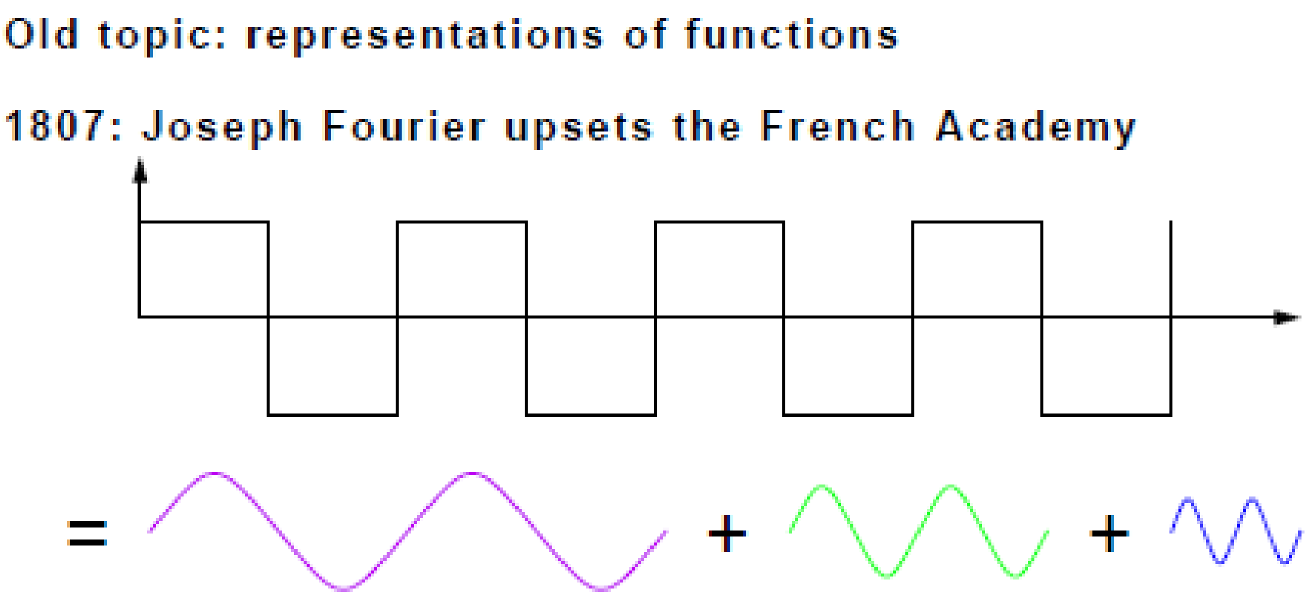
第十二讲 小波变换

第一讲：小波简介

- 小波起源
- 金字塔分解
- 多尺度分析
- 小波与滤波器组
- 小波算法
- 图像的小波分解
- 图像的小波域基本特性
- 隐马尔科夫模型

小波起源



傅立叶早在1807年就写成关于热传导的基本论文《热的传播》，向巴黎科学院呈交，但经拉格朗日、拉普拉斯和勒让德审阅后被科学院拒绝，1811年又提交了经修改的论文，该文获科学院大奖，却未正式发表。傅立叶在论文中推导出著名的热传导方程，并在求解该方程时发现解函数可以由三角函数构成的级数形式表示，从而提出任一函数都可以展成三角函数的无穷级数。傅立叶级数（即三角级数）、傅立叶分析等理论均由此创始。

小波起源

- Fourier 变换（级数）

$$F(f)(\omega)=\int_{-\infty}^{\infty} f(t)e^{-it\omega}dt$$

$$f(t)=\sum f_k e^{-i2\pi kt/T}, \qquad f_k=\frac{1}{T}\int_0^T f(t)e^{i2\pi kt/T}dt$$

  – Fourier分析的优点：能够准确地知道信号中某个特定频率的含量；得到信号的某个频率分量的过程就是分析（Analysis），对各个频率成分处理之后，用调和信号重建信号的过程称为合成。

小波起源（续）

- Fourier 分析的弱点：Fourier分析是一种整体分析，信号的任何局部的变动将导致所有频谱的变化；而且，从频谱上看不出这种变化来自于何处，比如，一首音乐，改动某个部分的乐谱，从Fourier分析中看不到改动的位置。
- 人们的期望：研究信号局部的频率成分：时频分析，即研究信号的在时刻t，频率为ω成分的含量，也就是说，给定一个信号$s(t)$，我们希望得到一个二元函数：$F(t, \omega)$，该函数数值就是表示信号$s(t)$在时间t，频率为ω分量的值。
- 一般来说：不可能

小波起源（续）

- 测不准原理
  - 给定信号s(t),

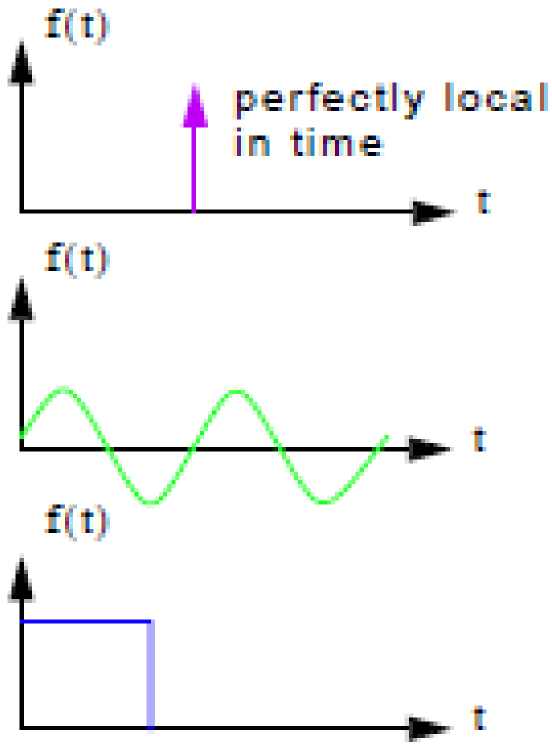
$$t_0 = \int t\,|s(t)|^2\,dt, \quad \omega_0 = \int \omega\,|F(s)(\omega)|^2\,d\omega$$

$$\Omega_t = \int (t - t_0)^2\,|s(t)|^2\,dt$$

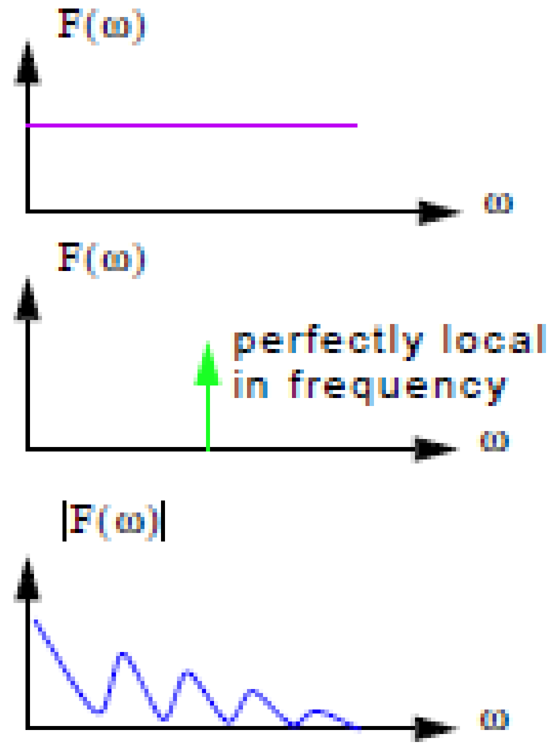
$$\Omega_\omega = \int (\omega - \omega_0)^2\,|F(s)(\omega)|^2\,d\omega$$

$$\Omega_t \Omega_\omega > c$$

1930: Heisenberg discovers that
you cannot have your cake and eat it too!

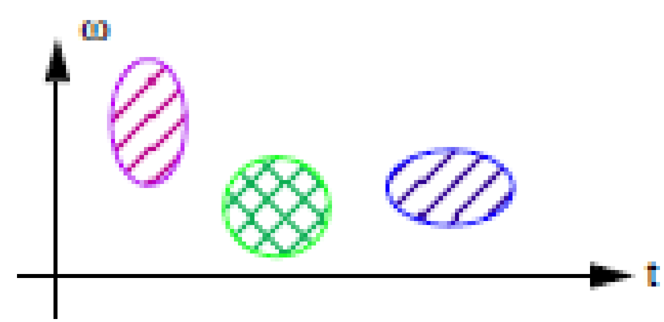


trade-off



Uncertainty principle

- lower bound on TF product



小波起源（续）

- 短时（加窗）Fourier变换
  - 为了分析信号在某个时刻的频率成分，引入了短时Fourier变换：
  - 函数g(u)称为窗函数：
  - 局部信号获取：
  - 局部信号分析：

$$\mathrm{supp}(g) \subset [-T, 0]$$

$$f_t(u) = g(u-t) f(u)$$

$$F(f_t) = \int_{-\infty}^{\infty} f_t(u) e^{-iu\omega} du = \int_{-\infty}^{\infty} f(u) g(u-t) e^{-iu\omega} du$$

小波起源（续）

- 定义：　$g_{t,\omega}(u)=g(u-t)e^{iu\omega}$
- 短时Fourier变换：　$F(t,\omega)=< f,g_{t,\omega} >$
- 逆变换：

$$f(u)=C\iint F(t,\omega)g_{t,\omega}(u)dtd\omega$$

- 离散算法：　$t=mt_0,\omega=n\omega_0$

$g_{m,n}(u)=g(u-mt_0)e^{inu\omega_0}$

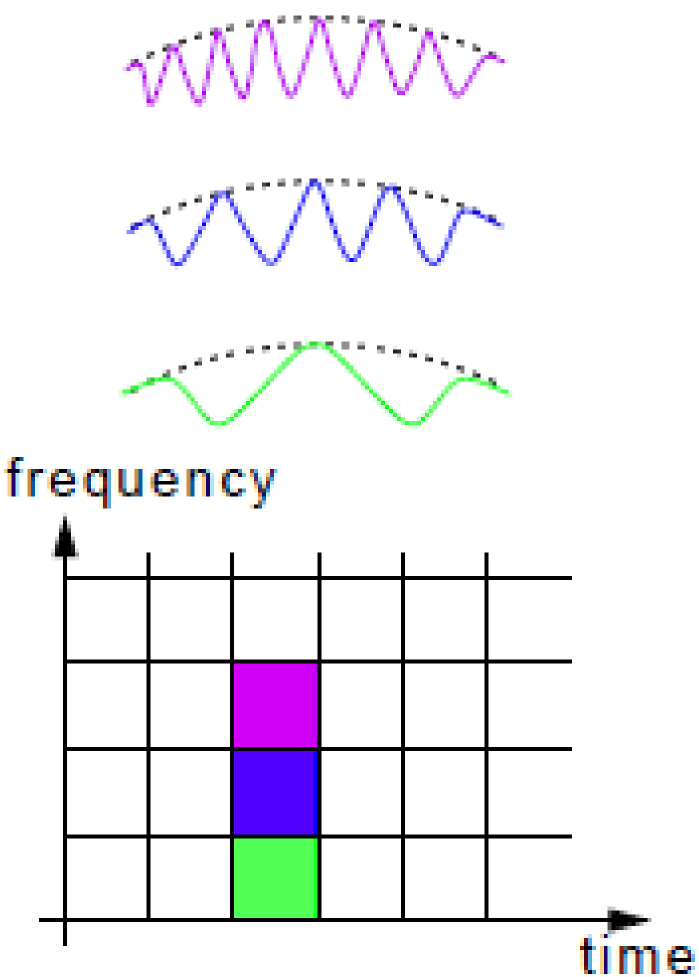
$\{g_{m,n}\}$构成了$L^2(R)$上的一个Frame.

假设$\{\tilde{g}_{m,n}\}$是其对偶Frame,

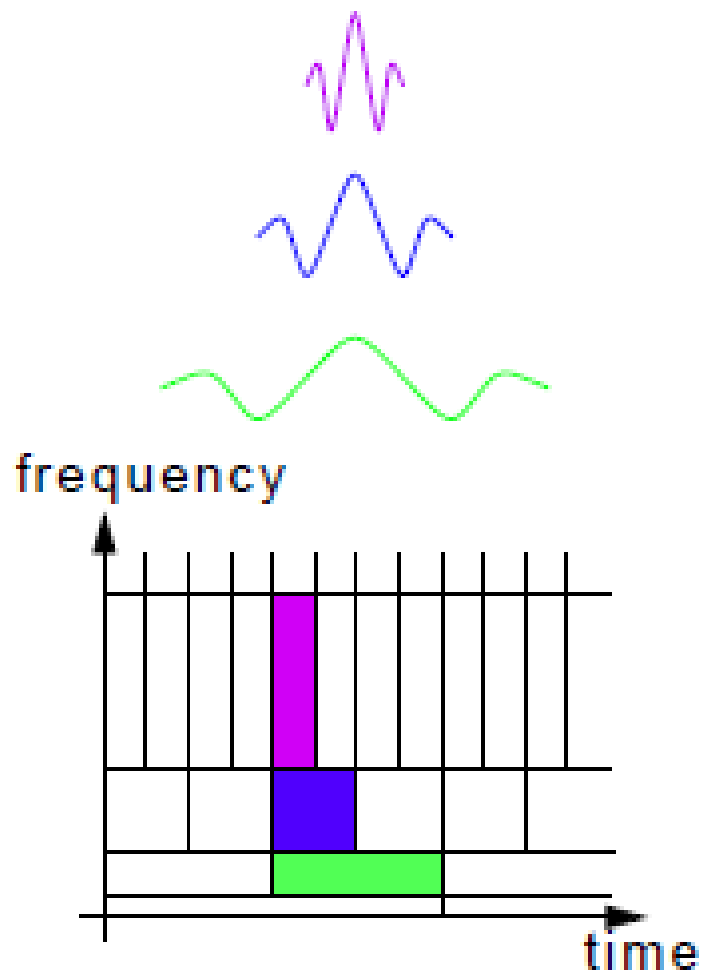
$$f(u)=\sum< f,g_{m,n} > \tilde{g}_{m,n}=\sum< f,\tilde{g}_{m,n} > g_{m,n}$$

1945: Gabor localizes the Fourier transform ⇒ STFT

1980: Morlet proposes the continuous wavelet transform



short-time Fourier transform



wavelet transform

小波起源（续）

- 短时Fourier变换的优点：
  - 能够分析信号的局部频谱
- 短时Fourier变换的缺点：
  - 窗口函数一旦确定，窗口的宽度不再变化，对于非平稳信号来说，不能适应，我们需要在信号变化比较快的地方用较小的窗口，避免信号叠加，同时在信号变化缓慢的地方用大的窗口，避免计算的重复。

小波起源（续）

- 小波变换

假设函数$\psi(t)$满足：

$$C=\int|\omega|^{-1}|\hat{\psi}(\omega)|^{2}\,d\omega<\infty$$

$$\psi_{a,b}(t)=|a|^{-1/2}\,\psi\left(\frac{t-b}{a}\right)$$

小波变换：$T(f)(a,b)=<f,\psi_{a,b}>$

$$=\int f(t)\,|a|^{-1/2}\,\overline{\psi\left(\frac{t-b}{a}\right)}dt$$

小波起源（续）

- 小波逆变换：

$$f(t) = C^{-1}\iint T(f)(a,b)\psi_{a,b}(t)\frac{dadb}{a^2}$$

- 离散化：

$$a = ma_0, b = nb_0$$

$$\psi_{m,n} = |ma_0|^{-1/2}\psi\left(\frac{t-nb_0}{ma_0}\right)$$

$\{\psi_{m,n}\}$构成$L^2(R)$的Frame,

假设$\{\tilde{\psi}_{m,n}\}$是其对偶Frame，

$$f(t) = \sum < f,\psi_{m,n} > \tilde{\psi}_{m,n} = \sum < f,\tilde{\psi}_{m,n} > \psi_{m,n}$$

小波起源（续）

- 连续小波变换的初步特点：
  - 窗口可变：不同的a对应不同的宽度
  - 奇异检测：

定理：

假设函数$\psi(\mathrm{t})$满足:$\int(1+|t|)|\psi(t)|dt<\infty,$

如果函数f是Hölder连续，即$|f(x)-f(y)|<C|x-y|^{\alpha}$

（$0<\alpha\leq 1$）那么

$|<f,\psi_{a,b}>|\leq C'|a|^{\alpha+1/2}$

小波起源（续）

- 连续小波变换弱点：
  - 没有快速算法
  - 选择合适的小波函数比较困难
  - 和传统的信号处理缺乏联系

Before MRA

- Burt, Adelson, The Laplacian pyramid as a compact image code, 1983， IEEE TRANSACTIONS ON COMMUNICATIONS （引用5397次）

The Laplacian Pyramid as a Compact Image Code

PETER J. BURT, MEMBER, IEEE, AND EDWARD H. ADELSON

Abstract—We describe a technique for image encoding in which local operators of many scales but identical shape serve as the basis functions. The representation differs from established techniques in that the code elements are localized in spatial frequency as well as in space.

Pixel-to-pixel correlations are first removed by subtracting a low-pass filtered copy of the image from the image itself. The result is a net data compression since the difference, or error, image has low variance and entropy, and the low-pass filtered image may represented at reduced sample density. Further data compression is achieved by quantizing the difference image. These steps are then repeated to compress the low-pass image. Iteration of the process at appropriately expanded scales generates a pyramid data structure.

The encoding process is equivalent to sampling the image with Laplacian operators of many scales. Thus, the code tends to enhance salient image features. A further advantage of the present code is that

does not permit simple sequential coding. Noncausal approaches to image coding typically involve image transforms, or the solution to large sets of simultaneous equations. Rather than encoding pixels sequentially, such techniques encode them all at once, or by blocks.

Both predictive and transform techniques have advantages. The former is relatively simple to implement and is readily adapted to local image characteristics. The latter generally provides greater data compression, but at the expense of considerably greater computation.

Here we shall describe a new technique for removing image correlation which combines features of predictive and transform methods. The technique is noncausal, yet computations

Before MRA

- Groosman, Morlet, **Decomposition of Hardy functions into square integrable wavelets of constant shape,** SIAM J. Math. Anal., 1984 (2313) .

SIAM J. MATH. ANAL.
Vol. 15, No. 4, July 1984

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DECOMPOSITION OF HARDY FUNCTIONS INTO SQUARE INTEGRABLE WAVELETS OF CONSTANT SHAPE*

A. GROSSMANN† AND J. MORLET‡

Abstract. An arbitrary square integrable real-valued function (or, equivalently, the associated Hardy function) can be conveniently analyzed into a suitable family of square integrable wavelets of constant shape, (i.e. obtained by shifts and dilations from any one of them.) The resulting integral transform is isometric and self-reciprocal if the wavelets satisfy an "admissibility condition" given here. Explicit expressions are obtained in the case of a particular analyzing family that plays a role analogous to that of coherent states (Gabor wavelets) in the usual L_2-theory. They are written in terms of a modified Γ-function that is introduced and studied. From the point of view of group theory, this paper is concerned with square integrable coefficients of an irreducible representation of the nonunimodular $ax+b$-group.

Before MRA

- Yves Meyer
  - Meyer wavelet, 1986, (Ondelettes et opérateurs 小波与算子,译本)

1987, S. Mallat, and Y. Meyer

Professor of
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Telecommunications(国立高等电信学院,), Paris, in 1985.
He receivedthe Ph.D. degree in electrical engineering from
the University of Pennsylvania, Philadelphia.
PA, m 1988.

Two papers

- **A theory for multiresolution signal decomposition : the wavelet representation,** IEEE Transaction on Pattern Analysis and Machine Intelligence, vol. 11, p. 674-693, July 1989. (引用18697次）
- **Multiresolution approximation and wavelet orthonormal bases of L2,** Transaction of the American Mathematical Society, vol. 315, p. 69-87, Sept. 1989 （引用2928次）

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 315, Number 1, September 1989

MULTIRESOLUTION APPROXIMATIONS AND WAVELET ORTHONORMAL BASES OF $L^2(R)$

STEPHANE G. MALLAT

A Theory for Multiresolution Signal Decomposition: The Wavelet Representation

STEPHANE G. MALLAT

Abstract—**Multiresolution representations are very effective for analyzing the information content of images. We study the properties of the operator which approximates a signal at a given resolution. We show that the difference of information between the approximation of a signal at the resolutions 2^{j+1} and 2^j can be extracted by decomposing this signal on a wavelet orthonormal basis of $L^2(R^n)$. In $L^2(R)$, a wavelet orthonormal basis is a family of functions $(\sqrt{2^j}\,\psi(2^jx - n))_{(j,n)\in Z^2}$, which is built by dilating and translating a unique function $\psi(x)$. This decomposition defines an orthogonal multiresolution representation called a wavelet representation. It is computed with a pyramidal algorithm based on convolutions with quadrature mirror filters. For images, the wavelet representation differentiates several spatial orientations. We study the application of this representation to data compression in image coding, texture discrimination and fractal analysis.**

A multiresolution decomposition enables us to have a scale-invariant interpretation of the image. The scale of an image varies with the distance between the scene and the optical center of the camera. When the image scale is modified, our interpretation of the scene should not change. A multiresolution representation can be partially scale-invariant if the sequence of resolution parameters $(r_j)_{j\in Z}$ varies exponentially. Let us suppose that there exists a resolution step $\alpha \in \boldsymbol{R}$ such that for all integers j, $r_j = \alpha^j$. If the camera gets α times closer to the scene, each object of the scene is projected on an area α^2 times bigger in the focal plane of the camera. That is, each object is measured at a resolution α times bigger. Hence, the de-

Ingrid Daubechies

- **Ingrid DAUBECHIES**
- Orthonormal bases of compactly supported wavelets, Comm. Pure & Appl. Math., 41 (7), pp. 909-996, 1988
- 1980 Ph.D. in Theoretical Physics; Vrije Universiteit Brussel
- Member of the American Academy of Arts and Sciences (elected 1993)
- National Academy of Sciences (elected 1998)
- Royal Netherlands Academy of Arts and Sciences (Foreign member; elected 1999)
- American Philosophical Society (elected 2003)
- London Mathematical Society (Honorary Member; elected 2007)
- Acad´ emie des Sciences, Paris, France (Foreign Member; elected 2009)

Orthonormal Bases of Compactly Supported Wavelets

INGRID DAUBECHIES
AT&T Bell Laboratories

Abstract

We construct orthonormal bases of compactly supported wavelets, with arbitrarily high regularity. The order of regularity increases linearly with the support width. We start by reviewing the concept of multiresolution analysis as well as several algorithms in vision decomposition and reconstruction. The construction then follows from a synthesis of these different approaches.

1. Introduction

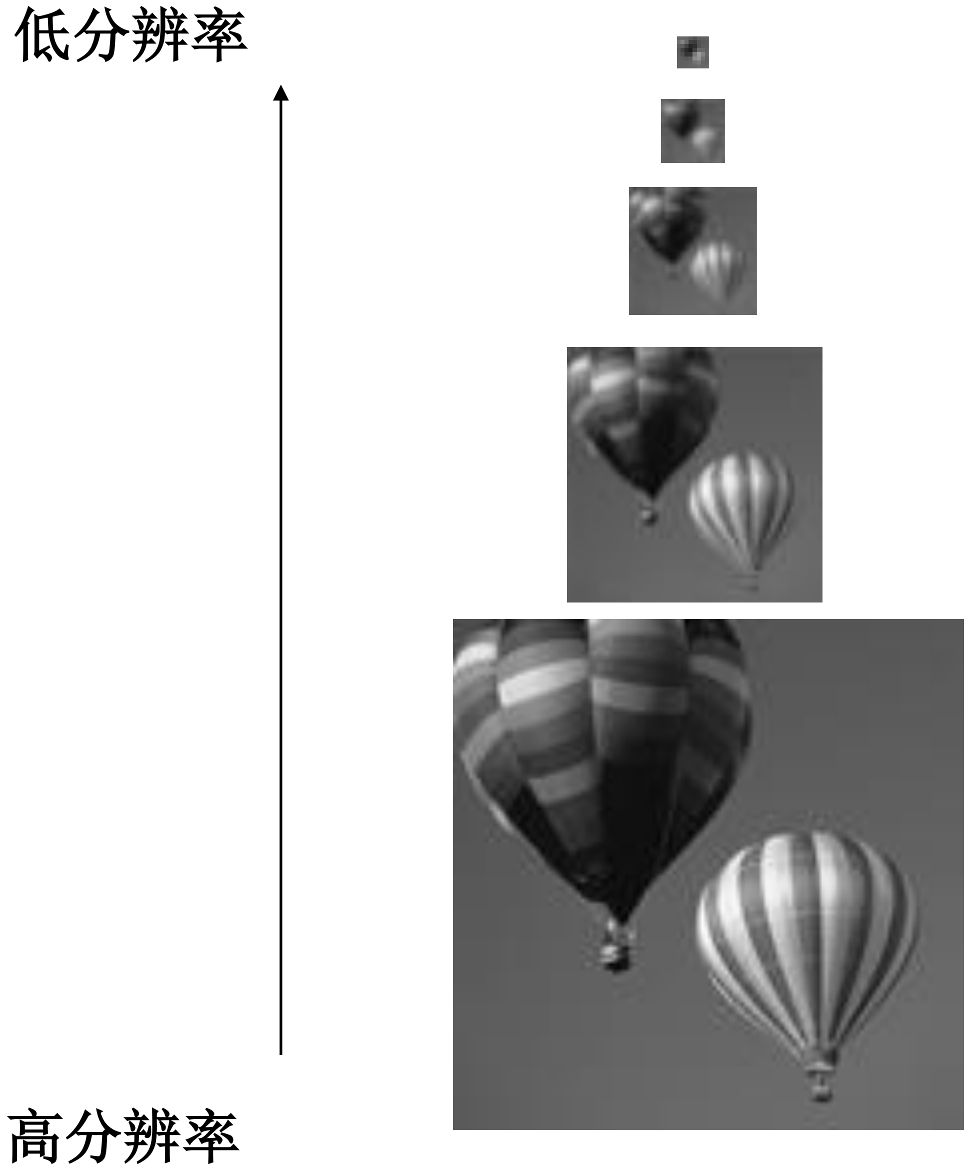
In recent years, families of functions $h_{a,b}$,

$$h_{a,b}(x) = |a|^{-1/2} h\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbb{R},\ a \neq 0, \tag{1.1}$$

Image Pyramids

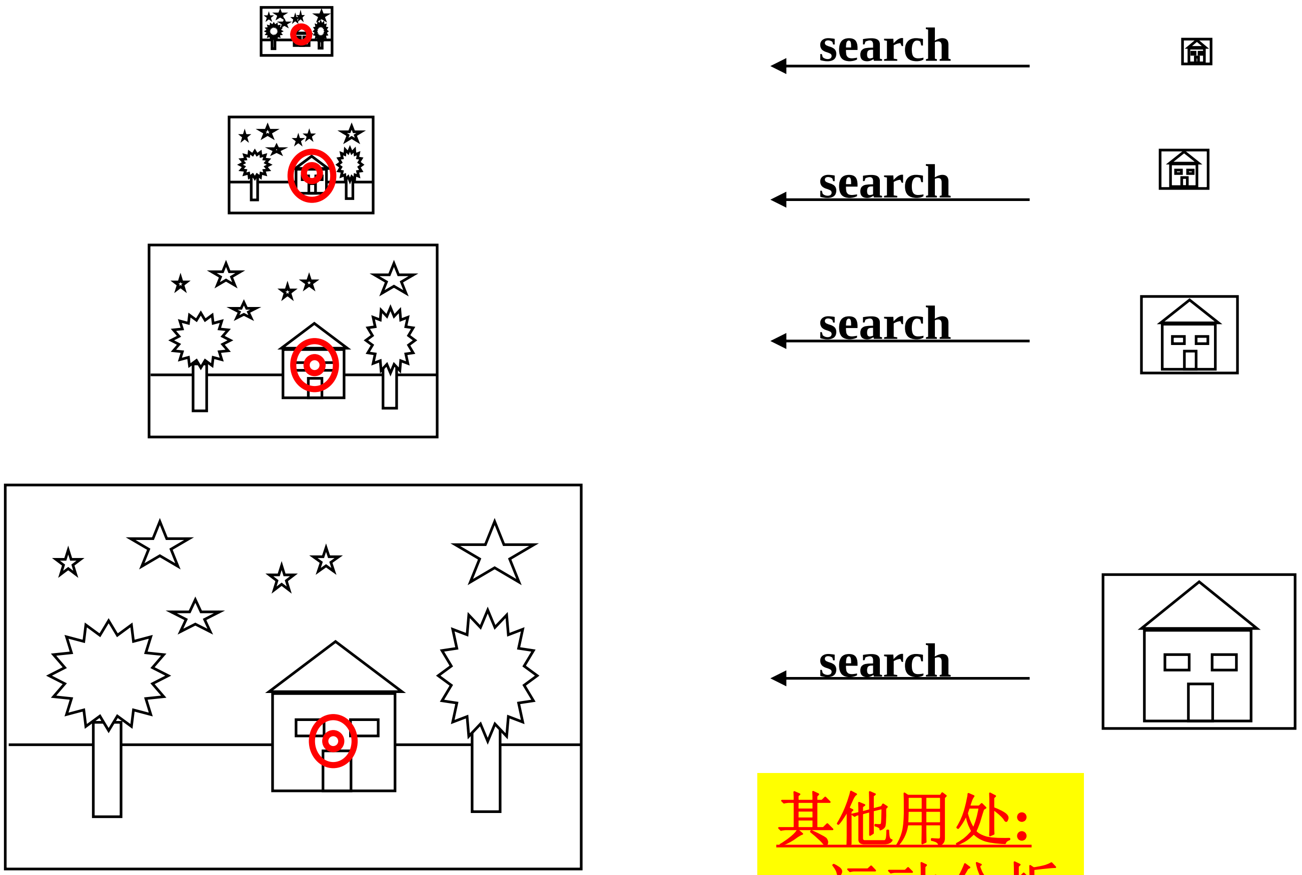
HCI/ComS 575X: Computational Perception
Iowa State University

图像金字塔结构



[www.wisdom.weizmann.ac.il/~mblank/CVfall04/handouts/lec4.ppt]

快速模板匹配

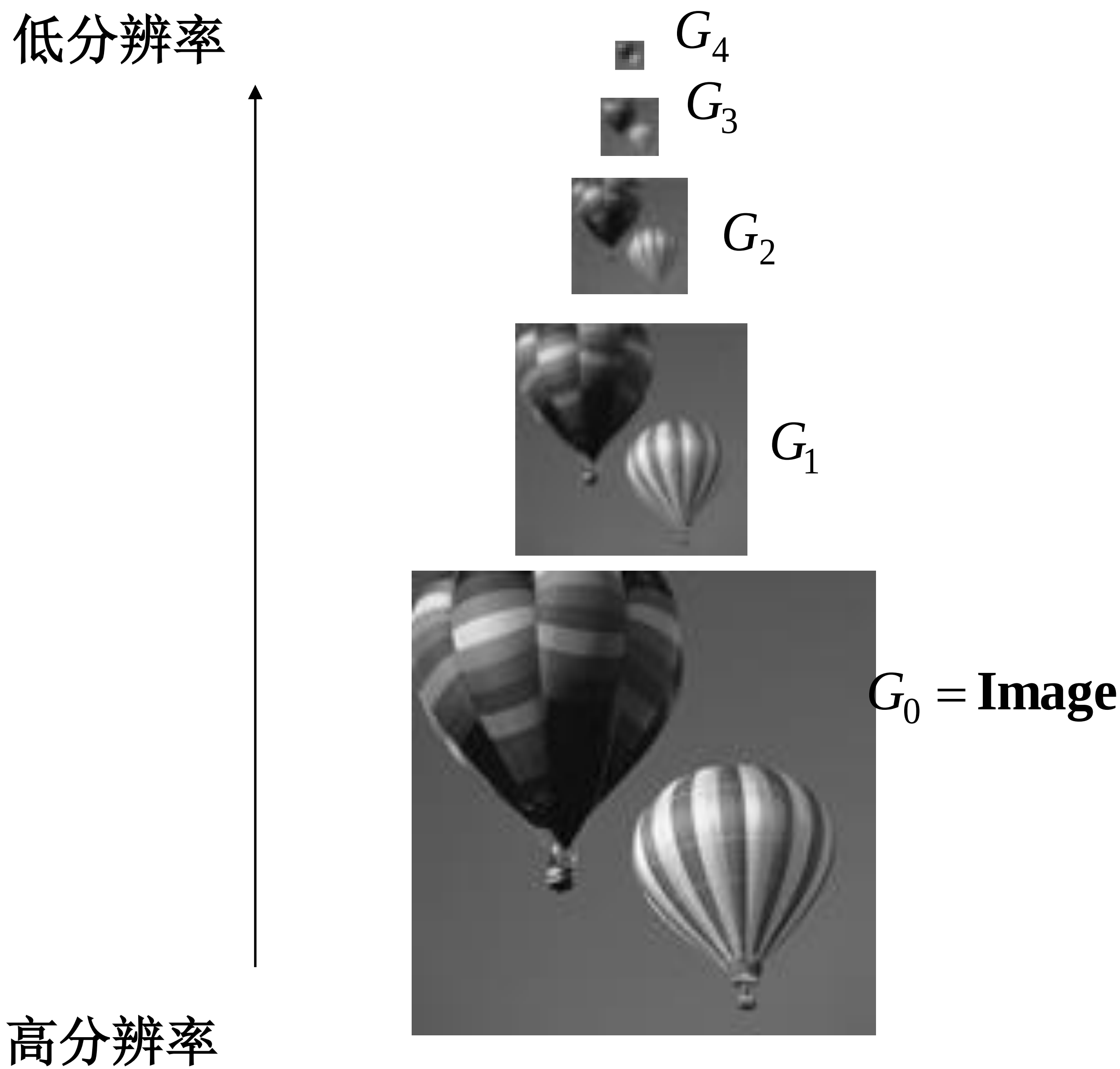


<u>其他用处:</u>

- 运动分析
- 图像压缩
- ……

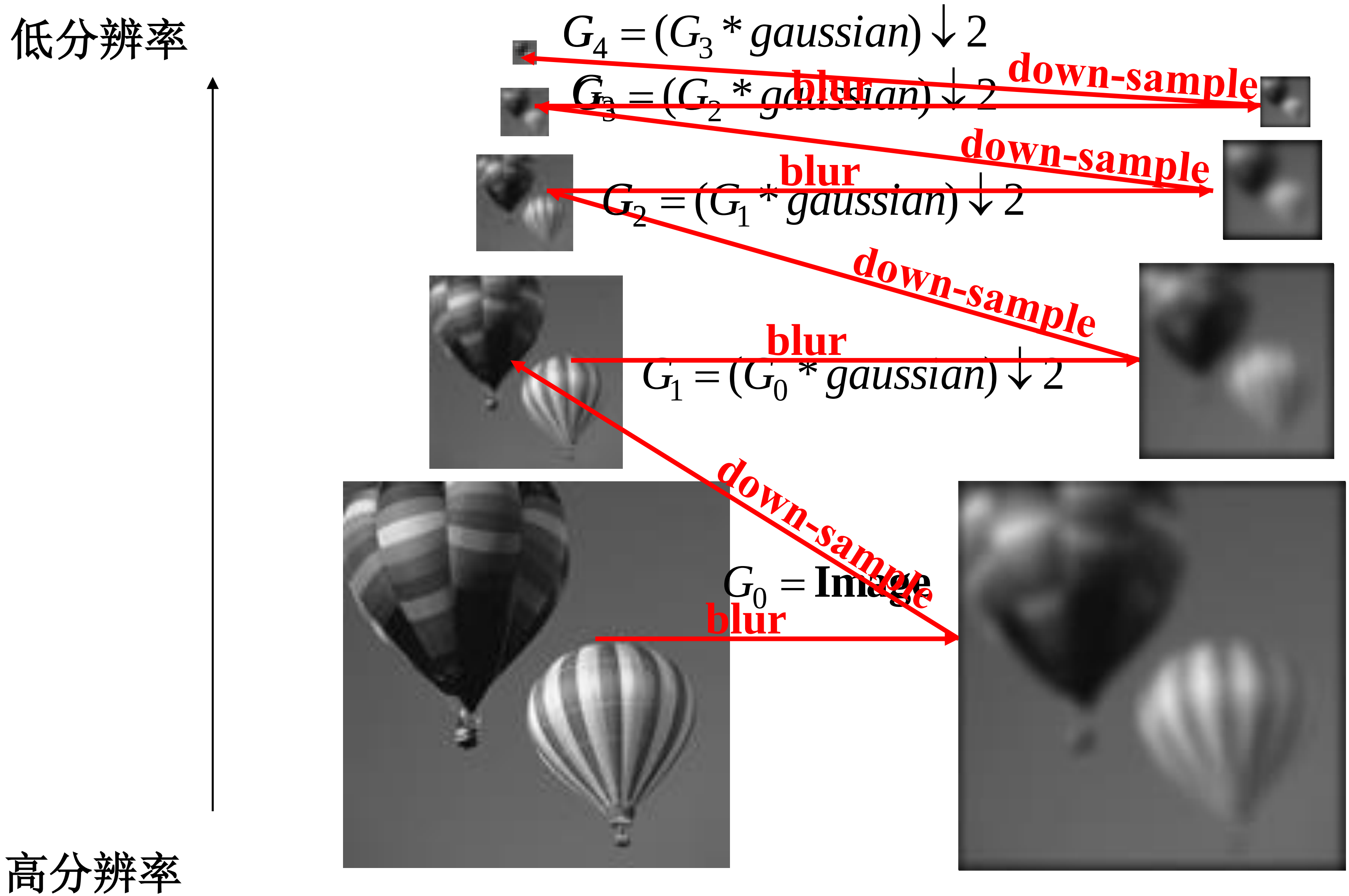
[www.wisdom.weizmann.ac.il/~mblank/CVfall04/handouts/lec4.ppt]

高斯金字塔



[www.wisdom.weizmann.ac.il/~mblank/CVfall04/handouts/lec4.ppt]

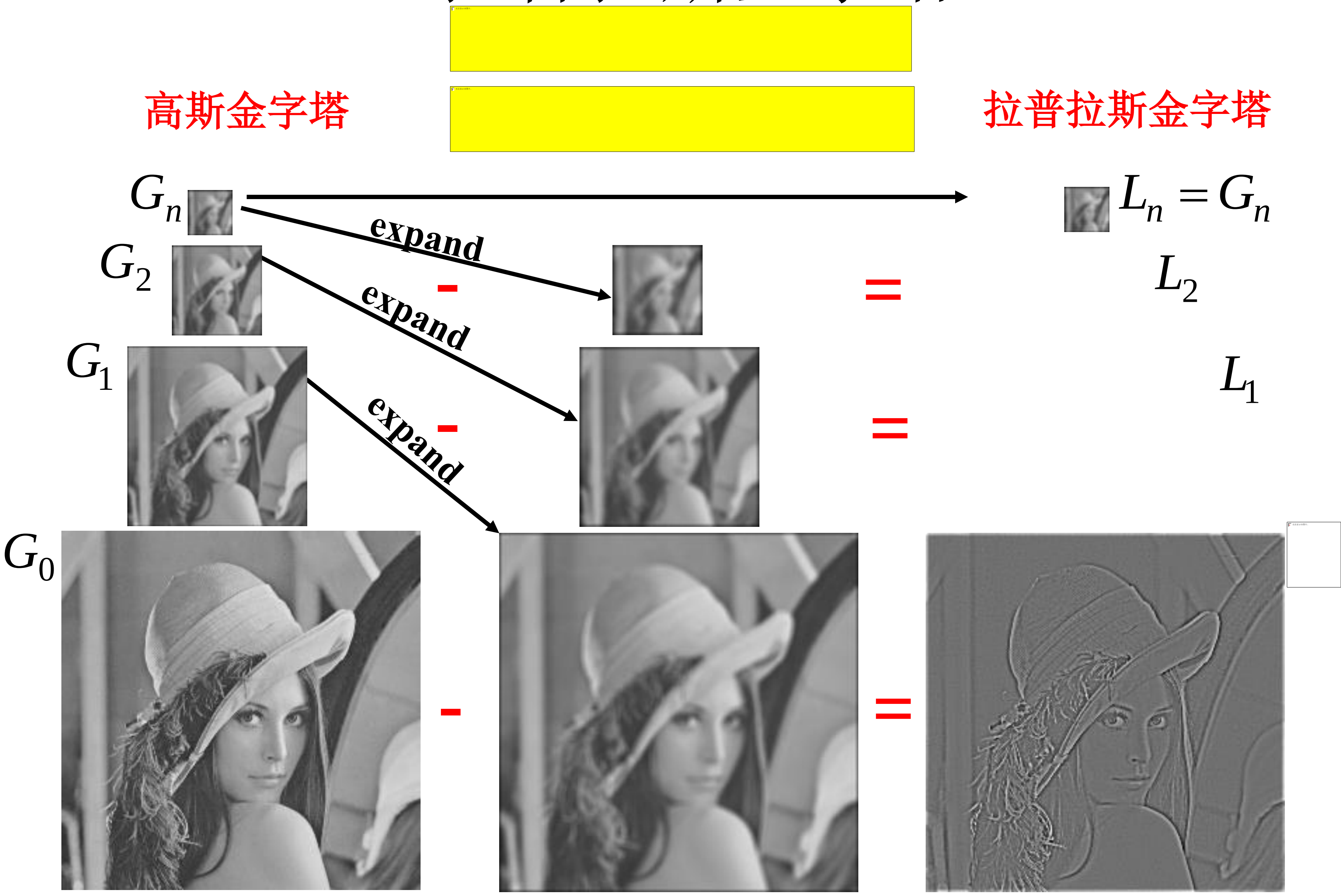
高斯金字塔



[www.wisdom.weizmann.ac.il/~mblank/CVfall04/handouts/lec4.ppt]

[www.wisdom.weizmann.ac.il/~mblank/CVfall04/handouts/lec4.ppt]

拉普拉斯金字塔



多尺度分析

- 1986年秋天， S. Mallat 正在跟Y. Meyer 读书，此前， Meyer构造了一组基， 即Meyer小波，同时， Mallat对图像分析比较感兴趣， 而且，图像在多个尺度上的信息利用在当时已经很广泛（典型的如Laplace金字塔编码），Mallat创造性地将二者结合， 提出了多尺度分析，一方面， 有助于进一步研究小波的性质， 同时， 建立了小波变换和信号处理的桥梁。

多尺度分析（续）

- Daubechies在《Ten Lectures on Wavelets》写道：
- “The history of the formulation of multiresolution analysis is a beautiful example of applications stimulating theoretical development”
- S. Mallat的两篇经典：
  - Multiresolution approximation and wavelets, Trans. Amer. Math. Soc., 1989, 315, 69-88
  - A theory for multiresolution signal decomposition: the wavelet representation, IEEE Trans. PAMI, 1989, 11, 674-693.

多尺度分析（续）

一串空间 V_j 和函数 $\varphi(\cdot)$ 如果满足如下条件，就称为一个正交多尺度分析：

1. $\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \cdots$
2. $\overline{\bigcup V_j} = L_2(R), \qquad \bigcap V_j = \{0\}$
3. $f(t) \in V_i \Leftrightarrow f(2^i t) \in V_0$
4. $f(t) \in V_0 \Rightarrow f(t-n) \in V_0, n \in Z$
5. V_0中存在一个标准正交基：　$\{\varphi(t-n) | n \in Z\}$

多尺度分析（续）

- 多尺度分析的有关结果：

  $\{\varphi(t-n)\}$ 是标准正交基当切仅当$\sum_{-\infty}^{+\infty}|\Phi(\omega+2\mathrm{k}\pi)|^2=1$

  $\{2^{i/2}\varphi(2^i t-n)\,|\,n\in Z\}$ 是 V_{-i}的标准正交基。

  $\varphi(t)$成为尺度函数，满足双尺度方程：

  $\varphi(t)=\sqrt{2}\sum g_0[n]\varphi(2t-n)$

  其中：$g_0[n]=<\varphi(t),\sqrt{2}\varphi(2t-n)>$

  定理：$|G_0(\omega)|^2+|G_0(\omega+\pi)|^2=2$

  $$G_0(0)=\sqrt{2},G_0(\pi)=0$$

多尺度分析（续）

- 多尺度分析的有关结果（续）：

定理: 给定一个 MRA $\left\{V_j,\varphi\right\}$, 定义 W_j 是 V_j 在 V_{j-1}中的正交补空间,那么存在一个函数ψ(t), 使得 $\left\{2^{\mathrm{i}/2}\psi(2^i t-n)\,|\,n\in Z\right\}$ 是 W_{-i}的标准正交基.

ψ(t)成为小波函数。

多尺度分析（续）

- 多尺度分析的有关结果（续）：

$$\psi(t)=\sqrt{2}\sum_k g_1[n]\varphi(2t-n) \quad \text{(小波方程)}$$

$$g_1[n]=<\psi(t),\sqrt{2}\varphi(2t-n)>$$

$$\Psi(\omega)=\frac{1}{\sqrt{2}}G_1(\omega/2)\Phi(\omega/2)$$

$$G_1(\omega)=\sum g_1[n]e^{-in\omega}$$

定理: $|G_1(\omega)|^2+|G_1(\omega+\pi)|^2=2$

$$G_1(\omega)\overline{G_0(\omega)}+G_1(\omega+\pi)\overline{G_0(\omega+\pi)}=0$$

多尺度分析（续）

- 多尺度分析的有关结果（续）

$V_{-1}=V_0\oplus W_0,\quad \left\{\sqrt{2}\varphi(2t-n)\right\}$ O.N.B V_{-1}

$\left\{\varphi(t-n)\right\}$ O.N.B $V_0,\quad \left\{\psi(t-n)\right\}$ O.N.B $W_0\Rightarrow$

$\left\{\varphi(t-n),\psi(t-n)\right\}$ O.N.B V_{-1}

V_{-1}中有两个基：

$f\in V_{-1},$

$$f(t)=\sum a_n^{-1}\sqrt{2}\varphi(2t-n)$$

$$=\sum a_n^0\varphi(t-n)+\sum b_n^0\psi(t-n)$$

多尺度分析（续）

- 多尺度分析的有关结果（续）

$$b_n^0 = \sum_k a_k^{-1} g_1[k-2n], \quad a_n^0 = \sum_k a_k^{-1} g_0[k-2n]$$

$$a_n^{-1} = \sum_k a_k^0 g_0[n-2k] + \sum_k b_k^0 g_1[n-2k]$$

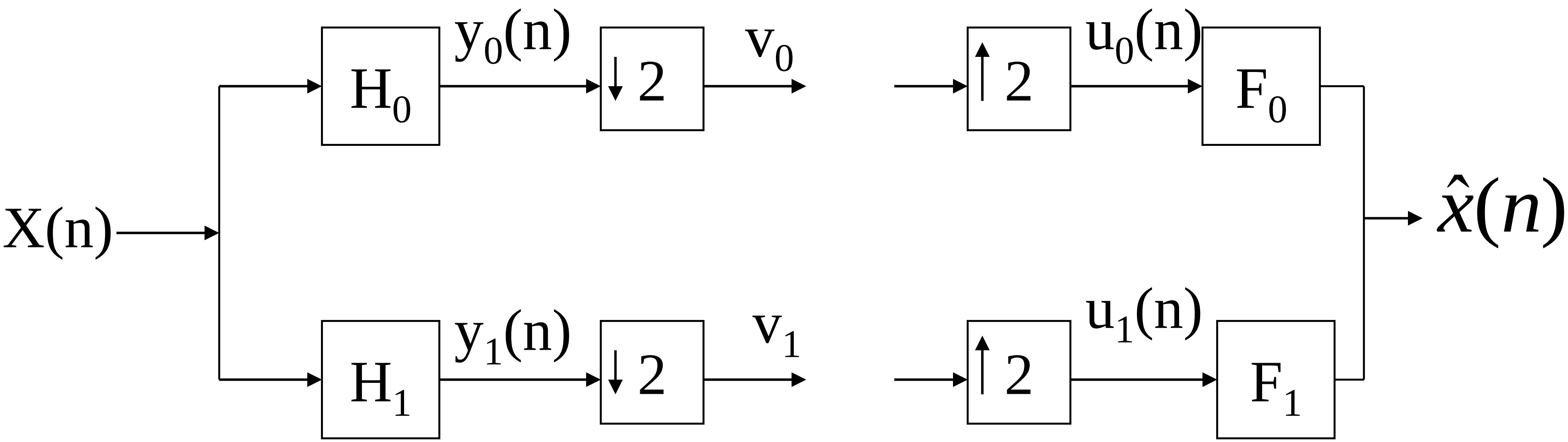
$$\boxed{a_k^{-1}} \to \boxed{\tilde{g}_0} \to \boxed{\downarrow 2} \to \boxed{a_n^0}$$

$$\downarrow \to \boxed{\tilde{g}_1} \to \boxed{\downarrow 2} \to \boxed{b_n^0}$$

$$\boxed{a_m^0} \to \boxed{\uparrow 2} \to \boxed{g_0} \to$$

$$\boxed{b_m^0} \to \boxed{\uparrow 2} \to \boxed{g_1} \to \downarrow \oplus \to \boxed{a_n^{-1}}$$

滤波器组



滤波器组

- 定理：

  完全重建条件：

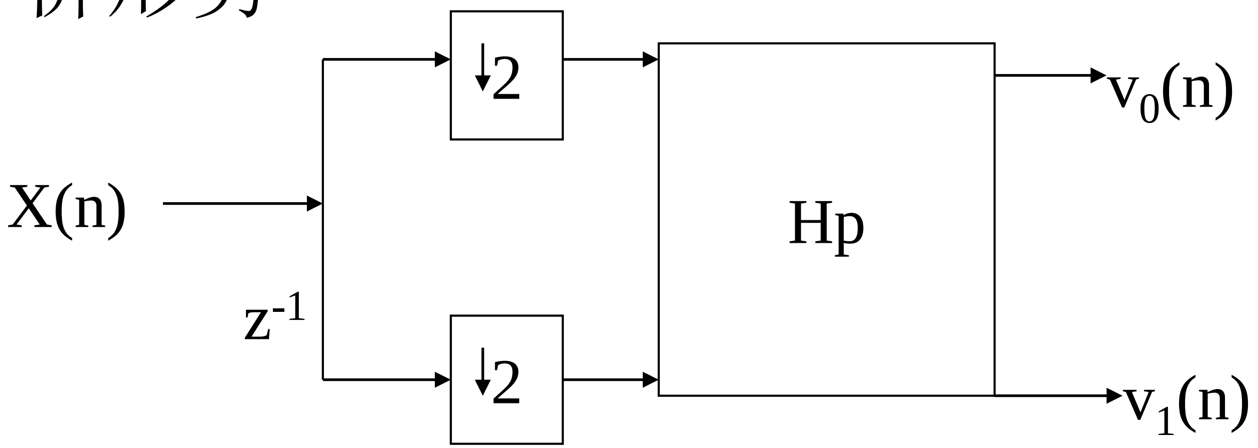
  – $F_0(z)H_0(z)+F_1(z)H_1(z)=2z^{-L}$

  – $F_0(z)H_0(-z)+F_1(z)H_1(-z)=0$

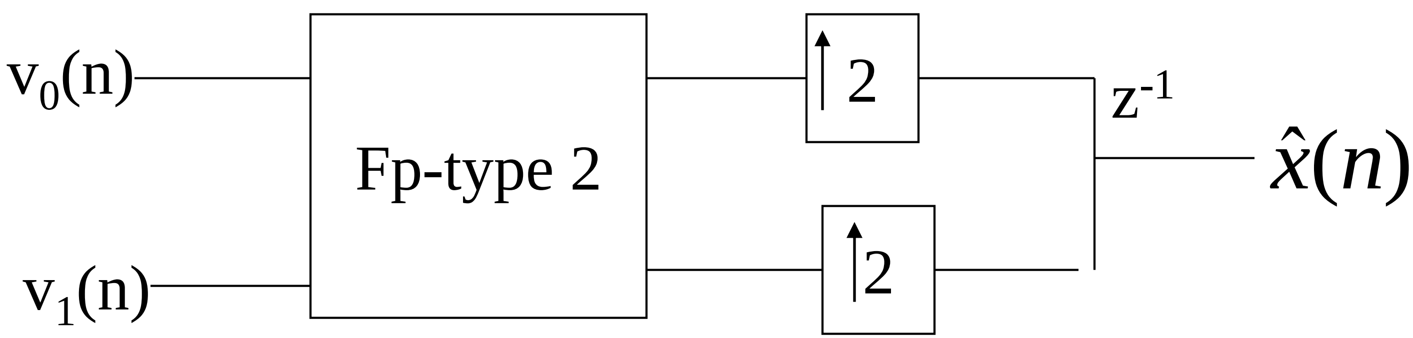
$$\begin{bmatrix} F_0(z) & F_1(z) \\ F_0(-z) & F_1(-z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \begin{bmatrix} 2z^{-l} & 0 \\ 0 & 2(-z)^{-l} \end{bmatrix}$$

滤波器组

- 等价形势



X(n)
$\downarrow 2$
$\downarrow 2$
z^{-1}
Hp
$v_0(n)$
$v_1(n)$



$v_0(n)$
$v_1(n)$
Fp-type 2
$\uparrow 2$
$\uparrow 2$
z^{-1}
$\hat{x}(n)$

滤波器组

- 其中：

$$H_p = \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} \quad \begin{matrix} H_0(z) = H_{00}(z^2) + H_{01}(z^2)z^{-1} \\ H_1(z) = H_{10}(z^2) + H_{11}(z^2)z^{-1} \end{matrix}$$

$$F_p^{II} = \begin{bmatrix} F_{0,odd}(z) & F_{1,odd}(z) \\ F_{0,even}(z) & F_{1,even}(z) \end{bmatrix}$$

$$F_p^{I} = \begin{bmatrix} F_{0,even}(z) & F_{1,even}(z) \\ F_{0,odd}(z) & F_{1,odd}(z) \end{bmatrix}$$

滤波器组

- 定理：

  如果：$\hat{X}(z)=X(z)z^{-2L-1}$

  - 完全重建条件
    - Fp(z)Hp(z)=I or z^{-L}I
  - Hp 是第一型, Fp 第二型。

滤波器组

- 正交滤波器组构造：

  如果:$H_p(z)H_p^T(z^{-1}) = I$

  那么

  $$H_p(z) = \Lambda(-1)R_l\Lambda(z)R_{l-1}...R_1\Lambda(z)R_0$$

  其中: $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $\Lambda(\text{z})=\begin{bmatrix} 1 & \\ & \text{z}^{-1} \end{bmatrix}$

- 一般情况

  只要$\det(H_p(z)) = cz^l, F_p = H_p^{-1}$

滤波器组

- Daubechies 小波

$$H(z)=\left(\frac{1+z^{-1}}{2}\right)^{N} R(z),\ R(z)\text{是次数小于}N\text{的多项式。}$$

$$P(z)=|H(z)|^{2}=2(1-y)^{p}\sum_{k=0}^{p-1}\binom{p+k-1}{k}y^{k}$$

$$y=\frac{1-\cos(\varpi)}{2}.$$

小波与滤波器组的关系

- 二者等价

  MRA可以得到滤波器组

  滤波器组可以得到相应的MRA；

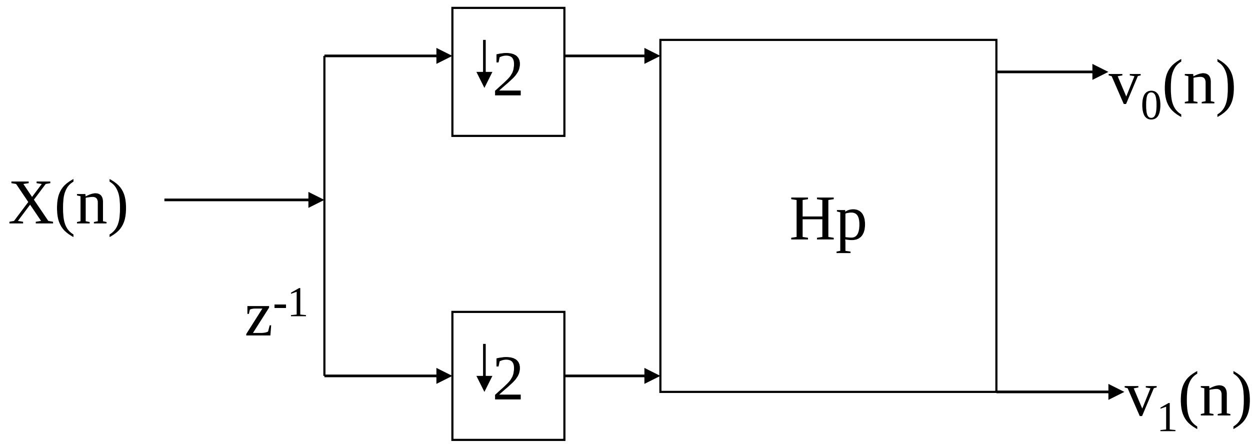
  连续信号 $f(t) \to$ 投影到 V_J

  $\to$ 映射到两个子空间 V_{J+1}, W_{J+1} 就是信号分解.

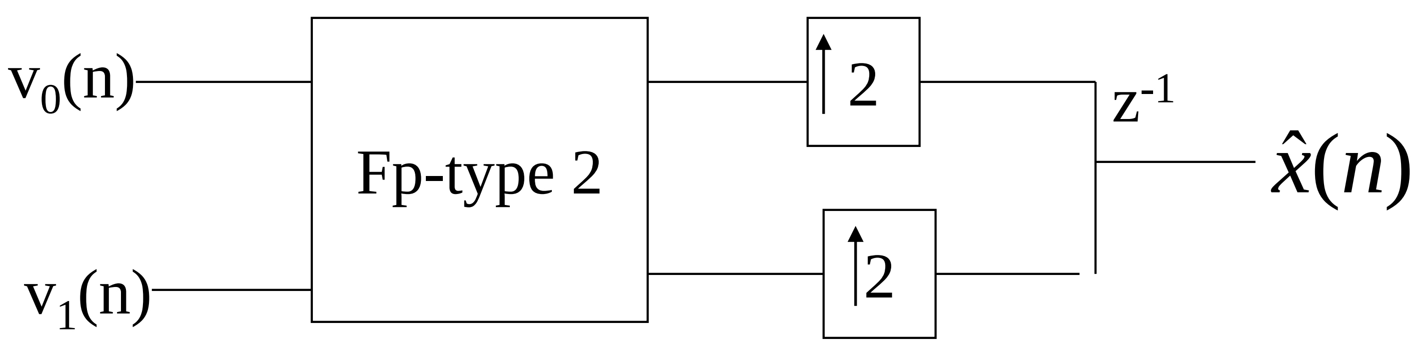
  离散信号 (Shannon 采样结果)

  $\to$ 几乎等同于连续信号在 V_J的投影系数

  $\to$ 滤波过程 (小波滤波器) = 映射到 V_{J+1}, W_{J+1}

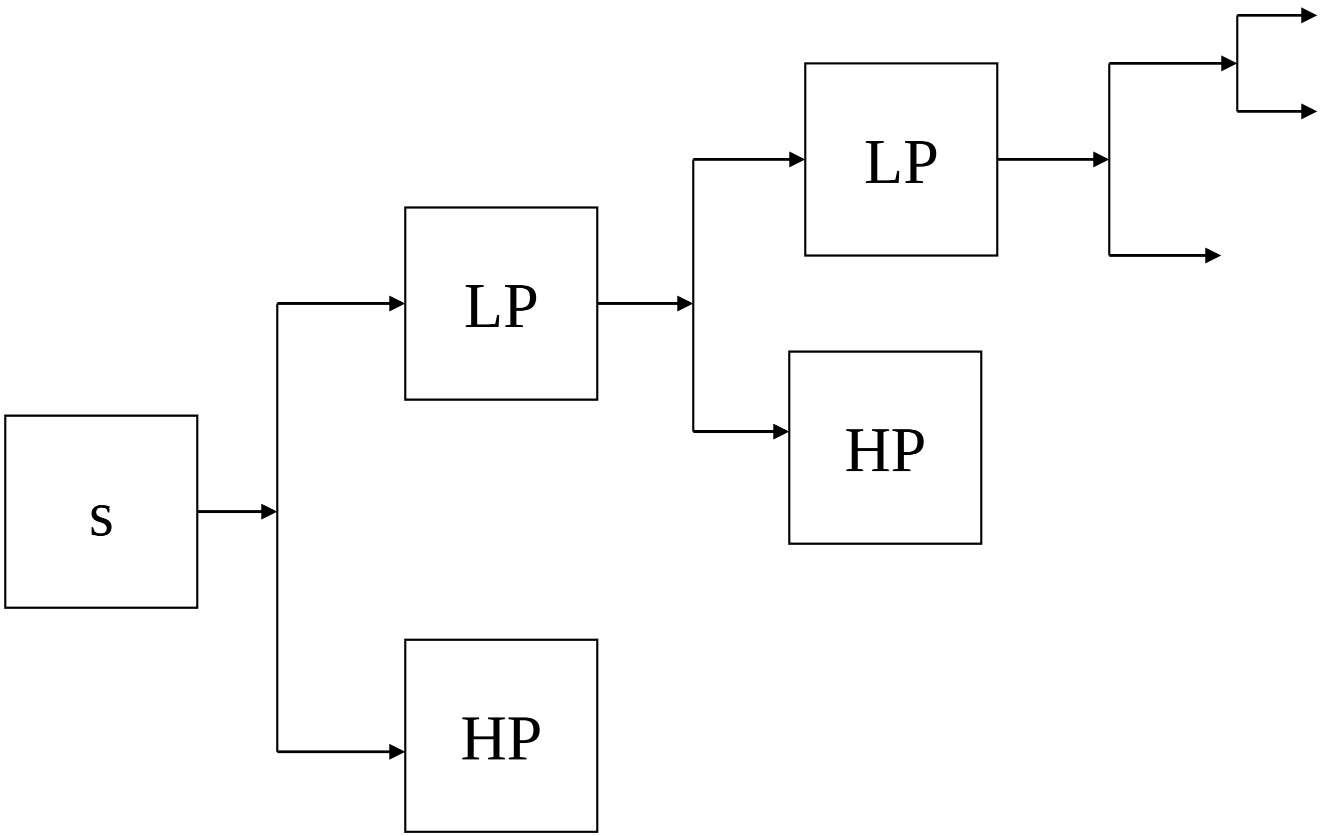


X(n)
z-1
2
2
Hp
v0(n)
v1(n)

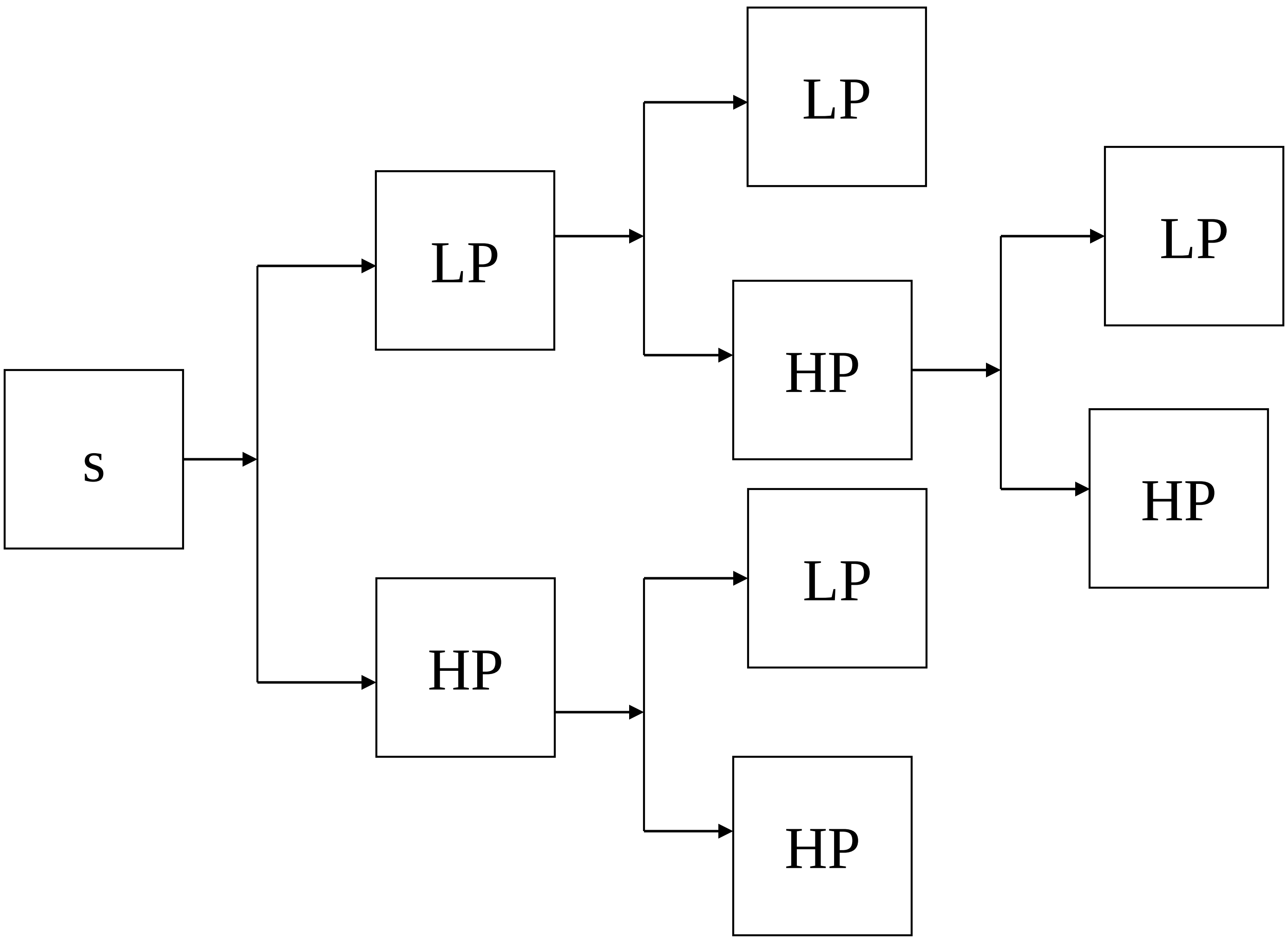


v0(n)
v1(n)
Fp-type 2
2
2
z-1
x̂(n)

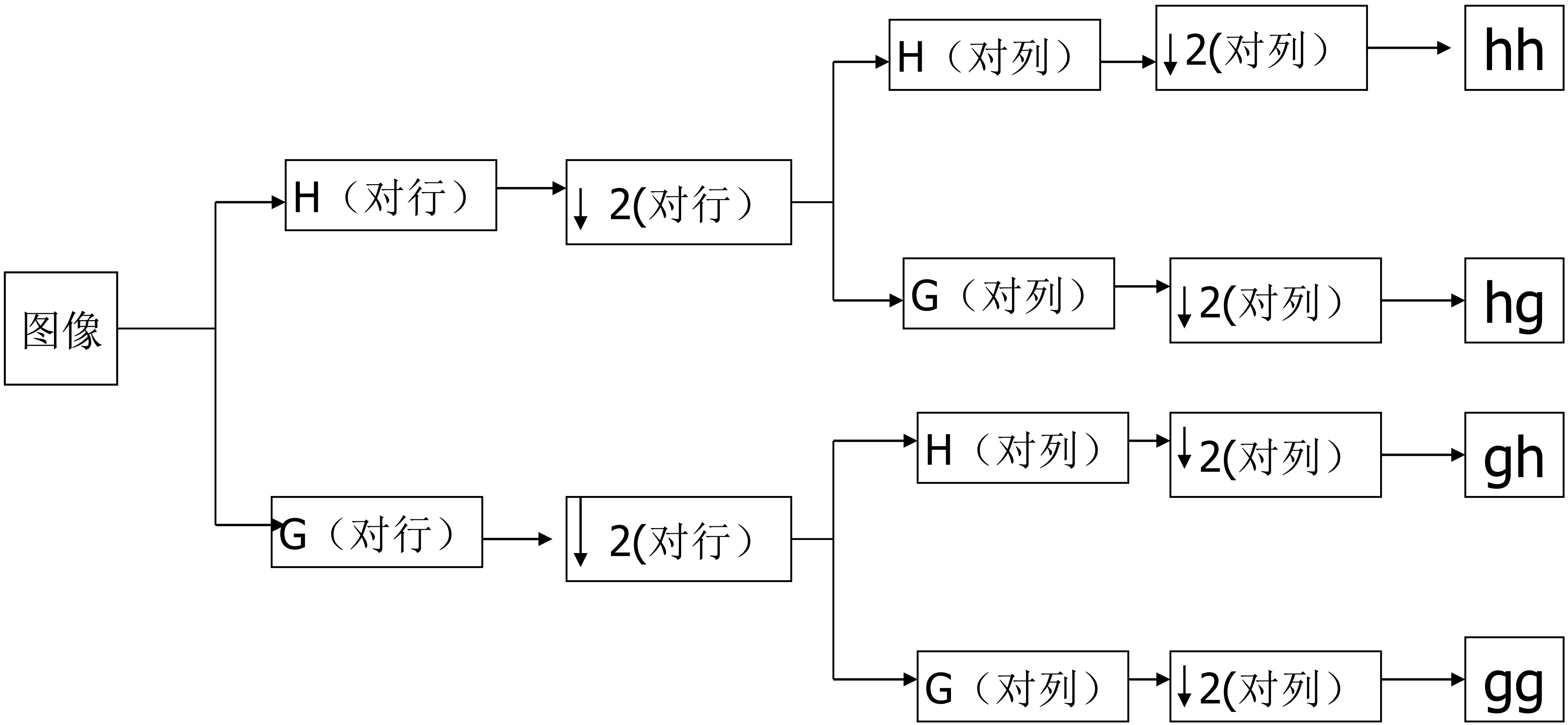
小波分解算法（一维）：Mallat 算法



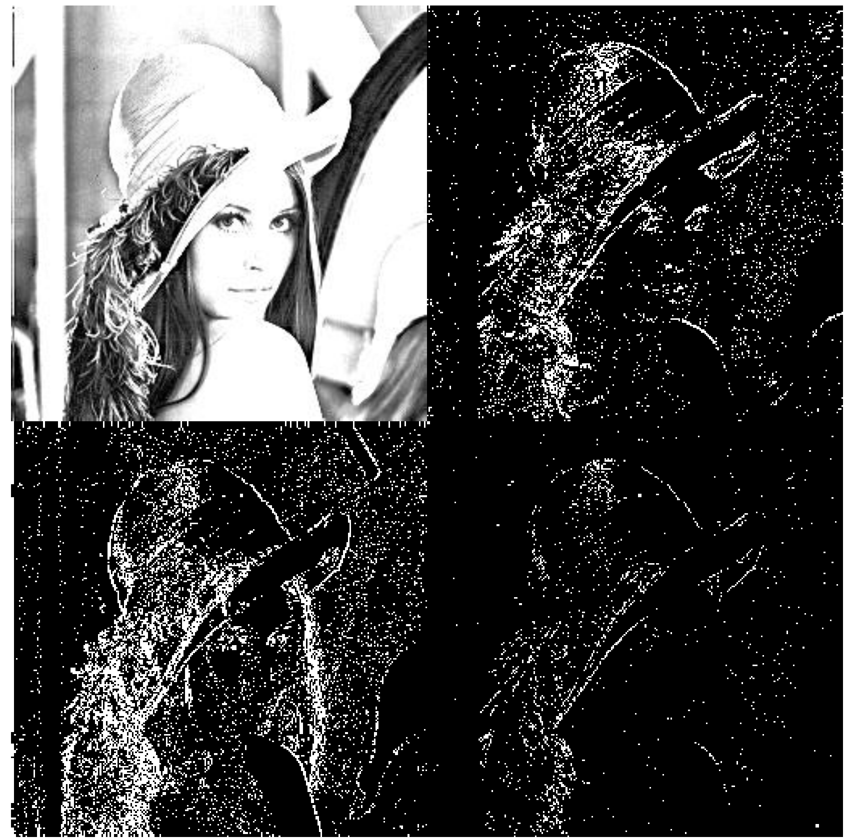
小波包（Wavelet Packet）



图像的小波分解

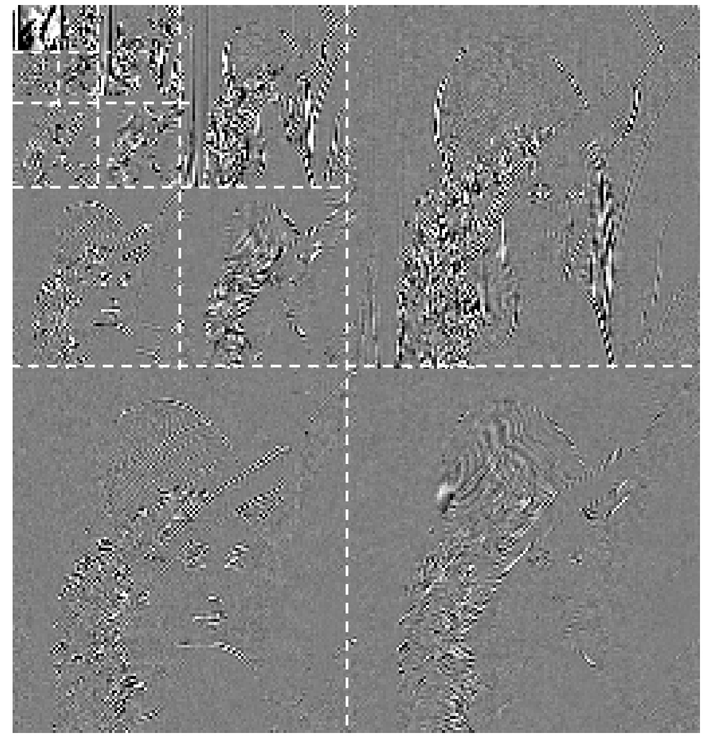


图像的小波分解例子：一次分解



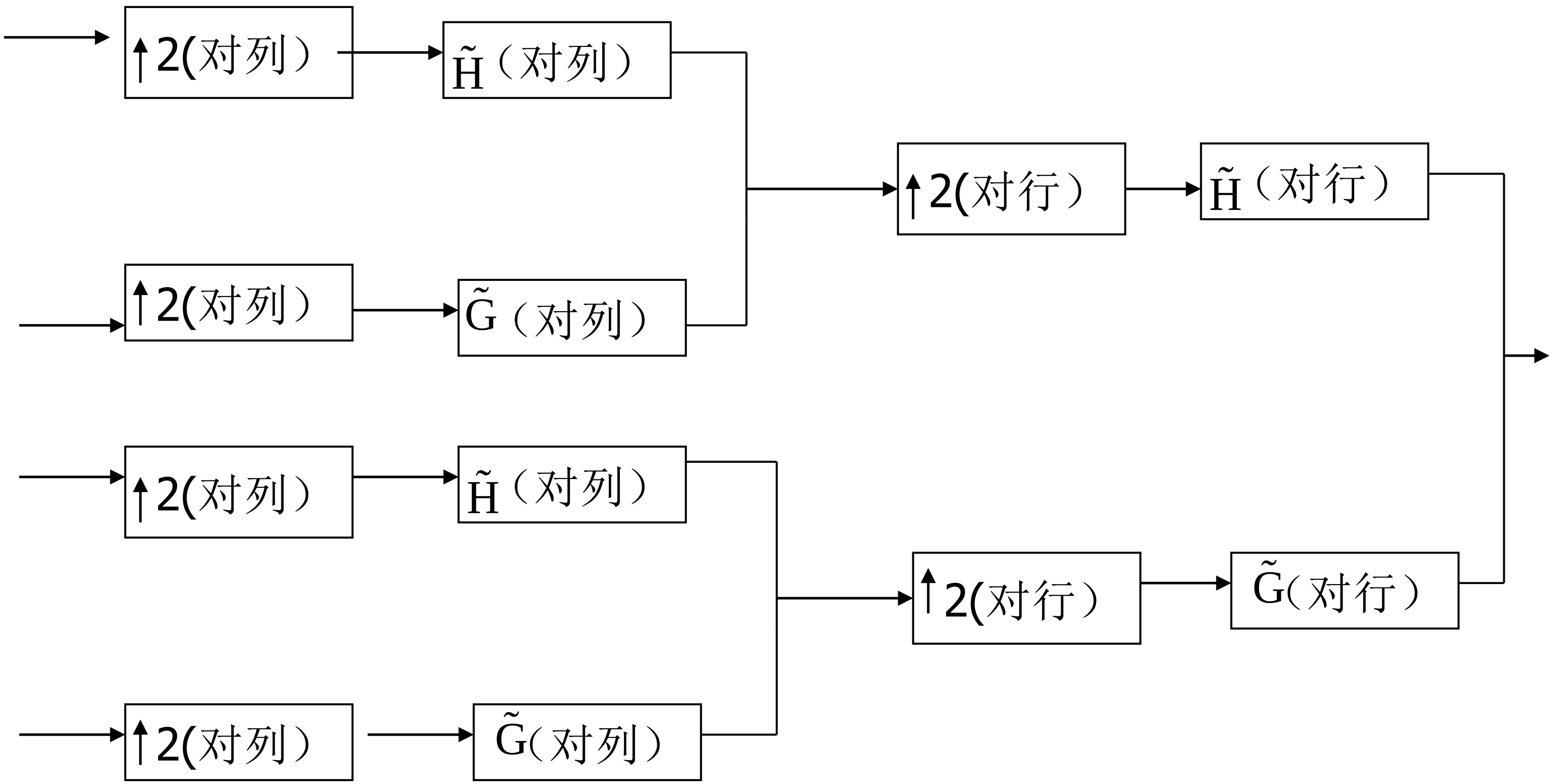
图像的小波分解例子：金字塔分解

Original
128, 129, 125, 64, 65, …



Transform Coeff.
4123, -12.4, -96.7, 4.5, …

图像重构



↑2(对列)
$\tilde{H}$（对列）
↑2(对列)
$\tilde{G}$（对列）
↑2(对行)
$\tilde{H}$（对行）
↑2(对列)
$\tilde{H}$（对列）
↑2(对列)
$\tilde{G}$（对列）
↑2(对行)
$\tilde{G}$（对行）

图像分解中的边界处理

- 对于图像来说，尺寸一般都不大，滤波器作用到图像的边界上时，需要用到边界之外的数据，如果给出这些未知的数据就是边界处理问题。
- 常用方法：
  - 补零：将未知数据补成零，这样会产生人造边缘；
  - 周期化：将图像当作一个周期信号在一个周期内的数据，然后按照周期的方法得到未知数据；适合于数值计算，同样产生人造边缘；

图像分解中的边界处理

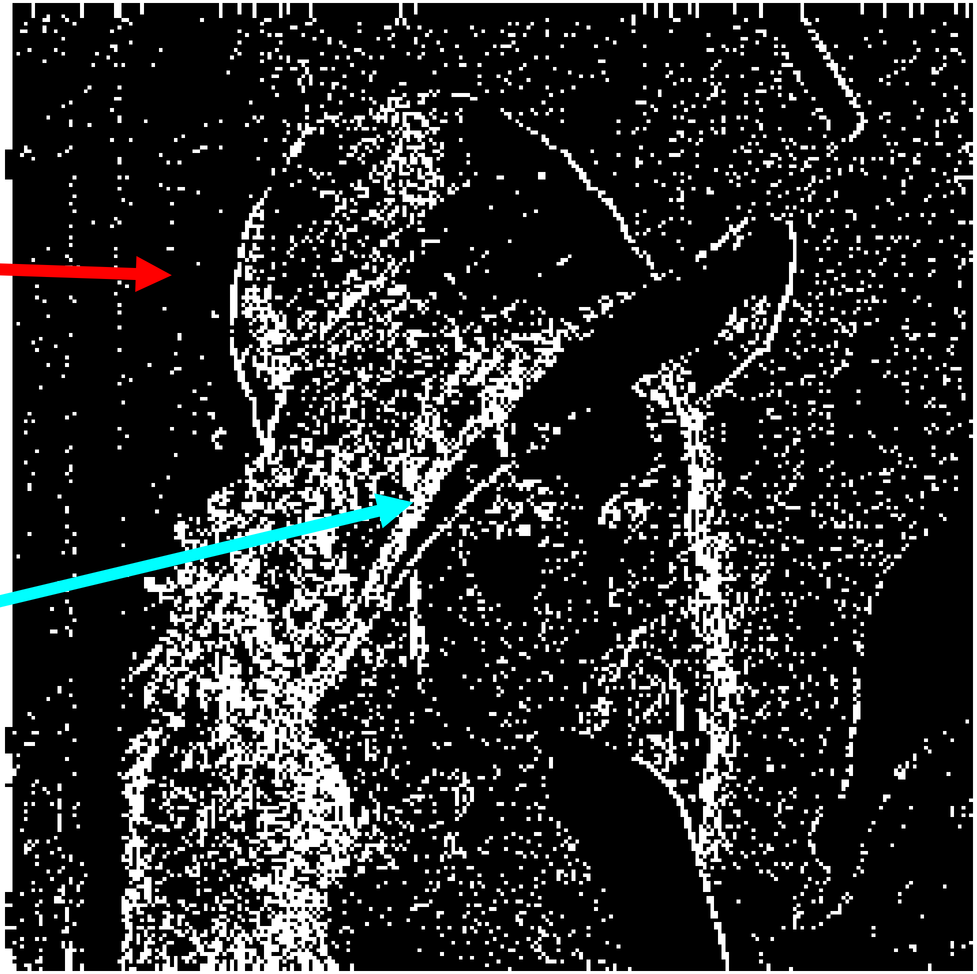
- 常用方法（续）
  - 对称化处理：就是以边界为中心，边界外的数据和边界内的数据形成中心对称。这是最常用的方法，尤其是当滤波器是对称的时候，能够在保持数据量为一半的情况下，保证数据的完整性；
  - 多项式外插法：将边界部分看作是多项式的一部分，外插得到相应的数据。该方法对Daubechies正交小波有效，但较为复杂，不太常用。

图像在小波域一般特性

- 能量聚集性

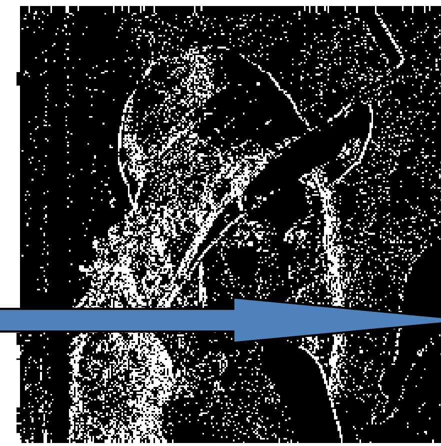
数据较小接近0

数据较大



图像在小波域一般特性

- 局部性和边缘检测性



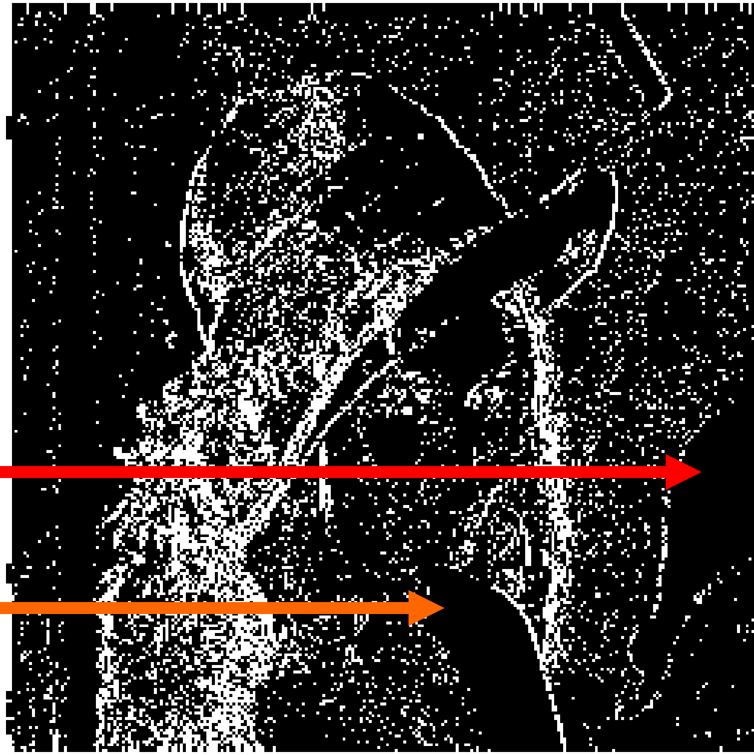
数据局部相关

存在边缘的地方数值大

图像在小波域一般特性

- 解相关性

强相关数据
变成弱相关



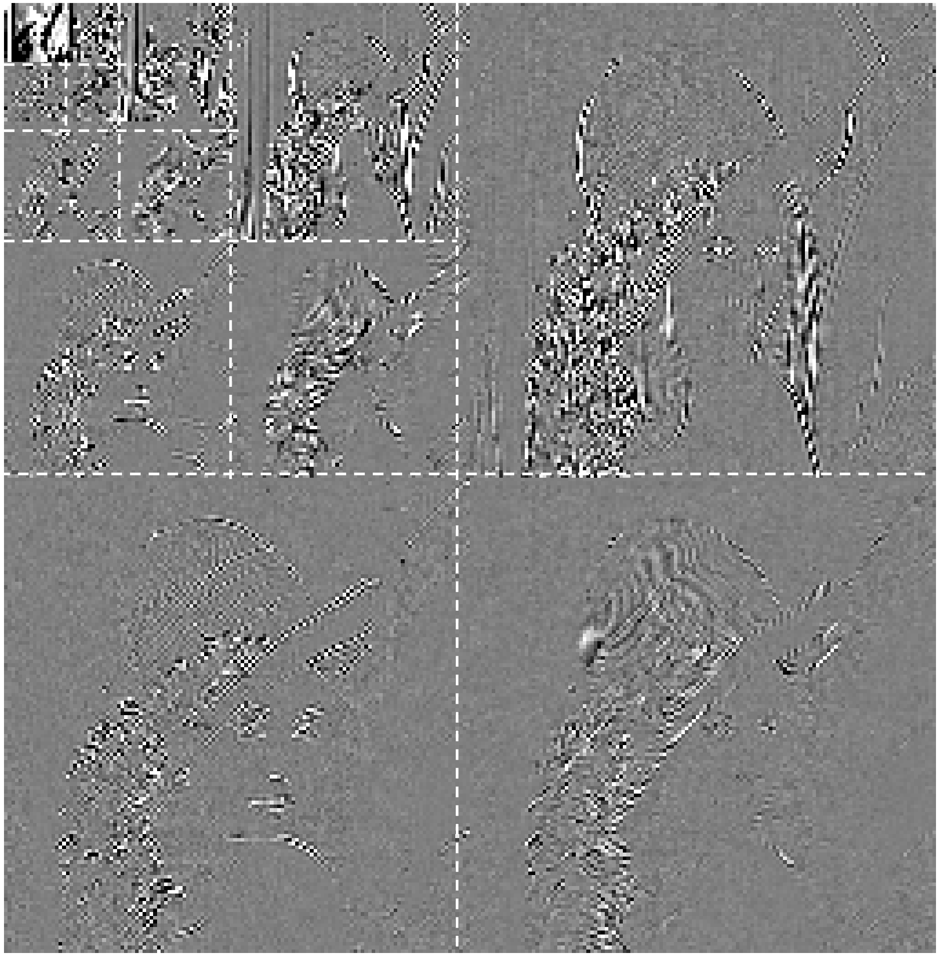
图像在小波域一般特性

- **多分辨率性**

第三次分解

第二次分解

第一次分解



图像在小波域一般特性

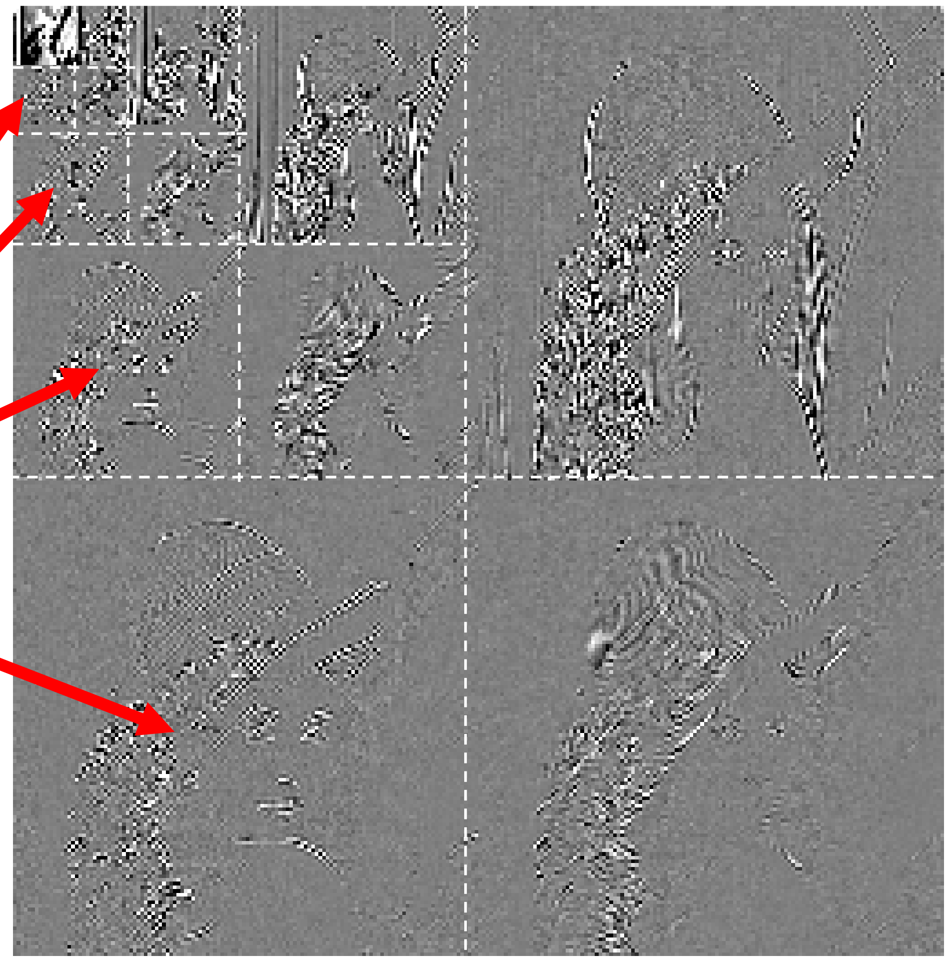
- 聚类性

边缘具有
连续性

图像在小波域一般特性

- 保持性

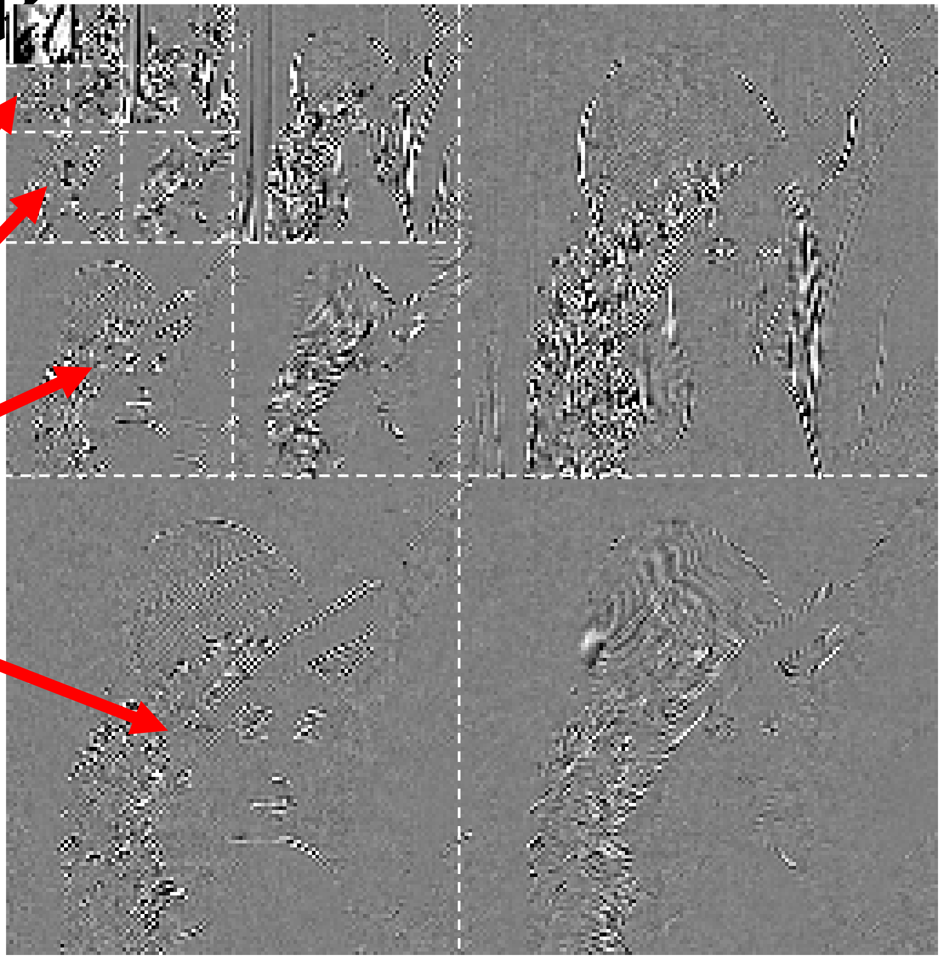
尺度间数据
分布的规律相似



图像在小波域一般特性

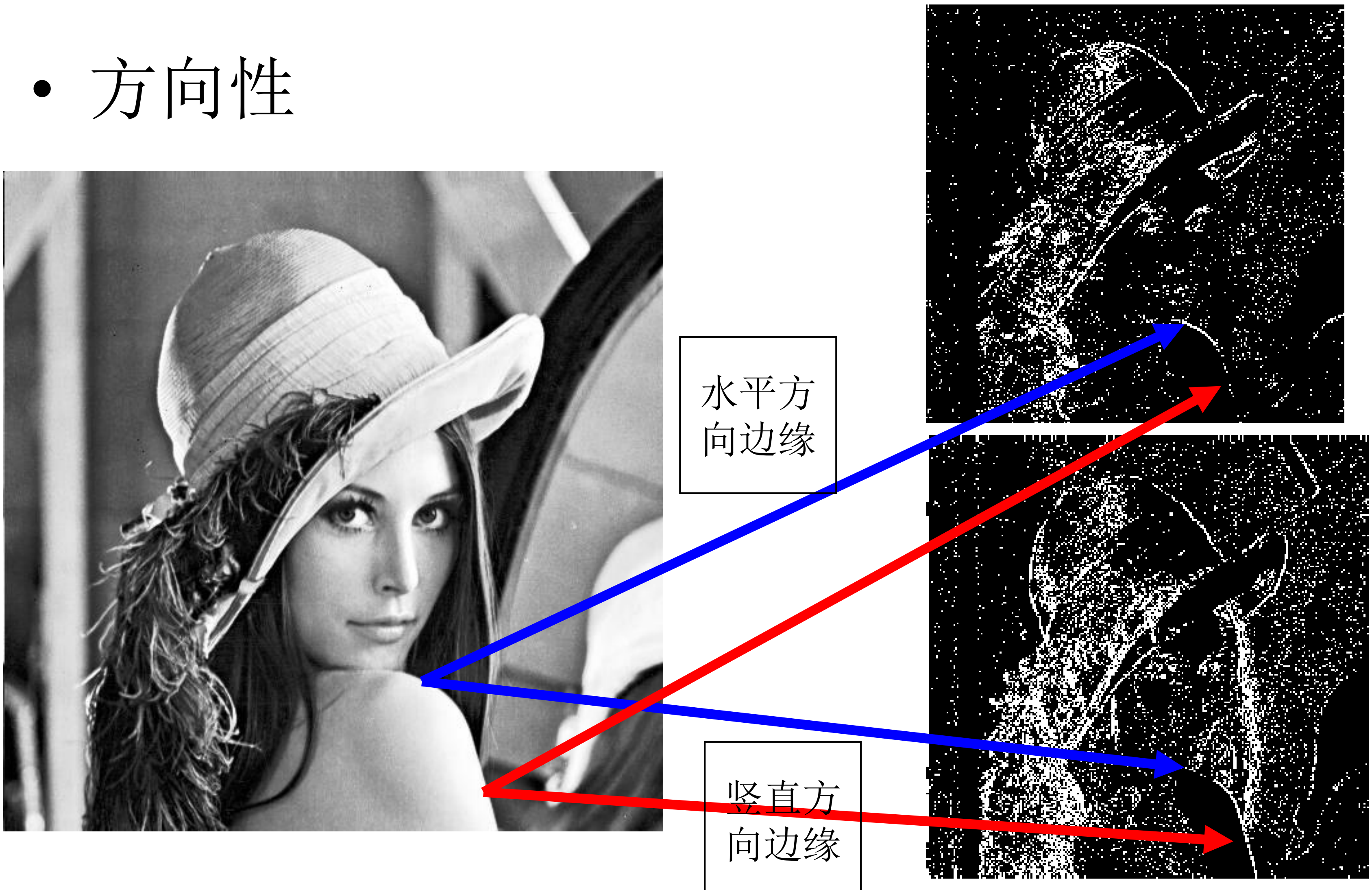
- **尺度间呈指数退化**

同一位置处数据指数退化



图像在小波域一般特性

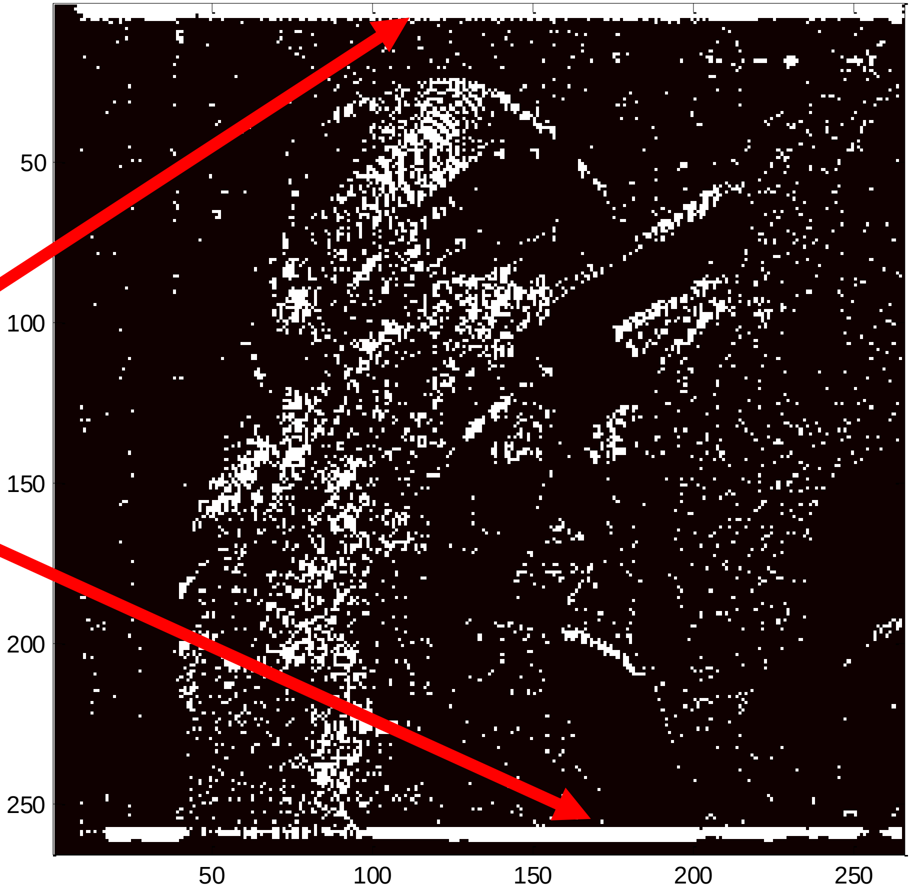
- 方向性



图像分解中的边界处理

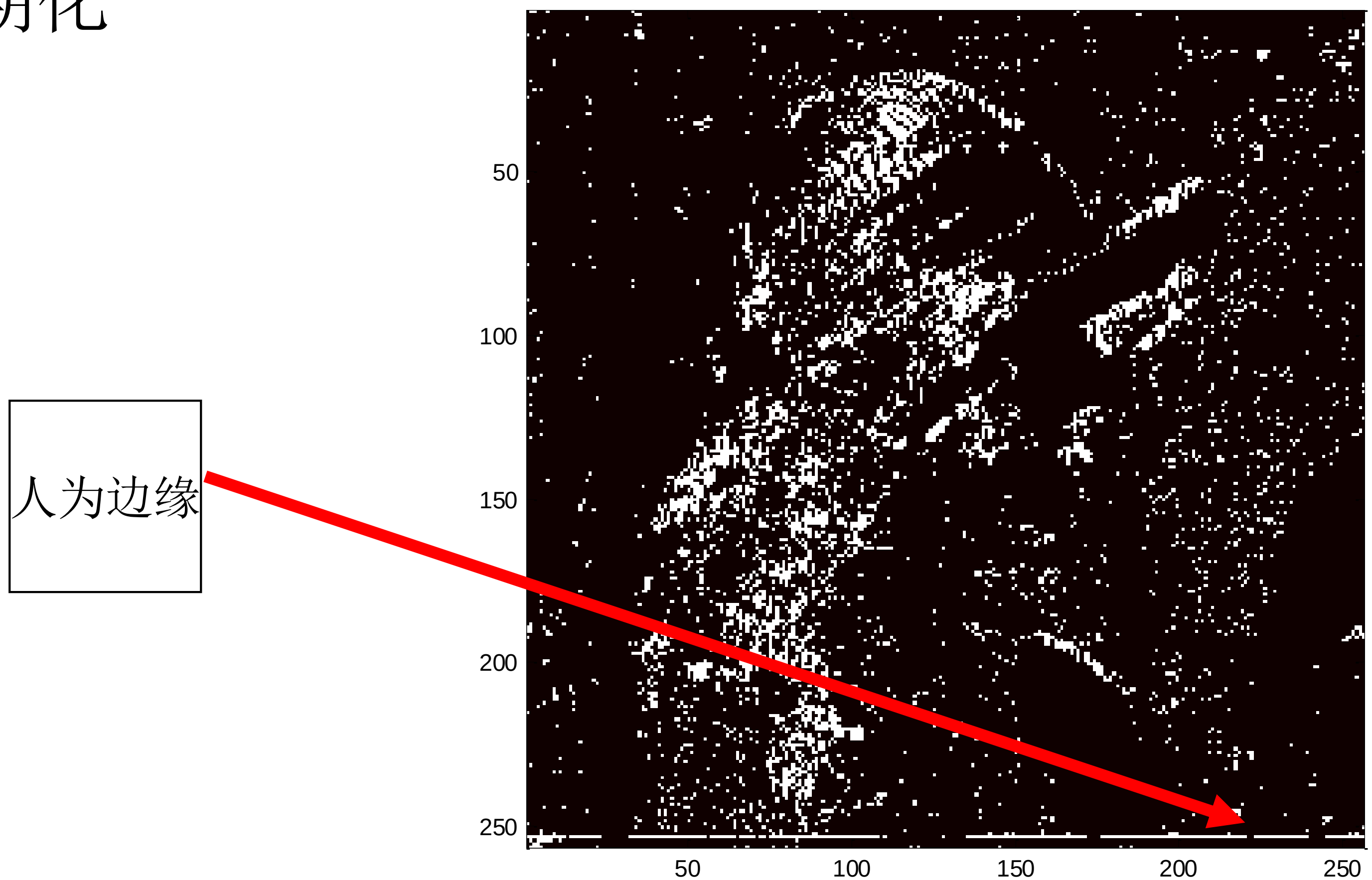
- 补零

人为边缘



图像分解中的边界处理

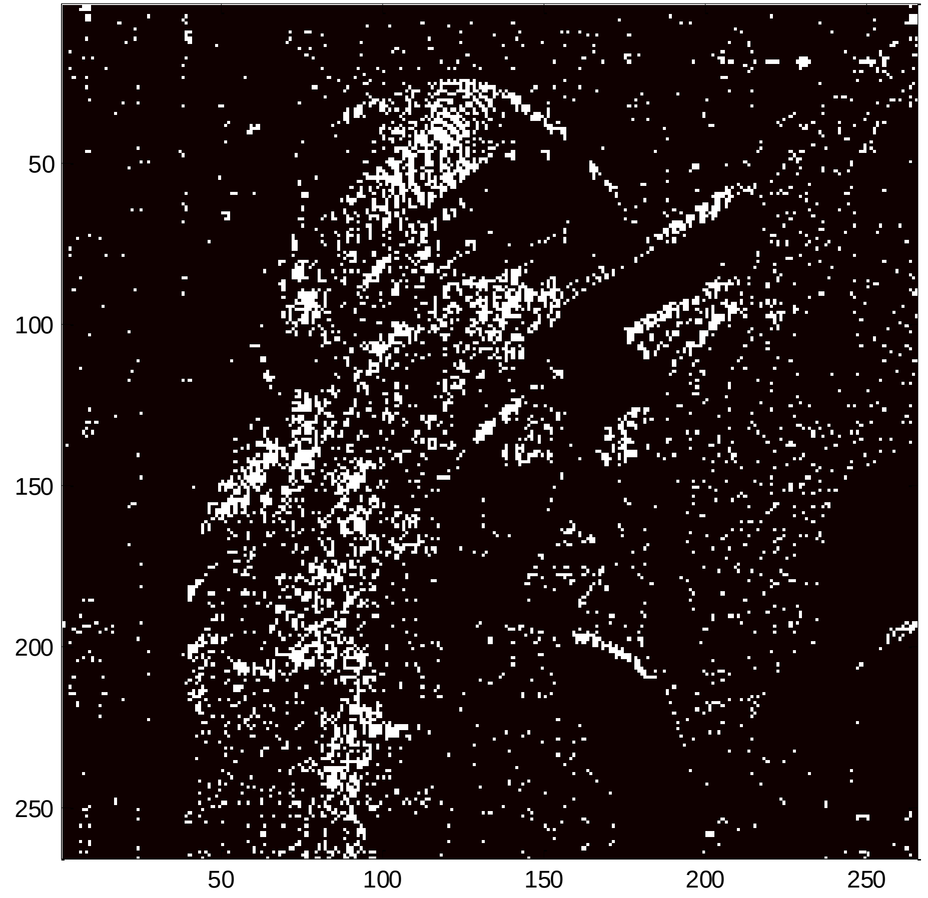
- 周期化



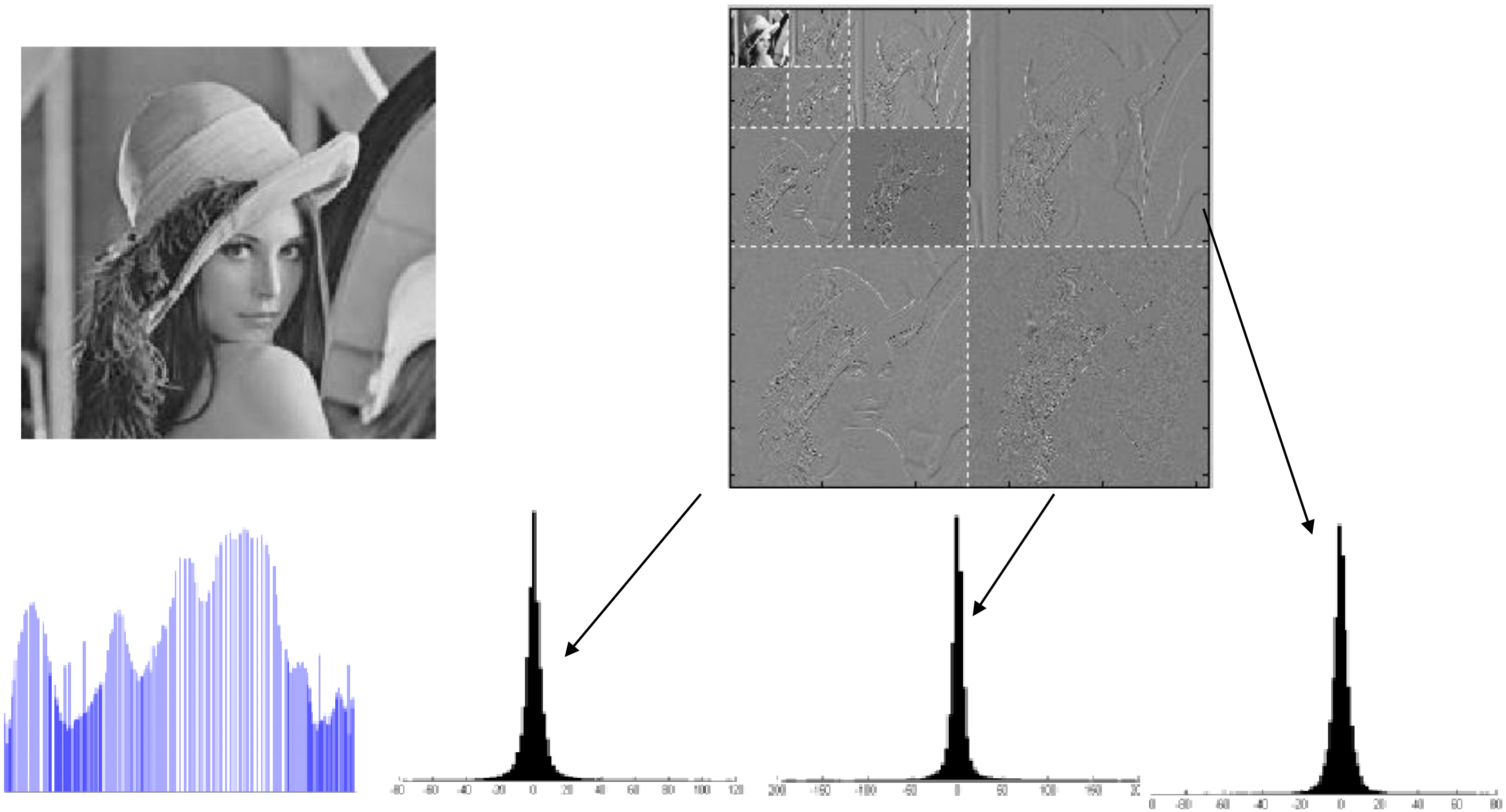
图像分解中的边界处理

- 对称化

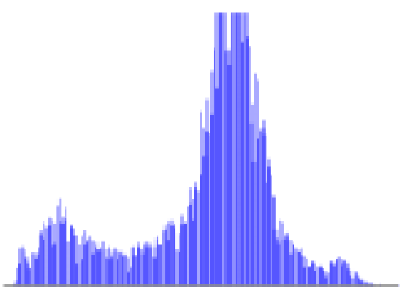
人为边缘很弱

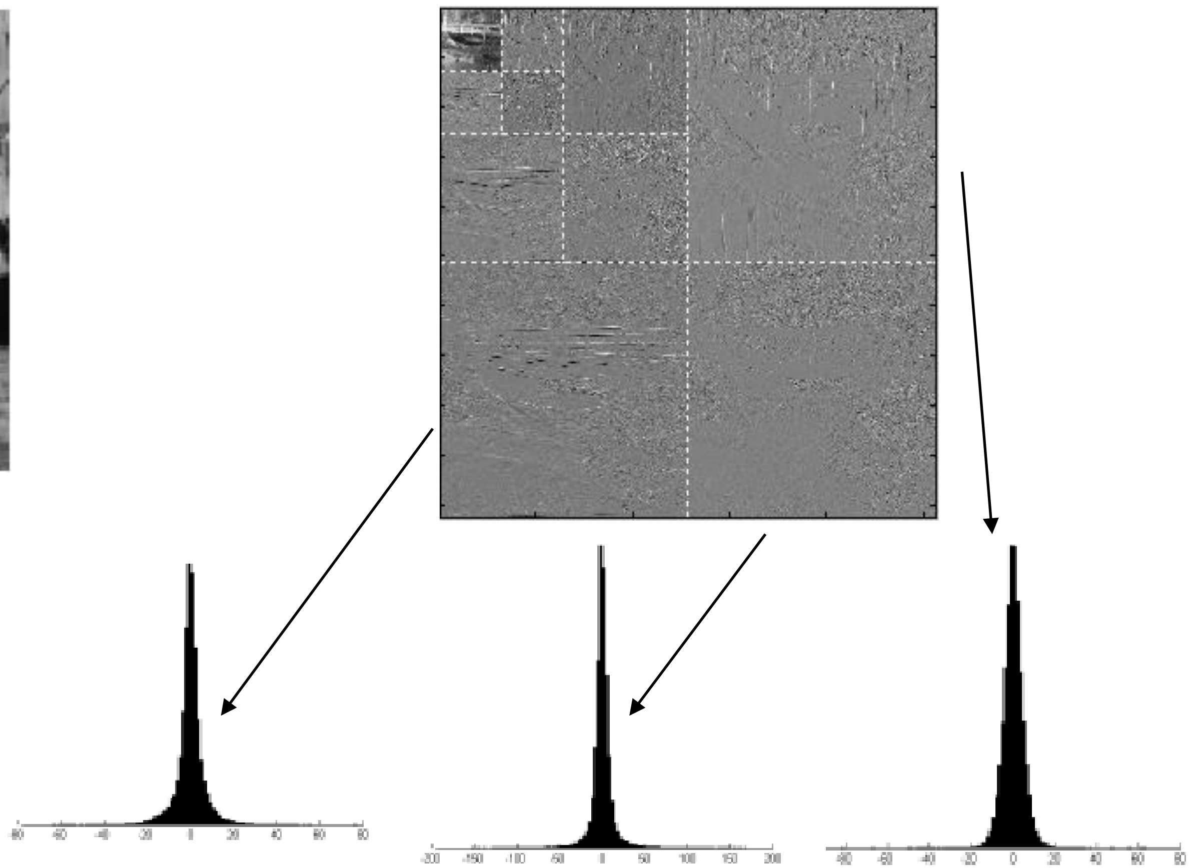


图像小波域简单统计特性



图像小波域简单统计特性





图像小波域简单统计特性

- **1. 广义高斯模型**

$$p(w;v,\sigma)=\left[\frac{v\eta(v,\sigma)}{2\Gamma(1/v)}\right]\exp(-\eta(v,\sigma)\,|\,w\,|^{v})$$

$$where \quad \eta(v,\sigma)=\sigma^{-1}\left[\frac{\Gamma(3/v)}{\Gamma(1/v)}\right]^{1/2}$$

- **2. Laplace:**

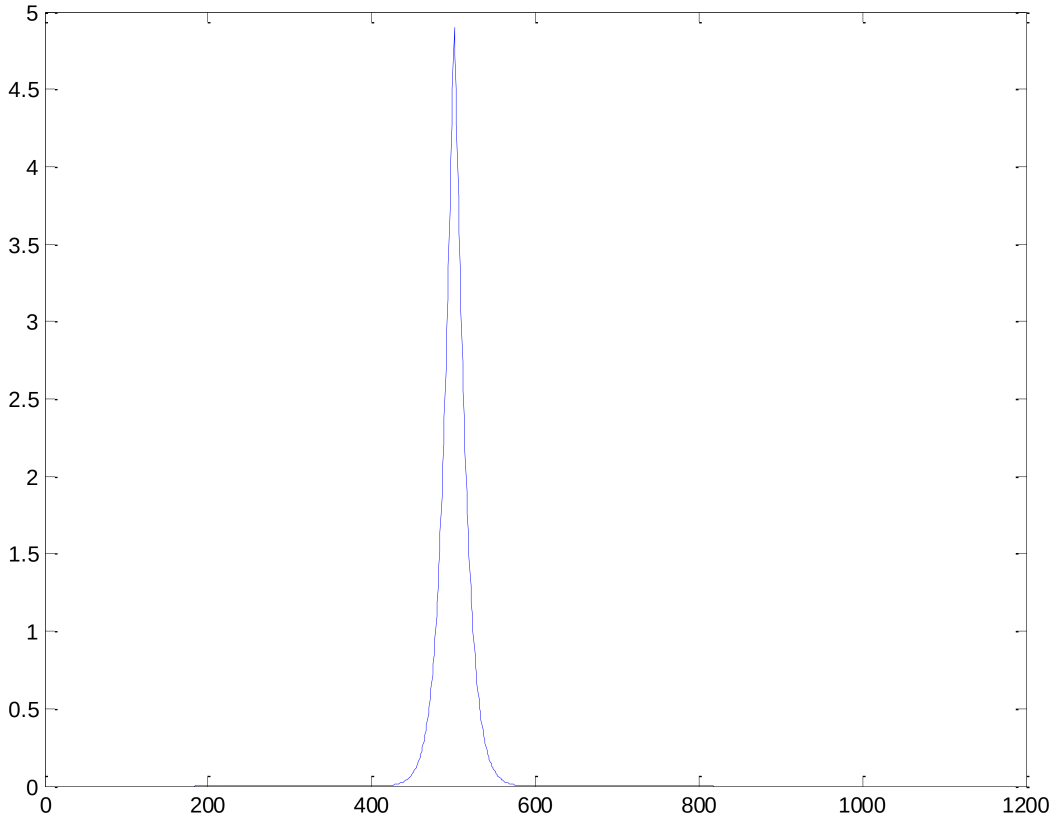
$$p(w;\sigma)=\frac{1}{\sqrt{2}\sigma}\exp\left(-\frac{\sqrt{2}\,|\,w\,|}{\sigma}\right)$$

- **3. 广义Laplace:**

$$p(w;s,p)=\frac{\exp\left(-\left|w/s\right|^{p}\right)}{Z(s,p)}, \quad where \quad Z(s,p)=2\frac{s}{p}\Gamma\left(\frac{1}{p}\right)$$

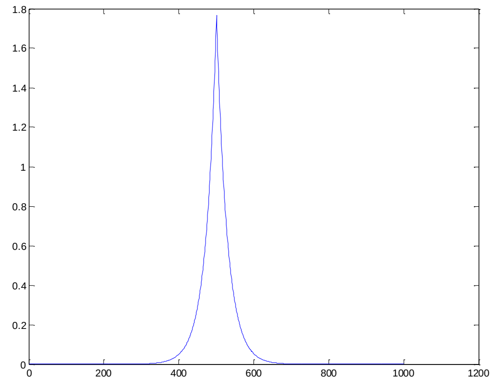
图像小波域简单统计特性

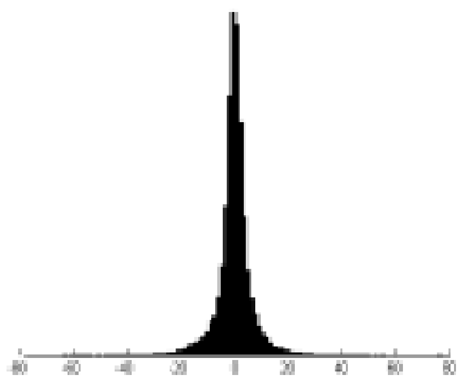
v=1.2
σ=0.1



图像小波域简单统计特性

$v=1, \sigma=0.2$





图像小波域简单统计特性

- **4. 高斯混合模型**

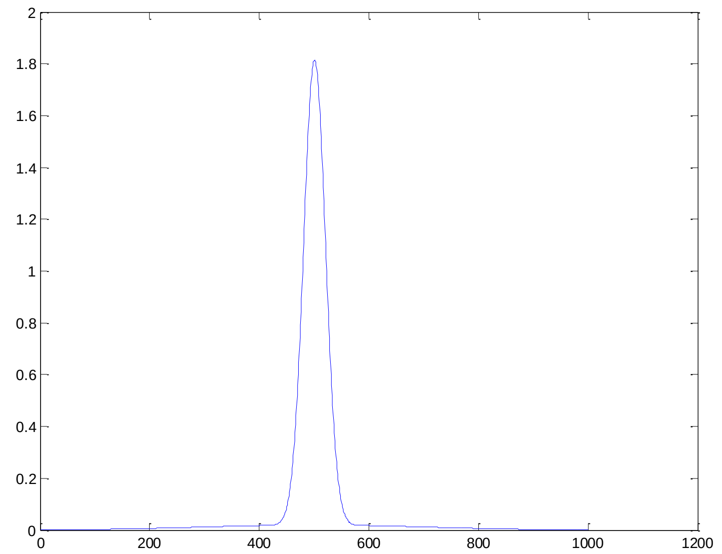
$$p(w) = \sum_i p(s = s_i)\, p(w \mid s = s_i)$$

$$where \qquad p(w \mid s = s_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(w - u_i)^2}{2\sigma_i^2}\right)$$

图像小波域简单统计特性

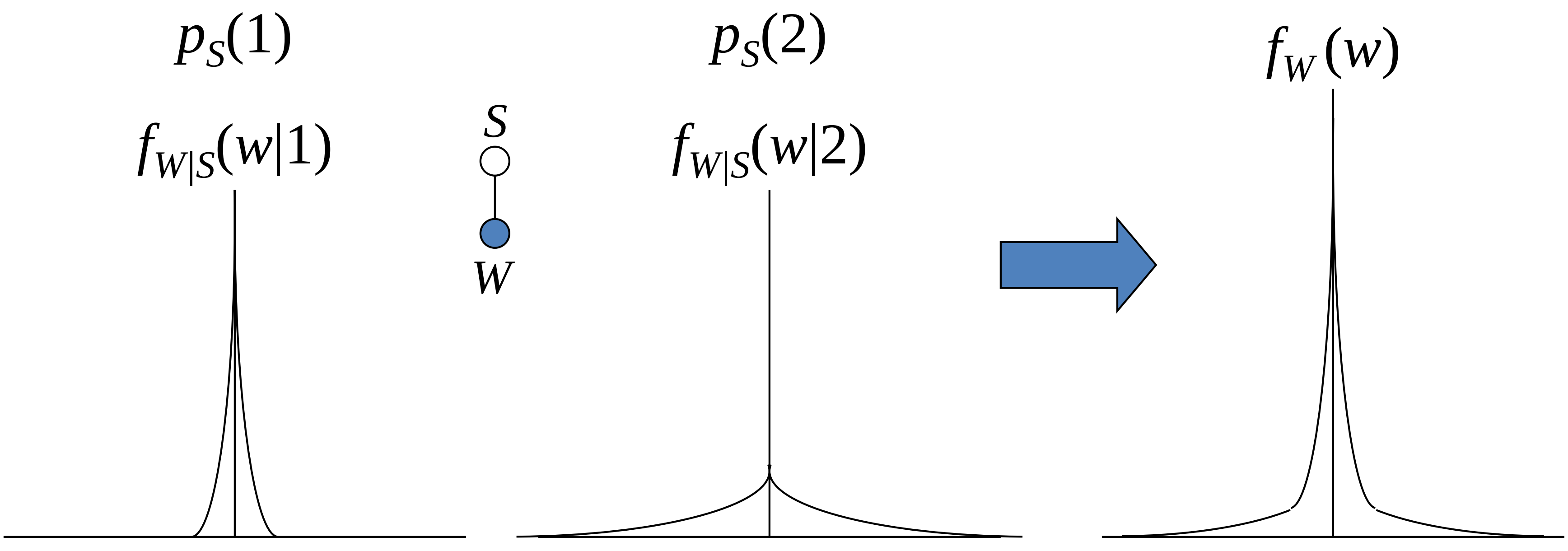
- 高斯混合模型

p1=0.8;p2=0.2
σ 1=0.2; σ 2=2;
μ =0

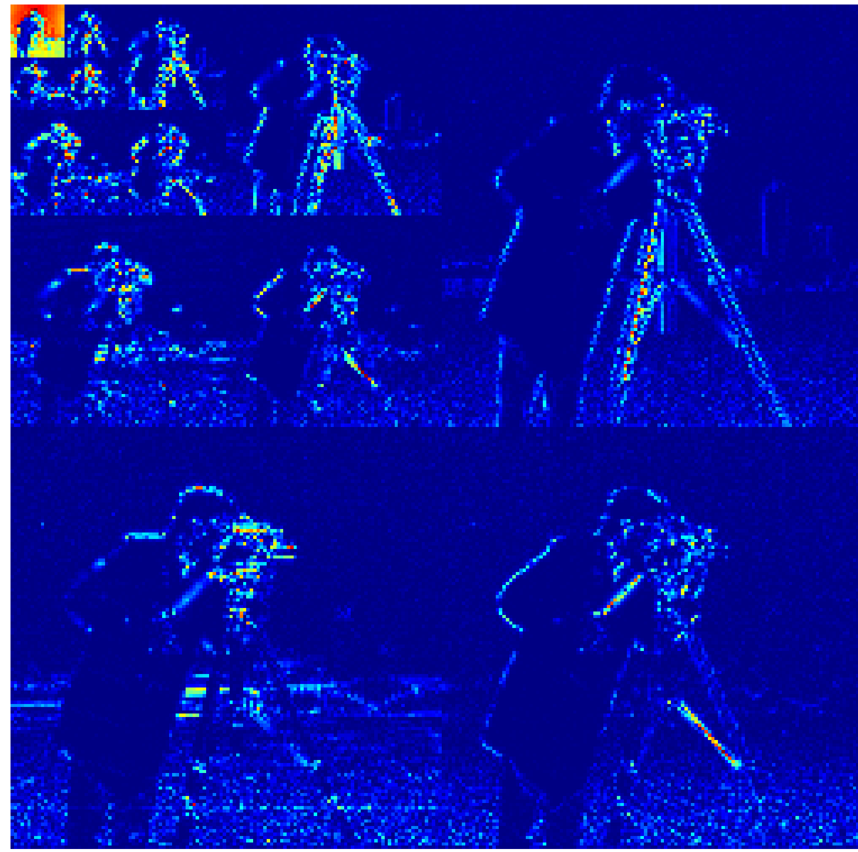


隐马尔可夫树模型

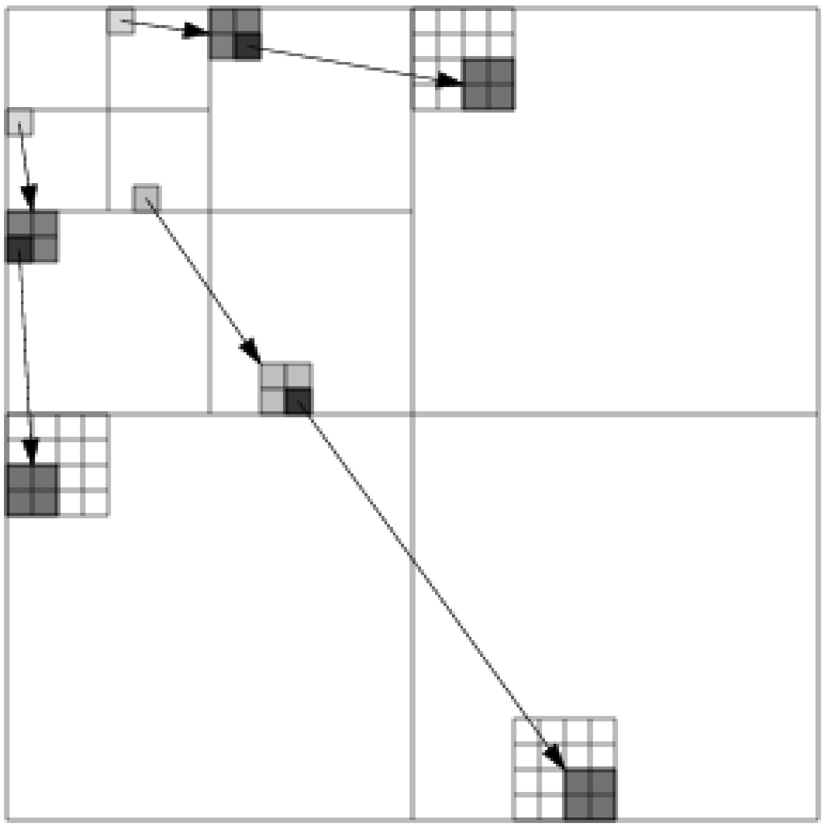
- 小波域图像数据的两个规律：
  - 小波系数分布曲线：长尾尖峰
    - 混合高斯模型： $p(w)=\sum_i p(s=s_i)p(w\,|\,s=s_i)$

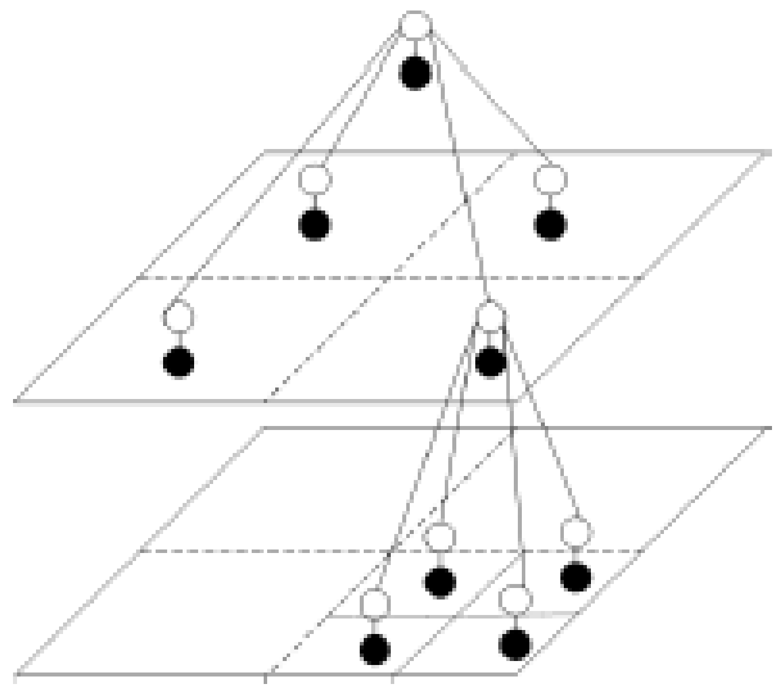


同一子带内保持性：



树结构+隐相关



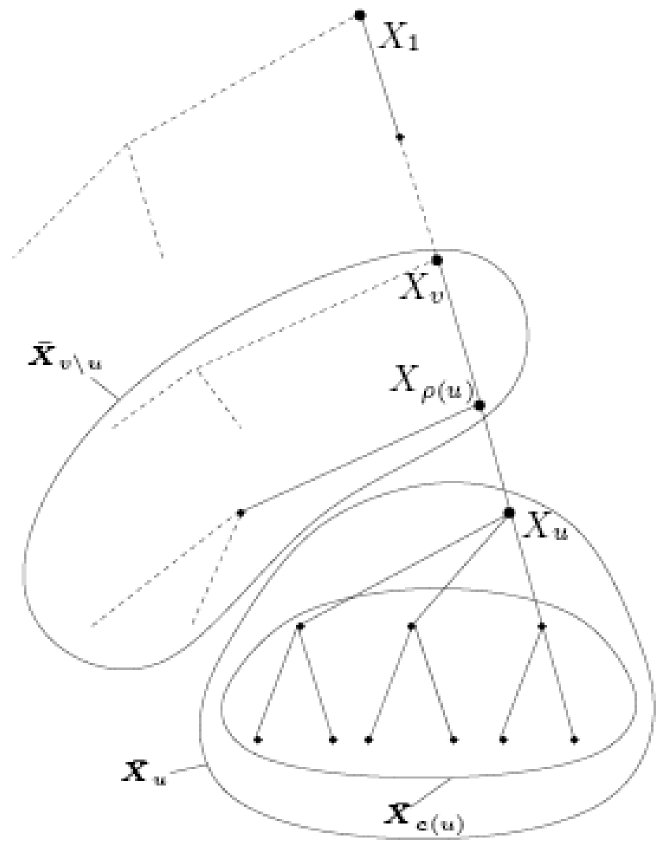


HMT (Hidden Markov Model)
（Crouse & Nowak ）

- 小波域树结构：$\bar{X}_1 = \{X_1, X_2, \cdots X_n\}, n$是图像像素个数
- 状态变量树结构：$\bar{S}_1 = \{S_1, S_2, \cdots, S_n\}$
- 状态空间: (1，2， …, K)
- 模型：计算出联合分布：$P(w, s)$

记号

- 节点：u
- 父节点：$\rho(u)$
- 子节点：$c(u)$
- $\bar{X}_u$： 以节点 u 为根节点的子树
- $\bar{X}_{c(u)}$：从$\bar{X}_u$去掉节点u：$\bar{X}_u \setminus \{u\}$
- $\bar{X}_u$是$\bar{X}_v$的子树，$\bar{X}_{v\setminus u}$是从$\bar{X}_v$去掉$\bar{X}_u$
- $\bar{X}_{1\setminus c(u)} = \bar{X}_1 \setminus \bar{X}_{c(u)}$



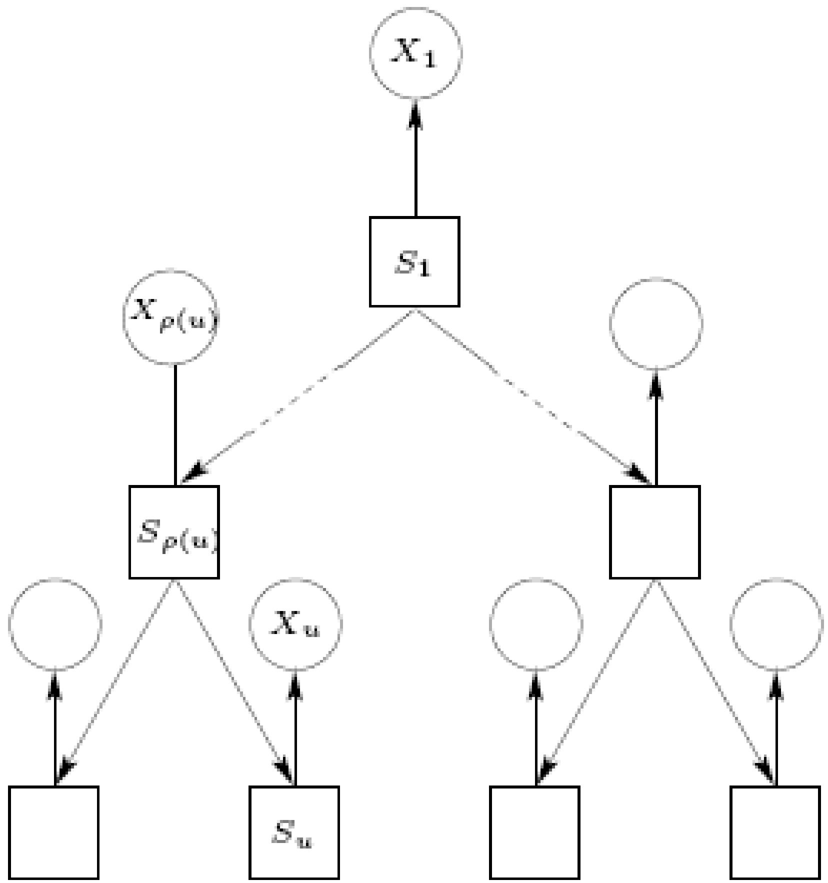
X_1
X_v
$\bar{X}_{v\setminus u}$
$X_{\rho(u)}$
X_u
$\bar{X}_u$
$X_{c(u)}$

两个假设：

- 高斯混合模型 $$P(X_u = x) = \sum_{j=1}^{K} P(S_u = j) P(X_u = x | S_u = j).$$
- 分解假设：

$$\forall\, (\bar{\boldsymbol{x}}_1, \bar{s}_1), \quad P(\bar{\boldsymbol{X}}_1 = \bar{\boldsymbol{x}}_1, \bar{\boldsymbol{S}}_1 = \bar{s}_1)$$
$$= P(S_1 = s_1) \left\{ \prod_{u \neq 1} P(S_u = s_u | S_{\rho(u)} = s_{\rho(u)}) \right\} \prod_u P(X_u = x_u | S_u = s_u).$$

- 即：根节点状态与其他节点状态独立；在父节点状态已知的时候，节点之间的状态互相独立；在节点状态已知的情况下，节点取值彼此独立。



X_1
S_1
$X_{\rho(u)}$
$S_{\rho(u)}$
X_u
S_u

模型参数：

- 根节点状态：

$$\pi = (\pi_j)_j = (P(S_1 = j))_j$$

- 状态转移矩阵：

$$P = (p_{ij})_{i,j},\ p_{ij} = (P(S_u = j \mid S_{\rho(u)} = i))$$

- 分布参数：

$$\theta = (\theta_1, \theta_2, \cdots, \theta_K), \theta_i = (\mu_i, \sigma_i)$$

$$P(X_u = x \mid S_u = i) = N(\theta_i) = N(\mu_i, \sigma_i)$$

模型训练算法：Upward-Downward

- 要计算出 $P(S_u = j, \bar{X}_1 = \bar{x}_1)$
- 一维HMM隐马尔可夫模型的forward-backward 算法（EM算法）的自然推广
- 两个过程：E (expectation)过程=upward和M（maximization) =downward过程

算法引入的记号：

- 中间变量：

$$
\begin{aligned}
\tilde{\beta}_u(j) &= P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_u = j); \\
\tilde{\beta}_{\rho(u),u}(j) &= P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_{\rho(u)} = j); \\
\tilde{\alpha}_u(j) &= P(S_u = j, \bar{\boldsymbol{X}}_{1\backslash u} = \bar{\boldsymbol{x}}_{1\backslash u}).
\end{aligned}
$$

- 基本公式：

$$
\begin{aligned}
P(S_u = j, \bar{\boldsymbol{X}}_1 = \bar{\boldsymbol{x}}_1) &= P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_u = j) P(S_u = j, \bar{\boldsymbol{X}}_{1\backslash u} = \bar{\boldsymbol{x}}_{1\backslash u}) \\
&= \tilde{\beta}_u(j) \tilde{\alpha}_u(j).
\end{aligned}
$$

Upward过程：给定状态变量初始值

$$
\begin{aligned}
\tilde{\beta}_u(j) &= P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_u = j) \\
&= \left\{ \prod_{v \in \boldsymbol{c}(u)} P(\bar{\boldsymbol{X}}_v = \bar{\boldsymbol{x}}_v | S_u = j) \right\} P(X_u = x_u | S_u = j) \\
&= \left\{ \prod_{v \in \boldsymbol{c}(u)} \tilde{\beta}_{u,v}(j) \right\} P_{\theta_j}(x_u);
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\beta}_{\rho(u),u}(j) &= P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_{\rho(u)} = j) \\
&= \sum_k P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_u = k) P(S_u = k | S_{\rho(u)} = j) \\
&= \sum_k \tilde{\beta}_u(k) p_{jk}.
\end{aligned}
$$

Downward过程：给定概率

$$
\begin{aligned}
\tilde{\alpha}_u(j) &= P(S_u = j, \bar{\boldsymbol{X}}_{1\backslash u} = \bar{\boldsymbol{x}}_{1\backslash u}) \\
&= \sum_i P(S_u = j, S_{\rho(u)} = i, \bar{\boldsymbol{X}}_{1\backslash\rho(u)} = \bar{\boldsymbol{x}}_{1\backslash\rho(u)}, \bar{\boldsymbol{X}}_{\rho(u)\backslash u} = \bar{\boldsymbol{x}}_{\rho(u)\backslash u}) \\
&= \sum_i P(S_u = j | S_{\rho(u)} = i) \frac{P(\bar{\boldsymbol{X}}_{\rho(u)} = \bar{\boldsymbol{x}}_{\rho(u)} | S_{\rho(u)} = i)}{P(\bar{\boldsymbol{X}}_u = \bar{\boldsymbol{x}}_u | S_{\rho(u)} = i)} \\
&\quad \times P(S_{\rho(u)} = i, \bar{\boldsymbol{X}}_{1\backslash\rho(u)} = \bar{\boldsymbol{x}}_{1\backslash\rho(u)}) \\
&= \sum_i \frac{p_{ij} \tilde{\beta}_{\rho(u)}(i) \tilde{\alpha}_{\rho(u)}(i)}{\tilde{\beta}_{\rho(u),u}(i)}.
\end{aligned}
$$

模型参数计算

$$
\begin{aligned}
p_{S_i}(m) &= \frac{1}{K}\sum_{k=1}^{K} p(S_i^k = m | W^k, \Theta^l) \\
\varepsilon_{i,\rho(i)}^{m,n} &= \frac{\sum_{k=1}^{K} p(S_i^k = m, S_{\rho(i)}^k = n | W^k, \Theta^l)}{K p_{S_{\rho(i)}}(n)} \\
\mu_{i,m} &= \frac{\sum_{k=1}^{K} w_i^k p(S_i^k = m | W^k, \Theta^l)}{K p_{S_{\rho(i)}}(m)} \\
\sigma_{i,m}^2 &= \frac{\sum_{k=1}^{K} (w_i^k - \mu_{i,m})^2 p(S_i^k = m | W^k, \Theta^l)}{K p_{S_{\rho(i)}}(m)}
\end{aligned}
$$

九参数模型

- HMT计算量太大：512x512需要一个小时
- 有没有必要每个像素点都要用不同的参数？
- 对于有些应用有没有必要这么精确？
- 有没有更简单的算法？
- 对于图像的更多先验知识能够加以利用？

模型化简：

一维情况（二维类似）

先看转移矩阵：节点 u 的状态和两个子节点c_1, c_2

之间的状态转移概率矩阵：

$$A = \begin{Bmatrix} p^{S\to S} & p^{S\to L} \\ p^{L\to S} & p^{L\to L} \end{Bmatrix}, \text{其中}: p^{S\to L} + p^{S\to S} = 1 (\text{两个状态})$$

$$p^{L\to L} + p^{L\to S} = 1$$

假设:

所有点的状态转移矩阵只与尺度有关;

所有点的不同状态下的均值和方差至于尺度有关;

模型化简

- 尺度间指数衰减性：

$$\sigma_{S,j} = C_{\sigma_S} 2^{-j\alpha_S}$$

$$\sigma_{L,j} = C_{\sigma_L} 2^{-j\alpha_L}$$

- 通用状态转移矩阵：

$$A_j = \begin{bmatrix} 1 - C_{SS} 2^{-\gamma_S j} & C_{SS} 2^{-\gamma_S j} \\ \frac{1}{2} - C_{LL} 2^{-\gamma_L j} & \frac{1}{2} + C_{LL} 2^{-\gamma_L j} \end{bmatrix}$$

通用参数：

$$\alpha_S = 3.1$$

$$C_{\sigma_S} = 2^{11}$$

$$\alpha_L = 2.25$$

$$C_{\sigma_L} = 2^{11}$$

$$\gamma_S = 1$$

$$C_{SS} = 2^{2.3}$$

$$\gamma_L = 0.4$$

$$C_{LL} = 2^{0.5}$$

$$p_{j_0}^L = \frac{1}{2}$$