatorname{Re} 4(a_5 + ib_5)z^3 \\ &= 16 \operatorname{Re} (a_5 + ib_5)[x^3 - 3xy^2 + i(3x^2y - y^3)] \\ &= 16a_5(x^3 - 3xy^2) - 16b_5(3x^2y - y^3) \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2[12(a_5 + ib_5)\bar{z}z^2 + 20(c_5 + id_5)z^3] \\ &= 24(a_5 + ib_5)(x - iy)(x + iy)^2 + 20(c_5 + id_5)(x + iy)^3 \\ &= [24a_5x(x^2 + y^2) - 24b_5y(x^2 + y^2) + 20c_5(x^3 - 3xy^2) \\ &\quad - 20d_5(3x^2y - y^3)] + i[24a_5y(x^2 + y^2) + 24b_5x(x^2 + y^2) \\ &\quad + 20c_6(3x^2y - y^3) + 20d_6(x^3 - 3xy^2)] \end{aligned}$$

The expressions in brackets give $\sigma_y - \sigma_x$ and $2\tau_{xy}$, respectively. The displacement components corresponding to this stress distribution are easily obtained from Eq. (99), which yields

$$2G(u + iv) = \frac{3 - v}{1 + v} (a_5 + ib_5)z^4 - 4(a_5 - ib_5)z\bar{z}^3 - 5(c_5 - id_5)\bar{z}^4$$

¹ These results and Eq. (99) were obtained by G. Kolosoff in his doctoral dissertation, Dorpat, 1909. See his paper in *Z. Math. Physik.*, vol. 62, 1914.

59. Resultant of Stress on a Curve. Boundary Conditions. In Fig. 125 AB is an arc of a curve drawn on the plate. The force acting on the arc ds , exerted by the material to the left on the material to the right, proceeding from A to B , may be represented by components $\bar{X} ds$ and $\bar{Y} ds$. Then, from Eqs. (12) of Art. 9,

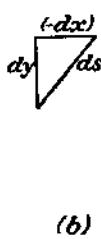
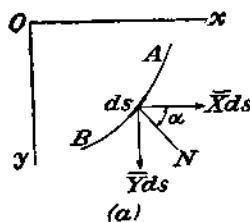


FIG. 125.

To ds correspond a dx and a dy as indicated in Fig. 125b. In traversing ds in the direction AB x decreases and dx will be a negative number. The length of the horizontal side of the elementary triangle in Fig. 125b is therefore $-dx$. Thus

$$\cos \alpha = \frac{dy}{ds}, \quad \sin \alpha = -\frac{dx}{ds} \quad (b)$$

Inserting these, together with

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

in Eqs. (a), we find

$$\begin{aligned} \bar{X} &= \frac{\partial^2 \phi}{\partial y^2} \cdot \frac{dy}{ds} + \frac{\partial^2 \phi}{\partial x \partial y} \cdot \frac{dx}{ds} = \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial y} \right) \frac{dy}{ds} + \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) \frac{dx}{ds} = \frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) \\ \bar{Y} &= -\frac{\partial^2 \phi}{\partial x^2} \frac{dx}{ds} - \frac{\partial^2 \phi}{\partial x \partial y} \frac{dy}{ds} = -\frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) \end{aligned} \quad (c)$$

The components of the resultant force on the arc AB are therefore

$$\begin{aligned} F_x &= \int_A^B \bar{X} ds = \int_A^B \frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) ds = \left[\frac{\partial \phi}{\partial y} \right]_A^B \\ F_y &= \int_A^B \bar{Y} ds = - \int_A^B \frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) ds = - \left[\frac{\partial \phi}{\partial x} \right]_A^B \end{aligned} \quad (d)$$

the square bracket representing the difference of the values of the enclosed quantity at B and at A .

The moment about O , clockwise, of the force on AB is, using Eqs. (c),

$$M = \int_A^B x \bar{Y} ds - \int_A^B y \bar{X} ds = - \int_A^B \left[x \frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) + y \frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) \right] ds$$

Integrating by parts yields¹

$$M = \left[\phi \right]_A^B - \left[x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right]_A^B \quad (e)$$

It will be evident from Eqs. (c) that if the curve AB represents an unloaded boundary, so that \bar{X} and \bar{Y} are zero, $\partial \phi / \partial x$ and $\partial \phi / \partial y$ must be constant along AB . If there are prescribed loads on AB , Eqs. (c) show that they can be specified by giving the values of $\partial \phi / \partial x$, $\partial \phi / \partial y$ along the boundary. This is equivalent to giving the derivatives $\partial \phi / \partial s$ along, and $\partial \phi / \partial n$ normal to, AB . These are known if ϕ and $\partial \phi / \partial n$ are given along AB .²

Now let the arc be continued to form a closed curve, so that B coincides with A , but is still regarded as the point reached after traversing the arc, now a closed circuit, AB . Then Eqs. (d) and (e) give the resultant force and moment of the stresses acting on the piece of the plate enclosed by the circuit. If these are not zero, $\partial \phi / \partial x$ and $\partial \phi / \partial y$ do not return to their starting values (A) after completing the circuit (B). They are therefore discontinuous functions, such as the angle θ of polar coordinates. This will be the case only when loads (equal and opposite to F_x , F_y , M) are applied to the piece of the plate within the closed circuit.

In terms of the complex potentials $\psi(z)$, $\chi(z)$ of Eq. (98) the two equations (d) may be written as

$$F_x + iF_y = \left[\frac{\partial \phi}{\partial y} - i \frac{\partial \phi}{\partial x} \right]_A^B = -i \left[\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right]_A^B$$

or, using Eq. (c) of Art. 58

$$F_x + iF_y = -i[\psi(z) + z\psi'(z) + \bar{x}\bar{\psi}'(\bar{z})]_A^B \quad (103)$$

Eq. (e) becomes

$$M = \operatorname{Re} [-z\bar{\psi}'(\bar{z}) + \chi(z) - \bar{z}\bar{\chi}'(\bar{z})]_A^B \quad (104)$$

Equations (103) and (104), applied to a complete circuit round the origin, show that if $\psi(z)$ and $\chi(z)$ are taken in the form z^n where n is a positive or negative integer, F_x , F_y , and M are zero, since the functions

¹ Equations (d) and (e) serve to establish an analogy between plane stress and the slow motion of a viscous fluid in two dimensions. See J. N. Goodier, *Phil. Mag.*, series 7, vol. 17, pp. 554 and 800, 1934.

² These boundary conditions lead to an analogy with the transverse deflections of elastic plates. An account of this analogy, with references, is given by R. D. Mindlin, *Quart. Applied Math.*, vol. 4, p. 279, 1946.

in brackets return to their initial values when the circuit is completed. These functions by themselves could not represent stress due to loads applied at the origin. The function $\log z = \log r + i\theta$ does not return to its initial value on completing a circuit round the origin, since θ increases by 2π . Thus if $\psi(z) = C \log z$, or $x(z) = Dz \log z$, where C and D are (complex) constants, Eq. (103) will yield a non-zero value for $F_x + iF_y$. Similarly $x(z) = D \log z$ will yield a non-zero value of M if D is imaginary, but a zero value if D is real.

60. Curvilinear Coordinates. Polar coordinates r, θ (Fig. 124) may be regarded as specifying the position of a point as the intersection of a circle (of radius r) and a radial line (at the angle θ from the initial line). A change from Cartesian to polar coordinates is effected by means of the equations

$$\sqrt{x^2 + y^2} = r, \quad \arctan \frac{y}{x} = \theta \quad (a)$$

The first, when r is given various constant values, represents the family of circles. The second, when θ is given various constant values, represents the family of radial lines.

Equations (a) are a special case of equations of the form

$$F_1(x, y) = \xi, \quad F_2(x, y) = \eta \quad (b)$$

Giving definite constant values to ξ and η , these equations will represent two curves which will intersect, when $F_1(x, y), F_2(x, y)$ are suitable functions. Different values of ξ and η will yield different curves and a different point of intersection. Thus each point in the xy -plane will be characterized by definite values of ξ and η —the values which make the two curves given by Eqs. (b) pass through it—and ξ, η may be regarded as "coordinates" of a point. Since given values of ξ, η define the point by means of two intersecting curves, they are called *curvilinear coordinates*.¹

Polar coordinates, with the associated stress components, proved very useful in Chap. 4 for problems of concentric circular boundaries. The stress and displacement on such a boundary become functions of θ only, since r is constant. If the boundaries consist of other curves, for instance ellipses, it is advantageous to use curvilinear coordinates one of which is constant on each boundary curve.

If Eqs. (b) are solved for x and y , we shall have two equations of the form

$$x = f_1(\xi, \eta), \quad y = f_2(\xi, \eta) \quad (c)$$

¹ The general theory of curvilinear coordinates was developed by Lamé in the book "Leçons sur les coordonnées curvilignes," Paris, 1859.

and it is usually most convenient to begin with these. Consider, for example, the two equations

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta \quad (d)$$

where c is a constant. Elimination of η yields

$$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1$$

If ξ is constant this is the equation of an ellipse with semiaxes $c \cosh \xi$, $c \sinh \xi$, and with foci at $x = \pm c$.

For different values of ξ we obtain different ellipses with the same foci—that is, a family of *confocal ellipses* (Fig. 126). On any one of these ellipses ξ is constant and η varies (through a range 2π), as on a circle in polar coordinates r is constant and θ varies. In fact in the present case η is the eccentric angle¹ of a point on the ellipse.

If on the other hand we eliminate ξ from Eqs. (d), by means of the equation $\cosh^2 \xi - \sinh^2 \xi = 1$, we have

$$\frac{x^2}{c^2 \cos^2 \eta} - \frac{y^2}{c^2 \sin^2 \eta} = 1 \quad (e)$$

For a constant value of η this represents a hyperbola having the same foci as the ellipses. Thus Eq. (e) represents a family of confocal hyperbolae, on any one of which η is constant and ξ varies. These coordinates are called *elliptic*.

The two equations (d) are equivalent to $x + iy = c \cosh(\xi + i\eta)$ or

$$z = c \cosh \xi \quad (f)$$

where $\xi = \xi + i\eta$. This is evidently a special case of the relation

$$z = f(\xi) \quad (g)$$

This, besides defining z as a function of ξ , may be solved to give ξ as a function of z . Then ξ and η are the real and imaginary parts of a func-

¹ If a, θ are the polar coordinates of a point on the circle circumscribing an ellipse of semiaxes a, b , the perpendicular from this point to the x -axis intersects the ellipse at the point $x = a \cos \theta, y = b \sin \theta$; θ is called the eccentric angle of this point on the ellipse.

tion of z , and therefore satisfy the Cauchy-Riemann equations (e) of Art. 55, also therefore the Laplace equations (f) and (g) of Art. 55.

The curvilinear coordinates to be used in this chapter will all be derived from equations of the form (g), and as a consequence will possess further special properties. The point x, y having the curvilinear coordinates ξ, η , a neighboring point $x + dx, y + dy$ will have curvilinear coordinates $\xi + d\xi, \eta + d\eta$, and since there will be two equations of the type (c) we may write

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta, \quad dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \quad (h)$$

If only ξ is varied, the increments dx, dy correspond to an element of arc ds_ξ along a curve $\eta = \text{constant}$, and

$$dx = \frac{\partial x}{\partial \xi} d\xi, \quad dy = \frac{\partial y}{\partial \xi} d\xi \quad (i)$$

Thus

$$(ds_\xi)^2 = (dx)^2 + (dy)^2 = \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \right] (d\xi)^2 \quad (j)$$

Since $z = f(\xi)$ we have

$$\frac{\partial z}{\partial \xi} = \frac{\partial x}{\partial \xi} + i \frac{\partial y}{\partial \xi} = \frac{d}{d\xi} f(\xi) \frac{\partial \xi}{\partial \xi} = f'(\xi) \quad (k)$$

where

$$f'(\xi) = \frac{df(\xi)}{d\xi}$$

Now any complex quantity can be written in the form $J \cos \alpha + iJ \sin \alpha$, or $Je^{i\alpha}$, where J and α are real. With

$$f'(\xi) = Je^{i\alpha} \quad (l)$$

Eq. (k) yields

$$\frac{\partial x}{\partial \xi} = J \cos \alpha, \quad \frac{\partial y}{\partial \xi} = J \sin \alpha \quad (m)$$

and then Eq. (j) gives

$$ds_\xi = J d\xi$$

The slope of ds_ξ is, using Eqs. (i) and (m),

$$\frac{dy}{dx} = \frac{\partial y / \partial \xi}{\partial x / \partial \xi} = \tan \alpha \quad (n)$$

Thus α , given by Eq. (l), is the angle between the tangent to the curve

$\eta = \text{constant}$, in the direction ξ -increasing, and the x -axis (Fig. 127). In the same way, if only η is varied, the increments dx and dy of Eqs. (h) correspond to an element of arc ds_η along a curve $\xi = \text{constant}$, and instead of Eqs. (i) we have

$$dx = \frac{\partial x}{\partial \eta} d\eta, \quad dy = \frac{\partial y}{\partial \eta} d\eta$$

Proceeding as above we shall find that

$$\frac{\partial x}{\partial \eta} = -J \sin \alpha, \quad \frac{\partial y}{\partial \eta} = J \cos \alpha$$

and that $ds_\eta = J d\eta$, and

$$dy/dx = -\cot \alpha$$

Comparing this last result with Eq. (n), we see that the curves $\xi = \text{constant}$, $\eta = \text{constant}$, intersect at right angles, the direction η -increasing making an angle $(\pi/2) + \alpha$ with the x -axis (Fig. 127).

Consider for instance the elliptic coordinates defined by Eq. (f). We have

$$f'(\xi) = c \sinh \xi = c \sinh \xi \cos \eta + i c \cosh \xi \sin \eta = Je^{i\alpha}$$

Comparing the real and imaginary parts of the last equality we find

$$J \cos \alpha = c \sinh \xi \cos \eta, \quad J \sin \alpha = c \cosh \xi \sin \eta$$

and therefore

$$J^2 = c^2 (\sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta) = \frac{1}{2} c^2 (\cosh 2\xi - \cos 2\eta) \quad (o)$$

$$\tan \alpha = \coth \xi \tan \eta \quad (p)$$

61. Stress Components in Curvilinear Coordinates. Equations (99), (100), and (102) give the Cartesian components of displacement and stress in terms of the complex potentials $\psi(z)$, $\chi(z)$. When curvilinear coordinates are used the complex potentials can be taken as functions of ξ , and z itself is given in terms of ξ by the equation of the type of Eq. (g) of Art. 60 defining the curvilinear coordinates. There is thus no difficulty in expressing $\sigma_x, \sigma_y, \tau_{xy}$ in terms of ξ and η . It is usually convenient, however, to specify the stress as

σ_ξ , the normal component on a curve $\xi = \text{constant}$;

σ_η , the normal component on a curve $\eta = \text{constant}$;

$\tau_{\xi\eta}$, the shear component on both curves.

These components are indicated in Fig. 128. Comparing this and Fig. 127 with Fig. 12, we see that σ_ξ and $\tau_{\xi\eta}$ correspond to σ and τ in

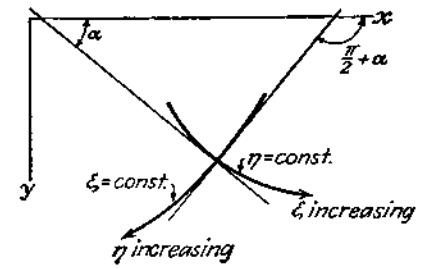


FIG. 127.

Fig. 12. We may therefore use Eqs. (13), and thus obtain

$$\begin{aligned}\sigma_\xi &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\alpha + \tau_{xy} \sin 2\alpha \\ \tau_{\xi\eta} &= -\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\alpha + \tau_{xy} \cos 2\alpha\end{aligned}$$

Replacing α by $(\pi/2) + \alpha$ we find similarly

$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\alpha - \tau_{xy} \sin 2\alpha$$

and from these we easily obtain the following equations:¹

$$\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y \quad (105)$$

$$\sigma_\eta - \sigma_\xi + 2i\tau_{\xi\eta} = e^{2i\alpha} (\sigma_y - \sigma_x + 2i\tau_{xy}) \quad (106)$$

The factor $e^{2i\alpha}$ for curvilinear coordinates defined by $z = f(\xi)$ can be found from Eq. (l) of Art. 60. This, with its conjugate, obtained by changing i to $-i$ throughout, gives

$$f'(\xi) = J e^{i\alpha}, \quad f'(\bar{\xi}) = J e^{-i\alpha}$$

so that

$$e^{2i\alpha} = \frac{f'(\xi)}{f'(\bar{\xi})} \quad (107)$$

For example, our elliptic coordinates give $f'(\xi) = c \sinh \xi$, and

$$e^{2i\alpha} = \frac{\sinh \xi}{\sinh \bar{\xi}} \quad (q)$$

With the value of $e^{2i\alpha}$ so determined, Eqs. (105) and (106) express σ_ξ , σ_η , $\tau_{\xi\eta}$ in terms of σ_x , σ_y , τ_{xy} .

The displacement in curvilinear coordinates is specified by means of a component u_ξ in the direction ξ -increasing (Fig. 127) and a component u_η in the direction η -increasing. If u and v are the Cartesian components of the displacement, we have

$$u_\xi = u \cos \alpha + v \sin \alpha, \quad u_\eta = v \cos \alpha - u \sin \alpha$$

and therefore

$$u_\xi + iu_\eta = e^{-i\alpha}(u + iv) \quad (108)$$

Using Eq. (99) with $z = f(\xi)$, and Eq. (107), this enables us to express u_ξ and u_η in terms of ξ and η when the complex potentials $\psi(z)$ and $\chi(z)$ have been chosen.

¹ Equations (105), (106), and (108) were obtained by Kolosoff, *loc. cit.*

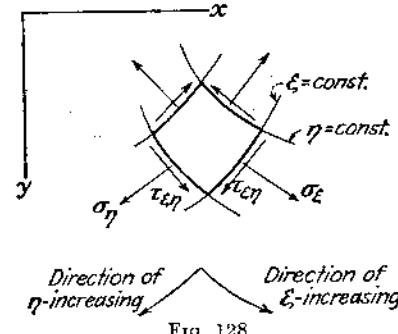


FIG. 128.

Combining Eqs. (99), (100), and (102) with (105), (106), and (108) we have the following equations for the stress and displacement components (with i replaced by $-i$ in the last):

$$\sigma_\xi + \sigma_\eta = 2[\psi'(z) + \bar{\psi}'(\bar{z})] = 4 \operatorname{Re} \psi'(z) \quad (109)$$

$$\sigma_\eta - \sigma_\xi + 2i\tau_{\xi\eta} = 2e^{2i\alpha}[\bar{z}\psi''(z) + \chi''(z)] \quad (110)$$

$$2G(u_\xi - iu_\eta) = e^{i\alpha} \left[\frac{3 - \nu}{1 + \nu} \bar{\psi}'(\bar{z}) - \bar{z}\psi'(z) - \chi'(z) \right] \quad (111)$$

We shall use these equations in the solution of several problems involving curved boundaries.

Problems

1. Show that for polar coordinates, given by $z = e^i$, Eq. (107) becomes $e^{2i\alpha} = e^{2ir}$, and $\alpha = \eta = \theta$.

2. Obtain the solutions of the following problems in polar coordinates by means of the complex potentials indicated. Evaluate the stress and displacement components. Capitals denote constants, not necessarily real.

(a) A ring ($a < r < b$) with equal and opposite couples M applied by means of shear stress to the two boundaries (Fig. 122). $\psi(z) = 0$, $\chi(z) = A \log z$.

(b) The ring under internal pressure p_i , external pressure p_o (see page 59). $\psi(z) = Az$, $\chi(z) = B \log z$.

(c) The pure bending of a curved bar, and the "rotational dislocation" of the ring, as in Arts. 27 and 29. $\psi(z) = Az \log z + Bz$, $\chi(z) = C \log z$.

(d) The problem solved in Art. 31. $\psi(z) = Az^2 + B \log z$, $\chi(z) = Cz \log z + D/z$.

(e) The plate under tension with a circular hole (Art. 32). $\psi(z) = Az + B/z$, $\chi(z) = C \log z + Dz^2 + F/z^2$.

(f) The radial stress distribution of Art. 33. $\psi(z) = A \log z$, $\chi(z) = Bz \log z$.

(g) The force at a point of an infinite plate (Art. 38). $\chi(z) = A \log z$,

$$x(z) = Bz \log z$$

62. Solutions in Elliptic Coordinates. Elliptic Hole in Uniformly Stressed Plate. The elliptic coordinates ξ , η , already considered in Art. 60 and shown in Fig. 126, were defined by

$$z = c \cosh \xi, \quad \xi = \xi + i\eta \quad (a)$$

which give

$$x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta$$

and

$$\frac{dz}{d\xi} = c \sinh \xi, \quad e^{2i\alpha} = \frac{\sinh \xi}{\sinh \bar{\xi}} \quad (b)$$

The coordinate ξ is constant and equal to ξ_0 on an ellipse of semiaxes $c \cosh \xi_0$, $c \sinh \xi_0$. If the semiaxes are given as a and b , c and ξ_0 can be found from

$$c \cosh \xi_0 = a, \quad c \sinh \xi_0 = b \quad (c)$$

and therefore if one member of the family of ellipses is given, the whole

family of ellipses and also the family of hyperbolae (see page 193) are definite. If ξ is very small the corresponding ellipse is very slender, and in the limit $\xi = 0$ it becomes a line of length $2c$ joining the foci. Taking larger and larger positive values of ξ the ellipse becomes larger and larger, approaching an infinite circle in the limit $\xi = \infty$. A point on any one ellipse goes once around the ellipse as η goes from zero (on the positive x -axis, Fig. 126) to 2π . In this respect η resembles the angle θ of polar coordinates. Continuity of displacement and stress components requires that they be periodic in η with period 2π , so that they will have the same values for $\eta = 2\pi$ as they have for $\eta = 0$.

Consider now an infinite plate in a state of uniform all-round tension S disturbed by an elliptical hole of semiaxes a and b , which is free from stress.¹ These conditions mean that

$$\sigma_x = \sigma_y = S \quad \text{at infinity } (\xi \rightarrow \infty) \quad (d)$$

$$\sigma_\xi = \tau_{\xi\eta} = 0 \quad \text{on the elliptical boundary of the hole, where } \xi \text{ has the value } \xi_0 \quad (e)$$

From Eqs. (100) and (102) we find that the condition (d) is satisfied if

$$2 \operatorname{Re} \psi'(z) = S, \quad \bar{z}\psi''(z) + x''(z) = 0 \text{ at infinity} \quad (f)$$

Since the stress and displacement components are, for continuity, to be periodic in η with period 2π , we are led to consider forms for $\psi(z)$ and $x(z)$ which will give a stress function with the same periodicity, and such forms are

$$\begin{aligned} \sinh n\xi, \text{ i.e., } \sinh n\xi \cos n\eta + i \cosh n\xi \sin n\eta \\ \cosh n\xi, \text{ i.e., } \cosh n\xi \cos n\eta + i \sinh n\xi \sin n\eta \end{aligned}$$

where n is an integer. The function $x(z) = Bc^2\xi$, B being a constant, is also suitable to the problem.

Taking $\psi(z) = Ac \sinh \xi$, with A a constant, and using the first of Eqs. (b) for $d\xi/dz$, which is the reciprocal of $dz/d\xi$, we find

$$\psi'(z) = Ac \cosh \xi \cdot \frac{d\xi}{dz} = A \frac{\cosh \xi}{\sinh \xi} = A \coth \xi \quad (g)$$

At an infinite distance from the origin ξ is infinite, and $\coth \xi$ has the

¹ Solutions for the plate with an elliptical hole were first given by Kolosoff, *loc. cit.*; and C. E. Inglis, *Trans. Inst. Naval Arch.*, London, 1913; *Engineering*, vol. 95, p. 415, 1913. See also T. Pöschl, *Math. Z.*, vol. 11, p. 95, 1921. The method employed here is that of Kolosoff. The same method was applied to several two-dimensional problems of elasticity by A. C. Stevenson, *Proc. Roy. Soc. (London)*, series A, vol. 184, pp. 129 and 218, 1945. Other references are given later in the chapter.

value unity. The first of conditions (f) is therefore satisfied if $2A = S$. From (g) we find further

$$\psi''(z) = -\frac{A}{c} \frac{1}{\sinh^3 \xi} \quad (h)$$

and

$$\bar{z}\psi''(z) = -A \frac{\cosh \xi}{\sinh^3 \xi} \quad (i)$$

Taking $x(z) = Bc^2\xi$, where B is a constant, we have

$$x'(z) = \frac{Bc}{\sinh \xi}, \quad x''(z) = -B \frac{\cosh \xi}{\sinh^3 \xi} \quad (j)$$

Equations (i) and (j) show that $\bar{z}\psi''(z)$ and $x''(z)$ each vanish at infinity. The second of conditions (f) is therefore now satisfied.

The condition (e) can be satisfied by suitable choice of the constant B . Subtracting Eq. (110) from Eq. (109) we have

$$\sigma_\xi - i\tau_{\xi\eta} = \psi'(z) + \bar{\psi}'(\bar{z}) - e^{2i\alpha} [\bar{z}\psi''(z) + x''(z)] \quad (k)$$

and $e^{2i\alpha}$ is given by the second of Eqs. (b). Thus

$$\begin{aligned} \sigma_\xi - i\tau_{\xi\eta} &= A \left(\frac{\cosh \xi}{\sinh \xi} + \frac{\cosh \bar{\xi}}{\sinh \bar{\xi}} \right) + \frac{\sinh \xi}{\sinh \bar{\xi}} \left(A \frac{\cosh \xi}{\sinh^3 \xi} + B \frac{\cosh \xi}{\sinh^3 \xi} \right) \\ &= \frac{1}{\sinh^2 \xi \sinh \bar{\xi}} \{ A [\sinh \xi \sinh (\xi + \bar{\xi}) + \cosh \bar{\xi}] \\ &\quad + B \cosh \xi \} \quad (l) \end{aligned}$$

At the boundary of the elliptical hole $\xi = \xi_0$ and $\xi + \bar{\xi} = 2\xi_0$, $\bar{\xi} = 2\xi_0 - \xi$. Then (l) reduces to

$$\frac{1}{\sinh^2 \xi \sinh \bar{\xi}} (A \cosh 2\xi_0 + B) \cosh \xi$$

Condition (e) is therefore satisfied if

$$B = -A \cosh 2\xi_0 = -\frac{1}{2}S \cosh 2\xi_0 \quad (m)$$

We now have

$$\psi(z) = \frac{1}{2}Sc \sinh \xi, \quad x(z) = -\frac{1}{2}Sc^2 \cosh 2\xi_0 \cdot \xi \quad (n)$$

All the boundary conditions have now been satisfied. But we cannot be sure that the complex potentials (n) represent the solution of our problem until we know that they imply no discontinuity in the displacement. The Cartesian components of displacement can be found from Eq. (99), which in the present case gives

$$2G(u + iv) = \frac{3 - \nu}{1 + \nu} Ac \sinh \xi - Ac \cosh \xi \coth \xi - \frac{Bc}{\sinh \xi} \quad (o)$$

with $A = S/2$, and B as given by Eq. (m). The hyperbolic functions have real and imaginary parts which are periodic in η . Thus a circuit round any ellipse $\xi = \text{constant}$, within the plate, will bring u and v back to the initial values. The complex potentials (n) therefore provide the solution of the problem.

The stress component σ_η at the hole is easily found from Eq. (109), since σ_ξ at the hole is zero. Inserting the value of $\psi'(z)$ from Eq. (g), with $A = S/2$ we have

$$\sigma_\xi + \sigma_\eta = 4 \operatorname{Re} \psi'(z) = 2S \operatorname{Re} \coth \xi$$

But

$$\coth \xi = \frac{e^{\xi+i\eta} + e^{-\xi-i\eta}}{e^{\xi+i\eta} - e^{-\xi-i\eta}} = \frac{(e^{\xi+i\eta} + e^{-\xi-i\eta})}{(e^{\xi+i\eta} - e^{-\xi-i\eta})} \cdot \frac{(e^{\xi-i\eta} - e^{-\xi+i\eta})}{(e^{\xi-i\eta} - e^{-\xi+i\eta})} \quad (p)$$

Multiplying out the brackets in numerator and denominator, this reduces to

$$\coth \xi = \frac{\sinh 2\xi - i \sin 2\eta}{\cosh 2\xi - \cos 2\eta}$$

Hence

$$\sigma_\xi + \sigma_\eta = \frac{2S \sinh 2\xi}{\cosh 2\xi - \cos 2\eta}$$

and at the boundary of the hole

$$(\sigma_\eta)_{\xi=\xi_0} = \frac{2S \sinh 2\xi_0}{\cosh 2\xi_0 - \cos 2\eta}$$

The greatest value, occurring at the ends of the major axis, where $\eta = 0$ and π , and $\cos 2\eta = 1$, is

$$(\sigma_\eta)_{\max.} = \frac{2S \sinh 2\xi_0}{\cosh 2\xi_0 - 1}$$

It is easily shown from Eqs. (c) that

$$c^2 = a^2 - b^2, \quad \sinh 2\xi_0 = \frac{2ab}{c^2}, \quad \cosh 2\xi_0 = \frac{a^2 + b^2}{c^2}$$

and with these we find that

$$(\sigma_\eta)_{\max.} = 2S \frac{a}{b}$$

which becomes larger and larger as the ellipse is made more and more slender.

The least value of $(\sigma_\eta)_{\xi=\xi_0}$ occurs at the ends of the minor axes where $\cos 2\eta = -1$. Thus

$$(\sigma_\eta)_{\min.} = \frac{2S \sinh 2\xi_0}{\cosh 2\xi_0 + 1} = 2S \frac{b}{a}$$

When $a = b$, so that the ellipse becomes a circle, both $(\sigma_\eta)_{\max.}$ and $(\sigma_\eta)_{\min.}$ reduce to $2S$, in agreement with the value for the circular hole under uniform all-round tension found on page 81.

The displacements can be evaluated from Eqs. (n) and (111) or (99). They are of course continuous, being represented by single-valued continuous functions.

The problem of uniform pressure S within an elliptical hole,¹ and zero stress at infinity, is easily obtained by combining the above solution with the state of uniform stress $\sigma_\xi = \sigma_\eta = -S$, derivable from the complex potential $\psi(z) = -Sz/2$.

63. Elliptic Hole in a Plate under Simple Tension. As a second problem, consider the infinite plate in a state of simple tensile stress S in a direction at an angle β below the positive x -axis (Fig. 129), disturbed by an elliptical hole, with its major axis along the x -axis, as in the preceding problem. The elliptical hole with major axis perpendicular or parallel to the tension² is a special case. The more general problem is, however, no more difficult by the present method.

From its solution we can find the effect of the elliptical hole on any state of uniform plane stress, specified by principal stresses at infinity in any orientation with respect to the hole.

Let Ox' , Oy' be Cartesian axes obtained by rotating Ox through the angle β so as to bring it parallel to the tension S . Then by Eqs. (105), (106)

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y, \quad \sigma_{y'} - \sigma_{x'} + 2i\tau_{x'y'} = e^{2i\beta}(\sigma_y - \sigma_x + 2i\tau_{xy})$$

Since at infinity $\sigma_{x'} = S$, $\sigma_{y'} = \tau_{x'y'} = 0$, we have

$$\sigma_x + \sigma_y = S, \quad \sigma_y - \sigma_x + 2i\tau_{xy} = -Se^{-2i\beta} \text{ at infinity}$$

and so, from Eqs. (100) and (102),

$$4 \operatorname{Re} \psi'(z) = S, \quad 2[\bar{z}\psi''(z) + \chi''(z)] = -Se^{-2i\beta} \text{ at infinity} \quad (a)$$

At the boundary of the hole $\xi = \xi_0$ we must have $\sigma_\xi = \tau_{\xi\eta} = 0$.

¹ Nonuniform pressure within an elliptical hole is considered by I. N. Sneddon and H. A. Elliott, *Quart. Applied Math.*, vol. 4, p. 262, 1946.

² See the papers cited in the footnote on p. 198.

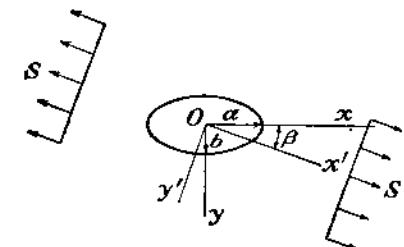


FIG. 129.

All these boundary conditions can be satisfied by taking $\psi(z)$, $x(z)$ in the forms¹

$$\begin{aligned}4\psi(z) &= Ac \cosh \xi + Bc \sinh \xi \\4x(z) &= Cc^2\xi + Dc^2 \cosh 2\xi + Ec^2 \sinh 2\xi\end{aligned}$$

where A, B, C, D, E are constants to be found.

Since $z = c \cosh \xi$, the term $Ac \cosh \xi$ in the expression for $4\psi(z)$ is simply Az . It will contribute to the stress function [Eq. (97)] a term $\operatorname{Re} A\bar{z}z$ or $\operatorname{Re} Ar^2$. This is zero if A is imaginary, and therefore A may at once be taken as real. The constant C must also be real. For if we insert the above expressions for $\psi(z)$, $x(z)$ in Eq. (104), taking for the curve AB a complete circuit round the hole, we find that all terms except the term in C yield zero because the hyperbolic functions are periodic in η with period 2π . The term in C is $\operatorname{Re} [Cc^2(\xi + i\eta)]$. This vanishes for a complete circuit only if C is real.

The constants B, D, E are complex, and we may write

$$B = B_1 + iB_2, \quad D = D_1 + iD_2, \quad E = E_1 + iE_2 \quad (b)$$

Substitution of the above forms for $\psi(z)$, $x(z)$ in the conditions (a) yields

$$A + B_1 = S, \quad 2(D + E) = -Se^{-2i\beta} \quad (c)$$

Subtracting Eq. (110) from Eq. (109) to obtain $\sigma_\xi - i\tau_{\xi\eta}$, we find

$$\begin{aligned}4(\sigma_\xi - i\tau_{\xi\eta}) &= \operatorname{cosech} \xi [(2A + B \coth \xi) \sinh \xi \\&\quad + (\bar{B} + B \operatorname{cosech}^2 \xi) \cosh \xi + (C + 2E) \operatorname{cosech} \xi \coth \xi \\&\quad - 4D \sinh \xi - 4E \cosh \xi]\end{aligned}$$

At the boundary of the hole $\xi = \xi_0$ and $\bar{\xi} = 2\xi_0 - \xi$. If this value of ξ is inserted in $\sinh \xi$ and $\cosh \xi$ in the above expression, and the functions $\sinh(2\xi_0 - \xi)$, $\cosh(2\xi_0 - \xi)$ expanded, the expression in square brackets reduces to

$$\begin{aligned}(2A \sinh 2\alpha - 2iB_2 \cosh 2\alpha - 4E) \cosh \xi \\- (2A \cosh 2\alpha - 2iB_2 \sinh 2\alpha + 4D) \sinh \xi \\+ (C + 2E + B \cosh 2\alpha) \coth \xi \operatorname{cosech} \xi\end{aligned}$$

This, and consequently $\sigma_\xi - i\tau_{\xi\eta}$ at the hole, vanishes if the coefficients of $\cosh \xi$, $\sinh \xi$, $\coth \xi$ $\operatorname{cosech} \xi$ vanish. We have thus three equations, together with the two equations (c), to be satisfied by the constants A, B, C, D, E . Since A and C are real, there are actually nine equa-

¹ Stevenson, *loc. cit.*

tions to be satisfied by eight constants— A, C , and $B_1, B_2, D_1, D_2, E_1, E_2$, which are the real and imaginary parts of B, D, E . They are consistent, and the solution is

$$\begin{aligned}A &= Se^{2\xi_0} \cos 2\beta & D &= -\frac{1}{2}Se^{2\xi_0} \cosh 2(\xi_0 + i\beta) \\B &= S(1 - e^{2\xi_0 - 2i\beta}) & E &= \frac{1}{2}Se^{2\xi_0} \sinh 2(\xi_0 + i\beta) \\C &= -S(\cosh 2\xi_0 - \cos 2\beta)\end{aligned}$$

The complex potentials of this problem are consequently given by

$$\begin{aligned}4\psi(z) &= Sc[e^{2\xi_0} \cos 2\beta \cosh \xi + (1 - e^{2\xi_0 - 2i\beta}) \sinh \xi] \\4x(z) &= -Sc^2[(\cosh 2\xi_0 - \cosh 2\beta)\xi + \frac{1}{2}e^{2\xi_0} - \cosh 2(\xi - \xi_0 - i\beta)]\end{aligned}$$

The displacements can now be determined from Eq. (111). It may be seen at once that they are single-valued.

The stress σ_η at the hole can be obtained from Eq. (109) since at the hole σ_ξ is zero. Then

$$(\sigma_\eta)_{\xi=\xi_0} = S \frac{\sinh 2\xi_0 + \cos 2\beta - e^{2\xi_0} \cos 2(\beta - \eta)}{\cosh 2\xi_0 - \cos 2\eta}$$

When the tension S is at right angles to the major axis ($\beta = \pi/2$),

$$(\sigma_\eta)_{\xi=\xi_0} = Se^{2\xi_0} \left[\frac{\sinh 2\xi_0(1 + e^{-2\xi_0})}{\cosh 2\xi_0 - \cos 2\eta} - 1 \right]$$

and the greatest value, occurring at the ends of the major axis ($\cos 2\eta = 1$), reduces to

$$S \left(1 + 2 \frac{a}{b} \right)$$

This increases without limit as the hole becomes more and more slender. When $a = b$ it agrees with the value $3S$ found for the circular hole on page 80. The least value of the stress round the elliptical hole is $-S$, at the ends of the minor axis. This is the same as for the circular hole.

When the tension S is parallel to the major axis ($\beta = 0$) the greatest value of σ_η round the hole is found at the ends of the minor axis, and is $S(1 + 2b/a)$. This approaches S when the ellipse is very slender. At the ends of the major axis the stress is $-S$ for any value of a/b .

The effect of the elliptical hole on a state of pure shear S parallel to the x - and y -axes is easily found by superposition of the two cases of tension S at $\beta = \pi/4$ and $-S$ at $\beta = 3\pi/4$. Then

$$(\sigma_\eta)_{\xi=\xi_0} = -2S \frac{e^{2\xi_0} \sin 2\eta}{\cosh 2\xi_0 - \cos 2\eta}$$

This vanishes at the ends of both the major and the minor axes and has the greatest values

$$\pm S \frac{(a+b)^2}{ab}$$

at the points determined by $\tan \eta = \tanh \xi_0 = b/a$. When the ellipse is very slender these values are very large, and the points at which they occur are close to the ends of the major axis.

Solutions have been found for the elliptical hole in a plate subject to pure flexure in its plane^{1,2} and to a parabolic distribution of shear as in a thin rectangular beam,² for an elliptical hole with equal and opposite concentrated forces at the ends of the minor diameter,³ and for rigid and elastic "inclusions" filling the hole in a plate under tension.⁴ More general series forms of the real stress function ϕ in elliptic coordinates have been considered.⁵ Their equivalent complex potentials can

be constructed from the functions used or mentioned here, together with the analogues of the simple functions quoted in the Problems on p. 197, when dislocations and concentrated forces and couples are to be included.

64. Hyperbolic Boundaries.
Notches. It was shown in Art. 60 that the curves $\eta = \text{constant}$ in elliptic coordinates are hyperbolas, and in Art. 62 that the range of η may be taken as 0 to 2π , that of ξ being 0 to ∞ .

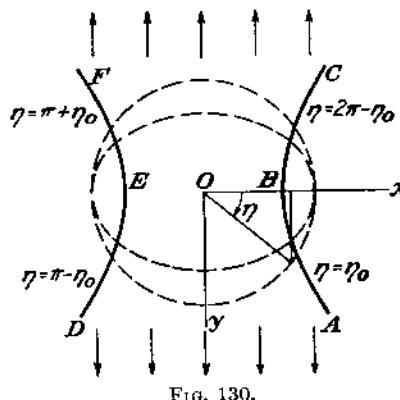


FIG. 130.

Let η_0 be the constant value of η along the hyperbolic arc BA of Fig. 130. It will be between 0 and $\pi/2$, since both x and y are positive along BA . Along the other half of this branch of the hyperbola, BC ,

¹ K. Wolf, *Z. tech. Physik*, 1922, p. 160.

² H. Neuber, *Ingenieur-Archiv*, vol. 5, p. 242, 1934. This solution and several others relating to ellipses and hyperbolae are given in Neuber's book "Kerbspannungslehre," Berlin, 1937.

³ P. S. Symonds, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 13, p. A-183, 1946. A solution in finite form is given by A. E. Green, *ibid.*, vol. 14, p. A-246, 1947.

⁴ N. I. Muschelisvili, *Zeit. angew. Math. Mech.*, vol. 13, p. 264, 1933; L. H. Donnell, "Theodore von Kármán Anniversary Volume," p. 293, Pasadena, 1941.

⁵ E. G. Coker and L. N. G. Filon, "Photo-elasticity," pp. 123, 535, Cambridge University Press, 1931; A. Timpe, *Math. Z.*, vol. 17, p. 189, 1923.

the value of η is $2\pi - \eta_0$. Along the half ED of the other branch, η is $\pi - \eta_0$, and along EF it is $\pi + \eta_0$.

Consider the plate $ABCDEF$ within these hyperbolic boundaries, in a state of tension in the direction Oy .¹ The tensile stress at infinity must fall to zero to preserve a finite tensile force across the waist EOB . Complex potentials which permit this, and satisfy the other necessary conditions of symmetry about Ox and Oy , and freedom of the hyperbolic boundaries, are

$$\psi(z) = -\frac{1}{2}Ai\xi, \quad \chi(z) = -\frac{1}{2}Ai\xi - Bci \sinh \xi \quad (a)$$

where A and B are real constants, and $z = c \cosh \xi$. These give

$$\psi'(z) = -\frac{iA}{2c \sinh \xi}, \quad \chi'(z) = -\frac{1}{2}Ai\xi - \left(\frac{1}{2}A + B\right)i \coth \xi \quad (b)$$

Equation (103) of Art. 59 shows that the hyperbolic boundary $\eta = \eta_0$ will be free from force provided the function

$$\psi(z) + z\psi'(\bar{z}) + \bar{\chi}'(\bar{z}) \quad (c)$$

is constant along it, or equivalently if the conjugate of this function is constant. The conjugate is, from Eqs. (a) and (b),

$$A\eta - \frac{1}{2}Ai \frac{\cosh \xi}{\sinh \xi} - \left(\frac{1}{2}A + B\right)i \coth \xi \quad (d)$$

On the hyperbola $\eta = \eta_0$ we have $\xi = \xi - 2i\eta_0$, and with this the expression becomes

$$A\eta_0 - \frac{1}{2}A \sin 2\eta_0 - \left(\frac{1}{2}A \cos 2\eta_0 + \frac{1}{2}A + B\right)i \coth \xi$$

which is a constant if the quantity in parentheses is made to vanish. Thus

$$B = -A \cos^2 \eta_0 \quad (e)$$

To find the resultant force transmitted we may apply Eq. (103) of Art. 59 to the narrow section EOB , Fig. 130, more precisely to the lower part of the limiting ellipse $\xi = 0$ between the hyperbolae $\eta = \eta_0$ and $\eta = \pi - \eta_0$. On this ellipse ξ becomes $i\eta$, $\bar{\xi}$ becomes $-i\eta$, and we have from Eq. (103), (c) and (d)

$$F_x - iF_y = i[A\eta - (A + B) \cot \eta]_{\eta=\pi-\eta_0}^{η=η_0} \\ = i[A(\pi - 2\eta_0 + 2 \cot \eta_0) + 2B \cot \eta_0]$$

¹ This problem (also the case of shear loading) was solved by A. A. Griffith, *Tech. Rept. Aeronaut. Research Comm. (Great Britain)*, 1927-1928, vol. II, p. 668; and H. Neuber, *Z. angew. Math. Mech.*, vol. 13, p. 439, 1933; or "Kerbspannungslehre," p. 35, Berlin, 1938.

Since A and B were taken as real, F_x is zero and, using Eq. (e)

$$F_y = -A(\pi - 2\eta_0 + \sin 2\eta_0)$$

which determines A when the total tension F_y is assigned. The stress and displacement components are easily found from Eqs. (109), (110), (111). The first gives

$$\sigma_\xi + \sigma_\eta = -\frac{4A}{c} \frac{\cosh \xi \sin \eta}{\cosh 2\xi - \cos 2\eta}$$

The value of σ_ξ along the hyperbolic boundary is found by setting $\eta = \eta_0$ in this expression. It has a maximum, $-2A/c \sin \eta_0$, at the waist where $\xi = 0$. Neuber¹ has expressed this as a function of the radius of curvature of the hyperbola at the waist. He has solved, by another method, the problems of bending and shear of the plate as well as tension.

65. Bipolar Coordinates. Problems involving two nonconcentric circular boundaries, including the special case of a circular hole in a semi-infinite plate, usually require the use of the *bipolar coordinates* ξ , η , defined by

$$z = ia \coth \frac{1}{2}\xi \quad \xi = \xi + i\eta \quad (a)$$

Replacing $\coth \frac{1}{2}\xi$ by $(e^{i\xi} + e^{-i\xi})/(e^{i\xi} - e^{-i\xi})$ and solving the first equation for e^ξ , it is easily shown that this is equivalent to

$$\xi = \log \frac{z + ia}{z - ia} \quad (b)$$

The quantity $z + ia$ is represented by the line joining the point $-ia$ to the point z in the xy -plane, in the sense that its projections on the axes give the real and imaginary parts. The same quantity may be represented by $r_1 e^{i\theta_1}$ where r_1 is the length of the line, and θ_1 the angle it makes with the x -axis (Fig. 131). Similarly $z - ia$ is the line joining the point ia to the point z , and may be represented by $r_2 e^{i\theta_2}$ (Fig. 131). Then Eq. (b) becomes

$$\xi + i\eta = \log \left(\frac{r_1}{r_2} e^{i\theta_1} e^{-i\theta_2} \right) = \log \frac{r_1}{r_2} + i(\theta_1 - \theta_2)$$

so that

$$\xi = \log \frac{r_1}{r_2}, \quad \eta = \theta_1 - \theta_2 \quad (c)$$

¹Loc. cit. For a comparison of Neuber's results with photoelastic and fatigue tests of notched plates and grooved shafts see R. E. Peterson and A. M. Wahl, "J. Applied Mechanics," vol. 3, p. 15, 1936, or S. Timoshenko, "Strength of Materials," 2d ed., vol. 2, p. 340. See also M. M. Frocht, "Photoelasticity," vol. 2.

It may be seen from Fig. 131 that $\theta_1 - \theta_2$ is the angle between the two lines joining the "poles" $-ia$, ia to the typical point z , when this point lies to the right of the y -axis, and is minus this angle when the point lies to the left. It follows that a curve $\eta = \text{constant}$ is an arc of a circle passing through the poles. Several such circles are drawn in Fig. 131. From Eqs. (c) it is clear that a curve $\xi = \text{constant}$ will be a curve for which $r_1/r_2 = \text{constant}$. Such a curve is also a circle. It surrounds the pole ia if r_1/r_2 exceeds unity, that is, if ξ is positive. It surrounds the other pole $-ia$ if ξ is negative. Several such circles are drawn in

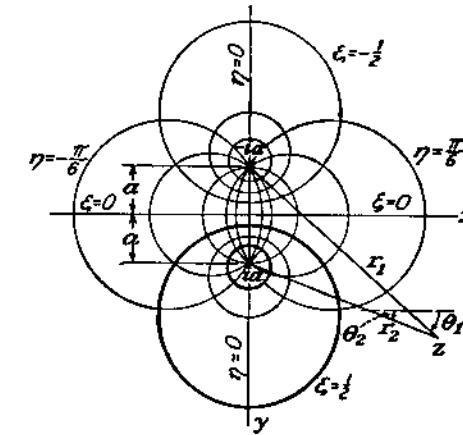


FIG. 131.

Fig. 131. They form a family of coaxal circles with the two poles as limiting points.

The coordinate η changes from π to $-\pi$ on crossing the segment of the y -axis joining the poles, its range for the whole plane being $-\pi$ to π . Stresses and displacements will be continuous across this segment if they are represented by periodic functions of η with period 2π .

Separation of real and imaginary parts in Eq. (a) leads to¹

$$x = \frac{a \sin \eta}{\cosh \xi - \cos \eta}, \quad y = \frac{a \sinh \xi}{\cosh \xi - \cos \eta} \quad (d)$$

Differentiation of Eq. (a) yields

$$Je^{i\alpha} = \frac{dz}{d\xi} = -\frac{1}{2}ia \operatorname{cosech}^2 \frac{1}{2}\xi \quad (e)$$

and

$$e^{2i\alpha} = \frac{dz/d\xi}{d\bar{z}/d\bar{\xi}} = -\sinh^2 \frac{1}{2}\xi \operatorname{cosech}^2 \frac{1}{2}\xi \quad (f)$$

¹See the derivation of Eq. (c) in Art. 54.

66. Solutions in Bipolar Coordinates. We now consider the problem of a circular disk with an eccentric hole, subject to pressure p_0 round the outside, and pressure p_1 round the hole.¹ The stress components obtained will also be valid for a circular thick-walled tube with eccentric bore.

Let the external boundary be that circle of the family $\xi = \text{constant}$, for which $\xi = \xi_0$, and let the hole be the circle $\xi = \xi_1$. Two such circles are drawn in heavy lines in Fig. 131. It follows from the expression for y in Eqs. (d) of Art. 65 that these circles have radii $a \operatorname{cosech} \xi_0$, $a \operatorname{cosech} \xi_1$, and that their centers are at the distances $a \coth \xi_0$, $a \coth \xi_1$ from the origin. Thus a , ξ_0 , and ξ_1 can be determined if the radii and distance between centers are given.

In going counterclockwise once round any circle $\xi = \text{constant}$, starting just to the left of the y -axis in Fig. 131, the coordinate η ranges from $-\pi$ to π . Thus the functions which are to give the stress and displacement components must have the same values at $\eta = \pi$ as they have at $\eta = -\pi$. This is ensured if they are periodic functions of η of period 2π . It is therefore appropriate to take the complex potentials $\psi(z)$ and $x(z)$ in the forms

$$\cosh n\xi, \quad \sinh n\xi \quad (a)$$

with n an integer, since these are in fact periodic functions of η of period 2π . So also are their derivatives with respect to z , since $d\xi/dz$ has the same property [Eq. (e), Art. 65].

If such functions are introduced into Eqs. (103), (104), applied to any circle $\xi = \text{constant}$ in the material, the corresponding force and couple will be zero, in virtue of the periodicity. This must hold for the complete solution, for equilibrium of the plate within the circle.

We shall require also the function $x(z) = aD\xi$, D being a constant. Considering this in Eqs. (103), (104) as above, we find that the moment of Eq. (104) will be zero only if D is real. We therefore take it to be so. Considering the displacement equation (99) we find that this function, as well as the functions (a) used as either $\psi(z)$ or $x(z)$, will give displacements free from discontinuity.

The state of uniform all-round tension or compression, which will be part of the solution, is obtained from the complex potential $\psi(z) = Az$, with A real. The corresponding real stress function is, from Eq. (97),

$$\phi = \operatorname{Re}(\bar{z}Az) = A\bar{z}z = A(x^2 + y^2)$$

¹ The original solution, in terms of the real stress function, is due to G. B. Jeffery, *Trans. Roy. Soc. (London)*, series A, vol. 221, p. 265, 1921.

This may be expressed in bipolar coordinates by means of Eqs. (d) of Art. 65, the result being

$$Aa^2 \frac{\cosh \xi + \cos \eta}{\cosh \xi - \cos \eta} \quad (b)$$

Considering functions of the form (a), with $n = 1$, we observe that since the stress distribution in the present problem is symmetrical about the y -axis, we must choose them so that the corresponding stress functions have the same symmetry. Thus we may take

$$\psi(z) = iB \cosh \xi, \quad x(z) = B' \sinh \xi \quad (c)$$

with B, B' real, and

$$\psi(z) = iC \sinh \xi, \quad x(z) = C' \cosh \xi \quad (d)$$

with C, C' real.

The real stress function corresponding to (c) is, from Eq. (97),

$$aB \frac{\sinh \xi \cosh \xi \cos \eta - \sinh \xi \sin^2 \eta}{\cosh \xi - \cos \eta} + B' \frac{\sinh \xi \cosh \xi \cos \eta - \sinh \xi \cos^2 \eta}{\cosh \xi - \cos \eta}$$

If we choose $B' = aB$ the terms in $\sin^2 \eta, \cos^2 \eta$ in the numerators become independent of η , and the complete numerator depends on η only in the term in $\cos \eta$, just as does the function (b). The same thing is true of the complex potentials (d), if we choose $C' = aC$. We thus obtain simpler, more restricted functions which turn out to be adequate for the present problem.

Taking therefore

$$\psi(z) = iB \cosh \xi, \quad x(z) = aB \sinh \xi \quad (e)$$

we find by means of Eqs. (109), (110) and (a), (f) of Art. 65, that the corresponding stress components are given by

$$a(\sigma_\xi + \sigma_\eta) = 2B(2 \sinh \xi \cos \eta - \sinh 2\xi \cos 2\eta) \quad (f)$$

$$a(\sigma_\eta - \sigma_\xi + 2ir_{\xi\eta}) = -2B[\sinh 2\xi - 2 \sinh 2\xi \cosh \xi \cos \eta + \sinh 2\xi \cos 2\eta - i(2 \cosh 2\xi \cosh \xi \sin \eta - \cosh 2\xi \sin 2\eta)] \quad (g)$$

Similarly the functions

$$\psi(z) = iC \sinh \xi, \quad x(z) = aC \cosh \xi \quad (h)$$

yield

$$\begin{aligned} a(\sigma_\xi + \sigma_\eta) &= -2C(1 - 2 \cosh \xi \cos \eta + \cosh 2\xi \cos 2\eta) \quad (i) \\ a(\sigma_\eta - \sigma_\xi + 2i\tau_{\xi\eta}) &= 2C[-\cosh 2\xi + 2 \cosh 2\xi \cosh \xi \cos \eta \\ &\quad - \cosh 2\xi \cos 2\eta + i(2 \sinh 2\xi \cosh \xi \sin \eta \\ &\quad - \sinh 2\xi \sin 2\eta)] \quad (j) \end{aligned}$$

The stress components arising from

$$x(z) = aD\xi \quad (k)$$

are given by

$$\sigma_\xi + \sigma_\eta = 0$$

$$a(\sigma_\eta - \sigma_\xi + 2i\tau_{\xi\eta}) = D[\sinh 2\xi - 2 \sinh \xi \cos \eta \\ - i(2 \cosh \xi \sin \eta - \sin 2\eta)] \quad (l)$$

The state of uniform all-round tension given by

$$\psi(z) = Az \quad (m)$$

yields

$$\sigma_\xi + \sigma_\eta = 4A, \quad \sigma_\eta - \sigma_\xi + 2i\tau_{\xi\eta} = 0$$

or

$$\sigma_\xi = \sigma_\eta = 2A, \quad \tau_{\xi\eta} = 0 \quad (n)$$

The solution of our problem can be obtained by superposition of the states of stress represented by the complex potentials (e), (h), (k), and (m). Collecting the terms representing $\tau_{\xi\eta}$ in Eqs. (g), (j), and (l) we find that the vanishing of $\tau_{\xi\eta}$ on the boundaries $\xi = \xi_o$, $\xi = \xi_1$ requires

$$\begin{aligned} D - 2B \cosh 2\xi_o - 2C \sinh 2\xi_o &= 0 \\ D - 2B \cosh 2\xi_1 - 2C \sinh 2\xi_1 &= 0 \end{aligned} \quad (o)$$

Solving these for B and C in terms of D we have

$$2B = D \frac{\cosh(\xi_1 + \xi_o)}{\cosh(\xi_1 - \xi_o)}, \quad 2C = -D \frac{\sinh(\xi_1 + \xi_o)}{\cosh(\xi_1 - \xi_o)} \quad (p)$$

The normal stress σ_ξ can be found by subtracting the real part of Eq. (g) from Eq. (f) and similarly for the other pairs. On the boundary $\xi = \xi_o$ it is to take the value $-p_o$, and on the boundary $\xi = \xi_1$ the value $-p_1$. Using the values of B and C given by Eqs. (p) these conditions lead to the two equations

$$2A + \frac{D}{a} \sinh^2 \xi_o \tanh(\xi_1 - \xi_o) = -p_o$$

$$2A - \frac{D}{a} \sinh^2 \xi_1 \tanh(\xi_1 - \xi_o) = -p_1$$

and therefore

$$\begin{aligned} A &= -\frac{1}{2} \frac{p_o \sinh^2 \xi_1 + p_1 \sinh^2 \xi_o}{\sinh^2 \xi_1 + \sinh^2 \xi_o} \\ D &= -a \frac{(p_o - p_1) \coth(\xi_1 - \xi_o)}{\sinh^2 \xi_1 + \sinh^2 \xi_o} \end{aligned}$$

These with Eqs. (p) complete the determination of the complex potentials. When there is internal pressure p_1 only ($p_o = 0$) the peripheral stress at the hole is found to be

$$(\sigma_\eta)_{\xi=\xi_1} = -p_1 + 2p_1(\sinh^2 \xi_1 + \sinh^2 \xi_o)^{-1}(\cosh \xi_1 \\ - \cos \eta)[\sinh \xi_1 \coth(\xi_1 - \xi_o) + \cos \eta]$$

An expression for the maximum value¹ of this has already been given on page 60.

A general series form of stress function in bipolar coordinates was given by G. B. Jeffery.² Its equivalent complex potentials are easily found, and involve the functions considered here together with the bipolar analogues of the simple functions quoted in the Problems on page 197, when dislocations and concentrated forces are included. It has been applied to the problems of a semi-infinite plate with a concentrated force at any point,³ a semi-infinite region with a circular hole, under tension parallel to the straight edge or plane boundary,⁴ and under its own weight,⁵ and to the infinite plate with two holes,⁶ or a hole formed by two intersecting circles.⁷

Solutions have been given for the circular disk subject to concentrated forces at any point,⁸ to its own weight when suspended at a point,⁹ or in rotation about an eccentric axis,¹⁰ with and without¹¹ the use of bipolar coordinates, and for the effect of a circular hole in a semi-infinite plate with a concentrated force on the straight edge.¹²

¹ An exhaustive discussion of the maximum value is given by Coker and Filon, *loc. cit.*

² *Loc. cit.*

³ E. Melan, *Z. angew. Math. Mech.*, vol. 5, p. 314, 1925.

⁴ See p. 82.

⁵ R. D. Mindlin, *Proc. A.S.C.E.*, p. 619, 1939.

⁶ T. Pöschl, *Z. angew. Math. Mech.*, vol. 1, p. 174, 1921, and vol. 2, p. 187, 1922. Also C. Weber, *ibid.*, vol. 2, p. 267, 1922; E. Weinel, *ibid.*, vol. 17, p. 276, 1937; Chih Bing Ling, *J. Applied Phys.*, vol. 19, p. 77, 1948.

⁷ Chih Bing Ling, *ibid.*, p. 405, 1948.

⁸ R. D. Mindlin, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 4, p. A-115, 1937.

⁹ R. D. Mindlin, *J. Applied Physics*, vol. 9, p. 714, 1938.

¹⁰ R. D. Mindlin, *Phil. Mag.*, series 7, vol. 26, p. 713, 1938.

¹¹ B. Sen, *Bull. Calcutta Math. Soc.*, vol. 36, pp. 58 and 83, 1944.

¹² A. Barjansky, *Quart. Applied Math.*, vol. 2, p. 16, 1944.

Other Curvilinear Coordinates. The equation

$$z = e^{\xi} + abe^{-\xi} + ac^3e^{-3\xi}$$

yielding

$$\begin{aligned}x &= (e^{\xi} + abe^{-\xi}) \cos \eta + ac^3e^{-3\xi} \cos 3\eta \\y &= (e^{\xi} - abe^{-\xi}) \sin \eta - ac^3e^{-3\xi} \sin 3\eta\end{aligned}$$

where a, b, c are constants, gives a family of curves $\xi = \text{constant}$ which can be made to include various oval shapes, including a square with rounded corners. The effect of a hole of such shape in a plate under tension has been evaluated (by means of the real stress function) by M. Greenspan.¹ By means of a generalization of these coordinates A. E. Green² has obtained solutions for a triangular hole with rounded corners, and, by means of another coordinate transformation, for an exactly rectangular hole. In the latter case the perfectly sharp corners introduce infinite stress concentration.

The curvilinear coordinates given by

$$z = \xi + ia_1e^{i\xi} + ia_2e^{i2\xi} + \dots + ia_ne^{in\xi}$$

a_1, a_2, \dots, a_n being real constants, have been applied by C. Weber to the semi-infinite plate with a serrated boundary,³ as in the example of evenly spaced semicircular notches which is worked out. When the distance between notch centers is twice the notch diameter, the stress concentration, for tension, is found to be 2.13. The value for a single notch is 3.07 (see page 89).

A method for determining the complex potentials from the boundary conditions, without the necessity of guessing their form in advance, has been developed by N. Muschelishvili.^{4,5}

¹ Quart. Applied Math., vol. 2, p. 60, 1944. See also V. Morkovin, *ibid.*, p. 350, 1945.

² Proc. Roy. Soc. (London), series A, vol. 184, p. 231, 1945.

³ C. Weber, Z. angew. Math. Mech., vol. 22, p. 29, 1942.

⁴ Math. Ann., vol. 107, pp. 282-312, 1932. Also Z. angew. Math. Mech., vol. 13, p. 264, 1933. An account of this method is given by I. S. Sokolnikoff, Lectures on the Theory of Elasticity, Brown University, 1941 (mimeographed notes).

⁵ It is used by Morkovin (see footnote 1). Most of the work of N. I. Muschelishvili and his associates is in Russian. His book "Singular Integral Equations" (2d ed.), which contains solutions of several two-dimensional problems of elasticity, in particular mixed boundary value problems, has been translated by the Aero-nautical Research Laboratories, Dept. of Supply and Development, Commonwealth of Australia (Translation No. 12, 1949). Another translation edited by J. R. M. Radok was published by P. Noordhoff, N.V., Groningen, Netherlands, 1953. Muschelishvili gives his methods in "Some Basic Problems of the Mathematical Theory of Elasticity," 3d ed., Moscow, 1949, translated by J. R. M. Radok, published by P. Noordhoff, Groningen, Netherlands, 1953. Results of the type noted under footnotes 1 and 2 were obtained earlier by Russian authors cited in the later book.

CHAPTER 8

ANALYSIS OF STRESS AND STRAIN IN THREE DIMENSIONS

67. Specification of Stress at a Point. Our previous discussions were limited to two-dimensional problems. Let us consider now the general case of stress distribution in three dimensions. It was shown (see Art. 4) that the stresses acting on the six sides of a cubic element can be described by six components of stress, namely the three normal stresses $\sigma_x, \sigma_y, \sigma_z$ and the three shearing stresses $\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$. If these components of stress at any point are known, the stress acting on any inclined plane through this point can be calculated from the equations of statics. Let O be a point of the stressed body, and suppose the stresses are known for the coordinate planes xy, xz, yz (Fig. 132). To get the stress for any inclined plane through O , we take a plane BCD parallel to it at a small distance from O , so that this latter plane together with the coordinate planes cuts out from the body a very small tetrahedron $BCDO$. Since the stresses vary continuously over the volume of the body, the stress acting on the plane BCD will approach the stress on the parallel plane through O as the element is made infinitesimal.

In considering the conditions of equilibrium of the elemental tetrahedron the body forces can be neglected (see page 4). Also as the element is very small we can neglect the variation of the stresses over the sides and assume that the stresses are uniformly distributed. The forces acting on the tetrahedron can therefore be determined by multiplying the stress components by the areas of the faces. If A denotes the area of the face BCD of the tetrahedron, then the areas of the three other faces are obtained by projecting A on the three coordinate planes. If N is the normal to the plane BCD , and we write

$$\cos (Nx) = l, \quad \cos (Ny) = m, \quad \cos (Nz) = n \quad (a)$$

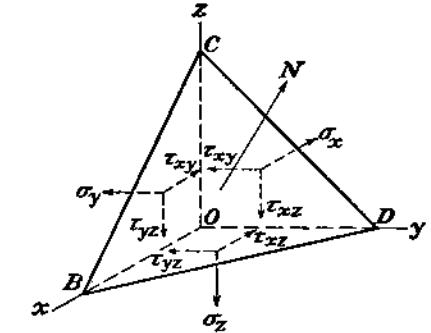


FIG. 132.

the areas of the three other faces of the tetrahedron are

$$Al, \quad Am, \quad An$$

If we denote by X, Y, Z the three components of stress, parallel to the coordinate axes, acting on the inclined face BCD , then the component of force acting on the face BCD in the direction of the x -axis is AX . Also the components of forces in the x -direction acting on the three other faces of the tetrahedron are $-Al\sigma_x, -Am\tau_{xy}, -An\tau_{xz}$. The corresponding equation of equilibrium of the tetrahedron is

$$AX - Al\sigma_x - Am\tau_{xy} - An\tau_{xz} = 0$$

In the same manner two other equations of equilibrium are obtained by projecting the forces on the y - and z -axes. After canceling the factor A , these equations of equilibrium of the tetrahedron can be written

$$\begin{aligned} X &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\ Y &= \tau_{xy} l + \sigma_y m + \tau_{yz} n \\ Z &= \tau_{xz} l + \tau_{yz} m + \sigma_z n \end{aligned} \quad (112)$$

Thus the components of stress on any plane, defined by the direction cosines l, m, n , can easily be calculated from Eqs. (112), provided the six components of stress $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$ at the point O are known.

68. Principal Stresses. Let us consider now the normal component of stress σ_n acting on the plane BCD (Fig. 132). Using the notations (a) for the direction cosines we find

$$\sigma_n = Xl + Ym + Zn$$

or, substituting the values of X, Y, Z from Eqs. (112),

$$\sigma_n = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{yz} mn + 2\tau_{xz} ln + 2\tau_{xy} lm \quad (113)$$

The variation of σ_n with the direction of the normal N can be represented geometrically as follows. Let us put in the direction of N a vector whose length, r , is inversely proportional to the square root of the absolute value of the stress σ_n , i.e.,

$$r = \frac{k}{\sqrt{|\sigma_n|}} \quad (b)$$

in which k is a constant factor. The coordinates of the end of this vector will be

$$x = lr, \quad y = mr, \quad z = nr \quad (c)$$

Substituting

$$\sigma_n = \pm \frac{k^2}{r^2} \quad (d)$$

from (b), and the values of l, m, n from (c) in Eq. (113), we find¹

$$\pm k^2 = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2\tau_{yz} mn + 2\tau_{xz} ln + 2\tau_{xy} lm \quad (114)$$

As the plane BCD rotates about the point O , the end of the vector r always lies on the surface of the second degree given by Eq. (114). This surface is completely defined by the stress condition at the point O , and, if the directions of the coordinate axes x, y, z are changed, the surface will remain entirely unchanged and only the components of stress $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$, and hence the coefficients in Eq. (114), will alter.

It is well known that in the case of a surface of the second degree, such as given by Eq. (114), it is always possible to find for the axes x, y, z such directions that the terms in this equation containing the products of coordinates vanish. This means that we can always find three perpendicular planes for which $\tau_{yz}, \tau_{xz}, \tau_{xy}$ vanish, i.e., the resultant stresses are perpendicular to the planes on which they act. We call these stresses the *principal stresses* at the point, their directions the *principal axes*, and the planes on which they act *principal planes*. It can be seen that the stress at a point is completely defined if the directions of the principal axes and the magnitudes of the three principal stresses are given.

69. Stress Ellipsoid and Stress-director Surface. If the coordinate axes x, y, z are taken in the directions of the principal axes, calculation of the stress on any inclined plane becomes very simple. The shearing stresses $\tau_{yz}, \tau_{xz}, \tau_{xy}$ are zero in this case, and Eqs. (112) become

$$X = \sigma_x l, \quad Y = \sigma_y m, \quad Z = \sigma_z n \quad (115)$$

Putting the values of l, m, n from these equations into the well-known relation $l^2 + m^2 + n^2 = 1$, we find

$$\frac{X^2}{\sigma_x^2} + \frac{Y^2}{\sigma_y^2} + \frac{Z^2}{\sigma_z^2} = 1 \quad (116)$$

¹ The plus-or-minus sign in Eq. (d) applies according as σ_n is tensile or compressive, and correspondingly in Eq. (114). When all three principal stresses have the same sign, only one of the alternative signs is needed, and the surface is an ellipsoid. When the principal stresses are not all of the same sign, both signs are needed and the surface, now represented by both Eqs. (114), consists of a hyperboloid of two sheets, together with a hyperboloid of one sheet, with a common asymptotic cone.

This means that, if for each inclined plane through a point O the stress is represented by a vector from O with the components X, Y, Z , the ends of all such vectors lie on the surface of the ellipsoid given by Eq. (116). This ellipsoid is called the *stress ellipsoid*. Its semiaxes give the principal stresses at the point. From this it can be concluded that the maximum stress at any point is the largest of the three principal stresses at this point.

If two of the three principal stresses are numerically equal the stress ellipsoid becomes an ellipsoid of revolution. If these numerically equal principal stresses are of the same sign the resultant stresses on all planes through the axis of symmetry of the ellipsoid will be equal and perpendicular to the planes on which they act. In this case the stresses on any two perpendicular planes through this axis can be considered as principal stresses. If all three principal stresses are equal and of the same sign, the stress ellipsoid becomes a sphere and any three perpendicular directions can be taken as principal axes. When one of the principal stresses is zero, the stress ellipsoid reduces to the area of an ellipse and the vectors representing the stresses on all the planes through the point lie in the same plane. This condition of stress is called *plane stress* and has already been discussed in previous sections. When two principal stresses are zero we have the cases of simple tension or compression.

Each radius vector of the stress ellipsoid represents, to a certain scale, the stress on one of the planes through the center of the ellipsoid. To find this plane we use, together with the stress ellipsoid (116), the *stress director surface* defined by the equation

$$\frac{x^2}{\sigma_x} + \frac{y^2}{\sigma_y} + \frac{z^2}{\sigma_z} = 1 \quad (117)$$

The stress represented by a radius vector of the stress ellipsoid acts on the plane parallel to the tangent plane to the stress-director surface at the point of its intersection with the radius vector. This can be shown as follows. The equation of the tangent plane to the stress-director surface (117) at any point x_0, y_0, z_0 is

$$\frac{xx_0}{\sigma_x} + \frac{yy_0}{\sigma_y} + \frac{zz_0}{\sigma_z} = 1 \quad (a)$$

Denoting by h the length of the perpendicular from the origin of coordinates to the above tangent plane, and by l, m, n the direction cosines of this perpendicular, the equation of this tangent plane can be written in the form

$$lx + my + nz = h \quad (b)$$

Comparing (a) and (b) we find

$$\sigma_x = \frac{x_0 h}{l}, \quad \sigma_y = \frac{y_0 h}{m}, \quad \sigma_z = \frac{z_0 h}{n} \quad (c)$$

Substituting these values in Eqs. (115) we find

$$X = x_0 h, \quad Y = y_0 h, \quad Z = z_0 h$$

i.e., the components of stress on the plane with direction cosines l, m, n are proportional to the coordinates x_0, y_0, z_0 . Hence the vector representing the stress goes through the point x_0, y_0, z_0 , as was stated above.¹

70. Determination of the Principal Stresses. If the stress components for three coordinate planes are known, we can determine the directions and magnitudes of the principal stresses by using the property that the principal stresses are perpendicular to the planes on which they act. Let l, m, n be the direction cosines of a principal plane and S the magnitude of the principal stress acting on this plane. Then the components of this stress are:

$$X = Sl, \quad Y = Sm, \quad Z = Sn$$

Substituting in Eqs. (112) we find

$$\begin{aligned} (S - \sigma_x)l - \tau_{xy}m - \tau_{xz}n &= 0 \\ -\tau_{xy}l + (S - \sigma_y)m - \tau_{yz}n &= 0 \\ -\tau_{xz}l - \tau_{yz}m + (S - \sigma_z)n &= 0 \end{aligned} \quad (a)$$

These are three homogeneous linear equations in l, m, n . They will give solutions different from zero only if the determinant of these equations is zero. Calculating this determinant and putting it equal to zero gives us the following cubic equation in S :

$$S^3 - (\sigma_x + \sigma_y + \sigma_z)S^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)S - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2) = 0 \quad (118)$$

The three roots of this equation give the values of the three principal stresses. By substituting each of these stresses in Eqs. (a) and using the relation $l^2 + m^2 + n^2 = 1$, we can find three sets of direction cosines for the three principal planes.

It may be noted that Eq. (118) for determining the principal stresses must be independent of the directions of the coordinates x, y, z ; hence the factors in parenthesis in this equation should remain constant for any change of directions of coordinates. Hence the coefficients of Eq. (118)

- (a) $\sigma_x + \sigma_y + \sigma_z$
- (b) $\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$
- (c) $\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2$

¹ Another method of representing the stress at a point, by using circles, has been developed by O. Mohr, "Technische Mechanik," 2d ed., p. 192, 1914. See also A. Föppl and L. Föppl, "Drang und Zwang," vol. 1, p. 9, and H. M. Westergaard, *Z. angew. Math. Mech.*, vol. 4, p. 520, 1924. Applications of Mohr's circles were made in discussing two-dimensional problems (see Art. 9).

do not vary with changing directions of the coordinates. This means that the sum $\sigma_x + \sigma_y + \sigma_z$ of the three normal components of stress at a point in three perpendicular directions remains constant and is equal to the sum of the principal stresses at this point.

71. Determination of the Maximum Shearing Stress. Let x, y, z be the principal axes so that $\sigma_x, \sigma_y, \sigma_z$ are principal stresses, and let l, m, n be the direction cosines for a given plane. Then, from Eqs. (115), the square of the total stress on this plane is

$$S^2 = X^2 + Y^2 + Z^2 = \sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 n^2$$

The square of the normal component of the stress on the same plane is, from Eq. (113),

$$\sigma_n^2 = (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2 \quad (a)$$

Then the square of the shearing stress on the same plane must be

$$\tau^2 = S^2 - \sigma_n^2 = \sigma_x^2 l^2 + \sigma_y^2 m^2 + \sigma_z^2 n^2 - (\sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2)^2 \quad (b)$$

We shall now eliminate one of the direction cosines, say n , from this equation by using the relation

$$l^2 + m^2 + n^2 = 1$$

and then determine l and m so as to make τ a maximum. After substituting $n^2 = 1 - l^2 - m^2$ in expression (b), calculating its derivatives with respect to l and m , and equating these derivatives to zero, we obtain the following equations for determining the direction cosines of the planes for which τ is a maximum or minimum:

$$\begin{aligned} l[(\sigma_x - \sigma_z)l^2 + (\sigma_y - \sigma_z)m^2 - \frac{1}{2}(\sigma_x - \sigma_z)] &= 0 \\ m[(\sigma_x - \sigma_z)l^2 + (\sigma_y - \sigma_z)m^2 - \frac{1}{2}(\sigma_y - \sigma_z)] &= 0 \end{aligned} \quad (c)$$

One solution of these equations is obtained by putting $l = m = 0$. We can also obtain solutions different from zero. Taking, for instance, $l = 0$, we find from the second of Eqs. (c) that $m = \pm \sqrt{\frac{1}{2}}$; and taking $m = 0$, we find from the first of Eqs. (c) that $l = \pm \sqrt{\frac{1}{2}}$. There are in general no solutions of Eqs. (c) in which l and m are both different from zero, for in this case the expressions in brackets cannot both vanish.

Repeating the above calculations by eliminating from expression (b) m and then l , we finally arrive at the following table of direction cosines making τ a maximum or minimum:

DIRECTION COSINES FOR PLANES OF τ_{\max} AND τ_{\min} .

$l =$	0	0	± 1	0	$\pm \sqrt{\frac{1}{2}}$	$\pm \sqrt{\frac{1}{2}}$
$m =$	0	± 1	0	$\pm \sqrt{\frac{1}{2}}$	0	$\pm \sqrt{\frac{1}{2}}$
$n =$	± 1	0	0	$\pm \sqrt{\frac{1}{2}}$	$\pm \sqrt{\frac{1}{2}}$	0

The first three columns give the directions of the planes of coordinates, coinciding, as was assumed originally, with the principal planes. For these planes the shearing stress is zero, i.e., expression (b) is a minimum. The three remaining columns give planes through each of the principal axes bisecting the angles between the two other principal axes. Substituting the direction cosines of these three planes into expression (b) we find the following values of the shearing stresses on these three planes:

$$\tau = \pm \frac{1}{2}(\sigma_y - \sigma_z), \quad \tau = \pm \frac{1}{2}(\sigma_x - \sigma_z), \quad \tau = \pm \frac{1}{2}(\sigma_x - \sigma_y) \quad (119)$$

This shows that the maximum shearing stress acts on the plane bisecting the angle between the largest and the smallest principal stresses and is equal to half the difference between these two principal stresses.

If the x -, y -, z -axes in Fig. 132 represent the directions of principal stress, and if $OB = OC = OD$, so that the normal N to the slant face of the tetrahedron has direction cosines $l = m = n = 1/\sqrt{3}$, the normal stress on this face is given by Eq. (a), or (113), as

$$\sigma_n = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \quad (d)$$

This is called the "mean stress." The shear stress on the face is given by Eq. (b) as

$$\tau^2 = \frac{1}{3}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{1}{9}(\sigma_x + \sigma_y + \sigma_z)^2$$

This can also be written

$$\tau^2 = \frac{1}{9}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2]$$

and also, by using (d), as

$$\tau^2 = \frac{1}{3}[(\sigma_x - \sigma_n)^2 + (\sigma_y - \sigma_n)^2 + (\sigma_z - \sigma_n)^2]$$

This shear stress is called the "octahedral shear stress," because the face on which it acts is one face of a regular octahedron with vertices on the axes. It occurs frequently in the theory of plasticity.

72. Homogeneous Deformation. We consider only small deformations, such as occur in engineering structures. The small displacements of the particles of a deformed body will usually be resolved into

components u, v, w parallel to the coordinate axes x, y, z , respectively. It will be assumed that these components are very small quantities varying continuously over the volume of the body.

Consider, as an example, simple tension of a prismatical bar fixed at the upper end (Fig. 133). Let ϵ be the unit elongation of the bar in the x -direction and $v\epsilon$ the unit lateral contraction. Then the components of displacement of a point with coordinates x, y, z are

$$u = \epsilon x, \quad v = -v\epsilon y, \quad w = -v\epsilon z$$

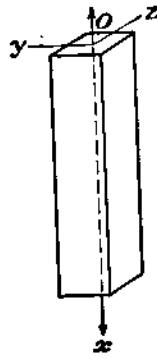


FIG. 133.

Denoting by x', y', z' the coordinates of the point after deformation,

$$\begin{aligned} x' &= x + u = x(1 + \epsilon), & y' &= y + v = y(1 - v\epsilon), \\ z' &= z + w = z(1 - v\epsilon) \end{aligned} \quad (a)$$

If we consider a plane in the bar before deformation such as that given by the equation

$$ax + by + cz + d = 0 \quad (b)$$

the points of this plane will still be in a plane after deformation. The equation of this new plane is obtained by substituting in Eq. (b) the values of x, y, z from Eq. (a). It can easily be proved in this manner that parallel planes remain parallel after deformation and parallel lines remain parallel.

If we consider a spherical surface in the bar before deformation such as given by the equation

$$x^2 + y^2 + z^2 = r^2 \quad (c)$$

this sphere becomes an ellipsoid after deformation, the equation of which can be found by substituting in Eq. (c) the expressions for x, y, z obtained from Eqs. (a). This gives

$$\frac{x'^2}{r^2(1 + \epsilon)^2} + \frac{y'^2}{r^2(1 - v\epsilon)^2} + \frac{z'^2}{r^2(1 - v\epsilon)^2} = 1 \quad (d)$$

Thus a sphere of radius r deforms into an ellipsoid with semiaxes $r(1 + \epsilon), r(1 - v\epsilon), r(1 - v\epsilon)$.

The simple extension, and lateral contraction, considered above, represent only a particular case of a more general type of deformation in which the components of displacement, u, v, w , are linear functions of the coordinates. Proceeding as before, it can be shown that this type of deformation has all the properties found above for the case of

simple tension. Planes and straight lines remain plane and straight after deformation. Parallel planes and parallel straight lines remain parallel after deformation. A sphere becomes, after deformation, an ellipsoid. This kind of deformation is called *homogeneous deformation*. It will be shown later that in this case the deformation in any given direction is the same at all points of the deformed body. Thus two geometrically similar and similarly oriented elements of a body remain geometrically similar after distortion.

In more general cases the deformation varies over the volume of a deformed body. For instance, when a beam is bent, the elongations and contractions of longitudinal fibers depend on their distances from the neutral surface; the shearing strain in elements of a twisted circular shaft is proportional to their distances from the axis of the shaft. In such cases of nonhomogeneous deformation an analysis of the strain in the neighborhood of a point is necessary.

73. Strain at a Point. In discussing strain in the neighborhood of a point O of a deformed body (Fig. 134), let us consider a small linear element OO_1 of length r , with the direction cosines l, m, n . The small projections of this element on the coordinate axes are

$$\delta x = rl, \quad \delta y = rm, \quad \delta z = rn \quad (a)$$

They represent the coordinates of the point O_1 with respect to the x -, y -, z -axes through O as an origin. If u, v, w are the components of the displacement of the point O during deformation of the body, the corresponding displacements of the neighboring point O_1 can be represented as follows:

$$\begin{aligned} u_1 &= u + \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z \\ v_1 &= v + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z \\ w_1 &= w + \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + \frac{\partial w}{\partial z} \delta z \end{aligned} \quad (b)$$

It is assumed here that the quantities $\delta x, \delta y, \delta z$ are small, and hence the terms with higher powers and products of these quantities can be neglected in (b) as small quantities of higher order. The coordinates of the point O_1 become, after deformation,

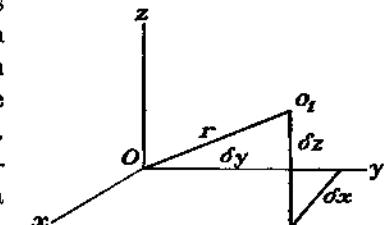


FIG. 134.

$$\begin{aligned}\delta x + u_1 - u &= \delta x + \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z \\ \delta y + v_1 - v &= \delta y + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z \\ \delta z + w_1 - w &= \delta z + \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + \frac{\partial w}{\partial z} \delta z\end{aligned}\quad (c)$$

It will be noticed that these coordinates are linear functions of the initial coordinates $\delta x, \delta y, \delta z$; hence the deformation in a very small element of a body at a point O can be considered as *homogeneous* (Art. 72).

Let us consider the elongation of the element r , due to this deformation. The square of the length of this element after deformation is equal to the sum of the squares of the coordinates (c). Hence, if ϵ is the unit elongation of the element, we find

$$\begin{aligned}(r + \epsilon r)^2 &= \left(\delta x + \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z \right)^2 \\ &\quad + \left(\delta y + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z \right)^2 \\ &\quad + \left(\delta z + \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + \frac{\partial w}{\partial z} \delta z \right)^2\end{aligned}$$

or, dividing by r^2 and using Eqs. (a),

$$\begin{aligned}(1 + \epsilon)^2 &= \left[l \left(1 + \frac{\partial u}{\partial x} \right) + m \frac{\partial u}{\partial y} + n \frac{\partial u}{\partial z} \right]^2 \\ &\quad + \left[l \frac{\partial v}{\partial x} + m \left(1 + \frac{\partial v}{\partial y} \right) + n \frac{\partial v}{\partial z} \right]^2 \\ &\quad + \left[l \frac{\partial w}{\partial x} + m \frac{\partial w}{\partial y} + n \left(1 + \frac{\partial w}{\partial z} \right) \right]^2\end{aligned}\quad (d)$$

Remembering that ϵ and the derivatives $\partial u / \partial x, \dots, \partial w / \partial z$ are small quantities whose squares and products can be neglected, and using $l^2 + m^2 + n^2 = 1$, Eq. (d) becomes

$$\begin{aligned}\epsilon &= l^2 \frac{\partial u}{\partial x} + m^2 \frac{\partial v}{\partial y} + n^2 \frac{\partial w}{\partial z} + lm \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + ln \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ &\quad + mn \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)\end{aligned}\quad (120)$$

Hence the elongation of an element r can be calculated provided the expressions $\frac{\partial u}{\partial x}, \dots, \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \right), \dots$ are known. Using the

notations

$$\begin{aligned}\frac{\partial u}{\partial x} &= \epsilon_x, & \frac{\partial v}{\partial y} &= \epsilon_y, & \frac{\partial w}{\partial z} &= \epsilon_z \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= \gamma_{xy}, & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= \gamma_{xz}, & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} &= \gamma_{yz}\end{aligned}\quad (e)$$

Eq. (120) can be presented in the form

$$\epsilon = \epsilon_x l^2 + \epsilon_y m^2 + \epsilon_z n^2 + \gamma_{xy} lm + \gamma_{xz} ln + \gamma_{yz} mn \quad (121)$$

The physical meaning of such quantities as $\epsilon_x, \dots, \gamma_{yz}, \dots$ has already been discussed (see Art. 5), and it was shown that $\epsilon_x, \epsilon_y, \epsilon_z$ are unit elongations in the x -, y -, z -directions and $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ the three unit shear strains related to the same directions. We now see that the elongation of any linear element through a point O can be calculated from Eq. (121), provided we know the six strain components.

In the particular case of homogeneous deformation the components u, v, w of displacement are linear functions of the coordinates, and from Eqs. (e) the components of strain are constant over the volume of the body, i.e., in this case each element of the body undergoes the same strain.

In investigating strain around a point O it is necessary sometimes to know the change in the angle between two linear elements through the point. Using Eqs. (c) and (a) and considering ϵ as a small quantity, the direction cosines of the element r (Fig. 134), after deformation, are

$$\begin{aligned}l_1 &= \frac{\delta x + u_1 - u}{r(1 + \epsilon)} = l \left(1 - \epsilon + \frac{\partial u}{\partial x} \right) + m \frac{\partial u}{\partial y} + n \frac{\partial u}{\partial z} \\ m_1 &= \frac{\delta y + v_1 - v}{r(1 + \epsilon)} = l \frac{\partial v}{\partial x} + m \left(1 - \epsilon + \frac{\partial v}{\partial y} \right) + n \frac{\partial v}{\partial z} \\ n_1 &= \frac{\delta z + w_1 - w}{r(1 + \epsilon)} = l \frac{\partial w}{\partial x} + m \frac{\partial w}{\partial y} + n \left(1 - \epsilon + \frac{\partial w}{\partial z} \right)\end{aligned}\quad (f)$$

Taking another element r' through the same point with direction cosines l', m', n' , the magnitudes of these cosines, after deformation, are given by equations analogous to (f). The cosine of the angle between the two elements after deformation is

$$\cos(r r') = l_1 l_1' + m_1 m_1' + n_1 n_1'$$

Considering the elongations ϵ and ϵ' in these two directions as small quantities and using Eqs. (f), we find

$$\begin{aligned}\cos(r r') &= (ll' + mm' + nn')(1 - \epsilon - \epsilon') + 2(\epsilon_x ll' + \epsilon_y mm' + \epsilon_z nn') \\ &\quad + \gamma_{xy}(mn' + m'n) + \gamma_{xz}(nl' + n'l) + \gamma_{yz}(lm' + l'm)\end{aligned}\quad (122)$$

If the directions of r and r' are perpendicular to each other, then

$$ll' + mm' + nn' = 0$$

and Eq. (122) gives the shearing strain between these directions.

74. Principal Axes of Strain. From Eq. (121) a geometrical interpretation of the variation of strain at a point can be obtained. For this purpose let us put in the direction of each linear element such as r (Fig. 134) a radius vector of the length

$$R = \frac{k}{\sqrt{|\epsilon|}} \quad (a)$$

Then, proceeding as explained in Art. 68, it can be shown that the ends of all these radii are on the surface given by the equation

$$\pm k^2 = \epsilon_x x^2 + \epsilon_y y^2 + \epsilon_z z^2 + \gamma_{xy}yz + \gamma_{xz}zx + \gamma_{yz}xy \quad (123)$$

The shape and orientation of this surface is completely determined by the state of strain at the point and is independent of the directions of coordinates. It is always possible to take such directions of orthogonal coordinates that the terms with products of coordinates in Eq. (123) disappear, i.e., the shearing strains for such directions become zero. These directions are called *principal axes of strain*, corresponding planes the *principal planes of strain*, and the corresponding strains the *principal strains*. From the above discussion it is evident that the principal axes of strain remain perpendicular to each other after deformation, and a rectangular parallelepiped with the sides parallel to the principal planes remains a rectangular parallelepiped after deformation. In general it will have undergone a small *rotation*.

If the x -, y -, and z -axes are principal axes of strain, then Eq. (123) becomes

$$\pm k^2 = \epsilon_x x^2 + \epsilon_y y^2 + \epsilon_z z^2$$

In this case the elongation of any linear element with the direction cosines l, m, n becomes, from Eq. (121),

$$\epsilon = \epsilon_x l^2 + \epsilon_y m^2 + \epsilon_z n^2 \quad (124)$$

and the shearing strain corresponding to two perpendicular directions r and r' becomes, from Eq. 122,

$$\gamma_{rr'} = 2(\epsilon_x ll' + \epsilon_y mm' + \epsilon_z nn') \quad (125)$$

It can thus be seen that the strain at a point is completely determined if we know the directions of the principal axes of strain and the magnitudes of the principal extensions. The determination of the principal axes of strain and the principal extensions can be done in the same manner as explained in Art. 70. It can also be shown that the sum

$\epsilon_x + \epsilon_y + \epsilon_z$ remains constant when the system of coordinates is rotated. This sum has, as we know, a simple physical meaning; it represents the unit volume expansion due to the strain at a point.

75. Rotation. In general during the deformation of a body, any element is changed in shape, translated and rotated. On account of the shear strain the edges do not rotate by equal amounts, and it is necessary to consider how the rotation of the whole element can be specified. Any rectangular element could have been brought into its final form, position, and orientation in the following three steps, beginning with the element in the undeformed body:

1. The strains $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$ are applied to the element, and the element is so oriented that the directions of principal strain have not rotated.

2. The element is translated until its center occupies its final position.

3. The element is rotated into its final orientation.

The rotation in step 3 is evidently the rotation of the directions of principal strain, and is therefore independent of our choice of x -, y -, z -axes. It must be possible to evaluate it when the displacements u, v, w are given. On the other hand it is clearly independent of the strain components.

Since the translation of the element is of no interest to us here, we may consider the displacement of a point O_1 , as in Art. 73 and Fig. 134, relative to the point O , the center of the element. This relative displacement is given by Eqs. (b) of Art. 73 as

$$\begin{aligned} u_1 - u &= \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z \\ v_1 - v &= \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z \\ w_1 - w &= \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + \frac{\partial w}{\partial z} \delta z \end{aligned} \quad (a)$$

Introducing the notation (e) of Art. 73 for the strain components, and also the notation¹

$$\frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \omega_x, \quad \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = \omega_y, \quad \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \omega_z \quad (126)$$

¹ A glance at Fig. 6 will show that $\partial v / \partial x$ and $-\partial u / \partial y$, occurring in the expression for ω_z , are the clockwise rotations of the line elements $O'A'$, $O'B'$ from their original positions OA , OB . Thus ω_z is the average of these rotations, and ω_x, ω_y have a similar significance in the yz - and xz -planes.

we can write Eqs. (a) in the form

$$\begin{aligned} u_1 - u &= \epsilon_x \delta x + \frac{1}{2} \gamma_{xy} \delta y + \frac{1}{2} \gamma_{xz} \delta z - \omega_x \delta y + \omega_y \delta z \\ v_1 - v &= \frac{1}{2} \gamma_{xy} \delta x + \epsilon_y \delta y + \frac{1}{2} \gamma_{yz} \delta z - \omega_x \delta z + \omega_z \delta x \\ w_1 - w &= \frac{1}{2} \gamma_{xz} \delta x + \frac{1}{2} \gamma_{yz} \delta y + \epsilon_z \delta z - \omega_y \delta x + \omega_x \delta y \end{aligned} \quad (b)$$

which express the relative displacement in two parts, one depending only on the strain components, the other depending only on the quantities $\omega_x, \omega_y, \omega_z$.

We can now show that $\omega_x, \omega_y, \omega_z$ are in fact the components of the rotation θ . Consider the surface given by Eq. (123). The square of the radius in any direction is inversely proportional to the unit elongation of a linear element in that direction. Equation (123) is of the form

$$F(x, y, z) = \text{constant} \quad (c)$$

If we consider a neighboring point $x + dx, y + dy, z + dz$ on the surface, we have the relation

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0 \quad (d)$$

The shift dx, dy, dz is in a direction whose direction cosines are proportional to dx, dy, dz . The three quantities $\partial F / \partial x, \partial F / \partial y, \partial F / \partial z$ also specify a direction, since we can take direction cosines proportional to them. The left-hand side of Eq. (d) is then proportional to the cosine of the angle between these two directions. Since it vanishes, the two directions are at right angles, and since dx, dy, dz represent a direction in the tangent plane to the surface at the point x, y, z , the direction represented by $\partial F / \partial x, \partial F / \partial y, \partial F / \partial z$ is normal to the surface given by Eq. (c).

Now $F(x, y, z)$ is in our case the function on the right-hand side of Eq. (123). Thus

$$\begin{aligned} \frac{\partial F}{\partial x} &= 2\epsilon_x x + \gamma_{xy} y + \gamma_{xz} z \\ \frac{\partial F}{\partial y} &= \gamma_{xy} x + 2\epsilon_y y + \gamma_{yz} z \\ \frac{\partial F}{\partial z} &= \gamma_{xz} x + \gamma_{yz} y + 2\epsilon_z z \end{aligned} \quad (e)$$

The surface given by Eq. (123) being drawn with the point O (Fig. 134) as center, we may identify $\delta x, \delta y, \delta z$ in Eqs. (b) with x, y, z in Eqs. (e).

We consider now the special case when $\omega_x, \omega_y, \omega_z$ are zero. Then the right-hand sides of Eqs. (e) are the same as the right-hand sides of Eqs. (b) but for a factor 2. Consequently the displacement given by Eqs. (b) is normal to the surface given by Eq. (123). Considering the point O_1 (Fig. 134) as a point on the surface, this means that the displacement of O_1 is normal to the surface at O_1 . Hence if OO_1 is one of the principal axes of strain, that is, one of the principal axes of the surface, the displacement of O_1 is in the direction of OO_1 , and therefore OO_1 does not rotate. The displacement in question will correspond to step 1.

In order to complete the displacement we must restore to Eqs. (b) the terms in $\omega_x, \omega_y, \omega_z$. But these terms correspond to a small rigid-body rotation having components $\omega_x, \omega_y, \omega_z$ about the x -, y -, z -axes. Consequently these quantities, given by (126), express the rotation of step 3—that is, the rotation of the principal axes of strain at the point O . They are called simply the *components of rotation*.

Problem

- What is the equation, of the type $f(x, y, z) = 0$, of the surface with center at O which becomes a sphere $x'^2 + y'^2 + z'^2 = r^2$ after the homogeneous deformation of Art. 72? What kind of surface is it?

CHAPTER 9
GENERAL THEOREMS

76. Differential Equations of Equilibrium. In the discussion of Art. 67 we considered the stress at a point of an elastic body. Let us consider now the variation of the stress as we change the position of the point. For this purpose the conditions of equilibrium of a small rectangular parallelepiped with the sides $\delta x, \delta y, \delta z$ (Fig. 135) must be studied. The components of stresses acting on the sides of this small element and their positive directions are indicated in the figure. Here

we take into account the small changes of the components of stress due to the small increases $\delta x, \delta y, \delta z$ of the coordinates. Thus designating the mid-points of the sides of the element by 1, 2, 3, 4, 5, 6 as in Fig. 135, we distinguish between the value of σ_x at point 1, and its value at point 2, writing these $(\sigma_x)_1$ and $(\sigma_x)_2$ respectively. The symbol σ_x itself denotes, of course, the value of this stress component at the point x, y, z . In calculating the forces acting on the element we consider the sides as very small, and the force is obtained by multiplying the stress at the centroid of a side by the area of this side.

It should be noted that the body force acting on the element, which was neglected as a small quantity of higher order in discussing the equilibrium of a tetrahedron (Fig. 132), must now be taken into account, because it is of the same order of magnitude as the terms due to variations of the stress components, which we are now considering. If we let X, Y, Z denote the components of this force per unit volume of the element, then the equation of equilibrium obtained by summing all the forces acting on the element in the x -direction is

$$[(\sigma_x)_1 - (\sigma_x)_2] \delta y \delta z + [(\tau_{xy})_3 - (\tau_{xy})_4] \delta x \delta z + [(\tau_{xz})_5 - (\tau_{xz})_6] \delta x \delta y + X \delta x \delta y \delta z = 0$$

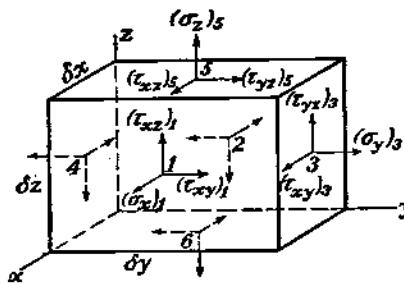


FIG. 135.

The two other equations of equilibrium are obtained in the same manner. After dividing by $\delta x \delta y \delta z$ and proceeding to the limit by shrinking the element down to the point x, y, z , we find

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z &= 0 \end{aligned} \quad (127)$$

Equations (127) must be satisfied at all points throughout the volume of the body. The stresses vary over the volume of the body, and when we arrive at the surface they must be such as to be in equilibrium with the external forces on the surface of the body. These conditions of equilibrium at the surface can be obtained from Eqs. (112). Taking a tetrahedron $OBCD$ (Fig. 132), so that the side BCD coincides with the surface of the body, and denoting by $\bar{X}, \bar{Y}, \bar{Z}$ the components of the surface forces per unit area at this point, Eqs. (112) become

$$\begin{aligned} \bar{X} &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\ \bar{Y} &= \sigma_y m + \tau_{yz} n + \tau_{xy} l \\ \bar{Z} &= \sigma_z n + \tau_{xz} l + \tau_{yz} m \end{aligned} \quad (128)$$

in which l, m, n are the direction cosines of the external normal to the surface of the body at the point under consideration.

If the problem is to determine the state of stress in a body submitted to the action of given forces it is necessary to solve Eqs. (127), and the solution must be such as to satisfy the boundary conditions (128). These equations, containing six components of stress, $\sigma_x, \dots, \tau_{yz}$, are not sufficient for the determination of these components. The problem is a statically indeterminate one, and in order to obtain the solution we must proceed as in the case of two-dimensional problems, i.e., the elastic deformations of the body must also be considered.

77. Conditions of Compatibility. It should be noted that the six components of strain at each point are completely determined by the three functions u, v, w , representing the components of displacement. Hence the components of strain cannot be taken arbitrarily as functions of x, y, z but are subject to relations which follow from Eqs. (2) (see page 6).

Thus, from Eqs. (2),

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}, \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

from which

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (a)$$

Two more relations of the same kind can be obtained by cyclical interchange of the letters x, y, z .

Calculating the derivatives

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial y \partial z} &= \frac{\partial^3 u}{\partial x \partial y \partial z}, & \frac{\partial \gamma_{yz}}{\partial x} &= \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial \gamma_{xz}}{\partial y} &= \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial x \partial y}, & \frac{\partial \gamma_{xy}}{\partial z} &= \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} \end{aligned}$$

we find that

$$2 \frac{\partial^2 \epsilon_z}{\partial y \partial z} = \frac{\partial}{\partial x} \left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \quad (b)$$

Two more relations of the kind (b) can be obtained by interchange of the letters x, y, z . We thus arrive at the following six differential relations between the components of strain, which must be satisfied by virtue of Eqs. (2):

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, & 2 \frac{\partial^2 \epsilon_z}{\partial y \partial z} &= \frac{\partial}{\partial x} \left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\ \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} &= \frac{\partial^2 \gamma_{xz}}{\partial y \partial z}, & 2 \frac{\partial^2 \epsilon_x}{\partial x \partial z} &= \frac{\partial}{\partial y} \left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\ \frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial z}, & 2 \frac{\partial^2 \epsilon_y}{\partial x \partial y} &= \frac{\partial}{\partial z} \left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \end{aligned} \quad (129)$$

These differential relations¹ are called the *conditions of compatibility*.

By using Hooke's law [Eqs. (3)] conditions (129) can be transformed into relations between the components of stress. Take, for instance, the condition

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad (c)$$

From Eqs. (3) and (4), using the notation (7), we find

$$\epsilon_y = \frac{1}{E} [(1 + \nu) \sigma_y - \nu \Theta]$$

$$\epsilon_z = \frac{1}{E} [(1 + \nu) \sigma_z - \nu \Theta]$$

$$\gamma_{yz} = \frac{2(1 + \nu) \tau_{yz}}{E}$$

¹ Proofs that these six equations are sufficient to ensure the existence of a displacement corresponding to a given set of functions $\epsilon_x, \dots, \gamma_{xy}, \dots$, may be found in A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 49, and I. S. Sokolnikoff, "Mathematical Theory of Elasticity," p. 24, 1946.

Substituting these expressions in (c), we obtain

$$(1 + \nu) \left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2} \right) - \nu \left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2} \right) = 2(1 + \nu) \frac{\partial^2 \tau_{yz}}{\partial y \partial z} \quad (d)$$

The right side of this equation can be transformed by using the equations of equilibrium (127). From these equations we find

$$\begin{aligned} \frac{\partial \tau_{yz}}{\partial y} &= -\frac{\partial \sigma_z}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - Z \\ \frac{\partial \tau_{yz}}{\partial z} &= -\frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - Y \end{aligned}$$

Differentiating the first of these equations with respect to z and the second with respect to y , and adding them together, we find

$$2 \frac{\partial^2 \tau_{yz}}{\partial y \partial z} = -\frac{\partial^2 \sigma_z}{\partial z^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial}{\partial x} \left(\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y} \right) - \frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial y}$$

or, by using the first of Eqs. (127),

$$2 \frac{\partial^2 \tau_{yz}}{\partial y \partial z} = \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

Substituting this in Eq. (d) and using, to simplify the writing, the symbol

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

we find

$$\begin{aligned} (1 + \nu) \left(\nabla^2 \Theta - \nabla^2 \sigma_x - \frac{\partial^2 \Theta}{\partial x^2} \right) - \nu \left(\nabla^2 \Theta - \frac{\partial^2 \Theta}{\partial z^2} \right) \\ = (1 + \nu) \left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z} \right) \end{aligned} \quad (e)$$

Two analogous equations can be obtained from the two other conditions of compatibility of the type (c).

Adding together all three equations of the type (e) we find

$$(1 - \nu) \nabla^2 \Theta = -(1 + \nu) \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right)$$

Substituting this expression for $\nabla^2 \Theta$ in Eq. (e),

$$\nabla^2 \sigma_x + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x^2} = -\frac{\nu}{1 - \nu} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) - 2 \frac{\partial X}{\partial x} \quad (f)$$

We can obtain three equations of this kind, corresponding to the first three of Eqs. (129). In the same manner the remaining three conditions (129) can be transformed into equations of the following kind:

$$\nabla^2 \tau_{yz} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y \partial z} = - \left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z} \right) \quad (g)$$

If there are no body forces or if the body forces are constant, Eqs. (f) and (g) become

$$\begin{aligned} (1+\nu)\nabla^2 \sigma_x + \frac{\partial^2 \Theta}{\partial x^2} &= 0, & (1+\nu)\nabla^2 \tau_{xz} + \frac{\partial^2 \Theta}{\partial y \partial z} &= 0 \\ (1+\nu)\nabla^2 \sigma_y + \frac{\partial^2 \Theta}{\partial y^2} &= 0, & (1+\nu)\nabla^2 \tau_{xy} + \frac{\partial^2 \Theta}{\partial x \partial z} &= 0 \\ (1+\nu)\nabla^2 \sigma_z + \frac{\partial^2 \Theta}{\partial z^2} &= 0, & (1+\nu)\nabla^2 \tau_{yz} + \frac{\partial^2 \Theta}{\partial x \partial y} &= 0 \end{aligned} \quad (130)$$

We see that in addition to the equations of equilibrium (127) and the boundary conditions (128) the stress components in an isotropic body must satisfy the six conditions of compatibility (f) and (g) or the six conditions (130). This system of equations is generally sufficient for determining the stress components without ambiguity (see Art. 82).

The conditions of compatibility contain only second derivatives of the stress components. Hence, if the external forces are such that the equations of equilibrium (127) together with the boundary conditions (128) can be satisfied by taking the stress components either as constants or as linear functions of the coordinates, the equations of compatibility are satisfied identically and this stress system is the correct solution of the problem. Several examples of such problems will be considered in Chap. 10.

78. Determination of Displacements. When the components of stress are found from the previous equations, the components of strain can be calculated by using Hooke's law [Eqs. (3) and (6)]. Then Eqs. (2) are used for the determination of the displacements u, v, w . Differentiating Eqs. (2) with respect to x, y, z we can obtain 18 equations containing 18 second derivatives of u, v, w , from which all these derivatives can be determined. For u , for instance, we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial \epsilon_x}{\partial x}, & \frac{\partial^2 u}{\partial y^2} &= \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \epsilon_y}{\partial x}, & \frac{\partial^2 u}{\partial z^2} &= \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \epsilon_z}{\partial x} \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial \epsilon_x}{\partial y}, & \frac{\partial^2 u}{\partial x \partial z} &= \frac{\partial \epsilon_x}{\partial z}, & \frac{\partial^2 u}{\partial y \partial z} &= \frac{1}{2} \left(\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} \right). \end{aligned} \quad (a)$$

The second derivatives for the two other components of displacement v and w can be obtained by cyclical interchange in Eqs. (a) of the letters x, y, z .

Now u, v, w can be obtained by double integration of these second derivatives. The introduction of arbitrary constants of integration will result in adding to the values of u, v, w linear functions in x, y, z , as it is evident that such functions can be added to u, v, w without affecting such equations as (a). To have the strain components (2) unchanged by such an addition, the additional linear functions must have the form

$$\begin{aligned} u' &= a + by - cz \\ v' &= d - bx + ez \\ w' &= f + cx - ey \end{aligned} \quad (b)$$

This means that the displacements are not entirely determined by the stresses and strains. On the displacements found from the differential Eqs. (127), (128), (130) a displacement like that of a rigid body can be superposed. The constants a, d, f in Eqs. (b) represent a translatory motion of the body, and the constants b, c, e are the three rotations of the rigid body around the coordinate axes. When there are sufficient constraints to prevent motion as a rigid body, the six constants in Eqs. (b) can easily be calculated so as to satisfy the conditions of constraint. Several examples of such calculations will be shown later.

79. Equations of Equilibrium in Terms of Displacements. One method of solution of the problems of elasticity is to eliminate the stress components from Eqs. (127) and (128) by using Hooke's law, and to express the strain components in terms of displacements by using Eqs. (2). In this manner we arrive at three equations of equilibrium containing only the three unknown functions u, v, w . Substituting in the first of Eqs. (127) from (11)

$$\sigma_x = \lambda e + 2G \frac{\partial u}{\partial x} \quad (a)$$

and from (6)

$$\begin{aligned} \tau_{xy} &= G \gamma_{xy} = G \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \tau_{xz} &= G \gamma_{xz} = G \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \end{aligned} \quad (b)$$

we find

$$(\lambda + G) \frac{\partial e}{\partial x} + G \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + X = 0$$

The two other equations can be transformed in the same manner.

Then, using the symbol ∇^2 (see page 231), the equations of equilibrium (127) become

$$\begin{aligned} (\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 u + X &= 0 \\ (\lambda + G) \frac{\partial e}{\partial y} + G \nabla^2 v + Y &= 0 \\ (\lambda + G) \frac{\partial e}{\partial z} + G \nabla^2 w + Z &= 0 \end{aligned} \quad (131)$$

and, when there are no body forces,

$$\begin{aligned} (\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 u &= 0 \\ (\lambda + G) \frac{\partial e}{\partial y} + G \nabla^2 v &= 0 \\ (\lambda + G) \frac{\partial e}{\partial z} + G \nabla^2 w &= 0 \end{aligned} \quad (132)$$

Differentiating these equations, the first with respect to x , the second with respect to y , and the third with respect to z , and adding them together, we find

$$(\lambda + 2G) \nabla^2 e = 0$$

i.e., the volume expansion e satisfies the differential equation

$$\frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial y^2} + \frac{\partial^2 e}{\partial z^2} = 0 \quad (133)$$

The same conclusion holds also when body forces are constant throughout the volume of the body.

Substituting from such equations as (a) and (b) into the boundary conditions (128) we find

$$\bar{X} = \lambda e l + G \left(\frac{\partial u}{\partial x} l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \right) + G \left(\frac{\partial u}{\partial x} l + \frac{\partial v}{\partial x} m + \frac{\partial w}{\partial x} n \right) \quad (134)$$

Equations (131) together with the boundary conditions (134) define completely the three functions u, v, w . From these the components of strain are obtained from Eqs. (2) and the components of stress from Eqs. (9) and (6). Applications of these equations will be shown in Chap. 15.

80. General Solution for the Displacements. It is easily verified by substitution that the differential equations (132) of equilibrium in terms of displacement are satisfied by¹

$$\begin{aligned} u &= \phi_1 - \alpha \frac{\partial}{\partial x} (\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) \\ v &= \phi_2 - \alpha \frac{\partial}{\partial y} (\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) \\ w &= \phi_3 - \alpha \frac{\partial}{\partial z} (\phi_0 + x\phi_1 + y\phi_2 + z\phi_3) \end{aligned}$$

where $4\alpha = 1/(1 - \nu)$ and the four functions $\phi_0, \phi_1, \phi_2, \phi_3$ are harmonic, i.e.,

$$\nabla^2 \phi_0 = 0, \quad \nabla^2 \phi_1 = 0, \quad \nabla^2 \phi_2 = 0, \quad \nabla^2 \phi_3 = 0$$

It can be shown that this solution is general, and that any one of the four functions may be dropped without loss of generality.

This form of solution has been adapted to curvilinear coordinates by Neuber, and applied by him in the solution of problems of solids of revolution² generated by hyperbolas (the hyperbolic groove on a cylinder) and ellipses (cavity in the form of an ellipsoid of revolution) transmitting tension, bending, torsion, or shear force transverse to the axis with accompanying bending.

81. The Principle of Superposition. The solution of a problem of a given elastic solid with given surface and body forces requires us to determine stress components, or displacements, which satisfy the differential equations and the boundary conditions. If we choose to work with stress components we have to satisfy: (a) the equations of equilibrium (127); (b) the compatibility conditions (129); (c) the boundary conditions (128). Let $\sigma_x, \tau_{xy}, \tau_{xz}$ be the stress components so determined, and due to surface forces $\bar{X}, \bar{Y}, \bar{Z}$, and body forces X, Y, Z .

Let $\sigma'_x, \tau'_{xy}, \tau'_{xz}$ be the stress components in the same elastic solid due to surface forces $\bar{X}', \bar{Y}', \bar{Z}'$ and body forces X', Y', Z' . Then

¹ This solution was given independently by P. F. Papkovitch, *Compt. rend.*, vol. 195, pp. 513 and 754, 1932, and by H. Neuber, *Z. angew. Math. Mech.*, vol. 14, p. 203, 1934. Other general solutions were given by B. Galerkin, *Compt. rend.*, vol. 190, p. 1047, 1930, and by Boussinesq and Kelvin—see Todhunter and Pearson, "History of Elasticity," vol. 2, pt. 2, p. 268. See also R. D. Mindlin, *Bull. Am. Math. Soc.*, 1936, p. 373.

² H. Neuber, "Kerbspannungslehre." This book also contains solutions of two-dimensional problems. See Chap. 7 above.

the stress components $\sigma_x + \sigma'_x, \dots, \tau_{xy} + \tau'_{xy}, \dots$, will represent the stress due to the surface forces $\bar{X} + \bar{X}', \dots$, and the body forces $X + X', \dots$. This holds because all the differential equations and boundary conditions are *linear*. Thus adding the first of Eqs. (127) to the corresponding equation

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + X' = 0$$

we find

$$\frac{\partial}{\partial x} (\sigma_x + \sigma'_x) + \frac{\partial}{\partial y} (\tau_{xy} + \tau'_{xy}) + \frac{\partial}{\partial z} (\tau_{xz} + \tau'_{xz}) + X + X' = 0$$

and similarly from the first of (128) and its counterpart we have by addition

$$\bar{X} + \bar{X}' = (\sigma_x + \sigma'_x)l + (\tau_{xy} + \tau'_{xy})m + (\tau_{xz} + \tau'_{xz})n$$

The compatibility conditions can be combined in the same manner. The complete set of equations shows that $\sigma_x + \sigma'_x, \dots, \tau_{xy} + \tau'_{xy}, \dots$, satisfy all the equations and conditions determining the stress due to forces $\bar{X} + \bar{X}', \dots, X + X', \dots$. This is the *principle of superposition*.

In deriving our equations of equilibrium (127) and boundary conditions (128) we made no distinction between the position and form of the element before loading, and its position and form after loading. As a consequence our equations, and the conclusions drawn from them, are valid only so long as the small displacements in the deformation do not affect substantially the action of the external forces. There are cases, however, in which the deformation must be taken into account. Then the justification of the principle of superposition given above fails. The beam under simultaneous thrust and lateral load affords an example of this kind, and many others arise in considering the elastic stability of thin-walled structures.

82. Uniqueness of Solution. We consider now whether our equations can have more than one solution corresponding to given surface and body forces.

Let $\sigma'_x, \dots, \tau'_{xy}, \dots$ represent a solution for loads \bar{X}, \dots, X, \dots , and let $\sigma''_x, \dots, \tau''_{xy}, \dots$ represent a second solution for the same loads \bar{X}, \dots, X, \dots

Then for the first solution we have such equations as

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + X = 0$$

$$\bar{X} = \sigma'_x l + \tau'_{xy} m + \tau'_{xz} n$$

and also the conditions of compatibility.

For the second solution we have

$$\frac{\partial \sigma''_x}{\partial x} + \frac{\partial \tau''_{xy}}{\partial y} + \frac{\partial \tau''_{xz}}{\partial z} + X = 0$$

$$\bar{X} = \sigma''_x l + \tau''_{xy} m + \tau''_{xz} n$$

and also the conditions of compatibility.

By subtraction we find that the stress distribution given by the differences $\sigma'_x - \sigma''_x, \dots, \tau'_{xy} - \tau''_{xy}$, satisfies the equations

$$\frac{\partial(\sigma'_x - \sigma''_x)}{\partial x} + \frac{\partial(\tau'_{xy} - \tau''_{xy})}{\partial y} + \frac{\partial(\tau'_{xz} - \tau''_{xz})}{\partial z} = 0$$

$$0 = (\sigma'_x - \sigma''_x)l + (\tau'_{xy} - \tau''_{xy})m + (\tau'_{xz} - \tau''_{xz})n$$

in which all external forces vanish. The conditions of compatibility (129) will also be satisfied by the corresponding strain components $\epsilon'_x - \epsilon''_x, \dots, \gamma'_{xy} - \gamma''_{xy}, \dots$

Thus this stress distribution is one which corresponds to zero surface and body forces. The work done by these forces during loading is zero, and it follows that $\iiint V_0 dx dy dz$ vanishes. But, as Eq. (85) shows, V_0 is positive for all states of strain, and therefore the integral can vanish only if V_0 vanishes at all points of the body. This requires that each of the strain components $\epsilon'_x - \epsilon''_x, \dots, \gamma'_{xy} - \gamma''_{xy}, \dots$ should be zero. The two states of strain $\epsilon'_x, \dots, \gamma'_{xy}, \dots$, and

$\epsilon_x'' \dots, \gamma_{xy}'' \dots$, and consequently the two states of stress $\sigma_x' \dots, \tau_{xy}' \dots$, and $\sigma_x'' \dots, \tau_{xy}'' \dots$, are therefore identical. That is, the equations can yield only one solution corresponding to given loads.¹

The proof of uniqueness of solution was based on the assumption that the strain energy, and hence stresses, in a body disappear when it is freed of external forces. However there are cases when *initial stresses* may exist in a body while external forces are absent. An example of this kind was encountered in studying the circular ring (see Art. 39). If a portion of the ring between two adjacent cross sections is cut out, and the ends of the ring are joined again by welding or other means, a ring with initial stresses is obtained.² Several examples of this kind were discussed in considering two-dimensional problems.

We can also have initial stresses in a simply connected body due to some nonelastic deformations during the process of forming the body. We may have, for instance, considerable initial stresses in large forgings due to nonuniform cooling and also in rolled metallic bars due to the plastic flow produced by cold work. For determining these initial stresses the equations of elasticity are not sufficient, and additional information regarding the process of forming the body is necessary.

It should be noted that in all cases in which the principle of superposition can be used the deformations and stresses produced by external forces are not affected by initial stresses and can be calculated in exactly the same manner as if there were no initial stresses. Then the total stresses are obtained by superposing the stresses produced by external forces on the initial stresses. In cases when the principle of superposition is not applicable, the stresses produced by external loads cannot be determined without knowing the initial stresses. We cannot, for instance, calculate bending stresses produced by lateral loads in a thin bar, if the bar has an initial axial tension or compression, without knowing the magnitude of this initial stress.

¹ This theorem is due to G. Kirchhoff. See his *Vorlesungen über Math. Phys., Mechanik*.

² The ring represents the simplest example of multiply-connected bodies. In the case of such bodies general equations of elasticity, expressed in terms of stress components, are not sufficient for determining stresses, and to get a complete solution an additional investigation of displacements is necessary. The first investigations of this kind were made by J. H. Michell, *Proc. London Math. Soc.*, vol. 31, p. 103, 1899. See also L. N. G. Filon, *Brit. Assoc. Advancement Sci. Rept.*, 1921, p. 305, and V. Volterra, *Sur l'équilibre des corps élastiques multiplement connexés*, *Ann. école norm.*, Paris, series 3, vol. 24, pp. 401–517, 1907. Further references on initial stresses are given in the paper by P. Neményi, *Z. angew. Math. Mech.*, vol. 11, p. 59, 1931.

83. The Reciprocal Theorem. Limiting ourselves to the two-dimensional case let us consider the plate under two different loading conditions, and denote by $\bar{X}_1, \bar{Y}_1, X_1$, and Y_1 the components of the boundary and the volume forces in the first case, and by $\bar{X}_2, \bar{Y}_2, X_2$, and Y_2 in the second case. For the displacements, the strain components, and stress components in the two cases we use the notation $u_1, v_1, \epsilon_x', \epsilon_y', \gamma_{xy}', \sigma_x', \sigma_y', \tau_{xy}'$ and $u_2, v_2, \epsilon_x'', \epsilon_y'', \gamma_{xy}'', \sigma_x'', \sigma_y'', \tau_{xy}''$. Let us consider now the work which would be produced by the forces of the first state of stress on the corresponding displacements of the second state. This work will be

$$T = \int \bar{X}_1 u_2 ds + \int \bar{Y}_1 v_2 ds + \iint X_1 u_2 dx dy + \iint Y_1 v_2 dx dy \quad (a)$$

where the first two integrals are extended around the entire boundary of the plate and the second two over the entire area of the plate. Substituting for \bar{X}_1 its expression from Eqs. (20), page 23, we can represent the first term on the right-hand side of Eq. (a) as follows:

$$\int \bar{X}_1 u_2 ds = \int l \sigma_x' u_2 ds + \int m \tau_{xy}' u_2 ds \quad (b)$$

Proceeding now as explained on page 164 we get

$$\begin{aligned} \int l \sigma_x' u_2 ds &= \iint u_2 \frac{\partial \sigma_x'}{\partial x} dx dy + \iint \frac{\partial u_2}{\partial x} \sigma_x' dx dy \\ \int m \tau_{xy}' u_2 ds &= \iint u_2 \frac{\partial \tau_{xy}'}{\partial y} dx dy + \iint \frac{\partial u_2}{\partial y} \tau_{xy}' dx dy \end{aligned}$$

Substituting this in (b) we find that the first and the third terms of (a) give us

$$\begin{aligned} \int \bar{X}_1 u_2 ds + \iint X_1 u_2 dx dy &= \iint \left(\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + X_1 \right) u_2 dx dy \\ &\quad + \iint \left(\frac{\partial u_2}{\partial x} \sigma_x' + \frac{\partial u_2}{\partial y} \tau_{xy}' \right) dx dy \quad (c) \end{aligned}$$

Similarly the second and the fourth terms give

$$\begin{aligned} \int \bar{Y}_1 v_2 ds + \iint Y_1 v_2 dx dy &= \iint \left(\frac{\partial \tau_{xy}'}{\partial x} + \frac{\partial \sigma_y'}{\partial y} + Y_1 \right) v_2 dx dy \\ &\quad + \iint \left(\frac{\partial v_2}{\partial y} \sigma_y' + \frac{\partial v_2}{\partial x} \tau_{xy}' \right) dx dy \quad (d) \end{aligned}$$

Observing now that the first terms on the right-hand side of equations (c) and (d) vanish in virtue of equilibrium equations (18), and sub-

stituting in Eq. (a), we obtain

$$\begin{aligned} T &= \int \int (\epsilon_x''\sigma_x' + \epsilon_y''\sigma_y' + \gamma_{xy}''\tau_{xy}') dx dy \\ &= \frac{1}{E} \int \int [\sigma_x''\sigma_x' + \sigma_y''\sigma_y' - \nu\sigma_y''\sigma_x' - \nu\sigma_x''\sigma_y' \\ &\quad + 2(1 + \nu)\tau_{xy}''\tau_{xy}'] dx dy \end{aligned}$$

Exactly the same result is obtained if we calculate the work done by the forces of the second state of stress on the displacements of the first state. Thus we can conclude, comparing two states of stress of an elastic body, that the work done by the forces of the first state on the corresponding displacements of the second is equal to the work done by the forces of the second state on the corresponding displacements of the first. This represents the *reciprocal theorem*. It can be easily extended also to bodies in motion or in vibration. It is only necessary to add the inertia forces to the external loads.

The reciprocal theorem finds an important application in the theory of structures in the construction of influence lines. It also has useful applications in the theory of elasticity.

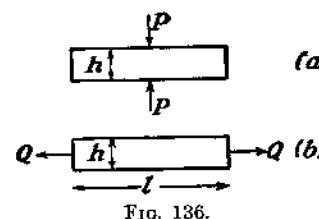


FIG. 136.

Take as a simple example the case of a prismatical bar compressed by two equal and opposite forces¹ P , Fig. 136a. The problem of finding the stresses produced by these forces is a complicated one; but assume that we are interested not in the stresses but in the total elongation δ of the bar. This question can be answered at once by using the theorem. For this purpose we consider in addition to the given stress condition represented in Fig. 136a the simple central tension of the bar shown in Fig. 136b. For this second case we find the lateral contraction, equal to $\delta_1 = \nu \frac{Qh}{AE}$, where A is the cross-sectional area of the bar. Then the reciprocal theorem gives us the equation

$$P \cdot \nu \frac{Qh}{AE} = Q\delta$$

¹ We may suppose that the forces are distributed over a small area, to avoid singularities. Ideally concentrated forces in two-dimensional problems usually result in infinite displacement, indicating that the actual displacement depends on the distribution.

and the elongation of the bar, produced by two forces P in Fig. 136a, is

$$\delta = \frac{\nu Ph}{AE}$$

and is independent of the shape of the cross section.

As a second example let us calculate the reduction Δ in volume of an elastic body produced by two equal and opposite forces P , Fig. 137a. As a second case of stress we take the same body submitted to the action of uniformly distributed pressure p . In this latter case we will have at each point of the body a uniform compression in all directions of the magnitude $(1 - 2\nu)p/E$ [sec Eq. (8), page 9] and the distance l between the points of application A and B will be diminished by the amount $(1 - 2\nu)pl/E$. The reciprocal theorem applied to the two stress conditions¹ of Fig. 137 will then give

$$P \cdot \frac{(1 - 2\nu)pl}{E} = \Delta p$$

and the reduction in the volume of the body is therefore

$$\Delta = \frac{Pl(1 - 2\nu)}{E}$$

84. Approximate Character of the Plane Stress Solutions. It was pointed out on page 25 that the set of equations we found sufficient for plane stress problems under the assumptions made ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$, $\sigma_x, \sigma_y, \tau_{xy}$ independent of z) did not ensure satisfaction of all the conditions of compatibility. These assumptions imply that $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}$ are independent of z , and that γ_{xz}, γ_{yz} are zero. The first of the conditions of compatibility (129) was included in the plane stress theory, as Eq. (21). It is easily verified that the other five are satisfied only if ϵ_z is a linear function of x and y , which is the exception rather than the rule in the plane stress solutions obtained in Chaps. 3 to 7. Evidently these solutions cannot be exact, but we shall now see that they are close approximations for thin plates.

Let us seek exact solutions of the three-dimensional equations for which²

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

¹ For other applications of this kind see A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., pp. 174-176, 1927.

² A. Clebsch, "Elasticität," Art. 39. See also A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 145, 1927.

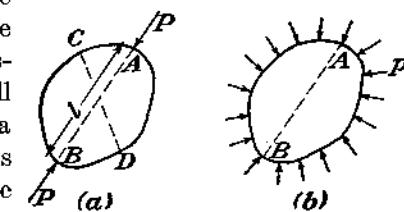


FIG. 137.

taking body force as zero. Such solutions must satisfy the equations of equilibrium (127) and the compatibility conditions (130).

Since σ_z , τ_{xz} , τ_{yz} are zero, the third, fourth, and fifth of Eqs. (130) (reading by columns) give

$$\frac{\partial}{\partial z} \left(\frac{\partial \Theta}{\partial z} \right) = 0, \quad \frac{\partial}{\partial y} \left(\frac{\partial \Theta}{\partial z} \right) = 0, \quad \frac{\partial}{\partial x} \left(\frac{\partial \Theta}{\partial z} \right) = 0$$

which mean that $\partial \Theta / \partial z$ is a constant. Writing this k , we have, by integration with respect to z ,

$$\Theta = kz + \Theta_0 \quad (a)$$

where Θ_0 is so far an arbitrary function of x and y .

The third of Eqs. (127) is identically satisfied, and the first two reduce to the two-dimensional forms

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

which are satisfied, as before, by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (b)$$

but ϕ is now a function of x , y , and z .

Returning to Eqs. (130) we observe that by addition of the three equations on the left, recalling that $\Theta = \sigma_z + \sigma_y + \sigma_z$, we have

$$\nabla^2 \Theta = 0 \quad (c)$$

and therefore, from (a)

$$\nabla_1^2 \Theta_0 = 0 \quad (d)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Also, since σ_z is zero, and σ_x and σ_y are given by the first two of Eqs. (b), we can write $\nabla_1^2 \phi = \Theta$, and therefore, using (a)

$$\nabla_1^2 \phi = kz + \Theta_0 \quad (e)$$

where Θ_0 is a function of x and y satisfying Eq. (d). Using (a) and the first of (b), the first of Eqs. (130) becomes

$$(1 + \nu) \nabla^2 \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \Theta_0}{\partial x^2} = 0 \quad (f)$$

But

$$\nabla^2 \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial y^2} \nabla^2 \phi = \frac{\partial^2}{\partial y^2} \left(\nabla_1^2 \phi + \frac{\partial^2 \phi}{\partial z^2} \right) = \frac{\partial^2}{\partial y^2} \left(\Theta_0 + \frac{\partial^2 \phi}{\partial z^2} \right)$$

where Eq. (e) has been used in the last step. Also, on account of (d), we can replace $\partial^2 \Theta_0 / \partial x^2$ in (f) by $-\partial^2 \Theta_0 / \partial y^2$. Then (f) becomes

$$(1 + \nu) \frac{\partial^2}{\partial y^2} \left(\Theta_0 + \frac{\partial^2 \phi}{\partial z^2} \right) - \frac{\partial^2 \Theta_0}{\partial y^2} = 0$$

or

$$\frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\nu}{1 + \nu} \Theta_0 \right) = 0 \quad (g)$$

This equation may be used in place of the first of (130). Similarly the second and last can be replaced by

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\nu}{1 + \nu} \Theta_0 \right) = 0, \quad \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\nu}{1 + \nu} \Theta_0 \right) = 0$$

These, with (g), show that all three second derivatives with respect to x and y of the function (of x , y , and z) in brackets vanish. Thus this function must be linear in x and y , and we can write

$$\frac{\partial^2 \phi}{\partial z^2} + \frac{\nu}{1 + \nu} \Theta_0 = a + bx + cy \quad (h)$$

where a , b , and c are arbitrary functions of z . Integrating this equation twice with respect to z , we find

$$\phi = -\frac{1}{2} \frac{\nu}{1 + \nu} \Theta_0 z^2 + A + Bx + Cy + \phi_1 z + \phi_0 \quad (i)$$

where A , B , C are functions of z obtained by repeated integration of a , b , c , and ϕ_1 , ϕ_0 are functions of x and y , as yet arbitrary.

If we evaluate σ_x , σ_y , τ_{xy} from (i) by means of the formulas (b), the terms

$$A + Bx + Cy$$

make no difference. We may therefore set A , B , and C equal to zero, corresponding to taking a , b , c , zero in (h).

If we restrict ourselves to problems in which the stress distribution is symmetrical about the middle plane of the plate, $z = 0$, the term $\phi_1 z$ must also be zero. So also must k in Eq. (a).

Then (i) reduces to

$$\phi = \phi_0 - \frac{1}{2} \frac{\nu}{1 + \nu} \Theta_0 z^2 \quad (j)$$

However ϕ and Θ_0 are related by (e) in which we can now take $k = 0$. Thus, substituting (j) in (e) and using (d), we have

$$\nabla_1^2 \phi_0 = \Theta_0 \quad (k)$$

and therefore, from (d),

$$\nabla_1^4 \phi_0 = 0 \quad (l)$$

The remaining equations of (130) are satisfied on account of Eq. (a) and the vanishing of σ_z , τ_{xz} , τ_{yz} .

We can now obtain a stress distribution by choosing a function ϕ_0 of x and y which satisfies Eq. (l), finding Θ_0 from Eq. (k), and ϕ from Eq. (j). The stresses are then found by the formulas (b). Each will consist of two parts, the first derived from ϕ_0 in Eq. (j), the second from the term $-\frac{1}{2} \frac{\nu}{1 + \nu} \Theta_0 z^2$. In view of Eq. (l), the first part is exactly like the plane stress components determined in Chaps. 3 to

7. The second part, being proportional to z^2 , may be made as small as we please compared with the first by restricting ourselves to plates which are sufficiently thin. Hence the conclusion that our solutions in Chaps. 3 to 7, which do not satisfy all the compatibility conditions, are nevertheless good approximations for thin plates.

The "exact" solutions, represented by stress functions of the form (j), will require that the stresses at the boundary, as elsewhere, have a parabolic variation over the thickness. However any change from this distribution, so long as it does not alter the intensity of force per unit length of boundary curve, will only alter the stress in the immediate neighborhood of the edge, by Saint-Venant's principle (page 33). The type of solution considered above will always represent the actual stress, and the components σ_z , τ_{xz} , τ_{yz} will in fact be zero, except close to the edges.

Problems

1. Show that

$$\epsilon_x = k(x^2 + y^2), \quad \epsilon_y = k(y^2 + z^2), \quad \gamma_{xy} = k'xyz \\ \epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

where k , k' are small constants, is not a possible state of strain.

2. A solid is heated nonuniformly to temperature T , a function of x , y , and z . If it is supposed that each element has unrestrained thermal expansion, the strain components will be

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha T, \quad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

where α is the constant coefficient of thermal expansion.

Prove that this can only occur when T is a linear function of x , y , and z . (The stress and consequent further strain arising when T is not linear are discussed in Chap. 14.)

3. A disk or cylinder of the shape shown in Fig. 137a is compressed by forces P at C and D , along CD , causing extension of AB . It is then compressed by forces P along AB (Fig. 137a) causing extension of CD . Show that these extensions are equal.

4. In the general solution of Art. 80 what choice of the functions ϕ_0 , ϕ_1 , ϕ_2 , ϕ_3 will give the general solution for plane strain ($w = 0$)?

CHAPTER 10

ELEMENTARY PROBLEMS OF ELASTICITY IN THREE DIMENSIONS

85. Uniform Stress. In discussing the equations of equilibrium (127) and the boundary conditions (128), it was stated that the true solution of a problem must satisfy not only Eqs. (127) and (128) but also the compatibility conditions (see Art. 77). These latter conditions contain, if no body forces are acting, or if the body forces are constant, only second derivatives of the stress components. If, therefore, Eqs. (127) and conditions (128) can be satisfied by taking the stress components either as constants or as linear functions of the coordinates, the compatibility conditions are satisfied identically and these stresses are the correct solution of the problem.

As a very simple example we may take tension of a prismatical bar in the axial direction (Fig. 138). Body forces are neglected. The equations of equilibrium are satisfied by taking

$$\sigma_x = \text{constant}, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \quad (a)$$

It is evident that boundary conditions (128) for the lateral surface of the bar, which is free of external forces, are satisfied, because all stress components, except σ_x , are zero. The boundary conditions for the ends reduce to

$$\sigma_x = \bar{X} \quad (b)$$

i.e., we have a uniform distribution of tensile stresses over cross sections of a prismatical bar if the tensile stresses are uniformly distributed over the ends. In this case solution (a) satisfies Eqs. (127) and (128) and is the correct solution of the problem because the compatibility conditions (130) are identically satisfied.

If the tensile stresses are not uniformly distributed over the ends, solution (a) is no longer the correct solution because it does not satisfy the boundary conditions at the ends. The true solution becomes more complicated because the stresses on a cross section are no longer uni-

formly distributed. Examples of such nonuniform distribution occurred in the discussion of two-dimensional problems (see pages 51 and 167).

As a second example consider the case of a uniform hydrostatic compression with no body forces. The equations of equilibrium (127) are satisfied by taking

$$\sigma_x = \sigma_y = \sigma_z = -p, \quad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \quad (c)$$

The ellipsoid of stress in this case is a sphere. Any three perpendicular directions can be considered as principal directions, and the stress on any plane is a normal compressive stress equal to p . The surface conditions (128) will evidently be satisfied if the pressure p is uniformly distributed over the surface of the body.

86. Stretching of a Prismatical Bar by Its Own Weight. If ρg is the weight per unit volume of the bar (Fig. 139), the body forces are

$$X = Y = 0, \quad Z = -\rho g \quad (a)$$

The differential equations of equilibrium (127) are satisfied by putting

$$\sigma_z = \rho g z, \quad \sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0 \quad (b)$$

i.e., by assuming that on each cross section we have a uniform tension produced by the weight of the lower portion of the bar.

It can easily be seen that the boundary conditions (128) at the lateral surface, which is free from forces, are satisfied. The boundary conditions give zero stresses for the lower end of the bar, and, for the upper end, the uniformly distributed tensile stress $\sigma_z = \rho g l$, in which l is the length of the bar.

The compatibility equations (130) are also satisfied by the solution (b), hence it is the correct solution of the problem for a uniform distribution of forces at the top. It coincides with the solution which is usually given in elementary books on the strength of materials.

Let us consider now the displacements (see Art. 78). From Hooke's law, using Eqs. (3) and (6), we find

$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{\sigma_z}{E} = \frac{\rho g z}{E} \quad (c)$$

$$\epsilon_x = \epsilon_y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\nu \frac{\rho g z}{E} \quad (d)$$

$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \quad (e)$$

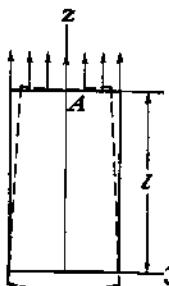


Fig. 139.

The displacements u , v , w can now be found by integrating Eqs. (c), (d), and (e). Integration of Eq. (c) gives

$$w = \frac{\rho g z^2}{2E} + w_0 \quad (f)$$

where w_0 is a function of x and y , to be determined later. Substituting (f) in the second and third of Eqs. (e), we find

$$\frac{\partial w_0}{\partial x} + \frac{\partial u}{\partial z} = 0, \quad \frac{\partial w_0}{\partial y} + \frac{\partial v}{\partial z} = 0$$

from which

$$u = -z \frac{\partial w_0}{\partial x} + u_0, \quad v = -z \frac{\partial w_0}{\partial y} + v_0 \quad (g)$$

in which u_0 and v_0 are functions of x and y only. Substituting expressions (g) into Eqs. (d), we find

$$-z \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = -\nu \frac{\rho g z}{E}, \quad -z \frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\nu \frac{\rho g z}{E} \quad (h)$$

Remembering that u_0 and v_0 do not depend on z , Eqs. (h) can be satisfied only if

$$\frac{\partial u_0}{\partial x} = \frac{\partial v_0}{\partial y} = 0, \quad \frac{\partial^2 w_0}{\partial x^2} = \frac{\partial^2 w_0}{\partial y^2} = \frac{\nu \rho g}{E} \quad (k)$$

Substituting expressions (g) for u and v into the first of Eqs. (e), we find

$$-2z \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0$$

and, since u_0 and v_0 do not depend on z , we must have

$$\frac{\partial^2 w_0}{\partial x \partial y} = 0, \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0 \quad (l)$$

From Eqs. (k) and (l) general expressions can now be written for the functions u_0 , v_0 , w_0 . It is easy to show that all these equations are satisfied by

$$u_0 = \delta y + \delta_1$$

$$v_0 = -\delta x + \gamma_1$$

$$w_0 = \frac{\nu \rho g}{2E} (x^2 + y^2) + \alpha x + \beta y + \gamma$$

in which $\alpha, \beta, \gamma, \delta, \delta_1, \gamma_1$ are arbitrary constants. Now, from Eqs. (f) and (g), the general expressions for the displacements are

$$\begin{aligned} u &= -\frac{\nu \rho g x z}{E} - \alpha z + \delta y + \delta_1 \\ v &= -\frac{\nu \rho g y z}{E} - \beta z - \delta x + \gamma_1 \\ w &= \frac{\rho g z^2}{2E} + \frac{\nu \rho g}{2E} (x^2 + y^2) + \alpha x + \beta y + \gamma \end{aligned} \quad (m)$$

The six arbitrary constants must be determined from the conditions at the support. The support must be such as to prevent any movement of the bar as a rigid body. To prevent a translatory motion of the bar, let us fix the centroid A of the upper end of the bar so that $u = v = w = 0$ for $x = y = 0$ and $z = l$. To eliminate rotation of the bar about axes through the point A , parallel to the x - and y -axes, let us fix an element of the z -axis at A . Then $\partial u / \partial z = \partial v / \partial z = 0$ at that point. The possibility of rotation about the z -axis is eliminated by fixing an elemental area through A , parallel to the zx -plane. Then $\partial v / \partial x = 0$ at the point A . Using Eqs. (m) the above six conditions at the point A become

$$\begin{aligned} -\alpha l + \delta_1 &= 0, & -\beta l + \gamma_1 &= 0, & \frac{\rho g l^2}{2E} + \gamma &= 0 \\ \alpha &= 0, & \beta &= 0, & \delta &= 0 \end{aligned}$$

Hence

$$\delta_1 = 0, \quad \gamma_1 = 0, \quad \gamma = -\frac{\rho g l^2}{2E}$$

and the final expressions for the displacements are

$$\begin{aligned} u &= -\frac{\nu \rho g x z}{E} \\ v &= -\frac{\nu \rho g y z}{E} \\ w &= \frac{\rho g z^2}{2E} + \frac{\nu \rho g}{2E} (x^2 + y^2) - \frac{\rho g l^2}{2E} \end{aligned}$$

It may be seen that points on the z -axis have only vertical displacements

$$w = -\frac{\rho g}{2E} (l^2 - z^2)$$

Other points of the bar, on account of lateral contraction, have not only vertical but also horizontal displacements. Lines which were

parallel to the z -axis before deformation become inclined to this axis after deformation, and the form of the bar after deformation is as indicated in Fig. 139 by dotted lines. Cross sections of the bar perpendicular to the z -axis after deformation are curved to the surface of a paraboloid. Points on the cross section $z = c$, for instance, after deformation will be on the surface

$$z = c + w = c + \frac{\rho g c^2}{2E} + \frac{\nu \rho g}{2E} (x^2 + y^2) - \frac{\rho g l^2}{2E}$$

This surface is perpendicular to all longitudinal fibers of the bar, these being inclined to the z -axis after deformation, so that there is no shearing strain γ_{xy} or γ_{xz} .

87. Twist of Circular Shafts of Constant Cross Section. The elementary theory of twist of circular shafts states that the shearing stress τ at any point of the cross section (Fig. 140) is perpendicular to the radius r and proportional to the length r and to the angle of twist θ per unit length of the shaft:

$$\tau = G \theta r \quad (a)$$

where G is the modulus of rigidity. Resolving this stress into two components parallel to the x - and y -axes, we find

$$\tau_{xy} = G \theta r \cdot \frac{x}{r} = G \theta x \quad (b)$$

$$\tau_{xz} = -G \theta r \cdot \frac{y}{r} = -G \theta y$$

The elementary theory also assumes that

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

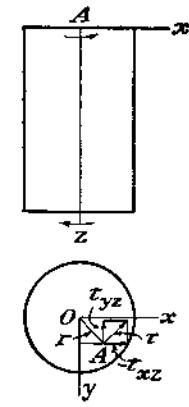


FIG. 140.

We can show that this elementary solution is the exact solution under certain conditions. Since the stress components are all either linear functions of the coordinates or zero, the equations of compatibility (130) are satisfied, and it is only necessary to consider the equations of equilibrium (127) and the boundary conditions (128). Substituting the above expressions for stress components into Eqs. (127) we find that these equations are satisfied, provided there are no body forces. The lateral surface of the shaft is free from forces, and the boundary conditions (128), remembering that for the cylindrical surface $\cos(Nz) = n = 0$, reduce to

$$0 = \tau_{xz} \cos(Nx) + \tau_{yz} \cos(Ny) \quad (c)$$

For the case of a circular cylinder we have also

$$\cos(Nx) = \frac{x}{r}, \quad \cos(Ny) = \frac{y}{r} \quad (d)$$

Substituting these and expressions (b) for the stress components into Eq. (c) it is evident that this equation is satisfied. It is also evident that for cross sections other than circular, for which Eqs. (d) do not hold, the stress components (b) do not satisfy the boundary condition (c), and therefore solution (a) cannot be applied. These more complicated problems of twist will be considered later (see Chap. 11).

Considering now the boundary conditions for the ends of the shaft, we see that the surface shearing forces must be distributed in exactly the same manner as the stresses τ_{xz} and τ_{yz} over any intermediate cross section of the shaft. Only for this case is the stress distribution given

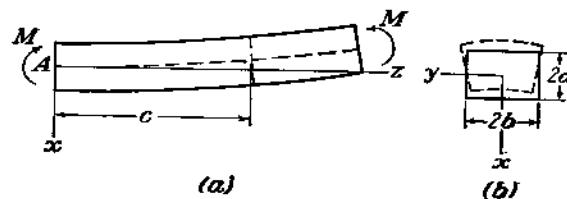


FIG. 141.

by Eqs. (b) an exact solution of the problem. But the practical application of the solution is not limited to such cases. From Saint-Venant's principle it can be concluded that in a long twisted bar, at a sufficient distance from the ends, the stresses depend only on the magnitude of the torque M , and are practically independent of the manner in which the forces are distributed over the ends.

The displacements for this case can be found in the same manner as in the previous article. Assuming the same condition of constraint at the point A as in the previous problem we find

$$u = -\theta yz, \quad v = \theta xz, \quad w = 0$$

This means that the assumption that cross sections remain plane and radii remain straight, which is usually made in the elementary derivation of the theory of twist, is correct.

88. Pure Bending of Prismatical Bars. Consider a prismatical bar bent in one of its principal planes by two equal and opposite couples M (Fig. 141). Taking the origin of the coordinates at the centroid of the

cross section and the xz -plane in the principal plane of bending, the stress components given by the usual elementary theory of bending are

$$\sigma_z = \frac{Ex}{R}, \quad \sigma_y = \sigma_x = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \quad (a)$$

in which R is the radius of curvature of the bar after bending. Substituting expressions (a) for the stress components in the equations of equilibrium (127), it is found that these equations are satisfied if there are no body forces. The boundary conditions (128) for the lateral surface of the bar, which is free from external forces, are also satisfied. The boundary conditions (128) at the ends require that the surface forces must be distributed over the ends in the same manner as the stresses σ_z . Only under this condition do the stresses (a) represent the exact solution of the problem. The bending moment M is given by the equation

$$M = \int \sigma_z x dA = \int \frac{Ex^2 dA}{R} = \frac{EI_y}{R}$$

in which I_y is the moment of inertia of the cross section of the beam with respect to the neutral axis parallel to the y -axis. From this equation we find

$$\frac{1}{R} = \frac{M}{EI_y}$$

which is a well-known formula of the elementary theory of bending.

Let us consider now the displacements for the case of pure bending. Using Hooke's law and Eqs. (2) we find, from solution (a),

$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{x}{R} \quad (b)$$

$$\epsilon_x = \frac{\partial u}{\partial x} = -\nu \frac{x}{R}, \quad \epsilon_y = \frac{\partial v}{\partial y} = -\nu \frac{x}{R} \quad (c)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \quad (d)$$

By using these differential equations, and taking into consideration the fastening conditions of the bar, the displacements can be obtained in the same manner as in Art. 86.

From Eq. (b) we find

$$w = \frac{xz}{R} + w_0$$

in which w_0 is a function of x and y only. The second and third of Eqs. (d) give

$$\frac{\partial u}{\partial z} = -\frac{z}{R} - \frac{\partial w_0}{\partial x}, \quad \frac{\partial v}{\partial z} = -\frac{\partial w_0}{\partial y}$$

from which

$$u = -\frac{z^2}{2R} - z \frac{\partial w_0}{\partial x} + u_0, \quad v = -z \frac{\partial w_0}{\partial y} + v_0 \quad (e)$$

Here u_0 and v_0 denote unknown functions of x and y , which will be determined later. Substituting expressions (e) in Eqs. (c),

$$-z \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial u_0}{\partial x} = -\frac{\nu x}{R}, \quad -z \frac{\partial^2 w_0}{\partial y^2} + \frac{\partial v_0}{\partial y} = -\nu \frac{x}{R}$$

These equations must be satisfied for any value of z , hence

$$\frac{\partial^2 w_0}{\partial x^2} = 0, \quad \frac{\partial^2 w_0}{\partial y^2} = 0 \quad (f)$$

and by integration

$$u_0 = -\frac{\nu x^2}{2R} + f_1(y), \quad v_0 = -\frac{\nu xy}{R} + f_2(x) \quad (g)$$

Now substituting (e) and (g) into the first of Eqs. (d), we find

$$2z \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial f_1(y)}{\partial y} - \frac{\partial f_2(x)}{\partial x} + \frac{\nu y}{R} = 0$$

Noting that only the first term in this equation depends on z , we conclude that it is necessary to have

$$\frac{\partial^2 w_0}{\partial x \partial y} = 0, \quad \frac{\partial f_1(y)}{\partial y} + \frac{\partial f_2(x)}{\partial x} - \frac{\nu y}{R} = 0$$

These equations and Eqs. (f) require that

$$w_0 = mx + ny + p$$

$$f_1(y) = \frac{\nu y^2}{2R} + \alpha y + \gamma$$

$$f_2(x) = -\alpha x + \beta$$

in which $m, n, p, \alpha, \beta, \gamma$ are arbitrary constants. The expressions for the displacements now become

$$u = -\frac{z^2}{2R} - mz - \frac{\nu x^2}{2R} + \frac{\nu y^2}{2R} + \alpha y + \gamma$$

$$v = -nz - \frac{\nu xy}{R} - \alpha x + \beta$$

$$w = \frac{xz}{R} + mx + ny + p$$

The arbitrary constants are determined from the conditions of fastening. Assuming that the point A , the centroid of the left end of the bar, together with an element of the z -axis and an element of the xz -plane, are fixed, we have for $x = y = z = 0$

$$u = v = w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial x} = 0$$

These conditions are satisfied by taking all the arbitrary constants equal to zero. Then

$$u = -\frac{1}{2R} [z^2 + \nu(x^2 - y^2)], \quad v = -\frac{\nu xy}{R}, \quad w = \frac{xz}{R} \quad (h)$$

To get the deflection curve of the axis of the bar we substitute in the above Eqs. (h) $x = y = 0$. Then

$$u = -\frac{z^2}{2R} = -\frac{Mz^2}{2EI_y}, \quad v = w = 0$$

This is the same deflection curve as is given by the elementary theory of bending.

Let us consider now any cross section $z = c$, a distance c from the left end of the bar. After deformation, the points of this cross section will be in the plane

$$z = c + w = c + \frac{cx}{R}$$

i.e., in pure bending the cross section remains plane as is assumed in the elementary theory. To examine the deformation of the cross section in its plane, consider the sides $y = \pm b$ (Fig. 141b). After bending we have

$$y = \pm b + v = \pm b \left(1 - \frac{\nu x}{R} \right)$$

The sides become inclined as shown in the figure by dotted lines.

The other two sides of the cross section $x = \pm a$ are represented after bending by the equations

$$x = \pm a + u = \pm a - \frac{1}{2R} [c^2 + \nu(a^2 - y^2)]$$

They are therefore bent to parabolic curves, which can be replaced with sufficient accuracy by an arc of a circle of radius R/ν , when the

deformation is small. In considering the upper or lower sides of the bar it is evident that while the curvature of these sides after bending is convex down in the lengthwise direction, the curvature in the crosswise direction is convex upward. Contour lines for this anticlastic surface will be as shown in Fig. 142a. By taking x and u constant in the first of Eqs. (h) we find that the equation for the contour lines is

$$z^2 - \nu y^2 = \text{constant}$$

They are therefore hyperbolae with the asymptotes

$$z^2 - \nu y^2 = 0$$

From this equation the angle α (Fig. 142a) is found from

$$\tan^2 \alpha = \frac{1}{\nu}$$

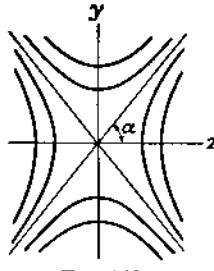


FIG. 142a.

This equation has been used for determining Poisson's ratio ν .¹ If the upper surface of the beam is polished and a glass plate put over it, there will be, after bending, an air gap of variable thickness between the glass plate and the curved surface of the beam. This variable thickness can be measured optically. A beam of monochromatic

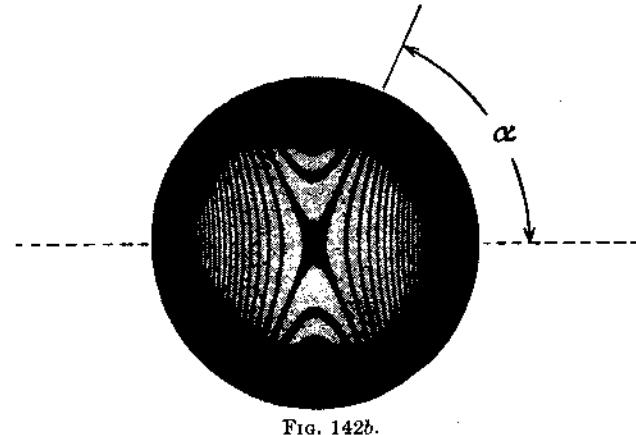


FIG. 142b.

light, say yellow sodium light, perpendicular to the glass plate, will be reflected partially by the plate and partially by the surface of the beam. The two reflected rays of light interfere with each other at points where the thickness of the air gap is such that the difference between the

¹ A. Cornu, *Compt. rend.*, vol. 69, p. 333, 1869. See also R. Straubel, *Wied. Ann.*, vol. 68, p. 369, 1899.

paths of the two rays is equal to an uneven number of half wave lengths of the light. The picture shown in Fig. 142b, representing the hyperbolic contour lines, was obtained by this means.

89. Pure Bending of Plates. The result of the previous article can be applied in discussing the bending of plates of uniform thickness. If stresses $\sigma_z = Ez/R$ are distributed over the edges of the plate parallel to the y -axis (Fig. 143), the surface of the plate will become¹ an anticlastic surface, the curvature of which in planes parallel to the xz -plane is $1/R$ and in the perpendicular direction is $-\nu/R$. If h denotes the thickness of the plate, M_1 the bending moment per unit length on the edges parallel to the y -axis and

$$I_y = \frac{1 \cdot h^3}{12}$$

the moment of inertia per unit length, the relation between M_1 and R , from the previous article, is

$$\frac{1}{R} = \frac{M_1}{EI_y} = \frac{12M_1}{Eh^3} \quad (a)$$

When we have bending moments in two perpendicular directions (Fig. 144), the curvatures of the deflection surface may be obtained by

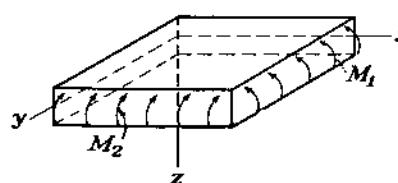


FIG. 143.

superposition. Let $1/R_1$ and $1/R_2$ be the curvatures of the deflection surface in planes parallel to the coordinate planes zx and zy , respectively; and let M_1 and M_2 be the bending moments per unit length on the edges parallel to the y - and x -

axes, respectively. Then, using Eq. (a) and applying the principle of superposition, we find

$$\begin{aligned} \frac{1}{R_1} &= \frac{12}{Eh^3} (M_1 - \nu M_2) \\ \frac{1}{R_2} &= \frac{12}{Eh^3} (M_2 - \nu M_1) \end{aligned} \quad (b)$$

¹ It is assumed that deflections are small in comparison with the thickness of the plate.

The moments are considered positive if they produce a deflection of the plate which is convex down. Solving Eqs. (b) for M_1 and M_2 , we find

$$\begin{aligned} M_1 &= \frac{Eh^3}{12(1-\nu^2)} \left(\frac{1}{R_1} + \nu \frac{1}{R_2} \right) \\ M_2 &= \frac{Eh^3}{12(1-\nu^2)} \left(\frac{1}{R_2} + \nu \frac{1}{R_1} \right) \end{aligned} \quad (c)$$

For small deflections we can use the approximations

$$\frac{1}{R_1} = -\frac{\partial^2 w}{\partial x^2}, \quad \frac{1}{R_2} = -\frac{\partial^2 w}{\partial y^2}$$

Then, writing

$$\frac{Eh^3}{12(1-\nu^2)} = D \quad (135)$$

we find

$$\begin{aligned} M_1 &= -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \\ M_2 &= -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \end{aligned} \quad (136)$$

The constant D is called the *flexural rigidity* of a plate. In the particular case when the plate is bent to a cylindrical surface with generators parallel to the y -axis we have $\partial^2 w / \partial y^2 = 0$, and, from Eqs. (136),

$$\begin{aligned} M_1 &= -D \frac{\partial^2 w}{\partial x^2} \\ M_2 &= -\nu D \frac{\partial^2 w}{\partial x^2} \end{aligned} \quad (137)$$

For the particular case in which $M_1 = M_2 = M$, we have

$$\frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{R}$$

The plate is bent to a spherical surface and the relation between the curvature and the bending moment is, from Eq. (c),

$$M = \frac{Eh^3}{12(1-\nu)} \cdot \frac{1}{R} = \frac{D(1+\nu)}{R} \quad (138)$$

We shall have use for these results later.

The formulas (136) are used in the theory of plates when the bending moments are not uniform, and are accompanied by shear forces and surface pressures. For these circumstances they can be deduced from the general equations of Chap. 9 as approximations valid when the plate is thin. The elementary theory of bending of bars can be related to the general equations in a similar manner.¹

¹ J. N. Goodier, *Trans. Roy. Soc. Can.*, 3d ser., sec. III, vol. 32, p. 65, 1938.

CHAPTER 11

TORSION

90. Torsion of Prismatical Bars. It has already been shown (Art. 87) that the exact solution of the torsional problem for a circular shaft is obtained if we assume that the cross sections of the bar remain plane and rotate without any distortion during twist. This theory, developed by Coulomb,¹ was applied later by Navier² to prismatical bars of noncircular cross sections. Making the above assumption he arrived at the erroneous conclusions that, for a given torque, the angle of twist of bars is inversely proportional to the centroidal polar moment of inertia of the cross section, and that the maximum shearing stress occurs at the points most remote from the centroid of the cross section.³

It is easy to see that the above assumption is in contradiction with the boundary conditions. Take, for instance, a bar of rectangular cross section (Fig. 145). From Navier's assumption it follows that at any point A on the boundary the shearing stress should act in the direction perpendicular to the radius OA . Resolving this stress into two components τ_{xz} and τ_{yz} , it is evident that there should be a complementary shearing stress, equal to τ_{yz} , on the element of the lateral surface of the bar at the point A (see page 4), which is in contradiction with the assumption that the lateral surface of the bar is free from external forces, the twist being produced by couples applied at the ends. A simple experiment with a rectangular bar, represented in Fig. 146, shows that the cross sections of the bar do not remain plane during torsion, and that the distortions of rectangular elements on the surface of the bar are greatest at the middles of the sides, i.e., at the points which are nearest to the axis of the bar.

¹ "Histoire de l'académie," 1784, pp. 229–269, Paris, 1787.

² Navier, "Résumé des leçons sur l'application de la mécanique," 3d ed., Paris, 1864, edited by Saint-Venant.

³ These conclusions are correct for a thin elastic layer, corresponding to a slice of the bar between two cross sections, attached to rigid plates. See J. N. Goodier, *J. Applied Phys.*, vol. 13, p. 167, 1942.

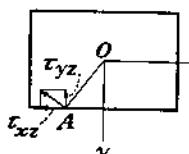


FIG. 145.

The correct solution of the problem of torsion of prismatical bars by couples applied at the ends was given by Saint-Venant.¹

He used the so-called *semi-inverse method*. That is, at the start he made certain assumptions as to the deformation of the twisted bar and showed that with these assumptions he could satisfy the equations of equilibrium (127) and the boundary conditions (128). Then from the uniqueness of solutions of the elasticity equations (Art. 82) it follows that the assumptions made at the start are correct and the solution obtained is the exact solution of the torsion problem.

Consider a prismatical bar of any cross section twisted by couples applied at the ends, Fig. 147. Guided by the solution for a circular shaft (page 249), Saint-Venant assumes that the deformation of the twisted shaft consists (a) of rotations of cross sections of the shaft as in the case of a circular shaft and (b) of *warping* of the cross sections which is the same for all cross sections. Taking the origin of coordinates in an end cross section (Fig. 147) we find that the displacements corresponding to rotation of cross sections are

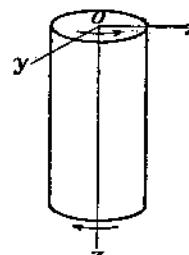


FIG. 147.

$$u = -\theta zy, \quad v = \theta zx \quad (a)$$

where θz is the angle of rotation of the cross section at a distance z from the origin.

The warping of cross sections is defined by a function

$$w = \theta \psi(x, y) \quad (b)$$

With the assumed displacements (a) and (b) we calculate the components of strain from Eqs. (2), which give

$$\begin{aligned} \epsilon_x &= \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \\ \gamma_{zz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \theta \left(\frac{\partial \psi}{\partial x} - y \right) \\ \gamma_{yy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \theta \left(\frac{\partial \psi}{\partial y} + x \right) \end{aligned} \quad (c)$$

¹ *Mém. savants étrangers*, vol. 14, 1855. See also Saint-Venant's note to Navier's book, *loc. cit.*, and I. Todhunter and K. Pearson, "History of the Theory of Elasticity," vol. 2.

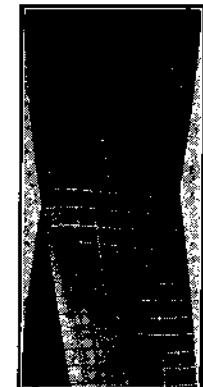


FIG. 146.

The corresponding components of stress, from Eqs. (3) and (6), are

$$\begin{aligned}\sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \\ \tau_{xz} &= G\theta \left(\frac{\partial \psi}{\partial x} - y \right) \\ \tau_{yz} &= G\theta \left(\frac{\partial \psi}{\partial y} + x \right)\end{aligned}\quad (d)$$

It can be seen that with the assumptions (a) and (b) regarding the deformation, there will be no normal stresses acting between the longitudinal fibers of the shaft or in the longitudinal direction of those fibers. There also will be no distortion in the planes of cross sections, since $\epsilon_x, \epsilon_y, \gamma_{xy}$ vanish. We have at each point pure shear, defined by the components τ_{xz} and τ_{yz} . The function $\psi(x, y)$, defining warping of

cross section, must now be determined in such a way that equations of equilibrium (127) will be satisfied. Substituting expressions (d) in these equations and neglecting body forces we find that the function ψ must satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (139)$$

Consider now the boundary conditions (128). For the lateral surface of the bar, which is free from external forces and has normals perpendicular to the z -axis, we have $\bar{X} = \bar{Y} = \bar{Z} = 0$ and $\cos(Nz) = n = 0$. The first two of Eqs. (128) are identically satisfied and the third gives

$$\tau_{xz}l + \tau_{yz}m = 0 \quad (e)$$

which means that the resultant shearing stress at the boundary is directed along the tangent to the boundary, Fig. 148. It was shown before (see page 258) that this condition must be satisfied if the lateral surface of the bar is free from external forces.

Considering an infinitesimal element abc at the boundary and assuming that s is increasing in the direction from c to a , we have

$$l = \cos(Nx) = \frac{dy}{ds}, \quad m = \cos(Ny) = -\frac{dx}{ds}$$

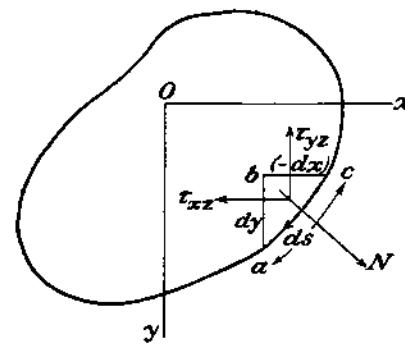


FIG. 148.

and Eq. (e) becomes

$$\left(\frac{\partial \psi}{\partial x} - y \right) \frac{dy}{ds} - \left(\frac{\partial \psi}{\partial y} + x \right) \frac{dx}{ds} = 0 \quad (140)$$

Thus each problem of torsion is reduced to the problem of finding a function ψ satisfying Eq. (139) and the boundary condition (140).

An alternative procedure, which has the advantage of leading to a simpler boundary condition, is as follows. In view of the vanishing of $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$ [Eqs. (d)], the equations of equilibrium (127) reduce to

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

The first two are already satisfied since τ_{xz} and τ_{yz} , as given by Eqs. (d), are independent of z . The third means that we can express τ_{xz} and τ_{yz} as

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x} \quad (141)$$

where ϕ is a function of x and y , called the *stress function*.¹

From Eqs. (141) and (d) we have

$$\frac{\partial \phi}{\partial y} = G\theta \left(\frac{\partial \psi}{\partial x} - y \right), \quad -\frac{\partial \phi}{\partial x} = G\theta \left(\frac{\partial \psi}{\partial y} + x \right) \quad (f)$$

Eliminating ψ by differentiating the first with respect to y , the second with respect to x , and subtracting from the first, we find that the stress function must satisfy the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F \quad (142)$$

where

$$F = -2G\theta \quad (143)$$

The boundary condition (e) becomes, introducing Eqs. (141),

$$\frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} = \frac{d\phi}{ds} = 0 \quad (144)$$

This shows that the stress function ϕ must be constant along the boundary of the cross section. In the case of singly connected boundaries, e.g., for solid bars, this constant can be chosen arbitrarily, and in the following discussion we shall take it equal to zero. Thus the determination of the stress distribution over a cross section of a twisted bar

¹ It was introduced by L. Prandtl. See *Physik. Z.*, vol. 4, 1903.

consists in finding the function ϕ which satisfies Eq. (142) and is zero at the boundary. Several applications of this general theory to particular shapes of cross sections will be shown later.

Let us consider now the conditions at the ends of the twisted bar. The normals to the end cross sections are parallel to the z -axis. Hence $l = m = 0$, $n = \pm 1$ and Eqs. (128) become

$$\bar{X} = \pm \tau_{xz}, \quad \bar{Y} = \pm \tau_{yz} \quad (g)$$

in which the + sign should be taken for the end of the bar for which the external normal has the direction of the positive z -axis, as for the lower end of the bar in Fig. 147. We see that over the ends the shearing forces are distributed in the same manner as the shearing stresses over the cross sections of the bar. It is easy to prove that these forces give us a torque. Substituting in Eqs. (g) from (141) and observing that ϕ at the boundary is zero, we find

$$\begin{aligned} \iint \bar{X} dx dy &= \iint \tau_{xz} dx dy = \iint \frac{\partial \phi}{\partial y} dx dy = \int dx \int \frac{\partial \phi}{\partial y} dy = 0 \\ \iint \bar{Y} dx dy &= \iint \tau_{yz} dx dy = - \iint \frac{\partial \phi}{\partial x} dx dy \\ &= - \int dy \int \frac{\partial \phi}{\partial x} dx = 0 \end{aligned}$$

Thus the resultant of the forces distributed over the ends of the bar is zero, and these forces represent a couple the magnitude of which is

$$\begin{aligned} M_t &= \iint (\bar{Y}x - \bar{X}y) dx dy = - \iint \frac{\partial \phi}{\partial x} x dx dy \\ &\quad - \iint \frac{\partial \phi}{\partial y} y dx dy \quad (h) \end{aligned}$$

Integrating this by parts, and observing that $\phi = 0$ at the boundary, we find

$$M_t = 2 \iint \phi dx dy \quad (145)$$

each of the integrals in the last member of Eqs. (h) contributing one half of this torque. Thus we find that half the torque is due to the stress component τ_{xz} and the other half to τ_{yz} .

We see that by assuming the displacements (a) and (b), and determining the stress components τ_{xz} , τ_{yz} from Eqs. (141), (142), and (144), we obtain a stress distribution which satisfies the equations of equilibrium (127), leaves the lateral surface of the bar free from external

forces, and sets up at the ends the torque given by Eq. (145). The compatibility conditions (130) need not be considered since the stress has been derived from the displacements (a) and (b). Thus all the equations of elasticity are satisfied and the solution obtained in this manner is the exact solution of the torsion problem.

It was pointed out that the solution requires that the forces at the ends of the bar should be distributed in a definite manner. But the practical application of the solution is not limited to such cases. From Saint-Venant's principle it follows that in a long twisted bar, at a sufficient distance from the ends, the stresses depend only on the magnitude of the torque M_t and are practically independent of the manner in which the tractions are distributed over the ends.

91. Bars with Elliptical Cross Section.

Let the boundary of the cross section (Fig. 149) be given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad (a)$$

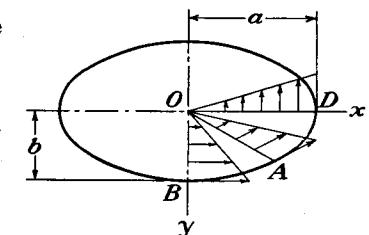


FIG. 149.

Then Eq. (142) and the boundary condition (144) are satisfied by taking the stress function in the form

$$\phi = m \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad (b)$$

in which m is a constant. Substituting (b) into Eq. (142), we find

$$m = \frac{a^2 b^2 F}{2(a^2 + b^2)} F$$

Hence

$$\phi = \frac{a^2 b^2 F}{2(a^2 + b^2)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad (c)$$

The magnitude of the constant F will now be determined from Eq. (145). Substituting in this equation from (c), we find

$$M_t = \frac{a^2 b^2 F}{a^2 + b^2} \left(\frac{1}{a^2} \iint x^2 dx dy + \frac{1}{b^2} \iint y^2 dx dy - \iint dx dy \right) \quad (d)$$

Since

$$\begin{aligned} \iint x^2 dx dy &= I_y = \frac{\pi b a^3}{4}, & \iint y^2 dx dy &= I_x = \frac{\pi a b^3}{4}, \\ \iint dx dy &= \pi ab \end{aligned}$$

we find, from (d),

$$M_t = -\frac{\pi a^3 b^3 F}{2(a^2 + b^2)}$$

from which

$$F = -\frac{2M_t(a^2 + b^2)}{\pi a^3 b^3} \quad (e)$$

Then, from (c),

$$\phi = -\frac{M_t}{\pi ab} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad (f)$$

Substituting in Eqs. (141), the stress components are

$$\tau_{xz} = -\frac{2M_t y}{\pi a b^3}, \quad \tau_{yz} = \frac{2M_t x}{\pi a^3 b} \quad (146)$$

The ratio of the stress components is proportional to the ratio y/x and hence is constant along any radius such as OA (Fig. 149). This means that the resultant shearing stress along any radius OA has a constant direction which evidently coincides with the direction of the tangent to the boundary at the point A . Along the vertical axis OB the stress component τ_{yz} is zero, and the resultant stress is equal to τ_{xz} . Along the horizontal axis OD the resultant shearing stress is equal to τ_{yz} . It is evident that the maximum stress is at the boundary, and it can easily be proved that this maximum occurs at the ends of the minor axis of the ellipse. Substituting $y = b$ in the first of Eqs. (146), we find that the absolute value of this maximum is

$$\tau_{\max.} = \frac{2M_t}{\pi a b^2} \quad (147)$$

For $a = b$ this formula coincides with the well-known formula for a circular cross section.

Substituting (e) in Eq. (143) we find the expression for the angle of twist

$$\theta = M_t \cdot \frac{a^2 + b^2}{\pi a^3 b^3 G} \quad (148)$$

The factor by which we divide torque to obtain the twist per unit length is called the *torsional rigidity*. Denoting it by C , its value for the elliptic cross section, from (148), is

$$C = \frac{\pi a^3 b^3 G}{a^2 + b^2} = \frac{G}{4\pi^2} \frac{(A)^4}{I_p} \quad (149)$$

in which

$$A = \pi ab, \quad I_p = \frac{\pi ab^3}{4} + \frac{\pi ba^3}{4}$$

are the area and centroidal moment of inertia of the cross section.

Having the stress components (146) we can easily obtain the displacements. The components u and v are given by Eqs. (a) of Art. 90. The displacement w is found from Eqs. (d) and (b) of Art. 90. Substituting from Eqs. (146) and (148) and integrating, we find

$$w = M_t \frac{(b^2 - a^2)xy}{\pi a^3 b^3 G} \quad (150)$$

This shows that the contour lines for the warped cross section are hyperbolas having the principal axes of the ellipse as asymptotes (Fig. 150).

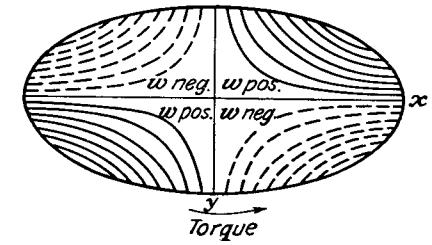


FIG. 150.

92. Other Elementary Solutions. In studying the torsional problem, Saint-Venant discussed several solutions of Eq. (142) in the form of polynomials. To solve the problem let us represent the stress function in the form

$$\phi = \phi_1 + \frac{F}{4} (x^2 + y^2) \quad (a)$$

Then, from Eq. (142),

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \quad (b)$$

and along the boundary, from Eq. (144),

$$\phi_1 + \frac{F}{4} (x^2 + y^2) = \text{constant} \quad (c)$$

Thus the torsional problem is reduced to obtaining solutions of Eq. (b) satisfying the boundary condition (c). To get solutions in the form of polynomials we take the function of the complex variable

$$(x + iy)^n \quad (d)$$

The real and the imaginary parts of this expression are each solutions of Eq. (b) (see page 182). Taking, for instance, $n = 2$ we obtain the solutions $x^2 - y^2$ and $2xy$. With $n = 3$ we obtain solutions $x^3 - 3xy^2$ and $3x^2y - y^3$. With $n = 4$, we arrive at solutions in the form of homogeneous functions of the fourth degree, and so on. Combining such solutions we can obtain various solutions in the form of polynomials.

Taking, for instance,

$$\phi = \frac{F}{4} (x^2 + y^2) + \phi_1 = \frac{F}{2} \left[\frac{1}{2} (x^2 + y^2) - \frac{1}{2a} (x^3 - 3xy^2) + b \right] \quad (e)$$

we obtain a solution of Eq. (142) in the form of a polynomial of the third degree with constants a and b which will be adjusted later. This polynomial is a solution of the torsional problem if it satisfies the boundary condition (144), i.e., if the boundary of the cross section of the bar is given by the equation

$$\frac{1}{2} (x^2 + y^2) - \frac{1}{2a} (x^3 - 3xy^2) + b = 0 \quad (f)$$

By changing the constant b in this equation, we obtain various shapes of the cross section.

Taking $b = -\frac{2}{27}a^2$ we arrive at the solution for the equilateral triangle. Equation (f) in this case can be presented in the form

$$(x - \sqrt{3}y - \frac{2}{3}a)(x + \sqrt{3}y - \frac{2}{3}a)(x + \frac{1}{3}a) = 0$$

which is the product of the three equations of the sides of the triangle shown in Fig. 151. Observing that $F = -2G\theta$ and substituting

$$\phi = -G\theta \left[\frac{1}{2} (x^2 + y^2) - \frac{1}{2a} (x^3 - 3xy^2) - \frac{2}{27} a^2 \right] \quad (g)$$

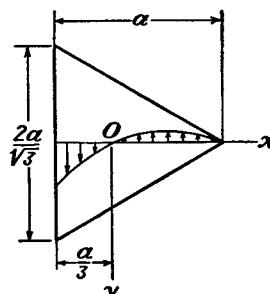


FIG. 151.

into Eqs. (141), we obtain the stress components τ_{xz} and τ_{yz} . Along the x -axis, $\tau_{xz} = 0$, from symmetry, and we find, from (g),

$$\tau_{yz} = \frac{3G\theta}{2a} \left(\frac{2ax}{3} - x^2 \right) \quad (h)$$

The largest stress is found at the middle of the sides of the triangle, where, from (h),

$$\tau_{\max.} = \frac{G\theta a}{2} \quad (k)$$

At the corners of the triangle the shearing stress is zero (see Fig. 151).

Substituting (g) into Eq. (145), we find

$$M_t = \frac{G\theta a^4}{15\sqrt{3}} = \frac{3}{5} \theta GI_p \quad (l)$$

Taking a solution of Eq. (142) in the form of a polynomial of the fourth degree containing only even powers of x and y , we obtain the stress function

$$\phi = -G\theta \left[\frac{1}{2} (x^2 + y^2) - \frac{a}{2} (x^4 - 6x^2y^2 + y^4) + \frac{1}{2} (a - 1) \right]$$

The boundary condition (144) is satisfied if the boundary of the cross section is given by the equation

$$x^4 + y^4 - a(x^4 - 6x^2y^2 + y^4) + a - 1 = 0$$

By changing a , Saint-Venant obtained the family of cross sections shown in Fig. 152a. Combining solutions in the form of polynomials of the fourth and eighth degrees, Saint-Venant arrived at the cross section shown in Fig. 152b.

On the basis of his investigations, Saint-Venant drew certain general conclusions of practical interest. He showed that, in the case of singly connected boundaries and for a given cross-sectional area, the torsional rigidity increases, if the polar moment of inertia of the cross section decreases. Thus for a given amount of material the circular shaft gives the largest torsional rigidity. Similar conclusions can be drawn regarding the maximum shearing stress. For a given torque and cross-

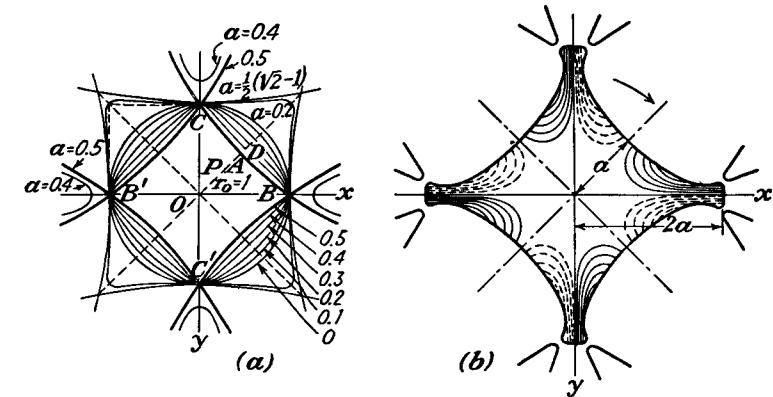


FIG. 152.

sectional area the maximum stress is the smallest for the cross section with the smallest polar moment of inertia.

Comparing various cross sections with singly connected boundaries, Saint-Venant found that the torsional rigidity can be calculated approximately by using Eq. (149), i.e., by replacing the given shaft by the shaft of an elliptic cross section having the same cross-sectional area and the same polar moment of inertia as the given shaft has.

The maximum stress in all cases discussed by Saint-Venant was obtained at the boundary at the points which are the nearest to the centroid of the cross section. A more detailed investigation of this question by Filon¹ showed that there are cases where the points of maximum stress, although always at the boundary, are not the nearest points to the centroid of the cross section.

¹L. N. G. Filon, *Trans. Roy. Soc. (London)*, series A, vol. 193, 1900. See also the paper by G. Polya, *Z. angew. Math. Mech.*, vol. 10, p. 353, 1930.

Taking $n = 1$ and $n = -1$ in expression (d), and using polar coordinates r and ψ , we obtain the following solutions of Eq. (b):

$$\phi_1 = r \cos \psi, \quad \phi_2 = \frac{1}{r} \cos \psi$$

Then the stress function (a) can be taken in the form

$$\phi = \frac{F}{4} (x^2 + y^2) - \frac{Fa}{2} r \cos \psi + \frac{Fb^2}{2} \frac{a}{r} \cos \psi - \frac{F}{4} b^2 \quad (m)$$

in which a and b are constants. It will satisfy the boundary condition (144) if at the boundary of the cross section we have $\phi = 0$, or, from (m),

$$r^2 - b^2 - 2a(r^2 - b^2) \frac{\cos \psi}{r} = 0 \quad (n)$$

or

$$(r^2 - b^2) \left(1 - \frac{2a \cos \psi}{r} \right) = 0 \quad (o)$$

which represents the equation of the boundary of the cross section shown in Fig. 153.¹ By taking

$$r^2 - b^2 = 0$$

we obtain a circle of radius b with the center at the origin; and by taking

$$1 - \frac{2a \cos \psi}{r} = 0$$

we have a circle of radius a touching the y -axis at the origin. The maximum shearing stress is at the point A and is

$$\tau_{\max.} = G\theta(2a - b) \quad (p)$$

When b is very small in comparison with a , i.e., when we have a semicircular longitudinal groove of very small radius, the stress at the bottom of the groove is twice as great as the maximum stress in the circular shaft of radius a without the groove.

93. Membrane Analogy. In the solution of torsional problems the *membrane analogy*, introduced by L. Prandtl,² has proved very valuable. Imagine a homogeneous membrane (Fig. 154) supported at the edges, with the same outline as that of the cross section of the twisted

¹ This problem was discussed by C. Weber, *Forschungsarbeiten*, No. 249, 1921.

² *Physik. Z.*, vol. 4, 1903. See also Anthes, *Dinglers polytech. J.*, p. 342, 1906. Further development of the analogy and applications in various cases are given in the papers by A. A. Griffith and G. I. Taylor, *Tech. Rept. Adv. Comm. Aeronautics*, vol. 3, pp. 910 and 938, 1917-1918.

bar, subjected to a uniform tension at the edges and a uniform lateral pressure. If q is the pressure per unit area of the membrane and S is the uniform tension per unit length of its boundary, the tensile forces acting on the sides ad and bc of an infinitesimal element $abcd$ (Fig. 154) give, in the case of small deflections of the membrane, a resultant in the upward direction $-S(\partial^2 z / \partial x^2) dx dy$. In the same manner the tensile

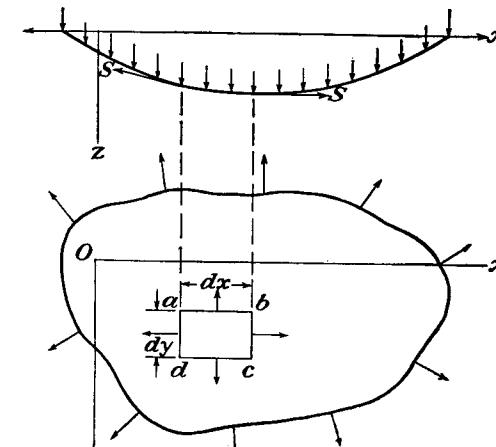


FIG. 154.

forces acting on the other two sides of the element give the resultant $-S(\partial^2 z / \partial y^2) dx dy$ and the equation of equilibrium of the element is

$$q dx dy + S \frac{\partial^2 z}{\partial x^2} dx dy + S \frac{\partial^2 z}{\partial y^2} dx dy = 0$$

from which

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{S} \quad (151)$$

At the boundary the deflection of the membrane is zero. Comparing Eq. (151) and the boundary condition for the deflections z of the membrane with Eq. (142) and the boundary condition (144) (see page 261) for the stress function ϕ , we conclude that these two problems are identical. Hence from the deflections of the membrane we can obtain values of ϕ by replacing the quantity $-(q/S)$ of Eq. (151) with the quantity $F = -2G\theta$ of Eq. (142).

Having the deflection surface of the membrane represented by contour lines (Fig. 155), several important conclusions regarding stress distribution in torsion can be obtained. Consider any point B on the

membrane. The deflection of the membrane along the contour line through this point is constant, and we have

$$\frac{\partial z}{\partial s} = 0$$

The corresponding equation for the stress function ϕ is

$$\frac{\partial \phi}{\partial s} = \left(\frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} \right) = \tau_{xz} \frac{dy}{ds} - \tau_{yz} \frac{dx}{ds} = 0$$

This expresses that the projection of the resultant shearing stress at a point B on the normal N to the contour line is zero and therefore we

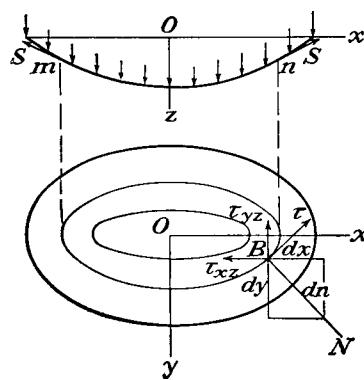


FIG. 155.

may conclude that the shearing stress at a point B in the twisted bar is in the direction of the tangent to the contour line through this point. The curves drawn in the cross section of a twisted bar, in such a manner that the resultant shearing stress at any point of the curve is in the direction of the tangent to the curve, are called *lines of shearing stress*. Thus the contour lines of the membrane are the lines of shearing stress for the cross section of the twisted bar.

The magnitude of the resultant stress τ at B (Fig. 155) is obtained by projecting on the tangent the stress components τ_{xz} and τ_{yz} . Then

$$\tau = \tau_{yz} \cos(Nx) - \tau_{xz} \cos(Ny)$$

Substituting

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}, \quad \cos(Nx) = \frac{dx}{dn}, \quad \cos(Ny) = \frac{dy}{dn}$$

we obtain

$$\tau = -\left(\frac{\partial \phi}{\partial x} \frac{dx}{dn} + \frac{\partial \phi}{\partial y} \frac{dy}{dn}\right) = -\frac{d\phi}{dn}$$

Thus the magnitude of the shearing stress at B is given by the maximum slope of the membrane at this point. It is only necessary in the expression for the slope to replace q/S by $2G\theta$. From this it can be concluded that the maximum shear acts at the points where the contour lines are closest to each other.

From Eq. (145) it can be concluded that double the volume bounded by the deflected membrane and the xy -plane (Fig. 155) represents the torque, provided q/S is replaced by $2G\theta$.

It may be observed that the form of the membrane, and therefore the stress distribution, is the same no matter what point in the cross section is taken for origin in the torsion problem. This point, of course, represents the axis of rotation of the cross sections. It is at first sight surprising that the cross sections can rotate about a different (parallel) axis when still subjected to the same torque. The difference, however, is merely a matter of rigid body rotation. Consider, for instance, a circular cylinder twisted by rotations about the central axis. A generator on the surface becomes inclined to its original direction, but can be brought back by a rigid body rotation of the whole cylinder about a diameter. The final positions of the cross sections then correspond to torsional rotations about this generator as a fixed axis. The cross sections remain plane but become inclined to their original planes in virtue of the rigid body rotation of the cylinder. In an arbitrary section there will be warping, and with a given choice of axis the inclination of a given element of area in the end section is definite, $\partial w/\partial x$ and $\partial w/\partial y$ being given by Eqs. (d) and (b) of Art. 90. Such an element can be brought back to its original orientation by a rigid body rotation about an axis in the end section. This rotation will change the axis of the torsional rotations to a parallel axis. Thus a definite axis or center of torsional rotation, or *center of torsion*, can be identified provided the final orientation of an element of area in the end section is specified—as for instance if the element is completely fixed.

Let us consider now the equilibrium condition of the portion mn of the membrane bounded by a contour line (Fig. 155). The slope of the membrane along this line is proportional at each point to the shearing stress τ and equal to $\tau \cdot q/S \cdot 1/2G\theta$. Then denoting by A the horizontal projection of the portion mn of the membrane, the equation of equilibrium of this portion is

$$\int S \left(\tau \frac{q}{S} \frac{1}{2G\theta} \right) ds = qA$$

or

$$\int \tau ds = 2G\theta A \quad (152)$$

From this the average value of the shearing stress along a contour line can be obtained.

By taking $q = 0$, i.e., considering a membrane without lateral load, we arrive at the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad (153)$$

which coincides with Eq. (b) of the previous article for the function ϕ_1 . Taking the ordinates of the membrane at the boundary so that

$$z + \frac{F}{4} (x^2 + y^2) = \text{constant} \quad (154)$$

the boundary condition (c) of the previous article is also satisfied. Thus we can obtain the function ϕ_1 from the deflection surface of an unloaded membrane, provided the ordinates of the membrane surface have definite values at the boundary. It will be shown later that both loaded and unloaded membranes can be used for determining stress distributions in twisted bars by experiment.

The membrane analogy is useful, not only when the bar is twisted within the elastic limit, but also when the material yields in certain portions of the cross section.¹ Assuming that the shearing stress remains constant during yielding, the stress distribution in the elastic zone of the cross section is represented by the membrane as before, but in the plastic zone the stress will be given by a surface having a constant

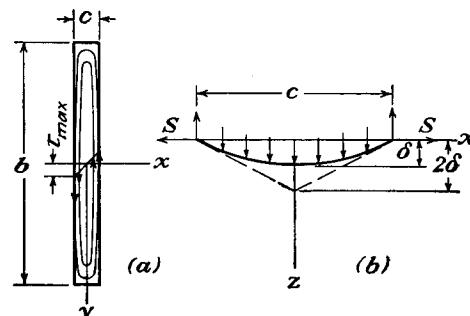


FIG. 156.

maximum slope corresponding to the yield stress. Imagine such a surface constructed as a roof on the cross section of the bar and the membrane stretched and loaded as explained before. On increasing the pressure we arrive at the condition when the membrane begins to touch the roof. This corresponds to the beginning of plastic flow in the twisted bar. As the pressure is increased, certain portions of the membrane come into contact with the roof. These portions of contact give us the regions of plastic flow in the twisted bar. Interesting experiments illustrating this theory were made by A. Nádai.²

94. Torsion of a Bar of Narrow Rectangular Cross Section. In the case of a narrow rectangular cross section the membrane analogy gives a very simple solution of the torsional problem. Neglecting the effect of the short sides of the rectangle and assuming that the surface of the slightly deflected membrane is cylindrical (Fig. 156), we obtain the

¹ This was indicated by L. Prandtl; see A. Nádai, *Z. angew. Math. Mech.*, vol. 3, p. 442, 1923. See also E. Trefftz, *ibid.*, vol. 5, p. 64, 1925.

² See *Trans. A.S.M.E.*, Applied Mechanics Division, 1930. See also A. Nádai, "Theory of Flow and Fracture of Solids," 1950, Chaps. 35 and 36.

deflection of the membrane from the elementary formula for the parabolic deflection curve of a uniformly loaded string¹ (Fig. 156b),

$$\delta = \frac{qc^2}{8S} \quad (a)$$

From the known properties of parabolic curves, the maximum slope, which occurs in the middle portions of the long sides of the rectangle, is equal to

$$\frac{4\delta}{c} = \frac{qc}{2S} \quad (b)$$

The volume bounded by the deflected membrane and the xy -plane, calculated as for a parabolic cylinder, is

$$V = \frac{2}{3} c \delta b = \frac{qbc^3}{12S} \quad (c)$$

Now using the membrane analogy and substituting $2G\theta$ for q/S in (b) and (c), we find

$$\tau_{\max.} = cG\theta, \quad M_t = \frac{1}{3}bc^3G\theta \quad (d)$$

from which

$$\theta = \frac{M_t}{\frac{1}{3}bc^3G} \quad (155)$$

$$\tau_{\max.} = \frac{M_t}{\frac{1}{3}bc^2} \quad (156)$$

From the parabolic deflection curve (Fig. 156b)

$$z = \frac{4\delta}{c^2} \left(\frac{c^2}{4} - x^2 \right)$$

and the slope of the membrane at any point is

$$\frac{dz}{dx} = - \frac{8\delta x}{c^2} = - \frac{q}{S} x$$

The corresponding stress in the twisted bar is

$$\tau_{yz} = 2G\theta x$$

The stress distribution follows a linear law as shown in Fig. 156a. Calculating the magnitude of the torque corresponding to this stress distribution we find

$$\frac{\tau_{\max.}}{4} c \cdot \frac{2}{3} c \cdot b = \frac{1}{6} bc^2 \tau_{\max.}$$

¹ See S. Timoshenko and D. H. Young, "Engineering Mechanics," p. 35.

This is only one half of the total torque given by Eq. (156). The second half is given by the stress components τ_{zz} , which were entirely neglected when we assumed that the surface of the deflected membrane is cylindrical. Although these stresses have an appreciable magnitude only near the short sides of the rectangle and their maximum values are smaller than τ_{\max} , as calculated above, they act at a greater distance from the axis of the bar and their moment represents the second half of the torque M_t .¹

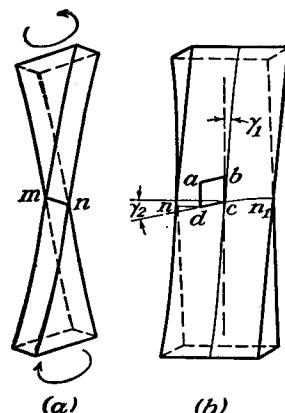


FIG. 157.

It is interesting to note that the τ_{\max} given by the first of Eqs. (d) is twice as great as in the case of a circular shaft with diameter equal to c and subjected to the same twist θ . This can be explained if we consider the warping of the cross sections. The sides of cross sections such as nn_1 (Fig. 157) remain normal to the longitudinal fibers of the bar at the corners, as is shown at the points n and n_1 . The total shear of an element such as $abcd$ consists of two parts: the part γ_1 due to rotation of the cross section about the axis of the bar and equal to the shear in the circular bar of diameter c ; and the part γ_2 due to warping of the cross section. In the case of a narrow rectangular cross section $\gamma_2 = \gamma_1$, and the resultant shear is twice as great as in the case of a circular cross section of the diameter c .

Equations (155) and (156), obtained above for a narrow rectangle, can also be used in the cases of thin-walled bars of such cross sections as shown in Fig. 158 by setting b equal to the developed length of the cross section. This follows from the fact that, if the thickness c of a slotted tube (Fig. 158a) is small in comparison with the diameter, the maximum slope of the membrane and the volume bounded by the membrane will be nearly the same as for a narrow rectangular cross section of the width c and of the same length as the circumference of the middle surface of the tube. An analogous conclusion can be made also for a channel (Fig. 158b). It should be noted that in this latter

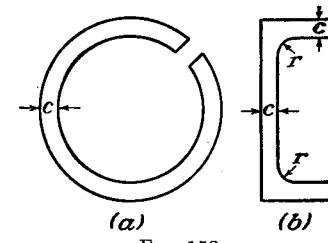


FIG. 158.

¹ This question was cleared up by Lord Kelvin; see Kelvin and Tait, "Natural Philosophy," vol. 2, p. 267.

case a considerable stress concentration takes place at the reentrant corners, depending on the magnitude of the radius r of the fillets, and Eq. (156) cannot be applied at these points. A more detailed discussion of this subject will be given in Art. 98.

95. Torsion of Rectangular Bars. Using the membrane analogy, the problem reduces to finding the deflections of a uniformly loaded rectangular membrane as shown in Fig. 159. These deflections must satisfy the Eq. (151)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{S} \quad (a)$$

and be zero at the boundary.

The condition of symmetry with respect to the y -axis and the boundary conditions at the sides $x = \pm a$ of the rectangle are satisfied by taking z in the form of a series,

$$z = \sum_{n=1,3,5,\dots}^{\infty} b_n \cos \frac{n\pi x}{2a} Y_n \quad (b)$$

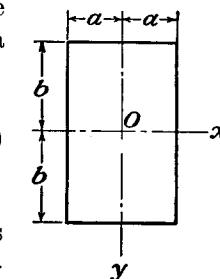


FIG. 159.

in which b_1, b_3, \dots are constant coefficients and Y_1, Y_3, \dots are functions of y only. Substituting (b) in Eq. (a), and observing that the right side of this equation can be presented in the form of a series,¹

$$-\frac{q}{S} = -\sum_{n=1,3,5,\dots}^{\infty} \frac{q}{S} \frac{4}{n\pi} (-1)^{\frac{n-1}{2}} \cos \frac{n\pi x}{2a} \quad (c)$$

we arrive at the following equation for determining Y_n :

$$Y_n'' - \frac{n^2\pi^2}{4a^2} Y_n = -\frac{q}{S} \frac{4}{n\pi b_n} (-1)^{\frac{n-1}{2}} \quad (d)$$

from which

$$Y_n = A \sinh \frac{n\pi y}{2a} + B \cosh \frac{n\pi y}{2a} + \frac{16qa^2}{Sn^3\pi^3b_n} (-1)^{\frac{n-1}{2}} \quad (e)$$

From the condition of symmetry of the deflection surface of the membrane with respect to the x -axis, it follows that the constant of integration A must be zero. The constant B is determined from the

¹ B. O. Peirce, "A Short Table of Integrals," p. 95, 1910.

condition that the deflections of the membrane are zero for $y = \pm b$, i.e., $(Y_n)_{y=\pm b} = 0$, which gives

$$Y_n = \frac{16qa^2}{S\pi^3 b_n} (-1)^{\frac{n-1}{2}} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)} \right] \quad (f)$$

and the general expression for the deflection surface of the membrane, from (b), becomes

$$z = \frac{16qa^2}{S\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)} \right] \cos \frac{n\pi x}{2a}$$

Replacing q/S by $2G\theta$, we obtain for the stress function

$$\phi = \frac{32G\theta a^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)} \right] \cos \frac{n\pi x}{2a} \quad (g)$$

The stress components are now obtained from Eqs. (141) by differentiation. For instance,

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = \frac{16G\theta a}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} (-1)^{\frac{n-1}{2}} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)} \right] \sin \frac{n\pi x}{2a} \quad (h)$$

Assuming that $b > a$, the maximum shearing stress, corresponding to the maximum slope of the membrane, is at the middle points of the long sides $x = \pm a$ of the rectangle. Substituting $x = a, y = 0$ in (h), we find

$$\tau_{\max.} = \frac{16G\theta a}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \left[1 - \frac{1}{\cosh(n\pi b/2a)} \right]$$

or, observing that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

we have

$$\tau_{\max.} = 2G\theta a - \frac{16G\theta a}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2 \cosh(n\pi b/2a)} \quad (157)$$

The infinite series on the right side, for $b > a$, converges very rapidly and there is no difficulty in calculating $\tau_{\max.}$ with sufficient accuracy for any particular value of the ratio b/a . For instance, in the case of a

very narrow rectangle, b/a becomes a large number, so that the sum of the infinite series in (157) can be neglected, and we find

$$\tau_{\max.} = 2G\theta a$$

This coincides with the first of the Eqs. (d) of the previous article.

In the case of a square cross section, $a = b$; and we find, from Eq. (157),

$$\begin{aligned} \tau_{\max.} &= 2G\theta a \left\{ 1 - \frac{8}{\pi^2} \left[\frac{1}{\cosh(\pi/2)} + \frac{1}{9 \cosh(3\pi/2)} + \dots \right] \right\} \\ &= 2G\theta a \left[1 - \frac{8}{\pi^2} \left(\frac{1}{2.509} + \frac{1}{9 \times 55.67} + \dots \right) \right] \\ &= 1.351G\theta a \end{aligned} \quad (158)$$

In general we obtain

$$\tau_{\max.} = k2G\theta a \quad (159)$$

in which k is a numerical factor depending on the ratio b/a . Several values of this factor are given in the table below.

TABLE OF CONSTANTS FOR TORSION OF A RECTANGULAR BAR

$\frac{b}{a}$	k	k_1	k_2	$\frac{b}{a}$	k	k_1	k_2
1.0	0.675	0.1406	0.208	3	0.985	0.263	0.267
1.2	0.759	0.166	0.219	4	0.997	0.281	0.282
1.5	0.848	0.196	0.231	5	0.999	0.291	0.291
2.0	0.930	0.229	0.246	10	1.000	0.312	0.312
2.5	0.968	0.249	0.258	∞	1.000	0.333	0.333

Let us calculate now the torque M_t as a function of the twist θ . Using Eq. (145) for this purpose, we find

$$\begin{aligned} M_t &= 2 \int_{-a}^a \int_{-b}^b \phi \, dx \, dy = \frac{64G\theta a^2}{\pi^3} \int_{-a}^a \int_{-b}^b \left\{ \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} (-1)^{\frac{n-1}{2}} \left[1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)} \right] \cos \frac{n\pi x}{2a} \right\} dx \, dy = \frac{32G\theta(2a)^3(2b)}{\pi^4} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4} \\ &\quad - \frac{64G\theta(2a)^4}{\pi^5} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^5} \tanh \frac{n\pi b}{2a} \end{aligned}$$

or, observing that¹

$$\frac{1}{1} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

¹ B. O. Peirce, "A Short Table of Integrals," p. 90, 1910.

we have

$$M_t = \frac{1}{3} G\theta(2a)^3(2b) \left(1 - \frac{192}{\pi^5} \frac{a}{b} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^5} \tanh \frac{n\pi b}{2a} \right) \quad (160)$$

The series on the right side converges very rapidly, and M_t can easily be evaluated for any value of the ratio a/b . In the case of a narrow rectangle we can take

$$\tanh \frac{n\pi b}{2a} = 1$$

Then

$$M_t = \frac{1}{3} G\theta(2a)^3(2b) \left(1 - 0.630 \frac{a}{b} \right) \quad (161)$$

In the case of a square, $a = b$; and (160) gives

$$M_t = 0.1406 G\theta(2a)^4 \quad (162)$$

In general the torque can be represented by the equation

$$M_t = k_1 G\theta(2a)^3(2b) \quad (163)$$

in which k_1 is a numerical factor depending on the magnitude of the ratio b/a . Several values of this factor are given in the table on page 277.

Substituting the value of θ from Eq. (163) into Eq. (159), we obtain the maximum shearing stress as a function of the torque in the form

$$\tau_{\max.} = \frac{M_t}{k_2(2a)^2(2b)} \quad (164)$$

where k_2 is a numerical factor the values of which can be taken from the table on page 277.

96. Additional Results. By using infinite series as in the previous article, the torsional problem can be solved for several other shapes of cross sections.

In the case of a sector of a circle¹ (Fig. 160) the boundaries are given by $\psi = \pm \alpha/2$, $r = 0$, $r = a$. We take a stress function in the form

$$\phi = \phi_1 + \frac{F}{4} (x^2 + y^2) = \phi_1 - \frac{G\theta r^2}{2}$$

¹ This problem was discussed by Saint-Venant, *Compt. rend.*, vol. 87, pp. 849 and 893, 1878. See also A. G. Greenhill, *Messenger of Math.*, vol. 9, p. 35, 1879. Another method of solution by using Bessel's function was given by A. Dinnik, *Bull. Don Polytech. Inst.*, Novotcherkassk, vol. 1, p. 309. See also A. Föppl and L. Föppl, "Drang und Zwang," p. 96, 1928.

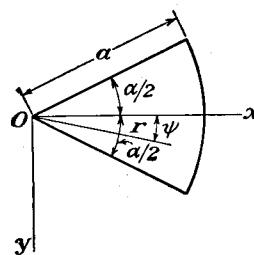


FIG. 160.

The function ϕ_1 must satisfy Laplace's equation (see Art. 92). Taking a solution of this equation in the form of the series

$$\phi_1 = \frac{G\theta}{2} \left[\frac{r^2 \cos 2\psi}{\cos \alpha} + a^2 \sum_{n=1,3,5,\dots}^{\infty} A_n \left(\frac{r}{a} \right)^{\frac{n\pi}{\alpha}} \cos \frac{n\pi\psi}{\alpha} \right]$$

we arrive at the stress function

$$\phi = \frac{G\theta}{2} \left[-r^2 \left(1 - \frac{\cos 2\psi}{\cos \alpha} \right) + a^2 \sum_{n=1,3,5,\dots}^{\infty} A_n \left(\frac{r}{a} \right)^{\frac{n\pi}{\alpha}} \cos \frac{n\pi\psi}{\alpha} \right]$$

This expression is zero at the boundaries:

$$\psi = \pm \frac{\alpha}{2}$$

To make it vanish also along the circular boundary $r = a$, we must put

$$\sum_{n=1,3,5,\dots}^{\infty} A_n \cos \frac{n\pi\psi}{\alpha} = 1 - \frac{\cos 2\psi}{\cos \alpha}$$

from which we obtain, in the usual way,

$$A_n = \frac{16\alpha^2}{\pi^3} (-1)^{\frac{n+1}{2}} \frac{1}{n \left(n + \frac{2\alpha}{\pi} \right) \left(n - \frac{2\alpha}{\pi} \right)}$$

The stress function is therefore

$$\phi = \frac{G\theta}{2} \left[-r^2 \left(1 - \frac{\cos 2\psi}{\cos \alpha} \right) + \frac{16a^2\alpha^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} (-1)^{\frac{n+1}{2}} \left(\frac{r}{a} \right)^{\frac{n\pi}{\alpha}} \frac{\cos \frac{n\pi\psi}{\alpha}}{n \left(n + \frac{2\alpha}{\pi} \right) \left(n - \frac{2\alpha}{\pi} \right)} \right]$$

Substituting into Eq. (145), we find $M_t = 2 \int \int \phi r d\psi dr = kGa^4\theta$, in which k is a factor depending on the angle α of the sector. Several values of k , calculated by Saint-Venant, are given below.

$\alpha =$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$k =$	0.0181	0.0349	0.0825	0.148	0.296	0.572 ¹	0.672 ¹	0.878 ²
$k_1 =$	0.452	0.622	0.719
$k_2 =$	0.490	0.652	0.849

¹ These figures have been corrected by M. Aisseu. See G. Pólya and G. Szegő, "Isoperimetric Inequalities in Mathematical Physics," p. 261, Princeton University Press, 1951.

² This figure has been corrected by Dinnik, *loc. cit.*

The maximum shearing stresses along the circular and along the radial boundaries are given by the formulas $k_1 G a \theta$ and $k_2 G a \theta$, respectively. Several values of k_1 and k_2 are given in the table on page 279.

The solution for a curvilinear rectangle bounded by two concentric circular arcs and two radii can be obtained in the same manner.¹

In the case of an isosceles right-angled triangle² the angle of twist is given by the equation

$$\theta = 38.3 \frac{M_t}{G a^4}$$

in which a is the length of the equal sides of the triangle. The maximum shearing stress is at the middle of the hypotenuse and is equal to

$$\tau_{\max.} = 18.02 \frac{M_t}{a^3}$$

By introducing curvilinear coordinates several other cross sections have been investigated. Taking elliptic coordinates (see page 193) and using conjugate functions ξ and η , determined by the equation

$$x + iy = c \cosh(\xi + i\eta)$$

we arrive at cross sections bounded by confocal ellipses and hyperbolae.³ By using the equation⁴

$$x + iy = \frac{1}{2}(\xi + i\eta)^2$$

we obtain cross sections bounded by orthogonal parabolas.

Solutions have been found for many other sections,⁵ solid and hollow, including polygons, angles, cardioids, lemniscates,⁶ and circles with one or several eccentric holes.⁷ When the section can be conformally mapped into the unit circle a solution can always be written down in terms of a complex integral.⁸

97. Solution of Torsional Problems by Energy Method.⁹ We have seen that the solution of torsional problems is reduced in each particu-

¹ Saint-Venant, *loc. cit.* See also A. E. H. Love, "Theory of Elasticity," 4th ed. p. 319, 1927.

² B. G. Galerkin, *Bull. acad. des sci. de Russ.*, p. 111, 1919; G. Kolosoff, *Compt. rend.*, vol. 178, p. 2057, 1924.

³ A. G. Greenhill, *Quart. J. Math.*, vol. 16, 1879. See also L. N. G. Filon, *Trans. Roy. Soc. (London)*, series A, vol. 193, 1900.

⁴ E. W. Anderson and D. L. Holl, *Iowa State Coll. J. Sci.*, vol. 3, p. 231, 1929.

⁵ A compilation is given by T. J. Higgins, *Am. J. Phys.*, vol. 10, p. 248, 1942.

⁶ References to papers giving exact solutions for such sections, too numerous to include here, may be found by consulting *Applied Mechanics Reviews*, *Science Abstracts A*, *Mathematical Reviews*, and *Zentralblatt für Mechanik*. Most of the references on p. 331 refer to or include the corresponding torsion problem.

⁷ See C. B. Ling, *Quart. Applied Math.*, vol. 5, p. 168, 1947.

⁸ Due to N. I. Muschel'svili. See I. S. Sokolnikoff, "Mathematical Theory of Elasticity," Chap. 4, 1946.

⁹ For a survey, with references, of this and other approximate methods see T. J. Higgins, *J. Applied Phys.*, vol. 14, p. 469, 1943.

lar case to the determination of the stress function satisfying the differential equation (142) and the boundary condition (144). In deriving an approximate solution of the problem it is useful, instead of working with the differential equation, to determine the stress function from the minimum condition of a certain integral,¹ which can be obtained from consideration of the strain energy of the twisted bar. The strain energy of the twisted bar per unit length, from (88), is

$$V = \frac{1}{2G} \int \int (\tau_{xz}^2 + \tau_{yz}^2) dx dy = \frac{1}{2G} \int \int \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] dx dy$$

If we give to the stress function ϕ any small variation $\delta\phi$, vanishing at the boundary,² the variation of the strain energy is

$$\frac{1}{2G} \delta \int \int \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] dx dy$$

and the variation of the torque is, from Eq. (145),

$$2 \int \int \delta \phi dx dy$$

Then by reasoning analogous to that used in developing equation (91) on page 164, we conclude that

$$\frac{1}{2G} \delta \int \int \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] dx dy = 2\theta \int \int \delta \phi dx dy$$

or

$$\delta \int \int \left\{ \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] - 2G\theta\phi \right\} dx dy = 0$$

Thus the true expression for the stress function ϕ is that which makes zero the variation of the integral

$$U = \int \int \left\{ \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right] - 2G\theta\phi \right\} dx dy \quad (165)$$

We come also to the same conclusion by using the membrane analogy and the principle of virtual work (Art. 48). If S is the uniform tension in the membrane, the increase in strain energy of the membrane due to deflection is obtained by multiplying the tension S by the increase of the surface of the membrane. In this manner we obtain

$$\frac{1}{2} S \int \int \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dx dy$$

¹ This method was proposed by W. Ritz, who used it in the solution of problems of bending and vibration of rectangular plates. See *J. reine angew. Math.*, vol. 135, 1908, and *Ann. Physik*, series 4, vol. 28, p. 737, 1909.

² If $\delta\phi$ is taken equal to zero at the boundary, no forces on the lateral surface of the bar will be introduced by variation of ϕ .

where z is the deflection of the membrane. If we take now a virtual displacement of the membrane from the position of equilibrium, the change in the strain energy of the membrane due to this displacement must be equal to the work done by the uniform load q on the virtual displacement. Thus we obtain

$$\frac{1}{2} S \delta \int \int \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] dx dy = \int \int q \delta z dx dy$$

and the determination of the deflection surface of the membrane is reduced to finding an expression for the function z which makes the integral

$$\int \int \left\{ \frac{1}{2} \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right] - \frac{q}{S} z \right\} dx dy$$

a minimum. If we substitute in this integral $2G\theta$ for q/S , we arrive at the integral (165) above.

In the approximate solution of torsional problems we replace the above problem of variational calculus by a simple problem of finding a minimum of a function. We take the stress function in the form of a series

$$\phi = a_0 \phi_0 + a_1 \phi_1 + a_2 \phi_2 + \dots \quad (a)$$

in which $\phi_0, \phi_1, \phi_2, \dots$ are functions satisfying the boundary condition, i.e., vanishing at the boundary. In choosing these functions we should be guided by the membrane analogy and take them in a form suitable for representing the function ϕ . The quantities a_0, a_1, a_2, \dots are numerical factors to be determined from the minimum condition of the integral (165). Substituting the series (a) in this integral we obtain, after integration, a function of the second degree in a_0, a_1, a_2, \dots , and the minimum condition of this function is

$$\frac{\partial U}{\partial a_0} = 0, \quad \frac{\partial U}{\partial a_1} = 0, \quad \frac{\partial U}{\partial a_2} = 0, \dots \quad (b)$$

Thus we obtain a system of linear equations from which the coefficients a_0, a_1, a_2, \dots can be determined. By increasing the number of terms in the series (a) we increase the accuracy of our approximate solution, and by using infinite series we may arrive at an exact solution of the torsional problem.¹

¹ The condition of convergency of this method of solution was investigated by Ritz, *loc. cit.* See also E. Trefftz, "Handbuch der Physik," vol. 6, p. 130, 1928.

Take as an example the case of a rectangular cross section¹ (Fig. 159). The boundary is given by the equations $x = \pm a$, $y = \pm b$, and the function $(x^2 - a^2)(y^2 - b^2)$ is zero at the boundary. The series (a) can be taken in the form

$$\phi = (x^2 - a^2)(y^2 - b^2) \sum a_{mn} x^m y^n \quad (c)$$

in which, from symmetry, m and n must be even.

Assuming that we have a square cross section and limiting ourselves to the first term of the series (c) we take

$$\phi = a_0 (x^2 - a^2)(y^2 - a^2) \quad (d)$$

Substituting this in (165) we find from the minimum condition that

$$a_0 = \frac{5}{8} \frac{G\theta}{a^2}$$

The magnitude of the torque, from Eq. (145), is then

$$M_t = 2 \int \int \phi dx dy = \frac{2}{9} G\theta a^4 = 0.1388(2a)^4 G\theta$$

Comparing this with the correct solution (162) we see that the error in the torque is about $1\frac{1}{3}$ per cent.

To get a closer approximation we take the three first terms in the series (c). Then, by using the condition of symmetry, we obtain

$$\phi = (x^2 - a^2)(y^2 - a^2)[a_0 + a_1(x^2 + y^2)] \quad (e)$$

Substituting this in (165) and using Eqs. (b), we find

$$a_0 = \frac{5}{8} \cdot \frac{259}{277} \cdot \frac{G\theta}{a^2}, \quad a_1 = \frac{5}{8} \cdot \frac{3}{2} \cdot \frac{35}{277} \cdot \frac{G\theta}{a^4}$$

Substituting in expression (145) for the torque, we obtain

$$M_t = \frac{2}{9} \left(\frac{259}{277} + \frac{3}{2} \cdot \frac{35}{277} \right) G\theta a^4 = 0.1404 G\theta (2a)^4$$

This value is only 0.15 per cent less than the correct value.

A much larger error is found in the magnitude of the maximum stress. Substituting (e) into expressions (141) for the stress components we find that the error in the maximum stress is about 4 per cent, and to get a better accuracy more terms of the series (c) must be taken.

It can be seen from the membrane analogy that in proceeding as explained above we generally get smaller values for the torque than the correct value. A perfectly flexible membrane, uniformly stretched at

¹ See S. Timoshenko, *Bull. Inst. Ways of Communication*, St. Petersburg, 1913, and *Proc. London Math. Soc.*, series 2, vol. 20, p. 389, 1921.

the boundary and uniformly loaded, is a system with an infinite number of degrees of freedom. Limiting ourselves to a few terms of the series (c) is equivalent to introducing into the system certain constraints, which reduce it to a system with a few degrees of freedom only. Such constraints can only reduce the flexibility of the system and diminish the volume bounded by the deflected membrane. Hence the torque, obtained from this volume, will generally be smaller than its true value.

E. Trefftz suggested¹ another method of approximate determination of the stress function ϕ . With this method the approximate magnitude of the torque is larger than its true value. Hence by using the Ritz and the Trefftz methods together the limits of error of the approximate solution can be established.

In using Ritz's method we are not limited to polynomials (c). We can take the functions $\phi_0, \phi_1, \phi_2, \dots$ of the series (a) in other forms suitable for the representation of the stress function ϕ . Taking, for instance, trigonometric functions, and observing the conditions of symmetry (Fig. 159), we obtain

$$\phi = \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} a_{mn} \cos \frac{m\pi x}{2a} \cos \frac{n\pi y}{2b} \quad (f)$$

Substituting in (165) and performing the integration, we find that

$$U = \frac{\pi^2 ab}{8} \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} a_{mn}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) - 2G\theta \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} a_{mn} \cdot \frac{16ab}{mn\pi^2} (-1)^{\frac{m+n-1}{2}}$$

Equations (b) become

$$\frac{\pi^2 ab}{4} a_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) - 2G\theta \cdot \frac{16ab}{mn\pi^2} (-1)^{\frac{m+n-1}{2}} = 0$$

and we find

$$a_{mn} = \frac{128G\theta b^2 (-1)^{\frac{m+n-1}{2}}}{\pi^4 mn(m^2\alpha^2 + n^2)}$$

¹ E. Trefftz, *Proc. Second Intern. Congr. Applied Mech.*, Zürich, 1926, p. 131. See also N. M. Basu, *Phil. Mag.*, vol. 10, p. 886, 1930.

where $\alpha = b/a$. Substituting in (f) we obtain the exact solution of the problem in the form of an infinite trigonometric series. The torque will then be

$$M_t = 2 \int_{-a}^a \int_{-b}^b \phi \, dx \, dy = \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{128G\theta b^2}{\pi^4 mn(m^2\alpha^2 + n^2)} \cdot \frac{32ab}{mn\pi^2} \quad (g)$$

This expression is brought into coincidence with expression (160) given before if we observe that

$$\frac{1}{m^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2(m^2\alpha^2 + n^2)} = \frac{\pi^4}{96m^2} \frac{\tanh \frac{m\alpha\pi}{2} - \frac{m\alpha\pi}{2}}{-\frac{1}{3}(m\alpha\pi/2)^3}$$

As another example, in the case of a narrow rectangle, when b is very large in comparison with a (Fig. 159), we may take, as a first approximation,

$$\phi = G\theta(a^2 - x^2) \quad (h)$$

which coincides with the solution discussed before (Art. 94). To get a better approximation satisfying the boundary condition at the short sides of the rectangle, we may take

$$\phi = G\theta(a^2 - x^2)[1 - e^{-\beta(b-y)}] \quad (k)$$

and choose the quantity β in such a manner as to make the integral (165) a minimum. In this way we find

$$\beta = \frac{1}{a} \sqrt{\frac{5}{2}}$$

Due to the exponential term in the brackets of expression (k) we obtain a stress distribution which practically coincides with that of the solution (h) at all points a considerable distance from the short sides of the rectangle. Near these sides the stress function (k) satisfies the boundary condition (144). Substituting (k) into equation (145) for the torque, we find

$$M_t = 2 \int_{-a}^a \int_{-b}^b \phi \, dx \, dy = \frac{1}{3} G\theta(2a)^3(2b) \left(1 - 0.632 \frac{a}{b} \right)$$

which is in very good agreement with Eq. (161) obtained before by using infinite series.

A polynomial expression for the stress function, analogous to expression (c) taken above for a rectangle, can be used successfully in all cases of cross sections bounded by a convex polygon. If

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0, \dots$$

are the equations of the sides of the polygon, the stress function can be taken in the form

$$\phi = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) \cdots (a_nx + b_ny + c_n) \sum a_{mn}x^ny^m$$

and the first few terms of the series are usually sufficient to get a satisfactory accuracy.

The energy method is also useful when the boundary of the cross section (Fig. 161) is given by two curves¹

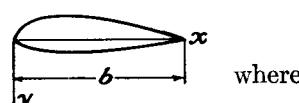


FIG. 161.

$$y = a\psi\left(\frac{x}{b}\right) \quad \text{and} \quad y = -a_1\psi\left(\frac{x}{b}\right)$$

$$\psi\left(\frac{x}{b}\right) = \psi(t) = (t)^m[1 - (t)^p]^q$$

The boundary conditions will be satisfied if we take for the stress function an approximate expression

$$\phi = A(y - a\psi)(y + a_1\psi)$$

Substituting into the integral (165) we find, from the equation $dI/dA = 0$,

$$A = -\frac{G\theta}{1 + \alpha(a^2 + a_1^2 + aa_1)/b^2}$$

where

$$\alpha = \frac{\int_0^1 \psi^3 (d\psi/dt)^2 dt}{\int_0^1 \psi^3 dt}$$

From Eq. (145) we find the torque

$$M_t = -A \frac{b(a + a_1)^3}{3} \int_0^1 \psi^3 dt$$

¹ Such problems were discussed by L. S. Leibenson. See his book "Variational Methods for Solving Problems of the Theory of Elasticity," Moscow, 1943. See also W. J. Duncan, *Phil. Mag.*, series 7, vol. 25, p. 634, 1938.

In the particular case when $m = \frac{1}{2}$, $p = q = 1$, $a = a_1$, we have $y = \pm a\psi(x/b) = \pm \sqrt{x/b}[1 - (x/b)]$, and we obtain

$$A = -\frac{G\theta}{1 + \frac{11}{13} \frac{a^2}{b^2}}, \quad M_t = 0.0736 \frac{G\theta ba^3}{1 + \frac{11}{13} \frac{a^2}{b^2}}$$

An approximate solution, and a comparison with tests, for sections bounded by a circle and a chord has been given by A. Weigand.¹ Numerical methods are discussed in the Appendix.

98. Torsion of Rolled Profile Sections. In investigating the torsion of rolled sections such as angles, channels, and I-beams, the formulas derived for narrow rectangular bars (Art. 94) can be used. If the cross section is of constant thickness, as in Fig. 162a, the angle of twist

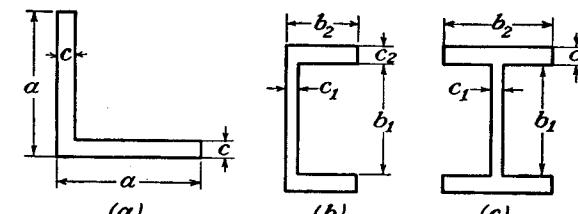


FIG. 162.

is obtained with sufficient accuracy from Eq. (155) by putting, instead of b , in this equation the developed length of the center line,² namely, $b = 2a - c$. In the case of a channel section (Fig. 162b) a rough approximation for the angle of twist is obtained by taking for the flanges an average thickness c_2 , subdividing the cross section into the three rectangles, and substituting in Eq. (155), $b_1c_1^3 + 2b_2c_2^3$ instead of bc^3 , i.e., assuming that the torsional rigidity of the channel is equal to the sum of the torsional rigidities of the three rectangles.³ Then

$$\theta = \frac{3M_t}{(b_1c_1^3 + 2b_2c_2^3)G} \quad (a)$$

¹ *Luftfahrt-forsch.*, vol. 20, 1944, translated as N.A.C.A. Tech. Mem. 1182, 1948.

² A more elaborate formula, taking account of the increased stiffness resulting from the junctions of the rectangles, was developed on the basis of soap film and torsion tests by G. W. Trayer and H. W. March, *Natl. Advisory Comm. Aeronaut.*, Rept. 334, 1930.

³ Comparison of torsional rigidities obtained in this manner with those obtained by experiments is given for several types of rolled sections and for various dimensions in the paper by A. Föppl, *Sitzber. Bayer. Akad. Wiss.*, p. 295, München, 1921. See also *Bauingenieur*, series 5, vol. 3, p. 42, 1922.

To calculate the stress at the boundary at points a considerable distance from the corners of the cross section we can use once more the equation for a narrow rectangle and take

$$\tau = c\theta G$$

Then, from Eq. (a), we obtain for the flanges of the channel

$$\tau = \frac{3M_t c_2}{b_1 c_1^3 + 2b_2 c_2^3} \quad (b)$$

The same approximate equations can be used for an I-beam (Fig. 162c).

At reentrant corners there is a considerable stress concentration, the magnitude of which depends on the radius of the fillets. A rough

approximation for the maximum stress at these fillets can be obtained from the membrane analogy. Let us consider a cross section in the form of an angle of constant thickness c (Fig. 163) and with radius a of the fillet of the reentrant corner. Assuming that the surface of the membrane at the bisecting line OO_1 of the fillet is approximately a surface of revolution, with axis perpendicular to the plane of the figure at O , and using polar coordinates, the Eq. (151) of the deflection surface of the membrane becomes (see page 57)

$$\frac{d^2z}{dr^2} + \frac{1}{r} \frac{dz}{dr} = -\frac{q}{S} \quad (c)$$

Remembering that the slope of the membrane dz/dr gives the shearing stress τ when q/S is replaced by $2G\theta$, we find from (c) the following equation for the shearing stress:

$$\frac{d\tau}{dr} + \frac{1}{r} \tau = -2G\theta \quad (d)$$

The corresponding equation in the arms of the angle at a considerable distance from the corners, where the membrane has a nearly cylindrical surface, is

$$\frac{d\tau}{dn} = -2G\theta \quad (e)$$

in which n is the normal to the boundary. Denoting by τ_1 the stress at the boundary we find from (e) the previously found solution for a

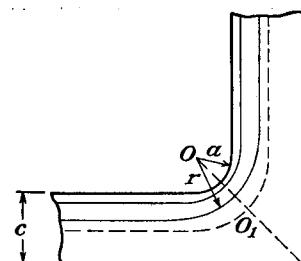


FIG. 163.

narrow rectangle $\tau_1 = G\theta c$. Using this, we obtain from (d)

$$\frac{d\tau}{dr} + \frac{1}{r} \tau = -\frac{2\tau_1}{c} \quad (d')$$

from which, by integration,

$$\tau = \frac{A}{r} - \frac{\tau_1 r}{c} \quad (f)$$

where A is a constant of integration. For the determination of this constant, let us assume that the shearing stress becomes zero at point O_1 at a distance $c/2$ from the boundary (Fig. 163). Then, from (f),

$$\frac{A}{a + (c/2)} - \frac{\tau_1[a + (c/2)]}{c} = 0$$

and

$$A = \frac{\tau_1}{c} \left(a + \frac{c}{2} \right)^2$$

Substituting in (f) and taking $r = a$, we find

$$\tau_{\max} = \tau_1 \left(1 + \frac{c}{4a} \right) \quad (g)$$

For $a = \frac{1}{2}c$, as in the Fig. 163, we have $\tau_{\max} = 1.5\tau_1$.

For a very small radius of fillet the maximum stress becomes very high. Taking, for instance, $a = 0.1c$ we find $\tau_{\max} = 3.5\tau_1$.

More accurate and complete results can be obtained by numerical calculations based on the method of finite differences (see Appendix). A curve of τ_{\max}/τ_1 as a function of a/c obtained by this method¹ is shown in Fig. 164 (curve A), together with the curve representing Eq. (g). It will be seen that this simple formula gives good results when a/c is less than 0.3.

99. The Use of Soap Films in Solving Torsion Problems. We have seen that the membrane analogy is very useful in enabling us to visualize the stress distribution over the cross section of a twisted bar.

¹ By J. H. Huth, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 17, p. 388, 1950. The rise of the curve towards the right is required by the limiting case as the fillet radius is increased in relation to the leg thickness. References to earlier attempts to solve this problem including soap-film measurements are given by I. Lyse and B. G. Johnston, *Proc. A.S.C.E.*, 1935, p. 469, and in the above paper.

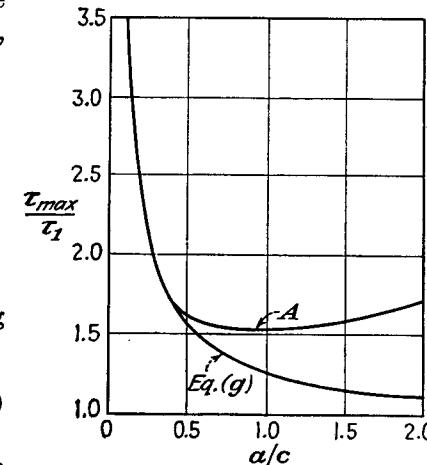


FIG. 164.

Membranes in the form of soap films have also been used for direct measurements of stresses.^{1,2} The films were formed on holes cut to the required shapes in flat plates. To make possible the direct determination of stresses, it was found necessary to have in the same plate a circular hole to represent a circular section for comparison. Submitting

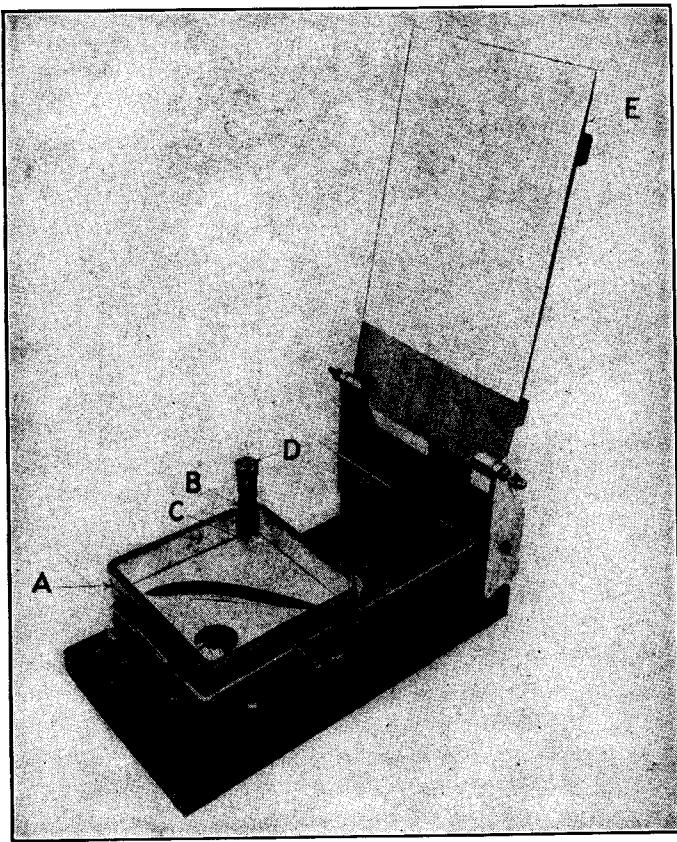


FIG. 165.

both films to the same pressure, we have the same values of q/S ,³ which correspond to the same values of $G\theta$ for the two bars under twist. Hence, by measuring the slopes of the two soap films we can compare the stresses in the bar of the given cross section with those in a circular

¹ See papers by Taylor and Griffith, *loc. cit.*; also the paper by Trayer and March, *loc. cit.*

² A survey of this and other analogies for torsion, with references, is given by T. J. Higgins, *Experimental Stress Analysis*, vol. 2, no. 2, p. 17, 1945.

³ It is assumed that the surface tension is the same in both films. This was proved with sufficient accuracy by the tests.

shaft under the condition that they have the same angle of twist θ per unit length and the same G . The corresponding ratio of the torques is determined by the ratio of the volumes between the soap films and the plane of the plate.

For obtaining the contour lines of the films the apparatus shown in Fig. 165 was used.¹ The aluminum plate with the holes is clamped between the two halves of the cast-iron box *A*. The lower part of the box, having the form of a shallow tray, is supported on leveling screws. The mapping of contour lines is done by using the screw *B* passing through a hole in a sheet of plate glass sufficiently large to cover the box in any possible position. The lower end of the screw carries a hard steel point whose distance from the glass plate is adjustable by the screw. The point is made to approach the film by moving the glass plate until the distortion of the image in the film shows that contact has occurred. The record is made on a sheet of paper attached to the board *E*, which can swing about a horizontal axis at the same height as the steel recording point *D*. To mark any position of the screw, it is only necessary to prick a dot on the paper by swinging it down on the recording point. After the point has been made to touch the film at a number of places, the dots recorded on the paper are used for drawing a contour line. By adjusting the screw *B* this can be repeated for as many contour lines as may be required. When these lines have been mapped, the volume and the corresponding torque can be obtained by summation. The slopes and the corresponding stresses are obtained by measuring the distances between neighboring contour lines. The slope can be obtained optically with much more accuracy by projecting a beam of light on to the surface of the film and measuring the angle of the reflected ray. The normal to the film is then half way between the incident and the reflected rays. A special instrument was constructed for this purpose by Griffith and Taylor. Figure 166 represents an example of contour lines obtained for a portion of an I-beam (wooden wing spar of an airplane). From the close grouping of the contour lines at the fillets of the reentrant corners and at the middle of the upper face, it follows that the shearing stresses are high at these places. The projecting parts of the flange are very lightly stressed. The

¹ See the paper by Taylor and Griffith, *loc. cit.*

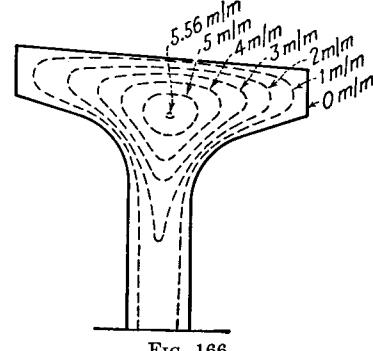


FIG. 166.

maximum stress in the middle portion of the web is practically constant along the side of the web and equal to that in a narrow rectangle for the same angle of twist. The application of soap-film measurements to such cross sections as ellipses and rectangles, for which exact solutions are known, shows that the maximum stress and the torque can be measured with an accuracy of 1 or 2 per cent. At the points of great stress concentration, as in the case of fillets of very small radii, an accuracy of the same order is not readily obtained.¹

100. Hydrodynamical Analogies. There are several analogies between the torsional problem and the hydrodynamical problem of the motion of fluid in a tube. Lord Kelvin² pointed out that the function [ϕ_1 see Eq. (a), Art. 92] which is sometimes used in the solution of torsional problems is identical with the *stream function* of a certain irrotational motion of "ideal fluid" contained in a vessel of the same cross section as the twisted bar.

Another analogy was indicated by J. Boussinesq.³ He showed that the differential equation and the boundary condition for determining the stress function ϕ (see Eqs. 142 and 144, Art. 90) are identical with those for determining velocities in a laminar motion of viscous fluid along a tube of the same cross section as the twisted bar.⁴

Greenhill showed that the stress function ϕ is mathematically identical with the stream function of a motion of ideal fluid circulating with uniform vorticity,⁵ in a tube of the same cross section as the twisted bar.⁶ Let u and v be the components of the velocity of the circulating fluid at a point A (Fig. 167). Then from the condition of incompressibility of the ideal fluid we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (a)$$

¹ See the paper by C. B. Biezeno and J. M. Rademaker, *De Ingenieur*, No. 52, 1931. See also papers by P. A. Cushman, *Trans. A.S.M.E.*, 1932, H. Quest, *Ingenieur-Archiv.*, vol. 4, p. 510, 1933, and J. H. Huth, *loc. cit.*

² Kelvin and Tait, "Natural Philosophy," pt. 2, p. 242.

³ J. Boussinesq, *J. math. pure et appl.*, series 2, vol. 16, 1871.

⁴ This analogy was used by M. Paschoud, *Compt. rend.*, vol. 179, p. 451, 1924. See also *Bull. tech. Suisse Rom.* (Lausanne), November, 1925.

⁵ The analytical expression for vorticity is the same as for rotation ω , discussed on p. 225, provided u and v denote the components of the velocity of the fluid.

⁶ A. G. Greenhill, *Hydromechanics*, an article in the *Encyclopaedia Britannica*, 11th ed., 1910.

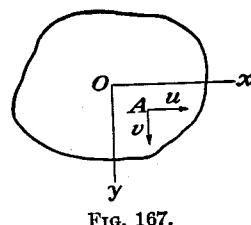


FIG. 167.

The condition of uniform vorticity is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{constant} \quad (b)$$

By taking

$$u = \frac{\partial \phi}{\partial y}, \quad v = -\frac{\partial \phi}{\partial x} \quad (c)$$

we satisfy Eq. (a), and from Eq. (b) we find

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \text{constant} \quad (d)$$

which coincides with Eq. (142) for the stress function in torsion.

At the boundary the velocity of the circulating fluid is in the direction of the tangent to the boundary and the boundary condition for the hydrodynamical problem is the same as the condition (144) for the torsional problem. Hence the velocity distribution in the hydrodynamical problem is mathematically identical with the stress distribution in torsion, and some practically important conclusions can be drawn by using the known solutions of hydrodynamics.

As a first example we take the case of a small *circular hole* in a twisted circular shaft¹ (Fig. 168). The effect of this hole on the stress distribution is similar to that of introducing a stationary solid cylinder of the same diameter as the hole into the stream of circulating fluid of the hydrodynamical model. Such a cylinder greatly changes the velocity of the fluid in its immediate neighborhood. The velocities at the front and rear points are reduced to zero, while those at the side points m and n are doubled. A hole of this kind therefore doubles the shearing stress in the portion of the shaft in which it is located. A small *semicircular groove* on the surface parallel to the length of the shaft (Fig. 168) has the same effect. The shearing stress at the bottom of the groove, the point m , is about twice the shearing stress at the surface of the shaft far away from the groove.

The same hydrodynamical analogy explains the effect of a small *hole of elliptic cross section* or of a groove of *semi-elliptic cross section*. If one of the principal axes a of the small elliptical hole is in the radial direction and the other principal axis is b , the stresses at the edge of the hole at the ends of the a -axis are increased in the proportion $(1 + \frac{a}{b}) : 1$.

¹ See J. Larmor, *Phil. Mag.*, vol. 33, p. 76, 1892.

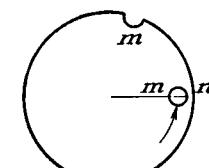


FIG. 168.

The maximum stress produced in this case thus depends upon the magnitude of the ratio a/b . The effect of the hole on the stress is greater when the major axis of the ellipse is in the radial direction than when it runs circumferentially. This explains why a radial crack has such a weakening effect on the strength of a shaft. Similar effects on the stress distribution are produced by a semi-elliptic groove on the surface, parallel to the axis of the shaft.

From the hydrodynamical analogy it can be concluded also that at the projecting corners of a cross section of a twisted bar the shearing stress becomes zero, and that at reentrant corners this stress becomes theoretically infinitely large, *i.e.*, even the smallest torque will produce yielding of material or a crack at such a corner. In the case of a rectangular keyway, therefore, a high stress concentration takes place at the reentrant corners at the bottom of the keyway. These high stresses can be reduced by rounding the corners.¹

101. Torsion of Hollow Shafts. So far the discussion has been limited to shafts whose cross sections are bounded by single curves. Let us consider now hollow shafts whose cross sections have two or more boundaries. The simplest problem of this kind is a hollow shaft with an inner boundary coinciding with one of the *stress lines* (see page 270) of the solid shaft, having the same boundary as the outer boundary of the hollow shaft.

Take, for instance, an elliptic cross section (Fig. 149). The stress function for the solid shaft is

$$\phi = \frac{a^2 b^2 F}{2(a^2 + b^2)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \quad (a)$$

The curve

$$\frac{x^2}{(ak)^2} + \frac{y^2}{(bk)^2} = 1 \quad (b)$$

is an ellipse which is geometrically similar to the outer boundary of the cross section. Along this ellipse the stress function (a) remains constant, and hence, for k less than unity, this ellipse is a stress line for the solid elliptic shaft. The shearing stress at any point of this line is in the direction of the tangent to the line. Imagine now a cylindrical surface generated by this stress line with its axis parallel to the axis of

¹ The stresses at the keyway were investigated by the soap-film method. See the paper by A. A. Griffith and G. I. Taylor, *Tech. Rept., Natl. Advisory Comm. Aeronaut.*, vol. 3, p. 938, 1917-1918. The same problem was discussed by the photoelastic method. See the paper by A. G. Solakian and G. B. Karelitz, *Trans. A.S.M.E.*, Applied Mechanics Division, 1931.

the shaft. Then, from the above conclusion regarding the direction of the shearing stresses, it follows that there will be no stresses acting across this cylindrical surface. We can imagine the material bounded by this surface removed without changing the stress distribution in the outer portion of the shaft. Hence the stress function (a) applies to the hollow shaft also.

For a given angle θ of twist the stresses in the hollow shaft are the same as in the corresponding solid shaft. But the torque will be smaller by the amount which in the case of the solid shaft is carried by the portion of the cross section corresponding to the hole. From Eq. (148) we see that the latter portion is in the ratio $k^4:1$ to the total torque. Hence, for the hollow shaft, instead of Eq. (148), we will have

$$\theta = \frac{M_t}{1 - k^4} \frac{a^2 + b^2}{\pi a^3 b^3 G}$$

and the stress function (a) becomes

$$\phi = - \frac{M_t}{\pi a b (1 - k^4)} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$

The formula for the maximum stress will be

$$\tau_{\max.} = \frac{2 M_t}{\pi a b^2} \frac{1}{1 - k^4}$$

In the membrane analogy the middle portion of the membrane, corresponding to the hole in the shaft (Fig. 169), must be replaced by the horizontal plate CD . We note that the uniform pressure distributed over the portion CFD of the membrane is statically equivalent to the pressure of the same magnitude uniformly distributed over the plate CD and the tensile forces S in the membrane acting along the edge of this plate are in equilibrium with the uniform load on the plate. Hence, in the case under consideration the same experimental soap-film method as before can be employed because the replacement of the portion CFD of the membrane by the plate CD causes no changes in the configuration and equilibrium conditions of the remaining portion of the membrane.

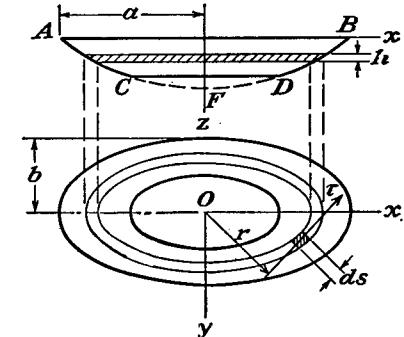


FIG. 169.

Let us consider now the more general case when the boundaries of the holes are no longer stress lines of the solid shaft. From the general theory of torsion we know (see Art. 90) that the stress function must be constant along each boundary, but these constants cannot be chosen arbitrarily. In discussing multiply-connected boundaries in two-dimensional problems it was shown that recourse must be had to the expressions for the displacements, and the constants of integration should be found in such a manner as to make these expressions single-valued. An analogous procedure is necessary in dealing with the torsion of hollow shafts. The constant values of the stress function along the boundaries should be determined in such a manner as to make the displacements single-valued. A sufficient number of equations for determining these constants will then be obtained.

From Eqs. (b) and (d) of Art. 90 we have

$$\tau_{xz} = G \left(\frac{\partial w}{\partial x} - \theta y \right), \quad \tau_{yz} = G \left(\frac{\partial w}{\partial y} + \theta x \right) \quad (c)$$

Let us now calculate the integral

$$\int \tau ds \quad (d)$$

along each boundary. Using (c) and resolving the total stress into its components we find

$$\begin{aligned} \int \tau ds &= \int \left(\tau_{xz} \frac{dx}{ds} + \tau_{yz} \frac{dy}{ds} \right) ds \\ &= G \int \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy \right) - \theta G \int (y dx - x dy) \end{aligned}$$

The first integral must vanish, from the condition that the integration is taken round a closed curve and that w is a single-valued function. Hence,

$$\int \tau ds = \theta G \int (y dx - x dy)$$

The integral on the right side is equal to double the area of the hole. Then

$$\int \tau ds = 2G\theta A \quad (e)$$

Thus we must determine the constant values of the stress function along the boundaries of the holes so as to satisfy Eq. (e) for each boundary. This equation is also valid for any closed curve drawn in the cross section, as may be seen by reexamining the proof.

A physical significance for Eq. (e) was discussed before [see Eq. (152), page 271]. It indicates that in using the membrane analogy the level of each plate, such as the plate CD (Fig. 169), must be taken so that the vertical load on the plate is equal and opposite to the vertical component of the resultant of the tensile forces on the plate produced by the membrane. If the boundaries of the holes coincide with the stress lines of the corresponding solid shaft, the above condition is sufficient to ensure the equilibrium of the plates. In the general case this condition is not sufficient, and to keep the plates in equilibrium in a horizontal position special guiding devices become necessary. This makes the soap-film experiments for hollow shafts more complicated.

To remove this difficulty the following procedure may be adopted.¹ We make a hole in the plate (Fig. 165) corresponding to the outer boundary of the shaft. The interior boundaries, corresponding to the holes, are mounted each on a vertical sliding column so that their heights can be easily adjusted. Taking these heights arbitrarily and stretching the film over the boundaries we obtain a surface which satisfies Eq. (142) and boundary conditions (144), but the Eq. (e) above generally will not be satisfied and the film does not represent the stress distribution in the hollow shaft. Repeating such an experiment as many times as the number of boundaries, each time with another adjustment of heights of the interior boundaries and taking measurements on the film each time, we obtain sufficient data for determining the correct values of the heights of the interior boundaries and can finally stretch the soap film in the required manner. This can be proved as follows: If i is the number of boundaries and $\phi_1, \phi_2, \dots, \phi_i$ are the film surfaces obtained with i different adjustments of the heights of the boundaries, then a function

$$\phi = m_1 \phi_1 + m_2 \phi_2 + \dots + m_i \phi_i \quad (f)$$

in which m_1, m_2, \dots, m_i are numerical factors, is also a solution of Eq. (142), provided that

$$m_1 + m_2 + \dots + m_i = 1$$

Observing now that the shearing stress is equal to the slope of the membrane, and substituting (f) into Eqs. (e), we obtain i equations of the form

$$\int \frac{\partial \phi}{\partial n} ds = 2G\theta A;$$

from which the i -factors m_1, m_2, \dots, m_i can be obtained as functions of θ . Then the true stress function is obtained from (f).² This method was applied by Griffith and Taylor in determining stresses in a hollow circular shaft having a

¹ A. A. Griffith, and G. I. Taylor, *Tech. Rept. Natl. Advisory Comm. Aeronaut.*, vol. 3, p. 938, 1917-1918.

² Griffith and Taylor concluded from their experiments that instead of *constant-pressure* films it is more convenient to use *zero-pressure* films (see p. 272) in studying the stress distribution in hollow shafts. A detailed discussion of the calculation of factors m_1, m_2, \dots is given in their paper.

keyway in it. It was shown in this manner that the maximum stress can be considerably reduced and the strength of the shaft increased by throwing the bore in the shaft off center.

The torque in the shaft with one or more holes is obtained using twice the volume under the membrane and the flat plates. To see this we calculate the torque produced by the shearing stresses distributed over an elemental ring between two adjacent stress lines, as in Fig. 169 (now taken to represent an arbitrary hollow section). Denoting by δ the variable width of the ring and considering an element such as that shaded in the figure, the shearing force acting on this element is $\tau\delta ds$ and its moment with respect to O is $r\tau\delta ds$. Then the torque on the elemental ring is

$$dM_t = \int r\tau\delta ds \quad (c)$$

in which the integration must be extended over the length of the ring. Denoting by A the area bounded by the ring and observing that τ is

the slope, so that $\tau\delta$ is the difference in level h of the two adjacent contour lines, we find, from (c),

$$dM_t = 2hA \quad (d)$$

i.e., the torque corresponding to the elemental ring is given by twice the volume shaded in the figure. The total torque is given by the sum of these volumes, i.e., twice

the volume between AB , the membrane AC and DB , and the flat plate CD . The conclusion follows similarly for several holes.

102. Torsion of Thin Tubes. An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 170) represent the levels of the outer and the inner boundaries, and AC and DB be the cross section of the membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that AC and DB are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \quad (a)$$

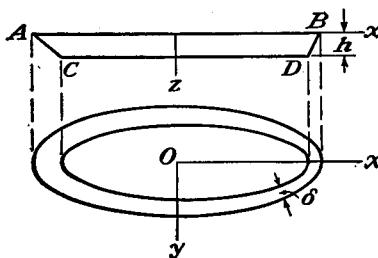


FIG. 170.

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t , we apply again the membrane analogy and calculate the torque from the volume $ACDB$. Then

$$M_t = 2Ah = 2A\delta\tau \quad (b)$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses,

$$\tau = \frac{M_t}{2A\delta} \quad (166)$$

For determining the angle of twist θ , we apply Eq. (152). Then

$$\tau ds = \frac{M_t}{2A} \int \frac{ds}{\delta} = 2G\theta A \quad (c)$$

from which¹

$$\theta = \frac{M_t}{4A^2G} \int \frac{ds}{\delta} \quad (167)$$

In the case of a tube of uniform thickness, δ is constant and (167) gives

$$\theta = \frac{M_t s}{4A^2G\delta} \quad (168)$$

in which s is the length of the center line of the ring section of the tube.

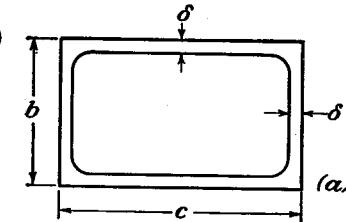


FIG. 171.

If the tube has reentrant corners, as in the case represented in Fig. 171, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (166) and depends on the radius a of the fillet of the reentrant corner (Fig. 171b). In calculating this maximum stress we shall use the membrane analogy as we did for the reentrant corners of rolled sections (Art. 98). The equation of the membrane at the reentrant corner may be taken in the form

$$\frac{d^2z}{dr^2} + \frac{1}{r} \frac{dz}{dr} = -\frac{q}{S}$$

¹ Equations (166) and (167) for thin tubular sections were obtained by R. Bredt, V.D.I., vol. 40, p. 815, 1896.

Replacing q/S by $2G\theta$ and noting that $\tau = -dz/dr$ (see Fig. 170), we find

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = 2G\theta \quad (d)$$

Assuming that we have a tube of a constant thickness δ and denoting by τ_0 the stress at a considerable distance from the corner calculated from Eq. (166), we find, from (c),

$$2G\theta = \frac{\tau_0 s}{A}$$

Substituting in (d),

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = \frac{\tau_0 s}{A} \quad (e)$$

The general solution of this equation is

$$\tau = \frac{C}{r} + \frac{\tau_0 sr}{2A} \quad (f)$$

Assuming that the projecting angles of the cross section have fillets with the radius a , as indicated in the figure, the constant of integration C can be determined from the equation

$$\int_a^{a+\delta} \tau dr = \tau_0 \delta \quad (g)$$

which follows from the hydrodynamical analogy (Art. 100), *viz.*: if an ideal fluid circulates in a channel having the shape of the ring cross section of the tubular member, the quantity of fluid passing each cross section of the channel must remain constant. Substituting expression (f) for τ into Eq. (g), and integrating, we find that

$$C = \tau_0 \delta \frac{1 - (s/4A)(2a + \delta)}{\log_e(1 + \delta/a)}$$

and, from Eq. (f), that

$$\tau = \frac{\tau_0 \delta}{r} \frac{1 - (s/4A)(2a + \delta)}{\log_e(1 + \delta/a)} + \frac{\tau_0 sr}{2A} \quad (h)$$

For a thin-walled tube the ratios $s(2a + \delta)/A$, sr/A , will be small, and (h) reduces to

$$\tau = \tau_0 \cdot \frac{\delta}{r} \left/ \log_e \left(1 + \frac{\delta}{a} \right) \right. \quad (i)$$

Substituting $r = a$ we obtain the stress at the reentrant corner. This is plotted in Fig. 172. The other curve¹ (A in Fig. 172) was obtained by the method of finite differences, without the assumption that the membrane at the corner has the form of a surface of revolution. It confirms the accuracy of Eq. (i) for small fillets—say up to $a/\delta = \frac{1}{4}$. For larger fillets the values given by Eq. (i) are too high.

Let us consider now the case when the cross section of a tubular member has more than two boundaries. Taking, for example, the case shown in Fig. 173 and assuming that the thickness of the wall is

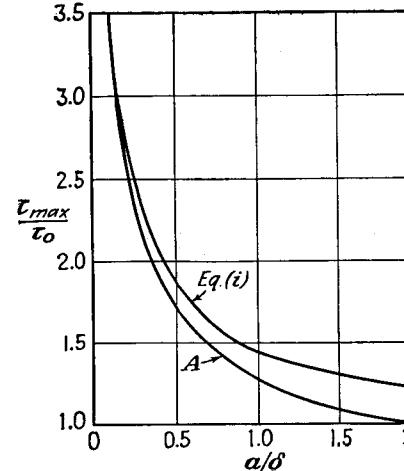


FIG. 172.

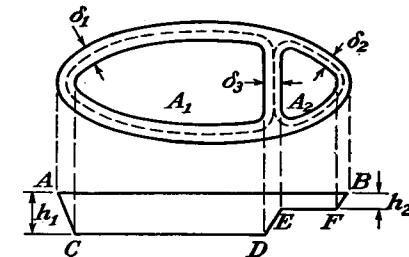


FIG. 173.

very small, the shearing stresses in each portion of the wall, from the membrane analogy, are

$$\begin{aligned} \tau_1 &= \frac{h_1}{\delta_1} \\ \tau_2 &= \frac{h_2}{\delta_2} \\ \tau_3 &= \frac{h_1 - h_2}{\delta_3} = \frac{\tau_1 \delta_1 - \tau_2 \delta_2}{\delta_3} \end{aligned} \quad (k)$$

in which h_1 and h_2 are the levels of the inner boundaries CD and EF .²

The magnitude of the torque, determined by the volume $ACDEFB$, is

$$M_t = 2(A_1 h_1 + A_2 h_2) = 2A_1 \delta_1 \tau_1 + 2A_2 \delta_2 \tau_2 \quad (l)$$

where A_1 and A_2 are areas indicated in the figure by dotted lines.

Further equations for the solution of the problem are obtained by applying Eq. (152) to the closed curves indicated in the figure by

¹ Huth, *loc. cit.*

² It is assumed that the plates are guided so as to remain horizontal (see p. 297).

dotted lines. Assuming that the thicknesses $\delta_1, \delta_2, \delta_3$ are constant and denoting by s_1, s_2, s_3 the lengths of corresponding dotted curves, we find, from Fig. 173,

$$\begin{aligned}\tau_1 s_1 + \tau_3 s_3 &= 2G\theta A_1 \\ \tau_2 s_2 - \tau_3 s_3 &= 2G\theta A_2\end{aligned}\quad (m)$$

By using the last of the Eqs. (k) and Eqs. (l) and (m), we find the stresses τ_1, τ_2, τ_3 as functions of the torque:

$$\tau_1 = \frac{M_t[\delta_3 s_2 A_1 + \delta_2 s_3 (A_1 + A_2)]}{2[\delta_1 \delta_3 s_2 A_1^2 + \delta_2 \delta_3 s_1 A_2^2 + \delta_1 \delta_2 s_3 (A_1 + A_2)^2]} \quad (n)$$

$$\tau_2 = \frac{M_t[\delta_3 s_1 A_2 + \delta_1 s_3 (A_1 + A_2)]}{2[\delta_1 \delta_3 s_2 A_1^2 + \delta_2 \delta_3 s_1 A_2^2 + \delta_1 \delta_2 s_3 (A_1 + A_2)^2]} \quad (o)$$

$$\tau_3 = \frac{M_t(\delta_1 s_2 A_1 - \delta_2 s_1 A_2)}{2[\delta_1 \delta_3 s_2 A_1^2 + \delta_2 \delta_3 s_1 A_2^2 + \delta_1 \delta_2 s_3 (A_1 + A_2)^2]} \quad (p)$$

In the case of a symmetrical cross section, $s_1 = s_2, \delta_1 = \delta_2, A_1 = A_2$, and $\tau_3 = 0$. In this case the torque is taken by the outer wall of the tube, and the web remains unstressed.¹

To get the twist for any section like that shown in Fig. 173, one substitutes the values of the stresses in one of the Eqs. (m). Thus θ can be obtained as a function of the torque M_t .

103. Torsion of a Bar in Which One Cross Section Remains Plane. In discussing torsional problems it has always been assumed that the torque is applied by means of shearing stresses distributed over the ends of a bar in a definite manner, obtained from the solution of Eq. (142) and satisfying the boundary condition (144). If the distribution of stresses at the ends is different from this, a local irregularity in stress distribution results and the solution of Eqs. (142) and (144) can be applied with satisfactory accuracy only in regions at some distance from the ends of the bar.²

A similar irregularity occurs if a cross section of a twisted bar is prevented from warping by some constraint. We encounter problems of this kind occasionally in engineering.³ A simple example is shown in Fig. 174. From symmetry it can be concluded that the middle cross section of the bar remains plane during torsion. Hence the stress distribution near this cross section must be different from that

¹ The small stresses corresponding to the change in slope of the membrane across the thickness of the web are neglected in this derivation.

² The local irregularities at the ends of a circular cylinder have been discussed by F. Purser, *Proc. Roy. Irish Acad.*, Dublin, vol. 26, series A, p. 54, 1906. See also K. Wolf, *Sitzber. Akad. Wiss., Wien*, vol. 125, p. 1149, 1916, and A. Timpe, *Math. Annalen*, vol. 71, p. 480, 1912.

³ Torsion of I-beams under such conditions was discussed by S. Timoshenko, *Z. Math. Physik*, vol. 58, p. 361, 1910. See also C. Weber, *Z. angew. Math. Mech.*, vol. 6, p. 85, 1926.

obtained above for rectangular bars (Art. 95). In discussing these stresses, let us consider first the case of a very narrow rectangle¹ and assume that the dimension a is large in comparison with b . If cross sections are free to warp, the stresses, from Art. 94, are

$$\tau_{xz} = -2G\theta y, \quad \tau_{yz} = 0 \quad (a)$$

and the corresponding displacements, from Eqs. (a), (b), and (d) of Art. 90, are

$$u = -\theta yz, \quad v = \theta zx, \quad w = -\theta xy \quad (b)$$

To prevent the warping of the cross sections, designated as displacement w , normal stresses σ_z must be distributed over the cross sections. We obtain an approximate solution by assuming that σ_z is proportional to w and that it diminishes with increase of distance z from the middle cross section. These assumptions are satisfied by taking

$$\sigma_z = -mE\theta e^{-mz}xy \quad (c)$$

in which m is a factor to be determined later. Due to the factor e^{-mz} the stress σ_z diminishes with increase of z and becomes negligible within a certain distance depending upon the magnitude of m .

The remaining stress components must now be chosen in such a manner as to satisfy the differential equations of equilibrium (127) and the boundary conditions. It is easy to prove that these requirements are satisfied by taking

$$\begin{aligned}\sigma_x &= \sigma_y = 0 \\ \tau_{xy} &= -\frac{1}{8}Em^3\theta e^{-mz}(a^2 - x^2)(b^2 - y^2) \\ \tau_{xz} &= \frac{1}{4}Em^2\theta e^{-mz}(a^2 - x^2)y - 2G\theta y \\ \tau_{yz} &= \frac{1}{4}Em^2\theta e^{-mz}(b^2 - y^2)x\end{aligned}\quad (d)$$

For large values of z this stress distribution approaches the stresses (a) for simple torsion. The stress component τ_{xy} becomes zero at the boundary $x = \pm a$ and $y = \pm b$; τ_{xz} and τ_{yz} are zero for $x = \pm a$ and $y = \pm b$, respectively. Hence the boundary conditions are satisfied and the lateral surface of the bar is free from forces.

For determining the factor m , we consider the strain energy of the bar and calculate m to make this energy a minimum. By using Eq. (84) on page 148, we find

$$V = \frac{1}{2G} \int_{-l}^l \int_{-a}^a \int_{-b}^b [\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 + \frac{1}{2(1+\nu)} \cdot \sigma_z^2] dx dy dz$$

Substituting from (d), and noting that for a long bar we can with sufficient accuracy put

$$\int_0^l e^{-mz} dz = \frac{1}{m}$$

¹ See S. Timoshenko, *Proc. London Math. Soc.*, series 2, vol. 20, p. 389, 1921.

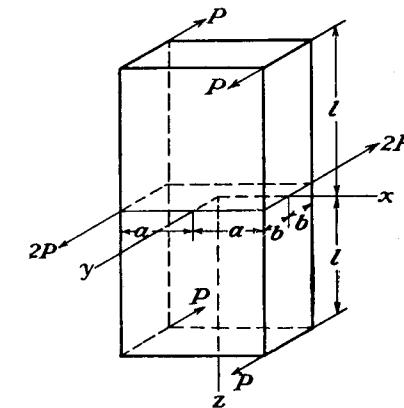


FIG. 174.

we get

$$V = \frac{1}{9} E\theta^2 a^3 b^3 \left\{ -3m + (1 + \nu) \left[\frac{2}{25} a^2 b^2 m^5 + \frac{1}{5} (a^2 + b^2) m^3 + \frac{12}{(1 + \nu)^2 a^2} l \right] \right\} \quad (e)$$

The minimum condition gives us the following equation for determining m :

$$(1 + \nu)[\frac{2}{5} a^2 b^2 m^4 + \frac{3}{5} (a^2 + b^2) m^2] = 3$$

which, for a narrow rectangle, reduces approximately to

$$m^2 = \frac{5}{(1 + \nu)a^2} \quad (f)$$

Substituting this value of m in (c) and (d), we find the stress distribution for the case when the middle cross section of the bar remains plane.

For calculating the angle of twist ψ , we put the potential energy (e) equal to the work done by the torque M_t ,

$$\frac{M_t \psi}{2} = V$$

from which the angle of twist is

$$\psi = \frac{3M_t}{16G a b^3} \left[l - \frac{\sqrt{5(1 + \nu)}}{6} a \right] \quad (g)$$

Comparing this result with Eq. (155) on page 273, we conclude that by preventing the middle cross section from warping we increase the rigidity of the bar with respect to torsion. The effect of the local irregularity in stress distribution on the value of ψ is the same as the influence of a diminution of the length l by

$$a \frac{\sqrt{5(1 + \nu)}}{6}$$

Taking $\nu = 0.30$, this reduction in l becomes $0.425a$. We see that the effect of the constraint of the middle cross section on the angle of twist is small if the dimension a is small in comparison with l .

The twist of a bar of an elliptic cross section can be discussed in an analogous manner.¹ Of greater effect is the constraint of the middle cross section in the case of torsion of a bar of I cross section. An approximate method for calculating the angle of twist in this case is obtained by considering bending of the flanges during torsion.²

104. Torsion of Circular Shafts of Variable Diameter. Let us consider a shaft in the form of a body of revolution twisted by couples applied at the ends (Fig. 175). We may take the axis of the shaft as

¹ A. Föppl, *Sitzber. Bayer. Akad. Wiss., Math.-phys. Klasse*, München, 1920, p. 261.

² See S. Timoshenko, *Z. Math. Physik.*, vol. 58, p. 361, 1910; or *Strength of Materials*, vol. 2, p. 287, 1941.

the z -axis and use polar coordinates r and θ for defining the position of an element in the plane of a cross section. The notations for stress components in such a case are σ_r , σ_θ , σ_z , τ_{rz} , $\tau_{r\theta}$, $\tau_{\theta z}$. The components of displacements in the radial and tangential directions we may denote by u and v and the component in the z -direction by w . Then, using the

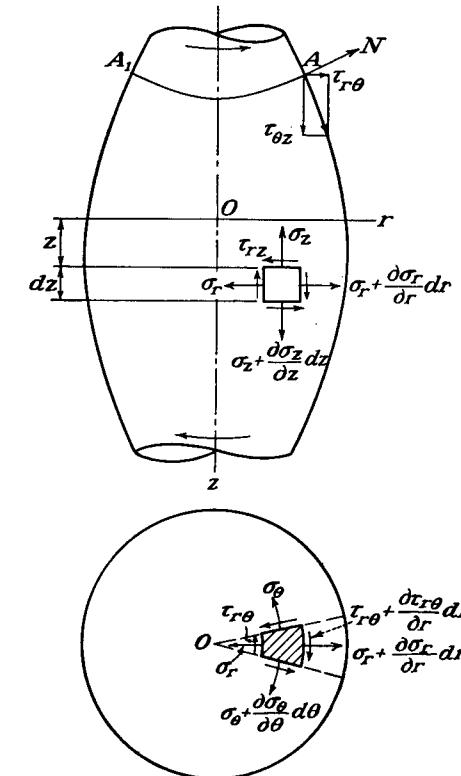


FIG. 175.

formulas obtained previously for two-dimensional problems (Art. 28), we find the following expressions for the strain components:

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r}, & \epsilon_\theta &= \frac{u}{r} + \frac{\partial v}{r \partial \theta}, & \epsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{r\theta} &= \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, & \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, & \gamma_{z\theta} &= \frac{\partial v}{\partial z} + \frac{\partial w}{r \partial \theta} \end{aligned} \quad (169)$$

Writing down the equations of equilibrium of an element (Fig. 175), as was done before for the case of two-dimensional problems (Art. 25),

and assuming that there are no body forces, we arrive at the following differential equations of equilibrium:¹

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} &= 0\end{aligned}\quad (170)$$

In the application of these equations to the torsional problem we use the *semi-inverse method* (see page 259) and assume that u and w are zero, i.e., that during twist the particles move only in tangential directions. This assumption differs from that for a circular shaft of constant diameter in that these tangential displacements are no longer proportional to the distance from the axis, i.e., the radii of a cross section become curved during twist. In the following pages it will be shown that the solution obtained on the basis of such an assumption satisfies all the equations of elasticity and therefore represents the true solution of the problem.

Substituting in (169) $u = w = 0$, and taking into account the fact that from symmetry the displacement v does not depend on the angle θ , we find that

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = 0, \quad \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{\theta z} = \frac{\partial v}{\partial z} \quad (a)$$

Hence, of all the stress components, only $\tau_{r\theta}$ and $\tau_{\theta z}$ are different from zero. The first two of Eqs. (170) are identically satisfied, and the third of these equations gives

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \quad (b)$$

This equation can be written in the form

$$\frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{\partial}{\partial z} (r^2 \tau_{\theta z}) = 0 \quad (c)$$

It is seen that this equation is satisfied by using a stress function ϕ of r and z , such that

$$r^2 \tau_{r\theta} = -\frac{\partial \phi}{\partial z}, \quad r^2 \tau_{\theta z} = \frac{\partial \phi}{\partial r} \quad (d)$$

¹ These equations were obtained by Lamé and Clapeyron; see *Crell's J.*, vol. 7, 1831.

To satisfy the compatibility conditions it is necessary to consider the fact that $\tau_{r\theta}$ and $\tau_{\theta z}$ are functions of the displacement v . From Eqs. (a) and (d) we find

$$\begin{aligned}\tau_{r\theta} &= G\gamma_{r\theta} = G\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = Gr \frac{\partial}{\partial r}\left(\frac{v}{r}\right) = -\frac{1}{r^2} \frac{\partial \phi}{\partial z} \\ \tau_{\theta z} &= G\gamma_{\theta z} = G\frac{\partial v}{\partial z} = Gr \frac{\partial}{\partial z}\left(\frac{v}{r}\right) = \frac{1}{r^2} \frac{\partial \phi}{\partial r}\end{aligned}\quad (e)$$

From these equations it follows that

$$\frac{\partial}{\partial r}\left(\frac{1}{r^3} \frac{\partial \phi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r^3} \frac{\partial \phi}{\partial r}\right) = 0 \quad (f)$$

or

$$\frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (g)$$

Let us consider now the boundary conditions for the function ϕ . From the condition that the lateral surface of the shaft is free from external forces we conclude that at any point A at the boundary of an axial section (Fig. 175) the total shearing stress must be in the direction of the tangent to the boundary and its projection on the normal N to the boundary must be zero. Hence

$$\tau_{r\theta} \frac{dz}{ds} - \tau_{\theta z} \frac{dr}{ds} = 0$$

where ds is an element of the boundary. Substituting from (d), we find that

$$\frac{\partial \phi}{\partial z} \frac{dz}{ds} + \frac{\partial \phi}{\partial r} \frac{dr}{ds} = 0 \quad (h)$$

from which we conclude that ϕ is constant along the boundary of the axial section of the shaft.

Equation (g) together with the boundary condition (h) completely determines the stress function ϕ , from which we may obtain the stresses satisfying the equations of equilibrium, the compatibility equations, and the condition at the lateral surface of the shaft.¹

¹ This general solution of the problem is due to J. H. Michell, *Proc. London Math. Soc.*, vol. 31, p. 141, 1899. See also A. Föppl, *Sitzber. Bayer. Akad. Wiss., München*, vol. 35, pp. 249 and 504, 1905. Also the book "Kerbspannungslehre" by H. Neuber, which gives solutions for the hyperboloid of revolution, and for a cavity in the form of an ellipsoid of revolution, by a different method. Reviews of the literature on the subject have been given by T. Pöschl, *Z. angew. Math. Mech.*, vol. 2, p. 137, 1922, and T. J. Higgins, *Experimental Stress Analysis*, vol. 3, no. 1, p. 94, 1945.

The magnitude of the torque is obtained by taking a cross section and calculating the moment given by the shearing stresses $\tau_{\theta z}$. Then

$$M_t = \int_0^a 2\pi r^2 \tau_{\theta z} dr = 2\pi \int_0^a \frac{\partial \phi}{\partial r} dr = 2\pi \left[\phi \right]_0^a \quad (k)$$

where a is the outer radius of the cross section. The torque is thus easily obtained if we know the difference between the values of the stress function at the outer boundary and at the center of the cross section.

In discussing displacements during twist of the shaft let us use the notation $\psi = v/r$ for the angle of rotation of an elemental ring of radius r in a cross section of the shaft. This ring can be considered as the cross section of one of a number of thin elemental tubes into which the shaft is subdivided. Then ψ is the angle of twist of such a tube. From the fact that the radii of the cross section become curved, it follows that ψ varies with r and the angles of twist of elemental tubes are not equal for the same cross section of the shaft. Equations (e) can now be written in the form

$$Gr^3 \frac{\partial \psi}{\partial r} = - \frac{\partial \phi}{\partial z}$$

$$Gr^3 \frac{\partial \psi}{\partial z} = \frac{\partial \phi}{\partial r}$$

from which

$$\frac{\partial}{\partial r} \left(r^3 \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial z} \left(r^3 \frac{\partial \psi}{\partial z} \right) = 0$$

or

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad (l)$$

A solution of this equation gives us the angle of twist as a function of r and z . If we put

$$\psi = \text{constant} \quad (m)$$

in this solution, we obtain a surface in which all the points have the same angle of twist. In Fig. 175, AA_1 represents the intersection of such a surface with the axial section of the shaft. From symmetry it follows that the surfaces given by Eq. (m) are surfaces of revolution and AA_1 is a meridian of the surface going through the point A . During twist these surfaces rotate about the z -axis without any distortion, exactly in the same manner as the plane cross sections in the case of circular cylindrical shafts. Hence the total strain at any point of the meridian AA_1 is pure shearing strain in the plane perpendicular to the

meridian, and the corresponding shearing stress in the axial section of the shaft has the direction normal to the meridian. At the boundary this stress is tangent to the boundary and the meridians are normal to the boundary of the axial section. If we go from the surface $\psi = \text{constant}$ to an adjacent surface the rate of change of ψ along the boundary of the axial section of the shaft is $d\psi/ds$, and in the same manner as for a cylindrical shaft of circular cross section (Art. 87) we have

$$\tau = Gr \frac{d\psi}{ds} \quad (n)$$

where

$$\tau = \tau_{rz} \frac{dr}{ds} + \tau_{\theta z} \frac{dz}{ds}$$

is the resultant shearing stress at the boundary. It is seen that the value of this shearing stress is easily obtained if we find by experiment the values of $d\psi/ds$.¹

Let us consider now a particular case of a conical shaft² (Fig. 176). In this case the ratio

$$\frac{z}{(r^2 + z^2)^{\frac{1}{2}}}$$

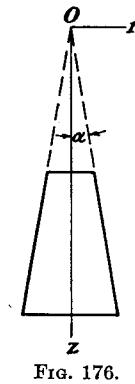


FIG. 176.

is constant at the boundary of the axial section and equal to $\cos \alpha$. Any function of this ratio will satisfy the boundary condition (h). In order to satisfy also Eq. (g) we take

$$\phi = c \left\{ \frac{z}{(r^2 + z^2)^{\frac{1}{2}}} - \frac{1}{3} \left[\frac{z}{(r^2 + z^2)^{\frac{1}{2}}} \right]^3 \right\} \quad (o)$$

where c is a constant. Then by differentiation we find

$$\tau_{\theta z} = \frac{1}{r^2} \frac{\partial \phi}{\partial r} = - \frac{crz}{(r^2 + z^2)^{\frac{3}{2}}} \quad (p)$$

The constant c is obtained from Eq. (k). Substituting (o) in this equation we find

$$c = - \frac{M_t}{2\pi(\frac{2}{3} - \cos \alpha + \frac{1}{3} \cos^3 \alpha)}$$

To calculate the angle of twist we use Eqs. (e), from which the expression for ψ , satisfying Eq. (l) and the boundary condition, is

$$\psi = \frac{c}{3G(r^2 + z^2)^{\frac{1}{2}}} \quad (q)$$

¹ Such experiments were made by R. Sonntag, *Z. angew. Math. Mech.*, vol. 9, p. 1, 1929.

² See Föppl, *loc. cit.*

It will be seen that the surfaces of equal angle of twist are spherical surfaces with their center at the origin O .

The case of a shaft in the form of an ellipsoid, hyperboloid, or paraboloid of revolution can be discussed in an analogous manner.¹

The problems encountered in practice are of a more complicated nature. The diameter of the shaft usually changes abruptly, as shown in Fig. 177a. The first investigation of such problems was made by A. Föppl. C. Runge suggested a numerical method for the approximate solution of these problems,² and it was shown that considerable

stress concentration takes place at such points as m and n , and that the magnitude of the maximum stress for a shaft of two different diameters d and D (Fig. 177a) depends on the ratio of the radius a of the fillet to the diameter d of the shaft and on the ratio d/D .

In the case of a semicircular groove of very small radius a , the maximum stress at the bottom of the groove

(Fig. 177b) is twice as great as at the surface of the cylindrical shaft without the groove.

In discussing stress concentration at the fillets and grooves of twisted circular shafts, an electrical analogy has proved very useful.³ The general equation for the flow of an electric current in a thin homogeneous plate of variable thickness is

$$\frac{\partial}{\partial x} \left(h \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial \psi}{\partial y} \right) = 0 \quad (r)$$

in which h is the variable thickness of the plate and ψ the potential function.

¹ See papers by E. Melan, *Tech. Blätter*, Prag, 1920; A. N. Dinnik, *Bull. Don Polytech. Inst., Novotcherkask*, 1912; W. Arndt, *Die Torsion von Wellen mit achsensymmetrischen Bohrungen und Hohlräumen*, Dissertation, Göttingen, 1916; A. Timpe, *Math. Annalen*, 1911, p. 480. Further references are given in a review by Higgins, *loc. cit.*

² See F. A. Willers, *Z. Math. Physik*, vol. 55, p. 225, 1907. Another approximate method was developed by L. Föppl, *Sitzber. Bayer. Akad. Wiss., München*, vol. 51, p. 61, 1921, and by R. Sonntag, *Z. angew. Math. Mech.*, vol. 9, p. 1, 1929.

³ See paper by L. S. Jacobsen, *Trans. A.S.M.E.*, vol. 47, p. 619, 1925, and the survey given by T. J. Higgins, *loc. cit.* Discrepancies between results obtained from this and other methods are discussed in the latter paper. For further comparisons and strain-gauge measurements extending Fig. 179 to $2a/d = 0.50$ see A. Weigand, *Luftfahrt-Forsch.*, vol. 20, p. 217, 1943, translated in *N.A.C.A. Tech. Mem.* 1179, September, 1947.

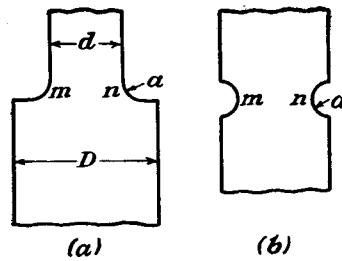


FIG. 177.

Let us assume that the plate has the same boundary as the axial section of the shaft (Fig. 178), that the x - and y -axes coincide with the z - and r -axes, and that the thickness of the plate is proportional to the cube of the radial distance r , so that $h = \alpha r^3$. Then Eq. (r) becomes

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{3}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} = 0$$

This coincides with equation (l), and we conclude that the equipotential lines of the plate are determined by the same equation as the lines of equal angles of twist in the case of a shaft of variable diameter.

Assuming that the ends of the plate, corresponding to the ends of the shaft, are maintained at a certain difference of potential so that the current flows along the z -axis, the equipotential lines are normal to the

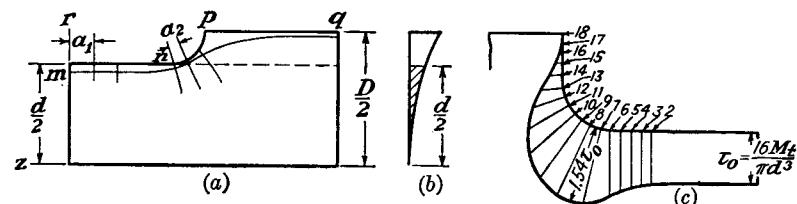


FIG. 178.

lateral sides of the plate, i.e., we have the same boundary conditions as for lines of constant angle of twist. If the differential equations and the boundary conditions are the same for these two kinds of lines, the lines are identical. Hence, by investigating the distribution of potential in the plate, valuable information regarding the stress distribution in the twisted shaft can be obtained.

The maximum stress is at the surface of the shaft and we obtain this stress by using Eq. (n). From this equation, by applying the electrical analogy, it follows that the stress is proportional to the rate of drop of potential along the edge of the plate.

Actual measurements were made on a steel model 24 in. long by 6 in. wide at the larger end and 1 in. maximum thickness (Fig. 178). The drop of potential along the edge $mnpq$ of the model was investigated by using a sensitive galvanometer, the terminals of which were connected to two sharp needles fastened in a block at a distance 2 mm. apart. By touching the plate with the needles the drop in potential over the distance between the needle points was indicated by the galvanometer. By moving the needles along the fillet it is possible to find the place of, and measure, the maximum voltage gradient. The ratio of this maximum to the voltage gradient at a remote point m (Fig. 178a) gives

the magnitude of the factor of stress concentration k^* in the equation

$$\tau_{\max} = k \frac{16M_t}{\pi d^3}$$

The results of such tests in one particular case are represented in Fig. 178c, in which the potential drop measured at each point is indicated

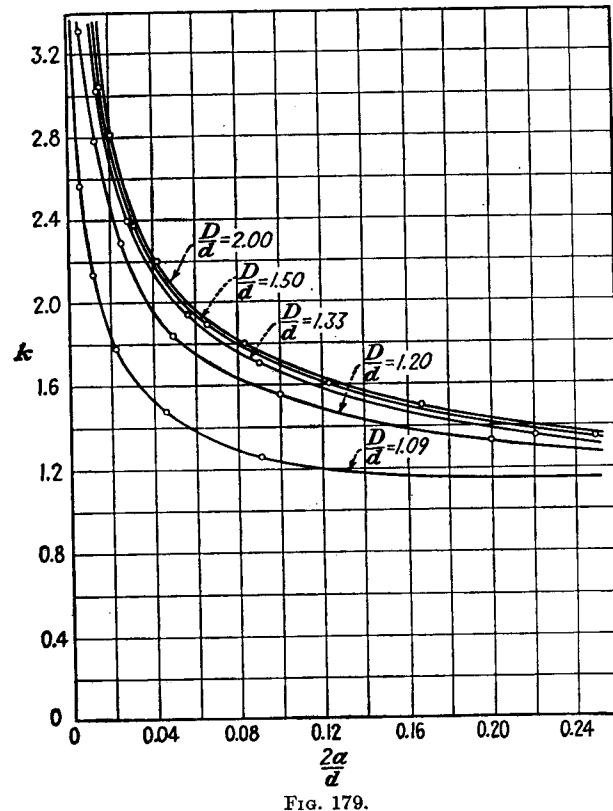


FIG. 179.

by the length on the normal to the edge of the plate at this point. From this figure the factor of stress concentration is found to be 1.54. The magnitudes of this factor obtained with various proportions of shafts are given in Fig. 179, in which the abscissas represent the ratios $2a/d$ of the radius of the fillet to the radius of the smaller shaft and the ordinates the factor of stress concentration k for various values of the ratio D/d (see Fig. 177). By interpolating from these curves the factor

* Small variations in radius r [Eq. (n)] can be neglected in this case.

of stress concentration for any particular case can be found with sufficient accuracy.

Problems

1. Show by considering the equilibrium of the whole bar that when all stress components vanish except τ_{xz} , τ_{yz} , the loading must consist of torsional couples only [cf. Eqs. (h), Art. 90].

2. Show that $\phi = A(r^2 - a^2)$ solves the torsion problem for the solid or hollow circular shaft. Determine A in terms of $G\theta$. Using Eqs. (141) and (145) evaluate the maximum shearing stress and the torsional rigidity in terms of M_t for the solid shaft, and verify that the results are in agreement with those given in any text on strength of materials.

3. Show that for the same twist, the elliptic section has a greater shearing stress than the inscribed circular section (radius equal to the minor axis b of the ellipse). Which takes the greater torque for the same allowable stress?

4. Use Eq. (g) of Art. 92 and Eq. (145) to evaluate the torsional rigidity of the equilateral triangle, and thus verify Eq. (l), Art. 92.

5. Using the stress function (m) of Art. 92 expressed in rectangular coordinates, find an expression for τ_{yz} along the middle line Ax of Fig. 153, and verify that the greatest value along this line is the value given by Eq. (p).

6. Evaluate the torsional rigidity of the section shown in Fig. 153. Is it appreciably different from that of the complete circular section when the groove is small?

7. Show that the expression for the stress function ϕ which corresponds to the parabolic membrane of Art. 94 is

$$\phi = -G\theta \left(x^2 - \frac{c^2}{4} \right)$$

In a narrow tapered section such as the triangle shown in Fig. 180, an approximate solution can be obtained by assuming that at any level y the membrane has the parabolic form appropriate to the width at that level. Prove that for the triangular section of height b

$$M_t = \frac{1}{12} G\theta b c_o^3$$

approximately.

8. Using the method indicated in Prob. 7, find an approximate expression for the torsional rigidity of the thin symmetrical section bounded by two parabolas shown in Fig. 181, for which the width c at a depth y below the center is given by

$$c = c_o \left(1 - \frac{y^2}{b^2} \right)$$

9. Show that the method indicated in Prob. 7 gives for a slender elliptical section the approximate stress function

$$\phi = -G\theta b^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$$



FIG. 180.

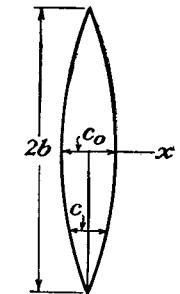


FIG. 181.

the ellipse being that of Fig. 149 with b/a small. Show that the exact solution of Art. 91 approaches this as b/a is made small.

Derive the approximate formulas

$$M_t = \pi ab^3 G\theta, \quad \tau_{\max.} = 2G\theta b = \frac{2M_t}{\pi ab^2}$$

for the slender elliptical section, and compare with the corresponding formulas for the thin rectangular section of length $2a$ and thickness $2b$.

10. Apply the method given at the end of Art. 97 to find an approximation to the torsional rigidity of the section described in Prob. 8.

11. A section has a single hole, and the stress function ϕ is determined so that it vanishes on the outside boundary and has a constant value ϕ_H on the boundary of the hole. By adapting the calculation indicated on page 262 for Eq. (145), prove that the total torque is given by twice the volume under the ϕ -surface plus twice the volume under a flat roof at height ϕ_H covering the hole (cf. page 298).

12. A closed thin-walled tube has a perimeter l and a uniform wall thickness δ . An open tube is made by making a fine longitudinal cut in it. Show that when the maximum shear stress is the same in both closed and open tubes,

$$\frac{M_{t\text{open}}}{M_{t\text{closed}}} = \frac{l\delta}{6A}, \quad \frac{\theta_{\text{open}}}{\theta_{\text{closed}}} = \frac{2A}{l\delta}$$

and that the ratio of the torsional rigidities is $l^2\delta^2/12A^2$, A being the area of the "hole."

Evaluate these ratios for a circular tube of 1 in. radius, $\frac{1}{16}$ in. wall thickness.

13. A thin-walled tube has the cross section shown in Fig. 182, with uniform wall thickness δ . Show that there will be no stress in the central web when the tube is twisted.

Find formulas for (a) the shear stress in the walls, away from the corners, (b) the unit twist θ , in terms of the torque.

14. Find expressions for the shear stresses in a tube of the section shown in Fig. 183, the wall thickness δ being uniform.

15. In discussing thin-walled closed sections, it was assumed that the shear stress is constant across the wall thickness, corresponding to constant membrane slope across the thickness. Show that this cannot be strictly true for a straight part of the wall (e.g., Fig. 171a) and that in general the correction to this shear stress consists of the shear stress in a tube made "open" by longitudinal cuts (cf. Prob. 12).

16. The theory of Art. 104 includes the uniform circular shaft as a special case. What are the corresponding forms for the functions ϕ and ψ ? Show that these functions give the correct relation between torque and unit twist.

17. Prove that

$$\phi = \frac{z}{R} + \frac{Az^3}{R^3} \quad \text{where} \quad R = (r^2 + z^2)^{\frac{1}{2}}$$

satisfies Eq. (g) of Art. 104 only if the constant A is $-\frac{1}{3}$ [cf. Eq. (o)].

18. At any point of an axial section of a shaft of variable diameter, line elements ds and dn (at right angles) in the section are chosen arbitrarily as shown in Fig. 184. The shear stress is expressed by components τ_s, τ_n along these. Show that

$$\tau_s = \frac{1}{r^2} \frac{\partial \phi}{\partial n}, \quad \tau_n = -\frac{1}{r^2} \frac{\partial \phi}{\partial s}, \quad \tau_s = Gr \frac{\partial \psi}{\partial s}, \quad \tau_n = Gr \frac{\partial \psi}{\partial n}$$

and deduce the boundary condition satisfied by ψ .

Show without calculation that the function given by Eq. (q) of Art. 104 satisfies this boundary condition for a conical boundary of any angle.

19. Verify that Eq. (q) of Art. 104 gives correctly the function ψ corresponding to the function ϕ in Eq. (o).

20. If the theory of Art. 104 is modified by discarding the boundary condition $\phi = \text{constant}$ the stress will be due to certain "rings of shear" on the boundary, as well as end torques. Considering the uniform circular shaft, describe the problem solved by $\phi = Czr^4$ where C is a constant, for $0 < z < l$.

21. Prove that the relative rotation of the ends of the (conical) tapered shaft shown in Fig. 185 due to torque M_t is

$$\frac{M}{2\pi(\frac{2}{3} - \cos \alpha + \frac{1}{3} \cos^3 \alpha)} \cdot \frac{1}{3G} \left(\frac{1}{a^3} - \frac{1}{b^3} \right)$$

If a and b are both made large, with $b - a = l$, and α is made small, the above result should approach the relative rotation of the ends of a uniform shaft of length l , and radius αa , due to torque M_t . Show that it does so.

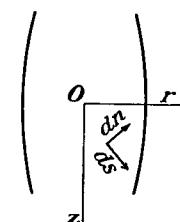


FIG. 184.

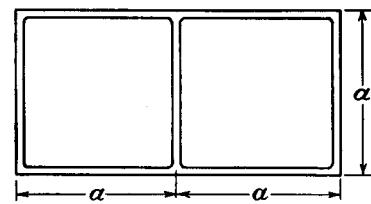


FIG. 182.

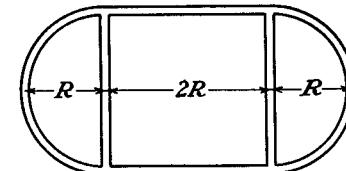


FIG. 183.

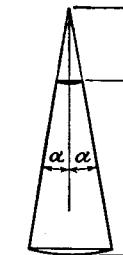


FIG. 185.

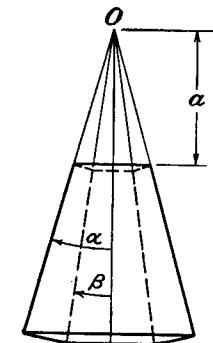


FIG. 186.

22. Use the functions given by Eqs. (o) and (q) of Art. 104 to find, in terms of M_t , the relative rotation of the ends of the hollow conical shaft shown in Fig. 186. The ends are spherical surfaces of radii a, b , center O .

CHAPTER 12
BENDING OF PRISMATICAL BARS

105. Bending of a Cantilever. In discussing pure bending (Art. 88) it was shown that, if a prismatical bar is bent in one of its principal planes by two equal and opposite couples applied at the ends, the deflection occurs in the same plane, and of the six components of stress only the normal stress parallel to the axis of the bar is different from zero. This stress is proportional to the distance from the neutral axis. Thus the exact solution coincides in this case with the elementary theory of bending. In discussing bending of a cantilever of narrow rectangular cross section by a force applied at the end (Art. 20), it was

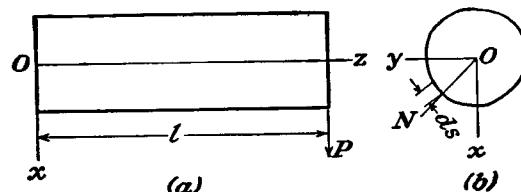


FIG. 187.

shown that in addition to normal stresses, proportional in each cross section to the bending moment, there will act also shearing stresses proportional to the shearing force.

Consider now a more general case of bending of a cantilever of a constant cross section of any shape by a force P applied at the end and parallel to one of the principal axes of the cross section¹ (Fig. 187). Take the origin of the coordinates at the centroid of the fixed end. The z -axis coincides with the center line of the bar, and the x - and y -axes coincide with the principal axes of the cross section. In the solution of the problem we apply Saint-Venant's semi-inverse method and at the very beginning make certain assumptions regarding stresses. We assume that normal stresses over a cross section at a distance z from the fixed end are distributed in the same manner as in the case of pure bending:

$$\sigma_z = -\frac{P(l-z)x}{I} \quad (a)$$

¹ This problem was solved by Saint-Venant, *J. mathémat.* (*Liouville*), series 2, vol. 1, 1856.

We assume also that there are shearing stresses, acting on the same cross sections, which we resolve at each point into components τ_{xz} and τ_{yz} . We assume that the remaining three stress components σ_x , σ_y , τ_{xy} are zero. It will now be shown that by using these assumptions we arrive at a solution which satisfies all of the equations of the theory of elasticity and which is hence the exact solution of the problem.

With these assumptions, neglecting body forces, the differential equations of equilibrium (127) become

$$\frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{yz}}{\partial z} = 0 \quad (b)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = -\frac{Px}{I} \quad (c)$$

From (b) we conclude that shearing stresses do not depend on z and are the same in all cross sections of the bar.

Considering now the boundary conditions (128) and applying them to the lateral surface of the bar, which is free from external forces, we find that the first two of these equations are identically satisfied and the third one gives

$$\tau_{xz}l + \tau_{yz}m = 0$$

From Fig. 187b we see that

$$l = \cos(Nx) = \frac{dy}{ds}, \quad m = \cos(Ny) = -\frac{dx}{ds}$$

in which ds is an element of the bounding curve of the cross section. Then the condition at the boundary is

$$\tau_{xz} \frac{dy}{ds} - \tau_{yz} \frac{dx}{ds} = 0 \quad (d)$$

Turning to the compatibility equations (130), we see that the first three of these equations, containing normal stress components, and the last equation, containing τ_{xy} , are identically satisfied. The system (130) then reduces to the two equations

$$\nabla^2 \tau_{yz} = 0, \quad \nabla^2 \tau_{xz} = -\frac{P}{I(1+\nu)} \quad (e)$$

Thus the solution of the problem of bending of a prismatical cantilever of any cross section reduces to finding, for τ_{xz} and τ_{yz} , functions of x and y which satisfy the equation of equilibrium (c), the boundary condition (d), and the compatibility equations (e).

106. Stress Function. In discussing the bending problems we shall again make use of a stress function $\phi(x, y)$. It is easy to see that the differential equations of equilibrium (b) and (c) of the previous article are satisfied by taking

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + f(y), \quad \tau_{yz} = -\frac{\partial \phi}{\partial x} \quad (171)$$

in which ϕ is the stress function of x and y , and $f(y)$ is a function of y only, which will be determined later from the boundary condition.

Substituting (171) in the compatibility equations (e) of the previous article, we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= 0 \\ \frac{\partial}{\partial y} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) &= \frac{\nu}{1+\nu} \frac{P}{I} - \frac{df}{dy^2} \end{aligned}$$

From these equations we conclude that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu} \frac{Py}{I} - \frac{df}{dy} + c \quad (a)$$

where c is a constant of integration. This constant has a very simple physical meaning. Consider the rotation of an element of area in the plane of a cross section of the cantilever. This rotation is expressed by the equation (see page 225)

$$2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The rate of change of this rotation in the direction of the z -axis can be written in the following manner:

$$\frac{\partial}{\partial z} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}$$

and, by using Hooke's law and expressions (171) for the stress components, we find

$$\frac{\partial}{\partial z} (2\omega_z) = \frac{1}{G} \left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} \right) = -\frac{1}{G} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{df}{dy} \right)$$

Substituting in Eq. (a),

$$-G \frac{\partial}{\partial z} (2\omega_z) = \frac{\nu}{1+\nu} \frac{Py}{I} + c \quad (b)$$

If the x -axis is an axis of symmetry of the cross section, bending by a force P in this axis will result in a symmetrical pattern of rotation ω_z of elements of the cross section (corresponding to anticlastic curvature), with a mean value of zero for the whole cross section. The mean value of $\partial \omega_z / \partial z$ will then also be zero, and this requires that c in Eq. (b) be taken as zero. If the cross section is not symmetrical we can define¹ bending without torsion by means of the zero mean value of $\partial \omega_z / \partial z$, again of course requiring the zero value for c . Then Eq. (b) shows that $\partial \omega_z / \partial z$ vanishes for the elements of cross sections at the centroids—that is, these elements along the axis have zero relative rotation, and if one is fixed the others have no rotation—about the axis. With c zero Eq. (a) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu} \frac{Py}{I} - \frac{df}{dy} \quad (172)$$

Substituting (172) in the boundary condition (d) of the previous article we find

$$\frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} = \frac{\partial \phi}{\partial s} = \left[\frac{Px^2}{2I} - f(y) \right] \frac{dy}{ds} \quad (173)$$

From this equation the values of the function ϕ along the boundary of the cross section can be calculated if the function $f(y)$ is chosen. Equation (172), together with the boundary condition (173), determines the stress function ϕ .

In the problems which will be discussed later we shall take function $f(y)$ in such a manner as to make the right side of Eq. (173) equal to zero.² ϕ is then constant along the boundary. Taking this constant equal to zero, we reduce the bending problem to the solution of the differential equation (172) with the condition $\phi = 0$ at the boundary. This problem is analogous to that of the deflection of a membrane uniformly stretched, having the same boundary as the cross section of the bent bar and loaded by a continuous load given by the right side of Eq. (172). Several applications of this analogy will now be shown.

107. Circular Cross Section. Let the boundary of the cross section be given by the equation

$$x^2 + y^2 = r^2 \quad (a)$$

¹ J. N. Goodier, *J. Aeronaut. Sci.*, vol. 11, p. 273, 1944. A different definition was proposed by E. Trefftz, *Z. angew. Math. Mech.*, vol. 15, p. 220, 1935.

² See S. Timoshenko, *Bull. Inst. Engineers of Ways of Communications*, St. Petersburg, 1913. See also *Proc London Math. Soc.*, series 2, vol. 20, p. 398, 1922.

The right side of the boundary condition (173) becomes zero if we take

$$f(y) = \frac{P}{2I} (r^2 - y^2) \quad (b)$$

Substituting this into Eq. (172), the stress function ϕ is then determined by the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1+2\nu}{1+\nu} \frac{Py}{I} \quad (c)$$

and the condition that $\phi = 0$ at the boundary. Thus the stress function is given by the deflections of a membrane with circular boundary of radius r , uniformly stretched and loaded by a transverse load of intensity proportional to

$$-\frac{1+2\nu}{1+\nu} \frac{Py}{I}$$

It is easy to see that Eq. (c) and the boundary condition are satisfied in this case by taking

$$\phi = m(x^2 + y^2 - r^2)y \quad (d)$$

where m is a constant factor. This function is zero at the boundary (a) and satisfies Eq. (c) if we take

$$m = \frac{(1+2\nu)P}{8(1+\nu)I}$$

Equation (d) then becomes

$$\phi = \frac{(1+2\nu)P}{8(1+\nu)I} (x^2 + y^2 - r^2)y \quad (e)$$

The stress components are now obtained from Eqs. (171):

$$\begin{aligned} \tau_{xz} &= \frac{(3+2\nu)P}{8(1+\nu)I} \left(r^2 - x^2 - \frac{1-2\nu}{3+2\nu} y^2 \right) \\ \tau_{yz} &= -\frac{(1+2\nu)Pxy}{4(1+\nu)I} \end{aligned} \quad (174)$$

The vertical shearing-stress component τ_{xz} is an even function of x and y , and the horizontal component τ_{yz} is an odd function of the same variables. Hence the distribution of stresses (174) gives a resultant along the vertical diameter of the circular cross section.

Along the horizontal diameter of the cross section, $x = 0$; and we find, from (174),

$$\tau_{xz} = \frac{(3+2\nu)P}{8(1+\nu)I} \left(r^2 - \frac{1-2\nu}{3+2\nu} y^2 \right), \quad \tau_{yz} = 0 \quad (f)$$

The maximum shearing stress is obtained at the center ($y = 0$), where

$$(\tau_{xz})_{\max.} = \frac{(3+2\nu)Pr^2}{8(1+\nu)I} \quad (g)$$

The shearing stress at the ends of the horizontal diameter ($y = \pm r$) is

$$(\tau_{xz})_{y=\pm r} = \frac{(1+2\nu)Pr^2}{4(1+\nu)I} \quad (h)$$

It will be seen that the magnitude of the shearing stresses depends on the magnitude of Poisson's ratio. Taking $\nu = 0.3$, (g) and (h) become

$$(\tau_{xz})_{\max.} = 1.38 \frac{P}{A}, \quad (\tau_{xz})_{y=\pm r} = 1.23 \frac{P}{A} \quad (k)$$

where A is the cross-sectional area of the bar. The elementary beam theory, based on the assumption that the shearing stress τ_{xz} is uniformly distributed along the horizontal diameter of the cross section, gives

$$\tau_{xz} = \frac{4}{3} \frac{P}{A}$$

The error of the elementary solution for the maximum stress is thus in this case about 4 per cent.

108. Elliptic Cross Section. The method of the previous article can also be used in the case of an elliptic cross section. Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \quad (a)$$

be the boundary of the cross section. The right side of Eq. (173) will vanish if we take

$$f(y) = -\frac{P}{2I} \left(\frac{a^2}{b^2} y^2 - a^2 \right) \quad (b)$$

Substituting into Eq. (172), we find

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{Py}{I} \left(\frac{a^2}{b^2} + \frac{\nu}{1+\nu} \right) \quad (c)$$

This equation together with the condition $\phi = 0$ at the boundary determines the stress function ϕ . The boundary condition and Eq. (c) are satisfied by taking

$$\phi = \frac{(1+\nu)a^2 + \nu b^2}{2(1+\nu)(3a^2 + b^2)} \cdot \frac{P}{I} \left(x^2 + \frac{a^2}{b^2} y^2 - a^2 \right) y \quad (d)$$

When $a = b$, this solution coincides with solution (c) of the previous article.

Substituting (b) and (d) in Eqs. (171), we find the stress components

$$\begin{aligned}\tau_{zz} &= \frac{2(1+\nu)a^2+b^2}{(1+\nu)(3a^2+b^2)} \cdot \frac{P}{2I} \left[a^2 - x^2 - \frac{(1-2\nu)a^2}{2(1+\nu)a^2+b^2} y^2 \right] \\ \tau_{yz} &= -\frac{(1+\nu)a^2+\nu b^2}{(1+\nu)(3a^2+b^2)} \frac{Pxy}{I}\end{aligned}\quad (175)$$

For the horizontal axis of the elliptic cross section ($x = 0$), we find

$$\begin{aligned}\tau_{zz} &= \frac{2(1+\nu)a^2+b^2}{(1+\nu)(3a^2+b^2)} \frac{P}{2I} \left[a^2 - \frac{(1-2\nu)a^2}{2(1+\nu)a^2+b^2} y^2 \right] \\ \tau_{yz} &= 0\end{aligned}$$

The maximum stress is at the center ($y = 0$) and is given by equation

$$(\tau_{zz})_{\max.} = \frac{Pa^2}{2I} \left[1 - \frac{a^2 + \nu b^2 / (1 + \nu)}{3a^2 + b^2} \right]$$

If b is very small in comparison with a , we can neglect the terms containing b^2/a^2 , in which case

$$(\tau_{zz})_{\max.} = \frac{Pa^2}{3I} = \frac{4P}{3A}$$

which coincides with the solution of the elementary beam theory. If b is very large in comparison with a , we obtain

$$(\tau_{zz})_{\max.} = \frac{2}{1+\nu} \frac{P}{A}$$

The stress at the ends of the horizontal diameter ($y = \pm b$) for this case is

$$\tau_{zz} = \frac{4\nu}{1+\nu} \frac{P}{A}$$

The stress distribution along the horizontal diameter is in this case very far from uniform and depends on the magnitude of Poisson's ratio ν . Taking $\nu = 0.30$, we find

$$(\tau_{zz})_{\max.} = 1.54 \frac{P}{A}, \quad (\tau_{zz})_{x=0, y=b} = 0.92 \frac{P}{A}$$

The maximum stress is about 14 per cent larger than that given by the elementary formula,

109. Rectangular Cross Section. The equation for the boundary line in the case of the rectangle shown in Fig. 188 is

$$(x^2 - a^2)(y^2 - b^2) = 0 \quad (a)$$

If we substitute into Eq. (173) the constant $Pa^2/2I$ for $f(y)$, the expression $Px^2/2I - Pa^2/2I$ becomes zero along the sides $x = \pm a$ of the rectangle. Along the vertical sides $y = \pm b$ the derivative dy/ds is zero. Thus the right side of Eq. (173) is zero along the boundary line and we can take $\phi = 0$ at the boundary. Differential equation (172) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu} \frac{Py}{I} \quad (b)$$

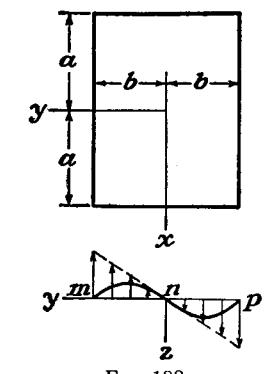


FIG. 188.

This equation, together with the boundary condition, determines completely the stress function. The problem reduces to the determination of the deflections of a uniformly stretched rectangular membrane produced by a continuous load, the intensity of which is proportional to

$$-\frac{\nu}{1+\nu} \frac{Py}{I}$$

The curve mnp in Fig. 188 represents the intersection of the membrane with the yz -plane.

From Eqs. (171) we see that shearing stresses can be resolved into the two following systems:

$$\begin{aligned}(1) \quad \tau_{zz}' &= \frac{P}{2I} (a^2 - x^2), & \tau_{yz}' &= 0 \\ (2) \quad \tau_{zz}'' &= \frac{\partial \phi}{\partial y}, & \tau_{yz}'' &= -\frac{\partial \phi}{\partial x}\end{aligned} \quad (c)$$

The first system represents the parabolic stress distribution given by the usual elementary beam theory. The second system, depending on the function ϕ , represents the necessary corrections to the elementary solution. The magnitudes of these corrections are given by the slopes of the membrane. Along the y -axis, $\partial \phi / \partial x = 0$, from symmetry, and the corrections to the elementary theory are vertical shearing stresses given by the slope $\partial \phi / \partial y$. From Fig. 188, τ_{zz}'' is positive at the points m and p and negative at n . Thus, along the horizontal

axis of symmetry, the stress τ_{xz} is not uniform as in the elementary theory but has maxima at the ends, m and p , and a minimum at the center n .

From the condition of loading of the membrane it can be seen that ϕ is an even function of x and an odd function of y . This requirement and also the boundary condition are satisfied by taking the stress function ϕ in the form of the Fourier series,

$$\phi = \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} A_{2m+1,n} \cos \frac{(2m+1)\pi x}{2a} \sin \frac{n\pi y}{b} \quad (d)$$

Substituting this into Eq. (b) and applying the usual method of calculating the coefficients of a Fourier series, we arrive at the equations

$$\begin{aligned} A_{2m+1,n} \pi^2 ab & \left[\left(\frac{2m+1}{2a} \right)^2 + \left(\frac{n}{b} \right)^2 \right] \\ & = - \frac{\nu}{1+\nu} \frac{P}{I} \int_{-a}^a \int_{-b}^b y \cos \frac{(2m+1)\pi x}{2a} \sin \frac{n\pi y}{b} dx dy \\ A_{2m+1,n} & = - \frac{\nu}{1+\nu} \frac{P}{I} \frac{8b(-1)^{m+n-1}}{\pi^4 (2m+1)n \left[\left(\frac{2m+1}{2a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]} \end{aligned}$$

Substituting in (d), we find

$$\phi = - \frac{\nu}{1+\nu} \frac{P}{I} \frac{8b^3}{\pi^4} \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} \frac{(-1)^{m+n-1} \cos \frac{(2m+1)\pi x}{2a} \sin \frac{n\pi y}{b}}{(2m+1)n \left[(2m+1)^2 \frac{b^2}{4a^2} + n^2 \right]}$$

Having this stress function, the components of shearing stress can be found from Eqs. (c).

Let us derive the corrections to the stress given by the elementary theory along the y -axis. It may be seen from the deflection of the membrane (Fig. 188) that along this axis the corrections have the largest values, and therefore the maximum stress occurs at the middle points of the sides $y = \pm b$. Calculating the derivative $\partial\phi/\partial y$ and taking $x = 0$, we find that

$$(\tau_{xz}'')_{x=0} = - \frac{\nu}{1+\nu} \frac{P}{I} \frac{8b^2}{\pi^3} \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} \frac{(-1)^{m+n-1} \cos \frac{n\pi y}{b}}{(2m+1) \left[(2m+1)^2 \frac{b^2}{4a^2} + n^2 \right]}$$

From this we find the following formulas for the center of the cross section ($y = 0$) and for the middle of the vertical sides of the rectangle:

$$\begin{aligned} (\tau_{xz}'')_{x=0, y=0} & = - \frac{\nu}{1+\nu} \frac{P}{I} \frac{8b^2}{\pi^3} \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} \frac{(-1)^{m+n-1}}{(2m+1) \left[(2m+1)^2 \frac{b^2}{4a^2} + n^2 \right]} \\ (\tau_{xz}'')_{x=0, y=b} & = \\ & - \frac{\nu}{1+\nu} \frac{P}{I} \frac{8b^2}{\pi^3} \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} \frac{(-1)^{m-1}}{(2m+1) \left[(2m+1)^2 \frac{b^2}{4a^2} + n^2 \right]} \end{aligned}$$

The summation of these series is greatly simplified if we use the known formulas

$$\begin{aligned} \sum_{n=1}^{n=\infty} \frac{1}{n^2} & = \frac{\pi^2}{6} \\ \sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^2} & = - \frac{\pi^2}{12} \\ \sum_{m=0}^{m=\infty} \frac{(-1)^m}{(2m+1)[(2m+1)^2 + k^2]} & = \frac{\pi^3}{32} \left(1 - \operatorname{sech} \frac{k\pi}{2} \right)^* \end{aligned}$$

* This formula can be obtained in the following manner: Using the trigonometric series (h) (p. 155) for the case of a tie rod loaded by the transverse force P and the direct tensile force S , we find that

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{n=1}^{n=\infty} \frac{\sin \frac{n\pi c}{l} \sin \frac{n\pi x}{l}}{n^2(n^2 + k^2)}$$

in which

$$k^2 = \frac{Sl^2}{EI\pi^2}$$

and c is the distance of the load P from the left support (Fig. 112). Substituting now $c = 0$ and $Pc = M$, we arrive at the following deflection curve, produced by the couple M applied at the left end

$$y = \frac{2Ml^3}{EI\pi^3} \sum_{n=1}^{n=\infty} \frac{\sin \frac{n\pi x}{l}}{n(n^2 + k^2)}$$

Then

$$\begin{aligned}(\tau_{xz}''|_{x=0, y=0}) &= -\frac{\nu}{1+\nu} \frac{3P}{2A} \cdot \frac{b^2}{a^2} \left[\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^2 \cosh \frac{n\pi a}{b}} \right] \\(\tau_{xz}''|_{x=0, y=b}) &= \frac{\nu}{1+\nu} \frac{3P}{2A} \frac{b^2}{a^2} \left[\frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{n^2 \cosh \frac{n\pi a}{b}} \right]\end{aligned}\quad (176)$$

in which $A = 4ab$ is the cross-sectional area. These series converge rapidly and it is not difficult to calculate corrections τ_{xz}'' for any value of the ratio a/b . These corrections must be added to the value

Point	$\frac{a}{b} =$	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$x = 0, y = 0$	Exact Approximate	0.983 0.981	0.940 0.936	0.856 0.856	0.805 0.826
$x = 0, y = b$	Exact Approximate	1.033 1.040	1.126 1.143	1.396 1.426	1.988 1.934

$3P/2A$ given by the elementary formula. In the first lines of the table above, numerical factors are given by which the approximate value of the shearing stress $3P/2A$ must be multiplied in order to obtain the exact values of the stress.¹ The Poisson's ratio ν is taken equal to one-fourth in this calculation. It is seen that the elementary formula gives very accurate values for these stresses when $a/b \geq 2$. For a square cross section the error in the maximum stress obtained by the elementary formula is about 10 per cent.

and the deflection at the middle is

$$\delta = \frac{2Ml^2}{EI\pi^3} \sum_{m=0}^{m=\infty} \frac{(-1)^m}{(2m+1)[(2m+1)^2 + k^2]} \quad (a)$$

The same deflection obtained by integration of the differential equation of the deflection curve is

$$\delta = \frac{Ml^2}{2EI\pi^2 k^2} \left(1 - \operatorname{sech} \frac{k\pi}{2} \right) \quad (b)$$

The above formula follows from comparison of (a) and (b).

¹ The figures of this table are somewhat different from those given by Saint-Venant. Checking of Saint-Venant's results showed that there is a numerical error in his calculations.

By using the membrane analogy useful approximate formulas for calculating these shearing stresses can be derived. If a is large in comparison with b (Fig. 188) we can assume that, at points sufficiently distant from the short sides of the rectangle, the surface of the membrane is practically cylindrical. Then Eq. (b) becomes

$$\frac{d^2\phi}{dy^2} = \frac{\nu}{1+\nu} \frac{Py}{I}$$

and we find

$$\phi = \frac{\nu}{1+\nu} \frac{P}{6I} (y^3 - b^2 y) \quad (e)$$

Substituting in Eqs. (c), the stresses along the y -axis are

$$\tau_{xz} = \frac{P}{2I} \left[a^2 + \frac{\nu}{1+\nu} \left(y^2 - \frac{b^2}{3} \right) \right] \quad (f)$$

It will be seen that for a narrow rectangle the correction to the elementary formula, given by the second term in the brackets, is always small.

If b is large in comparison with a , the deflections of the membrane at points distant from the short sides of the rectangle can be taken as a linear function of y , and from Eq. (b) we find

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^2} &= \frac{\nu}{1+\nu} \frac{Py}{I} \\ \phi &= \frac{\nu}{1+\nu} \frac{Py}{2I} (x^2 - a^2)\end{aligned}\quad (g)$$

Substituting in Eqs. (c), the shearing stress components are

$$\tau_{xz} = \frac{1}{1+\nu} \cdot \frac{P}{2I} (a^2 - x^2), \quad \tau_{yz} = -\frac{\nu}{1+\nu} \frac{P}{I} xy$$

At the centroid of the cross section ($x = y = 0$),

$$\tau_{xz} = \frac{1}{1+\nu} \frac{Pa^2}{2I}, \quad \tau_{yz} = 0$$

In comparison with the usual elementary solution the stress at this point is reduced in the ratio $1/(1+\nu)$.

To satisfy the boundary condition at the short sides of the rectangle we take, instead of expression (g), the following expression for the stress function:

$$\phi = \frac{\nu}{1+\nu} \frac{Py}{2I} (x^2 - a^2) [1 - e^{-(b-\nu)x}] \quad (h)$$

in which m must be determined from the condition of minimum energy (see Art. 97). In this manner we find

$$m = \frac{1}{2a} \sqrt{10}$$

With this value for m , and by using Eq. (h), we can calculate with sufficient accuracy the maximum shearing stress which occurs at the middle of the short sides of the rectangle.

If both sides of the rectangle are of the same order of magnitude we can obtain an approximate solution for the stress distribution in a polynomial form by taking the stress function in the form

$$\phi = (x^2 - a^2)(y^2 - b^2)(my + ny^3) \quad (k)$$

Calculating the coefficients m and n from the condition of minimum energy we find¹

$$m = -\frac{\nu}{1+\nu} \frac{P}{8Ib^2} \left(\frac{1}{7} + \frac{3a^2}{5b^2} \right) \left(\frac{1}{11} + \frac{8a^2}{b^2} \right) + \frac{1}{21} + \frac{9a^2}{35b^2}$$

$$n = -\frac{\nu}{1+\nu} \frac{P}{8Ib^4} \left(\frac{1}{7} + \frac{3a^2}{5b^2} \right) \left(\frac{1}{11} + \frac{8a^2}{b^2} \right) + \frac{1}{21} + \frac{9a^2}{35b^2}$$

The shearing stresses, calculated from (k), are

$$(\tau_{xz})_{x=0, y=0} = \frac{Pa^2}{2I} + ma^2b^2 \quad (l)$$

$$(\tau_{xz})_{x=0, y=b} = \frac{Pa^2}{2I} - 2a^2b^2(m + nb^2)$$

The approximate values of the shearing stresses given on the second lines of the table (see page 326) were calculated by using these formulas. It will be seen that the approximate formulas (l) give satisfactory accuracy in this range of values of a/b .

If the width of the rectangle is large in comparison with the depth maximum stresses much larger than the value $3P/2A$ of the elementary theory are found. Moreover if b/a exceeds 15 the maximum stress is no longer the component τ_{xz} at $x = 0, y = \pm b$, the mid-points of the vertical sides. It is the horizontal component τ_{yz} at points $x = a, y = \pm \eta$ on the top and bottom edges near the corners. Values of these stresses are given in the table² on page 329. The values of η are

¹ See Timoshenko, *loc. cit.*

² E. Reissner and G. B. Thomas, *J. Math. Phys.*, vol. 25, p. 241, 1946.

given in the form $(b - \eta)/2a$ in the last column, $b - \eta$ being the distance of the maximum point from the corner.

$\frac{b}{a}$	$\frac{(\tau_{xz})_{x=0, y=b}}{3P/2A}$	$\frac{(\tau_{yz})_{x=a, y=\eta}}{3P/2A}$	$\frac{b - \eta}{2a}$
0	1.000	0.000	0.000
2	1.39(4)	0.31(6)	0.31(4)
4	1.988	0.968	0.522
6	2.582	1.695	0.649
8	3.176	2.452	0.739
10	3.770	3.226	0.810
15	5.255	5.202	0.939
20	6.740	7.209	1.030
25	8.225	9.233	1.102
50	15.650	19.466	1.322

110. Additional Results. Let us consider a cross section the boundary of which consists of two vertical sides $y = \pm a$ (Fig. 189) and two hyperbolas¹

$$(1 + \nu)x^2 - \nu y^2 = a^2 \quad (a)$$

It is easy to show that this makes the right side of Eq. (173) on page 319 zero at the boundary if we take

$$f(y) = \frac{P}{2I} \left(\frac{\nu}{1+\nu} y^2 + \frac{a^2}{1+\nu} \right)$$

Substituting into Eq. (172), we find

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

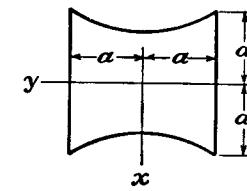


FIG. 189.

This equation and the boundary condition (173) are satisfied by taking $\phi = 0$. Then the shearing-stress components, from Eq. (171), are

$$\begin{aligned} \tau_{xz} &= \frac{P}{2I} \left(-x^2 + \frac{\nu}{1+\nu} y^2 + \frac{a^2}{1+\nu} \right) \\ \tau_{yz} &= 0 \end{aligned}$$

At each point of the cross section the shearing stress is vertical. The maximum of this stress is at the middle of the vertical sides of the cross section and is equal to

$$\tau_{\max.} = \frac{Pa^2}{2I}$$

The problem can also be easily solved if the boundary of the cross section is given by the equation

$$\left(\pm \frac{y}{b} \right)^{\frac{1}{\nu}} = \left(1 - \frac{x^2}{a^2} \right), \quad a > x > -a \quad (b)$$

¹ This problem was discussed by F. Grashof, "Elastizität und Festigkeit," p. 246, 1878.

For $\nu = \frac{1}{4}$, this cross-section curve has the shape shown in Fig. 190.

By taking

$$f(y) = \frac{Pa^2}{2I} \left[1 - \left(\pm \frac{y}{b} \right)^{\frac{1}{\nu}} \right]$$

the left side of the boundary condition (173) vanishes, i.e., ϕ must be constant along the boundary. Equation (172) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu} \frac{Py}{I} \pm \frac{Pa^2}{2bI\nu} \left(\pm \frac{y}{b} \right)^{\frac{1}{\nu}-1}$$

This equation and the boundary condition are satisfied by taking

$$\phi = \frac{Pa^2\nu}{2(1+\nu)I} \left[y \left(\frac{x^2}{a^2} - 1 \right) \pm b \left(\pm \frac{y}{b} \right)^{\frac{1}{\nu}+1} \right]$$

Substituting in Eqs. (171) we find

$$\tau_{xz} = \frac{P}{2(1+\nu)I} (a^2 - x^2), \quad \tau_{yz} = -\frac{P\nu}{(1+\nu)I} xy \quad (c)$$

We can arrive at the same result in a different way. In discussing stresses in a rectangular beam the width of which is large in comparison with the depth, we used as an approximate solution for the stress function [Eq. (g), Art. 109] the expression

$$\phi = \frac{\nu}{1+\nu} \frac{Py}{2I} (x^2 - a^2)$$

from which the expressions (c) for stress components may be derived. The equation of the boundary can now be found from the condition that at the boundary the direction of shearing stress coincides with the tangent to the boundary. Hence

$$\frac{dx}{\tau_{xz}} = \frac{dy}{\tau_{yz}}$$

Substituting from (c) and integrating, we arrive at the equation of the boundary,

$$y = b(a^2 - x^2)^{\nu}$$

By using the energy method (Art. 109) we may arrive at an approximate solution in many other cases. Let us consider, for instance, the cross section shown in Fig. 191. The vertical sides of the boundary are given by the equation $y = \pm b$, and the other two sides are arcs of the circle

$$x^2 + y^2 - r^2 = 0 \quad (d)$$

The right side of Eq. (173) vanishes if we take

$$f(y) = \frac{P}{2I} (r^2 - y^2)$$

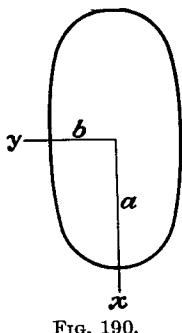


FIG. 190.

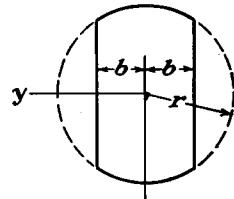


FIG. 191.

Then an approximate expression for the stress function is

$$\phi = (y^2 - b^2)(x^2 + y^2 - r^2)(Ay + By^3 + \dots)$$

in which the coefficients A, B, \dots are to be calculated from the condition of minimum energy.

Solutions for many shapes of cross section have been obtained by using polar and other curvilinear coordinates, and functions of the complex variable. These include sections bounded by two circles, concentric¹ or nonconcentric,² a circle with radial slits,³ a cardioid,⁴ a limacon,⁵ an elliptic limacon,⁶ two confocal ellipses,⁷ an ellipse and confocal hyperbolas,⁸ triangles and polygons⁹ including a rectangle with slits,¹⁰ and a sector of a circular ring.¹¹

111. Nonsymmetrical Cross Sections. As a first example let us consider the case of an isosceles triangle (Fig. 192). The boundary of the cross section is given by the equation

$$(y - a)[x + (2a + y) \tan \alpha][x - (2a + y) \tan \alpha] = 0$$

The right side of Eq. (173) is zero if we take

$$f(y) = \frac{P}{2I} (2a + y)^2 \tan^2 \alpha$$

Equation (172) for determining the stress function ϕ then becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu} \frac{Py}{I} - \frac{P}{I} (2a + y) \tan^2 \alpha \quad (a)$$

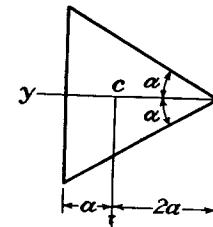


FIG. 192.

An approximate solution may be obtained by using the energy method. In the particular case when

$$\tan^2 \alpha = \frac{\nu}{1+\nu} = \frac{1}{3} \quad (b)$$

¹ A solution is given in A. E. H. Love's "Mathematical Theory of Elasticity," 4th ed. p. 335, and in I. S. Sokolnikoff's "Mathematical Theory of Elasticity," p. 253.

² B. R. Seth, *Proc. Indian Acad. Sci.*, vol. 4, sec. A, p. 531, 1936, and vol. 5, p. 23, 1937.

³ W. M. Shepherd, *Proc. Roy. Soc. (London)*, series A, vol. 138, p. 607, 1932; L. A. Wigglesworth, *Proc. London Math. Soc.*, series 2, vol. 47, p. 20, 1940, and *Proc. Roy. Soc. (London)*, series A, vol. 170, p. 365, 1939.

⁴ W. M. Shepherd, *Proc. Roy. Soc. (London)*, series A, vol. 154, p. 500, 1936.

⁵ D. L. Holl and D. H. Rock, *Z. angew. Math. Mech.*, vol. 19, p. 141, 1939.

⁶ A. C. Stevenson, *Proc. London Math. Soc.*, series 2, vol. 45, p. 126, 1939.

⁷ A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 336.

⁸ B. G. Galerkin, *Bull. Inst. Engineers of Ways of Communication*, St. Petersburg, vol. 96, 1927. See also S. Ghosh, *Bull. Calcutta Math. Soc.*, vol. 27, p. 7, 1935.

⁹ B. R. Seth, *Phil. Mag.*, vol. 22, p. 582, 1936, and vol. 23, p. 745, 1937.

¹⁰ D. F. Gunder, *Physics*, vol. 6, p. 38, 1935.

¹¹ M. Seegar and K. Pearson, *Proc. Roy. Soc. (London)*, series A, vol. 96, p. 211, 1920.

an exact solution of Eq. (a) is obtained by taking for the stress function the expression

$$\phi = \frac{P}{6I} \left[x^2 - \frac{1}{3} (2a + y)^2 \right] (y - a)$$

The stress components are then obtained from Eqs. (171):

$$\begin{aligned}\tau_{xz} &= \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + \frac{P}{6I} (2a + y)^2 = \frac{2\sqrt{3}P}{27a^4} [-x^2 + a(2a + y)] \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} = \frac{2\sqrt{3}P}{27a^4} x(a - y)\end{aligned}\quad (c)$$

Along the y -axis, $x = 0$, and the resultant shearing stress is vertical and is represented by the linear function

$$(\tau_{xz})_{x=0} = \frac{2\sqrt{3}P}{27a^3} (2a + y)$$

The maximum value of this stress, at the middle of the vertical side of the cross section, is

$$\tau_{\max.} = \frac{2\sqrt{3}P}{9a^2} \quad (d)$$

By calculating the moment with respect to the z -axis of the shearing forces given by the stresses (c), it can be shown that in this case the resultant shearing force passes through the centroid C of the cross section.

Let us consider next the more general case of a cross section with a horizontal axis of symmetry (Fig. 193), the lower and upper portions of the boundary being given by the equations



FIG. 193.

$$[x + \psi(y)][x - \psi(y)] = x^2 - [\psi(y)]^2$$

vanishes along the boundary and in our expressions for stress components (171) we can take

$$f(y) = \frac{P}{2I} [\psi(y)]^2$$

With this assumption the stress function has to satisfy the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\nu}{1+\nu} \frac{Py}{I} - \frac{P}{I} \psi(y) \frac{d\psi}{dy}$$

and be constant at the boundary. The problem is reduced to that of finding the deflections of a uniformly stretched membrane when the intensity of the load is given by the right-hand side of the above equation. This latter problem can usually be solved with sufficient accuracy by using the energy method as was shown in the case of the rectangular cross section (page 328).

The case shown in Fig. 194 can be treated in a similar manner. Assume, for example, that the cross section is a parabolic segment and that the equation of the parabola is

$$x^2 = A(y + a)$$

Then we take

$$f(y) = \frac{P}{2I} \cdot A(y + a)$$

With this expression for $f(y)$ the first factor on the right-hand side of Eq. (173) vanishes along the parabolic portion of the boundary. The factor dy/ds vanishes along the straight-line portion of the boundary. Thus we find again that the stress function is constant along the boundary and the problem can be treated by using the energy method.

112. Shear Center. In discussing the cantilever problem we chose for z -axis the centroidal axis of the bar and for x - and y -axes the principal centroidal axes of the cross section. We assumed that the force P is parallel to the x -axis and at such a distance from the centroid that twisting of the bar does not occur. This distance, which is of importance in practical calculations, can readily be found once the stresses represented by Eqs. (171) are known. For this purpose we evaluate the moment about the centroid produced by the shear stresses τ_{xz} and τ_{yz} . This moment evidently is

$$M_z = \iint (\tau_{xz}y - \tau_{yz}x) dx dy \quad (a)$$

Observing that the stresses distributed over the end cross section of the beam are statically equivalent to the acting force P we conclude that the distance d of the force P from the centroid of the cross section is

$$d = \frac{|M_z|}{P} \quad (b)$$

For positive M_z the distance d must be taken in the direction of positive y . In the preceding discussion the assumption was made that the force is acting parallel to the x -axis.

When the force P is parallel to the y -axis instead of the x -axis we can, by a similar calculation, establish the position of the line of action of P for which no rotation of centroidal elements of cross sections occurs. The intersection point of the two lines of action of the bending forces has an important significance. If a force, perpendicular to the axis of the beam, is applied at that point we can resolve it into two components parallel to the x - and y -axes and on the basis of the above discussion we conclude that it does not produce rotation of centroidal elements of

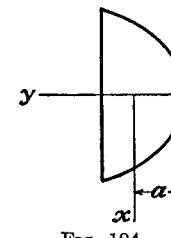


FIG. 194.

cross sections of the beam. This point is called the *shear center*—sometimes also the center of flexure, or flexural center.

If the cross section of the beam has two axes of symmetry we can conclude at once that the shear center coincides with the centroid of the cross section. When there is only one axis of symmetry we conclude, from symmetry, that the shear center will be on that axis. Taking the symmetry axis for y -axis, we calculate the position of the shear center from Eq. (b).

Let us consider, as an example, a semicircular cross section¹ as shown in Fig. 195. To find the shearing stresses we can utilize the solution developed for circular beams (see page 319). In that case there are no stresses acting on the vertical diametral section xz . We can imagine the beam divided by the xz -plane into two halves each of which represents a semicircular beam bent by the force $P/2$. The stresses are given by Eq. (174).

Substituting into Eq. (a), integrating, and dividing M_z by $P/2$, we find for the distance of the bending force from the origin O the value

$$e = \frac{2M_z}{P} = \frac{8}{15\pi} \frac{3+4\nu}{1+\nu} r$$

This defines the position of the force for which the cross-sectional element at point O , the center of the circle, does not rotate. At the same time an element at the centroid of the semicircular cross section will rotate by the amount [see Eq. (b) page 318]

$$\omega = \frac{\nu P(l-z)}{EI} \cdot 0.424r$$

where $0.424r$ is the distance from the origin O to the centroid of the semicircle. To eliminate this rotation a torque as shown in Fig. 195 must be applied. The magnitude of this torque is found by using the table on page 279, which gives for a semicircular cross section the angle of twist per unit length

$$\theta = \frac{M_t}{0.296Gr^4}$$

¹ See S. Timoshenko, *Bull. Inst. Engineers of Ways of Communications*, St. Petersburg, 1913. It seems that the displacement of the bending force from the centroid of the cross section was investigated in this paper for the first time.

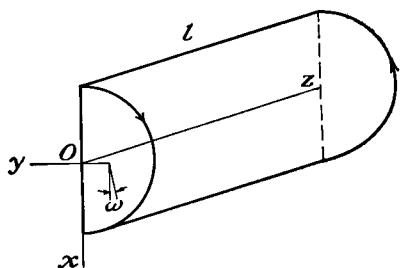


FIG. 195.

Then the condition that centroidal elements of cross sections do not rotate gives

$$\frac{M_t(l-z)}{0.296Gr^4} = \frac{\nu P(l-z)}{EI} \cdot 0.424r$$

and

$$M_t = \frac{\nu P \cdot 0.296r^4 \cdot 0.424r}{2(1+\nu)I}$$

This torque will be produced by shifting the bending force $P/2$ toward the z -axis by the amount

$$\delta = \frac{2M_t}{P} = \frac{8\nu \cdot 0.296 \cdot 0.424r}{2(1+\nu)\pi}$$

This quantity must be subtracted from the previously calculated distance e to obtain the distance of the shear center from the center O of the circle. Assuming $\nu = 0.3$, we obtain

$$e - \delta = 0.548r - 0.037r = 0.511r$$

In sections as in Fig. 193 the shearing-stress components are

$$\tau_{xz} = \frac{\partial \phi}{\partial y} - \frac{P}{2I} [x^2 - \psi^2(y)], \quad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

Hence

$$M_z = \iint \left(\frac{\partial \phi}{\partial y} y + \frac{\partial \phi}{\partial x} x \right) dx dy - \frac{P}{2I} \iint [x^2 - \psi^2(y)] y dx dy \quad (c)$$

Integrating by parts and observing that ϕ vanishes at the boundary $x = \pm\psi(y)$, we obtain

$$\begin{aligned} \iint \left(\frac{\partial \phi}{\partial y} y + \frac{\partial \phi}{\partial x} x \right) dx dy &= -2 \iint \phi dx dy \\ \iint [x^2 - \psi^2(y)] dx &= \frac{2}{3}\psi^3(y) - 2y\psi^3(y) = -\frac{4}{3}\psi^3(y) \\ \iint [x^2 - \psi^2(y)] y dx dy &= -\frac{4}{3}\int y\psi^3(y) dy \\ I &= \iint x^2 dx dy = \frac{2}{3}\int \psi^3(y) dy \end{aligned}$$

Substituting in (c) and dividing by P we find

$$d = \frac{|M_z|}{P} = \left| -\frac{2}{P} \iint \phi dx dy + \frac{\int y\psi^3(y) dy}{\int \psi^3(y) dy} \right|$$

Knowing $\psi(y)$ and using the membrane analogy for finding ϕ we can always calculate¹ with sufficient accuracy the position of the shear center for these cross sections.

¹ Examples of such calculations can be found in the book by L. S. Leibenson, "Variational Methods for Solving Problems of the Theory of Elasticity," Moscow, 1943.

The question of the shear center is especially important in the case of thin-walled open sections. Its position can be easily determined for such sections with sufficient accuracy by assuming that the shearing stresses are uniformly distributed over the thickness of the wall and are parallel to the middle surface of the wall.¹

The location of the shear center in the cross section is determined by the shape of the section only. On the other hand the location of the center of twist (see page 271) is dependent on the manner in which the bar is supported. By choosing this manner of support suitably the axis of twist can be made to coincide with the axis of shear centers. It can be shown that this occurs when the bar is so supported that the integral $\iint w^2 dx dy$ over the cross section is a minimum,² w being the warping displacement of torsion (indeterminate by a linear function of x and y before this condition is applied). In practice the fixing will usually disturb the stress distribution near the fixed end—as for instance when it prevents displacements in the end section completely. In that case, if we regard the bending force as a concentrated load at the shear center, producing zero rotation, the reciprocal theorem (page 239) shows that a torque will produce zero deflection of the shear center. This indicates that the center of twist will coincide with the shear center.³ The argument is of an approximate character since the existence of a center of twist depends on absence of deformation of cross sections in their planes, and this will not hold in the disturbed zone near the fixed end.

113. The Solution of Bending Problems by the Soap-film Method. The exact solutions of bending problems are known for only a few special cases in which the cross sections have certain simple forms. For practical purposes it is important to have means of solving the problem for any assigned shape of the cross section. This can be accomplished by numerical calculations based on equations of finite differences as explained in the Appendix, or experimentally by the soap-film method,⁴ analogous to that used in solving torsional problems (see page 289). For deriving the theory of the soap-film method we use Eqs. (171), (172), and (173) (see Art. 106). Taking

$$f(y) = \frac{\nu}{2(1+\nu)} \frac{Py^2}{I}$$

¹ References may be found in S. Timoshenko, "Strength of Materials," 2d ed., vol. 2, p. 55.

² R. Kappus, *Z. angew. Math. Mech.*, vol. 19, p. 347, 1939; A. Weinstein, *Quart. Applied Math.*, vol. 5, p. 79, 1947.

³ See R. V. Southwell, "Introduction to the Theory of Elasticity," p. 29; W. J. Duncan, D. L. Ellis, and C. Scruton, *Phil. Mag.*, vol. 16, p. 201, 1933.

⁴ This method was indicated first by Vening Meinesz, *De Ingenieur*, p. 108, Holland, 1911. It was developed independently by A. A. Griffith and G. I. Taylor, *Tech. Rept. Natl. Advisory Comm. Aeronaut.*, vol. 3, p. 950, 1917-1918. The results given here are taken from this paper.

Eq. (172) for the stress function is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (a)$$

This is the same equation as for an unloaded and uniformly stretched membrane (see page 271). The boundary condition (173) becomes

$$\frac{\partial \phi}{\partial s} = \left[\frac{Px^2}{2I} - \frac{\nu}{2(1+\nu)} \frac{Py^2}{I} \right] \frac{dy}{ds} \quad (b)$$

Integrating along the boundary s we find the expression

$$\phi = \frac{P}{I} \int \frac{x^2 dy}{2} - \frac{\nu}{2(1+\nu)} \frac{Py^3}{3I} + \text{constant} \quad (c)$$

from which the value of ϕ for every point of the boundary can be calculated. $\int(x^2/2) dy$ vanishes when taken around the boundary, since it represents the moment of the cross section with respect to the y -axis, which passes through the centroid of the cross section. Hence ϕ , calculated from (c), is represented along the boundary by a closed curve.

Imagine now that the soap film is stretched over this curve. Then the surface of the film satisfies Eq. (a) and boundary condition (c). Hence the ordinates of the film represent the stress function ϕ at all points of the cross section to the scale used for representation of the function ϕ along the boundary [Eq. (c)].

The photograph 196a illustrates one of the methods used for construction of the boundary of the soap film. A hole is cut in a plate of celluloid, of such a shape that after the plate is bent the projection of the edge of the hole on the horizontal plane has the same shape as the boundary of the cross section of the beam. The plate is fixed on vertical studs and adjusted by means of nuts and washers until the ordinates along the edge of the hole represent to a certain scale the values of ϕ given by expression (c). The photograph 196b illustrates another method for construction of the boundary by using thin sheets of annealed brass.¹ The small corrections of ordinates along the edge of the hole can be secured by slight bending of the boundary.

The analogy between the soap-film and the bending-problem equations holds rigorously only in the case of infinitely small deflections of the membrane. In experimenting it is desirable to have the total range of the ordinates of the film not more than one-tenth of the maximum

¹ See the paper by P. A. Cushman, *Trans. A.S.M.E.*, 1932.

horizontal dimension. If necessary the range of the function along the boundary can be reduced by introducing a new function ϕ_1 , instead of ϕ , by the substitution

$$\phi = \phi_1 + ax + by \quad (d)$$

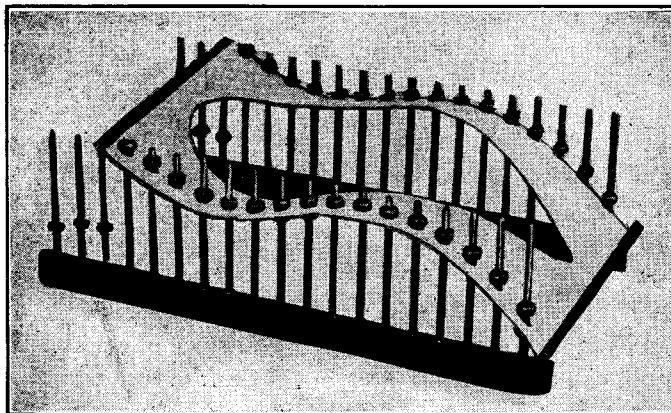


FIG. 196a.

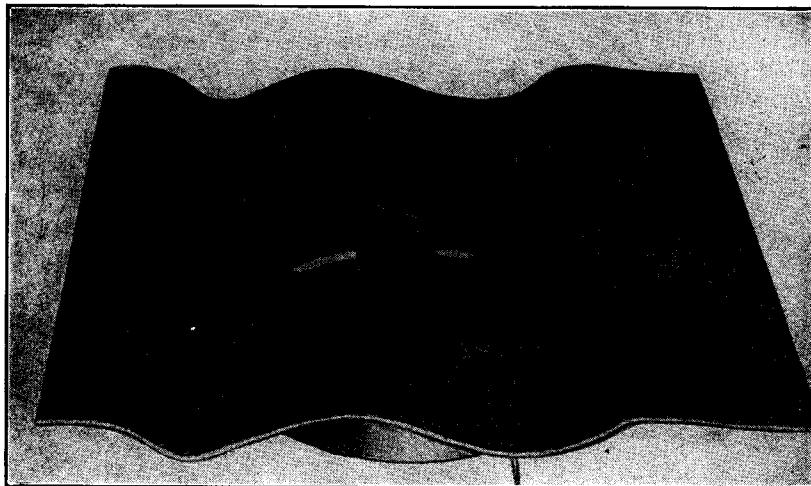


FIG. 196b.

where a and b are arbitrary constants. It may be seen that the function ϕ_1 also satisfies the membrane equation (a). The values of the function ϕ_1 along the boundary, from Eqs. (c) and (d), are given by

$$\phi_1 = \frac{P}{I} \int \frac{x^2}{2} dy - \frac{\nu}{2(1+\nu)} \frac{Py^3}{3I} - ax - by + \text{constant}$$

The reduction of the range of the function ϕ_1 at the boundary can usually be effected by a proper adjustment of the constants a and b .

When the function ϕ_1 is obtained from the soap film, the function ϕ is calculated from Eq. (d). Having the stress function ϕ , the shearing-stress components are obtained from Eqs. (171), which have now the form

$$\begin{aligned}\tau_{xz} &= \frac{\partial \phi}{\partial y} - \frac{Px^2}{2I} + \frac{\nu}{2(1+\nu)} \frac{Py^2}{I} \\ \tau_{yz} &= - \frac{\partial \phi}{\partial x}\end{aligned}\quad (e)$$

The stress components can now be easily calculated for every point of the cross section provided we know the values of the derivatives $\partial \phi / \partial y$ and $\partial \phi / \partial x$ at this point. These derivatives are given by the slopes of the soap film in the y - and x -directions. For determining slopes we proceed as in the case of torsional problems and first map contour lines of the film surface. From the contour map the slopes may be found by drawing straight lines parallel to the coordinate axes and constructing curves representing the corresponding sections of the soap film. The slopes found in this way must now be inserted in expressions (e) for shear-stress components. The accuracy of this procedure can be checked by calculating the resultant of all the shear stresses distributed over the cross section. This resultant should be equal to the bending force P applied at the end of the cantilever.

Experiments show that a satisfactory accuracy in determining stresses can be attained by using the soap-film method. The results obtained for an I-section¹ are shown in Figs. 197. From these figures it may be seen that the usual assumptions of the elementary theory, that the web of an I-beam takes most of the shearing force and that the shearing stresses are constant across the thickness of the web, are fully confirmed. The maximum shearing stress at the neutral plane is in very good agreement with that calculated from the elementary theory. The component τ_{yz} is practically zero in the web and reaches a maximum at the reentrant corner. This maximum should depend on the radius of the fillet rounding the reentrant corner. For the proportions taken, it is only about one-half of the maximum stress τ_{xz} at the neutral plane. The lines of equal shearing-stress components, giving the ratio of these components to the average shearing stress P/A , are shown in the figures.

The stress concentration at the reentrant corner has been studied for

¹ In this case of symmetry only one-quarter of the cross section need be investigated.

the case of a T-beam. The radius of the reentrant corner was increased in a series of steps, and contour lines were mapped for each case. It was shown in this manner that the maximum stress at the corner equals the maximum stress in the web when the radius of the fillet is about one-sixteenth of the thickness of the web.

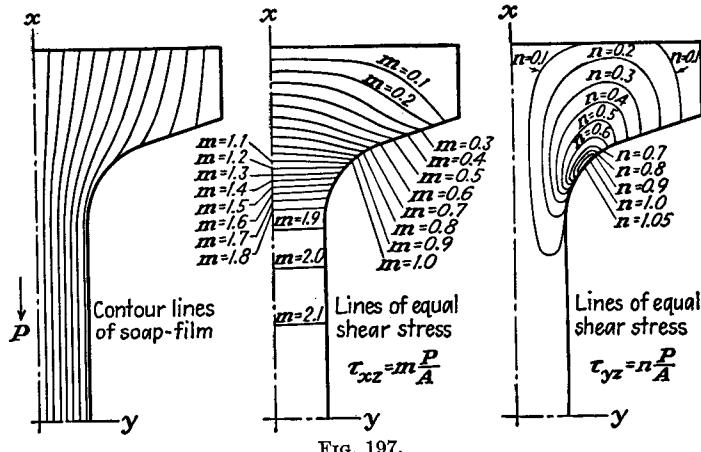


FIG. 197.

114. Displacements. When the stress components are found, the displacements u, v, w can be calculated in the same manner as in the case of pure bending (see page 250). Let us consider here the deflection curve of the cantilever. The curvatures of this line in the xz - and yz -planes are given with sufficient accuracy by the values of the derivatives $\partial^2 u / \partial z^2$ and $\partial^2 v / \partial z^2$ for $x = y = 0$. These quantities can be calculated from the equations

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \epsilon_z}{\partial x} = \frac{1}{G} \frac{\partial \tau_{xz}}{\partial z} - \frac{1}{E} \frac{\partial \sigma_z}{\partial x} = \frac{P(l-z)}{EI} \\ \frac{\partial^2 v}{\partial z^2} &= \frac{\partial \gamma_{yz}}{\partial z} - \frac{\partial \epsilon_z}{\partial y} = 0\end{aligned}\quad (a)$$

We see that the center line of the cantilever is bent in the xz -plane in which the load is acting, and the curvature at any point is proportional to the bending moment at this point, as is usually assumed in the elementary theory of bending. By integration of the first of Eqs. (a), we find

$$u = \frac{Plz^2}{2EI} - \frac{Pz^3}{6EI} + cz + d \quad (b)$$

where c and d are constants of integration which must be determined from the conditions at the fixed end of the cantilever. If the end of the

center line is built in, u and du/dz are zero when $z = 0$, and hence constants c and d in Eq. (b) are zero.

The cross sections of the beam do not remain plane. They become warped, owing to the action of shearing stresses. The angle of inclination of an element of the surface of the warped cross section at the centroid to the deflected center line is

$$\frac{\pi}{2} - \frac{(\tau_{xz})_{x=0, y=0}}{G}$$

and can be calculated if the shearing stresses at the centroid are known.

115. Further Investigations of Bending. In the foregoing articles we discussed the problem of bending of a cantilever fixed at one end and loaded by a transverse force on the other. The solutions obtained are the exact solutions of the bending problem, provided the external forces are distributed over the terminal cross sections in the same manner as the stresses $\sigma_z, \tau_{xz}, \tau_{yz}$ found in the solutions. If this condition is not fulfilled there will be local irregularities in the stress distribution near the ends of the beam, but on the basis of Saint-Venant's principle we can assume that at a sufficient distance from the ends, say at a distance larger than cross-sectional dimensions of the beam, our solutions are sufficiently accurate. By using the same principle we may extend the application of the above solutions to other cases of loading and supporting of beams. We may assume with sufficient accuracy that the stresses at any cross section of a beam, at sufficient distance from the loads, depend only on the magnitude of the bending moment and the shearing force at this cross section and can be calculated by superposition of the solutions obtained before for the cantilever.

If the bending forces are inclined to the principal axes of the cross section of the beam, they can always be resolved into two components acting in the direction of the principal axes and bending in each of the two principal planes can be discussed separately. The total stresses and displacements will then be obtained by using the principle of superposition.

Near the points of application of external forces there are irregularities in stress distribution which we discussed before for the particular case of a narrow rectangular cross section (see Art. 36). Analogous discussion for other shapes of cross section shows that these irregularities are of a local character.¹

¹ See L. Pochhammer's, "Untersuchungen über das Gleichgewicht des elastischen Stabes," Kiel, 1879. See also a paper by J. Dougall, *Trans. Roy. Soc. (Edinburgh)*, vol. 49, p. 895, 1914.

The problem of bending is solved also for certain cases of distributed load.¹ It is shown that in such cases the central line of the beam usually extends or contracts as in the case of the narrow rectangular cross section (see Art. 21) already discussed. The curvature of the center line in these cases is no longer proportional to the bending moment, but the necessary corrections are small and can be neglected in practical problems. For instance, in the case of a circular beam bent by its own weight,² the curvature at the fixed end is given by the equation

$$\frac{1}{r} = \frac{M}{EI} \left[1 - \frac{7 + 12\nu + 4\nu^2}{6(1+\nu)} \frac{a^2}{l^2} \right]$$

in which a is the radius of the cross section, and l the length of the cantilever. The second term in the brackets represents the correction to the curvature arising from the distribution of the load. It is small, of the order of a^2/l^2 . This conclusion holds also for beams of other shapes of cross section bent by their own weight.³

¹ J. H. Michell, *Quart. J. Math.*, vol. 32, 1901; also K. Pearson, *ibid.*, vol. 24, 1889, and K. Pearson and L. N. G. Filon, *ibid.*, vol. 31, 1900.

² This problem is discussed by A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 362, 1927.

³ The case of a cantilever of an elliptical cross section has been discussed by J. M. Klitchieff, *Bull. Polytech. Inst.*, St. Petersburg, p. 441, 1915.

CHAPTER 13

AXIALLY SYMMETRICAL STRESS DISTRIBUTION IN A SOLID OF REVOLUTION

116. General Equations. Many problems in stress analysis which are of practical importance are concerned with a solid of revolution deformed symmetrically with respect to the axis of revolution. The simplest examples are the circular cylinder strained by uniform internal or external pressure, and the rotating circular disk (see Arts. 26 and 30). For problems of this kind it is often convenient to use cylindrical coordinates [see Eqs. (170), page 306]. The deformation being symmetrical with respect to the z -axis, it follows that the stress components are independent of the angle θ , and all derivatives with respect to θ vanish. The components of shearing stress τ_{rz} and $\tau_{\theta z}$ also vanish on account of the symmetry. Thus Eqs. (170) reduce to

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \quad (177)$$

The strain components, for axially symmetrical deformation, are, from Eqs. (169),

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \quad (178)$$

It is again of advantage to introduce a stress function ϕ . It may be verified by substitution that Eqs. (177) are satisfied if we take

$$\begin{aligned} \sigma_r &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right) \\ \sigma_\theta &= \frac{\partial}{\partial z} \left(\nu \nabla^2 \phi - \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \\ \sigma_z &= \frac{\partial}{\partial z} \left[(2 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right] \\ \tau_{rz} &= \frac{\partial}{\partial r} \left[(1 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right] \end{aligned} \quad (179)$$

provided that the stress function ϕ satisfies the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right) = \nabla^2 \nabla^2 \phi = 0 \quad (180)$$

The symbol ∇^2 denotes the operation

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (a)$$

which corresponds to Laplace's operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in rectangular coordinates [see Eq. (d), page 57]. It should be noted that the stress function ϕ does not depend on θ , so that the third term in (a) gives zero when applied to ϕ .

We now transform the compatibility equations (130) (see page 232) to cylindrical coordinates. Denoting by θ the angle between r and the x -axis we have [see Eq. (13)]

$$\begin{aligned} \sigma_x &= \sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta \\ \sigma_y &= \sigma_r \sin^2 \theta + \sigma_\theta \cos^2 \theta \end{aligned} \quad (b)$$

unaffected by the presence of σ_z , τ_{rz} .

Then

$$\begin{aligned} \nabla^2 \sigma_x &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) (\sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta) \\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) (\sigma_r \cos^2 \theta + \sigma_\theta \sin^2 \theta) \\ &\quad - \frac{2}{r^2} \cos 2\theta (\sigma_r - \sigma_\theta) \end{aligned} \quad (c)$$

Using the symbol Θ for the sum of the three normal components of stress and applying Eq. (b) on page 57, we obtain for a symmetrical stress distribution

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial^2 \Theta}{\partial r^2} \cos^2 \theta + \frac{\partial \Theta}{\partial r} \frac{\sin^2 \theta}{r} \quad (d)$$

Substituting (c) and (d) in the first of Eqs. (130), we obtain

$$\begin{aligned} &\left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \sigma_r - \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial r^2} \right] \cos^2 \theta \\ &+ \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \sigma_\theta + \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{1}{r} \frac{\partial \Theta}{\partial r} \right] \sin^2 \theta = 0 \end{aligned}$$

This equation holds for any value of θ , hence

$$\begin{aligned} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \sigma_r - \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial r^2} &= 0 \\ \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \sigma_\theta + \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{1}{r} \frac{\partial \Theta}{\partial r} &= 0 \end{aligned} \quad (e)$$

The same result is obtained by considering the second of Eqs. (130), so that Eqs. (e) take the place of the first two equations of the system (130) for the case of a symmetrical deformation. The third equation of (130) retains the same form in cylindrical coordinates.

Consider now the remaining three equations of the system (130), containing shearing-stress components. In the case of symmetrical deformation only the shearing stress τ_{rz} is different from zero, and the stress components τ_{xz} and τ_{yz} , acting on a plane perpendicular to the z -axis, are obtained by resolving τ_{rz} into two components parallel to the x - and y -axes,

$$\tau_{xz} = \tau_{rz} \cos \theta, \quad \tau_{yz} = \tau_{rz} \sin \theta$$

We have also

$$\begin{aligned} \frac{\partial^2 \Theta}{\partial x \partial z} &= \frac{\partial^2 \Theta}{\partial r \partial z} \cos \theta \\ \nabla^2 \tau_{xz} &= \nabla^2 (\tau_{rz} \cos \theta) = \left(\nabla^2 \tau_{rz} - \frac{\tau_{rz}}{r^2} \right) \cos \theta \end{aligned}$$

Substituting in the fifth of Eqs. (130), we obtain

$$\nabla^2 \tau_{rz} - \frac{1}{r^2} \tau_{rz} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial r \partial z} = 0 \quad (f)$$

The same result is obtained by considering the fourth of Eqs. (130). The last equation of the system (130) can also be transformed to cylindrical coordinates by substituting

$$\tau_{xy} = \frac{1}{2} (\sigma_r - \sigma_\theta) \sin 2\theta$$

In this way we find

$$(1 + \nu) \nabla^2 \left[\frac{1}{2} (\sigma_r - \sigma_\theta) \sin 2\theta \right] + \frac{\sin 2\theta}{2} \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} \right) \Theta = 0$$

This equation follows at once from Eqs. (e) on subtracting one from the other. Hence the compatibility equations (130), in the case of a deformation symmetrical with respect to an axis, are, in cylindrical coordinates,

$$\begin{aligned}\nabla^2\sigma_r - \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial r^2} &= 0 \\ \nabla^2\sigma_\theta + \frac{2}{r^2}(\sigma_r - \sigma_\theta) + \frac{1}{1+\nu}\frac{1}{r}\frac{\partial\Theta}{\partial r} &= 0 \\ \nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} &= 0 \\ \nabla^2\tau_{rz} - \frac{1}{r^2}\tau_{rz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial r\partial z} &= 0\end{aligned}\quad (g)$$

It can be shown that all these equations are satisfied by the expressions for the stresses given in Eqs. (179) when the stress function satisfies Eq. (180). We see that the discussion of problems involving stress distributions symmetrical about an axis reduces to finding in each particular case the solution of Eq. (180), satisfying the boundary conditions of the problem.¹

In some cases it is useful to have Eq. (180) in polar coordinates R and ψ (Fig. 198) instead of cylindrical coordinates r and z . This transformation can easily be accomplished by using the formulas of Art. 25. We find

$$\begin{aligned}\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \psi^2} \\ \frac{1}{r}\frac{\partial}{\partial r} &= \frac{1}{R}\frac{\partial}{\partial R} \left(\sin \psi \frac{\partial}{\partial R} + \cos \psi \frac{\partial}{\partial \psi} \right) = \frac{1}{R}\frac{\partial}{\partial R} + \frac{\operatorname{ctn} \psi}{R^2}\frac{\partial}{\partial \psi}\end{aligned}$$

Substituting in Eq. (180),

$$\begin{aligned}\left(\frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\operatorname{ctn} \psi \frac{\partial}{\partial \psi} + \frac{1}{R^2}\frac{\partial^2}{\partial \psi^2} \right) \left(\frac{\partial^2\phi}{\partial R^2} + \frac{2}{R}\frac{\partial\phi}{\partial R} \right. \\ \left. + \frac{1}{R^2}\operatorname{ctn} \psi \frac{\partial\phi}{\partial \psi} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial \psi^2} \right) &= 0\end{aligned}\quad (181)$$

We shall apply several solutions of this equation in succeeding articles to the investigation of particular problems involving axial symmetry.

¹ This method of expressing all the stress components in terms of a single stress function, which satisfies Eq. (180), is given in detail by A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 274, 1927. Another method of expressing the problem in terms of a stress function has been given by J. H. Michell, *Proc. London Math. Soc.*, vol. 31, p. 144, 1900. The relation between the stress function of two-dimensional problems and the stress function discussed in this chapter has been considered by C. Weber, *Z. angew. Math. Mech.*, vol. 5, 1925.

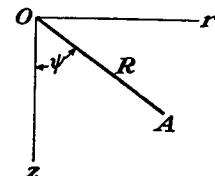


FIG. 198.

Another way of solving these problems is to consider explicitly the displacements. By using Eqs. (178) the stress components can be represented as functions of the displacements u and w . Substituting these functions in Eqs. (177) we arrive at two partial differential equations of the second order containing the two functions u and w . The problem is then reduced to the solution of these two equations.

117. Solution by Polynomials. Let us consider solutions of the Eq. (181), which are at the same time solutions of the equation

$$\frac{\partial^2\phi}{\partial R^2} + \frac{2}{R}\frac{\partial\phi}{\partial R} + \frac{1}{R^2}\operatorname{ctn} \psi \frac{\partial\phi}{\partial \psi} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial \psi^2} = 0\quad (182)$$

A particular solution of this latter equation can be taken in the form

$$\phi_n = R^n \Psi_n\quad (a)$$

in which Ψ_n is a function of the angle ψ only. Substituting (a) into Eq. (182) we find for Ψ_n the following ordinary differential equation:

$$\frac{1}{\sin \psi} \frac{\partial}{\partial \psi} \left(\sin \psi \frac{\partial \Psi_n}{\partial \psi} \right) + n(n+1)\Psi_n = 0\quad (b)$$

This equation can be simplified by introducing a new variable, $x = \cos \psi$. Then

$$\frac{\partial \Psi_n}{\partial \psi} = -\frac{\partial \Psi_n}{\partial x} \sin \psi, \quad \frac{\partial^2 \Psi_n}{\partial \psi^2} = \frac{\partial^2 \Psi_n}{\partial x^2} \sin^2 \psi - x \frac{\partial \Psi_n}{\partial x}$$

Substituting in Eq. (b), we obtain

$$(1-x^2) \frac{\partial^2 \Psi_n}{\partial x^2} - 2x \frac{\partial \Psi_n}{\partial x} + n(n+1)\Psi_n = 0\quad (183)$$

We shall solve this equation by series.¹ Assuming that

$$\Psi_n = a_1 x^{m_1} + a_2 x^{m_2} + a_3 x^{m_3} + \dots\quad (c)$$

and substituting in Eq. (183), we find

$$\begin{aligned}n(n+1)(a_1 x^{m_1} + a_2 x^{m_2} + a_3 x^{m_3} + \dots) &= m_1(m_1+1)a_1 x^{m_1} \\ &\quad - m_1(m_1-1)a_1 x^{m_1-2} + m_2(m_2+1)a_2 x^{m_2} - m_2(m_2-1)a_2 x^{m_2-2} \\ &\quad + \dots\end{aligned}\quad (d)$$

In order that this equation may be satisfied for any value of x , there must be the following relations between the exponents m_1, m_2, m_3, \dots .

$$m_2 = m_1 - 2, \quad m_3 = m_2 - 2, \quad \dots$$

¹ This is known as Legendre's equation. A complete discussion can be found in A. R. Forsyth, "A Treatise on Differential Equations," p. 155, 1903.

It follows that the series (c) is arranged in descending powers of x . The magnitude of m_1 will now be determined by equating coefficients of x^{m_1} in (d). Then

$$n(n+1) - m_1(m_1+1) = (n-m_1)(m_1+n+1) = 0$$

This gives for m_1 the two solutions

$$m_1 = n, \quad m_1 = -(n+1) \quad (e)$$

For the first of these solutions,

$$m_1 = n, \quad m_2 = n-2, \quad m_3 = n-4, \quad \dots$$

The coefficients a_2, a_3, \dots in Eq. (d) are found by equating to zero the coefficients of each power of x . Taking, for instance, the terms containing x^{m_1-2r+2} , we find for the calculation of the coefficient a_r the equation

$$n(n+1)a_r = (m_1-2r+2)(m_1-2r+3)a_r - (m_1-2r+4)(m_1-2r+3)a_{r-1}$$

from which, by substituting $m_1 = n$,

$$a_r = -\frac{(n-2r+4)(n-2r+3)}{2(r-1)(2n-2r+3)} a_{r-1}$$

The series (c) can now be put in the form

$$\Psi_n = a_1 \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} - \dots \right]. \quad (f)$$

which represents a solution of Eq. (183). Substituting this solution in (a) and remembering that

$$x = \cos \psi, \quad Rx = z, \quad R = \sqrt{r^2 + z^2}$$

we find, for n equal to 0, 1, 2, 3, ..., the following particular solutions of Eq. (182) in the form of polynomials:

$$\begin{aligned} \phi_0 &= A_0 \\ \phi_1 &= A_1 z \\ \phi_2 &= A_2 [z^2 - \frac{1}{3}(r^2 + z^2)] \\ \phi_3 &= A_3 [z^3 - \frac{2}{3}z(r^2 + z^2)] \\ \phi_4 &= A_4 [z^4 - \frac{6}{7}z^2(r^2 + z^2) + \frac{3}{35}(r^2 + z^2)^2] \\ \phi_5 &= A_5 [z^5 - \frac{10}{9}z^3(r^2 + z^2) + \frac{5}{21}z(r^2 + z^2)^2] \\ &\dots \end{aligned} \quad (184)$$

A_0, A_1, \dots are arbitrary constants. These polynomials are also solutions of the Eq. (181). From these solutions we can get new solutions of Eq. (181) which will no longer be solutions of Eq. (182). If $R^n \Psi_n$ is a solution of Eq. (182), it can be shown that $R^{n+2} \Psi_n$ is a solution of Eq. (181). Performing the operation indicated in the parentheses of Eq. (181),

$$\begin{aligned} \left(\frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \operatorname{ctn} \psi \frac{\partial}{\partial \psi} + \frac{1}{R^2} \frac{\partial^2}{\partial \psi^2} \right) R^{n+2} \Psi_n \\ = 2(2n+3)R^n \Psi_n \quad (g) \end{aligned}$$

Repeating the same operation again, as indicated in Eq. (181), we obtain zero, since (g) is a solution of Eq. (182). Hence $R^{n+4} \Psi_n$ is a solution of Eq. (181). It is seen that multiplying solutions (184) by $R^2 = r^2 + z^2$, we can obtain the following new solutions:

$$\begin{aligned} \phi_2 &= B_2(r^2 + z^2) \\ \phi_3 &= B_3 z(r^2 + z^2) \\ \phi_4 &= B_4(2z^2 - r^2)(r^2 + z^2) \\ \phi_5 &= B_5(2z^3 - 3r^2 z)(r^2 + z^2) \\ &\dots \end{aligned} \quad (185)$$

118. Bending of a Circular Plate. Several problems of practical interest can be solved with the help of the foregoing solutions. Among these are various cases of the bending of symmetrically loaded circular plates (Fig. 199). Taking, for instance, the polynomials of the third degree from (184) and (185), we obtain the stress function

$$\phi = a_3(2z^3 - 3r^2 z) + b_3(r^2 z + z^3) \quad (a)$$

Substituting in Eqs. (179), we find

$$\begin{aligned} \sigma_r &= 6a_3 + (10\nu - 2)b_3, & \sigma_\theta &= 6a_3 + (10\nu - 2)b_3 \\ \sigma_z &= -12a_3 + (14 - 10\nu)b_3, & \tau_{rz} &= 0 \end{aligned} \quad (186)$$

The stress components are thus constant throughout the plate. By a suitable adjustment of constants a_3 and b_3 we can get the stresses in a plate when any constant values of σ_z and σ_r at the surface of the plate are given.

Let us take now the polynomials of the fourth degree from (184) and (185), which gives us

$$\phi = a_4(8z^4 - 24r^2 z^2 + 3r^4) + b_4(2z^4 + r^2 z^2 - r^4) \quad (b)$$

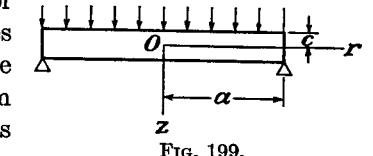


FIG. 199.

Substituting in Eqs. (179), we find

$$\begin{aligned}\sigma_r &= 96a_4z + 4b_4(14\nu - 1)z \\ \sigma_z &= -192a_4z + 4b_4(16 - 14\nu)z \\ \tau_{rz} &= 96a_4r - 2b_4(16 - 14\nu)r\end{aligned}\quad (187)$$

Taking

$$96a_4 - 2b_4(16 - 14\nu) = 0$$

we have

$$\sigma_z = \tau_{rz} = 0, \quad \sigma_r = 28(1 + \nu)b_4z \quad (c)$$

If z is the distance from the middle plane of the plate, the solution (c) represents pure bending of the plate by moments uniformly distributed along the boundary.

To get the solution for a circular plate uniformly loaded, we take the stress function in the form of a polynomial of the sixth power. Proceeding as explained in the previous article, we find

$$\phi = \frac{1}{8}a_6(16z^6 - 120z^4r^2 + 90z^2r^4 - 5r^6) + b_6(8z^6 - 16z^4r^2 - 21z^2r^4 + 3r^6)$$

Substituting in (179),

$$\begin{aligned}\sigma_r &= a_6(320z^3 - 720r^2z) + b_6[64(2 + 11\nu)z^3 + (504 - 48 \cdot 22\nu)r^2z] \\ \sigma_z &= a_6(-640z^3 + 960r^2z) + b_6\{[-960 + 32 \cdot 22(2 - \nu)]z^3 + [384 - 48 \cdot 22(2 - \nu)]r^2z\} \\ \tau_{rz} &= a_6(960rz^2 - 240r^3) + b_6[(-672 + 48 \cdot 22\nu)z^2r + (432 - 12 \cdot 22\nu)r^3]\end{aligned}$$

To these stresses we add the stresses

$$\sigma_r = 96a_4z, \quad \sigma_z = -192a_4z, \quad \tau_{rz} = 96a_4r$$

obtained from (187) by taking $b_4 = 0$, and a uniform tension in the z -direction $\sigma_z = b$, which can be obtained from (186). Thus we arrive at expressions for the stress components containing four constants a_6, b_6, a_4, b . These constants can be adjusted so as to satisfy the boundary conditions on the upper and lower surfaces of the plate (Fig. 199). The conditions are

$$\begin{aligned}\sigma_z &= 0 & \text{for} & \quad z = c \\ \sigma_z &= -q & \text{for} & \quad z = -c \\ \tau_{rz} &= 0 & \text{for} & \quad z = c \\ \tau_{rz} &= 0 & \text{for} & \quad z = -c\end{aligned}\quad (d)$$

Here q denotes the intensity of the uniform load and $2c$ is the thickness of the plate. Substituting the expressions for the stress components in these equations, we determine the four constants a_6, b_6, a_4, b . Using these values, the expressions for the stress components satisfying conditions (d) are

$$\begin{aligned}\sigma_r &= q \left[\frac{2 + \nu}{8} \frac{z^3}{c^3} - \frac{3(3 + \nu)}{32} \frac{r^2z}{c^3} - \frac{3}{8} \frac{z}{c} \right] \\ \sigma_z &= q \left(-\frac{z^3}{4c^3} + \frac{3}{4} \frac{z}{c} - \frac{1}{2} \right) \\ \tau_{rz} &= \frac{3qr}{8c^3} (c^2 - z^2)\end{aligned}\quad (e)$$

It will be seen that the stresses σ_z and τ_{rz} are distributed in exactly the same manner as in the case of a uniformly loaded beam of narrow rectangular cross section (Art. 21). The radial stresses σ_r are represented by an odd function of z , and at the boundary of the plate they give bending moments uniformly distributed along the boundary. To get the solution for a simply supported plate (Fig. 199), we superpose a pure bending stress (c) and adjust the constant b_4 so as to obtain for the boundary ($r = a$),

$$\int_{-c}^c \sigma_r z \, dz = 0$$

Then the final expression for σ_r becomes

$$\sigma_r = q \left[\frac{2 + \nu}{8} \frac{z^3}{c^3} - \frac{3(3 + \nu)}{32} \frac{r^2z}{c^3} - \frac{3}{8} \frac{2 + \nu}{5} \frac{z}{c} + \frac{3(3 + \nu)}{32} \frac{a^2z}{c^3} \right] \quad (188)$$

and at the center of the plate we have

$$(\sigma_r)_{r=0} = q \left[\frac{2 + \nu}{8} \frac{z^3}{c^3} - \frac{3}{8} \frac{2 + \nu}{5} \frac{z}{c} + \frac{3(3 + \nu)}{32} \frac{a^2z}{c^3} \right] \quad (f)$$

The elementary theory of bending of plates, based on the assumptions that linear elements of the plate perpendicular to the *middle plane* ($z = 0$) remain straight and normal to the deflection surface of the plate¹ during bending, gives for the radial stresses at the center

$$\sigma_r = \frac{3(3 + \nu)}{32} \frac{a^2z}{c^3} q \quad (g)$$

¹This assumption is analogous to the *plane cross sections* hypothesis in the theory of bending of beams. The exact theory of bending of plates was developed by J. H. Michell, *Proc. London Math. Soc.*, vol. 31, 1900, and A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 465, 1927.

Comparing this with (f), we see that the additional terms of the exact solution are small if the thickness of the plate, $2c$, is small in comparison with the radius a .

It should be noted that by superposing pure bending we eliminated bending moments along the boundary of the plate, but the radial stresses are not zero at the boundary but are

$$(\sigma_r)_{r=a} = q \left(\frac{2 + \nu}{8} \frac{z^3}{c^3} - \frac{3}{8} \frac{2 + \nu}{5} \frac{z}{c} \right) \quad (h)$$

The resultant of these stresses per unit length of the boundary line and their moment, however, are zero. Hence, on the basis of Saint-Venant's principle, we can say that the removal of these stresses does not affect the stress distribution in the plate at some distance from the edge.

By taking polynomials of higher order than the sixth for the stress function, we can investigate cases of bending of a circular plate by nonuniformly distributed loads. By taking, instead of solution (f) on page 348, the other solution of Eq. (182), we can also obtain solutions for a circular plate with a hole at the center.¹ All these solutions are satisfactory only if the deflection of the plate remains small in comparison with the thickness. For larger deflections the stretching of the middle plane of the plate must be considered.²

119. The Rotating Disk as a Three-dimensional Problem. In our previous discussion (Art. 30) it was assumed the stresses do not vary through the thickness of the disk. Let us now consider the same problem assuming only that the stress distribution is symmetrical with respect to the axis of rotation. The differential equations of equilibrium are obtained by including in Eqs. (177) the centrifugal force. Then

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \quad (189)$$

where ρ is the mass per unit volume, and ω the angular velocity of the disk.

The compatibility equations also must be changed. Instead of the system (130) we shall have three equations of the type (f) (see page 231) and three equations of the type (g). Substituting in these equations the components of body force,

$$X = \rho \omega^2 x, \quad Y = \rho \omega^2 y, \quad Z = 0 \quad (a)$$

¹ A number of solutions for a circular plate symmetrically loaded have been discussed by A. Korobov, *Bull. Polytech. Inst., Kiev*, 1913. Similar solutions were obtained independently by A. Timpe, *Z. angew. Math. Mech.*, vol. 4, 1924.

² See Kelvin and Tait, "Natural Philosophy," vol. 2, p. 171, 1903.

we find that the last three equations, containing shearing-stress components, remain the same as in the system (130), and the first three equations become [see Eqs. (e), Art. 116]

$$\begin{aligned} \nabla^2 \sigma_r - \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial r^2} &= - \frac{2\rho\omega^2}{1 - \nu} \\ \nabla^2 \sigma_\theta + \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1 + \nu} \frac{1}{r} \frac{\partial \Theta}{\partial r} &= - \frac{2\rho\omega^2}{1 - \nu} \\ \nabla^2 \sigma_z + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial z^2} &= - \frac{2\rho\omega^2}{1 - \nu} \end{aligned} \quad (b)$$

We begin with a particular solution of Eqs. (189), satisfying the compatibility equations. On this solution we superpose solutions in the form of polynomials (184) and (185) and adjust the constants of these polynomials so as to satisfy the boundary conditions of the problem. For the particular solution we take the expressions

$$\sigma_r = Br^2 + Dz^2, \quad \sigma_z = Ar^2, \quad \sigma_\theta = Cr^2 + Dz^2, \quad \tau_{rz} = 0 \quad (c)$$

It can be seen that these expressions satisfy the second of the equations of equilibrium. They also satisfy the compatibility equations which contain shearing-stress components [see Eqs. (f) and (g), Art. 116]. It remains to determine the constants A , B , C , D , so as to satisfy the remaining four equations, namely the first of Eqs. (189) and Eqs. (b). Substituting (c) in these equations, we find

$$A = \frac{\rho\omega^2(1 + 3\nu)}{6\nu}, \quad B = -\frac{\rho\omega^2}{3}, \quad C = 0, \quad D = -\frac{\rho\omega^2(1 + 2\nu)(1 + \nu)}{6\nu(1 - \nu)}$$

The particular solution is then

$$\begin{aligned} \sigma_r &= -\frac{\rho\omega^2}{3} r^2 - \frac{\rho\omega^2(1 + 2\nu)(1 + \nu)}{6\nu(1 - \nu)} z^2 \\ \sigma_z &= \frac{\rho\omega^2(1 + 3\nu)}{6\nu} r^2 \\ \sigma_\theta &= -\frac{\rho\omega^2(1 + 2\nu)(1 + \nu)}{6\nu(1 - \nu)} z^2 \\ \tau_{rz} &= 0 \end{aligned} \quad (190)$$

This solution can be used in discussing the stresses in any body rotating about an axis of generation.

In the case of a circular disk of constant thickness we superpose on the solution (190) the stress distribution derived from a stress function having the form of a polynomial of the fifth degree [see Eqs. (184), (185)],

$$\phi = a_5(8z^5 - 40r^2z^3 + 15r^4z) + b_5(2z^5 - r^2z^3 - 3r^4z) \quad (d)$$

Then, from Eqs. (179), we find

$$\begin{aligned} \sigma_r &= -a_5(180r^2 - 240z^2) + b_5[(36 - 54\nu)r^2 + (1 + 18\nu)6z^2] \\ \sigma_z &= -a_5(-240r^2 + 480z^2) + b_5[(96 - 108\nu)z^2 + (-102 + 54\nu)r^2] \\ \sigma_\theta &= a_5(-60r^2 + 240z^2) + b_5[(6 + 108\nu)z^2 + (12 - 54\nu)r^2] \\ \tau_{rz} &= 480a_5rz - b_5(96 - 108\nu)rz \end{aligned} \quad (e)$$

Adding this to the stresses (190) and determining the constants a_5 and b_5 so as to make the resultant stresses τ_{rz} and σ_z vanish, we find

$$\begin{aligned}\sigma_r &= -\rho\omega^2 \left[\frac{\nu(1+\nu)}{2(1-\nu)} z^2 + \frac{3+\nu}{8} r^2 \right] \\ \sigma_\theta &= -\rho\omega^2 \left[\frac{(1+3\nu)}{8} r^2 + \frac{\nu(1+\nu)}{2(1-\nu)} z^2 \right]\end{aligned}\quad (f)$$

To eliminate the resultant radial compression along the boundary, *i.e.*, to make

$$\left(\int_{-c}^c \sigma_r dz \right)_{r=a} = 0$$

we superpose on (f) a uniform radial tension of magnitude

$$\frac{\rho\omega^2}{8} (3+\nu)a^2 + \rho\omega^2 \frac{\nu(1+\nu)}{2(1-\nu)} \frac{c^2}{3}$$

Then the final stresses are

$$\begin{aligned}\sigma_r &= \rho\omega^2 \left[\frac{3+\nu}{8} (a^2 - r^2) + \frac{\nu(1+\nu)}{6(1-\nu)} (c^2 - 3z^2) \right] \\ \sigma_\theta &= \rho\omega^2 \left[\frac{3+\nu}{8} a^2 - \frac{1+3\nu}{8} r^2 + \frac{\nu(1+\nu)}{6(1-\nu)} (c^2 - 3z^2) \right] \\ \sigma_z &= 0, \quad \tau_{rz} = 0\end{aligned}\quad (191)$$

Comparing this with the previous solution (55), we have here additional terms with the factor¹ $(c^2 - 3z^2)$. The corresponding stresses are small in the case of a thin disk and their resultant over the thickness of the disk is zero. If the rim of the disk is free from external forces, solution (191) represents the state of stress in parts of the disk some distance from the edge.

The stress distribution in a rotating disk having the shape of a flat ellipsoid of revolution has been discussed by C. Chree.²

120. Force at a Point of an Indefinitely Extended Solid. In discussing this problem we use again Eq. (182) on page 347. By taking $m_1 = -(n+1)$ [see Eq. (e), page 348], we obtain the second integral of Eq. (183) in the form of the following series:

$$\Psi_n = a \left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-(n+3)} \right. \\ \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} x^{-(n+5)} + \dots \right]$$

Taking n equal to $-1, -2, -3, \dots$, we obtain from this the following particular solutions of Eq. (182):

¹ These terms are of the same nature as the terms in z^2 found in Art. 84. Equations (191) represent a state of plane stress since σ_z and τ_{rz} vanish. Body force (here centrifugal force), not included in Art. 84, does not alter the general conclusions so long as it is independent of z .

² C. Chree, Proc. Roy. Soc. (London), vol. 58, p. 39, 1895.

$$\begin{aligned}\phi_1 &= A_1(r^2 + z^2)^{-\frac{1}{2}} \\ \phi_2 &= A_2 z(r^2 + z^2)^{-\frac{1}{2}} \\ \phi_3 &= A_3 z^2(r^2 + z^2)^{-\frac{1}{2}} - \frac{1}{3}(r^2 + z^2)^{-\frac{1}{2}} \\ \dots &\dots \dots \dots \dots \dots\end{aligned}\quad (192)$$

which are also solutions of Eq. (181). By multiplying expressions (192) by $r^2 + z^2$ (see page 349), we obtain another series of solutions of Eq. (181), namely,

$$\begin{aligned}\phi_1 &= B_1(r^2 + z^2)^{\frac{1}{2}} \\ \phi_2 &= B_2 z(r^2 + z^2)^{-\frac{1}{2}} \\ \dots &\dots \dots \dots \dots\end{aligned}\quad (193)$$

Each of the solutions (192) and (193), and any linear combination of them, can be taken as a stress function, and, by a suitable adjustment of the constants $A_1, A_2, \dots, B_1, B_2, \dots$, solutions of various problems may be found.

For the case of a concentrated force we take the first of the solutions (193) and assume that the stress function is

$$\phi = B(r^2 + z^2)^{\frac{1}{2}}$$

where B is a constant to be adjusted later. Substituting in Eqs. (179), we find the corresponding stress components

$$\begin{aligned}\sigma_r &= B[(1-2\nu)z(r^2 + z^2)^{-\frac{1}{2}} - 3r^2 z(r^2 + z^2)^{-\frac{3}{2}}] \\ \sigma_\theta &= B(1-2\nu)z(r^2 + z^2)^{-\frac{1}{2}} \\ \sigma_z &= -B[(1-2\nu)z(r^2 + z^2)^{-\frac{1}{2}} + 3z^2(r^2 + z^2)^{-\frac{1}{2}}] \\ \tau_{rz} &= -B[(1-2\nu)r(r^2 + z^2)^{-\frac{1}{2}} + 3rz^2(r^2 + z^2)^{-\frac{1}{2}}]\end{aligned}\quad (194)$$

All these stresses approach infinity when we approach the origin of coordinates, where the concentrated force is applied. To avoid the necessity of considering infinite stresses we suppose the origin to be the center of a small spherical cavity (Fig. 200), and consider forces over the surface of the cavity as calculated from Eqs. (194). It can be shown that the resultant of these forces represents a force applied at the origin in the direction of the z -axis. From the condition of equilibrium of a ring-shaped element, adjacent to the cavity (Fig. 200), the component of surface forces in the z -direction is

$$\bar{Z} = -(\tau_{rz} \sin \psi + \sigma_z \cos \psi)$$

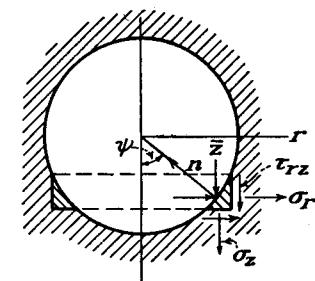


Fig. 200.

Using Eqs. (194) and the formulas

$$\sin \psi = r(r^2 + z^2)^{-\frac{1}{2}}, \quad \cos \psi = z(r^2 + z^2)^{-\frac{1}{2}}$$

we find that

$$\bar{Z} = B[(1 - 2\nu)(r^2 + z^2)^{-1} + 3z^2(r^2 + z^2)^{-2}]$$

The resultant of these forces over the surface of the cavity is

$$2 \int_0^{\frac{\pi}{2}} \bar{Z} \sqrt{r^2 + z^2} \cdot d\psi \cdot 2\pi r = 8B\pi(1 - \nu)$$

The resultant of the surface forces in a radial direction is zero, from symmetry. If P is the magnitude of the applied force, we have

$$P = 8B\pi(1 - \nu)$$

Substituting

$$B = \frac{P}{8\pi(1 - \nu)} \quad (195)$$

into Eqs. (194), we obtain the stresses produced by a force P applied at the origin in the z -direction.¹ This solution is the three-dimensional analogue of the solution of the two-dimensional problem discussed in Art. 38.

Substituting $z = 0$ in Eqs. (194), we find that there are no normal stresses acting on the coordinate plane $z = 0$. The shearing stresses over the same plane are

$$\tau_{rz} = -\frac{B(1 - 2\nu)}{r^2} = -\frac{P(1 - 2\nu)}{8\pi(1 - \nu)r^2} \quad (a)$$

These stresses are inversely proportional to the square of the distance r from the point of application of the load.

121. Spherical Container under Internal or External Uniform Pressure. By superposition we can get from the solution of the previous

article some new solutions of practical interest. We begin with the case of two equal and opposite forces, a small distance d apart, applied to an indefinitely extended elastic body (Fig. 201). The stresses pro-

¹ The solution of this problem was given by Lord Kelvin, *Cambridge and Dublin Math. J.*, 1848. See also his "Mathematical and Physical Papers," vol. 1, p. 37. From his solution it follows that the displacements corresponding to the stresses (194) are single-valued, which proves that (194) is the correct solution of the problem (see Art. 82).

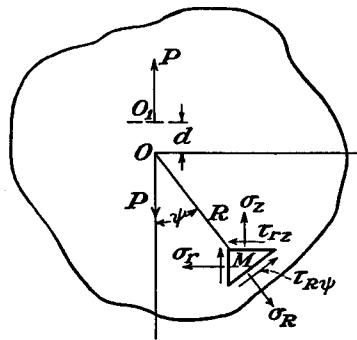


FIG. 201.

duced at any point by the force P applied at the origin O are determined by Eqs. (194) and (195) of the previous article. By using the same equations, the stresses produced by the force P at O_1 can also be calculated. Remembering that the second force is acting in the opposite direction and considering the distance d as an infinitely small quantity, any term $f(r, z)$ in expressions (194) should be replaced by $-[f + (\partial f / \partial z)d]$. Superposing the stresses produced by the two forces and using the symbol A for the product Bd , we find

$$\begin{aligned} \sigma_r &= -A \frac{\partial}{\partial z} [(1 - 2\nu)z(r^2 + z^2)^{-\frac{1}{2}} - 3r^2z(r^2 + z^2)^{-\frac{3}{2}}] \\ \sigma_\theta &= -A \frac{\partial}{\partial z} [(1 - 2\nu)z(r^2 + z^2)^{-\frac{1}{2}}] \\ \sigma_z &= A \frac{\partial}{\partial z} [(1 - 2\nu)z(r^2 + z^2)^{-\frac{1}{2}} + 3z^2(r^2 + z^2)^{-\frac{3}{2}}] \\ \tau_{rz} &= A \frac{\partial}{\partial z} [(1 - 2\nu)r(r^2 + z^2)^{-\frac{1}{2}} + 3rz^2(r^2 + z^2)^{-\frac{3}{2}}] \end{aligned} \quad (196)$$

Let us consider (Fig. 201) the stress components σ_R and $\tau_{R\psi}$ acting at a point M on an elemental area perpendicular to the radius OM , the length of which is denoted by R . From the condition of equilibrium of a small triangular element such as indicated in the figure we find¹

$$\begin{aligned} \sigma_R &= \sigma_r \sin^2 \psi + \sigma_z \cos^2 \psi + 2\tau_{rz} \sin \psi \cos \psi \\ \tau_{R\psi} &= (\sigma_r - \sigma_z) \sin \psi \cos \psi - \tau_{rz}(\sin^2 \psi - \cos^2 \psi) \end{aligned} \quad (a)$$

Using (196), and taking

$$\sin \psi = r(r^2 + z^2)^{-\frac{1}{2}} = \frac{r}{R}, \quad \cos \psi = z(r^2 + z^2)^{-\frac{1}{2}} = \frac{z}{R}$$

we obtain

$$\begin{aligned} \sigma_R &= -\frac{2(1 + \nu)A}{R^3} \left[-\sin^2 \psi + \frac{2(2 - \nu)}{1 + \nu} \cos^2 \psi \right] \\ \tau_{R\psi} &= -\frac{2(1 + \nu)A}{R^3} \sin \psi \cos \psi \end{aligned} \quad (b)$$

The distribution of these stresses is symmetrical with respect to the z -axis and with respect to the coordinate plane perpendicular to z .

Imagine now that we have at the origin, in addition to the system of two forces P acting along the z -axis, an identical system along the r -axis and another one along the axis perpendicular to the rz -plane.

¹ The stress components σ_θ , acting on the sides of the element in the meridional sections of the body, give a small resultant of higher order and can be neglected in deriving the equations of equilibrium.

By virtue of the symmetry stated above, we obtain in this way a stress distribution symmetrical with respect to the origin. If we consider a sphere with center at the origin, there will be only a normal uniformly distributed stress acting on the surface of this sphere. The magnitude of this stress can be calculated by using the first of Eqs. (b). Considering the stress at points on the circle in the rz -plane, the first of Eqs. (b) gives the part of this stress due to the double force along the z -axis. By interchanging $\sin \psi$ and $\cos \psi$, we obtain the normal stress round the same circle produced by the double force along the r -axis. The normal stress due to the double force perpendicular to the rz -plane is obtained by substituting $\psi = \pi/2$ in the same equation. Combining the actions of the three perpendicular double forces we find the following normal stress acting on the surface of the sphere:

$$\sigma_R = -\frac{4(1-2\nu)A}{R^3} \quad (c)$$

The combination of the three perpendicular double forces is called a *center of compression*. We see from (c) that the corresponding compression stress in the radial direction depends only on the distance from the center of compression and is inversely proportional to the cube of this distance.

This solution can be used for calculating stresses in a spherical container submitted to the action of internal or external uniform pressure. Let a and b denote the inner and outer radii of the sphere (Fig. 202), and p_i and p_o the internal and the external uniform pressures. Superposing on (c) a uniform tension or compression in all directions, we can take a general expression for the radial normal stress in the form

$$\sigma_R = \frac{C}{R^3} + D \quad (d)$$

C and D are constants, the magnitudes of which are determined from the conditions on the inner and outer surfaces of the container, which are

$$\frac{C}{a^3} + D = -p_i, \quad \frac{C}{b^3} + D = -p_o$$

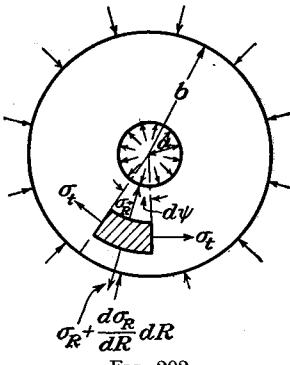


FIG. 202.

Then

$$\begin{aligned} C &= \frac{(p_i - p_o)a^3b^3}{a^3 - b^3} \\ D &= \frac{p_o b^3 - p_i a^3}{a^3 - b^3} \\ \sigma_R &= \frac{p_o b^3(R^3 - a^3)}{R^3(a^3 - b^3)} + \frac{p_i a^3(b^3 - R^3)}{R^3(a^3 - b^3)} \end{aligned} \quad (197)$$

The pressures p_o and p_i also produce in the sphere normal stresses σ_t in a tangential direction, the magnitude of which we find from the condition of equilibrium of an element cut out from the sphere by the two concentric spherical surfaces of radii R and $R + dR$ and by a circular cone with a small angle $d\psi$ (Fig. 202). The equation of equilibrium is

$$\sigma_t \frac{\pi R}{2} dR(d\psi)^2 = \frac{d\sigma_R}{dR} \frac{\pi R^2}{4} dR(d\psi)^2 + \sigma_R \frac{\pi R}{2} dR(d\psi)^2$$

from which

$$\sigma_t = \frac{d\sigma_R}{dR} \frac{R}{2} + \sigma_R \quad (e)$$

Using expression (197) for σ_R this becomes

$$\sigma_t = \frac{p_o b^3(2R^3 + a^3)}{2R^3(a^3 - b^3)} - \frac{p_i a^3(2R^3 + b^3)}{2R^3(a^3 - b^3)} \quad (198)$$

If $p_o = 0$, then

$$\sigma_t = \frac{p_i a^3 (2R^3 + b^3)}{2R^3 (b^3 - a^3)}$$

It will be seen that the greatest tangential tension in this case is at the inner surface, at which

$$(\sigma_t)_{\max} = \frac{p_i 2a^3 + b^3}{2 b^3 - a^3}$$

All these results are due to Lamé.¹

122. Local Stresses around a Spherical Cavity.

As a second example consider the stress distribution around a small spherical cavity in a bar submitted to uniform tension of magnitude S (Fig. 203).² In the case of a solid bar in tension, the normal and shearing components of stress acting on a spherical surface are

¹ Lamé, "Leçons sur la théorie . . . de l'élasticité," Paris, 1852.

² Solution of this problem is due to R. V. Southwell, *Phil. Mag.*, 1926; see also J. N. Goodier, *Trans. A.S.M.E.*, vol. 55, p. 39, 1933. The triaxial ellipsoidal cavity is considered by E. Sternberg and M. Sadowsky, *J. Applied Mechanics* (*Trans. A.S.M.E.*), vol. 16, p. 149, 1949.

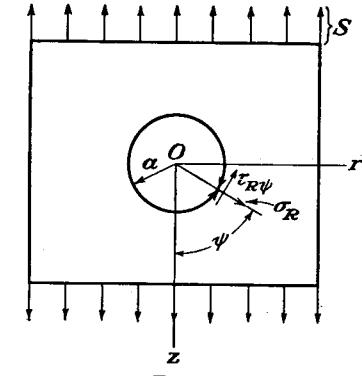


FIG. 203.

$$\sigma_R = S \cos^2 \psi, \quad \tau_{R\psi} = -S \sin \psi \cos \psi \quad (a)$$

To get the solution for the case of a small spherical cavity of radius a , we must superpose on the simple tension a stress system which has stress components on the spherical surface equal and opposite to those given by the Eqs. (a), and which vanishes at infinity.

Taking from the previous article the stresses (b), due to a double force in the z -direction, and the stresses (c), due to a center of compression, the corresponding stresses acting on the spherical surface of radius a can be presented in the following form:

$$\sigma_R' = -\frac{2(1+\nu)A}{a^3} \left(-1 + \frac{5-\nu}{1+\nu} \cos^2 \psi \right), \quad \tau_{R\psi}' = -\frac{2(1+\nu)A}{a^3} \sin \psi \cos \psi \quad (b)$$

$$\sigma_R'' = \frac{B}{a^3}, \quad \tau_{R\psi}'' = 0 \quad (c)$$

where A and B are constants to be adjusted later. It is seen that, combining stresses (b) and (c), the stresses (a) produced by tension cannot be made to vanish and that an additional stress system is necessary.

Taking, from solutions (192), a stress function

$$\phi = Cz(r^2 + z^2)^{-\frac{3}{2}}$$

the corresponding stress components, from Eqs. (179), are

$$\begin{aligned} \sigma_z &= \frac{3C}{R^5} (1 - 5 \cos^2 \psi - 5 \sin^2 \psi + 35 \sin^2 \psi \cos^2 \psi) \\ \sigma_x &= \frac{3C}{R^5} (3 - 30 \cos^2 \psi + 35 \cos^4 \psi) \\ \sigma_\theta &= \frac{3C}{R^5} (1 - 5 \cos^2 \psi) \\ \tau_{rz} &= \frac{15C}{R^5} (-3 \sin \psi \cos \psi + 7 \sin \psi \cos^3 \psi) \end{aligned} \quad (d)$$

Using now Eqs. (a) of the previous article, the stress components acting on a spherical surface of radius a are

$$\sigma_R''' = \frac{12C}{a^5} (-1 + 3 \cos^2 \psi), \quad \tau_{R\psi}''' = \frac{24C}{a^5} \sin \psi \cos \psi \quad (e)$$

Combining stress systems (b), (c), (e) we find

$$\begin{aligned} \sigma_R &= \frac{2(1+\nu)A}{a^3} - 2(5-\nu) \frac{A}{a^3} \cos^2 \psi + \frac{B}{a^3} - \frac{12C}{a^5} + \frac{36C}{a^5} \cos^2 \psi \\ \tau_{R\psi} &= -\frac{2(1+\nu)A}{a^3} \sin \psi \cos \psi + \frac{24C}{a^5} \sin \psi \cos \psi \end{aligned} \quad (f)$$

Superposing these stresses on the stresses (a), the spherical surface of the cavity becomes free from forces if we satisfy the conditions

$$\begin{aligned} \frac{2(1+\nu)A}{a^3} + \frac{B}{a^3} - \frac{12C}{a^5} &= 0 \\ -2(5-\nu) \frac{A}{a^3} + \frac{36C}{a^5} &= -S \\ -\frac{2(1+\nu)A}{a^3} + \frac{24C}{a^5} &= S \end{aligned} \quad (g)$$

from which

$$\frac{A}{a^3} = \frac{5S}{2(7-5\nu)}, \quad \frac{B}{a^3} = \frac{S(1-5\nu)}{7-5\nu}, \quad \frac{C}{a^5} = \frac{S}{2(7-5\nu)} \quad (h)$$

The complete stress at any point is now obtained by superposing on the simple tension S the stresses given by Eqs. (d), the stresses (196) due to the double force, and the stresses due to the center of pressure given by Eqs. (c) and (e) of the previous article.

Consider, for instance, the stresses acting on the plane $z = 0$. From the condition of symmetry there are no shearing stresses on this plane. From Eqs. (d), substituting $\psi = \pi/2$ and $R = r$,

$$\sigma_z' = \frac{9C}{r^5} = \frac{9Sa^5}{2(7-5\nu)r^5} \quad (k)$$

From Eqs. (196), for $z = 0$,

$$\sigma_z'' = \frac{A(1-2\nu)}{r^3} = \frac{5(1-2\nu)S}{2(7-5\nu)} \frac{a^3}{r^3} \quad (l)$$

From Eq. (e) of the previous article,

$$\sigma_z''' = (\sigma_z)_{z=0} = -\frac{B}{2r^3} = -\frac{S(1-5\nu)}{2(7-5\nu)} \frac{a^3}{r^3} \quad (m)$$

The total stress on the plane $z = 0$ is

$$\sigma_z = \sigma_z' + \sigma_z'' + \sigma_z''' + S = S \left[1 + \frac{4-5\nu}{2(7-5\nu)} \frac{a^3}{r^3} + \frac{9}{2(7-5\nu)} \frac{a^5}{r^5} \right] \quad (n)$$

At $r = a$, we find

$$(\sigma_z)_{\max.} = \frac{27-15\nu}{2(7-5\nu)} S \quad (o)$$

Taking $\nu = 0.3$,

$$(\sigma_z)_{\max.} = \frac{45}{22} S$$

The maximum stress is thus about twice as great as the uniform tension S applied to the bar. This increase in stress is of a highly localized character. With increase of r , the stress (n) rapidly approaches the value S . Taking, for instance, $r = 2a$, $\nu = 0.3$, we find $\sigma_z = 1.054S$.

In the same manner we find, for points in the plane $z = 0$,

$$(\sigma_\theta)_{z=0} = \frac{3C}{r^5} - \frac{A(1-2\nu)}{r^3} - \frac{B}{2r^3}$$

Substituting from Eqs. (h) and taking $r = a$, we find that the tensile stress along the equator ($\psi = \pi/2$) of the cavity is

$$(\sigma_\theta)_{z=0, r=a} = \frac{15\nu - 3}{2(7-5\nu)} S$$

At the pole of the cavity ($\psi = 0$ or $\psi = \pi$) we have

$$\sigma_r = \sigma_\theta = \frac{2(1-2\nu)A}{a^3} - \frac{12C}{a^5} - \frac{B}{2a^3} = -\frac{3+15\nu}{2(7-5\nu)} S$$

Thus the longitudinal tension S produces compression at this point.

Combining a tension S in one direction with compression S in the perpendicular direction we can obtain the solution for the stress distribution around a spherical

cavity in the case of pure shear.¹ It can be shown in this way that the maximum shearing stress is

$$\tau_{\max.} = \frac{15(1-\nu)}{7-5\nu} S \quad (p)$$

The results of this article may be of some practical interest in discussing the effect of small cavities² on the endurance limit of specimens submitted to the action of cyclical stresses.

123. Force on Boundary of a Semi-infinite Body. Imagine that the plane $z = 0$ is the boundary of a semi-infinite solid and that a force P is acting on this plane along the z -axis (Fig. 204).³ It was shown in

Art. 120 that the stress distribution given by Eqs. (194) and (195) can be produced in a semi-infinite body by a concentrated force at the origin and by shearing forces on the boundary plane $z = 0$, given by the equation

$$\tau_{rz} = -\frac{B(1-2\nu)}{r^2} \quad (a)$$

To eliminate these forces and arrive at the solution of the problem shown in Fig. 204, we

use the stress distribution corresponding to the center of compression (see page 358). In polar coordinates this stress distribution is

$$\sigma_R = \frac{A}{R^3}, \quad \sigma_t = \frac{d\sigma_R}{dR} \frac{R}{2} + \sigma_R = -\frac{1}{2} \frac{A}{R^3}$$

in which A is a constant. In cylindrical coordinates (Fig. 204) we have the following expressions for the stress components:

$$\begin{aligned} \sigma_r &= \sigma_R \sin^2 \psi + \sigma_t \cos^2 \psi = A(r^2 - \frac{1}{2}z^2)(r^2 + z^2)^{-\frac{3}{2}} \\ \sigma_z &= \sigma_R \cos^2 \psi + \sigma_t \sin^2 \psi = A(z^2 - \frac{1}{2}r^2)(r^2 + z^2)^{-\frac{3}{2}} \quad (199) \\ \tau_{rz} &= \frac{1}{2}(\sigma_R - \sigma_t) \sin 2\psi = \frac{3}{2}Arz(r^2 + z^2)^{-\frac{5}{2}} \\ \sigma_\theta &= \sigma_t = -\frac{1}{2} \frac{A}{R^3} = -\frac{1}{2} A(r^2 + z^2)^{-\frac{3}{2}} \end{aligned}$$

¹ This problem was discussed by J. Larmor, *Phil. Mag.*, series 5, vol. 33, 1892. See also A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 252, 1927.

² Such cavities are, for instance, present in a weld, and fatigue experiments show that cracks usually begin at these cavities.

³ The solution of this problem was given by J. Boussinesq, see "Application des potentiels . . .," Paris, 1885. The solution for a force at an internal point of the semi-infinite body was found by R. D. Mindlin, *Physics*, vol. 7, p. 195, 1936.

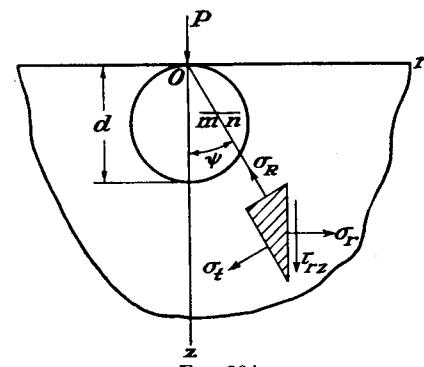


FIG. 204.

Assume now that centers of pressure are uniformly distributed along the z -axis from $z = 0$ to $z = -\infty$. Then, using the principle of superposition, the stress components produced in an indefinitely extended solid are, from Eqs. (199),

$$\begin{aligned} \sigma_r &= A \int_z^\infty (r^2 - \frac{1}{2}z^2)(r^2 + z^2)^{-\frac{3}{2}} dz \\ &= \frac{A}{2} \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} - z(r^2 + z^2)^{-\frac{1}{2}} \right] \\ \sigma_z &= A \int_z^\infty \left(z^2 - \frac{1}{2}r^2 \right) (r^2 + z^2)^{-\frac{3}{2}} dz = \frac{A}{2} z(r^2 + z^2)^{-\frac{3}{2}} \quad (200) \\ \tau_{rz} &= \frac{3}{2} A \int_z^\infty rz(r^2 + z^2)^{-\frac{5}{2}} dz = \frac{A}{2} r(r^2 + z^2)^{-\frac{3}{2}} \\ \sigma_\theta &= -\frac{1}{2} A \int_z^\infty (r^2 + z^2)^{-\frac{3}{2}} dz = -\frac{A}{2} \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} \right] \end{aligned}$$

Considering the plane $z = 0$ we find that the normal stress on this plane is zero, and the shearing stress is

$$(\tau_{rz})_{z=0} = \frac{1}{2} \frac{A}{r^2} \quad (b)$$

It appears now that by combining solutions (194) and (200), we can, by a suitable adjustment of the constants A and B , obtain such a stress distribution that the plane $z = 0$ will be free from stresses and a concentrated force P will act at the origin. From (a) and (b) we see that the shearing forces on the boundary plane are eliminated if

$$-B(1-2\nu) + \frac{A}{2} = 0$$

from which

$$A = 2B(1-2\nu)$$

Substituting in expressions (200) and adding together the stresses (194) and (200), we find

$$\begin{aligned} \sigma_r &= B \left\{ (1-2\nu) \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} \right] - 3r^2z(r^2 + z^2)^{-\frac{3}{2}} \right\} \\ \sigma_z &= -3Bz^2(r^2 + z^2)^{-\frac{3}{2}} \\ \sigma_\theta &= B(1-2\nu) \left[-\frac{1}{r^2} + \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} + z(r^2 + z^2)^{-\frac{3}{2}} \right] \\ \tau_{rz} &= -3Brz^2(r^2 + z^2)^{-\frac{5}{2}} \quad (c) \end{aligned}$$

This stress distribution satisfies the boundary conditions, since $\sigma_z = \tau_{rz} = 0$ for $z = 0$. It remains now to determine the constant B so that the forces distributed over a hemispherical surface with center at the origin are statically equivalent to the force P acting along the z -axis. Considering the equilibrium of an element such as shown in Fig. 200, the component in the z -direction of forces on the hemispherical surface is

$$\bar{Z} = -(\tau_{rz} \sin \psi + \sigma_z \cos \psi) = 3Bz^2(r^2 + z^2)^{-2}$$

For determining B we obtain the equation

$$P = 2\pi \int_0^{\frac{\pi}{2}} \bar{Z}r(r^2 + z^2)^{\frac{1}{2}} d\psi = 6\pi B \int_0^{\frac{\pi}{2}} \cos^2 \psi \sin \psi d\psi = 2\pi B$$

from which

$$B = \frac{P}{2\pi}$$

Finally, substituting in (c) we obtain the following expressions for the stress components due to a normal force P acting on the plane boundary of a semi-infinite solid:

$$\begin{aligned} \sigma_r &= \frac{P}{2\pi} \left\{ (1 - 2\nu) \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} \right] - 3r^2z(r^2 + z^2)^{-\frac{3}{2}} \right\} \\ \sigma_z &= -\frac{3P}{2\pi} z^3(r^2 + z^2)^{-\frac{3}{2}} \\ \sigma_\theta &= \frac{P}{2\pi} (1 - 2\nu) \left\{ -\frac{1}{r^2} + \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} + z(r^2 + z^2)^{-\frac{3}{2}} \right\} \\ \tau_{rz} &= -\frac{3P}{2\pi} rz^2(r^2 + z^2)^{-\frac{3}{2}} \end{aligned} \quad (201)$$

This solution is the three-dimensional analogue of the solution for the semi-infinite plate (see Art. 33).

If we take an elemental area mn perpendicular to z -axis (Fig. 204), the ratio of the normal and shearing components of the stress on this element, from Eqs. (201), is

$$\frac{\sigma_z}{\tau_{rz}} = \frac{z}{r} \quad (d)$$

Hence the direction of the resultant stress passes through the origin O . The magnitude of this resultant stress is

$$S = \sqrt{\sigma_z^2 + \tau_{rz}^2} = \frac{3P}{2\pi} \frac{z^2}{(r^2 + z^2)^2} = \frac{3P}{2\pi} \frac{\cos^2 \psi}{(r^2 + z^2)} \quad (202)$$

The stress is thus inversely proportional to the square of the distance from the point of application of the load P . Imagine a spherical surface of diameter d , tangent to the plane $z = 0$ at the origin O . For each point of this surface,

$$r^2 + z^2 = d^2 \cos^2 \psi \quad (e)$$

Substituting in (202) we conclude that for points of the sphere the total stress on horizontal planes is constant and equal to $3P/2\pi d^2$.

Consider now the displacements produced in the semi-infinite solid by the load P . From Eqs. (178) for strain components,

$$u = \epsilon_\theta r = \frac{r}{E} [\sigma_\theta - \nu(\sigma_r + \sigma_z)]$$

Substituting the values for the stress components from Eqs. (201),

$$u = \frac{(1 - 2\nu)(1 + \nu)P}{2\pi Er} \left[z(r^2 + z^2)^{-\frac{1}{2}} - 1 + \frac{1}{1 - 2\nu} r^2 z(r^2 + z^2)^{-\frac{3}{2}} \right] \quad (203)$$

For determining vertical displacements w , we have, from Eqs. (178),

$$\begin{aligned} \frac{\partial w}{\partial z} &= \epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_r + \sigma_\theta)] \\ \frac{\partial w}{\partial r} &= \gamma_{rz} - \frac{\partial u}{\partial z} = \frac{2(1 + \nu)\tau_{rz}}{E} - \frac{\partial u}{\partial z} \end{aligned}$$

Substituting for the stress components, and for the displacement u the values found above, we obtain

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{P}{2\pi E} \{ 3(1 + \nu)r^2 z(r^2 + z^2)^{-\frac{3}{2}} - [3 + \nu(1 - 2\nu)]z(r^2 + z^2)^{-\frac{1}{2}} \} \\ \frac{\partial w}{\partial r} &= -\frac{P(1 + \nu)}{2\pi E} [2(1 - \nu)r(r^2 + z^2)^{-\frac{1}{2}} + 3rz^2(r^2 + z^2)^{-\frac{3}{2}}] \end{aligned}$$

from which, by integration,

$$w = \frac{P}{2\pi E} [(1 + \nu)z^2(r^2 + z^2)^{-\frac{1}{2}} + 2(1 - \nu^2)(r^2 + z^2)^{-\frac{1}{2}}] \quad (204)$$

For the boundary plane ($z = 0$) the displacements are

$$(u)_{z=0} = -\frac{(1 - 2\nu)(1 + \nu)P}{2\pi Er}, \quad (w)_{z=0} = \frac{P(1 - \nu^2)}{\pi Er} \quad (205)$$

showing that the product wr is constant at the boundary. Hence the radii drawn from the origin on the boundary surface, after deformation, are hyperbolae with the asymptotes Or and Oz . At the origin the displacements and stresses become infinite. To eliminate the difficulties in applying our equations we can imagine the material near the origin cut out by a hemispherical surface of small radius and the concentrated force P replaced by the statically equivalent forces distributed over this surface.

124. Load Distributed over a Part of the Boundary of a Semi-infinite Solid. Having the solution for a concentrated force acting on the boundary of a semi-infinite solid we can find the displacements and the stresses produced by a distributed load by superposition. Take, as a simple example, the case of a uniform load distributed over the area of a circle of radius a (Fig. 205), and consider the deflection, in the direction of the load, of a point M on the surface of the body at a distance r from the center of the circle. Taking a small element of the loaded area shown shaded in the figure, bounded by two radii including the angle $d\psi$ and two arcs of circle with the radii s and

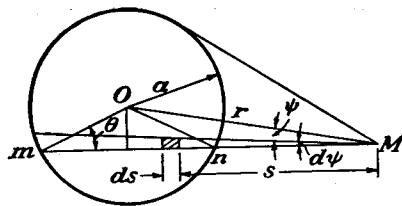


FIG. 205.

$s + ds$, all drawn from M , the load on this element is $qs d\psi ds$ and the corresponding deflection at M , from Eq. (205), is

$$\frac{(1 - \nu^2)q}{\pi E} \cdot \frac{s d\psi ds}{s} = \frac{(1 - \nu^2)q}{\pi E} d\psi ds$$

The total deflection is now obtained by double integration,

$$w = \frac{(1 - \nu^2)q}{\pi E} \int \int d\psi ds$$

Integrating with respect to s and taking into account the fact that the length of the chord mn is equal to $2\sqrt{a^2 - r^2 \sin^2 \psi}$ we find

$$w = \frac{4(1 - \nu^2)q}{\pi E} \int_0^{\psi_1} \sqrt{a^2 - r^2 \sin^2 \psi} d\psi \quad (a)$$

in which ψ_1 is the maximum value of ψ , i.e., the angle between r and the tangent to the circle. The calculation of the integral (a) is simplified by introducing, instead of the variable ψ , the variable angle θ . From the figure we have

$$a \sin \theta = r \sin \psi$$

from which

$$d\psi = \frac{a \cos \theta d\theta}{r \cos \psi} = \frac{a \cos \theta d\theta}{r \sqrt{1 - \frac{a^2}{r^2} \sin^2 \theta}}$$

Substituting in Eq. (a) and remembering that θ varies from 0 to $\pi/2$, when ψ changes from 0 to ψ_1 , we find

$$w = \frac{4(1 - \nu^2)q}{\pi E} \int_0^{\frac{\pi}{2}} \frac{a^2 \cos^2 \theta d\theta}{r \sqrt{1 - (a^2/r^2) \sin^2 \theta}} = \frac{4(1 - \nu^2)qr}{\pi E} \left[\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{a^2}{r^2} \sin^2 \theta} d\theta - \left(1 - \frac{a^2}{r^2} \right) \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - (a^2/r^2) \sin^2 \theta}} \right] \quad (206)$$

The integrals in this equation are known as *elliptic integrals*, and their values for any value of a/r can be taken from tables.¹

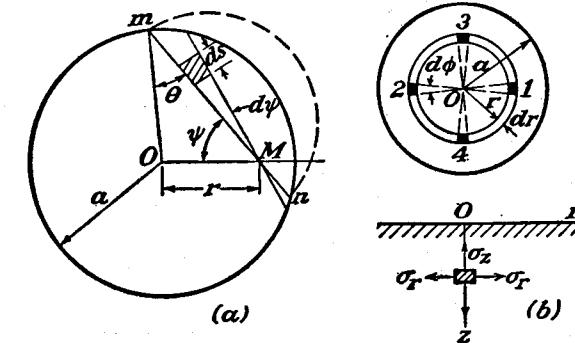


FIG. 206.

To get the deflection at the boundary of the loaded circle, we take $r = a$ in Eq. (206) and find

$$(w)_{r=a} = \frac{4(1 - \nu^2)qa}{\pi E} \quad (207)$$

If the point M is within the loaded area (Fig. 206a), we again consider the deflection produced by a shaded element on which the load $qs d\psi ds$ acts. Then the total deflection is

$$w = \frac{(1 - \nu^2)q}{\pi E} \int \int ds d\psi$$

¹ See for instance, E. Jahnke and F. Emde, "Funktionentafeln," Berlin, 1909; or Peirce, "Short Table of Integrals," 1910.

The length of the chord mn is $2a \cos \theta$, and ψ varies from zero to $\pi/2$, so

$$w = \frac{4(1 - \nu^2)q}{\pi E} \int_0^{\frac{\pi}{2}} a \cos \theta d\psi$$

or, since $a \sin \theta = r \sin \psi$, we have

$$w = \frac{4(1 - \nu^2)qa}{\pi E} \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{r^2}{a^2} \sin^2 \psi} d\psi \quad (208)$$

Thus the deflection can easily be calculated for any value of the ratio r/a by using tables of elliptic integrals. The maximum deflection occurs, of course, at the center of the circle. Substituting $r = 0$ in Eq. (208), we find

$$(w)_{\max.} = \frac{2(1 - \nu^2)qa}{E} \quad (209)$$

Comparing this with the deflection at the boundary of the circle we find that the latter is $2/\pi$ times the maximum deflection.¹ It is interesting to note that for a given intensity of the load q the maximum deflection is not constant but increases in the same ratio as the radius of the loaded circle.

By using superposition the stresses can also be calculated. Consider, for example, the stresses at a point on the z -axis (Fig. 206b). The stress σ_z produced at such a point by a load distributed over a ring area of radius r and width dr is obtained by substituting in the second of Eqs. (201) $2\pi r dr q$ instead of P . Then the stress σ_z produced by the uniform load distributed over the entire circular area of radius a is

$$\begin{aligned} \sigma_z &= - \int_0^a 3qr dr z^3 (r^2 + z^2)^{-\frac{3}{2}} = qz^3 \left| (r^2 + z^2)^{-\frac{3}{2}} \right|_0 \\ &= q \left[-1 + \frac{z^3}{(a^2 + z^2)^{\frac{3}{2}}} \right] \quad (b) \end{aligned}$$

This stress is equal to $-q$ at the surface of the body and gradually decreases with increase of distance z . In calculating the stresses σ_r and

¹ The solution of this problem was given by Boussinesq, *loc. cit.* See also H. Lamb, *Proc. London Math. Soc.*, vol. 34, p. 276, 1902; K. Terazawa, *J. Coll. Sci., Univ. Tokyo*, vol. 37, 1916; F. Schleicher, *Bauingenieur*, vol. 7, 1926, and *Bauingenieur*, vol. 14, p. 242, 1933. A complete investigation of this problem, also of the case in which the load is distributed over a rectangle, is given in the paper by A. E. H. Love, *Trans. Roy. Soc. (London)*, series A, vol. 228, 1929. Special properties of the deformation and stress in the general case are pointed out by S. Way, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 7, p. A-147, 1940.

σ_θ at the same point, consider the two elements 1 and 2 of the loaded area (Fig. 206b) with the loads $qr d\phi dr$. The stresses produced by these two elemental loads at a point on the z -axis, from the first and third of Eqs. (201), are

$$\begin{aligned} d\sigma_r' &= \frac{qr d\phi dr}{\pi} \left\{ (1 - 2\nu) \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} \right] - 3r^2 z (r^2 + z^2)^{-\frac{3}{2}} \right\} \\ d\sigma_\theta' &= \frac{qr d\phi dr}{\pi} (1 - 2\nu) \left[-\frac{1}{r^2} + \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} + z(r^2 + z^2)^{-\frac{1}{2}} \right] \end{aligned} \quad (c)$$

The normal stresses produced on the same planes by the elemental loads at points 3 and 4 are

$$\begin{aligned} d\sigma_r'' &= \frac{qr d\phi dr}{\pi} (1 - 2\nu) \left[-\frac{1}{r^2} + \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} + z(r^2 + z^2)^{-\frac{1}{2}} \right] \\ d\sigma_\theta'' &= \frac{qr d\phi dr}{\pi} \left\{ (1 - 2\nu) \left[\frac{1}{r^2} - \frac{z}{r^2} (r^2 + z^2)^{-\frac{1}{2}} \right] - 3r^2 z (r^2 + z^2)^{-\frac{3}{2}} \right\} \end{aligned} \quad (d)$$

By summation of (c) and (d) we find that the four elemental loads, indicated in the figure, produce the stresses

$$\begin{aligned} d\sigma_r &= d\sigma_\theta = \frac{qr d\phi dr}{\pi} [(1 - 2\nu)z(r^2 + z^2)^{-\frac{1}{2}} - 3r^2 z(r^2 + z^2)^{-\frac{3}{2}}] \\ &= \frac{qr d\phi dr}{\pi} [-2(1 + \nu)z(r^2 + z^2)^{-\frac{1}{2}} + 3z^3(r^2 + z^2)^{-\frac{3}{2}}] \end{aligned} \quad (e)$$

To get the stresses produced by the entire load uniformly distributed over the area of a circle of radius a we must integrate expression (e) with respect to ϕ between the limits 0 and $\pi/2$, and with respect to r , from 0 to a . Then

$$\begin{aligned} \sigma_r &= \sigma_\theta = \frac{q}{2} \int_0^a [-2(1 + \nu)z(r^2 + z^2)^{-\frac{1}{2}} + 3z^3(r^2 + z^2)^{-\frac{3}{2}}] r dr \\ &= \frac{q}{2} \left[-(1 + 2\nu) + \frac{2(1 + \nu)z}{\sqrt{a^2 + z^2}} - \left(\frac{z}{\sqrt{a^2 + z^2}} \right)^3 \right] \end{aligned} \quad (f)$$

For the point O , the center of the loaded circle, we find, from Eqs. (b) and (f),

$$\sigma_z = -q, \quad \sigma_r = \sigma_\theta = -\frac{q(1 + 2\nu)}{2}$$

Taking $\nu = 0.3$, we have $\sigma_r = \sigma_\theta = -0.8q$. The maximum shearing stress at the point O , on planes at 45 deg. to the z -axis, is equal to $0.1q$. Assuming that yielding of the material depends on the maximum

shearing stress, it can be shown that the point O , considered above, is not the most unfavorable point on the z -axis. The maximum shearing stress at any point on the z -axis (Fig. 206b), from Eqs. (b) and (f), is

$$\frac{1}{2}(\sigma_\theta - \sigma_z) = \frac{q}{2} \left[\frac{1-2\nu}{2} + (1+\nu) \frac{z}{\sqrt{a^2+z^2}} - \frac{3}{2} \left(\frac{z}{\sqrt{a^2+z^2}} \right)^3 \right] \quad (g)$$

This expression becomes a maximum when

$$\frac{z}{\sqrt{a^2+z^2}} = \frac{1}{3} \sqrt{2(1+\nu)}$$

from which

$$z = a \sqrt{\frac{2(1+\nu)}{7-2\nu}} \quad (h)$$

Substituting in expression (g),

$$\tau_{\max.} = \frac{q}{2} \left[\frac{1-2\nu}{2} + \frac{2}{9}(1+\nu) \sqrt{2(1+\nu)} \right] \quad (k)$$

Assuming $\nu = 0.3$, we find, from Eqs. (h) and (k),

$$z = 0.638a, \quad \tau_{\max.} = 0.33q$$

This shows that the maximum shearing stress for points on the z -axis is at a certain depth, approximately equal to two-thirds of the radius of the loaded circle, and the magnitude of this maximum is about one-third of the applied uniform pressure q .

For the case of a uniform pressure distributed over the surface of a square with sides $2a$, the maximum deflection at the center is

$$w_{\max.} = \frac{8}{\pi} \log_n (\sqrt{2} + 1) \frac{qa(1-\nu^2)}{E} = 2.24 \frac{qa(1-\nu^2)}{E} \quad (210)$$

The deflection at the corners of the square is only half the deflection at the center, and the average deflection is

$$w_{\text{aver.}} = 1.90 \frac{qa(1-\nu^2)}{E} \quad (211)$$

Analogous calculations have also been made for uniform pressure distribution over rectangles with various ratios, $\alpha = a/b$, of the sides.

All the results can be put in the form¹

$$w_{\text{aver.}} = m \frac{P(1-\nu^2)}{E \sqrt{A}} \quad (212)$$

¹ See Schleicher, *loc. cit.*

in which m is a numerical factor depending on α , A is the magnitude of this area, and P is the total load.

TABLE OF FACTORS m IN EQ. (212)

	Circle	Square	Rectangles with various $\alpha = \frac{a}{b}$					
			1.5	2	3	5	10	100
$m =$	0.96	0.95	0.94	0.92	0.88	0.82	0.71	0.37

Several values of the factor m are given in the table. It will be seen that for a given load P and area A deflections increase when the ratio of the perimeter of the loaded area to the area decreases. Equation (212) is sometimes used in discussing deflections of foundations¹ of engineering structures. In order to have equal deflections of various portions of the structure the average pressure on the foundation must be in a certain relation to the shape and the magnitude of the loaded area.

It was assumed in the previous discussion that the load was given, and we found the displacements produced. Consider now the case when the displacements are given and it is necessary to find the corresponding distribution of pressures on the boundary plane. Take, as an example, the case of an absolutely rigid die in the form of a circular cylinder pressed against the plane boundary of a semi-infinite elastic solid. In such a case the displacement w is constant over the circular base of the die. The distribution of pressures is not constant and its intensity is given by the equation²

$$q = \frac{P}{2\pi a \sqrt{a^2 - r^2}} \quad (213)$$

in which P is the total load on the die, a the radius of the die, and r the distance from the center of the circle on which the pressure acts. This distribution of pressures is obviously not uniform and its smallest value is at the center ($r = 0$), where

$$q_{\min.} = \frac{P}{2\pi a^2}$$

i.e., it is equal to half the average pressure on the circular area of contact. At the boundary of the same area ($r = a$) the pressure becomes

¹ See Schleicher, *loc. cit.*

² This solution was given by Boussinesq, *loc. cit.*

infinite. In actual cases we shall have yielding of material along the boundary. This yielding however is of local character and does not substantially affect the distribution of pressures (213) at points some distance from the boundary of the circle.

The displacement of the die is given by the equation

$$w = \frac{P(1 - \nu^2)}{2aE} \quad (214)$$

We see that, for a given value of the average unit pressure on the boundary plane, the deflection is not constant but increases in the same ratio as the radius of the die.

For comparison we give also the average deflection for the case of a uniform distribution of pressures [Eq. (208)]:

$$w_{\text{aver.}} = \frac{\int_0^a w 2\pi r dr}{\pi a^2} = \frac{16}{3\pi^2} \frac{P(1 - \nu^2)}{aE} = 0.54 \frac{P(1 - \nu^2)}{aE} \quad (215)$$

This average deflection is not very much different from the displacement (214) for an absolutely rigid die.

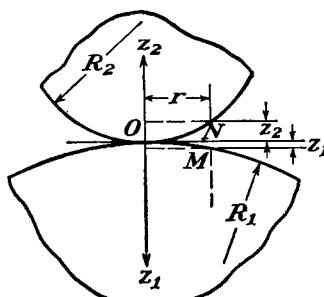


FIG. 207.

125. Pressure between Two Spherical Bodies in Contact. The results of the previous article can be used in discussing the pressure distribution between two bodies in contact.¹ We assume that at the point of contact these bodies have spherical surfaces with the radii R_1 and R_2 (Fig. 207). If there is no pressure between the bodies we have contact at one point O . The distances from the plane tangent at O of points such as M and N , on a meridian section of the

spheres at a very small distance² r from the axes z_1 and z_2 , can be represented with sufficient accuracy by the formulas

$$z_1 = \frac{r^2}{2R_1}, \quad z_2 = \frac{r^2}{2R_2} \quad (a)$$

and the mutual distance between these points is

$$z_1 + z_2 = r^2 \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right) = \frac{r^2(R_1 + R_2)}{2R_1R_2} \quad (b)$$

¹ This problem was solved by H. Hertz, *J. Math. (Crelle's J.)*, vol. 92, 1881. See also H. Hertz, "Gesammelte Werke," vol. 1, p. 155, Leipzig, 1895.

² r is small in comparison with R_1 and R_2 .

In the particular case of contact between a sphere and a plane (Fig. 208a), $R_1 = \infty$; and Eq. (b) for the distance MN gives

$$\frac{r^2}{2R_2} \quad (c)$$

In the case of contact between a ball and a spherical seat (Fig. 208b), R_1 is negative in Eq. (b), and

$$z_2 - z_1 = \frac{r^2(R_1 - R_2)}{2R_1R_2} \quad (c')$$

If the bodies are pressed together along the normal at O by a force P , there will be a local deformation near the point of contact producing contact over a small surface with a circular boundary, called the *surface of contact*. Assuming that the radii of curvature R_1 and R_2 are very large in comparison with the radius of the boundary of the surface of

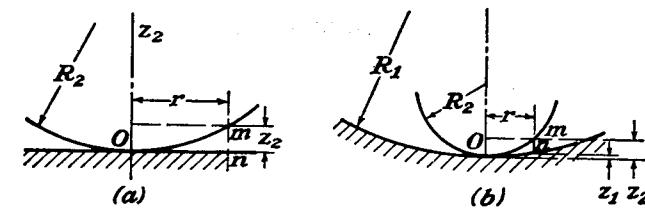


FIG. 208.

contact, we can apply, in discussing local deformation, the results obtained before for semi-infinite bodies. Let w_1 denote the displacement due to the local deformation in the direction z_1 of a point such as M on the surface of the lower ball (Fig. 207), and w_2 denote the same displacement in the direction z_2 for a point such as N of the upper ball. Assuming that the tangent plane at O remains immovable during local compression, then, due to this compression, any two points of the bodies on the axes z_1 and z_2 at large distances¹ from O will approach each other by a certain amount α , and the distance between two points such as M and N (Fig. 207) will diminish by $\alpha - (w_1 + w_2)$. If finally, due to local compression, the points M and N come inside the surface of contact, we have

$$\alpha - (w_1 + w_2) = z_1 + z_2 = \beta r^2 \quad (d)$$

in which β is a constant depending on the radii R_1 and R_2 and given by Eq. (b), (c), or (c'). Thus from geometrical considerations we find for

¹ Such distances that deformations due to the compression at these points can be neglected.

any point of the surface of contact,

$$w_1 + w_2 = \alpha - \beta r^2 \quad (e)$$

Let us now consider local deformations. From the condition of symmetry it can be concluded that the intensity q of pressure between the bodies in contact and the corresponding deformation are symmetrical with respect to the center O of the surface of contact. Taking Fig. 206a to represent the surface of contact, and M as a point on the surface of contact of the lower ball, the displacement w_1 of this point, from the previous article, is

$$w_1 = \frac{(1 - \nu_1^2)}{\pi E_1} \int \int q \, ds \, d\psi \quad (f)$$

in which ν_1 and E_1 are the elastic constants for the lower ball, and the integration is extended over the entire area of contact. An analogous formula is obtained also for the upper ball. Then

$$w_1 + w_2 = (k_1 + k_2) \iint q \, ds \, d\psi \quad (g)$$

in which

$$k_1 = \frac{1 - \nu_1^2}{\pi E_1}, \quad k_2 = \frac{1 - \nu_2^2}{\pi E_2} \quad (216)$$

From Eqs. (e) and (g),

$$(k_1 + k_2) \iint q \, ds \, d\psi = \alpha - \beta r^2 \quad (h)$$

Thus we must find an expression for q to satisfy Eq. (h). It will now be shown that this requirement is satisfied by assuming that the distribution of pressures q over the contact surface is represented by the ordinates of a hemisphere of radius a constructed on the surface of contact. If q_0 is the pressure at the center O of the surface of contact, then

$$q_0 = ka$$

in which $k = q_0/a$ is a constant factor representing the scale of our representation of the pressure distribution. Along a chord mn the pressure q varies, as indicated in Fig. 206 by the dotted semicircle. Performing the integration along this chord we find

$$\int q \, ds = \frac{q_0}{a} A$$

in which A is the area of the semicircle indicated by the dotted line and is equal to $\frac{\pi}{2} (a^2 - r^2 \sin^2 \psi)$. Substituting in Eq. (h), we find that

$$\frac{\pi(k_1 + k_2)q_0}{a} \int_0^{\frac{\pi}{2}} (a^2 - r^2 \sin^2 \psi) \, d\psi = \alpha - \beta r^2$$

or

$$(k_1 + k_2) \frac{q_0 \pi^2}{4a} (2a^2 - r^2) = \alpha - \beta r^2$$

This equation will be fulfilled for any value of r , and hence the assumed pressure distribution is the correct one if the following relations exist for the displacement α and the radius a of the surface of contact:

$$\begin{aligned} \alpha &= (k_1 + k_2)q_0 \frac{\pi^2 a}{2} \\ a &= (k_1 + k_2) \frac{\pi^2 q_0}{4\beta} \end{aligned} \quad (217)$$

The value of the maximum pressure q_0 is obtained by equating the sum of the pressures over the contact area to the compressive force P . Then, for the hemispherical pressure distribution this gives

$$\frac{q_0}{a} \cdot \frac{2}{3} \pi a^3 = P$$

from which

$$q_0 = \frac{3P}{2\pi a^2} \quad (218)$$

i.e., the maximum pressure is $1\frac{1}{2}$ times the average pressure on the surface of contact. Substituting in Eqs. (217) and taking, from Eq. (b),

$$\beta = \frac{R_1 + R_2}{2R_1 R_2}$$

we find for two balls in contact (Fig. 207)

$$\begin{aligned} a &= \sqrt[3]{\frac{3\pi P(k_1 + k_2)R_1 R_2}{4(R_1 + R_2)}} \\ \alpha &= \sqrt[3]{\frac{9\pi^2 P^2 (k_1 + k_2)^2 (R_1 + R_2)}{16 R_1 R_2}} \end{aligned} \quad (219)$$

Assuming that both balls have the same elastic properties and taking $\nu = 0.3$, this becomes

$$\begin{aligned} a &= 1.109 \sqrt[3]{\frac{P}{E} \frac{R_1 R_2}{R_1 + R_2}} \\ \alpha &= 1.23 \sqrt[3]{\frac{P^2}{E^2} \frac{R_1 + R_2}{R_1 R_2}} \end{aligned} \quad (220)$$

The corresponding maximum pressure is

$$q_0 = \frac{3}{2} \frac{P}{\pi a^2} = 0.388 \sqrt[3]{\frac{PE^2}{R_1^2 R_2^2} \frac{(R_1 + R_2)^2}{R_1 R_2}} \quad (221)$$

In the case of a ball pressed into a plane surface, and assuming the same elastic properties of material for both bodies, we find, by substituting $R_1 = \infty$ in Eqs. (220) and (221),

$$a = 1.109 \sqrt[3]{\frac{PR_2}{E}}, \quad \alpha = 1.23 \sqrt[3]{\frac{P^2}{E^2 R_2}}, \quad q_0 = 0.388 \sqrt[3]{\frac{PE^2}{R_2^2}} \quad (222)$$

By taking R_1 negative we can write down also equations for a ball in a spherical seat (Fig. 208b).

Having the magnitude of the surface of contact and the pressures acting on it, the stresses can be calculated by using the method developed in the previous article.¹ The results of these calculations for points along the axes Oz_1 and Oz_2 are shown in Fig. 209. The maximum pressure q_0 at the center of the surface of contact is taken as a unit of stress. In measuring the distances along the z -axis, the radius a of the surface of contact is taken as the unit. The greatest stress is the compressive stress σ_z at the center of the surface of contact, but the two other principal stresses σ_r and σ_θ , at the same point, are equal to $\frac{1+2\nu}{2} \sigma_z$. Hence the maximum

shearing stress, on which the yielding of such material as steel depends, is comparatively small at this point.

FIG. 209.

The point with maximum shearing stress is on the z -axis at a depth equal to about a half of the radius of the surface of contact. This point must be considered as the weakest point in such material as steel. The maximum shearing stress at this point (for $\nu = 0.3$) is about $0.31q_0$.

In the case of brittle materials, such as glass, failure is produced by maximum tensile stress. This stress occurs at the circular boundary of the surface of contact. It acts in a radial direction and has the magnitude

$$\sigma_r = \frac{(1-2\nu)}{3} q_0$$

¹ Such calculations were made by A. N. Dinnik, *Bull. Polytech. Inst., Kiev*, 1909. See also M. T. Huber, *Ann. Physik*, vol. 14, 1904, p. 153; S. Fuchs, *Physik Z.*, vol. 14, p. 1282, 1913; M. C. Huber and S. Fuchs, *Physik Z.*, vol. 15, p. 298, 1914; W. B. Morton and L. J. Close, *Phil. Mag.*, vol. 43, p. 320, 1922.

The other principal stress, acting in the circumferential direction, is numerically equal to the above radial stress but of opposite sign. Hence along the boundary of the surface of contact, where normal pressure on the surface becomes equal to zero, we have pure shear of the amount $q_0(1-2\nu)/3$. Taking $\nu = 0.3$, this shear becomes equal to $0.133q_0$. This stress is much smaller than the maximum shearing stress calculated above, but it is larger than the shearing stress at the center of the surface of contact, where the normal pressure is the largest.

Many experiments have been made which verify the theoretical results of Hertz for materials which follow Hooke's law and stress within the elastic limit.¹

126. Pressure between Two Bodies in Contact. More General Case.² The general case of compression of elastic bodies in contact may be treated in the same manner as the case of spherical bodies discussed in the previous article. Consider the tangent plane at the point of contact O as the xy -plane (Fig. 207). The surfaces of the bodies near the point of contact, by neglecting small quantities of higher order, can be represented by the equations³

$$\begin{aligned} z_1 &= A_1 x^2 + A_2 xy + A_3 y^2 \\ z_2 &= B_1 x^2 + B_2 xy + B_3 y^2 \end{aligned} \quad (a)$$

The distance between two points such as M and N is then

$$z_1 + z_2 = (A_1 + B_1)x^2 + (A_2 + B_2)xy + (A_3 + B_3)y^2 \quad (b)$$

We can always take for x and y such directions as to make the term containing the product xy disappear. Then

$$z_1 + z_2 = Ax^2 + By^2 \quad (c)$$

in which A and B are constants depending on the magnitudes of the principal curvatures of the surfaces in contact and on the angle between the planes of principal curvatures of the two surfaces. If R_1 and R_1' denote the principal radii of curvature at the point of con-

¹ References to the corresponding literature can be found in the paper by G. Berndt, *Z. tech. Physik*, vol. 3, p. 14, 1922. See also "Handbuch der physikalischen und technischen Mechanik," vol. 3, p. 120.

² This theory is due to Hertz, *loc. cit.* Tangential force and twisting couple at the contact are considered by R. D. Mindlin, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 16, p. 259, 1949.

³ It is assumed that point O is not a point of singularity on the surfaces of the bodies, but the surface adjacent to the point of contact is rounded and may be considered as a surface of the second degree.

tact of one of the bodies, and R_2 and R_2' those of the other,¹ and ψ the angle between the normal planes containing the curvatures $1/R_1$ and $1/R_2$, then the constants A and B are determined from the equations

$$\begin{aligned} A + B &= \frac{1}{2} \left(\frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'} \right) \\ B - A &= \frac{1}{2} \left[\left(\frac{1}{R_1} - \frac{1}{R_1'} \right)^2 + \left(\frac{1}{R_2} - \frac{1}{R_2'} \right)^2 \right. \\ &\quad \left. + 2 \left(\frac{1}{R_1} - \frac{1}{R_1'} \right) \left(\frac{1}{R_2} - \frac{1}{R_2'} \right) \cos 2\psi \right] \end{aligned} \quad (d)$$

It can be shown that A and B in Eq. (c) both have the same sign, and it can therefore be concluded that all points with the same mutual distance $z_1 + z_2$ lie on one ellipse. Hence, if we press the bodies together in the direction of the normal to the tangent plane at O , the surface of contact will have an elliptical boundary.

Let α , w_1 , w_2 have the same meaning as in the previous article. Then, for points on the surface of contact, we have

$$w_1 + w_2 + z_1 + z_2 = \alpha \quad (e)$$

or

$$w_1 + w_2 = \alpha - Ax^2 - By^2$$

This is obtained from geometrical considerations. Consider now the local deformation at the surface of contact. Assuming that this surface is very small and applying Eq. (205), obtained for semi-infinite bodies, the sum of the displacements w_1 and w_2 for points of the surface of contact is

$$w_1 + w_2 = \left(\frac{1 - \nu_1^2}{\pi E_1} + \frac{1 - \nu_2^2}{\pi E_2} \right) \iint \frac{q \, dA}{r} \quad (f)$$

where $q \, dA$ is the pressure acting on an infinitely small element of the surface of contact, and r is the distance of this element from the point under consideration. The integration must be extended over the entire surface of contact. Using notations (216), we obtain, from (e) and (f),

$$(k_1 + k_2) \iint \frac{q \, dA}{r} = \alpha - Ax^2 - By^2 \quad (g)$$

¹ The curvature of a body is considered as positive if the corresponding center of curvature is within the body. In Fig. 207 the curvatures of the bodies are positive. In Fig. 208b the spherical seat has a negative curvature.

The problem now is to find a distribution of pressures q to satisfy Eq. (g). H. Hertz showed that this requirement is satisfied by assuming that the intensity of pressures q over the surface of contact is represented by the ordinates of a semi-ellipsoid constructed on the surface of contact. The maximum pressure is then clearly at the center of the surface of contact. Denoting it by q_0 and denoting by a and b the semiaxes of the elliptic boundary of the surface of contact the magnitude of the maximum pressure is obtained from the equation

$$P = \iint q \, dA = \frac{2}{3} \pi ab q_0$$

from which

$$q_0 = \frac{3}{2} \frac{P}{\pi ab} \quad (223)$$

We see that the maximum pressure is $1\frac{1}{2}$ times the average pressure on the surface of contact. To calculate this pressure we must know the magnitudes of the semiaxes a and b . From an analysis analogous to that used for spherical bodies we find that

$$\begin{aligned} a &= m \sqrt[3]{\frac{3\pi P(k_1 + k_2)}{4(A + B)}} \\ b &= n \sqrt[3]{\frac{3\pi P(k_1 + k_2)}{4(A + B)}} \end{aligned} \quad (224)$$

in which $A + B$ is determined from Eqs. (d) and the coefficients m and n are numbers depending on the ratio $(B - A):(A + B)$. Using the notation

$$\cos \theta = \frac{B - A}{A + B} \quad (h)$$

the values of m and n for various values of θ are given in the table below.¹

$\theta =$	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
$m =$	2.731	2.397	2.136	1.926	1.754	1.611	1.486	1.378	1.284	1.202	1.128	1.061	1.000
$n =$	0.493	0.530	0.567	0.604	0.641	0.678	0.717	0.759	0.802	0.846	0.893	0.944	1.000

Considering, for instance, the contact of a wheel with a cylindrical rim of radius $R_1 = 15.8$ in. and of a rail with the radius of the head $R_2 = 12$ in., we find, by substituting $R_1' = R_2' = \infty$ and $\psi = \pi/2$ into Eqs. (d),

$$A + B = 0.0733, \quad B - A = 0.0099, \quad \cos \theta = 0.135, \quad \theta = 82^\circ 15'$$

¹ The table is taken from the paper by H. L. Whittemore and S. N. Petrenko, U. S. Bur. Standards, Tech. Paper 201, 1921.

Then, by interpolation, we find from the above table that

$$m = 1.098, \quad n = 0.918$$

Substituting in Eqs. (224) and taking $E = 30.10^6$ p.s.i. and $\nu = 0.25$,¹ we find

$$a = 0.00946 \sqrt[3]{P}, \quad b = 0.00792 \sqrt[3]{P}$$

For a load $P = 1,000$ lb.,

$$a = 0.0946 \text{ in.}, \quad b = 0.0792 \text{ in.}, \quad \text{area of contact } \pi ab = 0.0236 \text{ sq. in.}$$

and the maximum pressure at the center is

$$q_0 = \frac{3}{2} \frac{P}{\pi ab} = 63,600 \text{ p.s.i.}$$

Knowing the distribution of pressure, the stresses at any point can be calculated.² It was shown in this manner that the point of maximum shearing stress is on the z -axis at a certain small depth z_1 , depending on the magnitude of the semiaxes a and b . For instance: $z_1 = 0.47a$, when $b/a = 1$; and $z_1 = 0.24a$, when $b/a = 0.34$. The corresponding values of maximum shearing stress (for $\nu = 0.3$) are $\tau_{\max.} = 0.31q_0$ and $\tau_{\max.} = 0.32q_0$ respectively.

Considering points on the elliptical surface of contact and taking the x - and y -axes in the direction of the semiaxes a and b respectively, the principal stresses at the center of the surface of contact are

$$\begin{aligned} \sigma_x &= -2\nu q_0 - (1 - 2\nu)q_0 \frac{b}{a + b} \\ \sigma_y &= -2\nu q_0 - (1 - 2\nu)q_0 \frac{a}{a + b} \\ \sigma_z &= -q_0 \end{aligned} \quad (k)$$

For the ends of the axes of the ellipse we find $\sigma_x = -\sigma_y$ and $\tau_{xy} = 0$. The tensile stress in the radial direction is equal to the compressive stress in the circumferential direction. Thus at these points there

¹ If ν is increased from 0.25 to 0.30 the semiaxes (224) decrease about 1 per cent and the maximum pressure q_0 increases about 2 per cent.

² Such investigations have been made by Prof. N. M. Belajef, see *Bull. Inst. Engineers of Ways of Communication*, St. Petersburg, 1917, and "Memoirs on Theory of Structures," St. Petersburg, 1924; see also H. R. Thomas and V. A. Hoersch, *Univ. Illinois Eng. Expt. Sta., Bull.* 212, 1930, and G. Lundberg and F. K. G. Odqvist, *Proc. Ingénieurs Vetenskaps Akad.*, No. 116, Stockholm, 1932.

exists pure shear. The magnitude of this shear for the ends of the major axis ($x = \pm a, y = 0$) is

$$\tau = (1 - 2\nu)q_0 \frac{\beta}{e^2} \left(\frac{1}{e} \operatorname{arctanh} e - 1 \right) \quad (l)$$

and for the ends of minor axis ($x = 0, y = \pm b$) is

$$\tau = (1 - 2\nu)q_0 \frac{\beta}{e^2} \left(1 - \frac{\beta}{e} \operatorname{arctan} \frac{e}{\beta} \right) \quad (m)$$

where $\beta = b/a$, $e = (1/a) \sqrt{a^2 - b^2}$. When b approaches a and the boundary of the surface of contact approaches a circular shape, the stresses given by (k), (l), and (m) approach the stresses given in the previous article for the case of compression of balls.

A more detailed investigation of stresses for all points in the surface of contact shows¹ that for $e < 0.89$ the maximum shearing stress is given by Eq. (l). For $e > 0.89$ the maximum shearing stress is at the center of the ellipse and can be calculated from Eqs. (k) above.

By increasing the ratio a/b we obtain narrower and narrower ellipses of contact, and at the limit $a/b = \infty$ we arrive at the case of contact of two cylinders with parallel axes.² The surface of contact is now a narrow rectangle. The distribution of pressure q along the width of the surface of contact (Fig. 210) is represented by a semi-ellipse. If the x -axis is perpendicular to the plane of the figure, b is half the width of the surface of contact, and P' the load per unit length of the surface of contact, we obtain, from the semi-elliptic pressure distribution,

$$P' = \frac{1}{2} \pi b q_0$$

from which

$$q_0 = \frac{2P'}{\pi b} \quad (225)$$

¹ See Belajef, *loc. cit.*

² A direct derivation of this case, with consideration of tangential force at the contact, is given by H. Poritsky, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 17, p. 191, 1950.

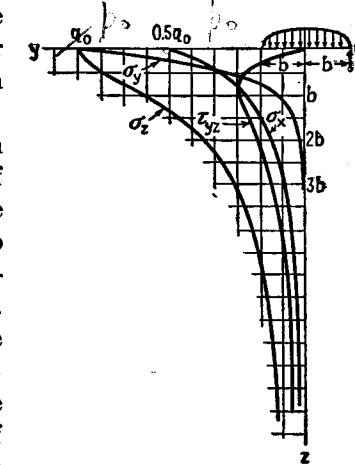


FIG. 210.

The investigation of local deformation gives for the quantity b the expression

$$b = \sqrt{\frac{4P'(k_1 + k_2)R_1 R_2}{R_1 + R_2}} \quad (226)$$

in which R_1 and R_2 are the radii of the cylinders and k_1 and k_2 are constants defined by Eqs. (216). If both cylinders are of the same material and $\nu = 0.3$, then

$$b = 1.52 \sqrt{\frac{P'R_1 R_2}{E(R_1 + R_2)}} \quad (227)$$

In the case of two equal radii, $R_1 = R_2 = R$,

$$b = 1.08 \sqrt{\frac{P'R}{E}} \quad (228)$$

For the case of contact of a cylinder with a plane surface,

$$b = 1.52 \sqrt{\frac{P'R}{E}} \quad (229)$$

Substituting b from Eq. (226) into Eq. (225), we find

$$q_0 = \sqrt{\frac{P'(R_1 + R_2)}{\pi^2(k_1 + k_2)R_1 R_2}} \quad (230)$$

If the materials of both cylinders are the same and $\nu = 0.3$,

$$q_0 = 0.418 \sqrt{\frac{P'E(R_1 + R_2)}{R_1 R_2}} \quad (231)$$

In the case of contact of a cylinder with a plane surface,

$$q_0 = 0.418 \sqrt{\frac{P'E}{R}} \quad (232)$$

Knowing q_0 and b , the stress at any point can be calculated. These calculations show¹ that the point with maximum shearing stress is on the z -axis at a certain depth. The variation of stress components with the depth, for $\nu = 0.3$, is shown in Fig. 210. The maximum shearing stress is at the depth $z_1 = 0.78b$ and its magnitude is $0.304q_0$.

¹ See Belajef, *loc. cit.*

127. Impact of Spheres. The results of the last two articles can be used in investigating impact of elastic bodies. Consider, as an example, the impact of two spheres (Fig. 211). As soon as the spheres, in their motion toward one another, come in contact at a point O ,¹ the compressive forces P begin to act and to change the velocities of the spheres. If v_1 and v_2 are the values of these velocities, their rates of change during impact are given by the equations

$$m_1 \frac{dv_1}{dt} = -P, \quad m_2 \frac{dv_2}{dt} = -P \quad (a)$$

in which m_1 and m_2 denote the masses of the spheres. Let α be the distance the two spheres approach one another due to local compression at O . Then the velocity of this approach is

$$\dot{\alpha} = v_1 + v_2$$

and we find, from Eqs. (a), that

$$\ddot{\alpha} = -P \frac{m_1 + m_2}{m_1 m_2} \quad (b)$$

Investigations show that the duration of impact, *i.e.*, the time during which the spheres remain in contact, is very long in comparison with the period of lowest mode of vibration of the spheres.² Vibrations can therefore be neglected, and it can be assumed that Eq. (219), which was established for statical conditions, holds during impact. Using the notations

$$n = \sqrt{\frac{16}{9\pi^2} \frac{R_1 R_2}{(k_1 + k_2)^2(R_1 + R_2)}}, \quad n_1 = \frac{m_1 + m_2}{m_1 m_2} \quad (c)$$

we find, from (219),

$$P = n\alpha^{\frac{3}{2}} \quad (d)$$

and Eq. (b) becomes

$$\ddot{\alpha} = -nn_1\alpha^{\frac{3}{2}} \quad (e)$$

Multiplying both sides of this equation by $\dot{\alpha}$,

$$\frac{1}{2}d(\dot{\alpha})^2 = -nn_1\alpha^{\frac{3}{2}} d\alpha \quad (f)$$

from which, by integration,

$$\frac{1}{2}(\dot{\alpha}^2 - v^2) = -\frac{2}{5}nn_1\alpha^{\frac{5}{2}} \quad (f)$$

where v is the velocity of approach of the two spheres at the beginning of impact. If we substitute $\dot{\alpha} = 0$ in this equation, we find the value of the approach at the instant of maximum compression, α_1 , as

$$\alpha_1 = \left(\frac{5}{4} \frac{v^2}{nn_1} \right)^{\frac{2}{5}} \quad (g)$$

With this value we can calculate, from Eqs. (219), the value of the maximum compressive force P acting between the spheres during impact, and the corresponding radius a of the surface of contact.

¹ We assume motions along the line joining the centers of the spheres.

² Lord Rayleigh, *Phil. Mag.*, series 6, vol. 11, p. 283, 1906.

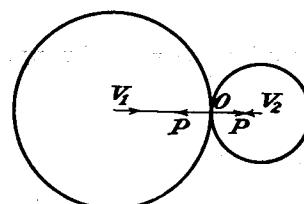


FIG. 211.

For calculating the duration of impact we write Eq. (f) in the following form:

$$dt = \frac{d\alpha}{\sqrt{v^2 - \frac{4}{3}nn_1\alpha^2}}$$

or writing $\alpha/\alpha_1 = x$ and using Eq. (g), we find that

$$dt = \frac{\alpha_1}{v} \frac{dx}{\sqrt{1 - (x)^{\frac{2}{3}}}}$$

from which the duration of impact is

$$t = \frac{2\alpha_1}{v} \int_0^1 \frac{dx}{\sqrt{1 - (x)^{\frac{2}{3}}}} = 2.94 \frac{\alpha_1}{v} \quad (233)$$

In the particular case of two equal spheres of the same material and radius R , we have, from (g),

$$\begin{aligned} \alpha_1 &= \left(\frac{5\sqrt{2}\pi\rho}{4} \frac{1 - \nu^2}{E} v^2 \right)^{\frac{1}{3}} R \\ t &= 2.94 \left(\frac{5\sqrt{2}\pi\rho}{4} \frac{1 - \nu^2}{E} \right)^{\frac{1}{3}} \frac{R}{(v)^{\frac{1}{3}}} \end{aligned} \quad (234)$$

where ρ denotes the mass per unit volume of the spheres.

We see that the duration of impact is proportional to the radius of the spheres and inversely proportional to $(v)^{\frac{1}{3}}$. This result was verified by several experimenters.¹ In the case of long bars with spherical ends, the period of the fundamental mode of vibration may be of the same order as the duration of impact, and in investigating local compression at the point of contact these vibrations should be considered.²

128. Symmetrical Deformation of a Circular Cylinder. For a circular cylinder submitted to the action of forces applied to the lateral surface and distributed symmetrically with respect to the axis of the cylinder, we introduce a stress function ϕ in cylindrical coordinates and apply Eq. (180).³ This equation is satisfied if we take for the stress function ϕ a solution of the equation

¹ M. Hamburger, *Wied. Ann.*, vol. 28, p. 653, 1886; A. Dinnik, *J. Russ. Phys.-Chem. Soc.*, vol. 38, p. 242, 1906, and vol. 41, p. 57, 1909. Further references to the literature of the subject are given in "Handbuch der physikalischen und technischen Mechanik," vol. 3, p. 448, 1927.

² See p. 452. Longitudinal impact of rods with spherical surfaces at the ends has been discussed by J. E. Sears, *Proc. Cambridge Phil. Soc.*, vol. 14, p. 257, 1908, and *Trans. Cambridge Phil. Soc.*, vol. 21, p. 49, 1912. Lateral impact of rods with consideration of local compression was discussed by S. Timoshenko, *Z. Math. Physik*, vol. 62, p. 198, 1914.

³ The problem of the deformation of a circular cylinder under the action of forces applied to the surface was discussed first by L. Pochhammer, *Crelle's J.*, vol. 81, 1876. Several problems of symmetrical deformation of cylinders were discussed by C. Chree, *Trans. Cambridge Phil. Soc.*, vol. 14, p. 250, 1889. See also the paper by L. N. G. Filon, *Trans. Roy. Soc. (London)*, series A, vol. 198, 1902, which contains solutions of several problems of practical interest relating to symmetrical deformation in a cylinder.

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (a)$$

This solution can be taken in the form

$$\phi = f(r) \sin kz \quad (b)$$

in which f is a function of r only. Substituting (b) into Eq. (a), we arrive at the following ordinary differential equation for determining $f(r)$:

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - k^2 f = 0 \quad (c)$$

We take an integral of this equation in the form of a series,

$$f(r) = a_0 + a_1 r^2 + a_2 r^4 + a_3 r^6 + \dots \quad (d)$$

Substituting this series in Eq. (c) we find the following relation between the consecutive coefficients:

$$(2n)^2 a_n - k^2 a_{n-1} = 0$$

from which

$$a_1 = \frac{k^2}{2^2} a_0, \quad a_2 = \frac{k^2}{4^2} a_1 = \frac{k^4}{2^2 \cdot 4^2} a_0, \dots$$

Substituting these in the series (d), we have

$$f(r) = a_0 \left(1 + \frac{k^2 r^2}{2^2} + \frac{k^4 r^4}{2^2 \cdot 4^2} + \frac{k^6 r^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \right) \quad (e)$$

The second integral of Eq. (c) can also be obtained in the form of a series, and it can be shown that this second integral becomes infinite when $r = 0$, and hence should not be considered when we are discussing deformation of a solid cylinder. The series in the parentheses of Eq. (e) is the Bessel function of zero order and of the imaginary argument ikr .¹ In the following we shall use for this function the notation $J_0(ikr)$ and write the stress function (b) in the form

$$\phi_1 = a_0 J_0(ikr) \sin kz \quad (f)$$

Equation (180) also has solutions different from solutions of Eq. (a). One of these solutions can be derived from the above function $J_0(ikr)$. By differentiation,

$$\frac{dJ_0(ikr)}{d(ikr)} = -\frac{ikr}{2} \left(1 + \frac{k^2 r^2}{2 \cdot 4} + \frac{k^4 r^4}{2 \cdot 4^2 \cdot 6} + \frac{k^6 r^6}{2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots \right) \quad (g)$$

This derivative with negative sign is called Bessel's function of the first order and is denoted by $J_1(ikr)$. Consider now the function

$$f_1(r) = r \frac{d}{dr} J_0(ikr) = -ikr J_1(ikr) = \frac{k^2 r^2}{2} \left(1 + \frac{k^2 r^2}{2 \cdot 4} + \frac{k^4 r^4}{2 \cdot 4^2 \cdot 6} + \dots \right) \quad (h)$$

¹ Discussion of the differential equation (c) and of Bessel's functions can be found in the following books: A. R. Forsyth, "A Treatise on Differential Equations," and A. Gray and G. B. Mathews, "A Treatise on Bessel Functions." Numerical tables for Bessel's functions can be found in E. Jahnke and F. Emde, "Funktionentafeln mit Formeln und Kurven," Berlin, 1909.

By differentiation it can be shown that

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - k^2 \right) f_1(r) = 2k^2 J_0(ikr)$$

Then, remembering that $J_0(ikr)$ is a solution of Eq. (c), it follows that $f_1(r)$ is a solution of the equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - k^2 \right) \left(\frac{d^2 f_1}{dr^2} + \frac{1}{r} \frac{df_1}{dr} - k^2 f_1 \right) = 0$$

Hence a solution of Eq. (180) can be taken in the form

$$\phi_2 = a_1 \sin kz(ikr) J_1(ikr) \quad (i)$$

Combining solutions (f) and (i), we can take the stress function in the form

$$\phi = \sin kz[a_0 J_0(ikr) + a_1(ikr) J_1(ikr)] \quad (j)$$

Substituting this stress function in Eqs. (179) we find the following expressions for the stress components:

$$\begin{aligned} \sigma_r &= \cos kz[a_0 F_1(r) + a_1 F_2(r)] \\ \tau_{rz} &= \sin kz[a_0 F_3(r) + a_1 F_4(r)] \end{aligned} \quad (k)$$

in which $F_1(r), \dots, F_4(r)$ are certain functions of r containing $J_0(ikr)$ and $J_1(ikr)$. By using tables of Bessel functions, the values of $F_1(r), \dots, F_4(r)$ can easily be calculated for any value of r .

Denoting by a the external radius of the cylinder, the forces applied to the surface of the cylinder, from Eqs. (k), are given by the following values of the stress components:

$$\begin{aligned} \sigma_r &= \cos kz[a_0 F_1(a) + a_1 F_2(a)] \\ \tau_{rz} &= \sin kz[a_0 F_3(a) + a_1 F_4(a)] \end{aligned} \quad (l)$$

By a suitable adjustment of the constants k, a_0, a_1 , various cases of symmetrical loading of a cylinder can be discussed. Denoting the length of the cylinder by l and taking

$$k = \frac{n\pi}{l}$$

$$\begin{aligned} a_0 F_1(a) + a_1 F_2(a) &= -A_n \\ a_0 F_3(a) + a_1 F_4(a) &= 0 \end{aligned}$$

we obtain the values of the constants a_0 and a_1 for the case when normal pressures $A_n \cos(n\pi z/l)$ act on the lateral surface of the cylinder. The case when $n = 1$ is represented in Fig. 212. In an

analogous manner we can get a solution for the case when tangential forces of intensity $B_n \sin(n\pi z/l)$ act on the surface of the cylinder.

By taking $n = 1, 2, 3, \dots$, and using the superposition principle, we arrive at solutions of problems in which the normal pressures on the surface of the cylinder can be represented by the series

$$A_1 \cos \frac{\pi z}{l} + A_2 \cos \frac{2\pi z}{l} + A_3 \cos \frac{3\pi z}{l} + \dots \quad (m)$$

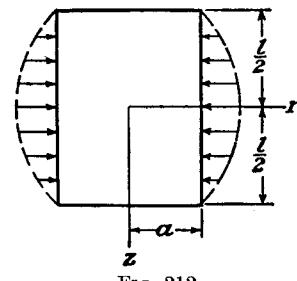


FIG. 212.

and the shearing forces by the series

$$B_1 \sin \frac{\pi z}{l} + B_2 \sin \frac{2\pi z}{l} + B_3 \sin \frac{3\pi z}{l} + \dots \quad (n)$$

If we take for the stress function ϕ , instead of expression (b), the expression

$$\phi = f(r) \cos kz$$

and proceed as before, we find, instead of expression (j), the stress function

$$\phi = \cos kz[b_0 J_0(ikr) + b_1(ikr) J_1(ikr)] \quad (o)$$

By a suitable adjustment of the constants k, b_0, b_1 , we obtain the solution for the case in which normal pressures on the cylinder are represented by a sine series and the shearing forces by cosine series. Hence, by combining solutions (j) and (o), we can get any axially symmetrical distribution of normal and shearing forces over the surface of the cylinder. At the same time there will also be certain forces distributed over the ends of the cylinder. By superposing a simple tension or compression we can always arrange that the resultant of these forces is zero, and their effect on stresses at some distance from the ends becomes negligible by virtue of Saint-Venant's principle. Several examples of symmetrical loading of cylinders are discussed by L. N. G. Filon in the paper already mentioned.¹ We give here final results from his solution for the case shown in Fig. 213. A cylinder, the length of which is equal to πa , is submitted to the tensile action of shearing forces uniformly distributed over the shaded portion of the surface of the cylinder indicated in the figure. The distribution of the normal stress σ_z over cross sections of the cylinder is of practical interest, and the table below gives the ratios of these stresses to the average tensile stress, obtained by dividing the total tensile force by the cross-sectional area of the cylinder. It can be seen that local tensile stresses near the loaded portions of the surface diminish rapidly with increase of distance from these portions and approach the average value.

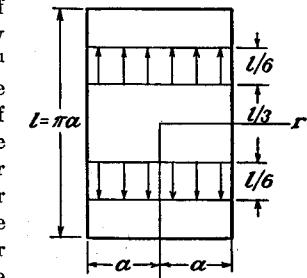


FIG. 213.

z	$r = 0$	$r = 0.2a$	$r = 0.4a$	$r = 0.6a$	$r = a$
0	0.689	0.719	0.810	0.962	1.117
0.05l	0.673	0.700	0.786	0.937	1.163
0.10l	0.631	0.652	0.720	0.859	1.344
0.15l	0.582	0.594	0.637	0.737	2.022
0.20l	0.539	0.545	0.565	0.617	1.368

¹ Loc. cit. See also G. Pickett, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 11, p. 176, 1944.

Another application of the general solution of the problem in terms of Bessel's functions is given by A. Nádai in discussing the bending of circular plates by a force concentrated at the middle¹ (Fig. 214).

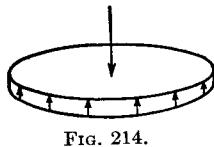


FIG. 214.

normal pressure p acting on the band $ABCD$ of the surface.

The required solution can evidently be obtained by superposing the effects of the two pressure distributions indicated in Fig. 215b. The basic problem is therefore that of pressure $p/2$ on the lower half of the cylindrical surface and $-p/2$ on

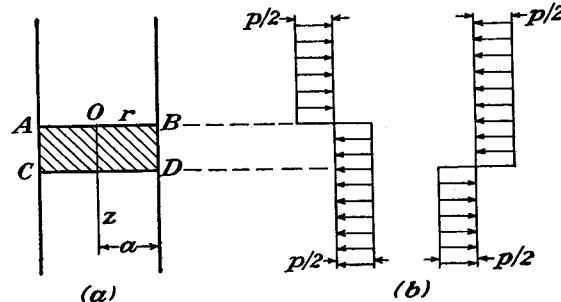


FIG. 215.

the upper half, the length of the cylinder being infinite, and its solution will now be given.

We begin with the stress function given by Eq. (a) of Art. 128, writing $I_0(kr)$ for $J_0(ikr)$ and $iI_1(kr)$ for $J_1(ikr)$. We also write $b_0 = \rho b_1$. Then

$$\phi = [\rho I_0(kr) - kr I_1(kr)] b_1 \cos kz \quad (a)$$

This satisfies Eq. (180) no matter what value is given to k . If we consider k to take a range of values we can allow b_1 to depend on k and an increment dk by writing

$$b_1 = f(k) dk$$

Putting this in (a) and adding up all such stress functions we obtain a more general stress function in the form

$$\phi = \int_0^\infty [\rho I_0(kr) - kr I_1(kr)] f(k) \cos kz dk \quad (b)$$

We shall now see how it is possible to select the function $f(k)$ so that this stress function will give the solution to our problem.

¹ A. Nádai, "Elastische Platten," p. 315, 1925.

² M. V. Barton, *J. Applied Mechanics (Trans. A.S.M.E.)*, vol. 8, p. A-97, 1941.

A. W. Rankin, *ibid.*, vol. 11, p. A-77, 1944.

³ The pressure in the shrink fit is not uniform in the axial direction.

From Eqs. (179) we find that the shear stress will be

$$\tau_{rz} = \int_0^\infty [\rho k I_0'(kr) - k^2 r I_1'(kr) - k I_1(kr) - 2k(1-\nu) I_0'(kr)] k^2 f(k) \cos kz dk \quad (c)$$

where primes denote differentiation with respect to kr . This must vanish at the surface $r = a$. Putting $r = a$ in the expression in square brackets, and equating this bracket to zero, we obtain an equation for ρ which gives

$$\rho = 2(1-\nu) + ka \frac{I_0(ka)}{I_1(ka)} \quad (d)$$

The remaining boundary condition is

$$\begin{aligned} \sigma_r &= \frac{p}{2} && \text{for } r = a, z > 0 \\ \sigma_r &= -\frac{p}{2} && \text{for } r = a, z < 0 \end{aligned} \quad (e)$$

The value of σ_r obtained from (b) by Eqs. (179) is

$$\sigma_r = - \int_0^\infty \left[(1-2\nu-\rho) I_0(kr) + \left(kr + \frac{\rho}{kr} \right) I_1(kr) \right] k^3 f(k) \sin kz dk \quad (f)$$

We now make use of the fact that¹

$$\int_0^\infty \frac{\sin kz}{k} dk = \begin{cases} \frac{\pi}{2} & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ -\frac{\pi}{2} & \text{for } z < 0 \end{cases} \quad (g)$$

If we multiply this by p/π , we obtain

$$\frac{p}{\pi} \int_0^\infty \frac{\sin kz}{k} dk = \begin{cases} \frac{p}{2} & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ -\frac{p}{2} & \text{for } z < 0 \end{cases} \quad (h)$$

in which the values on the right correspond to the boundary values for σ_r given by (e). The boundary conditions (e) are therefore satisfied if we make the right-hand side of Eq. (f), with $r = a$, identical with the left-hand side of Eq. (h). This requires

$$-\left[(1-2\nu-\rho) I_0(ka) + \left(ka + \frac{\rho}{ka} \right) I_1(ka) \right] k^3 f(k) = \frac{p}{\pi} \cdot \frac{1}{k} \quad (i)$$

and this equation determines $f(k)$. The stress components are then found from the stress function (b) by means of the formulas (179), and will be integrals of the same general nature as that of Eq. (f), which gives σ_r . Values, obtained by numerical integration, are given by Rankin in the paper cited on page 388. The curves in Fig. 216 show the variation of the stresses in the axial direction for various radial distances, and also the surface displacements.

¹ See for instance I. S. Sokolnikoff, "Advanced Calculus," 1st ed., p. 362.

They are reproduced from the paper of Barton (see page 388) and were obtained by a different method using Fourier series. From these curves results can be obtained for the problem of Fig. 215 by superposition, as explained at the beginning of this article. Curves for the stresses and displacement for pressure bands of several widths are given in the papers cited. When the width is equal to the radius of the cylinder the tangential stress σ_θ at the surface and at the middle of the pressure band reaches a value about 10 per cent higher than the applied pressure, and is, of course, compressive. The axial stress σ_z in the surface just outside the pres-

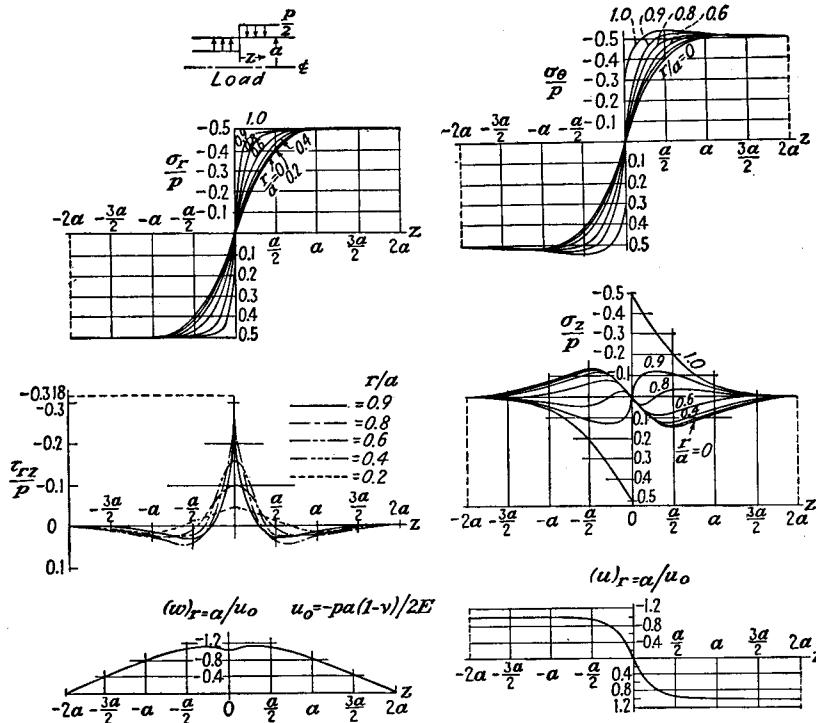


FIG. 216.

sure band reaches a tensile value of about 45 per cent of the applied pressure. The shear stress τ_{rz} attains a greatest value, equal to 31.8 per cent of the applied pressure, at the edges of the pressure band AB and CD in Fig. 215 and just below the surface.

When the pressure is applied all over the curved surface of the cylinder, of any length, we have simply compressive σ_r and σ_θ equal to the applied pressure, and σ_z and τ_{rz} zero.

Solutions have been obtained in a similar manner for a band of pressure in a hole in an infinite solid,¹ and for a band of pressure near one end of a solid cylinder.²

¹ C. J. Tranter, *Quart. Applied Math.*, vol. 4, p. 298, 1946; O. L. Bowie, *ibid.*, vol. 5, p. 100, 1947.

² C. J. Tranter and J. W. Craggs, *Phil. Mag.*, vol. 38, p. 214, 1947.

130. Twist of a Circular Ring Sector. This problem is of practical interest in connection with the calculation of stresses in close-coiled helical springs. Consider a ring sector under the action of two equal and opposite forces P along the axis through the center of the ring and perpendicular to the plane of the ring (Fig. 217). These forces produce the same torque $M_t = PR$ in all cross sections of the ring. If the cross-sectional dimensions of the ring are small in comparison with the radius R , formulas derived for the torsion of prismatical bars can be used with sufficient accuracy in calculating the stresses. In the case of heavy helical springs the cross-sectional dimensions are no longer small, and the difference in length of outer and inner circumferential fibers must be considered. In this manner it can be shown that at inner points, such as i , the shearing stress is considerably larger than that given by the theory of torsion of straight bars.¹ For a more rigorous solution of the problem we apply the general equations of the theory of elasticity in cylindrical coordinates² [Eqs. (170), page 306]. Assuming that in this case of torsion only the shearing-stress components τ_{rz} and $\tau_{r\theta}$ are different from zero (Fig. 218), we find, from Eqs. (170),

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} = 0 \quad (a)$$

Consider now the compatibility equations (130). From Fig. 219 we find

$$\begin{aligned} \tau_{yz} &= \tau_{\theta z} \cos \theta \\ \tau_{xy} &= \tau_{r\theta} (\cos^2 \theta - \sin^2 \theta) = \tau_{r\theta} \cos 2\theta \end{aligned}$$

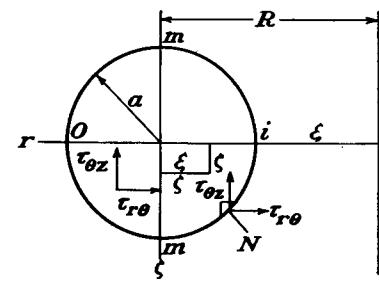


FIG. 217.

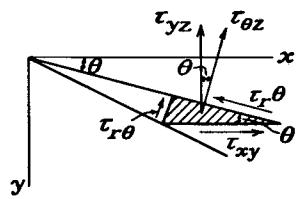


FIG. 218.

Substituting in the fourth and sixth of Eqs. (130) and remembering that

$$\Theta = \sigma_r + \sigma_\theta + \sigma_z = 0$$

¹ An elementary theory of twist of a ring sector was given by V. Roever, *V.D.I.*, vol. 57, 1913. See also M. Pilgram, *Arteil. Monatshefte*, 1913. An experimental determination of maximum stress by measuring strain at the surface of the coil was made by A. M. Wahl, *Trans. A.S.M.E.*, 1928.

² This solution is due to O. Göhner, *Ingenieur-Archiv*, vol. 1, p. 619, 1930; vol. 2, pp. 1 and 381, 1931; vol. 9, p. 355, 1938.

we find

$$\begin{aligned}\frac{\partial^2 \tau_{\theta z}}{\partial r^2} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial r} + \frac{\partial^2 \tau_{\theta z}}{\partial z^2} - \frac{\tau_{\theta z}}{r^2} &= 0 \\ \frac{\partial^2 \tau_{r\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial^2 \tau_{r\theta}}{\partial z^2} - \frac{4\tau_{r\theta}}{r^2} &= 0\end{aligned}\quad (b)$$

The remaining four of the compatibility equations [see Eqs. (g), page 346] are satisfied by virtue of our assumption that $\sigma_r = \sigma_\theta = \sigma_z = \tau_{rz} = 0$. Thus the problem reduces to the solution of Eqs. (a) and (b). For this solution we use a stress function ϕ . We satisfy Eq. (a) by taking

$$\tau_{r\theta} = \frac{GR^2}{r^2} \frac{\partial \phi}{\partial z}, \quad \tau_{\theta z} = -\frac{GR^2}{r^2} \frac{\partial \phi}{\partial r} \quad (c)$$

where G is the modulus of rigidity and R the radius of the ring. Substituting (c) in Eqs. (b), we find

$$\begin{aligned}\frac{\partial}{\partial r} \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{r} \frac{\partial \phi}{\partial r} \right) &= 0 \\ \frac{\partial}{\partial z} \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{r} \frac{\partial \phi}{\partial r} \right) &= 0\end{aligned}$$

from which we conclude that the expression in the parentheses must be a constant. Denoting this constant by $-2c$, the equation for determining the stress function ϕ is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{3}{r} \frac{\partial \phi}{\partial r} + 2c = 0 \quad (d)$$

We introduce now, instead of coordinates r and z (Fig. 218), new coordinates ξ and ζ ,

$$\xi = R - r, \quad \zeta = z$$

and Eq. (d) becomes

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \zeta^2} + \frac{3}{R \left(1 - \frac{\xi}{R} \right)} \frac{\partial \phi}{\partial \xi} + 2c = 0 \quad (e)$$

Considering ξ/R as a small quantity, and using the expansion

$$\frac{1}{1 - \frac{\xi}{R}} = 1 + \frac{\xi}{R} + \frac{\xi^2}{R^2} \dots \quad (f)$$

we shall now solve Eq. (e) by successive approximations. Assume

$$\phi = \phi_0 + \phi_1 + \phi_2 + \dots \quad (g)$$

and determine $\phi_0, \phi_1, \phi_2, \dots$ in such a manner as to satisfy the equations

$$\begin{aligned}\frac{\partial^2 \phi_0}{\partial \xi^2} + \frac{\partial^2 \phi_0}{\partial \zeta^2} + 2c &= 0 \\ \frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial^2 \phi_1}{\partial \zeta^2} + \frac{3}{R} \frac{\partial \phi_0}{\partial \xi} &= 0 \\ \frac{\partial^2 \phi_2}{\partial \xi^2} + \frac{\partial^2 \phi_2}{\partial \zeta^2} + \frac{3}{R} \frac{\partial \phi_1}{\partial \xi} + \frac{3\xi}{R^2} \frac{\partial \phi_0}{\partial \xi} &= 0 \\ \dots &\dots\end{aligned}\quad (h)$$

Then, as the number of terms in the series (g) increases, the sum of Eqs. (h) approaches more and more closely Eq. (e), and the series (g) approaches the exact solution for the stress function ϕ . Consider now the boundary conditions. The resultant shearing stress at the boundary (Fig. 218) must be in the direction of the tangent to the boundary, hence

$$\tau_{r\theta} \cos(N\xi) - \tau_{\theta z} \cos(N\zeta) = 0$$

or, by using Eqs. (c),

$$\frac{GR^2}{r^2} \left(\frac{\partial \phi}{\partial \zeta} \frac{d\xi}{ds} + \frac{\partial \phi}{\partial \xi} \frac{d\zeta}{ds} \right) = 0$$

This shows that ϕ must be constant at the boundary, and we satisfy this condition by taking solutions of Eqs. (h) such that $\phi_0, \phi_1, \phi_2, \dots$ are zero at the boundary.

Having obtained ϕ_0, ϕ_1, \dots the successive approximations for the stress components are now obtained from Eqs. (c). Introducing the new variables ξ and ζ , these equations can be represented in the following form:

$$\tau_{r\theta} = \frac{G}{\left(1 - \frac{\xi}{R} \right)^2} \frac{\partial \phi}{\partial \zeta}, \quad \tau_{\theta z} = \frac{G}{\left(1 - \frac{\xi}{R} \right)^2} \frac{\partial \phi}{\partial \xi} \quad (i)$$

Using now the expansion

$$\frac{1}{\left(1 - \frac{\xi}{R} \right)^2} = 1 + \frac{2\xi}{R} + \frac{3\xi^2}{R^2} + \dots$$

and the series (g), we find as the first approximation

$$(\tau_{r\theta})_0 = G \frac{\partial \phi_0}{\partial \zeta}, \quad (\tau_{\theta z})_0 = G \frac{\partial \phi_0}{\partial \xi} \quad (j)$$

For the second approximation we find from Eqs. (i)

$$\begin{aligned}(\tau_{r\theta})_1 &= G \left[\left(1 + \frac{2\xi}{R} \right) \frac{\partial \phi_0}{\partial \zeta} + \frac{\partial \phi_1}{\partial \zeta} \right] \\ (\tau_{\theta z})_1 &= G \left[\left(1 + \frac{2\xi}{R} \right) \frac{\partial \phi_0}{\partial \xi} + \frac{\partial \phi_1}{\partial \xi} \right]\end{aligned}\quad (k)$$

For the third approximation,

$$\begin{aligned}(\tau_{r\theta})_2 &= G \left[\left(1 + \frac{2\xi}{R} + \frac{3\xi^2}{R^2} \right) \frac{\partial \phi_0}{\partial \zeta} + \left(1 + \frac{2\xi}{R} \right) \frac{\partial \phi_1}{\partial \zeta} + \frac{\partial \phi_2}{\partial \zeta} \right] \\ (\tau_{\theta z})_2 &= G \left[\left(1 + \frac{2\xi}{R} + \frac{3\xi^2}{R^2} \right) \frac{\partial \phi_0}{\partial \xi} + \left(1 + \frac{2\xi}{R} \right) \frac{\partial \phi_1}{\partial \xi} + \frac{\partial \phi_2}{\partial \xi} \right]\end{aligned}\quad (l)$$

We apply this general discussion to the particular case of a ring of circular cross section of radius a . The equation of the boundary (Fig. 218) is

$$\xi^2 + \zeta^2 - a^2 = 0 \quad (m)$$

and the solution of the first of Eqs. (h), satisfying the boundary condition, is

$$\phi_0 = -\frac{c}{2} (\xi^2 + \zeta^2 - a^2)$$

The first approximation for the stress components, from Eqs. (j), is

$$(\tau_{r\theta})_0 = -cG\xi, \quad (\tau_{\theta z})_0 = -cG\xi \quad (n)$$

This is the same stress distribution as for a circular shaft. The corresponding value of the torque is

$$M_t = -\iint (\tau_{r\theta}\xi + \tau_{z\theta}\xi) d\xi d\zeta \quad (o)$$

Substituting from Eqs. (n),

$$(M_t)_0 = \frac{cG\pi a^4}{2}, \quad c = \frac{2(M_t)_0}{G\pi a^4}$$

To get the second approximation we use the second equation of (h). Substituting for ϕ_0 the expression above we find

$$\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{\partial^2 \phi_1}{\partial \zeta^2} - \frac{3c\xi}{R} = 0$$

The solution of this equation, satisfying the condition that ϕ_1 vanishes at the boundary, is

$$\phi_1 = \frac{3}{8} \frac{c\xi}{R} (\xi^2 + \zeta^2 - a^2)$$

Substituting this in Eqs. (k) we find the second approximation for the stress components

$$\begin{aligned} (\tau_{r\theta})_1 &= -cG \left(\xi + \frac{5}{4} \frac{\xi \zeta}{R} \right) \\ (\tau_{\theta z})_1 &= -cG \left[\xi + \frac{7}{8} \frac{\xi^2}{R} - \frac{3}{8R} (\xi^2 - a^2) \right] \end{aligned} \quad (p)$$

Substituting ϕ_0 and ϕ_1 in the third of Eqs. (h), we find

$$\frac{\partial^2 \phi_2}{\partial \xi^2} + \frac{\partial^2 \phi_2}{\partial \zeta^2} + \frac{3}{8} \frac{c}{R^2} (\xi^2 + 3\xi^2 - 3a^2) = 0$$

The solution of this equation satisfying the boundary condition is

$$\phi_2 = -\frac{c}{64R^2} (\xi^2 + 5\xi^2 - 15a^2)(\xi^2 + \zeta^2 - a^2)$$

By using Eqs. (l) we find the third approximation for the stress components,

$$\begin{aligned} (\tau_{r\theta})_2 &= -cG \left[\xi + \frac{5}{4} \frac{\xi \zeta}{R} + \frac{\zeta}{16R^2} (27\xi^2 + 5\xi^2 - 10a^2) \right] \\ (\tau_{\theta z})_2 &= -cG \left[\xi + \frac{7}{8} \frac{\xi^2}{R} - \frac{3}{8R} (\xi^2 - a^2) + \frac{13}{16} \frac{\xi^3}{R^2} - \frac{9}{16} \frac{\xi \zeta^2}{R^2} + \frac{1}{4} \frac{a^2 \xi}{R^2} \right] \end{aligned} \quad (q)$$

Substituting these expressions for the stress components into Eq. (o) the corresponding torque is

$$(M_t)_2 = \frac{cG\pi a^4}{2} \left(1 + \frac{3}{16} \frac{a^2}{R^2} \right) \quad (r)$$

By determining from this the constant c , and substituting it in expressions (q), we can find the stress components as functions of the applied torque $(M_t)_1$. Along the

horizontal diameter of the cross section of the ring (Fig. 218) $\xi = 0$, $\tau_{r\theta} = 0$, and from the second of the Eqs. (q), we find

$$(\tau_{\theta z})_2 = -cG \left(\xi + \frac{7}{8} \frac{\xi^2}{R} + \frac{13}{16} \frac{\xi^3}{R^2} + \frac{3a^2}{8R} + \frac{a^2 \xi}{4R^2} \right)$$

For the inner point i , $\xi = a$, and we have

$$(\tau_{\theta z})_i = -cGa \left(1 + \frac{5}{4} \frac{a}{R} + \frac{17}{16} \frac{a^2}{R^2} \right)$$

For the outer point O , $\xi = -a$, and

$$(\tau_{\theta z})_o = cGa \left(1 - \frac{5}{4} \frac{a}{R} + \frac{17}{16} \frac{a^2}{R^2} \right)$$

Using Eq. (r), the values of these stresses become

$$\begin{aligned} (\tau_{\theta z})_i &= -\frac{2M_t}{\pi a^3} \frac{1 + \frac{5}{4} \frac{a}{R} + \frac{17}{16} \frac{a^2}{R^2}}{1 + \frac{3}{16} \frac{a^2}{R^2}} = -\frac{2M_t}{\pi a^3} \left(1 + \frac{5}{4} \frac{a}{R} + \frac{7}{8} \frac{a^2}{R^2} \right) \\ (\tau_{\theta z})_o &= \frac{2M_t}{\pi a^3} \left(1 - \frac{5}{4} \frac{a}{R} + \frac{7}{8} \frac{a^2}{R^2} \right) \end{aligned}$$

The calculation of further approximations shows that the final expression for the greatest shearing stress can be put in the form¹

$$(\tau_{\theta z})_i = -\frac{2PR}{\pi a^3} \cdot \frac{\frac{1}{1-(a/R)} + \frac{1}{4} \frac{a}{R} + \frac{1}{16} \left(\frac{a}{R} \right)^2}{1 - \frac{3}{16} \frac{(a/R)^2}{1-(a/R)^2}} \quad (235)$$

The distribution of shearing stresses along the horizontal diameter for a particular case, $a/R = \frac{1}{5}$, is shown in Fig. 220. For comparison the first approximation obtained by applying the formula for a circular shaft is shown by a dotted line.²

The method described has also been applied to the torsion problem for ring sectors of elliptic and rectangular cross sections.³ For a square cross section with sides of length $2a$, the third approximation gives for the stress at the inner point

$$(\tau_{\theta z})_i = -\frac{0.6PR}{a^3} \left(1 + 1.20 \frac{a}{R} + 0.56 \frac{a^2}{R^2} \right) \quad (236)$$

131. Pure Bending of a Circular Ring Sector. The method of successive approximations used in the previous article can be applied also in discussing pure

¹ This formula was communicated to S. Timoshenko in a letter from O. Göhner.

² The elementary solutions mentioned before (see p. 391) give for $(\tau_{\theta z})_i$ values which are in good agreement with the results calculated from Eq. (235).

³ Göhner, loc. cit.

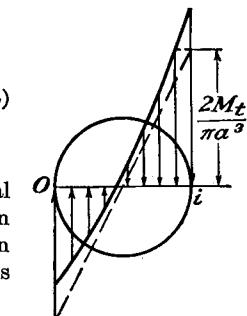


FIG. 220.

bending of a sector of a circular ring.¹ If two equal and opposite couples M are applied at the ends of a circular ring sector in the plane of the center line of the ring (Fig. 221), they produce strain symmetrical with respect to the z -axis, and the shearing stresses τ_{rz} and $\tau_{\theta z}$ in the meridional cross sections of the ring are zero. The remaining four stress components must satisfy the equations of equilibrium for the case of symmetrical strain (see Art. 116)

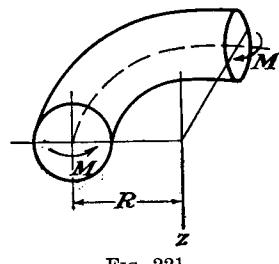


FIG. 221.

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \quad (a)$$

and the corresponding compatibility equations [see Eqs. (g), Art. 116]

$$\begin{aligned} \nabla^2 \sigma_r - \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial r^2} &= 0 \\ \nabla^2 \sigma_\theta + \frac{2}{r^2} (\sigma_r - \sigma_\theta) + \frac{1}{1+\nu} \frac{1}{r} \frac{\partial \Theta}{\partial r} &= 0 \\ \nabla^2 \sigma_z + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial z^2} &= 0 \\ \nabla^2 \tau_{rz} - \frac{1}{r^2} \tau_{rz} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial r \partial z} &= 0 \end{aligned} \quad (b)$$

Taking, as an example, a ring of constant circular cross section and introducing, instead of r and z , the new coordinates (Fig. 218)

$$\xi = R - r, \quad \zeta = z \quad (c)$$

Eqs. (a) and (b) become

$$\begin{aligned} \frac{\partial \sigma_\xi}{\partial \xi} + \frac{\partial \tau_{\xi\xi}}{\partial \zeta} - \frac{\sigma_\xi - \sigma_\theta}{R - \xi} &= 0 \\ \frac{\partial \tau_{\xi\xi}}{\partial \xi} + \frac{\partial \sigma_\zeta}{\partial \zeta} - \frac{\tau_{\xi\xi}}{R - \xi} &= 0 \\ \frac{\partial^2 \sigma_\xi}{\partial \xi^2} + \frac{\partial^2 \sigma_\xi}{\partial \zeta^2} - \frac{1}{R - \xi} \frac{\partial \sigma_\xi}{\partial \xi} - \frac{2}{(R - \xi)^2} (\sigma_\xi - \sigma_\theta) + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial \xi^2} &= 0 \\ \frac{\partial^2 \sigma_\theta}{\partial \xi^2} + \frac{\partial^2 \sigma_\theta}{\partial \zeta^2} - \frac{1}{R - \xi} \frac{\partial \sigma_\theta}{\partial \xi} + \frac{2}{(R - \xi)^2} (\sigma_\xi - \sigma_\theta) - \frac{1}{1+\nu} \frac{1}{R - \xi} \frac{\partial \Theta}{\partial \xi} &= 0 \\ \frac{\partial^2 \sigma_\zeta}{\partial \xi^2} + \frac{\partial^2 \sigma_\zeta}{\partial \zeta^2} - \frac{1}{R - \xi} \frac{\partial \sigma_\zeta}{\partial \xi} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial \zeta^2} &= 0 \\ \frac{\partial^2 \tau_{\xi\xi}}{\partial \xi^2} + \frac{\partial^2 \tau_{\xi\xi}}{\partial \zeta^2} - \frac{1}{R - \xi} \frac{\partial \tau_{\xi\xi}}{\partial \xi} - \frac{1}{(R - \xi)^2} \tau_{\xi\xi} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial \xi \partial \zeta} &= 0 \end{aligned} \quad (d)$$

As a first approximation we take the same stress distribution as occurs in pure bending of prismatical bars. Then

$$\begin{aligned} (\sigma_\xi)_0 &= (\sigma_\zeta)_0 = (\tau_{\xi\xi})_0 = 0 \\ (\sigma_\theta)_0 &= -cE\xi \end{aligned} \quad (f)$$

¹ Göhner, loc. cit.

where

$$c = \frac{4M}{\pi a^4 E}$$

To get the second approximation we consider ξ as small in comparison with R and neglect the products of ξ/R and of small corrections in the stresses as small quantities of higher order. Equations (d) and (e) then become

$$\begin{aligned} \frac{\partial(\sigma_\xi)_1}{\partial \xi} + \frac{\partial(\tau_{\xi\xi})_1}{\partial \zeta} - \frac{cE\xi}{R} &= 0 \\ \frac{\partial(\tau_{\xi\xi})_1}{\partial \xi} + \frac{\partial(\sigma_\zeta)_1}{\partial \zeta} &= 0 \end{aligned} \quad (d')$$

$$\begin{aligned} \Delta(\sigma_\xi)_1 + \frac{1}{1+\nu} \frac{\partial^2(\Theta)_1}{\partial \xi^2} &= 0 \\ \Delta(\sigma_\theta)_1 + \frac{1}{R} \left(1 + \frac{1}{1+\nu}\right) cE &= 0 \\ \Delta(\sigma_\zeta)_1 + \frac{1}{1+\nu} \frac{\partial^2(\Theta)_1}{\partial \zeta^2} &= 0 \\ \Delta(\tau_{\xi\xi})_1 + \frac{1}{1+\nu} \frac{\partial^2(\Theta)_1}{\partial \xi \partial \zeta} &= 0 \end{aligned} \quad (e')$$

where the symbol Δ means $\partial^2/\partial \xi^2 + \partial^2/\partial \zeta^2$.

We introduce now a stress function ϕ_1 . By taking

$$\begin{aligned} (\sigma_\xi)_1 &= \frac{cE}{2R} (\xi^2 + \zeta^2 - a^2) + \frac{cE}{R} \frac{\partial^2 \phi_1}{\partial \xi^2} \\ (\sigma_\zeta)_1 &= \frac{cE}{R} \frac{\partial^2 \phi_1}{\partial \zeta^2} \\ (\tau_{\xi\xi})_1 &= -\frac{cE}{R} \frac{\partial^2 \phi_1}{\partial \xi \partial \zeta} \end{aligned} \quad (g)$$

we satisfy Eqs. (d'). Substituting (g) in Eqs. (e'), we find that the stress function ϕ_1 should satisfy the equation

$$\Delta \Delta \phi_1 = -\frac{1+2\nu}{1+\nu} \quad (h)$$

The boundary conditions for ϕ_1 are obtained from Eqs. (12). As the first term in the expression for $(\sigma_\xi)_1$ is zero at the circular boundary and

$$l = \frac{d\xi}{ds}, \quad m = -\frac{d\zeta}{ds}$$

we find that

$$\frac{d}{ds} \left(\frac{\partial \phi_1}{\partial \zeta} \right) = 0, \quad \frac{d}{ds} \left(\frac{\partial \phi_1}{\partial \xi} \right) = 0$$

Thus $\partial \phi_1 / \partial \zeta$ and $\partial \phi_1 / \partial \xi$ are constant along the boundary, and we can assume that ϕ_1 and $d\phi_1/dn$ are zero at the boundary. Equation (h) together with these boundary conditions completely determines the stress function ϕ_1 . It is interesting to note that Eq. (h) and the above boundary conditions are identical with the equa-

tions for the deflection of a plate clamped at the edges and uniformly loaded. In the case of a circular plate we know the deflection surface. This deflection gives us the expression for the stress function

$$\phi_1 = -\frac{1+2\nu}{64(1+\nu)} (\xi^2 + \zeta^2 - a^2)^2 \quad (k)$$

Substituting in Eqs. (g) we find the following expressions for the stress components:

$$\begin{aligned} (\sigma_\xi)_1 &= \frac{cE}{16R(1+\nu)} \{(7+6\nu)(\xi^2 - a^2) + (5+2\nu)\zeta^2\} \\ (\sigma_\zeta)_1 &= -\frac{cE(1+2\nu)}{16R(1+\nu)} (3\xi^2 + \zeta^2 - a^2) \\ (\tau_{\xi\zeta})_1 &= \frac{cE}{8R} \frac{1+2\nu}{1+\nu} \xi\zeta \end{aligned} \quad (l)$$

Substituting these expressions in Eqs. (e'), we find

$$\begin{aligned} \frac{\partial^2(\sigma_\theta)_1}{\partial\xi^2} &= -\frac{cE(4+5\nu+2\nu^2)}{2R(1+\nu)} \\ \frac{\partial^2(\sigma_\theta)_1}{\partial\zeta^2} &= \frac{cE(3\nu+2\nu^2)}{2R(1+\nu)} \\ \frac{\partial^2(\sigma_\theta)_1}{\partial\xi\partial\zeta} &= 0 \end{aligned}$$

Integrating these and adjusting the constants of integration so as to make the distribution of normal stresses over cross sections of the ring statically equivalent to the bending moment M , we find

$$(\sigma_\theta)_1 = -\frac{4M}{\pi a^4} \left[\xi + \frac{(8+10\nu+4\nu^2)\xi^2 - (6\nu+4\nu^2)\zeta^2 - (2+\nu)a^2}{8(1+\nu)R} \right] \quad (m)$$

Taking $\xi = 0$ and $\zeta = a$, the stress at the inner point i (Fig. 218) is

$$(\sigma_\theta)_1 = -\frac{4M}{\pi a^3} \left\{ 1 + \frac{6+9\nu+4\nu^2}{8(1+\nu)} \frac{a}{R} \right\}$$

For $\nu = 0.3$, the above equation becomes

$$(\sigma_\theta)_1 = -\frac{4M}{\pi a^3} \left(1 + 0.87 \frac{a}{R} \right) \quad (n)$$

Calculations of further approximations result in the following expression for the stress at the inner point¹ ($\xi = a$):

$$\sigma_\theta = -\frac{4M}{\pi a^3} \left[1 + 0.87 \frac{a}{R} + \frac{0.64(a/R)^2}{1-(a/R)} \right] \quad (p)$$

The elementary theory of bending of curved bars, based on the assumption that cross sections remain plane and neglecting the stresses σ_r , gives in this case²

$$\sigma_\theta = -\frac{4M}{\pi a^3} \left[1 + 0.75 \frac{a}{R} + 0.50 \frac{a^2}{R^2} + \dots \right]$$

¹ This formula was communicated to S. Timoshenko in a letter from O. Göhner.

² See S. Timoshenko, "Strength of Materials," 2d ed., vol. 2, p. 73.

CHAPTER 14

THERMAL STRESS

132. The Simplest Cases of Thermal Stress Distribution. One of the causes of initial stresses in a body is nonuniform heating. With rising temperature the elements of a body expand. Such an expansion generally cannot proceed freely in a continuous body, and stresses due to the heating are set up. In many cases of machine design, such as in the design of steam turbines and Diesel engines, *thermal stresses* are of great practical importance and must be considered in more detail.

The simpler problems of thermal stress can easily be reduced to problems of boundary force of types already considered. As a first example

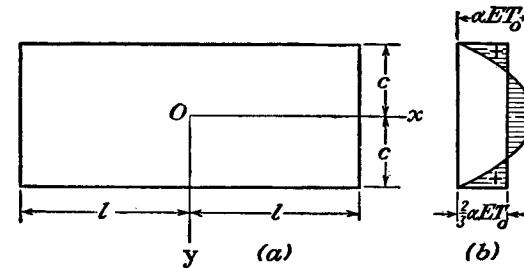


FIG. 222.

let us consider a thin rectangular plate of uniform thickness in which the temperature T is an even function of y (Fig. 222) and is independent of x and z . The longitudinal thermal expansion αT will be entirely suppressed by applying to each element of the plate the longitudinal compressive stress

$$\sigma_x' = -\alpha TE \quad (a)$$

Since the plate is free to expand laterally the application of the stresses (a) will not produce any stresses in the lateral directions and to maintain the stresses (a) throughout the plate it will be necessary to distribute compressive forces of the magnitude (a) at the ends of the plate only. These compressive forces will completely suppress any expansion of the plate in the direction of the x -axis due to the temperature rise T . To get the *thermal stresses* in the plate, which is free from external forces, we have to superpose on the stresses (a) the stresses

produced in the plate by tensile forces of intensity αTE distributed at the ends. These forces have the resultant

$$\int_{-c}^{+c} \alpha TE dy$$

and at a sufficient distance from the ends they will produce approximately uniformly distributed tensile stress of the magnitude

$$\frac{1}{2c} \int_{-c}^{+c} \alpha TE dy$$

so that the thermal stresses in the plate with free ends at a considerable distance from the ends will be

$$\sigma_x = \frac{1}{2c} \int_{-c}^{+c} \alpha TE dy - \alpha TE \quad (b)$$

Assuming, for example, that the temperature is distributed parabolically and is given by the equation

$$T = T_0 \left(1 - \frac{y^2}{c^2}\right)$$

we get, from Eq. (b),

$$\sigma_x = \frac{2}{3} \alpha T_0 E - \alpha T_0 E \left(1 - \frac{y^2}{c^2}\right) \quad (c)$$

This stress distribution is shown in Fig. 222b. Near the ends the stress distribution produced by the tensile forces is not uniform and can be calculated by the method explained on page 167. Superposing these stresses on the compressive stresses (a), the thermal stresses near the end of the plate will be obtained.

If the temperature T is not symmetrical with respect to the x -axis, we begin again with compressive stresses (a) suppressing the strain ϵ_x . In the nonsymmetrical cases these stresses give rise not only to a resultant force $-\int_{-c}^{+c} \alpha ET dy$ but also to a resultant couple $-\int_{-c}^{+c} \alpha ETy dy$, and in order to satisfy the conditions of equilibrium we must superpose on the compressive stresses (a) a uniform tension, determined as before, and bending stresses $\sigma''_x = \sigma y/c$ determined from the condition that the moment of the forces distributed over a cross section must be zero. Then

$$\int_{-c}^{+c} \frac{\sigma y^2 dy}{c} - \int_{-c}^{+c} \alpha ETy dy = 0$$

from which

$$\frac{\sigma}{c} = \frac{3}{2c^3} \int_{-c}^{+c} \alpha ETy dy, \quad \sigma''_x = \frac{3y}{2c^3} \int_{-c}^{+c} \alpha ETy dy$$

Then the total stress is

$$\sigma_x = -\alpha ET + \frac{1}{2c} \int_{-c}^{+c} \alpha ET dy + \frac{3y}{2c^3} \int_{-c}^{+c} \alpha ETy dy \quad (d)$$

In this discussion it was assumed that the plate was thin in the z -direction. Suppose now that the dimension in the z -direction is large. We have then a plate with the xz -plane as its middle plane, and a thickness $2c$. Let the temperature T be, as before, independent of x and z , and so a function of y only.

The free thermal expansion of an element of the plate in the x - and z -directions will be completely suppressed by applying stresses σ_x, σ_z obtained from Eqs. (3), page 7, by putting $\epsilon_x = \epsilon_z = -\alpha T, \sigma_y = 0$. These equations then give

$$\sigma_x = \sigma_z = -\frac{\alpha ET}{1 - \nu} \quad (e)$$

The elements can be maintained in this condition by applying the distributions of compressive force given by (e) to the edges ($x = \text{constant}, z = \text{constant}$). The thermal stress in the plate free from external force is obtained by superposing on the stresses (e) the stresses due to application of equal and opposite distributions of force on the edges. If T is an even function of y such that the mean value over the thickness of the plate is zero, the resultant force per unit run of edge is zero, and by Saint-Venant's principle (Art. 18) it produces no stress except near the edge.

If the mean value of T is not zero, uniform tensions in the x - and z -directions corresponding to the resultant force on the edge must be superposed on the compressive stresses (e). If in addition to this the temperature is not symmetrical with respect to the xz -plane, we must add the bending stresses. In this manner we finally arrive at the equation

$$\begin{aligned} \sigma_x = \sigma_z = & -\frac{\alpha TE}{1 - \nu} + \frac{1}{2c(1 - \nu)} \int_{-c}^{+c} \alpha TE dy \\ & + \frac{3y}{2c^3(1 - \nu)} \int_{-c}^{+c} \alpha TEy dy \end{aligned} \quad (f)$$

which is analogous to the Eq. (d) obtained before. By using Eq. (f) we can easily calculate thermal stresses in a plate, if the distribution of temperature T over the thickness of the plate is known.

Consider, as an example, a plate which has initially a uniform temperature T_0 and which is being cooled down by maintaining the surfaces $y = \pm c$ at a constant temperature T_1 .¹ By Fourier's theory the distribution of temperature at any instant t is

$$T = T_1 + \frac{4}{\pi} (T_0 - T_1) \left(e^{-p_1 t} \cos \frac{\pi y}{2c} - \frac{1}{3} e^{-p_3 t} \cos \frac{3\pi y}{2c} + \dots \right) \quad (g)$$

in which $p_1, p_3 = 3^2 p_1, \dots, p_n = n^2 p_1, \dots$, are certain constants. Substituting in Eq. (f), we find

$$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1 - \nu)} \left[e^{-p_1 t} \left(\frac{2}{\pi} - \cos \frac{\pi y}{2c} \right) + \frac{1}{3} e^{-p_3 t} \left(\frac{2}{3\pi} + \cos \frac{3\pi y}{2c} \right) + \frac{1}{5} e^{-p_5 t} \left(\frac{2}{5\pi} - \cos \frac{5\pi y}{2c} \right) + \dots \right] \quad (h)$$

After a moderate time the first term acquires dominant importance, and we can assume

$$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1 - \nu)} e^{-p_1 t} \left(\frac{2}{\pi} - \cos \frac{\pi y}{2c} \right)$$

For $y = \pm c$ we have tensile stresses

$$\sigma_x = \sigma_z = \frac{4\alpha E(T_0 - T_1)}{\pi(1 - \nu)} e^{-p_1 t} \frac{2}{\pi}$$

At the middle plane $y = 0$ we obtain compressive stresses

$$\sigma_x = \sigma_z = - \frac{4\alpha E(T_0 - T_1)}{\pi(1 - \nu)} e^{-p_1 t} \left(1 - \frac{2}{\pi} \right)$$

The points with zero stresses are obtained from the equation

$$\frac{2}{\pi} - \cos \frac{\pi y}{2c} = 0$$

from which

$$y = \pm 0.560c$$

If the surfaces $y = \pm c$ of a plate are maintained at two different temperatures T_1, T_2 , a steady state of heat flow is established after a certain time and the temperature is then given by the linear function

$$T = \frac{1}{2} (T_1 + T_2) + \frac{1}{2} (T_1 - T_2) \frac{y}{c} \quad (i)$$

¹ This problem was discussed by Lord Rayleigh, *Phil. Mag.*, series 6, vol. 1, p. 169, 1901.

Substitution in Eq. (f) shows that the thermal stresses are zero,¹ provided, of course, that the plate is not restrained. If the edges are perfectly restrained against expansion and rotation, the stress induced by the heating is given by Eqs. (e). For instance if $T_2 = -T_1$ we have from (i)

$$T = T_1 \frac{y}{c} \quad (j)$$

and Eqs. (e) give

$$\sigma_x = \sigma_z = - \frac{\alpha T}{1 - \nu} T_1 \frac{y}{c} \quad (k)$$

The maximum stress is

$$(\sigma_x)_{\max.} = (\sigma_z)_{\max.} = \frac{\alpha E T_1}{1 - \nu} \quad (l)$$

The thickness of the plate does not enter in this formula, but in the case of a thicker plate a greater difference of temperature between the two surfaces usually exists. Thus a thick plate of a brittle material is more liable to break due to thermal stresses than a thin one.

As a last example let us consider a sphere of large radius and assume that there occurs a temperature rise T in a small spherical element of radius a at the center of the large sphere. Since the element is not free to expand a pressure p will be produced at the surface of the element. The radial and the tangential stresses due to this pressure at any point of the sphere at a radius $r > a$ can be calculated from formulas (197) and (198) (see page 359). Assuming the outer radius of the sphere as very large in comparison with a we obtain from these formulas

$$\sigma_r = - \frac{pa^3}{r^3}, \quad \sigma_t = \frac{pa^3}{2r^3} \quad (m)$$

At the radius $r = a$ we obtain

$$\sigma_r = -p, \quad \sigma_t = \frac{1}{2}p$$

and the increase of this radius, due to pressure p , is

$$\Delta r = (ae_t)_{r=a} = \frac{a}{E} [\sigma_t - \nu(\sigma_r + \sigma_t)]_{r=a} = \frac{pa}{2E} (1 + \nu)$$

This increase must be equal to the increase of the radius of the heated spherical element produced by temperature rise and pressure p . Thus

¹ In general, when T is a linear function of x, y, z , the strain corresponding to free thermal expansion of each element, *viz.*,

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha T, \quad \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$$

satisfies the conditions of compatibility (129) and there will be no thermal stress.

we obtain the equation

$$\alpha Ta - \frac{pa}{E} (1 - 2\nu) = \frac{pa}{2E} (1 + \nu)$$

from which

$$p = \frac{2}{3} \frac{\alpha TE}{1 - \nu} \quad (n)$$

Substituting in equations (m) we obtain the formulas for the stresses outside the heated element

$$\sigma_r = -\frac{2}{3} \frac{\alpha TE a^3}{(1 - \nu)r^3}, \quad \sigma_t = \frac{1}{3} \frac{\alpha TE a^3}{(1 - \nu)r^3} \quad (o)$$

133. Some Problems of Plane Thermal Stress. Suppose that a strip of thin plate (Fig. 223) is nonuniformly heated so that the temperature T is a function of the longitudinal coordinate x only, being

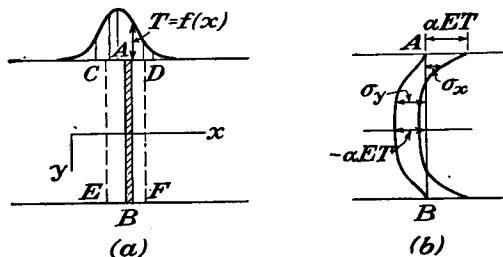


FIG. 223.

uniform across any given cross section. If the plate is cut into strips such as AB (Fig. 223), these strips expand vertically by different amounts. Due to the mutual restraint there will be stresses set up when they are in fact attached as in the plate.

Considering the unattached strips, their vertical expansion is suppressed if they are subjected to compressive stress

$$\sigma_y = -\alpha ET \quad (a)$$

by applying such stress at the ends A and B of each strip. The strips fit together as in the unheated plate. To arrive at the thermal stress we must superpose on (a) the stress due to the application of equal and opposite forces, i.e., tension of intensity αET , along the edges $y = \pm c$ of the strip.

If the heating is confined to a length of the strip short in comparison with its width $2c$, such as $CDFE$ in Fig. 223, the effect of the tensions αET will be felt only in the neighborhood of CD on the top edge, and of EF on the bottom edge. Each of these neighborhoods can then be considered as presenting a problem of the type considered in Art. 34.

It was pointed out on page 95 that a normal stress on a straight boundary produces a like normal stress parallel to and at the boundary. Hence the tensions αET will produce tensile stress αET in the x -direction. Both normal stresses die away as we proceed into the plate normal to the edge. On superposing these stresses on the compressive stress (a) in the y -direction, we obtain curves¹ for σ_x and σ_y along a line such as AB in the hottest part of the plate of the character shown in Fig. 223b. Near the edges the prevailing stress is σ_x , with the value αET , tensile when T is positive, and near the middle the prevailing stress is σ_y , a compressive stress of magnitude αET when T is positive. The maximum stresses are of magnitude αET_{\max} .

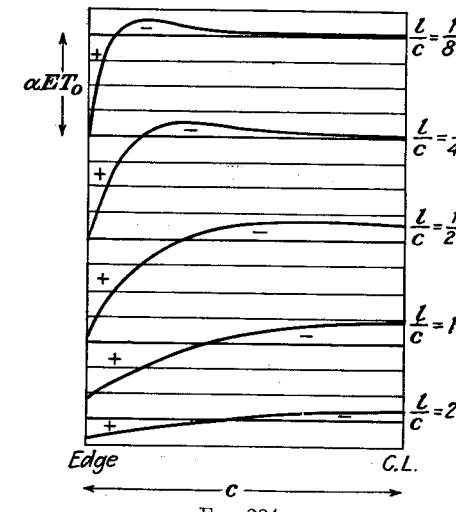


FIG. 224.

If the temperature T is a periodic function of x the application of edge tensions αET presents a problem of the type considered in Art. 23. When

$$T = T_0 \sin \alpha x \quad (b)$$

we find from Eqs. (k) of Art. 23, putting $A = B = -\alpha ET_0$ in accordance with Eq. (f),

$$\sigma_x = -2\alpha ET_0 \frac{(\alpha c \cosh \alpha c - \sinh \alpha c) \cosh \alpha y - \alpha y \sinh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \sin \alpha x$$

$$\sigma_y = 2\alpha ET_0 \frac{(\alpha c \cosh \alpha c + \sinh \alpha c) \cosh \alpha y - \alpha y \sinh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \sin \alpha x$$

$$\tau_{xy} = 2\alpha ET_0 \frac{\alpha c \cosh \alpha c \sinh \alpha y - \alpha y \cosh \alpha y \sinh \alpha c}{\sinh 2\alpha c + 2\alpha c} \cdot \cos \alpha x$$

¹ J. N. Goodier, *Physics*, vol. 7, p. 156, 1936.

Together with the compressive stress $\sigma_r = -\alpha ET$ from Eq. (a), these give the thermal stress in the plate.¹ In Fig. 224 the distributions of σ_θ along the lines of maximum temperature for various wave lengths $2l = 2\pi/\alpha$ are shown. We see that the maximum stress increases as the wave length diminishes and approaches the value αET_0 . Having the solution for a sinusoidal temperature distribution, other cases in which the temperature is a periodic function of x can be treated. It can be concluded also that the maximum stress in plates of finite length can differ only slightly from the value αET_0 obtained for an infinite strip.

134. The Thin Circular Disk: Temperature Symmetrical about Center. When the temperature T does not vary over the thickness of the disk, we may assume that the stress and displacement due to the heating also do not vary over the thickness. The stresses σ_r and σ_θ satisfy the equation of equilibrium

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (a)$$

obtained from Eq. (40), page 58, by putting $R = 0$. The shear stress $\tau_{r\theta}$ is zero on account of the symmetry.

The ordinary stress-strain relations, Eqs. (52), page 66, for plane stress, require modification since now the strain is partly due to thermal stress, partly due to stress. If ϵ_r represents the actual radial expansion, $\epsilon_r - \alpha T$ represents the part due to stress, and we have

$$\epsilon_r - \alpha T = \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \quad (b)$$

and similarly

$$\epsilon_\theta - \alpha T = \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \quad (c)$$

Solving (b) and (c) for σ_r , σ_θ we find

$$\begin{aligned} \sigma_r &= \frac{E}{1 - \nu^2} [\epsilon_r + \nu \epsilon_\theta - (1 + \nu) \alpha T], \\ \sigma_\theta &= \frac{E}{1 - \nu^2} [\epsilon_\theta + \nu \epsilon_r - (1 + \nu) \alpha T] \end{aligned} \quad (d)$$

and with these Eq. (a) becomes

$$r \frac{d}{dr} (\epsilon_r + \nu \epsilon_\theta) + (1 - \nu)(\epsilon_r - \epsilon_\theta) = (1 + \nu) \alpha r \frac{dT}{dr} \quad (e)$$

¹ The problem was discussed by J. P. Den Hartog, *J. Franklin Inst.*, vol. 222, p. 149, 1936, in connection with the thermal stress produced in the process of welding.

If u denotes the radial displacement we have, from Art. 28,

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r} \quad (f)$$

Substituting these in (e) we obtain

$$\frac{d^2u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = (1 + \nu) \alpha \frac{dT}{dr}$$

which may be written

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d(ru)}{dr} \right] = (1 + \nu) \alpha \frac{dT}{dr} \quad (g)$$

Integration of this equation yields

$$u = (1 + \nu) \alpha \frac{1}{r} \int_a^r Tr dr + C_1 r + \frac{C_2}{r} \quad (h)$$

where the lower limit a in the integral can be chosen arbitrarily. For a disk with a hole it may be the inner radius. For a solid disk we may take it as zero.

The stress components are now found by using the solution (h) in Eqs. (f), and substituting the results in Eqs. (d). Then

$$\sigma_r = -\alpha E \cdot \frac{1}{r^2} \int_a^r Tr dr + \frac{E}{1 - \nu^2} \left[C_1(1 + \nu) - C_2(1 - \nu) \frac{1}{r^2} \right] \quad (i)$$

$$\sigma_\theta = \alpha E \cdot \frac{1}{r^2} \int_a^r Tr dr - \alpha ET + \frac{E}{1 - \nu^2} \left[C_1(1 + \nu) + C_2(1 - \nu) \frac{1}{r^2} \right] \quad (j)$$

The constants C_1 , C_2 are determined by the boundary conditions.

For a solid disk, we take a as zero, and observing that

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r Tr dr = 0$$

we see from Eq. (h) that C_2 must vanish in order that u may be zero at the center. At the edge $r = b$ we must have $\sigma_r = 0$, and therefore from Eq. (i)

$$C_1 = (1 - \nu) \frac{\alpha}{b^2} \int_0^b Tr dr$$

The final expressions for the stresses are consequently

$$\sigma_r = \alpha E \left(\frac{1}{b^2} \int_0^b Tr dr - \frac{1}{r^2} \int_0^r Tr dr \right) \quad (237)$$

$$\sigma_\theta = \alpha E \left(-T + \frac{1}{b^2} \int_0^b Tr dr + \frac{1}{r^2} \int_0^r Tr dr \right) \quad (238)$$

These give finite values at the center since

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \int_0^r Tr dr = \frac{1}{2} T_0$$

where T_0 is the temperature at the center.

135. The Long Circular Cylinder. The temperature is taken to be symmetrical about the axis, and independent of the axial coordinate z .¹ We shall suppose first that w , the axial displacement, is zero throughout, and then modify the solution to the case of free ends.

We shall now have three components of stress, σ_r , σ_θ , σ_z , all three shear strains and stresses being zero on account of the symmetry about the axis and the uniformity in the axial direction. The stress-strain relations are

$$\begin{aligned}\epsilon_r - \alpha T &= \frac{1}{E} [\sigma_r - \nu(\sigma_\theta + \sigma_z)] \\ \epsilon_\theta - \alpha T &= \frac{1}{E} [\sigma_\theta - \nu(\sigma_r + \sigma_z)] \\ \epsilon_z - \alpha T &= \frac{1}{E} [\sigma_z - \nu(\sigma_r + \sigma_\theta)]\end{aligned}\quad (239)$$

But since $w = 0$, $\epsilon_z = 0$, and the third of Eqs. (239) gives

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) - \alpha ET \quad (a)$$

On substituting this into the first two of Eqs. (239), these equations become

$$\begin{aligned}\epsilon_r - (1 + \nu)\alpha T &= \frac{1 - \nu^2}{E} \left(\sigma_r - \frac{\nu}{1 - \nu} \sigma_\theta \right) \\ \epsilon_\theta - (1 + \nu)\alpha T &= \frac{1 - \nu^2}{E} \left(\sigma_\theta - \frac{\nu}{1 - \nu} \sigma_r \right)\end{aligned}\quad (b)$$

It may be seen at once that these equations can be obtained from the corresponding equations of plane stress, Eqs. (b) and (c) of the preceding article, by putting, in the latter equations, $E/(1 - \nu^2)$ for E , $\nu/(1 - \nu)$ for ν , and $(1 + \nu)\alpha$ for α .

Equations (a) and (f) of the preceding article remain valid here. The solution for u , σ_r , and σ_θ proceeds in just the same way. We may therefore write down the results by making the above substitutions in Eqs. (h), (i), and (j). Thus for the present problem

¹ The first solution of this problem is that of J. M. C. Duhamel, *Mémoires . . . par divers savants*, vol. 5, p. 440, Paris, 1838.

$$u = \frac{1 + \nu}{1 - \nu} \cdot \alpha \cdot \frac{1}{r} \int_a^r Tr dr + C_1 r + \frac{C_2}{r} \quad (c)$$

$$\sigma_r = - \frac{\alpha E}{1 - \nu} \cdot \frac{1}{r^2} \int_a^r Tr dr + \frac{E}{1 + \nu} \left(\frac{C_1}{1 - 2\nu} - \frac{C_2}{r^2} \right) \quad (d)$$

$$\sigma_\theta = \frac{\alpha E}{1 - \nu} \cdot \frac{1}{r^2} \int_a^r Tr dr - \frac{\alpha ET}{1 - \nu} + \frac{E}{1 + \nu} \left(\frac{C_1}{1 - 2\nu} + \frac{C_2}{r^2} \right) \quad (e)$$

and, from Eq. (a),

$$\sigma_z = - \frac{\alpha ET}{1 - \nu} + \frac{2\nu EC_1}{(1 + \nu)(1 - 2\nu)} \quad (f)$$

Normal force distributed according to Eq. (f) must be applied to the ends of the cylinder in order to keep $w = 0$ throughout. If we superpose a uniform axial stress $\sigma_z = C_3$, we can choose C_3 so that the resultant force on the ends is zero. The self-equilibrating distribution remaining on each end will, by Saint-Venant's principle (see page 33), give rise only to local effects at the ends.

The stresses σ_r , σ_θ will still be given by Eqs. (d) and (e). The displacement u , however, is affected by the axial stress C_3 . A term $-\nu C_3 r/E$ must be added on the right of Eq. (c). The axial displacement is that corresponding to the uniform stress C_3 .

Solid Cylinder. In this case we may take a , the lower limit of the integrals in Eqs. (c), (d), and (e), as zero. The displacement u must vanish when $r = 0$. This requires that $C_2 = 0$.

The constant C_1 is found from the condition that the curved surface $r = b$ is free from force, so that $(\sigma_r)_{r=b} = 0$. Thus from Eq. (d), putting $C_2 = 0$, $a = 0$ we find

$$\frac{C_1}{(1 + \nu)(1 - 2\nu)} = \frac{\alpha}{1 - \nu} \frac{1}{b^2} \int_0^b Tr dr \quad (g)$$

The resultant of the axial stress (f) is

$$\int_0^b \sigma_z 2\pi r dr = - \frac{2\pi\alpha E}{1 - \nu} \int_0^b Tr dr + \frac{2\nu EC_1}{(1 + \nu)(1 - 2\nu)} \pi b^2$$

and the resultant of the uniform axial stress C_3 is $C_3 \cdot \pi b^2$. The value of C_3 making the total axial force zero is therefore given by

$$C_3 \cdot \pi b^2 = \frac{2\pi\alpha E}{1 - \nu} \int_0^b Tr dr - \frac{2\nu EC_1}{(1 + \nu)(1 - 2\nu)} \pi b^2 \quad (h)$$

The final expressions for u , σ_r , σ_θ , σ_z are, from Eqs. (c), (d), (e), (f), (g), and (h),

$$u = \frac{1+\nu}{1-\nu} \cdot \alpha \left[(1-2\nu) \frac{r}{b^2} \int_0^b Tr dr + \frac{1}{r} \int_0^r Tr dr \right] \quad (240)$$

$$\sigma_r = \frac{\alpha E}{1-\nu} \left(\frac{1}{b^2} \int_0^b Tr dr - \frac{1}{r^2} \int_0^r Tr dr \right) \quad (241)$$

$$\sigma_\theta = \frac{\alpha E}{1-\nu} \left(\frac{1}{b^2} \int_0^b Tr dr + \frac{1}{r^2} \int_0^r Tr dr - T \right) \quad (242)$$

$$\sigma_z = \frac{\alpha E}{1-\nu} \left(\frac{2\nu}{b^2} \int_0^b Tr dr - T \right) \quad (243)$$

Take, for example, a long cylinder with a constant initial temperature T_0 . If beginning from an instant $t = 0$, the lateral surface of the cylinder is maintained at a temperature zero,¹ the distribution of temperature at any instant t is given by the series²

$$T = T_0 \sum_{n=1}^{\infty} A_n J_0 \left(\beta_n \frac{r}{b} \right) e^{-p_n t} \quad (i)$$

in which $J_0(\beta_n r/b)$ is the Bessel function of zero order (see page 385), and the β 's are the roots of the equation $J_0(\beta) = 0$. The coefficients of the series (i) are

$$A_n = \frac{2}{\beta_n J_1(\beta_n)}$$

and the constants p_n are given by the equation

$$p_n = \frac{k}{c\rho} \cdot \frac{\beta_n^2}{b^2} \quad (j)$$

in which k is the thermal conductivity, c the specific heat of the material, and ρ the density. Substituting series (i) into Eq. (241) and taking into account the fact that³

$$\int_0^r J_0 \left(\beta_n \frac{r}{b} \right) r dr = \frac{br}{\beta_n} J_1 \left(\beta_n \frac{r}{b} \right)$$

we find that

$$\sigma_r = \frac{2\alpha ET_0}{1-\nu} \sum_{n=1}^{\infty} e^{-p_n t} \left\{ \frac{1}{\beta_n^2} - \frac{1}{\beta_n^2} \frac{b}{r} \frac{J_1(\beta_n r/b)}{J_1(\beta_n)} \right\} \quad (k)$$

¹ It is assumed that the surface of the cylinder suddenly assumes the temperature zero. If the temperature of the surface is T_1 instead of zero, then $T_0 - T_1$ must be put instead of T_0 in our equations.

² See Byerly, "Fourier Series and Spherical Harmonics," p. 229. The calculation of thermal stresses for this case is given by A. Dinnik, "Applications of Bessel's Function to Elasticity Problems," pt. 2, p. 95, Ekaterinoslav, 1915. See also C. H. Lees, *Proc. Roy. Soc. (London)*, vol. 101, p. 411, 1922.

³ See E. Jahnke und F. Emde, "Funktionentafeln," p. 165, Berlin, 1909.

In the same manner, substituting series (i) in Eq. (242), we obtain

$$\sigma_\theta = \frac{2\alpha ET_0}{1-\nu} \sum_{n=1}^{\infty} e^{-p_n t} \left\{ \frac{1}{\beta_n^2} + \frac{1}{\beta_n^2} \frac{b}{r} \frac{J_1(\beta_n r/b)}{J_1(\beta_n)} - \frac{J_0(\beta_n r/b)}{\beta_n J_1(\beta_n)} \right\} \quad (l)$$

Substituting series (i) in Eq. (243) we find

$$\sigma_z = \frac{2\alpha ET_0}{1-\nu} \sum_{n=1}^{\infty} e^{-p_n t} \left\{ \frac{2}{\beta_n^2} - \frac{J_0(\beta_n r/b)}{\beta_n J_1(\beta_n)} \right\} \quad (m)$$

Formulas (k), (l), and (m) represent the complete solution of the problem. Several numerical examples can be found in the papers by A. Dinnik and C. H. Lees, mentioned above.¹

Figure 225 represents² the distribution of temperature in a steel cylinder. It is assumed that the cylinder had a uniform initial temperature equal to zero and that beginning from an instant $t = 0$ the surface of the cylinder is maintained at a temperature T_1 . The temperature distributions along the radius, for various values of t/b^2 (t is measured in seconds and b in centimeters), are represented by curves. It will be seen from Eqs. (i) and (j) that the temperature distribution for cylinders of various diameters is the same if the time of heating t is proportional to the square of the diameter. From the figure, the average temperature of the whole cylinder and also of an inner portion of the cylinder of radius r can be calculated. Having these temperatures we find the thermal stresses from Eqs. (241), (242), and (243). If we

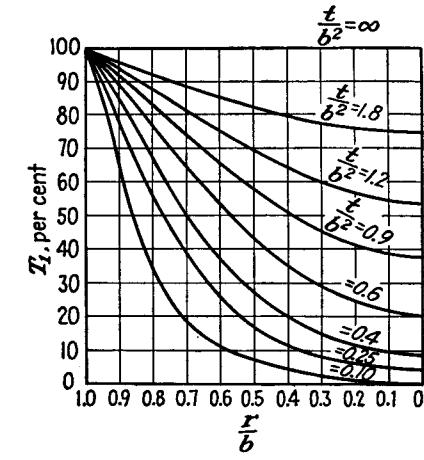


FIG. 225.

¹ Temperature distribution in solids during heating and cooling was discussed by Williamson and Adams, *Phys. Rev.*, series 2, vol. 14, p. 99, 1919. An experimental investigation of the effects of fire and water on columns has been made by Ingberg, Griffin, Robinson, and Wilson. See *U. S. Bur. Standards, Tech. Paper* 184, 1921.

² The figure is taken from A. Stodola, "Dampf- und Gasturbinen," 6th ed., p. 961, 1924.

take a very small value for t , the average temperatures, mentioned above, approach zero and we find at the surface

$$\sigma_r = 0, \quad \sigma_\theta = \sigma_z = -\frac{\alpha ET_1}{1-\nu}$$

This is the numerical maximum of the thermal stress produced in a cylinder by heating. It is equal to the stress necessary for entire suppression of thermal expansion at the surface. During heating this stress is compression, during cooling it is tension. In order to reduce the maximum stresses it is the usual practice to begin the heating of shafts and rotors with a somewhat lower temperature than the final temperature T_1 , and to increase the time of heating in proportion to the square of the diameter.

*Cylinder with a Concentric Circular Hole.*¹ The radius of the hole being a , and the outer radius of the cylinder b , the constants C_1, C_2 in Eqs. (c), (d), (e) are determined so that σ_r will be zero at these two radii. Then

$$\begin{aligned} \frac{C_1}{1-2\nu} - \frac{C_2}{a^2} &= 0 \\ -\frac{\alpha E}{1-\nu} \cdot \frac{1}{b^2} \int_a^b Tr dr + \frac{E}{1+\nu} \left(\frac{C_1}{1-2\nu} - \frac{C_2}{b^2} \right) &= 0 \end{aligned}$$

and from these

$$\begin{aligned} \frac{EC_2}{1+\nu} &= \frac{\alpha E}{1-\nu} \frac{a^2}{b^2-a^2} \int_a^b Tr dr \\ \frac{EC_1}{(1+\nu)(1-2\nu)} &= \frac{\alpha E}{1-\nu} \frac{1}{b^2-a^2} \int_a^b Tr dr \end{aligned}$$

Substituting these values in (d), (e), and (f), and adding to the last the axial stress C_3 required to make the resultant axial force zero, we find the formulas

$$\sigma_r = \frac{\alpha E}{1-\nu} \frac{1}{r^2} \left(\frac{r^2-a^2}{b^2-a^2} \int_a^b Tr dr - \int_a^r Tr dr \right) \quad (244)$$

$$\sigma_\theta = \frac{\alpha E}{1-\nu} \frac{1}{r^2} \left(\frac{r^2+a^2}{b^2-a^2} \int_a^b Tr dr + \int_a^r Tr dr - Tr^2 \right) \quad (245)$$

$$\sigma_z = \frac{\alpha E}{1-\nu} \left(\frac{2}{b^2-a^2} \int_a^b Tr dr - T \right) \quad (246)$$

Consider, as an example, a steady heat flow. If T_i is the temperature on the inner surface of the cylinder and the temperature on the outer

¹ See R. Lorenz, Z. Ver. deutsch. Ing., vol. 51, p. 743, 1907.

surface is zero, the temperature T at any distance r from the center is represented by the expression

$$T = \frac{T_i}{\log(b/a)} \log \frac{b}{r} \quad (n)$$

Substituting this in Eqs. (244), (245), and (246), we find the following expressions for the thermal stresses:¹

$$\begin{aligned} \sigma_r &= \frac{\alpha ET_i}{2(1-\nu) \log(b/a)} \left[-\log \frac{b}{r} - \frac{a^2}{(b^2-a^2)} \left(1 - \frac{b^2}{r^2} \right) \log \frac{b}{a} \right] \\ \sigma_\theta &= \frac{\alpha ET_i}{2(1-\nu) \log(b/a)} \left[1 - \log \frac{b}{r} - \frac{a^2}{(b^2-a^2)} \left(1 + \frac{b^2}{r^2} \right) \log \frac{b}{a} \right] \quad (247) \\ \sigma_z &= \frac{\alpha ET_i}{2(1-\nu) \log(b/a)} \left[1 - 2 \log \frac{b}{r} - \frac{2a^2}{(b^2-a^2)} \log \frac{b}{a} \right] \end{aligned}$$

If T_i is positive, the radial stress is compressive at all points and becomes zero at the inner and outer surfaces of the cylinder. The

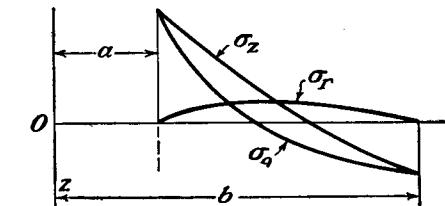


FIG. 226.

stress components σ_θ and σ_z have their largest numerical values at the inner and outer surfaces of the cylinder. Taking $r = a$, we find that

$$(\sigma_\theta)_{r=a} = (\sigma_z)_{r=a} = \frac{\alpha ET_i}{2(1-\nu) \log \frac{b}{a}} \left(1 - \frac{2b^2}{b^2-a^2} \log \frac{b}{a} \right) \quad (248)$$

For $r = b$ we obtain

$$(\sigma_\theta)_{r=b} = (\sigma_z)_{r=b} = \frac{\alpha ET_i}{2(1-\nu) \log \frac{b}{a}} \left(1 - \frac{2a^2}{b^2-a^2} \log \frac{b}{a} \right) \quad (249)$$

The distribution of thermal stresses over the thickness of the wall for a particular case $a/b = 0.3$ is shown in the Fig. 226. If T_i is positive, the stresses are compressive at the inner surface and tensile at the outer surface. In the case of such materials as stone, brick, or concrete, which are weak in tension, cracks are likely to start on the outer surface of the cylinder under the above conditions.

¹ Charts for the rapid calculation of stresses from Eqs. (247) are given by L. Barker, Engineering, vol. 124, p. 443, 1927.

If the thickness of the wall is small in comparison with the outer radius of the cylinder, we can simplify Eqs. (248) and (249) by putting

$$\frac{b}{a} = 1 + m, \quad \log \frac{b}{a} = m - \frac{m^2}{2} + \frac{m^3}{3} - \dots$$

and considering m as a small quantity. Then

$$(\sigma_\theta)_{r=a} = (\sigma_z)_{r=a} = -\frac{\alpha ET_i}{2(1-\nu)} \left(1 + \frac{m}{3}\right) \quad (248')$$

$$(\sigma_\theta)_{r=b} = (\sigma_z)_{r=b} = \frac{\alpha ET_i}{2(1-\nu)} \left(1 - \frac{m}{3}\right) \quad (249')$$

If the temperature at the outer surface of the cylinder is different from zero, the above results can be used by substituting the difference between the inner and the outer temperatures, $T_i - T_o$, in all our equations instead of T_i .

In the case of a very thin wall we can make a further simplification and neglect the term $m/3$ in comparison with unity in Eqs. (248') and (249'). Then

$$(\sigma_\theta)_{r=a} = (\sigma_z)_{r=a} = -\frac{\alpha ET_i}{2(1-\nu)} \quad (250)$$

$$(\sigma_\theta)_{r=b} = (\sigma_z)_{r=b} = \frac{\alpha ET_i}{2(1-\nu)}$$

and the distribution of thermal stresses over the thickness of the wall is the same as in the case of a flat plate of thickness $2c = b - a$, when the temperature is given by the equation (Fig. 222)

$$T = \frac{T_i y}{(b-a)}$$

and the edges are clamped, so that bending of the plate, due to non-uniform heating, is prevented [see Eq. (k), Art. 132].

If a high-frequency fluctuation of temperature is superposed on a steady heat flow, the thermal stresses produced by the fluctuation can be calculated in the same manner as explained for the case of flat plates (see Art. 132).¹

¹ Thermal stresses in cylinder walls are of great practical importance in the design of Diesel engines. A graphical solution of the problem, when the thickness of the wall of the cylinder and the temperature vary along the length of the cylinder, was developed by G. Eichelberg, *Forschungsarbeiten*, No. 263, 1923. Some information regarding temperature distribution in Diesel engines can be found in the following papers: H. F. G. Letson, *Proc. Mech. Eng.*, p. 19, London, 1925; A. Nägel, *Engineering*, vol. 127, pp. 59, 179, 279, 466, 626, 1929.

In the foregoing discussion it was assumed that the cylinder is very long and that we are considering stresses far away from the ends. Near the ends, the problem of thermal-stress distribution is more complicated due to local irregularities. Let us consider this problem for the case of a cylinder with a thin wall. Solution (250) requires that the normal forces shown in Fig. 227a should be distributed over the ends of the cylinder. To find the stresses in a cylinder with free ends we must superpose on the stresses (250) the stresses produced by forces equal and opposite to those shown in Fig. 227a. In the case of a

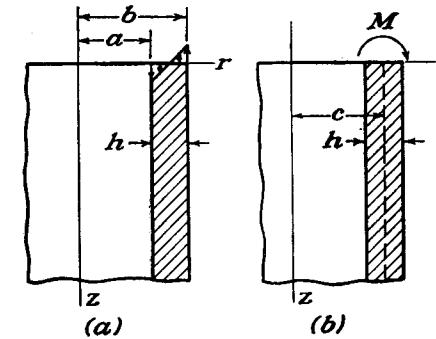


FIG. 227.

thin wall of thickness h these forces can be reduced to bending moments M , as shown in Fig. 227b, uniformly distributed along the edge of the cylinder and equal to

$$M = \frac{\alpha ET_i}{2(1-\nu)} \frac{h^2}{6} \quad (o)$$

per unit length of the edge. To estimate the stresses produced by these moments, consider a longitudinal strip, of width equal to unity, cut out from the cylindrical shell. Such a strip can be treated as a bar on an elastic foundation. The deflection curve of this strip is given by the equation¹

$$u = \frac{Me^{-\beta z}}{2\beta^2 D} (\cos \beta z - \sin \beta z) \quad (p)$$

in which

$$\beta = \sqrt{\frac{3(1-\nu^2)}{c^2 h^2}}, \quad D = \frac{Eh^3}{12(1-\nu^2)} \quad (q)$$

and c is the middle radius of the cylindrical shell. Having this deflection curve, the corresponding bending stresses σ_z and the tangential

¹ See S. Timoshenko, "Strength of Materials," 2d ed., vol. 2, p. 166.

stresses σ_θ can be calculated for any value of z . The maximum deflection of the strip is evidently at the end $z = 0$, where

$$(u)_{z=0} = \frac{M}{2\beta^2 D} = \frac{\alpha c T_i \sqrt{1 - \nu^2}}{2\sqrt{3}(1 - \nu)}$$

The corresponding strain component in the tangential direction is

$$\epsilon_\theta = \frac{u}{c} = \frac{\alpha T_i \sqrt{1 - \nu^2}}{2\sqrt{3}(1 - \nu)} \quad (r)$$

The stress component in the tangential direction at the outer surface of the wall is then obtained, using Hooke's law, from the equation

$$\sigma_\theta = E\epsilon_\theta + \nu\sigma_z = \frac{\alpha ET_i \sqrt{1 - \nu^2}}{2\sqrt{3}(1 - \nu)} - \frac{\nu\alpha ET_i}{2(1 - \nu)}$$

Adding this stress to the corresponding stress calculated from Eqs. (250), the maximum tangential stress for a thin-walled cylinder at the free end is

$$(\sigma_\theta)_{\max.} = \frac{\alpha ET_i}{2(1 - \nu)} \left(\frac{\sqrt{1 - \nu^2}}{\sqrt{3}} - \nu + 1 \right) \quad (251)$$

Assuming $\nu = 0.3$, we find

$$(\sigma_\theta)_{\max.} = 1.25 \frac{\alpha ET_i}{2(1 - \nu)}$$

Thus the maximum tensile stress at the free end of the cylinder is 25 per cent greater than that obtained from Eqs. (250) for the stress at points remote from the ends. From Eq. (p) it can be seen that the increase of stress near the free ends of the cylinder, since it depends on the deflection u , is of a local character and diminishes rapidly with increase of distance z from the end.

The approximate method of calculating thermal stresses in thin-walled cylinders, by using the deflection curve of a bar on an elastic foundation, can also be applied in the case in which the temperature varies along the axis of the cylindrical shell.¹

136. The Sphere. We consider here the simple case of a temperature symmetrical with respect to the center, and so a function of r , the radial distance, only.²

¹ S. Timoshenko and J. M. Lessells, "Applied Elasticity," p. 147, 1925, and C. H. Kent, *Trans. A.S.M.E.*, Applied Mechanics Division, vol. 53, p. 167, 1931.

² The problem was solved by Duhamel, *loc. cit.*; F. Neuman, *Abhandl. Akad. Wiss.*, Berlin, 1841; see also his "Vorlesungen über die Theorie der Elastizität der

On account of the symmetry there will be three non-zero stress components, the radial component σ_r , and two tangential components σ_t , as in Art. 121, and these must satisfy the condition of equilibrium, in the radial direction, of an element [Fig. 202, Eq. (e), page 359]

$$\frac{d\sigma_r}{dr} + \frac{2}{r} (\sigma_r - \sigma_t) = 0 \quad (a)$$

The stress-strain relations are

$$\epsilon_r - \alpha T = \frac{1}{E} (\sigma_r - 2\nu\sigma_t) \quad (b)$$

$$\epsilon_t - \alpha T = \frac{1}{E} [\sigma_t - \nu(\sigma_r + \sigma_t)] \quad (c)$$

and, u being the radial displacement, we have

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_t = \frac{u}{r} \quad (d)$$

From (b) and (c) we find

$$\sigma_r = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_r + 2\nu\epsilon_t - (1 + \nu)\alpha T] \quad (e)$$

$$\sigma_t = \frac{E}{(1 + \nu)(1 - 2\nu)} [\epsilon_t + \nu\epsilon_r - (1 + \nu)\alpha T] \quad (f)$$

Substituting these in (a), then replacing ϵ_r , ϵ_t by the values given in (d) we find the differential equation for u

$$\frac{d^2u}{dr^2} + \frac{2}{r} \frac{du}{dr} - \frac{2u}{r^2} = \frac{1 + \nu}{1 - \nu} \cdot \alpha \frac{dT}{dr} \quad (g)$$

which can be written

$$\frac{d}{dr} \left[\frac{1}{r^2} \frac{d}{dr} (r^2 u) \right] = \frac{1 + \nu}{1 - \nu} \cdot \alpha \frac{dT}{dr}$$

The solution is

$$u = \frac{1 + \nu}{1 - \nu} \cdot \alpha \cdot \frac{1}{r^2} \int_a^r Tr^2 dr + C_1 r + \frac{C_2}{r^2} \quad (h)$$

where C_1 and C_2 are constants of integration to be determined later from boundary conditions, and a is any convenient lower limit for the integral, such as the inner radius of a hollow sphere.

festen Körper," Leipzig, 1885; J. Hopkinson, *Messenger of Math.*, vol. 8, p. 168, 1879. Nonsymmetrical temperatures were considered by C. W. Borchardt, *Monatsber. Akad. Wiss.*, Berlin, 1873, p. 9.

This solution can be substituted in Eqs. (d), and the results used in Eqs. (e) and (f). Then

$$\sigma_r = -\frac{2\alpha E}{1-\nu} \cdot \frac{1}{r^3} \int_a^r Tr^2 dr + \frac{EC_1}{1-2\nu} - \frac{2EC_2}{1+\nu} \cdot \frac{1}{r^3} \quad (i)$$

$$\sigma_t = \frac{\alpha E}{1-\nu} \cdot \frac{1}{r^3} \int_a^r Tr^2 dr + \frac{EC_1}{1-2\nu} + \frac{EC_2}{1+\nu} \cdot \frac{1}{r^3} - \frac{\alpha ET}{1-\nu} \quad (j)$$

We shall now consider several particular cases.

Solid Sphere. In this case the lower limit a of the integrals may be taken as zero. We must have $u = 0$ at $r = 0$, and, from Eq. (h) this requires that $C_2 = 0$, since

$$\lim_{r \rightarrow 0} \frac{1}{r^3} \int_0^r Tr^2 dr = 0$$

The stress components given by (i) and (j) will now be finite at the center since

$$\lim_{r \rightarrow 0} \frac{1}{r^3} \int_0^r Tr^2 dr = \frac{T_0}{3}$$

where T_0 is the temperature at the center. The constant C_1 is determined from the condition that the outer surface $r = b$ is free from force, so that $\sigma_r = 0$. Then from Eq. (i), putting $\sigma_r = 0$, $a = 0$, $C_2 = 0$, $r = b$, we find

$$\frac{EC_1}{1-2\nu} = \frac{2\alpha E}{1-\nu} \cdot \frac{1}{b^3} \int_0^b Tr^2 dr$$

and the stress components become

$$\begin{aligned} \sigma_r &= \frac{2\alpha E}{1-\nu} \left(\frac{1}{b^3} \int_0^b Tr^2 dr - \frac{1}{r^3} \int_0^r Tr^2 dr \right) \\ \sigma_t &= \frac{\alpha E}{1-\nu} \left(\frac{2}{b^3} \int_0^b Tr^2 dr + \frac{1}{r^3} \int_0^r Tr^2 dr - T \right) \end{aligned} \quad (252)$$

The mean temperature of the sphere within the radius r is

$$\frac{4\pi}{8\pi r^3} \int_0^r Tr^2 dr = \frac{3}{r^3} \int_0^r Tr^2 dr$$

Therefore the stress σ_r at any radius r is proportional to the difference between the mean temperature of the whole sphere and the mean tem-

perature of a sphere of radius r . The tangential stress at any point is

$$\frac{2\alpha E}{3(1-\nu)}$$

multiplied by the expression

[The mean temperature of the whole sphere + ($\frac{1}{2}$ the mean temperature within the sphere of radius r) - $\frac{3}{2}T$]

If the distribution of temperature is known, the calculation of stresses in each particular case can be carried out without difficulty.¹ An interesting example of such calculations was made by G. Grünberg² in connection with an investigation of the strength of isotropic materials subjected to equal tension in three perpendicular directions. If a solid sphere at a uniform initial temperature T_0 is put in a liquid at a higher temperature T_1 , the outer portion of the sphere expands and produces at the center of the sphere a uniform tension in all directions. The maximum value of this tension occurs after a time

$$t = 0.0574 \frac{b^2 c \rho}{k} \quad (k)$$

Here b is the radius of the sphere, k the thermal conductivity, c the specific heat of the material, and ρ the density. The magnitude of this maximum tensile stress is³

$$\sigma_r = \sigma_t = 0.771 \frac{\alpha E}{2(1-\nu)} (T_1 - T_0) \quad (l)$$

The maximum compressive stress occurs at the surface of the sphere at the moment of application of the temperature T_1 and is equal to $\alpha E(T_1 - T_0)/(1-\nu)$. This is the same as we found before for a cylinder (see page 412). Applying Eqs. (k) and (l) to the case of steel, and taking $b = 10$ cm. and $T_1 - T_0 = 100^\circ\text{C}$., we find $\sigma_r = \sigma_t = 1,270$ kg. per square centimeter, and $t = 33.4$ sec.

Sphere with a Hole at the Center. Denoting by a and b the inner and outer radii of the sphere, we determine the constants C_1 and C_2 in (i)

¹ Several examples of such calculations are given in the paper by E. Honegger, Festschrift Prof. A. Stodola, Zürich, 1929. A table for calculating the distribution of temperature during cooling of a solid sphere is given by Adams and Williamson, *loc. cit.*

² G. Grünberg, *Z. Physik*, vol. 35, p. 548, 1925.

³ It was assumed in the analysis that the surface of the sphere takes at once the temperature T_1 of the fluid.

and (j) from the conditions that σ_r is zero on the inner and outer surfaces. Then from (i) we have

$$\begin{aligned} \frac{EC_1}{1-2\nu} - \frac{2EC_2}{1+\nu} \cdot \frac{1}{a^3} &= 0 \\ -\frac{2\alpha E}{1-\nu} \frac{1}{b^3} \int_a^b Tr^2 dr + \frac{EC_1}{1-2\nu} - \frac{2EC_2}{1+\nu} \cdot \frac{1}{b^3} &= 0 \end{aligned}$$

Solving for C_1 and C_2 and inserting the results in (i) and (j) we find

$$\begin{aligned} \sigma_r &= \frac{2\alpha E}{1-\nu} \left[\frac{r^3 - a^3}{(b^3 - a^3)r^3} \int_a^b Tr^2 dr - \frac{1}{r^3} \int_a^r Tr^2 dr \right] \quad (253) \\ \sigma_t &= \frac{2\alpha E}{1-\nu} \left[\frac{2r^3 + a^3}{2(b^3 - a^3)r^3} \int_a^b Tr^2 dr + \frac{1}{2r^3} \int_a^r Tr^2 dr - \frac{1}{2} T \right] \end{aligned}$$

Thus the stress components can be calculated if the distribution of temperature is given.

Consider, as an example, the case of steady heat flow. We denote the temperature at the inner surface by T_i and the temperature at the outer surface we take as zero. Then the temperature at any distance r from the center is

$$T = \frac{T_i a}{b-a} \left(\frac{b}{r} - 1 \right) \quad (m)$$

Substituting this in expressions (253), we find

$$\begin{aligned} \sigma_r &= \frac{\alpha ET_i}{1-\nu} \frac{ab}{b^3 - a^3} \left[a + b - \frac{1}{r} (b^2 + ab + a^2) + \frac{a^2 b^2}{r^3} \right] \\ \sigma_t &= \frac{\alpha ET_i}{1-\nu} \frac{ab}{b^3 - a^3} \left[a + b - \frac{1}{2r} (b^2 + ab + a^2) - \frac{a^2 b^2}{2r^3} \right] \end{aligned}$$

We see that the stress σ_r is zero for $r = a$ and $r = b$. It becomes a maximum or minimum when

$$r^2 = \frac{3a^2 b^2}{a^2 + ab + b^2}$$

The stress σ_t , for $T_i > 0$, increases as r increases. When $r = a$, we have

$$\sigma_t = -\frac{\alpha ET_i}{2(1-\nu)} \frac{b(b-a)(a+2b)}{b^3 - a^3} \quad (n)$$

When $r = b$, we obtain

$$\sigma_t = \frac{\alpha ET_i}{2(1-\nu)} \frac{a(b-a)(2a+b)}{b^3 - a^3} \quad (o)$$

In the case of a spherical shell of small thickness we put

$$b = a(1+m)$$

where m is a small quantity. Substituting in (n) and (o) and neglecting higher powers of m , we obtain

$$\begin{aligned} \text{for } r = a, \quad \sigma_t &= -\frac{\alpha ET_i}{2(1-\nu)} \left(1 + \frac{2}{3} m \right) \\ \text{for } r = b, \quad \sigma_t &= \frac{\alpha ET_i}{2(1-\nu)} \left(1 - \frac{2}{3} m \right) \end{aligned}$$

If we neglect the quantity $\frac{2}{3}m$ we arrive at the same values for the tangential stresses as we obtained before for a thin cylindrical shell [see Eqs. (250)] and for a thin plate with clamped edges.

137. General Equations. The differential equations (132) of equilibrium in terms of displacements can be extended to cover thermal stress and strain. The stress-strain relations for three-dimensional problems are

$$\begin{aligned} \epsilon_x - \alpha T &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y - \alpha T &= \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z - \alpha T &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \end{aligned} \quad (a)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G} \quad (b)$$

Equations (b) are not affected by the temperature because free thermal expansion does not produce angular distortion in an isotropic material.

Adding Eqs. (a) and using the notation given in (7) we find

$$e = \frac{1}{E} (1-2\nu)\Theta + 3\alpha T$$

Using this, and solving for the stresses from Eqs. (a), we find

$$\sigma_x = \lambda e + 2G\epsilon_x - \frac{\alpha ET}{1-2\nu} \quad (c)$$

Substituting from this and Eqs. (6) into the equations of equilibrium (127), and assuming there are no body forces, we find

$$(\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 u - \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial x} = 0 \quad (254)$$

These equations replace Eqs. (131) in the calculation of thermal stresses. The boundary equations (128), after substituting from Eqs. (c) and (6) and assuming that there are no surface forces, become

$$\frac{\alpha ET}{1-2\nu} l = \lambda el + G \left(\frac{\partial u}{\partial x} l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \right) + G \left(\frac{\partial u}{\partial x} l + \frac{\partial v}{\partial x} m + \frac{\partial w}{\partial x} n \right) \quad (255)$$

.

Comparing Eqs. (254) and (255) with Eqs. (131) and (134) it is seen that terms

$$-\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial x}, \quad -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial y}, \quad -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial z}$$

take the places of components X, Y, Z of the body forces, and terms

$$\frac{\alpha ET}{1-2\nu} l, \quad \frac{\alpha ET}{1-2\nu} m, \quad \frac{\alpha ET}{1-2\nu} n$$

replace components $\bar{X}, \bar{Y}, \bar{Z}$ of the surface forces. Thus the displacements u, v, w , produced by the temperature change T , are the same as the displacements produced by the body forces

$$X = -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial x}, \quad Y = -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial y}, \quad Z = -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial z} \quad (d)$$

and normal tensions

$$\frac{\alpha ET}{1-2\nu} \quad (e)$$

distributed over the surface.

If the solution of Eqs. (254) satisfying the boundary conditions (255) is found, giving the displacements u, v, w , the shearing stresses can be calculated from Eqs. (b) and the normal stresses from Eqs. (c). From the latter equations we see that the normal stress components consist of two parts: (a) a part derived in the usual way by using the strain components, and (b) a "hydrostatic" pressure of the amount

$$\frac{\alpha ET}{1-2\nu} \quad (f)$$

proportional at each point to the temperature change at that point. Thus, the total stress produced by nonuniform heating is obtained by

superposing hydrostatic pressure (f) on the stresses produced by body forces (d) and surface forces (e).

The same conclusion may be reached in another way. Imagine that the body undergoing nonuniform heating is subdivided into infinitely small elements and assume that the thermal strains $\epsilon_x = \epsilon_y = \epsilon_z = \alpha T$ of these elements are counteracted by applying to each element a uniform pressure p , the magnitude of which, from Eq. (8), is given by (f). In this way thermal strain is removed, and the elements fit one another and form a continuous body of the initial shape. The pressure distribution (f) can be realized by applying certain body forces and surface pressures to the above body formed by the elements. These forces must satisfy the equations of equilibrium (127) and the boundary conditions (128). Substituting in these equations

$$\sigma_x = \sigma_y = \sigma_z = -p = -\frac{\alpha ET}{1-2\nu}, \quad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \quad (g)$$

we find that, to keep the body formed by the elements in its initial shape, the necessary body forces are

$$X = \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial x}, \quad Y = \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial y}, \quad Z = \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial z} \quad (h)$$

and that the pressure (f) should be applied to the surface.

We now assume that the elements are joined together, and remove the forces (h) and the surface pressure (f). Then the thermal stresses are evidently obtained by superposing on the pressures (f) the stresses which are produced in the elastic body by the body forces

$$X = -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial x}, \quad Y = -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial y}, \quad Z = -\frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial z}$$

and a normal tension on the surface equal to

$$\frac{\alpha ET}{1-2\nu}$$

These latter stresses satisfy the equations of equilibrium

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial x} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} - \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial y} &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} - \frac{\alpha E}{1-2\nu} \frac{\partial T}{\partial z} &= 0 \end{aligned} \quad (256)$$

and the boundary conditions

$$\begin{aligned}\sigma_x + \tau_{xy}m + \tau_{xz}n &= \frac{\alpha ET}{1 - 2\nu} \\ \sigma_y m + \tau_{yz}n + \tau_{xy}l &= \frac{\alpha ET}{1 - 2\nu} m \\ \sigma_z n + \tau_{xz}l + \tau_{yz}m &= \frac{\alpha ET}{1 - 2\nu} n\end{aligned}\quad (257)$$

together with the compatibility conditions discussed in Art. 77. The solution of these equations, superposed on the pressure (f), gives the thermal stresses in a body undergoing temperature change.

Plane strain will occur in a long cylindrical or prismatic body when the temperature, although varying over a cross section, does not vary along lines parallel to the axis of the cylinder or prism (the z -axis). Then T is independent of z .

Beginning again with the stresses (g) which result in zero strain, the necessary body forces are given by (h) where now $Z = 0$, and the pressure (f) must be applied to the surface, including the ends.

Then supposing the elements joined together, we remove the body forces, and the surface pressure *on the curved surface only*, keeping the axial strain ϵ_z zero. The effects of this removal are obtained by solving the problem of applying body force

$$X = -\frac{\alpha E}{1 - 2\nu} \frac{\partial T}{\partial x}, \quad Y = -\frac{\alpha E}{1 - 2\nu} \frac{\partial T}{\partial y} \quad (i)$$

and a normal tensile stress of amount

$$\frac{\alpha ET}{1 - 2\nu} \quad (j)$$

on the curved surface only, as a problem in plane strain ($\epsilon_z = 0$). This problem is of the type considered at the end of Art. 16, except that we have to convert Eq. (32) from plane stress to plane strain by replacing ν by $\nu/1 - \nu$. Thus instead of Eqs. (31) and (32) we shall have

$$\sigma_x - \frac{\alpha ET}{1 - 2\nu} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y - \frac{\alpha ET}{1 - 2\nu} = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (k)$$

and

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -\frac{\alpha E}{1 - \nu} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (l)$$

The required stress function is that which satisfies Eq. (l) and gives the normal boundary tension (j). The stresses are then calculated from (k). On these we have to superpose the stresses (g).

The axial stress σ_z will consist of the term from (g) together with $\nu(\sigma_x + \sigma_y)$ obtained from (k). The resultant axial force and bending couple on the ends can be removed by superposition of simple tension and bending.

Plane stress will occur in a thin plate when the temperature does not vary through the thickness. Taking the x and y plane as the middle plane of the plate we may

assume that $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. We may also regard each element as free to expand in the z -direction. It will be sufficient, to ensure fitting of the elements, to suppress the expansion in the x and y directions only. This requires

$$\sigma_x = \sigma_y = -\frac{\alpha ET}{1 - \nu}, \quad \tau_{xy} = 0 \quad (m)$$

Substituting these in the equations of equilibrium (18), page 22, we find that the required body forces are

$$X = \frac{\alpha E}{1 - \nu} \frac{\partial T}{\partial x}, \quad Y = \frac{\alpha E}{1 - \nu} \frac{\partial T}{\partial y} \quad (n)$$

and the normal pressure $\alpha ET/(1 - \nu)$ should be applied to the edges of the plate.

Removing these forces, we conclude that the thermal stress consists of (m) together with the plane stress due to body forces

$$X = -\frac{\alpha E}{1 - \nu} \frac{\partial T}{\partial x}, \quad Y = -\frac{\alpha E}{1 - \nu} \frac{\partial T}{\partial y} \quad (o)$$

and to normal tensile stress $\alpha ET/(1 - \nu)$ applied round the edges. The determination of this plane stress again presents us with a problem of the type considered in Art. 16. We have only to put in Eqs. (31) and (32)

$$V = \frac{\alpha ET}{1 - \nu}$$

this being the potential corresponding to the forces (o).

When the edges are fixed the problem reduces to finding the stress due to the body forces (o). A method for solving this problem for the rectangular plate was explained on page 156.

138. Initial Stresses. The method used above for calculating thermal stresses can be applied in the more general problem of *initial stresses*. Imagine a body subdivided into small elements and suppose that each of these elements undergoes a certain permanent plastic deformation or change in shape produced by metallographical transformation. Let this deformation be defined by the strain components

$$\epsilon_x', \quad \epsilon_y', \quad \epsilon_z', \quad \gamma_{xy}', \quad \gamma_{xz}', \quad \gamma_{yz}' \quad (a)$$

We assume that these strain components are small and are represented by continuous functions of the coordinates. If they also satisfy the compatibility conditions (129), the elements into which the body is subdivided fit each other after the permanent set (a) and there will be no initial stresses produced.

Let us consider now a general case when the strain components (a) do not satisfy the compatibility conditions so that the elements into which the body is subdivided do not fit each other after permanent set, and forces must be applied to the surface of these elements in order to make them satisfy the compatibility equations. Assuming that after the permanent set (a) the material remains perfectly elastic, and applying Hooke's law, we find from Eqs. (11) and (6) that the permanent set (a) can be eliminated by applying to each element the surface forces

$$\sigma_x' = -(\lambda e' + 2G\epsilon_x'), \dots, \quad \tau_{xy}' = -G\gamma_{xy}', \dots \quad (b)$$

where

$$e' = \epsilon_x' + \epsilon_y' + \epsilon_z'$$

The surface forces (b) can be induced by applying certain body and surface forces to the body formed by the small elements. These forces must satisfy the equations of equilibrium (127) and the boundary conditions (128). Substituting the stress of equilibrium (127) and the boundary conditions (128). Substituting the stress components (b) in these equations we find that the necessary body forces are

$$X = \frac{\partial}{\partial x} (\lambda e' + 2G\epsilon_x') + \frac{\partial}{\partial y} (G\gamma_{xy}') + \frac{\partial}{\partial z} (G\gamma_{zz}') \quad (c)$$

and the surface forces are

$$\bar{X} = -(\lambda e' + 2G\epsilon_x')l - G\gamma_{xy}'m - G\gamma_{zz}'n \quad (d)$$

By applying the body forces (c) and the surface forces (d) we remove the initial permanent set (a) so that the elements fit one another and form a continuous body. We now assume that the elements into which the body was subdivided are joined together and remove the forces (c) and (d). Then evidently the initial stresses are obtained by superposing on the stresses (b) the stresses which are produced in the elastic body by the body forces

$$X = -\frac{\partial}{\partial x} (\lambda e' + 2G\epsilon_x') - \frac{\partial}{\partial y} (G\gamma_{xy}') - \frac{\partial}{\partial z} (G\gamma_{zz}') \quad (e)$$

and the surface forces

$$\bar{X} = (\lambda e' + 2G\epsilon_x')l + G\gamma_{xy}'m + G\gamma_{zz}'n \quad (f)$$

Thus the problem of determining the initial stresses is reduced to the usual system of equations of the theory of elasticity in which the magnitudes of the fictitious body and surface forces are completely determined if the permanent set (a) is given.

In the particular case when $\epsilon_x' = \epsilon_y' = \epsilon_z' = \alpha T$ and $\gamma_{xy}' = \gamma_{xz}' = \gamma_{yz}' = 0$, the above equations coincide with those obtained before in calculating thermal stresses.

Let us consider now the reversed problem when the initial stresses are known and it is desired to determine the permanent set (a) which produces these stresses. In the case of transparent materials, such as glass, the initial stresses can be investigated by the photoelastic method (Chap. 5). In other cases these stresses can be determined by cutting the body into small elements and measuring the strains which occur as the result of freeing these elements from surface forces representing initial stresses in the uncut body. From the previous discussion it is clear that the initial deformation produces initial stresses only if the strain components (a) do not satisfy the compatibility equations; otherwise these strains may exist without producing initial stresses. From this it follows that knowledge of the initial stresses is not sufficient for determining the strain components (a). If a solution for these components is obtained, any permanent strain system satisfying

the compatibility equations can be superposed on this solution without affecting the initial stresses.¹

Initial stresses, producing doubly-refracting properties in glass, present great difficulties in manufacturing optical instruments. To diminish these stresses it is the usual practice to anneal the glass. The elastic limit of glass at high temperatures is very low, and the material yields under the action of the initial stresses. If a sufficient time is given, the yielding of the material at a high temperature results in a considerable release from initial stresses. Annealing has an analogous effect in the case of various metallic castings and forgings.

Cutting of large bodies into smaller pieces releases initial stresses along the surfaces of cutting and diminishes the total amount of strain energy due to initial stresses, but the magnitude of the maximum initial stress is not always diminished by such cutting. For example, suppose a circular ring (Fig. 228) has initial stresses symmetrically distributed with respect to the center and the initial stress component σ_θ' varies along a cross section mn according to a linear law (ab in the figure). Cutting the ring radially, as shown in the figure by dotted lines, releases the stresses σ_θ' along these cuts. This is equivalent to the application to the ends of each portion of the ring of two equal and opposite couples producing pure bending. The distribution of the stress σ_θ along mn , due to this bending, is nearly hyperbolic (see Art. 27), as shown by the curve cde . The residual stress along mn after cutting is then given by $\sigma_\theta + \sigma_\theta'$ and is shown in the figure by the shaded area. If the inner radius of the ring is small there is a high stress concentration at the inner boundary, and the maximum initial stress after cutting, represented in Fig. 228 by bc , may be larger than the maximum initial stress before cutting. This or similar reasoning explains why glass sometimes cracks after cutting.²

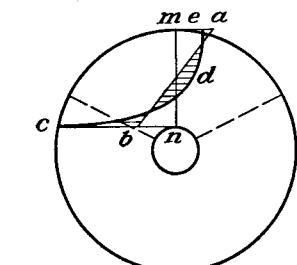


FIG. 228.

139. Two-dimensional Problems with Steady Heat Flow. In steady heat flow parallel to the xy -plane, as in a thin plate or in a long cylinder with no variation of temperature in the axial (z)-direction, the temperature T will satisfy the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (a)$$

Consider a cylinder (not necessarily circular) in a state of plane strain, with $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$. The stress-strain relations in Carte-

¹ The fact that permanent set (a) is not completely determined by the magnitudes of initial stresses is discussed in detail in the paper by H. Reissner; see *Z. angew. Math. Mech.*, vol. 11, p. 1, 1931.

² Several examples of the calculation of initial stresses in portions cut out from a circular plate are discussed in the paper by M. V. Laue, *Z. tech. Physik*, vol. 11, p. 385, 1930. Various methods of calculating residual stresses in cold-drawn tubes are discussed in the paper by N. Dawidenkow, *Z. Metallkunde*, vol. 24, p. 25, 1932.

sian coordinates are analogous to Eqs. (a) and (b) of Art. 135 in the case of plane strain. Corresponding to Eqs. (b) we shall have

$$\begin{aligned}\epsilon_x - (1 + \nu)\alpha T &= \frac{1 - \nu^2}{E} \left(\sigma_x - \frac{\nu}{1 - \nu} \sigma_y \right) \\ \epsilon_y - (1 + \nu)\alpha T &= \frac{1 - \nu^2}{E} \left(\sigma_y - \frac{\nu}{1 - \nu} \sigma_x \right)\end{aligned}\quad (b)$$

We now inquire whether it is possible to have σ_x , σ_y and τ_{xy} zero. Putting $\sigma_x = \sigma_y = 0$ in Eqs. (b) we find

$$\epsilon_x = (1 + \nu)\alpha T, \quad \epsilon_y = (1 + \nu)\alpha T \quad (c)$$

and of course $\gamma_{xy} = 0$.

Such strain components are possible only if they satisfy the conditions of compatibility (129). Since $\epsilon_z = 0$ and the other components of strain are independent of z , all of these conditions except the first are satisfied.

The first reduces to

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = 0$$

But on account of Eqs. (c) and (a), this equation also is satisfied. We find, therefore, that in steady heat flow the equations of equilibrium, the boundary condition that the curved surface be free from force, and the compatibility conditions, are all satisfied by taking

$$\sigma_x = \sigma_y = \tau_{xy} = 0, \quad \sigma_z = -\alpha ET \quad (d)$$

For a solid cylinder the above equations and conditions are complete, and we can conclude that in a steady state of two-dimensional heat conduction there is no thermal stress except the axial stress σ_z given by Eqs. (d), required to maintain the plane strain condition $\epsilon_z = 0$. In the case of a long cylinder with unrestrained ends we obtain an approximate solution valid except near the ends by superposing simple tension, or compression, and simple bending so as to reduce the resultant force and couple on the ends, due to σ_z , to zero.

For a hollow cylinder, however, we cannot conclude that the plane strain problem is solved by Eqs. (d). It is necessary to examine the corresponding displacements. It is quite possible that they will prove to have discontinuities, analogous to those discussed on pages 68 and 120.

For instance suppose the cylinder is a tube and a longitudinal slit is cut, as indicated in Fig. 229b. If it is hotter on the inside than on

the outside it will tend to uncurl, and the slit will open up. There will be a discontinuity of displacement between the two faces of the slit. Thus the displacement *should* be represented by discontinuous functions of θ . The cross section is solid, i.e., singly connected, and Eqs. (d) give the stress correctly for plane strain. But if the tube has no slit (Fig. 229a) discontinuities of displacement are physically impossible. This indicates that the assumed temperature distribution will in fact give rise to stress components σ_x , σ_y , τ_{xy} , representing the stress produced by suitably drawing together again the separated faces of the

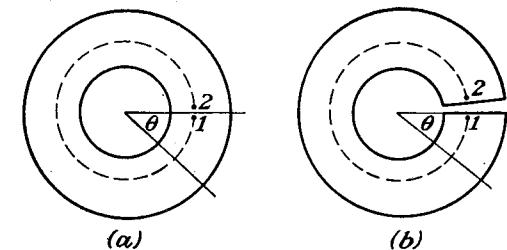


FIG. 229.

slit tube and joining them (cf. Art. 39 and Fig. 82). The component σ_z will also be affected by this operation.

To investigate this question further, we rewrite Eqs. (c) as

$$\frac{\partial u}{\partial x} = \epsilon', \quad \frac{\partial v}{\partial y} = \epsilon' \quad (e)$$

where $\epsilon' = (1 + \nu)\alpha T$. Since $\gamma_{xy} = 0$ we can write

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (f)$$

and

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega_z \quad (g)$$

ω_z being a component of rotation (see page 225). Equations (f) and (g) yield

$$\frac{\partial u}{\partial y} = -\omega_z, \quad \frac{\partial v}{\partial x} = \omega_z \quad (h)$$

and these with (e) give

$$\frac{\partial \epsilon'}{\partial x} = \frac{\partial \omega_z}{\partial y}, \quad \frac{\partial \epsilon'}{\partial y} = -\frac{\partial \omega_z}{\partial x} \quad (i)$$

Equations (i) are *Cauchy-Riemann* equations, discussed in Art. (55). They show that $\epsilon' + i\omega_z$ is an analytic function of the complex variable $x + iy$. Denoting this function by Z , we have

$$Z = \epsilon' + i\omega_z \quad (j)$$

If u_1, v_1, u_2, v_2 are the values of u, v at two points 1, 2 in the cross section of the cylinder the differences $u_2 - u_1, v_2 - v_1$ can be expressed as

$$u_2 - u_1 = \int_1^2 \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right), \quad v_2 - v_1 = \int_1^2 \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right)$$

where the integrals are taken along any curve joining the two points and lying entirely in the material. Multiplying the second by i and adding to the first we find

$$u_2 - u_1 + i(v_2 - v_1) = \int_1^2 \left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + i \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) \right] \quad (k)$$

and it is easily verified from Eqs. (e) and (h) that the integral on the right-hand side is the same as $\int_1^2 (\epsilon' + i\omega_z)(dx + i dy)$ or $\int_1^2 Z dz$. Thus Eq. (k) becomes

$$u_2 - u_1 + i(v_2 - v_1) = \int_1^2 Z dz \quad (l)$$

The displacements will be single-valued when this integral vanishes for a complete circuit of any closed curve, such as the broken-line circle in Fig. 229a, which can be drawn in the material of the cross section. We shall use this result later in solving a thermal-stress problem of the hollow circular cylinder.

Not only the displacements, but also the rotation ω_z must be continuous. We have

$$(\omega_z)_2 - (\omega_z)_1 = \int_1^2 \left(\frac{\partial \omega_z}{\partial x} dx + \frac{\partial \omega_z}{\partial y} dy \right)$$

and using Eqs. (i) this becomes

$$(\omega_z)_2 - (\omega_z)_1 = \int_1^2 \left(-\frac{\partial \epsilon'}{\partial y} dx + \frac{\partial \epsilon'}{\partial x} dy \right)$$

Since ϵ' is proportional to T , this integral is proportional to the amount of heat flowing per unit time, per unit axial distance, across the curve joining points 1 and 2. If this is a closed curve $(\omega_z)_2 - (\omega_z)_1$ must vanish, and therefore the total heat flow across the curve must be zero. If a pipe has heat flow from inside to outside or vice versa this condition is not fulfilled and the stress is not correctly given by Eqs. (d).

But if the pipe is slit, as indicated in Fig. 229b, the displacement or rotation at point 2 can differ from that at point 1, for instance if the heating causes the slit to open up. The simple state of stress given by Eqs. (d) is then correct. To arrive at the state of stress which exists in the pipe when it is not slit, we have to superpose the stress due to closing the gap. The determination of this dislocational stress¹ involves problems of the types illustrated by Figs. 45 and 82.

¹ The relation between thermal stress in steady heat flow and dislocational stress was established by N. Muschelišvili, *Bull. Elec. Tech. Inst.*, St. Petersburg, vol. 13, p. 23, 1916, and independently by M. A. Biot, *Phil. Mag.*, series 7, vol. 19, p. 540, 1935. Thermal stresses in a hollow circular cylinder, and in a square cylinder with a circular hole, have been determined photoelastically by this method by E. E. Weibel, *Proc. Fifth Intern. Cong. Applied Mechanics*, Cambridge, Mass., 1938, p. 213.

Consider, for example, a hollow circular cylinder of outer radius b , with a concentric bore of radius a . If the temperature T_i at the inner surface is uniform, and the temperature at the outer surface is zero, the temperature T at any radius r is given by Eq. (n) of Art. 135. We may write this as

$$T = -A \log b + A \log r \quad (m)$$

where

$$A = -\frac{T_i}{\log(b/a)} \quad (n)$$

The constant term $-A \log b$ in Eq. (m) can be ignored, since a uniform change of temperature does not cause thermal stress. Then, since $\log z = \log r + i\theta$,

$$\begin{aligned} Z &= \epsilon' + i\omega_z = (1 + \nu)\alpha T + i\omega_z \\ &= (1 + \nu)\alpha A \log r + i\omega_z = (1 + \nu)\alpha A \log z \end{aligned}$$

Writing B for $(1 + \nu)\alpha A$, we have from Eq. (l)

$$u_2 - u_1 + i(v_2 - v_1) = B \int_1^2 \log z dz = B [z(\log z - 1)]_1^2 \quad (o)$$

This equation applies to any curve between points 1 and 2 lying wholly in the material. It gives the relative displacement of the two points when the temperature is given by Eq. (m) and the stress by Eqs. (d).

Applying it to a circular path of radius r starting at 1 (Fig. 229), going round the hole, and ending at 2, we have, since $\theta_1 = 0, \theta_2 = 2\pi$,

$$[z(\log z - 1)]_1^2 = re^{i2\pi}(\log r + i \cdot 2\pi) - re^{i0}(\log r + i \cdot 0) = i \cdot 2\pi r$$

Inserting this in Eq. (o) we find

$$u_2 - u_1 = 0, \quad v_2 - v_1 = B \cdot 2\pi r \quad (p)$$

The relative displacement is not zero, and therefore it is necessary to consider the cylinder as slit, so that the point 2 can move away from the point 1 by the vertical displacement $2\pi r B$ (Fig. 229b). The movement of the upper face of the slit relative to the lower face is equivalent to a rotation $2\pi B$ in the clockwise sense about the center of the cylinder. However, B is negative if T_i is positive. Then the slit opens to a gap subtending an angle $-2\pi B$ at the center. The problem of closing such a gap was solved on page 69 for the case of plane stress. The solution can be converted to plane strain by the substitutions given on page 34. The stress components which result, combined with the axial stress $\sigma_z = -\alpha ET$ from Eqs. (d), are identical with those given by Eqs. (247).

Inside and outside temperatures which vary round the boundary circles can be represented by Fourier series

$$\begin{aligned} T_i &= A_0 + A_1 \cos \theta + A_2 \cos 2\theta + \dots + B_1 \sin \theta + B_2 \sin 2\theta + \dots \\ T_0 &= A'_0 + A'_1 \cos \theta + A'_2 \cos 2\theta + \dots + B'_1 \sin \theta + B'_2 \sin 2\theta + \dots \end{aligned} \quad (q)$$

The thermal stress due to the several terms can be treated separately, that due to the uniform terms A_0, A'_0 being covered by the preceding case, with $T_i = A_0 - A'_0$. Corresponding to the terms $\cos \theta, \sin \theta; \cos 2\theta, \sin 2\theta$; etc., the function Z will have terms proportional to

$$z, z^{-1}; \quad z^2, z^{-2}; \quad \text{etc.} \quad (r)$$

Solutions of equations of this type are considered in the theory of potential.¹ A solution can be written down as the gravitational potential of a distribution of matter of density $-(1 + \nu)\alpha T / 4\pi(1 - \nu)$, which is²

$$\psi = -\frac{(1 + \nu)\alpha}{4\pi(1 - \nu)} \int \int \int T(\xi, \eta, \zeta) \frac{1}{r'} d\xi d\eta d\zeta \quad (e)$$

where $T(\xi, \eta, \zeta)$ is the temperature at a typical point ξ, η, ζ at which there is an element of volume $d\xi d\eta d\zeta$, and r' is the distance between this point and the point x, y, z . Equation (e) gives the complete solution of the thermal-stress problem of an infinite solid at temperature zero except for a heated (or cooled) region.³ The cases of such a region in the form of an ellipsoid of revolution and a semi-infinite circular cylinder, uniformly hot, have been worked out.⁴ For the ellipsoid the maximum stress which can occur is $\alpha ET/1 - \nu$, and is normal to the surface of the ellipsoid at the points of sharpest curvature of the generating ellipse. This value occurs only for the two extreme cases of a very flat or very elongated ellipsoid of revolution. Intermediate cases have smaller maximum stress. For a spherical region the value is two-thirds as great.

When T is independent of z , and $w = 0$, we shall have plane strain, with ψ, u , and v independent of z . Equation (d) becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{1 + \nu}{1 - \nu} \alpha T \quad (f)$$

A particular solution is given by the *logarithmic potential*

$$\psi = \frac{1}{2\pi} \cdot \frac{1 + \nu}{1 - \nu} \cdot \alpha \int \int T(\xi, \eta) \log r' d\xi d\eta \quad (g)$$

where

$$r' = [(x - \xi)^2 + (y - \eta)^2]^{\frac{1}{2}}$$

For a thin plate, with no variation of T through the thickness, we may assume plane stress, with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, and $u, v, \sigma_x, \sigma_y, \tau_{xy}$ independent of z . We have then the stress-strain relations [cf. Eqs. (d) of Art. 134]

$$\begin{aligned} \sigma_x &= \frac{E}{1 - \nu^2} \left[\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} - (1 + \nu)\alpha T \right] \\ \sigma_y &= \frac{E}{1 - \nu^2} \left[\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} - (1 + \nu)\alpha T \right] \\ \tau_{xy} &= \frac{E}{2(1 + \nu)} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned} \quad (h)$$

¹ See, for instance, "Theory of the Potential," by W. D. MacMillan, New York, 1930.

² This potential function was used by C. W. Borchardt in the problem of the sphere. See *Monatsber. königl. Preuss. Akad. Wiss.*, Berlin, 1873, p. 9.

³ J. N. Goodier, *Phil. Mag.*, vol. 23, p. 1017, 1937. The semi-infinite solid is considered by R. D. Mindlin and D. H. Cheng, *J. Applied Phys.*, vol. 21, pp. 926, 931, 1950.

⁴ N. O. Myklestad, *J. Applied Mechanics (Trans. A.S.M.E.)*, 1942, p. A-131.

Substituting these in the two equations of equilibrium (18) (with zero body force) we find the equations

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1 - \nu}{1 + \nu} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) = 2\alpha \frac{\partial T}{\partial x} \quad (i)$$

These are satisfied by

$$u = \frac{\partial \psi}{\partial x}, \quad v = \frac{\partial \psi}{\partial y} \quad (j)$$

provided that ψ is a solution of

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = (1 + \nu)\alpha T \quad (k)$$

Comparing with Eq. (f) we see that a particular solution is given by the logarithmic potential (g) with the factor $1 - \nu$ in the denominator omitted. This gives the complete solution for local heating in an infinite plate, where the stress and deformation must tend to zero at infinity.

As a first example of this kind we consider an infinite plate at temperature zero except for a rectangular region $ABCD$ of sides $2a, 2b$ (Fig. 230) within which the temperature is T , and uniform.¹ The required logarithmic potential is

$$\psi = \frac{1}{2\pi} (1 + \nu)\alpha T \int_{-b}^b \int_{-a}^a \frac{1}{2} \log [(x - \xi)^2 + (y - \eta)^2] d\xi d\eta \quad (l)$$

The displacements are obtained by differentiation according to (j) and then the stress components can be found from (h). The results for σ_x and τ_{xy} at points such as P outside the hot rectangle can be reduced to

$$\sigma_x = E\alpha T \frac{1}{2\pi} (\psi_1 - \psi_2), \quad \tau_{xy} = E\alpha T \frac{1}{4\pi} \log \frac{r_1 r_3}{r_2 r_4} \quad (m)$$

the angles ψ_1, ψ_2 and the distances r_1, r_2, r_3, r_4 being those indicated in Fig. 230. The angles are those subtended at P by the two sides AD, BC of the rectangle parallel to the x -axis. The expression for σ_y is obtained from the first of Eqs. (m) by using instead of ψ_1 and ψ_2 the angles subtended at P by the other two sides AB, DC of the rectangle.

The value of σ_x just below AD and just to the left of A is

$$E\alpha T \cdot \frac{1}{2\pi} \left(\pi - \arctan \frac{a}{b} \right)$$

and is greatest for a rectangle infinitely long in the y -direction, when it becomes $\frac{1}{2}E\alpha T$. Both normal stress components change sharply on turning a corner of the

¹ Goodier, *loc. cit.*

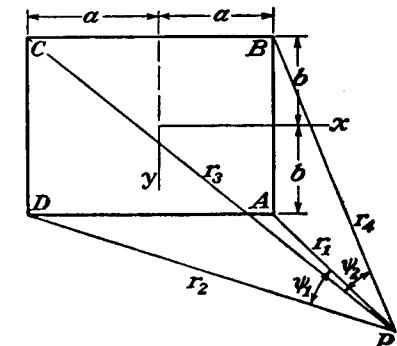


FIG. 230.

rectangle. The shear stress τ_{xy} approaches infinity as a corner is approached. These peculiarities are, of course, a consequence of the ideally sharp corners of the heated rectangle.

If the heated area is elliptical¹ instead of rectangular, the ellipse being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the value of the stress σ_y just outside the ellipse, near an end of the major axis, is

$$\frac{E\alpha T}{1 + (b/a)}$$

which approaches $E\alpha T$ for a very slender ellipse. If the heated area is circular it becomes $\frac{1}{2}E\alpha T$. The stress σ_x just beyond an end of the minor axis is

$$\frac{E\alpha T}{1 + (a/b)}$$

and approaches zero for a very slender ellipse.

The method of the present article becomes particularly simple when the temperature varies with time, and satisfies the differential equation of heat conduction²

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T \quad (n)$$

where κ is the thermal conductivity divided by the specific heat and by the density. Differentiating Eq. (d) with respect to t and then substituting for $\partial T / \partial t$ from Eq. (n) we find that the function ψ must satisfy the equation

$$\nabla^2 \frac{\partial \psi}{\partial t} = \frac{1 + \nu}{1 - \nu} \alpha \kappa \nabla^2 T$$

We may therefore take

$$\frac{\partial \psi}{\partial t} = \frac{1 + \nu}{1 - \nu} \alpha \kappa T$$

The integral of this which is appropriate for a temperature which approaches zero as time goes on is

$$\psi = -\frac{1 + \nu}{1 - \nu} \alpha \kappa \int_t^\infty T dt \quad (o)$$

as may be verified by substitution in Eq. (d), making use of Eq. (n).

Consider for instance a long circular cylinder (plane strain) which is cooling or being heated toward a steady state of heat conduction. The temperature is not symmetrical about the axis, but is independent of the axial coordinate z . The temperature is then representable by a series of terms of the form

$$T_{sn} = e^{-\kappa s^2 t} J_n(sr) e^{in\theta} \quad (p)$$

¹ Goodier, *loc. cit.*

² See, for instance, "Theory of Heat Conduction," by Ingersoll and Zobel.

where the real or imaginary parts of $e^{in\theta}$ may be taken to obtain $\cos n\theta$ or $\sin n\theta$. From Eq. (o) the function ψ corresponding to this temperature term will be

$$\psi_{sn} = -\frac{1 + \nu}{1 - \nu} \alpha \kappa \frac{1}{s^2} T_{sn} \quad (q)$$

A series of such terms, corresponding to the series for T , will represent a particular solution of the general equations (b). The displacements may be calculated according to Eqs. (a), or their polar equivalents,

$$u = \frac{\partial \psi}{\partial r}, \quad v = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

u and v here being the radial and tangential components. The axial component w is zero in plane strain.

The strain components follow from the results of Art. 28, page 65. The stress components can then be found from the plane strain formulas (a) and (b) of Art. 135, together with the last of Eqs. 52, page 66, for the shear stress $\tau_{r\theta}$.

When such a solution has been obtained it will, in general, be found that it gives non-zero boundary forces ($\sigma_r, \tau_{r\theta}$) on the curved surface of the cylinder. The effects of removing these are found by solving an ordinary plane strain problem, using the general stress function in polar coordinates given in Art. 39.¹

¹ This problem is worked out for a hollow cylinder, with temperature corresponding to Eq. (p), in the paper by J. N. Goodier cited above.

CHAPTER 15

THE PROPAGATION OF WAVES IN ELASTIC SOLID MEDIA

141. In the preceding chapters it was usually assumed that the elastic body was at rest under the action of external forces, and the resulting problems were problems of statics. There are cases, however, in which motion produced in an elastic body by suddenly applied forces or by variable forces should be considered. The action of a suddenly applied force is not transmitted at once to all parts of the body. At the beginning the remote portions of the body remain undisturbed, and deformations produced by the force are propagated through the body in the form of elastic waves. If the dimensions of the body are large, the time taken by the waves to traverse the body becomes of practical importance and should be considered. We have such problems, for instance, in discussing the effect of impact or waves produced by earthquakes. The investigation of the propagation of waves in an elastic medium is the subject of the following discussion.¹ We begin

with the simple problem of the propagation of longitudinal waves in a long prismatical bar.

142. Longitudinal Waves in Prismatical Bars. Taking the axis of the bar for x -axis (Fig. 231) and assuming that cross sections of the bar remain

plane during deformation, the unit elongation at any cross section plane during deformation, the unit elongation at any cross section mn , due to a longitudinal displacement u , is equal to $\partial u / \partial x$ and the corresponding tensile force in the bar is $AE(\partial u / \partial x)$, where A is the cross-sectional area.² Considering an element of the bar between the

¹ Problems of steady vibrations of elastic bodies (standing waves) have been discussed in S. Timoshenko, "Vibration Problems in Engineering," New York, 1928.

² It is assumed that we have here a simple tension in the x -direction and the elongation $\partial u / \partial x$ is accompanied by lateral contraction of the amount $v(\partial u / \partial x)$. Inertia forces corresponding to motion of particles in lateral direction are neglected in our derivation. This approximate solution is accurate enough so long as the length of waves is not small in comparison with cross-sectional dimensions of the bar. In the case of short waves, motion of particles in the direction perpendicular to the axis of the bar should be considered. See Lord Rayleigh, "Theory of

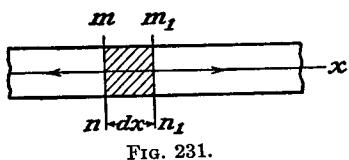


FIG. 231.

two adjacent cross sections mn and m_1n_1 , the difference of forces acting on the sides mn and m_1n_1 is

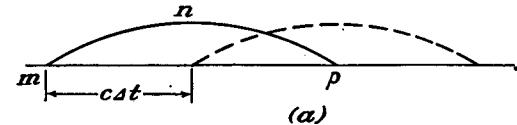
$$AE \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx \right) - AE \frac{\partial u}{\partial x} = AE \frac{\partial^2 u}{\partial x^2} dx$$

and the equation of motion of the element is

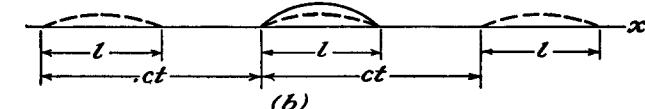
$$A\rho dx \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial^2 u}{\partial x^2} dx$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (258)$$



(a)



(b)

FIG. 232.

in which ρ is the mass per unit volume of the bar and

$$c = \sqrt{\frac{E}{\rho}} \quad (259)$$

It can be shown by substitution that any function $f(x + ct)$ is a solution of Eq. (258). Any function $f_1(x - ct)$ is also a solution, and the general solution of the Eq. (258) can be represented in the form

$$u = f(x + ct) + f_1(x - ct) \quad (260)$$

This solution has a very simple physical interpretation, which can easily be explained in the following manner. Consider the second term on the right side of Eq. (260). For a definite instant t , this term is a function of x only and can be represented by a certain curve such as mnp (Fig. 232a), the shape of which depends on the function f_1 . After an interval of time Δt , the argument of the function f_1 becomes

Sound," Chap. 7; L. Pochhammer, *J. Math. (Crelle's J.)*, vol. 81, 1876; C. Chree, *Quart. J. Math.*, vol. 21, 1886, and vol. 24, 1890; J. Prescott, *Phil. Mag.*, vol. 33, p. 703, 1942; R. M. Davies, *Trans. Roy. Soc. (London)*, series A, vol. 240, p. 375, 1948.

$x - c(t + \Delta t)$. The function f_1 will remain unchanged provided that simultaneously with the increase of t by Δt the abscissas are increased by an amount Δx equal to $c \Delta t$. This means that the curve mnp , constructed for the moment t , can also be used for the instant $t + \Delta t$, if it is displaced in the x -direction by the distance $\Delta x = c \Delta t$, as shown by the dotted line in the figure. From this consideration it can be seen that the second term of the solution (260) represents a wave traveling in the direction of the x -axis with a constant speed c . In the same manner it can be shown that the first term of the solution (260) represents a wave traveling in the opposite direction. Thus the general solution (260) represents two waves traveling along the x -axis in two opposite directions with the constant velocity c given by Eq. (259). This velocity depends only on the modulus E and the density of the material of the bar. In the case of steel, for instance, we can assume $c = 16,850$ ft. per second.

The functions f and f_1 should be determined in each particular case from the initial conditions at the instant $t = 0$. For this instant we have, from Eq. (260),

$$\begin{aligned} (u)_{t=0} &= f(x) + f_1(x) \\ \left(\frac{\partial u}{\partial t} \right)_{t=0} &= c[f'(x) - f'_1(x)] \end{aligned} \quad (a)$$

Assume, for instance, that the initial velocity along the length of the rod is zero and there is an initial displacement given by the equation

$$(u)_{t=0} = F(x)$$

Conditions (a) are satisfied by taking

$$f(x) = f_1(x) = \frac{1}{2}F(x)$$

Thus in this case the initial displacement will be split into halves which will be propagated as waves in two opposite directions (Fig. 232b).

The velocity of propagation of waves in prismatical bars can be obtained from elementary considerations. Assume that a uniformly distributed compressive stress is suddenly applied to the left end of a

prismatical bar (Fig. 233). It will produce at the first instant a uniform compression of an infinitely thin layer at the end of the bar. This compression will be transmitted to the adjacent layer, and so on. A wave of compression begins to travel along the bar with a certain velocity c , and, after a time interval t , a portion of the bar of length ct

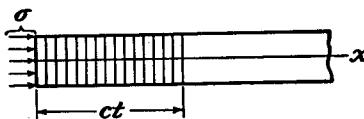


FIG. 233.

will be compressed and the remaining portion will be at rest in an unstressed condition.

The velocity of wave propagation c should be distinguished from the velocity v , given to the particles in the compressed zone of the bar by the compressive forces. The velocity of the particles v can be found by taking into account the fact that the compressed zone (shaded in the figure) shortens due to compressive stress σ by the amount $(\sigma/E)ct$. Hence the velocity of the left end of the bar, equal to the velocity of particles in the compressed zone, is

$$v = \frac{c\sigma}{E} \quad (b)$$

The velocity c of wave propagation can be found by applying the equation of momentum. At the beginning the shaded portion of the bar was at rest. After the elapse of the time t it has velocity v and momentum $Act\rho v$. Putting this equal to the impulse of the compressive force, we find

$$Act = Act\rho v \quad (c)$$

Using Eq. (b), we find for c the value given by Eq. (259)¹ and for the velocity of particles we find

$$v = \frac{\sigma}{\sqrt{E\rho}} \quad (261)$$

It will be seen that, while c is independent of the compressive force, the velocity v of particles is proportional to the stress σ .

If, instead of compression, a tensile force is suddenly applied at the end of the bar, a tension is propagated along the bar with the velocity c . The velocity of particles again is given by Eq. (261). But the direction of this velocity will be opposite to the direction of the x -axis. Thus in a compressive wave the velocity v of particles is in the same direction as the velocity of wave propagation, but in a tension wave the velocity v is in the opposite direction from that of the wave.

From Eqs. (259) and (261) we have

$$\sigma = E \frac{v}{c} \quad (262)$$

The stress in the wave is thus determined by the ratio of the two velocities and by the modulus E of the material. If an absolutely rigid body, moving with a velocity v , strikes longitudinally a pris-

¹ This elementary derivation of the formula for the velocity of wave propagation is due to Babinet; see Clebsch, Théorie de l'élasticité des corps solides, traduite par Saint-Venant, p. 480d, 1883.

matical bar, the compressive stress on the surface of contact at the first instant is given by Eq. (262).¹ If the velocity v of the body is above a certain limit, depending on the mechanical properties of the material of the bar, a permanent set will be produced in the bar although the mass of the striking body may be very small.²

Consider now the energy of the wave shown shaded in Fig. 233. This energy consists of two parts: strain energy of deformation equal to

$$\frac{Act\sigma^2}{2E}$$

and kinetic energy equal to

$$\frac{Act\rho v^2}{2} = \frac{Act\sigma^2}{2E}$$

It will be seen that the total energy of the wave, equal to the work done by the compressive force $A\sigma$ acting over the distance $(\sigma/E) ct$, is half potential and half kinetic.

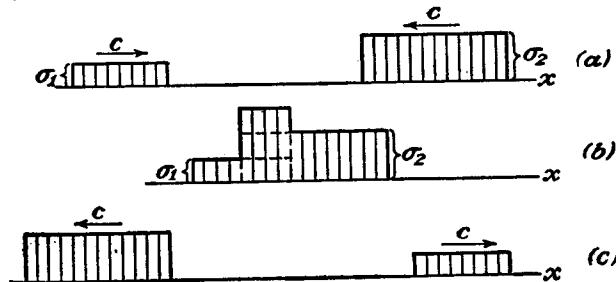


FIG. 234.

Equation (258), governing the wave propagation, is linear, so that, if we have two solutions of the equation, their sum will also be a solution of this equation. From this it follows that in discussing waves traveling along a bar we may use the method of superposition. If two waves traveling in opposite directions (Fig. 234) come together, the resulting stress and the resulting velocity of particles are obtained by superposition. If both waves are, for instance, compressive waves, the resultant compression is obtained by simple addition, as shown in Fig. 234b, and the resultant velocity of particles by subtraction. After passing, the waves return to their initial shape, as shown in Fig. 234c.

¹ This conclusion is due to Thomas Young; see his "Course of Lectures on Natural Philosophy . . .," vol. 1, pp. 135 and 144, 1807.

² It is assumed that contact occurs simultaneously at all points of the end section of the bar.

Assume that a compression wave is moving along the bar in the x -direction and a tension wave of the same length and with the same magnitude of stress is moving in the opposite direction (Fig. 235). When the waves come together, tension and compression annul each

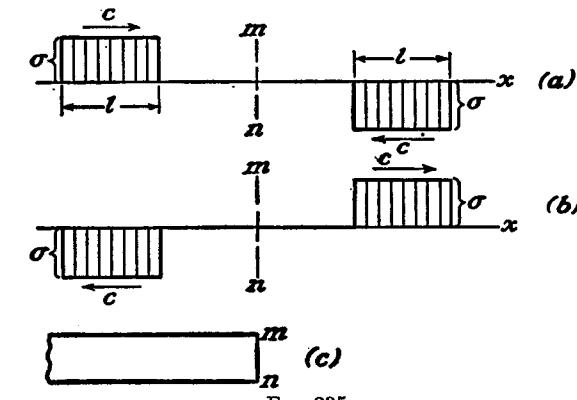


FIG. 235.

other, and in the portion of the bar in which the two waves are superposed we have zero stress. At the same time the velocity of particles in this portion of the bar is doubled and equal to $2v$. After passing, the waves return to their initial shape, as shown in Fig. 235b. At the middle cross section mn there will be at all times zero stress and we may consider it as a *free end* of a bar (Fig. 235c). Then comparing Figs. 235a and 235b it can be concluded that in the case of a free end a compressive wave is reflected as a similar tension wave, and vice versa.

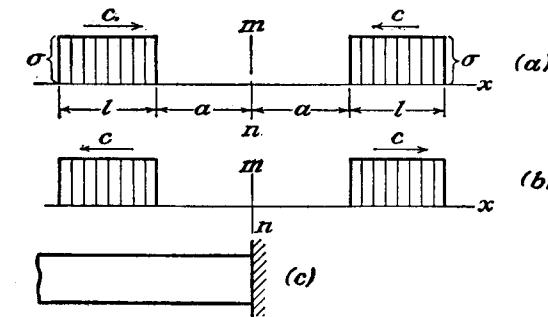


FIG. 236.

If two identical waves, moving toward one another (Fig. 236a), come together, there will be doubled stress and zero velocity in the portion of the bar in which the waves are superposed. At the middle cross section mn we always have zero velocity. This section remains immova-

ble during passage of the waves and we may consider it as a fixed end of the bar (Fig. 236c). Then, from comparison of Figs. 236a and 236b, it can be concluded that a wave is reflected from a fixed end entirely unchanged.

Up to now we have considered waves produced by constant forces. The stress σ and the velocity of particles v were constant along the length of the wave. In the case of a variable force, a wave will be produced in which σ and v vary along the length. Conclusions obtained before regarding propagation, superposition, and reflection of waves can be applied also in this more general case.

143. Longitudinal Impact of Bars. If two equal rods of the same material strike each other longitudinally with the same velocity v (Fig. 237a), the plane of contact mn will not move during the impact¹ and

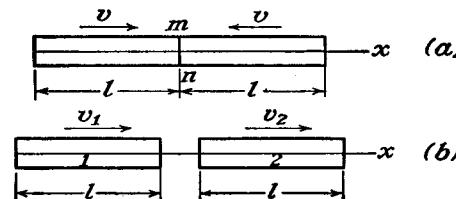


FIG. 237.

two identical compression waves start to travel along both bars with equal velocities c . The velocities of particles in the waves, superposed on the initial velocities of the bars, bring the zones of waves to rest, and at the instant when the waves reach the free ends of the bars ($t = l/c$), both bars will be uniformly compressed and at rest. Then the compression waves will be reflected from the free ends as tension waves which will travel back toward the cross section of contact mn . In these waves the velocities of particles, equal to v , will now be in the direction away from mn , and when the waves reach the plane of contact the bars separate with a velocity equal to their initial velocity v . The duration of impact in this case is evidently equal to $2l/c$ and the compressive stress, from Eq. (261), is equal to $v\sqrt{E\rho}$.

Consider now a more general case when the bars 1 and 2 (Fig. 237b) are moving² with the velocities v_1 and v_2 ($v_1 > v_2$). At the instant of impact two identical compression waves start to travel along both bars. The corresponding velocities of particles relative to the unstressed por-

¹ It is assumed that contact takes place at the same instant over the whole surface of the ends of the rods.

² Velocities are considered positive if they are in the direction of the x -axis.

tions of the moving bars are equal and are directed in each bar away from the surface of contact. The magnitude of these velocities must be equal to $(v_1 - v_2)/2$ in order to have the absolute velocities of particles of the two bars at the surface of contact equal. After an interval of time equal to l/c , the compression waves arrive at the free ends of the bars. Both bars are at this instant in a state of uniform compression, and the absolute velocities of all particles of the bars are

$$v_1 - \frac{v_1 - v_2}{2} = v_2 + \frac{v_1 - v_2}{2} = \frac{v_1 + v_2}{2}$$

The compression waves will then be reflected from the free ends as tension waves and at the instant $t = 2l/c$, when these waves arrive at the surface of contact of the two bars, the velocities of bars 1 and 2 become

$$\frac{v_1 + v_2}{2} - \frac{v_1 - v_2}{2} = v_2, \quad \frac{v_1 + v_2}{2} + \frac{v_1 - v_2}{2} = v_1$$

Thus the bars, during impact, exchange their velocities.

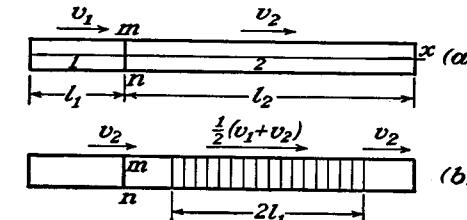


FIG. 238.

If the above bars have different lengths, l_1 and l_2 (Fig. 238a), the conditions of impact at first will be the same as in the previous case. But after a time interval $2l_1/c$, when the reflected wave of the shorter bar 1 arrives at the surface of contact mn , it is propagated through the surface of contact along the longer bar and the conditions will be as shown in Fig. 238b. The tension wave of the bar l_1 annuls the pressure between the bars, but they remain in contact until the compression wave in the longer bar (shaded in the figure) returns, after reflection, to the surface of contact (at $t = 2l_2/c$).

In the case of two bars of equal length, each of them, after rebounding, has the same velocity in all points and moves as a rigid body. The total energy is the energy of translatory motion. In the case of the bars of different lengths, the longer bar, after rebounding, has a

traveling wave in it, and in calculating the total energy of the bar the energy of this wave must be considered.¹

Consider now a more complicated problem of a bar with a fixed end struck by a moving mass at the other end² (Fig. 239). Let M be the mass of the moving body per unit area of the cross section of the bar and v_0 the initial velocity of this body. Considering the body as absolutely rigid the velocity of particles at the end of the bar at the instant of impact ($t = 0$) is v_0 , and the initial compressive stress, from Eq. (261), is

$$\sigma_0 = v_0 \sqrt{E\rho} \quad (a)$$

Owing to the resistance of the bar the velocity of the moving body and hence the pressure on the bar will gradually decrease, and we obtain a

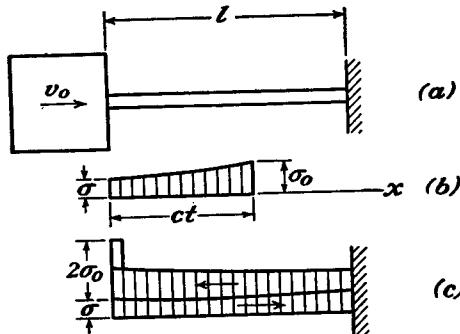


FIG. 239.

compression wave with a decreasing compressive stress traveling along the length of the bar (Fig. 239b). The change in compression with the time can easily be found from the equation of motion of the body. Denoting by σ the variable compressive stress at the end of the bar and by v the variable velocity of the body, we find

$$M \frac{dv}{dt} + \sigma = 0 \quad (b)$$

¹ The question of the lost kinetic energy of translatory motion in the case of longitudinal impact of bars was discussed by Cauchy, Poisson, and finally by Saint-Venant; see *Compt. rend.*, p. 1108, 1866, and *J. mathémat.* (*Liouville*), pp. 257 and 376, 1867.

² This problem was discussed by several authors. The final solution was given by J. Boussinesq, *Compt. rend.*, p. 154, 1883. A history of the problem can be found in "Théorie de l'élasticité des corps solides" Clebsch, traduite par Saint-Venant, see note of par. 60. The problem was also discussed by L. H. Donnell. By using the laws of wave propagation he simplified the solution and extended it to the case of a conical bar. See *Trans. A.S.M.E.*, Applied Mechanics Division, 1930.

or, substituting for v its expression from Eq. (261),

$$\frac{M}{\sqrt{E\rho}} \frac{d\sigma}{dt} + \sigma = 0$$

from which

$$\sigma = \sigma_0 e^{-\frac{t\sqrt{E\rho}}{M}} \quad (c)$$

This equation can be used so long as $t < 2l/c$. When $t = 2l/c$, the compressive wave with the front pressure σ_0 returns to the end of the bar which is in contact with the moving body. The velocity of the body cannot change suddenly, and hence the wave will be reflected as from a fixed end and the compressive stress at the surface of contact suddenly increases by $2\sigma_0$, as is shown in Fig. 239c. Such a sudden increase of pressure occurs during impact at the end of every interval of time $T = 2l/c$, and we must obtain a separate expression for σ for each one of these intervals. For the first interval, $0 < t < T$, we use Eq. (c). For the second interval, $T < t < 2T$, we have the conditions represented by Fig. 239c, and the compressive stress σ is produced by two waves moving away from the end struck and one wave moving toward this end. We designate by $s_1(t)$, $s_2(t)$, $s_3(t)$, . . . the total compressive stress produced at the end struck by all waves moving away from this end, after the intervals of time T , $2T$, $3T$, . . . The waves coming back toward the end struck are merely the waves sent out during the preceding interval, delayed a time T , due to their travel across the bar and back. Hence the compression produced by these waves at the end struck is obtained by substituting $t - T$, for t , in the expression for the compression produced by waves sent out during the preceding interval. The general expression for the total compressive stress during any interval $nT < t < (n+1)T$ is therefore

$$\sigma = s_n(t) + s_{n-1}(t - T) \quad (d)$$

The velocity of particles at the end struck is obtained as the difference between the velocity due to the pressure $s_n(t)$ of the waves going away, and the velocity due to the pressure $s_{n-1}(t - T)$ of the waves going toward the end. Then, from Eq. (261),

$$v = \frac{1}{\sqrt{E\rho}} [s_n(t) - s_{n-1}(t - T)] \quad (e)$$

The relation between $s_n(t)$ and $s_{n-1}(t - T)$ will now be obtained by using the equation of motion (b) of the striking body. Denoting by α

the ratio of the mass of the bar to the mass of the striking body, we have

$$\alpha = \frac{l\rho}{M}, \quad \frac{\sqrt{E\rho}}{M} = \frac{cl\rho}{Ml} = \frac{2\alpha}{T} \quad (f)$$

Using this, with (d) and (e), Eq. (b) becomes

$$\frac{d}{dt}[s_n(t) - s_{n-1}(t - T)] + \frac{2\alpha}{T}[s_n(t) + s_{n-1}(t - T)] = 0$$

Multiplying by $e^{\frac{2\alpha t}{T}}$,

$$\begin{aligned} e^{\frac{2\alpha t}{T}} \frac{ds_n(t)}{dt} + \frac{2\alpha}{T} e^{\frac{2\alpha t}{T}} s_n(t) &= e^{\frac{2\alpha t}{T}} \frac{ds_{n-1}(t - T)}{dt} \\ &\quad + \frac{2\alpha}{T} e^{\frac{2\alpha t}{T}} s_{n-1}(t - T) - \frac{4\alpha}{T} e^{\frac{2\alpha t}{T}} s_{n-1}(t - T) \end{aligned}$$

or

$$\frac{d}{dt}[e^{\frac{2\alpha t}{T}} s_n(t)] = \frac{d}{dt}[e^{\frac{2\alpha t}{T}} s_{n-1}(t - T)] - \frac{4\alpha}{T} e^{\frac{2\alpha t}{T}} s_{n-1}(t - T)$$

from which

$$s_n(t) = s_{n-1}(t - T) - \frac{4\alpha}{T} e^{-\frac{2\alpha t}{T}} \left[\int e^{\frac{2\alpha t}{T}} s_{n-1}(t - T) dt + C \right] \quad (g)$$

in which C is a constant of integration. This equation will now be used for deriving expressions for the consecutive values s_1, s_2, \dots . During the first interval $0 < t < T$ the compressive stress is given by Eq. (c), and we can put

$$s_0 = \sigma_0 e^{-\frac{2\alpha t}{T}} \quad (h)$$

Substituting this for s_{n-1} in Eq. (g),

$$\begin{aligned} s_1(t) &= \sigma_0 e^{-2\alpha(\frac{t}{T}-1)} - \frac{4\alpha}{T} e^{-\frac{2\alpha t}{T}} \left(\int \sigma_0 e^{2\alpha} dt + C \right) \\ &= \sigma_0 e^{-2\alpha(\frac{t}{T}-1)} \left(1 - \frac{4\alpha t}{T} \right) - C \frac{4\alpha}{T} e^{-\frac{2\alpha t}{T}} \end{aligned} \quad (k)$$

The constant of integration C is found from the condition that at the instant $t = T$ the compressive stress at the end struck increases suddenly by $2\sigma_0$ (Fig. 239c). Hence, using Eq. (d),

$$\left[\sigma_0 e^{-\frac{2\alpha t}{T}} \right]_{t=T} + 2\sigma_0 = \left[\sigma_0 e^{-2\alpha(\frac{t}{T}-1)} + \sigma_0 e^{-2\alpha(\frac{t}{T}-1)} \left(1 - \frac{4\alpha t}{T} \right) - C \frac{4\alpha}{T} e^{-\frac{2\alpha t}{T}} \right]_{t=T}$$

from which

$$C = -\frac{\sigma_0 T}{4\alpha} (1 + 4\alpha e^{2\alpha})$$

Substituting in Eq. (k),

$$s_1 = s_0 + \sigma_0 e^{-2\alpha(\frac{t}{T}-1)} \left[1 + 4\alpha \left(1 - \frac{t}{T} \right) \right] \quad (l)$$

Proceeding further in the same manner and substituting s_1 , instead of s_{n-1} , into Eq. (g), we find

$$s_2 = s_1 + \sigma_0 e^{-2\alpha(\frac{t}{T}-2)} \left[1 + 2 \cdot 4\alpha \left(2 - \frac{t}{T} \right) + 2 \cdot 4\alpha^2 \left(2 - \frac{t}{T} \right)^2 \right] \quad (m)$$

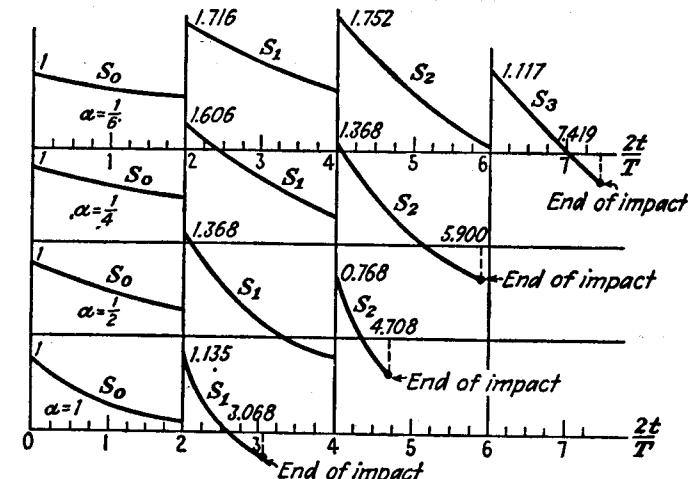


FIG. 240.

Continuing in the same way,

$$\begin{aligned} s_3 &= s_2 + \sigma_0 e^{-2\alpha(\frac{t}{T}-3)} \left[1 + 2 \cdot 6\alpha \left(3 - \frac{t}{T} \right) \right. \\ &\quad \left. + 2 \cdot 3 \cdot 4\alpha^2 \left(3 - \frac{t}{T} \right)^2 + \frac{2 \cdot 2 \cdot 3}{3 \cdot 3} 8\alpha^3 \left(3 - \frac{t}{T} \right)^3 \right] \end{aligned} \quad (n)$$

and so on. In Fig. 240 the functions s_0, s_1, s_2, \dots are represented graphically for $\sigma_0 = 1$ and for four different ratios,¹ $\alpha = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1$. By using these curves the compressive stress σ at the end struck can easily

¹ These curves were calculated by Saint-Venant and Flamant. See *Compt. rend.*, pp. 127, 214, 281, and 353, 1883.

be calculated from Eq. (d). In Fig. 241 this stress is represented graphically for $\sigma_0 = 1$ and for $\alpha = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1$. It changes at intervals T , by jumps. The maximum value of this stress depends on the ratio α . For $\alpha = \frac{1}{6}$ and $\alpha = 1$ the stress has its maximum value at $t = T$. In the case of $\alpha = \frac{1}{4}$, the maximum stress occurs at $t = 2T$.

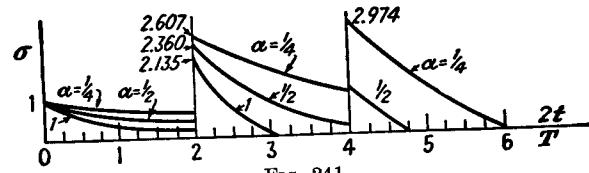


FIG. 241.

The instant when σ becomes equal to zero indicates the end of the impact. It will be seen that the duration of the impact increases when α decreases. Calculations of Saint-Venant give the following values for this duration:

$\alpha =$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	1
$\frac{2t}{T} =$	7.419	5.900	4.708	3.068

For a very small α the time of contact can be calculated from the elementary formula

$$t = \frac{\pi l}{c} \sqrt{\frac{1}{\alpha}} \quad (p)$$

which is obtained by neglecting the mass of the rod entirely and assuming that the duration of the impact is equal to half the period of simple harmonic oscillation of the body attached to the rod.

Functions s_1, s_2, s_3, \dots calculated above can also be used for determining the stresses in any other cross section of the bar. The total stress is always the sum of two values of s [Eq. (d)], one value in the resultant wave going toward the fixed end and one in the resultant wave going in the opposite direction. When the portion of the wave corresponding to the maximum value of s (the highest peak of one of the curves in Fig. 240) arrives at the fixed end and is reflected there, both of the waves mentioned above will have this maximum value: the total compressive stress at this point and at this instant is as great as can occur during the impact. From this we see that the maximum stress during impact occurs at the fixed end and is equal to twice the maxi-

mum value of s . From Fig. 240 it can be concluded at once that for $\alpha = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, 1$, the maximum compressive stresses are $2 \times 1.752\sigma_0$, $2 \times 1.606\sigma_0$, $2 \times 1.368\sigma_0$, and $2 \times 1.135\sigma_0$, respectively. In Fig. 242

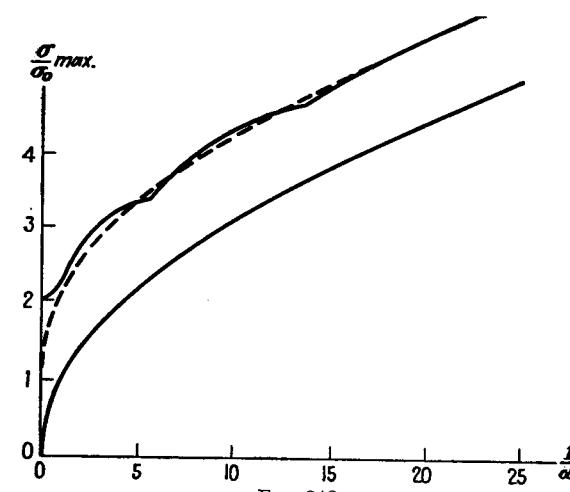


FIG. 242.

the values of σ_{\max}/σ_0 for various values of the ratio $\alpha = pl/M$ are given.¹ For comparison there is also shown the lower parabolic curve calculated from the equation

$$\sigma = \sigma_0 \sqrt{\frac{M}{\rho l}} = \frac{\sigma_0}{\sqrt{\alpha}} \quad (q)$$

which can be obtained at once in an elementary way by neglecting the mass of the rod entirely and equating the strain energy of the rod to the kinetic energy of the striking body. The dotted line shown is a parabolic curve² defined by the equation

$$\sigma = \sigma_0 \left(\sqrt{\frac{M}{\rho l}} + 1 \right) \quad (r)$$

It will be seen that for large values of $1/\alpha$, it always gives a very good approximation.

The theory of impact developed above is based on the assumption that contact takes place at the same instant over the whole surface of the end of the rod. This condition is difficult to realize in practice, and

¹ See papers by Saint-Venant and Flamant, *loc. cit.*

² This curve was proposed by Boussinesq; see *Compt. rend.*, p. 154, 1883.

experiments do not agree satisfactorily with the theory.¹ A much better agreement with the theory was obtained by using helical springs instead of rods.² In such a case the velocity of propagation of longitudinal waves is small, and the time T taken by the wave to travel across the rod and back is large in comparison with the time required for flattening small unevennesses of the ends.

Another way of making experiments under definite conditions is to use rods with spherical ends and to consider the local deformation, which can be found from Hertz's formula³ (see page 372).

144. Waves of Dilatation and Waves of Distortion in Isotropic Elastic Media. In discussing the propagation of waves in an elastic medium it is of advantage to use differential equations in terms of displacements [Eqs. (131), page 234]. To obtain the equations of small motion from these equations of equilibrium, it is only necessary to add the inertia forces. Then, assuming that there are no body forces, the equations of motion are

$$\begin{aligned} (\lambda + G) \frac{\partial e}{\partial x} + G \nabla^2 u - \rho \frac{\partial^2 u}{\partial t^2} &= 0 \\ (\lambda + G) \frac{\partial e}{\partial y} + G \nabla^2 v - \rho \frac{\partial^2 v}{\partial t^2} &= 0 \\ (\lambda + G) \frac{\partial e}{\partial z} + G \nabla^2 w - \rho \frac{\partial^2 w}{\partial t^2} &= 0 \end{aligned} \quad (263)$$

in which e is the volume expansion, and the symbol ∇^2 represents the operation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Assume first that the deformation produced by the waves is such that the volume expansion is zero, the deformation consisting of shearing distortion and rotation only. Then Eqs. (263) become

$$\begin{aligned} G \nabla^2 u - \rho \frac{\partial^2 u}{\partial t^2} &= 0 \\ \dots & \\ \dots & \end{aligned} \quad (264)$$

These are equations for waves called *waves of distortion*.

¹ See W. Voigt, *Ann. Physik*, vol. 19, p. 44, 1883, and vol. 46, p. 657, 1915. For a complete review of literature on impact see the article by T. Pöschl, "Handbuch der Physik," vol. 6, p. 525, Berlin, 1928.

² Such experiments were made by C. Ramsauer, *Ann. Physik*, vol. 30, p. 416, 1909.

³ Such an investigation was made by J. E. Sears, *Trans. Cambridge Phil. Soc.*, vol. 21, p. 49, 1908. See also J. E. P. Wagstaff, *Proc. Roy. Soc. (London)*, series A, vol. 105, p. 544, 1924; and W. A. Prowse, *Phil. Mag.*, vol. 22, p. 209, 1936.

Consider now the case when the deformation produced by the waves is not accompanied by rotation. The rotation of an element with respect to the z -axis is (see Art. 75)

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (a)$$

Analogous expressions give the rotation about the x - and y -axes. The conditions that the deformation is *irrotational* can therefore be represented in the form

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \quad \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \quad (b)$$

These equations are satisfied if the displacements u, v, w are derived from a single function ϕ as follows:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z} \quad (c)$$

Then

$$e = \nabla^2 \phi, \quad \frac{\partial e}{\partial x} = \frac{\partial}{\partial x} \nabla^2 \phi = \nabla^2 u$$

Substituting these in Eqs. (263), we find that

$$\begin{aligned} (\lambda + 2G) \nabla^2 u - \rho \frac{\partial^2 u}{\partial t^2} &= 0 \\ \dots & \\ \dots & \end{aligned} \quad (265)$$

These are equations for *irrotational waves, or waves of dilatation*.

A more general case of propagation of waves in an elastic medium is obtained by superposing waves of distortion and waves of dilatation. For both kinds of waves the equations of motion have the common form

$$\frac{\partial^2 \psi}{\partial t^2} = a^2 \nabla^2 \psi \quad (266)$$

in which

$$a = c_1 = \sqrt{\frac{\lambda + 2G}{\rho}} \quad (267)$$

for the case of waves of dilatation, and

$$a = c_2 = \sqrt{\frac{G}{\rho}} \quad (268)$$

for the case of waves of distortion. We shall now show that c_1 and c_2 are velocities of propagation of waves of dilatation and of distortion.

145. Plane Waves. If a disturbance is produced at a point of an elastic medium, waves radiate from this point in all directions. At a great distance from the center of disturbance, however, such waves can be considered as *plane waves*, and it may be assumed that all particles are moving parallel to the direction of wave propagation (longitudinal waves), or perpendicular to this direction (transverse waves). In the first case we have *waves of dilatation*; in the second, *waves of distortion*.

Considering *longitudinal waves*, if we take the x -axis in the direction of wave propagation, then $v = w = 0$ and u is a function of x only. Equations (265) then give

$$\frac{\partial^2 u}{\partial t^2} = c_1^2 \frac{\partial^2 u}{\partial x^2} \quad (a)$$

This is the same equation as we had before in discussing longitudinal waves in prismatical bars [see Eq. (258), page 439] except that the quantity

$$c = \sqrt{\frac{E}{\rho}}$$

is replaced by the quantity

$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho}}$$

Substituting for λ and G their expressions in terms of E and Poisson's ratio (see pages 10 and 9), this latter quantity can be represented in the form

$$c_1 = \sqrt{\frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)\rho}} \quad (269)$$

It can be seen that c_1 is larger than c . This result is due to the fact that lateral displacement in this case is suppressed, while in the case of a bar it was assumed that longitudinal strain is accompanied by lateral contraction or expansion. The ratio c_1/c depends on the magnitude of Poisson's ratio. For $\nu = 0.25$, $c_1/c = 1.095$; for $\nu = 0.30$, $c_1/c = 1.16$. All the conclusions obtained before regarding the propagation and superposition of longitudinal waves can also be applied in this case.

Consider now *transverse waves*. Assuming that the x -axis is in the direction of wave propagation, and the y -axis in the direction of transverse displacement, we find that the displacements u and w are zero and the displacement v is a function of x and t . Then, from Eqs. (264),

$$\frac{\partial^2 v}{\partial t^2} = c_2^2 \frac{\partial^2 v}{\partial x^2} \quad (b)$$

Again we have an equation of the same form as before, and we can conclude that waves of distortion are being propagated along the x -axis with the velocity

$$c_2 = \sqrt{\frac{G}{\rho}}$$

or, by (269),

$$c_2 = c_1 \sqrt{\frac{1 - 2\nu}{2(1 - \nu)}} \quad (270)$$

For $\nu = 0.25$, the above equation gives

$$c_2 = \frac{c_1}{\sqrt{3}}$$

Any function

$$f(x - c_2 t) \quad (c)$$

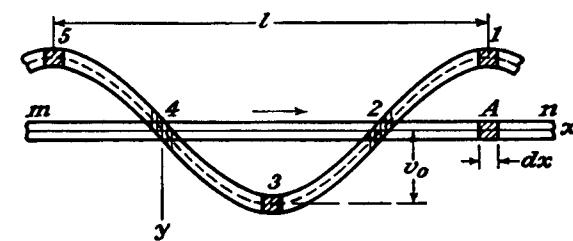


FIG. 243.

is a solution of Eq. (b) and represents a wave traveling in the x -direction with the velocity c_2 . Take, for example, solution (c) in the form

$$v = v_0 \sin \frac{2\pi}{l} (x - c_2 t) \quad (d)$$

The wave has in this case a sinusoidal form. The length of the wave is l and the amplitude v_0 . The velocity of transverse motion is

$$\frac{\partial v}{\partial t} = - \frac{2\pi c_2}{l} v_0 \cos \frac{2\pi}{l} (x - c_2 t) \quad (e)$$

It is zero when the displacement (d) is a maximum and has its largest value when the displacement is zero. The shearing strain produced by the wave is

$$\gamma_{xy} = \frac{\partial v}{\partial x} = \frac{2\pi v_0}{l} \cos \frac{2\pi}{l} (x - c_2 t) \quad (f)$$

It will be seen that the maximum distortion (f) and the maximum of the absolute value of the velocity (e) occur at a given point simultaneously.

We can represent this kind of wave propagation as follows: Let mn (Fig. 243) be a thin thread of an elastic medium. When a sinusoidal

wave (*d*) is being propagated along the *x*-axis, an element *A* of the thread undergoes displacements and distortions, the consecutive values of which are indicated by the shaded elements 1, 2, 3, 4, 5 At the instant $t = 0$, the element *A* has a position as indicated by 1. At this moment its distortion and its velocity are zero. Then it acquires a positive velocity and after an interval of time equal to $\frac{1}{2}l/c_2$ its distortion is as indicated by 2. At this instant the displacement of the element is zero and its velocity is a maximum. After an interval of time equal to $\frac{1}{2}l/c_2$ the conditions are as indicated by 3, and so on.

Assuming that the cross-sectional area of the thread is equal to unity, the kinetic energy of the element *A* is

$$\frac{\rho}{2} \frac{dx}{dt} \left(\frac{\partial v}{\partial t} \right)^2 = \frac{\rho}{2} \frac{dx}{dt} \cdot \frac{4\pi^2 c_2^2}{l^2} v_0^2 \cos^2 \frac{2\pi}{l} (x - c_2 t)$$

and its strain energy is

$$\frac{1}{2} G \gamma_{xy}^2 dx = \frac{G}{2} \frac{4\pi^2 v_0^2}{l^2} \cos^2 \frac{2\pi}{l} (x - c_2 t)$$

Remembering that $c_2^2 = G/\rho$, it can be concluded that the kinetic and the potential energies of the element at any instant are equal. This is the same conclusion as we had before in discussing longitudinal waves in prismatical bars (see page 442).

In the case of an earthquake both kinds of waves, those of dilatation and those of distortion, are propagated through the earth with velocities c_1 and c_2 . They can be recorded by a *seismograph*, and the interval of time between the arrival of these two kinds of waves gives some indication regarding the distance of the recording station from the center of disturbance.¹

146. Propagation of Waves over the Surface of an Elastic Solid Body. In the previous article we discussed propagation of waves in an elastic medium at a distance from the surface. On the surface of an elastic body is it possible to have waves of a different type, which are propagated over the surface and which penetrate but a little distance into the interior of the body. They are similar to waves produced on a smooth surface of water when a stone is thrown into it. Lord Rayleigh, who was the first to investigate these waves,² remarked:

¹ The waves produced in the earth by reciprocating engines are discussed in the following papers: L. Mintrop, Dissertation, Göttingen, 1911; A. Heinrich, Dissertation, Breslau, 1930; G. Bornitz, "Über die Ausbreitung von Bodenschwingungen," Berlin, 1932. See also E. Reissner and H. F. Sagoci, *J. Applied Phys.*, vol. 15, p. 652, 1944; H. F. Sagoci, *ibid.*, p. 655.

² See *Proc. London Math. Soc.*, vol. 17, 1887.

"It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance." The study of records of seismic waves supports Rayleigh's expectation.

At a great distance from the source, the deformation produced by these waves may be considered as a two-dimensional one. We assume that the body is bounded by the plane $y = 0$, and take the positive sense of the *y*-axis in the direction toward the interior of the body and the positive direction of the *x*-axis in the direction of wave propagation. Expressions for the displacements are obtained by combining dilatation waves [Eqs. (265)] and distortion waves [Eqs. (264)]. Assuming in both cases that $w = 0$, the solution of Eqs. (265) representing waves of dilatation can be taken in the form

$$u_1 = se^{-ry} \sin (pt - sx), \quad v_1 = -re^{-ry} \cos (pt - sx) \quad (a)$$

in which p , r , and s are constants. The exponential factor in these expressions indicates that for real positive values of r the amplitude of waves rapidly diminishes with increase of the depth y . The argument $pt - sx$ of the trigonometrical functions shows that the waves are propagated in the *x*-direction with the velocity

$$c_3 = \frac{p}{s} \quad (271)$$

Substituting expressions (a) into Eqs. (265), we find that these equations are satisfied if

$$r^2 = s^2 - \frac{\rho p^2}{\lambda + 2G}$$

or, by using the notation

$$\frac{\rho p^2}{\lambda + 2G} = \frac{p^2}{c_1^2} = h^2 \quad (b)$$

we have

$$r^2 = s^2 - h^2 \quad (c)$$

We take solutions of Eqs. (264), representing waves of distortion in the form

$$u_2 = Abe^{-by} \sin (pt - sx), \quad v_2 = -Ase^{-by} \cos (pt - sx) \quad (d)$$

in which A is a constant and b a positive number. It can be shown that the volume expansion corresponding to the displacements (d) is zero and that Eqs. (264) are satisfied if

$$b^2 = s^2 - \frac{\rho p^2}{G}$$

or, by using the notation

$$\frac{\rho p^2}{G} = \frac{v^2}{c_2^2} = k^2 \quad (e)$$

we obtain

$$b^2 = s^2 - k^2 \quad (f)$$

Combining solutions (a) and (d) and taking $u = u_1 + u_2, v = v_1 + v_2$, we now determine the constants A, b, p, r, s , so as to satisfy the boundary conditions. The boundary of the body is free from external forces, hence, for $y = 0, \bar{X} = 0$ and $\bar{Y} = 0$. Substituting this in Eqs. (134) on page 234 and taking $l = n = 0, m = -1$, we obtain

$$\begin{aligned} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 0 \\ \lambda e + 2G \frac{\partial v}{\partial y} &= 0 \end{aligned} \quad (g)$$

The first of these equations indicates that the shearing stresses, and the second that the normal stresses on the surface of the body, are zero. Substituting the above expressions for u and v in these equations we find that

$$2rs + A(b^2 + s^2) = 0 \quad (h)$$

$$\left(\frac{k^2}{h^2} - 2\right)(r^2 - s^2) + 2(r^2 + Abs) = 0$$

where

$$\frac{k^2}{h^2} - 2 = \frac{\lambda}{G}$$

from (b) and (e).

Eliminating the constant A from Eqs. (h) and using (c) and (f), we obtain

$$(2s^2 - k^2)^2 = 4brs^2 \quad (k)$$

or, by (c) and (f),

$$\left(\frac{k^2}{s^2} - 2\right)^4 = 16 \left(1 - \frac{h^2}{s^2}\right) \left(1 - \frac{k^2}{s^2}\right)$$

By using Eqs. (b), (e), and (271) all the quantities of this equation can be expressed by the velocities c_1 of waves of dilatation, c_2 of waves of distortion, and c_3 of surface waves, and we obtain

$$\left(\frac{c_3^2}{c_2^2} - 2\right)^4 = 16 \left(1 - \frac{c_3^2}{c_1^2}\right) \left(1 - \frac{c_3^2}{c_2^2}\right) \quad (l)$$

Using the notation

$$\frac{c_3}{c_2} = \alpha$$

and remembering that

$$\frac{c_2^2}{c_1^2} = \frac{1 - 2\nu}{2(1 - \nu)}$$

Eq. (l) becomes

$$\alpha^6 - 8\alpha^4 + 8 \left(3 - \frac{1 - 2\nu}{1 - \nu}\right) \alpha^2 - 16 \left[1 - \frac{1 - 2\nu}{2(1 - \nu)}\right] = 0 \quad (m)$$

Taking, for example, $\nu = 0.25$, we obtain

$$3\alpha^6 - 24\alpha^4 + 56\alpha^2 - 32 = 0$$

or

$$(\alpha^2 - 4)(3\alpha^4 - 12\alpha^2 + 8) = 0$$

The three roots of this equation are

$$\alpha^2 = 4, \quad \alpha^2 = 2 + \frac{2}{\sqrt{3}}, \quad \alpha^2 = 2 - \frac{2}{\sqrt{3}}$$

Of these three roots only the last one satisfies the conditions that the quantities r^2 and b^2 , given by Eqs. (c) and (f), are positive numbers. Hence,

$$c_3 = \alpha c_2 = 0.9194 \sqrt{\frac{G}{\rho}}$$

Taking, as an extreme case, $\nu = \frac{1}{2}$, Eq. (m) becomes

$$\alpha^6 - 8\alpha^4 + 24\alpha^2 - 16 = 0$$

and we find

$$c_3 = 0.9553 \sqrt{\frac{G}{\rho}}$$

In both cases the velocity of surface waves is slightly less than the velocity of waves of distortion propagated through the body. Having α , the ratio between the amplitudes of the horizontal and vertical displacements at the surface of the body can easily be calculated. For $\nu = \frac{1}{4}$, this ratio is 0.681. The above velocity of propagation of surface waves can also be obtained by a consideration of the vibrations of a body bounded by two parallel planes.¹

¹ See H. Lamb, *Proc. Roy. Soc. (London)*, Series A, vol. 93, p. 114, 1917. See also S. Timoshenko. *Phil. Mag.*, vol. 43, p. 125, 1922.

APPENDIX

THE APPLICATION OF FINITE-DIFFERENCE EQUATIONS IN ELASTICITY

1. Derivation of Finite-difference Equations. We have seen that the problems of elasticity usually require solution of certain partial differential equations with given boundary conditions. Only in the case of simple boundaries can these equations be treated in a rigorous manner.

Very often we cannot obtain a rigorous solution and must resort to approximate methods. As one of such methods we will discuss here the numerical method, based on the replacement of differential equations by the corresponding finite-difference equations.¹

If a smooth function $y(x)$ is given by a series of equidistant values y_0, y_1, y_2, \dots for $x = 0, x = \delta, x = 2\delta, \dots$, we can, by subtraction, calculate the *first differences* $(\Delta_1 y)_{x=0} = y_1 - y_0, (\Delta_1 y)_{x=\delta} = y_2 - y_1, (\Delta_1 y)_{x=2\delta} = y_3 - y_2, \dots$. Dividing them by the value δ of the interval, we obtain approximate values for the first derivatives of $y(x)$ at the corresponding points:

$$\left(\frac{dy}{dx} \right)_{x=0} \approx \frac{y_1 - y_0}{\delta}, \quad \left(\frac{dy}{dx} \right)_{x=\delta} \approx \frac{y_2 - y_1}{\delta}, \dots \quad (1)$$

¹ It seems that the first application of finite-difference equations in elasticity is due to C. Runge, who used this method in solving torsional problems. (*Z. Math. Phys.*, vol. 56, p. 225, 1908.) He reduces the problem to the solution of a system of linear algebraic equations. Further progress was made by L. F. Richardson, *Trans. Roy. Soc. (London)*, series A, vol. 210, p. 307, 1910, who used for the solution of such algebraic equations a certain iteration process, and so obtained approximate values of the stresses produced in dams by gravity forces and water pressure. Another iteration process and the proof of its convergence was given by H. Liebmann, *Sitzber. Bayer. Akad. Wiss.*, 1918, p. 385. The convergence of this iteration process in the case of harmonic and biharmonic equations was further discussed by F. Wolf, *Z. angew. Math. Mech.*, vol. 6, p. 118, 1926, and R. Courant, *Z. angew. Math. Mech.*, vol. 6, p. 322. The finite-differences method was applied successfully in the theory of plates by H. Marcus, *Armerter Beton*, 1919, p. 107; H. Hencky, *Z. angew. Math. Mech.*, vol. 1, p. 81, 1921, and vol. 2, p. 58, 1922. In recent times the finite-differences method has found very wide application in publications by R. V. Southwell and his pupils. See R. V. Southwell, "Relaxation Methods," vols. I, II, and III.

Using the first differences we calculate the second differences as follows:

$$(\Delta_2 y)_{x=\delta} = (\Delta_1 y)_{x=\delta} - (\Delta_1 y)_{x=0} = y_2 - 2y_1 + y_0$$

With second differences we obtain the approximate values of second derivatives such as

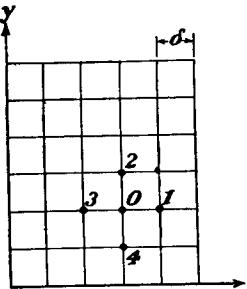


FIG. 1.

$$\left(\frac{d^2 y}{dx^2} \right)_{x=\delta} \approx \frac{(\Delta_2 y)_{x=\delta}}{\delta^2} = \frac{y_2 - 2y_1 + y_0}{\delta^2} \quad (2)$$

If we have a smooth function $w(x, y)$ of two variables, we can use for approximate calculations of partial derivatives equations similar to Eqs. (1) and (2). Suppose, for example, that we are dealing with a rectangular boundary, Fig. 1, and that the numerical values of a function w at the nodal points of a regular *square net* with *mesh side* δ are known to us. Then we can use as approximate values of the partial derivatives of w at a point O the following expressions

$$\begin{aligned} \frac{\partial w}{\partial x} &\approx \frac{w_1 - w_0}{\delta}, & \frac{\partial w}{\partial y} &\approx \frac{w_2 - w_0}{\delta} \\ \frac{\partial^2 w}{\partial x^2} &\approx \frac{w_1 - 2w_0 + w_3}{\delta^2}, & \frac{\partial^2 w}{\partial y^2} &\approx \frac{w_2 - 2w_0 + w_4}{\delta^2} \end{aligned} \quad (3)$$

In a similar manner we can derive also the approximate expressions for partial derivatives of higher order. Having such expressions we can transform partial differential equations into equations of finite differences.

Take as a first example the torsion of prismatical bars. The problem can be reduced, as we have seen,¹ to the integration of the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \quad (4)$$

in which ϕ is the stress function, which must be constant along the boundary of the cross section, θ is the angle of twist per unit length of the bar, and G is the modulus of shear. Using formulas (3) we can transform the above equation into the finite-difference equation

$$\frac{1}{\delta^2} (\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0) = -2G\theta \quad (5)$$

¹ See Eq. (142), p. 261.

In this way every torsional problem reduces to the finding of a set of numerical values for the stress function ϕ which satisfy equation (5) at every nodal point within the boundary of the cross section and become constant along the boundary.

As the simplest example, let us consider a bar of a square cross section $a \times a$, Fig. 2, and use a square net with mesh side $\delta = \frac{1}{4}a$. From symmetry we conclude that in this case it is sufficient to consider only one-eighth of the cross section, shaded in the figure. If we determine the values α, β, γ of the function ϕ at the three points shown in Fig. 2, we shall know ϕ at all nodal points of the net within the boundary. Along the boundary we can assume it equal to zero. Thus the problem reduces to the calculation of three quantities α, β, γ , for which we can write three equations of the form (5). Observing the conditions of symmetry we obtain

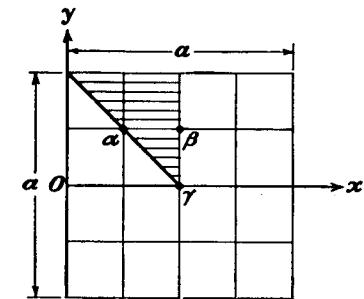


FIG. 2.

$$\begin{aligned} 2\beta - 4\alpha &= -2G\theta\delta^2 \\ 2\alpha + \gamma - 4\beta &= -2G\theta\delta^2 \\ 4\beta - 4\gamma &= -2G\theta\delta^2 \end{aligned}$$

Solving these equations, we find

$$\alpha = 1.375G\theta\delta^2, \quad \beta = 1.750G\theta\delta^2, \quad \gamma = 2.250G\theta\delta^2$$

The required stress function is thus determined by the above numerical values at all nodal points within the boundary and by zero values at the boundary.

To calculate partial derivatives of the stress function we imagine a smooth surface having as ordinates at the nodal points the calculated numerical values. The slope of this surface at any point will then give us the corresponding approximate value of the torsional stress. Maximum stress occurs at the middle of the sides of the boundary square. To get some idea of the accuracy which can be obtained with the assumed small number of nodal points of the net, let us calculate torsional stress at point O , Fig. 2. To get the necessary slope we take a smooth curve having at the nodal points of the x -axis the calculated ordinates β, γ, β . These values, divided by $\frac{1}{4}G\theta\delta^2$, are given in the

second line of the table below. The remaining lines of the table give the values of the consecutive finite differences.¹ The required smooth curve is then given by Newton's interpolation formula:

$x =$	0	δ	2δ	3δ	4δ
$\phi =$	0	7	9	7	0
$\Delta_1 =$	7	2	-2	-7	
$\Delta_2 =$	-5	-4	-5		
$\Delta_3 =$	1	-1			
$\Delta_4 =$	-2				

$$\begin{aligned}\phi = \phi_0 + x \frac{\Delta_1}{\delta} + x(x - \delta) \frac{\Delta_2}{1 \cdot 2 \cdot \delta^2} + x(x - \delta)(x - 2\delta) \frac{\Delta_3}{1 \cdot 2 \cdot 3 \cdot \delta^3} \\ + x(x - \delta)(x - 2\delta)(x - 3\delta) \frac{\Delta_4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \delta^4}\end{aligned}$$

Taking the derivative of ϕ and substituting for $\Delta_1, \Delta_2, \dots$ their values from the table multiplied by $G\theta\delta^2/4$ we obtain, for $x = 0$,

$$\left(\frac{\partial \phi}{\partial x}\right)_{x=0} = \frac{124}{48} G\theta\delta = 0.646 \cdot G\theta\delta$$

Comparing this result with the correct value given on page 277, we see that the error in this case is about 4.3 per cent. To get better accuracy we have to use a *finer net*. Taking, for example, $\delta = a/6$, Fig. 3, we

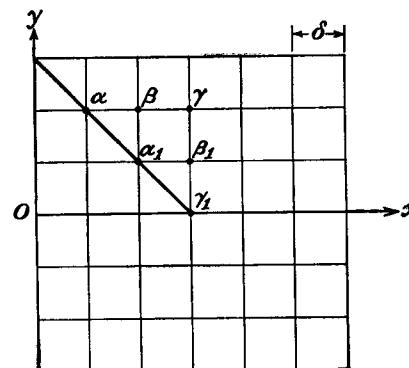


FIG. 3.

have to solve six equations and we obtain

$$\begin{aligned}\alpha &= 0.952 \times 2G\theta\delta^2, & \beta &= 1.404 \times 2G\theta\delta^2, & \gamma &= 1.539 \times 2G\theta\delta^2 \\ \alpha_1 &= 2.125 \times 2G\theta\delta^2, & \beta_1 &= 2.348 \times 2G\theta\delta^2, & \gamma_1 &= 2.598 \times 2G\theta\delta^2\end{aligned}$$

¹ We consider here the differences as all existing at one end of the set of quantities and use them in Newton's formula

Using now seven ordinates along the x -axis and calculating¹ the slope at point O , we obtain the maximum shearing stress

$$\left(\frac{\partial \phi}{\partial x}\right)_{x=0} = 0.661G\theta a$$

The error of this result is about 2 per cent. Having the results for $\delta = \frac{1}{4}a$ and $\delta = \frac{1}{6}a$ a better approximation can be obtained by extrapolation.¹ It can be shown² that the error of the derivative of the stress function ϕ , due to the use of finite difference rather than differential equations, is proportional to the square of the mesh side, when this is small. If the error in maximum stress for $\delta = \frac{1}{4}a$ is denoted by Δ , then for $\delta = \frac{1}{6}a$ it can be assumed equal to $\Delta(\frac{2}{3})^2$. Using the values of maximum stress calculated above we obtain Δ from the equation

$$\Delta - \Delta(\frac{2}{3})^2 = 0.015G\theta a$$

from which

$$\Delta = 0.027G\theta a$$

The more accurate value of the stress is then

$$0.646G\theta a + 0.027G\theta a = 0.673G\theta a$$

which differs from the exact value $0.675G\theta a$ by less than $\frac{1}{2}$ per cent.

2. Methods of Successive Approximation. From the simple example of the preceding article it is seen that, to increase the accuracy of the finite-difference method, we must go to finer and finer nets. But then the number of equations which must be solved becomes larger and larger and the time required for their solution will be so great that the method becomes unpractical.³ The solution of the equations can be greatly simplified by using a method of successive approximations. To illustrate this let us consider the equation⁴

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad (6)$$

¹ The calculation of derivatives of an interpolation curve is greatly simplified by using the tables calculated by W. G. Bickley. These tables are given in the book "Relaxation Methods in Theoretical Physics," by R. V. Southwell.

² See L. F. Richardson, *loc. cit.*

³ In the previously mentioned paper by C. Runge a system of 42 equations was used and, due to the simplicity of these equations, the solution was obtained without much difficulty.

⁴ It was shown on p. 265 that torsional problems can be reduced to the solution of this equation with prescribed values of ϕ at the boundary.

The corresponding finite-difference equation, from Eq. (5), will be

$$\phi_0 = \frac{1}{4}(\phi_1 + \phi_2 + \phi_3 + \phi_4) \quad (7)$$

It shows that the true value of the function ϕ at any nodal point O of the square net is equal to the average value of the function at the four adjacent nodal points. This fact will now be utilized for calculation of the values of ϕ by successive approximations. Let us take again, as the simplest example, the case of a square boundary, Fig. 4, and assume that the boundary values of ϕ are such as shown in the figure. From the symmetry of these values with respect to the vertical central axis we conclude that ϕ also will be symmetrical with respect to the

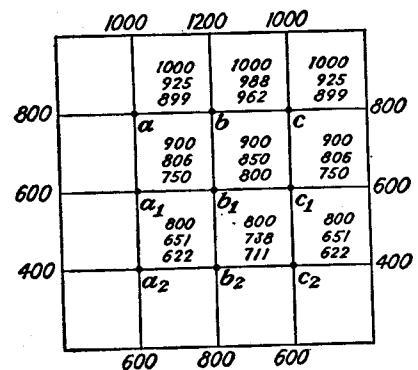


FIG. 4.

same axis. Thus we have to calculate only six nodal values, a , b , a_1 , b_1 , a_2 , b_2 , of ϕ . This can easily be done by writing and solving six equations (7), which are simple in this case and which give $\phi_a = 854$, $\phi_b = 914$, $\phi_{a_1} = 700$, $\phi_{b_1} = 750$, $\phi_{a_2} = 597$, $\phi_{b_2} = 686$.¹ Instead of this we can proceed as follows: We assume some values of ϕ , for instance, those given by the top numbers in each column written in Fig. 4. To get a better

approximation for ϕ , we use for each nodal point Eq. (7). Considering point a we take, as a first approximation, the value

$$\phi_a' = \frac{1}{4}(800 + 1,000 + 1,000 + 900) = 925$$

In calculating a first approximation for point b we use the calculated value ϕ_a' and also the condition of symmetry, which requires that $\phi_c' = \phi_a'$. Equation (7) then gives

$$\phi_b' = \frac{1}{4}(925 + 1,200 + 925 + 900) = 988$$

Making similar calculations for all inner nodal points, we obtain the first approximations given by the second (from the top) numbers in each column. Using these numbers we can calculate the second approximations such as

$$\begin{aligned}\phi_a'' &= \frac{1}{4}(800 + 1,000 + 988 + 806) = 899 \\ \phi_b'' &= \frac{1}{4}(899 + 1,200 + 899 + 850) = 962\end{aligned}$$

¹ We make the calculation with three figures only and neglect decimals.

The second approximations are also written in Fig. 4 and we can see how the successive approximations gradually approach the correct values given above. After repeating such calculations 10 times we obtain in this case results differing from the true values by no more than one unit in the last figure and we can accept this approximation.

Generally, the number of repetitions of the calculation necessary to get a satisfactory approximation depends very much on the selection of the initial values of the function ϕ . The better the starting set of values the less will be the labor of subsequent corrections.

It is advantageous to begin with a *coarse net* having only few internal nodal points. The values of ϕ at these points can be obtained by direct solution of Eqs. (7) or by the iteration process described above. After this we can advance to a finer net, as illustrated in Fig. 5, in which heavy lines represent the coarser net. Having the values of ϕ for the nodal points, shown by small circles, and applying Eq. (7), we calculate the values for the points marked by crosses. Using now both sets of values marked by circles and by crosses and applying again Eq. (7) we obtain values for points marked by small black circles. In this way all values for the nodal points of the finer net, shown by thin lines, will be determined, and we can begin the iteration process on the finer net.

Instead of calculating the values of ϕ , we can calculate the corrections ψ to the initially assumed values ϕ^0 of the function ϕ .¹ In such a case

$$\phi = \phi^0 + \psi$$

Since the function ϕ satisfies Eq. (6) the sum $\phi^0 + \psi$ also must satisfy it, and we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = - \left(\frac{\partial^2 \phi^0}{\partial x^2} + \frac{\partial^2 \phi^0}{\partial y^2} \right) \quad (8)$$

At the boundary the values of ϕ are given to us, which means that there the corrections ψ are zero. Thus the problem is now to find a function ψ satisfying Eq. (8) at each internal point and vanishing at

¹ This method simplifies the calculations since we will have to deal with comparatively small numbers.

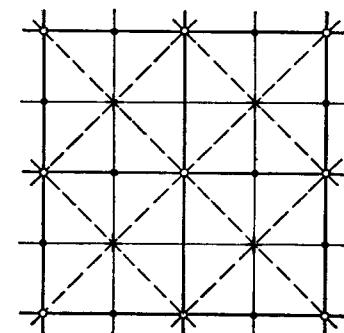


FIG. 5.

the boundary. Replacing Eq. (8) by the corresponding finite-difference equation we obtain for any point O of a square net (Fig. 1)

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0 = -(\phi_1^0 + \phi_2^0 + \phi_3^0 + \phi_4^0 - 4\phi_0^0) \quad (8')$$

The right-hand side of this equation can be evaluated for each internal nodal point by using the assumed values ϕ^0 of the function ϕ . Thus the problem of calculating the corrections ψ reduces to the solution of a system of equations similar to Eqs. (5) of the preceding article, and these can be treated by the iteration method.

3. Relaxation Method. A useful method for treating difference equations, such as Eqs. (8') in the preceding article, was developed by R. V. Southwell and was called by him the *relaxation method*. Southwell begins with Prandtl's membrane analogy,¹ which is based on the fact that the differential equation (4) for torsional problems has the same form as the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{q}{S} \quad (9)$$

for the deflection of a uniformly stretched and laterally loaded membrane. In this equation w denotes deflection from the initially horizontal plane surface of the membrane, q is the intensity of the distributed load, and S is the constant tensile force per unit length of the boundary of the membrane. The problem is to find the deflection w as a function of x and y which satisfies Eq. (9) at every point of the membrane and which vanishes at the boundary.

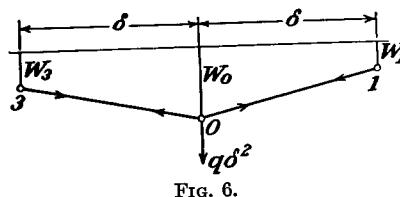


FIG. 6.

Let us derive now the corresponding finite-difference equation. For this purpose we replace the membrane by the square net of uniformly stretched strings, Fig. 1. Considering point O and denoting by $S\delta$ the tensile force in

the strings, we see that the strings $O-1$ and $O-3$ exert on the node O , Fig. 6, a force in the upward direction, equal to²

$$S\delta \left(\frac{w_0 - w_1}{\delta} + \frac{w_0 - w_3}{\delta} \right) \quad (10)$$

A similar expression can be written for the force exerted by the two other strings, $O-2$ and $O-4$. Replacing the continuous load acting on

¹ See p. 268.

² We consider the deflections as very small.

the membrane by concentrated forces $q\delta^2$ applied at nodal points, we can now write the equation of equilibrium of a nodal point O as follows:

$$q\delta^2 + S(w_1 + w_2 + w_3 + w_4 - 4w_0) = 0 \quad (11)$$

This is the finite-difference equation, corresponding to the differential equation (9). To solve the problem, we have to find such a set of values of the deflections w that equation (11) will be satisfied at every nodal point.

We begin with some starting values $w_0^0, w_1^0, w_2^0, w_3^0, w_4^0, \dots$ of the deflection. Substituting them into Eqs. (11) we shall usually find that the conditions of equilibrium are not satisfied and that, to maintain the assumed deflections of the net, we need to introduce supports at the nodal points. The quantities such as

$$R_0 = q\delta^2 + S(w_1^0 + w_2^0 + w_3^0 + w_4^0 - 4w_0^0) \quad (12)$$

will then represent the portions of the load transmitted to the supports. We call these forces *residual forces*, or residuals. Imagine now that the supports are of the *screw-jack* type, so that a controlled displacement may be imposed at any desired nodal point. Then by proper displacements of the supports we can ultimately make all residual forces (12) vanish. Such displacements will then represent the corrections which must be added to the initially assumed deflections $w_0^0, w_1^0, w_2^0, \dots$ to get the true values of w .

The procedure which Southwell follows in manipulating the displacements of the supports is similar to that developed by Calisev¹ in handling highly statically indeterminate frames. We first displace one of the supports, say support O , Fig. 6, keeping the other supports fixed. From such equations as (11) we can see that to a downward displacement w_0' will correspond a vertical force $-4Sw_0'$ acting on the nodal point O . The minus sign indicates that the force acts upward. Adjusting the displacement so that

$$R_0 - 4Sw_0' = 0, \quad \text{that is,} \quad w_0' = \frac{R_0}{4S} \quad (13)$$

we make the residual force (12) vanish and there will no longer be a pressure transmitted to the support O , but at the same time pressures Sw_0' will be transferred to the adjacent supports and their residual forces will be increased by this amount. Proceeding in the same way

¹ K. A. Calisev, Tehnicki List, 1922 and 1923, Zagreb. German translation in Pubs. Intern. Assoc. Bridge and Structural Engineering, vol. 4, p. 199, 1936. A similar method was developed in this country by Hardy Cross.

with all the other supports and repeating the procedure several times, we can reduce all residual forces to small quantities, which can be neglected. The total displacements of the supports, accumulated in this procedure, represent then the corrections which must be added with the proper signs to the starting values $w_0^0, w_1^0, w_2^0, \dots$ in order to obtain the true deflections of the stretched square net.

To simplify the calculations required by the procedure described, we first put Eq. (11) in nondimensional form by substituting

$$w = \frac{q\delta^2}{S} \psi \quad (14)$$

In this way we obtain

$$1 + (\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0) = 0 \quad (15)$$

where ψ_0, ψ_1, \dots are pure numbers.

The problem then reduces to finding such a set of values of ψ that Eq. (15) will be satisfied at all inner points of the net. At the boundary ψ is zero. To get the solution we proceed in the manner described above and take some starting values $\psi_0^0, \psi_1^0, \psi_2^0, \dots$. They will not satisfy the equilibrium equations (15) and we shall have residuals

$$r_0 = 1 + (\psi_1^0 + \psi_2^0 + \psi_3^0 + \psi_4^0 - 4\psi_0^0) \quad (16)$$

which in this case are pure numbers.

Our problem now is to add to the assumed values $\psi_0^0, \psi_1^0, \psi_2^0, \dots$ such corrections as to annul all residuals. Adding to ψ_0^0 a correction ψ_0' we add to the residual r_0 the quantity $-4\psi_0'$, and to the residuals of the adjacent nodal points the quantities ψ_0' . Taking $\psi_0' = r_0/4$ we shall annul the residual at the nodal point O and shall somewhat change the residuals at the adjacent nodal points. Proceeding in the same way with all nodal points and repeating the procedure many times we shall in due course reduce the residual forces to negligible values and so obtain the values of ψ with sufficient accuracy. The corresponding values of w will then be obtained from Eq. (14).

To illustrate the procedure let us consider the problem of torsion of a square bar, already discussed in Art. 1. In this case we have the differential equation (4). To bring it to nondimensional form let us put

$$\phi = \frac{2G\delta^2}{1,000} \psi \quad (17)$$

The finite-difference equation (5) then becomes

$$1,000 + (\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0) = 0 \quad (18)$$

The denominator 1,000 is introduced into Eq. (17) for the purpose of making the ψ 's such large numbers that half a unit of the last figure can be neglected. Thus we have to deal with integer numbers only. To make our example as simple as possible we will start with the coarse net of Fig. 2. Then we have to find values of ψ only for three internal points for which we already have the correct answers (see page 463). We make our square net to a large scale to have enough space to put on the sketch the results of all intermediate calculations (Fig. 7). The calculation starts with assumed initial values of ψ , which we write to

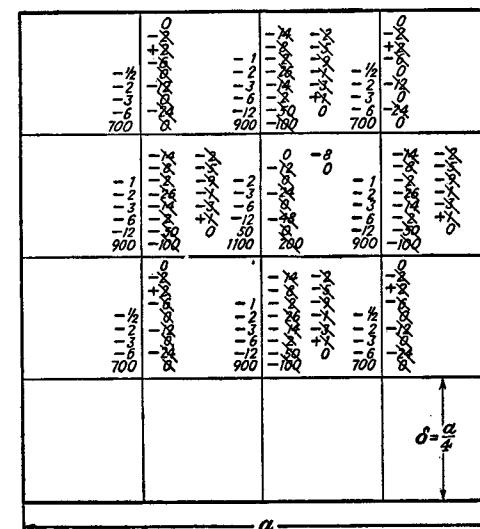


FIG. 7.

the left above each nodal point. The values 700, 900, and 1,100 are intentionally taken somewhat different from the previously calculated correct values. Substituting these values together with the zero values at the boundary into the left-hand side of Eq. (18) we calculate the residual forces for all nodal points. These forces are written above each nodal point to the right. The largest residual force, equal to 200, occurs at the center of the net, and we start our relaxation process from this point. Adding to the assumed value 1,100 a correction 50, which is written in the sketch above the number 1,100, we eliminate entirely the residual force at the center. Thus we cross out the number 200 in the sketch and put zero instead. Now we have to change the residuals in the adjacent nodes. We add 50 to each of those residuals and write the new values -50 of the residuals above the original values as shown in the figure. This finishes the operation with the central point of the

net. We have now four symmetrically located nodal points with residuals -50 and it is of advantage to make corrections to all of them simultaneously. Let us take for all these points the same correction, equal to -12 .¹ These corrections are written in the sketch above the initial values, 900 . With these corrections the values $12 \times 4 = 48$ must be added to the previous residuals, equal to -50 , and we will obtain residuals equal to -2 , as shown in the sketch. At the same time the forces -12 will be added to the residuals in the adjacent points. Thus, as it is easy to see, $-12 \times 4 = -48$ must be added to the residual at the center and $-12 \times 2 = -24$ must be added at the points closest to the corners of the figure. This finishes the first round of our calculations. The second round we again begin with the point at the center and make the correction -12 , which eliminates the residual at this point and adds -12 to the residuals of the adjacent points. Taking now the points near the corners and introducing corrections -6 , we eliminate the residuals at these points and make the residuals equal to -26 at the four symmetrically located points. To finish the second round we introduce corrections -6 at these points. The sketch shows three more rounds of calculations which result in further reduction of the residuals. The required values of ψ will be obtained by adding to the starting values all the corrections introduced. Thus we obtain

$$700 - 6 - 3 - \frac{1}{2} = 688.5, \quad 900 - 12 - 6 - 3 - 2 - 1 = 876,$$

$$1,100 + 50 - 12 - 6 - 3 - 2 = 1,127$$

Equation (17) then gives for ϕ the values

$$\frac{688.5}{500} G\theta\delta^2 = 1.377G\theta\delta^2 = 0.0861G\theta a^2$$

$$\frac{876}{500} G\theta\delta^2 = 1.752G\theta\delta^2 = 0.1095G\theta a^2$$

$$\frac{1,127}{500} G\theta\delta^2 = 2.254G\theta\delta^2 = 0.1409G\theta a^2$$

which are in very good agreement with the results previously obtained (see page 463).

It is seen that Southwell's method gives us a physical picture of the iteration process of solving Eqs. (15) which may be helpful in selecting the proper order in which the nodes of the net should be manipulated.

¹ We take the correction -12 , instead of $-\frac{50}{4} = -12.5$, since it is preferable to work with integer numbers.

To get a better approximation we must advance to a finer net. Using the method illustrated in Fig. 5, we get starting values of ψ for a square net with mesh side $\delta = \frac{1}{8}a$. Applying to these values the standard relaxation process the values of ψ for the finer net can be obtained and a more accurate value of the maximum stress can be calculated. With the two values of maximum stress found for $\delta = \frac{1}{4}a$ and $\delta = \frac{1}{8}a$ a better approximation can be found by extrapolation, as explained in Art. 1 (see page 465).

4. Triangular and Hexagonal Nets. In our previous discussion a square net was used, but sometimes it is preferable to use a triangular or hexagonal net, Figs. 8a and 8b. Considering the triangular net,

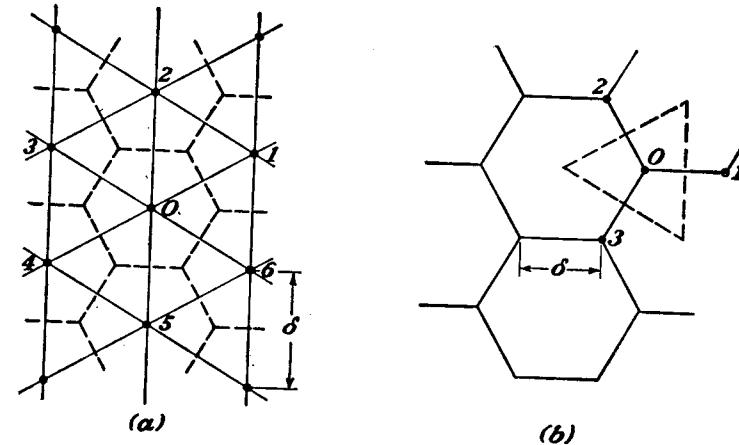


FIG. 8.

Fig. 8a, we see that the distributed load within the hexagon shown by dotted lines will be transferred to the nodal point O . If δ denotes the mesh side, the side of the above hexagon will be equal to $\delta/\sqrt{3}$ and the area of the hexagon is $\sqrt{3}\delta^2/2$, so that the load transferred to each nodal point will be $\sqrt{3}\delta^2q/2$. This load will be balanced by forces in the strings $0-1, 0-2, \dots, 0-6$. To make the string net correspond to the uniformly stretched membrane, the tensile force in each string must be equal to the tensile force in the membrane transmitted through one side of the hexagon, i.e., equal to $S\delta/\sqrt{3}$. Proceeding now as in the preceding article, we obtain for the nodal point O the following equation of equilibrium:

$$\frac{w_1 + w_2 + \dots + w_6 - 6w_0}{\delta} \cdot \frac{S\delta}{\sqrt{3}} + \frac{\sqrt{3}q\delta^2}{2} = 0$$

or

$$w_1 + w_2 + \dots + w_6 - 6w_0 + \frac{3}{2} \frac{q\delta^2}{S} = 0 \quad (19)$$

We introduce a nondimensional function ψ defined by the equation

$$w = \frac{3}{2} \frac{q\delta^2}{S} \psi \quad (20)$$

and the finite-difference equation becomes

$$\psi_1 + \psi_2 + \dots + \psi_6 - 6\psi_0 + 1 = 0 \quad (21)$$

Such an equation can be written for each internal nodal point, and for the solution of these equations we can, as before, use iteration or relaxation methods.

In the case of a hexagonal net, Fig. 8b, the load distributed over the equilateral triangle, shown in the figure by dotted lines, will be transferred to the nodal point O . Denoting by δ the length of the mesh side, we see that the side of the triangle will be $\delta\sqrt{3}$ and its area $3\sqrt{3}\delta^2/4$. The corresponding load is $3\sqrt{3}q\delta^2/4$. This load will be balanced by the tensile forces in the three strings, 0-1, 0-2, 0-3. To make the string net correspond to the stretched membrane we take the tensile forces in the strings equal to $S\delta\sqrt{3}$.

The equation of equilibrium will then be

$$\frac{w_1 + w_2 + w_3 - 3w_0}{\delta} \cdot S\delta\sqrt{3} + \frac{3\sqrt{3}q\delta^2}{4} = 0$$

or

$$w_1 + w_2 + w_3 - 3w_0 + \frac{3}{4} \frac{q}{S} \delta^2 = 0 \quad (22)$$

To get the finite-difference equations for torsional problems we have to substitute in Eqs. (19) and (22) $2G\theta$ instead of q/S .

As an example let us consider torsion of a bar the cross section of which is an equilateral triangle,¹ Fig. 9. The rigorous solution for this case is given on page 266.

Using the relaxation method it is natural to select for this case a triangular net. Starting with a coarse net we take the mesh side δ equal to one-third of the length a of the side of the triangle. Then

This example is discussed in detail in Southwell's book, referred to above.

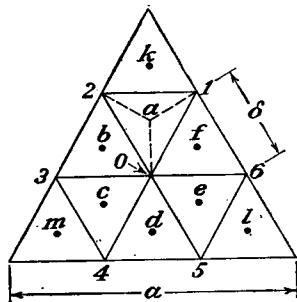


FIG. 9.

there will be only one internal point O of the net, and the values of the required stress function ϕ are zero at all adjacent nodal points 1, 2, ..., 6 since these points are on the boundary. The finite-difference equation for point O is then obtained from Eq. (19) by substituting ϕ_0 for w_0 and $2G\theta$ for q/S , which gives

$$6\phi_0 = 3G\theta\delta^2 = \frac{G\theta a^2}{3}$$

and

$$\phi_0 = \frac{G\theta a^2}{18} \quad (23)$$

Let us now advance to a finer net. To get some starting values for such a net let us consider point a — the centroid of the triangle 1-2-0. Assume that this point is connected to the nodal points 0, 1, and 2 by the three strings $a-0$, $a-1$, $a-2$ of length $\delta/\sqrt{3}$. Considering the point a as a nodal point of a hexagonal net, Fig. 8b, substituting into Eq. (22) $\delta/\sqrt{3}$ for δ , $2G\theta$ for q/S , and taking $w_1 = w_2 = 0$, $w_3 = \phi_0$, $w_0 = \phi_a$, we obtain

$$\phi_a = \frac{1}{3} \left(\phi_0 + \frac{G\theta\delta^2}{2} \right) = \frac{G\theta a^2}{27} \quad (24)$$

The same values of the stress function may be taken also for the points b , c , d , e , and f in Fig. 9. To get the values of the stress function at points k , l , m , we use again Eq. (22) and, observing that in this case $w_1 = w_2 = w_3 = 0$, we find

$$\phi_k = \phi_m = \phi_l = \frac{G\theta a^2}{54} \quad (25)$$

In this way we find the values of ϕ at all nodal points marked by small circles in Fig. 10. It is seen that at each of the nodal points a , c , and e there are six strings as required in a triangular net, Fig. 8a. But at the remaining nodes the number of strings is smaller than six. To satisfy the conditions of a triangular net at all internal points we proceed as indicated by dotted lines in the upper portion of Fig. 10. In this way the cross section will be divided into equilateral triangles with sides $\delta = a/9$. From symmetry we conclude that it is sufficient to consider only one-sixth of the cross section, which is shown in Fig. 11a. The values of ϕ at the nodal points O , a , b , and k are already determined. The values at the points 1, 2, and 3 will now be determined, as before,

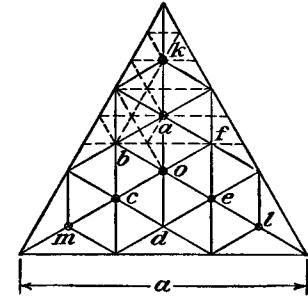


FIG. 10.

by using Eq. (22) and the values of ϕ at three adjacent points. For point 1, for example, we will get

$$\phi_0 + \phi_b + \phi_a - 3\phi_1 + \frac{3}{4} \cdot 2G\theta \left(\frac{a}{9}\right)^2 = 0$$

and substituting for ϕ_a , ϕ_b , ϕ_0 the previously calculated values, we obtain

$$\phi_1 = \frac{4}{81} \cdot G\theta a^2 \quad (26)$$

In a similar way ϕ_2 and ϕ_3 are calculated. All these values are written down to the left of the corresponding nodal points in Fig. 11a.¹ They will now be taken as starting values in the relaxation process.

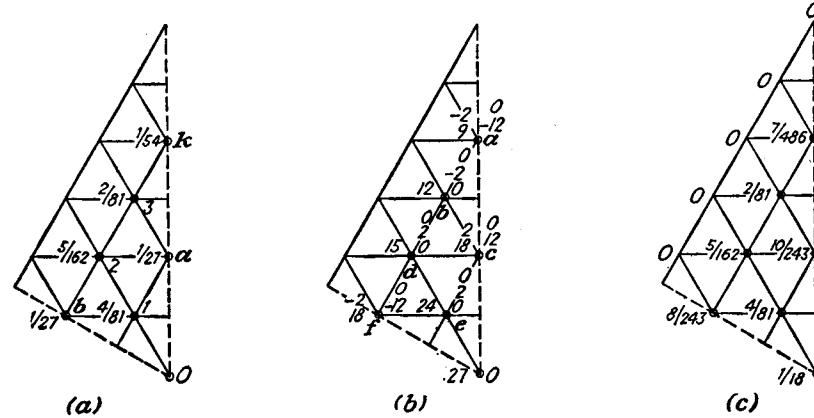


FIG. 11.

In the case of torsion, Eq. (19) will be replaced by the equation

$$\phi_1 + \phi_2 + \cdots + \phi_6 - 6\phi_0 + 3G\theta \frac{a^2}{81} = 0$$

To bring it into purely numerical form we introduce the notation

$$\phi = \frac{G\theta a^2 \psi^*}{486} \quad \text{or} \quad \psi = \frac{486\phi}{G\theta a^2} \quad (27)$$

and obtain

$$\psi_1 + \psi_2 + \cdots + \psi_6 - 6\psi_0 + 18 = 0 \quad (28)$$

The starting values of ψ , calculated from Eq. (27), are written to the left of the nodal points in Fig. 11b. Substituting these values into the

¹ The constant factor $G\theta a^2$ is omitted in the figure.

* The number 486 is introduced in this formula so that we may work with integers only.

left-hand side of Eq. (28), we find the corresponding residuals

$$R_0 = \psi_1^0 + \psi_2^0 + \cdots + \psi_6^0 - 6\psi_0^0 + 18 \quad (29)$$

The residuals, calculated in this way, are written to the right of each nodal point in Fig. 11b. The liquidation of these residuals is begun with point a . Giving to this point a displacement $\psi'_a = -2$ we add [see Eq. (29)] +12 to the residual at a and -2 to the residuals at all adjacent points. Thus the residual at a is liquidated and a residual -2 appears at point b . We are not concerned with residuals at the boundary, since there we have permanent supports. Considering now the point c and introducing there a displacement +2 we bring to zero the residual there and add +2 to the residuals at b , d , and e . All the remaining residuals will now be brought to zero by imposition of a dis-

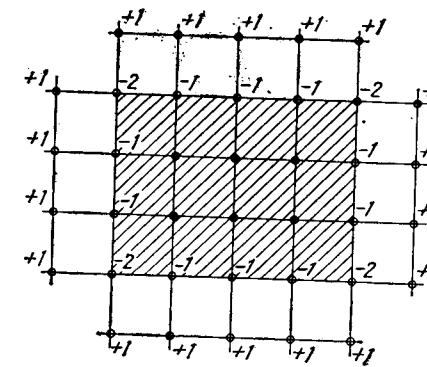


FIG. 12.

placement -2 at point f . Adding to the starting values of ψ all recorded corrections, we obtain the required values of ψ , and from Eq. (27) we obtain the values of ϕ . These values, divided by $G\theta a^2$, are shown in Fig. 11c. They coincide with the values which can be obtained from the rigorous solution (g) on page 266.

5. Block and Group Relaxation. The operation used up to now in liquidating the residuals consisted in manipulation of single nodal points, considering the rest of the points as fixed. Sometimes it is better to move a group of nodal points simultaneously. Assume, for example, that Fig. 12 represents a portion of a square net and that we give to all points within the shaded area a displacement equal to unity while the rest of the nodal points remain fixed. We can imagine that all nodal points of the shaded area are attached to an absolutely rigid weightless plate and that this plate is given a unit displacement, perpendicular to the plate. From considerations of equilibrium (Fig. 6),

we conclude that the displacement described will produce changes of residuals at the end points of the strings attaching the shaded plate to the remaining portion of the net. If O and 1 denote the nodes at the ends of one string, the contributions to the residuals due to displacement w_0 and w_1 are

$$R_0 = -S\delta \frac{w_0 - w_1}{\delta} \quad \text{and} \quad R_1 = S\delta \frac{w_0 - w_1}{\delta}$$

If now we keep point 1 fixed and give to point O an additional displacement Δw_0 , we get the increments of the residual forces

$$\Delta R_0 = -S \Delta w_0, \quad \Delta R_1 = S \Delta w_0$$

Introducing dimensionless quantities according to our previous notation,

$$\frac{R}{q\delta^2} = r, \quad w = \frac{q\delta^2}{S} \psi$$

we find

$$\Delta r_0 = -\Delta\psi_0, \quad \Delta r_1 = \Delta\psi_0$$

We see that unit increment in ψ_0 produces changes in the residuals equal to

$$\Delta r_0 = -1, \quad \Delta r_1 = +1$$

These changes are shown in the figure. The residuals of the rest of the nodal points of the net remain unchanged. If n denotes the number of strings attaching the shaded plate to the rest of the net, the unit displacement of the plate results in diminishing by n the resultant of the residual forces of the shaded portion of the net. Choosing the displacement so that the resultant vanishes we get residual forces which are self-equilibrating and as such lend themselves more readily to liquidation by subsequent point relaxation of the normal kind. In

practical applications it is advantageous to alternate sequences of block displacement with sequences of point relaxation. Assume, for example, that the shaded area in Fig. 13 represents a portion of the triangular net. The number n of strings attaching this portion to the rest of the net is 16 and the resultant of the residuals shown in the figure is 8.8. Consequently an appropriate block displacement in this case will be $8.8/16 = 0.55$. After such a displacement the resultant of the residual forces, acting on the shaded portion of the net, vanishes

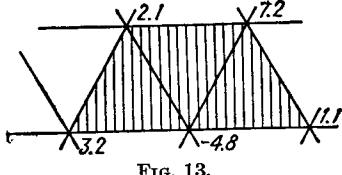


FIG. 13.

and the liquidation of the residuals by subsequent point relaxation will proceed more rapidly.

Instead of giving the fictitious plate a displacement perpendicular to the plate and constant for all the nodal points attached to the plate, we can rotate the plate about an axis lying in its plane. The corresponding displacements of nodal points and changes of residuals can be readily calculated. So we can liquidate not only the resultant residual force sustained by the fictitious plate but also the resultant moment about any axis chosen in the plane of the plate.

We can also discard the notion of the fictitious plate and assign to a group of points arbitrary selected displacements. If we have some idea of the shape of the deflection surface of the net we can select group displacements which may result in acceleration of the liquidation process.

6. Torsion of Bars with Multiply-connected Cross Sections. It was shown¹ that in the case of bars with multiply-connected cross sections the stress function ϕ must not only satisfy Eq. (4), but along the boundary of each hole we must have

$$-\int \frac{\partial \phi}{\partial n} ds = 2G\theta A \quad (30)$$

where A denotes the area of the hole.

In using the membrane analogy the corresponding equation is

$$-S \int \frac{\partial w}{\partial n} ds = qA \quad (31)$$

which means that the load uniformly distributed over the area of the hole² is balanced by the tensile forces in the membrane. Now applying finite-difference equations and considering a square net, we put $S\delta$ for the tension in the strings, w_0 for the deflection of the boundary of the hole, and w_i for the deflection of a nodal point i adjacent to the hole. Instead of Eq. (31) we then have

$$S\delta \sum \frac{(w_i - w_0)}{\delta} + qA = 0$$

or

$$S \left(\sum_{i=1}^n w_i - nw_0 \right) + qA = 0 \quad (32)$$

¹ See p. 296.

² The hole is represented by a weightless absolutely rigid plate which can move perpendicularly to the initial plane of the stretched membrane.

where n is the number of strings attaching the area of the hole to the rest of the net. The equilibrium equation (11) is only a particular case of Eq. (32) in which $n = 4$.

We can write as many equations (32) as there are holes in the cross section. These equations together with Eqs. (11) written for each nodal point of the square net are sufficient for determining the deflections of all nodal points of the net, and of all the boundaries of the holes.

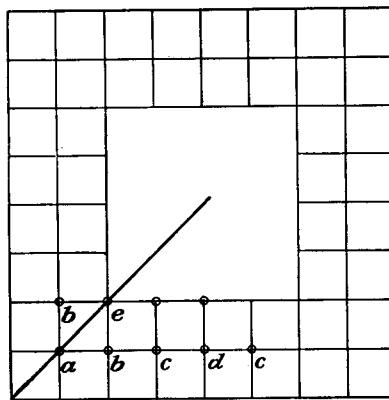


FIG. 14.

Consider as an example the case of a square tube, the cross section of which is represented in Fig. 14. Taking the coarse square net, shown in the figure, and considering the conditions of symmetry, we observe that it is necessary in this case to calculate only five values, a , b , c , d , and e , of the stress function. The necessary equations will be obtained by using Eq. (32) and the four Eqs. (11) written for the nodal points a , b , c , d . Substituting $2G\theta$ for q/S and observing that $n = 20$

and $A = 16\delta^2$ we write these equations as follows:

$$\begin{aligned} 20e - 8b - 8c - 4d &= 16 \cdot 2G\theta\delta^2 \\ 2b - 4a &= -2G\theta\delta^2 \\ a - 4b + c + e &= -2G\theta\delta^2 \\ b - 4c + d + e &= -2G\theta\delta^2 \\ 2c - 4d + e &= -2G\theta\delta^2 \end{aligned}$$

These equations can be readily solved and in this way we obtain

$$e = \frac{1,170}{488} \cdot 2G\theta\delta^2$$

and also the values a , b , c , and d .

These values, obtained with a coarse net, do not give us the stresses with sufficient accuracy and an advance to a finer net is necessary. The results of such finer calculations, made by the relaxation method, can be found in Southwell's book.¹

7. Points Near the Boundary. In our previous examples the nodal points of the net fall exactly on the boundary and the same standard

¹ R. V. Southwell, "Relaxation Methods in Theoretical Physics," p. 60, Oxford University Press, New York, 1946.

relaxation procedure is used for all the points. Very often the points close to the boundary are connected with it by shorter strings. Due to difference in lengths of strings some changes in equilibrium equations (11) and (19) should be introduced. The necessary changes will now be discussed in connection with the example shown in Fig. 15. A flat specimen with semicircular grooves is submitted to the action of tensile forces uniformly distributed over the ends. Suppose that the difference of principal stresses at each point has been determined by the photoelastic method, as explained in Chap. 5, and that we have to determine the sum of the principal stresses, which, as we have seen (page 465), must satisfy the differential equation (6). For the points at the boundary one of the two principal stresses is known, and using the results of the photoelastic tests, the second principal stress can be calculated, so that the sum of these two stresses along the boundary is known. Thus we have to solve the differential equation (6), the values of ϕ along the boundary being known. In using the finite-difference method and taking a square net we conclude, from symmetry, that only one-quarter of the specimen should be considered. This portion

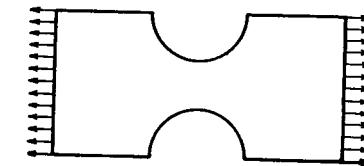


FIG. 15.

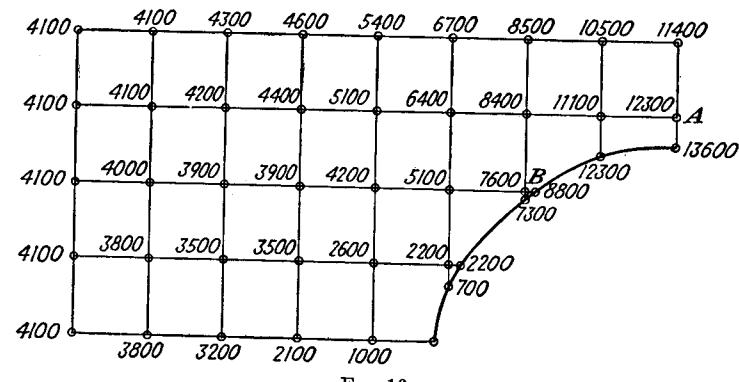


FIG. 16.

with the boundary values of ϕ is shown in Fig. 16. Considering point A of this figure, we see that three strings at that point have standard length δ while the fourth is shorter, say of length $m\delta$ ($m \approx 0.4$ in this case). This must be taken into consideration in the derivation of the equation of equilibrium of point A . This equation must be written as follows:

$$S\delta \left(\frac{\phi_a - \phi_1}{\delta} + \frac{\phi_a - \phi_2}{\delta} + \frac{\phi_a - \phi_3}{\delta} + \frac{\phi_a - \phi_4}{m\delta} \right) = 0$$

or

$$\phi_1 + \phi_2 + \phi_3 + \frac{1}{m} \phi_4 - \left(3 + \frac{1}{m}\right) \phi_a = 0$$

In applying to point *A* the standard relaxation process and giving to ϕ_a an increment equal to unity, we will introduce the changes in residuals shown in Fig. 17a. This pattern must be used in liquidating residuals at point *A*. In considering point *B* we see that there are two shorter strings. Denoting their lengths by $m\delta$ and $n\delta$ we find that in the liquidation of residuals at *B* the pattern shown in Fig. 17b should be used. Introducing these changes at the points near the boundary and using the standard relaxation process at all other points the values of ϕ , shown in Fig. 16, will be obtained.¹

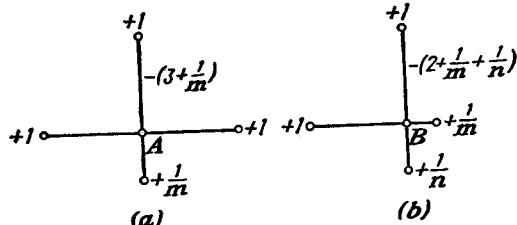


FIG. 17.

In a more general case, when we are dealing with Eq. (9) and there is external load at the nodal points, we denote by $m\delta$, $n\delta$, $r\delta$, $s\delta$ the lengths of the strings at an irregular point *O* of a square net and we assume that the load transmitted to this point is equal to $\frac{q\delta^2}{4}(m+n+r+s)$. The equation of equilibrium then will be

$$\begin{aligned} & \frac{q\delta^2}{4}(m+n+r+s) \\ & + S \left[\frac{w_1}{m} + \frac{w_2}{n} + \frac{w_3}{r} + \frac{w_4}{s} - w_0 \left(\frac{1}{m} + \frac{1}{n} + \frac{1}{r} + \frac{1}{s} \right) \right] = 0 \quad (33) \end{aligned}$$

For $m = n = r = s = 1$ this equation coincides with our previous Eq. (11) derived for a regular point. Using Eq. (33) the proper pattern, similar to those shown in Fig. 17, can be developed in each particular case. An equation, similar to Eq. (33), can be derived also for a triangular net.

¹ This example was treated by R. Weller and G. H. Shortley; see *J. Applied Mechanics*, 1939, p. A-71. Since the boundary values of ϕ , obtained from photoelastic tests, are known only with a comparatively low accuracy, the values of ϕ at internal points are given with no more than two or three significant figures.

With the changes discussed in this article the relaxation process is extended to cases in which we have irregular points near the boundary.

8. Biharmonic Equation. We have seen (page 26) that in the case of two-dimensional problems of elasticity, in the absence of volume forces and with given forces at the boundary, the stresses are defined by a stress function ϕ , which satisfies the biharmonic equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad (34)$$

and the boundary conditions (20) on p. 23, which, in this case, become

$$\begin{aligned} l \frac{\partial^2 \phi}{\partial y^2} - m \frac{\partial^2 \phi}{\partial x \partial y} &= \bar{X} \\ m \frac{\partial^2 \phi}{\partial x^2} - l \frac{\partial^2 \phi}{\partial x \partial y} &= \bar{Y} \end{aligned} \quad (35)$$

Knowing the forces distributed along the boundary we may calculate ϕ at the boundary by integration¹ of Eqs. (35). Then the problem is reduced to that of finding a function ϕ which satisfies Eq. (34) at every point within the boundary and at the boundary has, together with its first derivatives, the prescribed values. Using the finite difference method, let us take a square net (Fig. 18) and transform Eq. (34) to a finite-difference equation. Knowing the second derivatives,

$$\begin{aligned} \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 &\approx \frac{1}{\delta^2} (\phi_1 - 2\phi_0 + \phi_3) \\ \left(\frac{\partial^2 \phi}{\partial x^2} \right)_1 &\approx \frac{1}{\delta^2} (\phi_5 - 2\phi_1 + \phi_0) \\ \left(\frac{\partial^2 \phi}{\partial x^2} \right)_3 &\approx \frac{1}{\delta^2} (\phi_0 - 2\phi_3 + \phi_9) \end{aligned}$$

we conclude that

$$\begin{aligned} \left(\frac{\partial^4 \phi}{\partial x^4} \right)_0 &= \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial x^2} \right) \approx \frac{1}{\delta^2} \left[\left(\frac{\partial^2 \phi}{\partial x^2} \right)_1 - 2 \left(\frac{\partial^2 \phi}{\partial x^2} \right)_0 + \left(\frac{\partial^2 \phi}{\partial x^2} \right)_3 \right] \\ &\approx \frac{1}{\delta^4} (6\phi_0 - 4\phi_1 - 4\phi_3 + \phi_5 + \phi_9) \end{aligned}$$

¹ We consider here only simply connected regions.

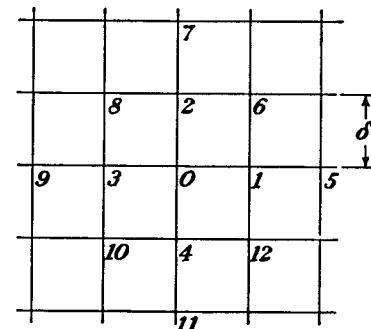


FIG. 18.

Similarly we find

$$\begin{aligned}\frac{\partial^4 \phi}{\partial y^4} &\approx \frac{1}{\delta^4} (6\phi_0 - 4\phi_2 - 4\phi_4 + \phi_7 + \phi_{11}) \\ \frac{\partial^4 \phi}{\partial x^2 \partial y^2} &\approx \frac{1}{\delta^4} [4\phi_0 - 2(\phi_1 + \phi_2 + \phi_3 + \phi_4) + \phi_6 + \phi_8 + \phi_{10} + \phi_{12}]\end{aligned}$$

Substituting into Eq. (34) we obtain the required finite-difference equation

$$20\phi_0 - 8(\phi_1 + \phi_2 + \phi_3 + \phi_4) + 2(\phi_6 + \phi_8 + \phi_{10} + \phi_{12}) + \phi_5 + \phi_7 + \phi_9 + \phi_{11} = 0 \quad (36)$$

This equation must be satisfied at every nodal point of the net within the boundary of the plate. To find the boundary values of the stress function ϕ we integrate Eqs. (35). Observing (Fig. 20, p. 23) that

$$l = \cos \alpha = \frac{dy}{ds} \quad \text{and} \quad m = \sin \alpha = -dx/ds$$

we write Eqs. (35) in the following form:

$$\begin{aligned}\frac{dy}{ds} \frac{\partial^2 \phi}{\partial y^2} + \frac{dx}{ds} \frac{\partial^2 \phi}{\partial x \partial y} &= \frac{d}{ds} \left(\frac{\partial \phi}{\partial y} \right) = \bar{X} \\ -\frac{dx}{ds} \frac{\partial^2 \phi}{\partial x^2} - \frac{dy}{ds} \frac{\partial^2 \phi}{\partial x \partial y} &= -\frac{d}{ds} \left(\frac{\partial \phi}{\partial x} \right) = \bar{Y}\end{aligned} \quad (37)$$

and by integration we obtain

$$\begin{aligned}-\frac{\partial \phi}{\partial x} &= \int \bar{Y} ds \\ \frac{\partial \phi}{\partial y} &= \int \bar{X} ds\end{aligned} \quad (38)$$

To find ϕ we use the equation

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x} \frac{dx}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds}$$

which, after integration by parts, gives

$$\phi = x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} - \int \left(x \frac{\partial^2 \phi}{\partial s \partial x} + y \frac{\partial^2 \phi}{\partial s \partial y} \right) ds \quad (39)$$

Substituting in this equation the values of the derivatives given by Eqs. (37) and (38), we can calculate the boundary values of ϕ . It should be noted that in calculating the first derivatives (38), two constants of integration, say A and B , will appear and the integration in

Eq. (39) will introduce a third constant, say C , so that the final expression for ϕ will contain a linear function $Ax + By + C$. Since the stress components are represented by the second derivatives of ϕ , this linear function will not affect the stress distribution and the constants A , B , C can be taken arbitrarily.

From the boundary values of ϕ and its first derivatives we can calculate the approximate values of ϕ at the nodal points of the net adjacent to the boundary, such as points A , C , E in Fig. 19. Having, for example, at point B the values ϕ_B and $(\partial \phi / \partial x)_B$, we obtain

$$\phi_C = \phi_B + \left(\frac{\partial \phi}{\partial x} \right)_B \delta, \quad \phi_A = \phi_B - \left(\frac{\partial \phi}{\partial x} \right)_B \delta$$

Similar formulas can be written also for point E . A better approximation for these quantities can be obtained later when, by further calculation, the shape of the surface representing the stress function ϕ becomes approximately known.

Having found the approximate values of ϕ at the nodal points adjacent to the boundary, and writing for the remaining nodal point within the boundary the equations of the form (36), we shall have a system of linear equations sufficient for calculating all the nodal values of ϕ . The second differences of ϕ can then be used for approximate calculation of stresses.

The system of Eqs. (36) may be solved directly, or we can find an approximate solution by one of the processes already described. The various methods of solution will now be illustrated by the simple example of a square plate loaded as shown¹ in Fig. 20.

Taking coordinate axes as shown in the figure,² we calculate the boundary values of ϕ starting from the origin. From $x = 0$ to $x = 0.4a$ we have no forces applied to the boundary, hence

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

¹ This is one of many cases discussed by P. M. Varvak, "Collection of Papers on Structural Mechanics," Kiev Structural Institute, vol. 3, p. 143, 1936 (Russian). A solution, also numerical, of a similar problem is given in the book by K. Beyer, "Die Statik im Eisenbetonbau," 2d ed., vol. 2, p. 733, Berlin, 1934.

² The system is obtained by rotating clockwise by π the axes used in Fig. 20, p. 23.

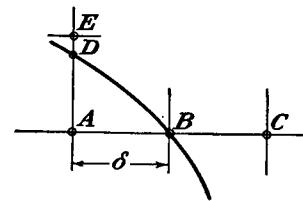


FIG. 19.

and integration gives

$$\frac{\partial \phi}{\partial x} = A, \quad \phi = Ax + B, \quad \frac{\partial \phi}{\partial y} = C$$

Here A, B, C are constants along the x -axis, which, as mentioned before, can be chosen arbitrarily. We assume $A = B = C = 0$. Then ϕ vanishes along the unloaded portion of the bottom side of the

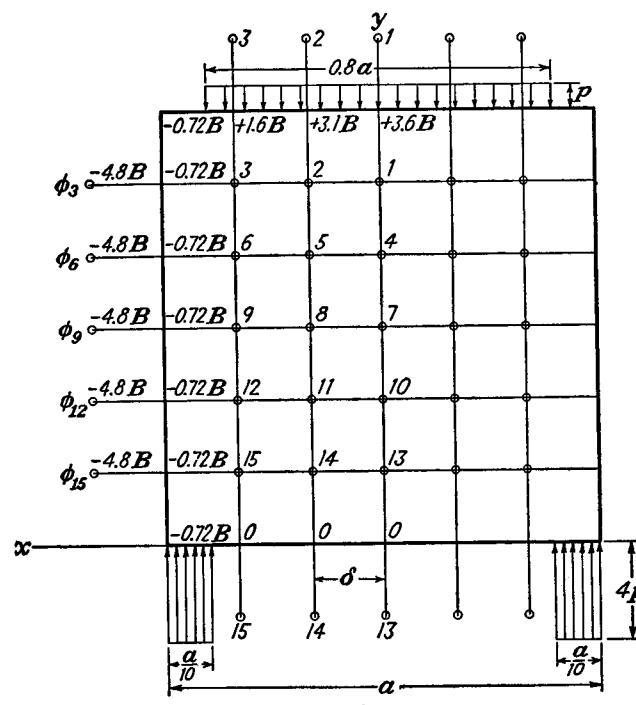


FIG. 20.

plate, which ensures the symmetry of ϕ with respect to the y -axis. From $x = 0.4a$ to $x = 0.5a$ there acts a uniformly distributed load of intensity $4p$ and Eqs. (38) give

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= - \int 4p dx = -4px + C_1 \\ \frac{\partial \phi}{\partial y} &= 0\end{aligned}$$

The second integration gives

$$\phi = -2px^2 + C_1x + C_2$$

The constants of integration will be calculated from the conditions that for the point $x = 0.4a$, the common point of the two parts of the boundary, the values of ϕ and $\partial\phi/\partial x$ must have the same values for both parts. Hence

$$(-4px + C_1)_{x=0.4a} = 0, \quad (-2px^2 + C_1x + C_2)_{x=0.4a} = 0$$

and we find

$$C_1 = 1.6pa, \quad C_2 = -0.32pa^2$$

The stress function ϕ , from $x = 0.4a$ to $x = 0.5a$, will be represented by the parabola

$$\phi = -2px^2 + 1.6pax - 0.32pa^2 \quad (a)$$

At the corner of the plate we obtain

$$(\phi)_{x=0.5a} = -0.02pa^2, \quad \left(\frac{\partial \phi}{\partial x}\right)_{x=0.5a} = -0.4pa \quad (b)$$

Along the vertical side of the plate there are no forces applied and, from Eqs. (38), we conclude that along this side the values of $\partial\phi/\partial x$ and of $\partial\phi/\partial y$ must be the same as those at the lower corner, i.e.,

$$\frac{\partial \phi}{\partial x} = -0.4pa, \quad \frac{\partial \phi}{\partial y} = 0 \quad (c)$$

From this it follows that ϕ remains constant along the vertical side of the plate. This constant must be equal to $-0.02pa^2$, as calculated above for the lower corner.

Along the unloaded portion of the upper side of the plate the first derivatives of ϕ remain constant and will have the same values (c) as calculated for the upper corner. Thus the stress function will be

$$\phi = -0.4pax + C$$

Since at the upper corner to the left ϕ must have the previously calculated value, equal to $-0.02pa^2$, we conclude that $C = 0.18pa^2$ and the stress function is

$$\phi = -0.4pax + 0.18pa^2 \quad (d)$$

Taking now the loaded portion of the upper side of the plate and observing that for this portion $ds = -dx$ and $\bar{Y} = -p$, $\bar{X} = 0$, we obtain, from Eqs. (38),

$$\frac{\partial \phi}{\partial x} = -px + C_1$$

$$\frac{\partial \phi}{\partial y} = C_2$$

For $x = 0.4a$ these values must coincide with the values (c). Hence, $C_1 = C_2 = 0$, and the stress function must have the form

$$\phi = -\frac{px^2}{2} + C$$

For $x = 0.4a$ it must have a value equal to that obtained from Eq. (d). We conclude that $C = 0.1pa^2$ and

$$\phi = -\frac{px^2}{2} + 0.1pa^2 \quad (e)$$

The stress function is represented by a parabola symmetrical with respect to the y -axis. This finishes the calculation of the boundary values of ϕ and its first derivatives, since for the right-hand portion of the boundary all these values are obtained from symmetry.

With the notation

$$\frac{pa^2}{36} = B$$

we can now write all the calculated boundary values of ϕ as shown in Fig. 20.

Next, by extrapolation, we calculate the values of ϕ for the nodal points taken outside the boundary. Starting again with the bottom side of the plate and observing that $\partial\phi/\partial y$ vanishes along this side we can take for the outside points the same values $\phi_{13}, \phi_{14}, \phi_{15}$ as for the inside points adjacent to the boundary.¹ We proceed similarly along the upper side of the plate. Along the vertical side of the plate we have the slope

$$\left(\frac{\partial\phi}{\partial x}\right)_{x=0.5a} = -0.4pa$$

and we can, as an approximation, obtain the values for the outside points by subtracting the quantity

$$0.4pa \cdot 2\delta = \frac{0.4p\gamma^2}{3} = 4.8B$$

from the inside points adjacent to the boundary, as shown in Fig. 20.

Now we can start the calculation of ϕ values for the inside nodal points of the net. Using the method of direct solution of the difference

¹ This manner of extrapolation, used in Varvak's paper, is different from that described on p. 485.

equations, we have to write in this symmetrical case the Eqs. (36) for the 15 points shown in Fig. 20. The solution of these equations gives for ϕ the values shown in the table below.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ϕ/B	3.356	2.885	1.482	2.906	2.512	1.311	2.306	2.024	1.097	1.531	1.381	0.800	0.634	0.608	0.396

Let us calculate the normal stress σ_x along the y -axis. The values of this stress are given by the second derivative $\partial^2\phi/\partial y^2$. Using finite differences we obtain for the upper point ($y = a$)

$$(\sigma_x)_{y=a} \approx \frac{(3.356 - 2 \cdot 3.600 + 3.356)B}{\delta^2} = -\frac{0.488pa^2}{36\delta^2} = -0.488p$$

For the lower point ($y = 0$) we find

$$(\sigma_x)_{y=0} = \frac{(0.634 - 0 + 0.634)B}{\delta^2} = 1.268p$$

If we consider the plate as a beam on two supports and assume a linear distribution of σ_x over the middle cross section ($x = 0$) we find $(\sigma_x)_{\max.} = 0.60p$. We can see that for a plate of such proportions the usual beam formula gives a very unsatisfactory result.

To solve the finite-difference equations (36) by iteration, we assume some starting values $\phi_1, \phi_2, \dots, \phi_{15}$ for the stress function. Substituting these into Eqs. (36) we obtain residual forces for all internal nodal points which can be liquidated by a relaxation process. The proper pattern, as obtained from Eq. (36), is shown in Fig. 21, in which the changes in residuals due to unit change of ϕ_0 are given. In applying this method to the square plate discussed above it must be observed that the ϕ values along the boundary are restricted by the boundary conditions, which means that the residual forces at points on the boundary need not be liquidated.

We can next advance to a finer net, obtaining starting values of ϕ from the results of the calculation on the coarse net.

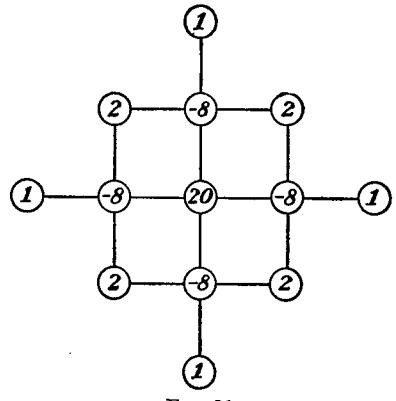


FIG. 21.

In the case of a nonsymmetrical loading such as shown in Fig. 22a, we can split the load as shown in Figs. 22b and 22c into symmetrical and antisymmetrical loadings. In both latter cases we have to consider one-half of the plate, since $\phi(x,y) = \phi(-x,y)$ for the symmetrical case and $\phi(x,y) = -\phi(-x,y)$ for the antisymmetrical loading.

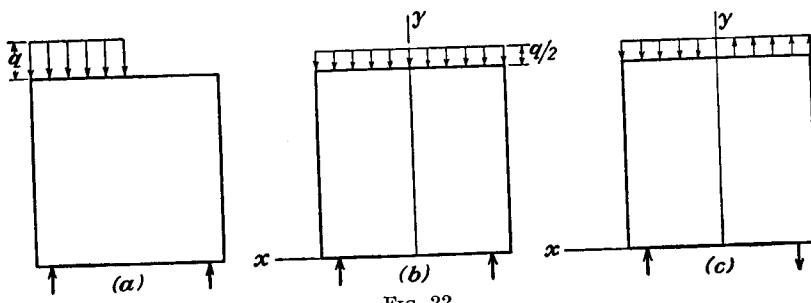


FIG. 22.

The work can be further reduced by considering also the horizontal axis of symmetry of the rectangular plate. The load shown in Fig. 20 can be resolved into symmetrical and antisymmetrical cases as shown in Fig. 23. For each of these cases only one-quarter of the plate should be considered in calculating numerical values of the stress function.

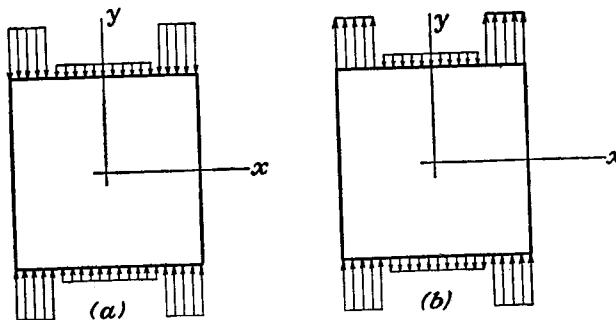


FIG. 23.

9. Torsion of Circular Shafts of Variable Diameter. In this case, as we have seen (page 307), it is necessary to find a stress function which satisfies the differential equation

$$\frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (40)$$

at every point of the axial section of the shaft, Fig. 24, and is constant along the boundary of that section. Only in a few simple cases have

we a rigorous solution of the problem, and in practical cases we must usually resort to approximate methods.

Using the finite-difference method, we shall take a square net. Considering a nodal point O , Fig. 24, we can treat the second derivatives in Eq. (40) as before. For the first derivative we can take

$$\left(\frac{\partial \phi}{\partial r} \right)_{r=r_0} \approx \frac{1}{2} \left(\frac{\phi_1 - \phi_0}{\delta} + \frac{\phi_0 - \phi_3}{\delta} \right) = \frac{\phi_1 - \phi_3}{2\delta}$$

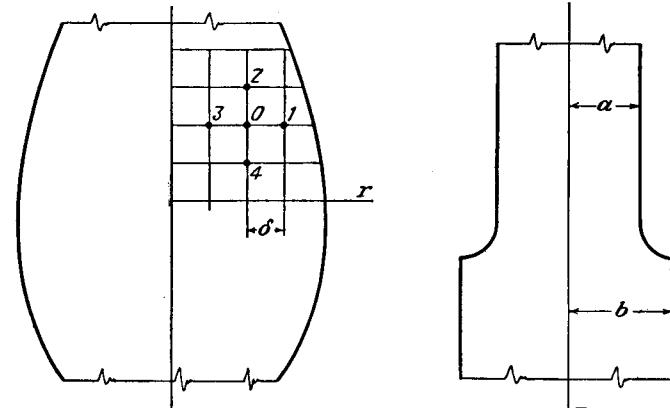


FIG. 24.

FIG. 25.

Then the finite-difference equation, corresponding to Eq. (40), is

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 - 4\phi_0 - \frac{3\delta}{2r_0} (\phi_1 - \phi_3) = 0 \quad (41)$$

The problem then is to find such a set of values of ϕ that Eq. (41) will be satisfied at every nodal point of the net and ϕ will be equal to the assumed constant value at the boundary. This problem can be treated either by direct solution of Eqs. (41) or by one of the iteration methods.

As an example, let us consider the case shown in Fig. 25. In the region of rapid change in the diameter there will be a complicated stress distribution, but at sufficient distances from the fillets a simple Coulomb solution will hold with sufficient accuracy, and the stress distribution will be independent of z . Equation (40) for such points becomes

$$\frac{d^2 \phi}{dr^2} - \frac{3}{r} \frac{d\phi}{dr} = 0 \quad (42)$$

The general solution of this equation is

$$\phi = Ar^4 + B \quad (43)$$

and the corresponding stresses are (see page 306)

$$\tau_{z\theta} = \frac{1}{r^2} \frac{d\phi}{dr} = 4Ar, \quad \tau_{r\theta} = 0$$

Comparing this result with the Coulomb solution, we find

$$4A = \frac{M_t}{I_p}$$

where M_t is the applied torque and I_p is the polar moment of inertia of the shaft. Omitting the constant B in the general solution (43) as having no effect on the stress distribution, we find for the stress function at sufficient distances from the fillet the expressions

$$\phi_a = \frac{M_t}{2\pi a^4} \cdot r^4, \quad \phi_b = \frac{M_t}{2\pi b^4} \cdot r^4 \quad (44)$$

These expressions vanish at the axis of the shaft and assume at the boundary a common value $M_t/2\pi$. Since ϕ is constant along the boundary, the value $M_t/2\pi$ holds also for the fillets. Thus selection of the constant at the boundary in solving Eqs. (41) is equivalent to assuming a definite value for the torque.

In solving Eqs. (41) we can again apply the membrane analogy. We begin with points where Eq. (42) holds. The corresponding finite-differences equation is

$$\phi_1 + \phi_3 - 2\phi_0 - \frac{3\delta}{2r_0} (\phi_1 - \phi_3) = 0 \quad (45)$$

This equation is of the same form as that for deflections to a cylindrical form of a membrane with tension varying inversely as r^3 . To show this let us consider three consecutive points of the net, Fig. 26. The corresponding deflections we denote by w_3, w_0, w_1 .

FIG. 26.

The tension at the middle of the strings 3-0 and 0-1 will be

$$\left(\frac{S\delta}{r_0 - \frac{\delta}{2}} \right)^3 \approx \frac{S\delta}{r_0^3} \left(1 + \frac{3\delta}{2r_0} \right)$$

and

$$\left(\frac{S\delta}{r_0 + \frac{\delta}{2}} \right)^3 \approx \frac{S\delta}{r_0^3} \left(1 - \frac{3\delta}{2r_0} \right)$$

The equation of equilibrium for the point O is then

$$\begin{aligned} \frac{S\delta}{r_0^3} \left(1 - \frac{3\delta}{2r_0} \right) \frac{w_1 - w_0}{\delta} + \frac{S\delta}{r_0^3} \left(1 + \frac{3\delta}{2r_0} \right) \frac{w_3 - w_0}{\delta} &= 0 \\ w_1 - 2w_0 + w_3 - \frac{3\delta}{2r_0} (w_1 - w_3) &= 0 \end{aligned}$$

This is the same as Eq. (45).

Similarly, in the general case, observing that the tension in the membrane does not depend on z , we obtain the equation

$$w_1 + w_2 + w_3 + w_4 - 4w_0 - \frac{3\delta}{2r_0} (w_1 - w_3) = 0 \quad (46)$$

which agrees with Eq. (41). It is seen that we can calculate the stress function as the deflection of a membrane with nonuniform tension having constant deflection $M_t/2\pi$ along the boundary and deflections (44) at the points at large distances from the fillets. We assume some starting values for w at the nodal points, substitute them into the left-hand sides of Eqs. (46), and calculate the residuals. Now the problem is to liquidate all these residuals by the relaxation process. From Fig. 26 we see that by giving to point O a displacement unity we add to the residuals at points 1 and 3 the quantities

$$\frac{S}{r_0^3} \left(1 - \frac{3\delta}{2r_0} \right) \quad \text{and} \quad \frac{S}{r_0^3} \left(1 + \frac{3\delta}{2r_0} \right)$$

which indicates that the pattern for the relaxation process is as shown in Fig. 27. It varies from point to point with variation of the radial

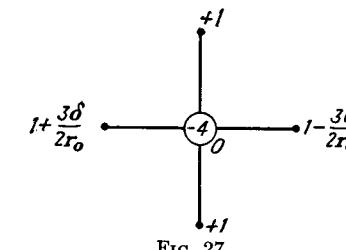


FIG. 27.

distance r_0 . Calculations of this kind were carried out by R. V. Southwell and D. N. de G. Allen.¹

¹ Proc. Roy. Soc. (London), series A, vol. 183, pp. 125-134. See also Southwell's book "Relaxation Methods in Theoretical Physics," p. 152.

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