# Lecture 22: Systems of Nonlinear First-Order DE

## 12.1 Autonomous Systems

Recall that a **first-order** system is a system of first-order ODEs that has the normal form

$$\begin{aligned} \frac{dx_1}{dt} &= g_1(t, x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} &= g_2(t, x_1, x_2, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(t, x_1, x_2, \ldots, x_n) \end{aligned} \qquad \text{or} \qquad X' = g(t, X).$$

### Autonomous systems

A system of first-order ODEs is said to be **autonomous** when it can be written in the form

$$\begin{aligned} \frac{dx_1}{dt} &= g_1(x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} &= g_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= g_n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{1}$$

or, in compact notation,

$$X' = g(X).$$

Note that the independent variable $t$ does not appear explicitly on the right-hand side of the system of ODEs.

When the independent variable $t$ is interpreted as time, we call (1) a **dynamical system** whose solution $X(t)$ is the **state of the system** or the **response of the system** at time $t$.

Thus, a dynamical system is autonomous when the rate $X'(t)$ at which the system changes depends only on the system's present state $X(t)$.

**Example 1.** Consider the second-order ODE: $x'' = h(x, x')$. This equation can be written as the autonomous system

$$x' = y$$
$$y' = h(x, y).$$

For example, if $h(x, x') = -\frac{g}{l}\sin x$, we see that the pendulum ODE can be expressed as an autonomous system.

### Vector Field Interpretation

In the case $n = 2$ the system (1) is called a **plane autonomous system**, and we write it as

$$X' = g(X) \qquad \text{or} \qquad \begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y). \end{aligned} \tag{2}$$

The vector $g(X) = [P(x, y), Q(x, y)]^T$ defines a **vector field** in a region of the plane, and a solution $X = X(t)$ to the system may be viewed as the resulting path of a particle that is initially placed at position $X(0) = X_0$ as it moves through the region with a velocity vector $g$.

### Types of Solutions

If $P(x, y)$, $Q(x, y)$ and $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ are continuous in a region $R$ of the plane, then a solution of (2) that satisfies $X(0) = X_0$ is unique and of one of the three basic types:

(i) A **constant solution** $X(t) = X_0$ for all $t$. This solution is also called a **critical** or **stationary point** or an **equilibrium solution**. Since $X'(t) = 0$, a critical point is a solution of the system of algebraic equations

$$P(x, y) = 0$$
$$Q(x, y) = 0.$$

(ii) A solution $X(t)$ that defines an **arc** (a plane curve that does *not* cross itself).

(iii) A **periodic solution** or **cycle** $X(t)$ so that $X(t + p) = X(t)$, with $p$ being the period of the solution.

## 12.2 Stability of Linear Systems

Suppose that $X_c$ is a critical point of the plane autonomous system (2) and $X = X(t)$ is a solution of the system that satisfies $X(0) = X_0$. If the solution is interpreted as a path of a moving particle, what happens when $X_0$ is placed near $X_c$? We have the following cases.

(i) If $X_0 = X_c$, then the particle remains stationary.

(ii) If when $X_0$ is placed in some neighborhood of $X_c$, the particle always returns to $X_c$ (that is, $\lim_{t\to\infty} X(t) = X_c$) or remains close to it, we call the critical point $X_c$ **locally stable**.

(iii) If, however, an $X_0$ can be found in any given neighborhood of $X_c$ so that the particle moves away from $X_c$, we call the critical point **unstable.**

We have the following theorem.

**Theorem 1** (Stability Criteria for Linear Systems)**.** *For a linear plane autonomous system $X' = AX$ with $\det A \neq 0$, let $X = X(t)$ be the solution that satisfies the initial condition $X(0) = X_0$, where $X_0 \neq 0$.*

*(a)* $\lim_{t\to\infty} X(t) = 0$ *if and only if the eigenvalues of $A$ have negative real parts. This will happen when* $\det A > 0$ *and* $\operatorname{tr} A < 0$.

*(b)* $X(t)$ *is periodic if and only if the eigenvalues of $A$ are pure imaginary. This will happen when* $\det A > 0$ *and* $\operatorname{tr} A = 0$.

*(c) In all other cases, given any neighborhood of the origin, there is at least one $x_0$ in the neighborhood for which $X(t)$ becomes unbounded as $t$ increases.*

Here the eigenvalues of $A$ satisfy the of *characteristic equation*

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0, \tag{3}$$

whise roots are

$$\lambda = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2} \tag{4}$$

**Remark 1.** Assuming that $\det A \neq 0$ ensures that $X_0 = 0$ is the only critical point of the linear plane autonomous system.

## Clasifying Critical Points

Figure 1 summarizes the results of this section

**Example 2.** For each of the following systems of ODEs, use the trace-determinant plane to classify the type of equilibrium that exists at the origin.

$$(a) \qquad \begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 8x + 3y. \end{aligned}$$

$$(b)\quad \begin{aligned} \frac{dx}{dt} &= -10x + 6y \\ \frac{dy}{dt} &= 15x - 19y. \end{aligned}$$

$$(c)\quad \begin{aligned} \frac{dx}{dt} &= -x + 2y \\ \frac{dy}{dt} &= -x + y. \end{aligned}$$

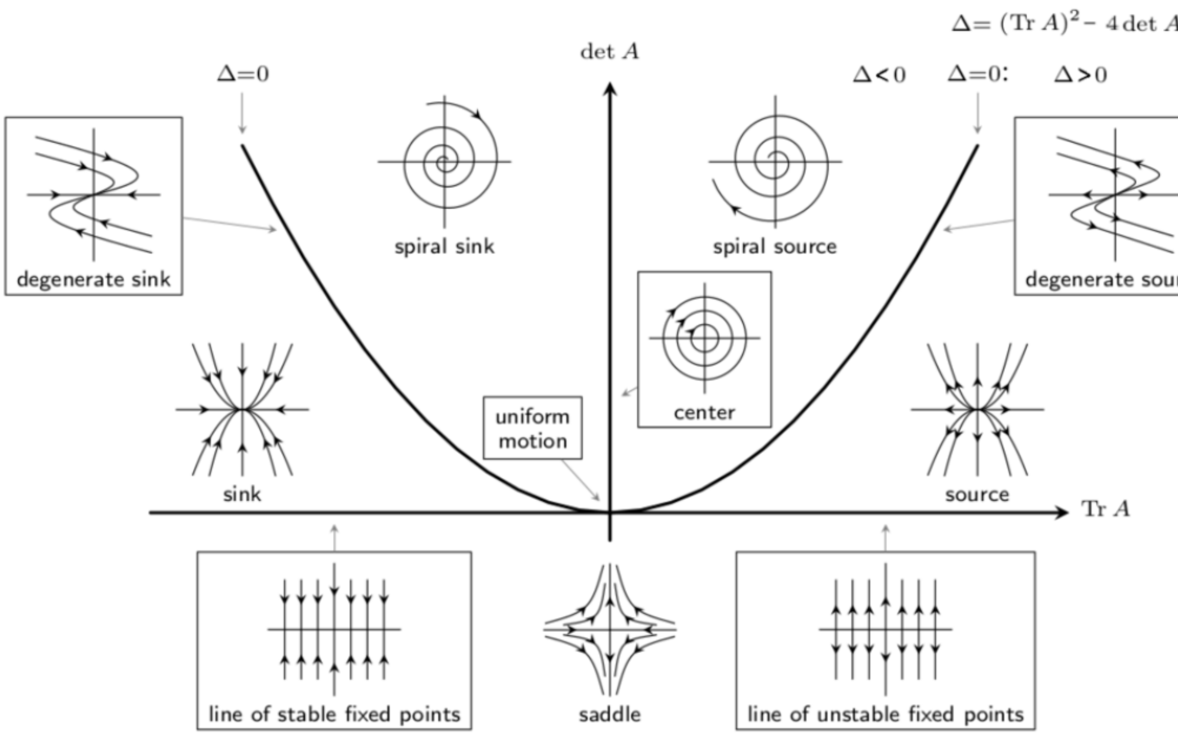


**Figure 1**: Poincaré Diagram: Classification of Phase Portraits in the (Tr $A$, det $A$)-plane. (Image from blog.nekomath.com )

**Remark 2.** Phase Portrait and Field Directions of Two-Dimensional Linear Systems of ODEs https://demonstrations.wolfram.com/PhasePortraitAndFieldDirectionsOfTwoDimensionalLinearSystems/

## 12.3 Linearization and Local Stability

Let us analyze now two dimensional systems of ODEs, which have the general form (2), namely,

$$X' = g(X) \qquad \text{or} \qquad \begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y), \end{aligned}$$

where $P(x,y)$, $Q(x,y)$ and $\partial P/\partial x$, $\partial P/\partial y$, $\partial Q/\partial x$, $\partial Q/\partial y$ are continuous in a region $R$ of the plane.

As before, our goal is to do some basic qualitative analysis of this system by learning to

(i) find the equilibria of the system; and

(ii) determine their stability.

Let $X' = g(X)$ and suppose that $g(X_c) = 0$, that is, $X_c$ is an equilibrium point of the autonomous system. If we make the change of coordinates $\widetilde{X} = X - X_c$ then the new system $\widetilde{X}' = g(X_c + \widetilde{X})$ has an equilibrium point at $\widetilde{X} = 0$. The linear system of differential equations

$$\widetilde{X}' = J_g(X_c)\widetilde{X},$$

where $J_g(X_c)$ is the Jacobian matrix of $g$ at $X_c$, is called the *linearized system near* $X_c$.

(This is obtained by expanding $g(X_c + \widetilde{X})$ as a Taylor series around $X_c$ up to linear order, that is $g(X_c + \widetilde{X}) = g(X_c) + J_g(X_c)\widetilde{X} = J_g(X_c)\widetilde{X}$ since $g(X_c) = 0$.)

An equilibrium point $X_c$ of a nonlinear system is said to be *hyperbolic* if all of the eigenvalues of $J_g(X_c)$ have nonzero real parts. When the linearization of the system at an equilibrium point is hyperbolic, we can readily determine the stability of that point. However, many important equilibrium points that arise in applications are nonhyperbolic.

There are three basic steps to carry out to attain our desired level of qualitative analysis.

1. Find all equilibria, that is, all pairs $(x,y)$ such that $P(x,y) = 0 = Q(x,y)$.
2. Compute the Jacobian. $J(x,y) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}$.
3. For each equilibrium, evaluate the Jacobian at that point and use this as the linear dynamics matrix to classify which type the equilibrium is.

**Example 3.** Find the equilibria of the following system of ODEs and classify them

$$\begin{aligned} x' &= 1 - e^y \\ y' &= -xy + x^2 - 1. \end{aligned}$$

**Solution:**

**(1) Find all equilibria.** We set $x' = y' = 0$ and solve the system

$$0 = 1 - e^y$$
$$0 = -xy + x^2 - 1.$$

From the first equation, we have $e^y = 1$ so that $y = 0$. Substituting this value into the second equation, gives $x^2 = 1$ so that $x = \pm 1$. Hence, there are two equilibria: $(1, 0)$ and $(-1, 0)$.

**(2) Compute the Jacobian.**

$$J(x, y) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & -e^y \\ -y + 2x & -x \end{bmatrix}.$$

**(3) Evaluate the Jacobian at the equilibria and classify.**

*Equilibrium point* $(1, 0)$. Evaluating the Jacobian at the equilibrium gives

$$A_1 = J(1, 0) = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}.$$

Thus $\mathrm{tr}(A_1) = -1$, $\det(A_1) = 2$, and $\Delta_1 = (\mathrm{tr}(A_1))^2 - 4\det(A_1) < 0$. From Figure 1, we classify the equilibrium as a `stable spiral`.

(Alternatively, by (4) the eigenvalues of $A_1$ are

$$\lambda = \frac{-1 \pm \sqrt{(-1)^2 - 4(2)}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}.$$

Since the roots of the characteristic equation are complex with negative real part, by Theorem 1 we classify the equilibrium as a `stable spiral`.)

*Equilibrium point* $(-1, 0)$. Evaluating the Jacobian at the equilibrium gives

$$A_2 = J(-1, 0) = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}.$$

Thus $\mathrm{tr}(A_2) = 1$, $\det(A_2) = -2$, and $\Delta_2 = (\mathrm{tr}(A_2))^2 - 4\det(A_2) > 0$. From Figure 1, we classify the equilibrium as a `saddle node`.

(Alternatively, by (4) the eigenvalues of $A_2$ are

$$\lambda = \frac{1 \pm \sqrt{(1)^2 - 4(-2)}}{2} = \frac{1}{2} \pm \frac{3}{2}.$$

Since the roots of the characteristic equation are real and distinct with opposite signs, by Theorem 1 we classify the equilibrium as a `saddle node`.)

## 12.4 Autonomous Systems as Mathematical Models

### Rabbit vs sheep

Let $x(t)$ and $y(t)$ denote the population of rabbits and sheep respectively. When the species do not interact, their populations evolve according to the logistic equation

$$\begin{aligned} x' &= r_1 x\left(1 - \frac{x}{k_1}\right) \\ y' &= r_2 y\left(1 - \frac{y}{k_2}\right). \end{aligned}$$

If the animals share the same territory, however, they compete for the same resources and so have a negative influence on each other. This is accounted for by adding a nonlinear term into each equation:

$$\begin{aligned} x' &= r_1 x\left(1 - \frac{x}{k_1}\right) - axy \\ y' &= r_2 y\left(1 - \frac{y}{k_2}\right) - bxy. \end{aligned}$$

**Example 4.** An example of the above plane autonomous system is

$$\begin{aligned} x' &= x(3 - x - 2y) \\ y' &= y(2 - x - y). \end{aligned}$$

Find the fixed points of this system and classify them.