矩阵行列式

Determinants

Baobin Li

Email:libb@ucas.ac.cn

School of Computer and Control, UCAS

Introduction

- The ancient Chinese counting board on which colored bamboo rods were manipulated according to prescribed “rules of thumb” in order to solve a system of linear equations.
- Seki Kowa (1642—1708), a great Japanese mathematician, synthesized the ancient Chinese ideas of array manipulation.
- Kowa formulated the concept of what we now call the determinant to facilitate solving linear systems—his definition is thought to have been made some time before 1683.
- About the same time—somewhere between 1678 and 1693—Gottfried W. Leibniz (1646—1716), a German mathematician, was independently developing his own concept of the determinant together with applications of array manipulation to solve systems of linear equations.
- It appears that Leibniz’ s early work dealt with only three equations in three unknowns, whereas Seki Kowa gave a general treatment for n equations in n unknowns.

- These men had something else in common — their ideas concerning the solution of linear systems were never adopted by the mathematical community of their time, and their discoveries quickly faded into oblivion.
- Eventually the determinant was rediscovered, and much was written on the subject between 1750 and 1900.
- During this era, determinants became the major tool used to analyze and solve linear systems, while the theory of matrices remained relatively undeveloped.
- The study and use of determinants eventually gave way to Cayley’s matrix algebra, and today matrix and linear algebra are in the main stream of applied mathematics, while the role of determinants has been relegated to a minor backwater position.
- Nevertheless, it is still important to understand what a determinant is and to learn a few of its fundamental properties.
- Our goal is not to study determinants for their own sake, but rather to explore those properties that are useful in the further development of matrix theory and its applications.

Determinants

- Over the years there have various ways to define the determinant.
- We are going to opt for expedience over elegance and proceed with the classical treatment.
- A **permutation** $p = (p_1, p_2, \cdots, p_n)$ of the numbers $(1, 2, \cdots, n)$ is simply any rearrangement.
- For example, the set

$$\{(1,2,3) \quad (1,3,2) \quad (2,1,3) \quad (2,3,1) \quad (3,1,2) \quad (3,2,1)\}$$

  contains the six distinct permutations of $(1, 2, 3)$.
- In general, the sequence $(1, 2, \cdots, n)$ has $n! = n(n-1)(n-2)\cdots 1$ different permutations.
- Given a permutation, consider the problem of restoring it to natural order by a sequence of pairwise interchanges.

- For example, $(1, 4, 3, 2)$ can be restored to natural order with a single interchange of 2 and 4 or three adjacent interchanges can be used.

- The important thing here is that both 1 and 3 are odd.
- Try to restore $(1, 4, 3, 2)$ to natural order by using an even number of interchanges, and you will discover that it is impossible.
- The parity of a permutation is unique—i.e., if a permutation p can be restored to natural order by an even (odd) number of interchanges, then every other sequence of interchanges that restores p to natural order must also be even (odd).

- Accordingly, the sign of a permutation p is defined to be the number

$$\sigma(p) = \begin{cases} +1 & \text{if } p \text{ can be restored to natural order by an} \\ & \textit{even} \text{ number of interchanges,} \\ -1 & \text{if } p \text{ can be restored to natural order by an} \\ & \textit{odd} \text{ number of interchanges.} \end{cases}$$

Definition of Determinant

For an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, the ***determinant*** of $\mathbf{A}$ is defined to be the scalar

$$\det(\mathbf{A}) = \sum_{p} \sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n},$$

where the sum is taken over the $n!$ permutations $p = (p_1, p_2, \ldots, p_n)$ of $(1, 2, \ldots, n)$.

- The determinant of $\mathbf{A}$ can be denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$.
- Note: the determinant of a nonsquare matrix is not defined.

- For example, When $\mathbf{A}$ is 2×2, there are 2 permutations of $(1, 2)$, namely, $\{(1, 2)\ (2, 1)\}$, so det $(\mathbf{A})$ contains the two terms

$$\sigma(1,2)a_{11}a_{22} \quad \textit{and} \quad \sigma(2,1)a_{12}a_{21}.$$

Since $\sigma(1, 2) = +1$ and $\sigma(2, 1) = -1$, we obtain the familiar formula

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Triangular Determinants

The determinant of a triangular matrix is the product of its diagonal entries. In other words,

$$\begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{vmatrix} = t_{11}t_{22}\cdots t_{nn}.$$

Transposition Doesn't Alter Determinants

- $\det\left(\mathbf{A}^T\right) = \det\left(\mathbf{A}\right)$ for all $n \times n$ matrices.

Effects of Row Operations

Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}_{n\times n}$ by one of the three elementary row operations:

Type I: Interchange rows i and j.

Type II: Multiply row i by $\alpha \neq 0$.

Type III: Add α times row i to row j.

The value of $\det\left(\mathbf{B}\right)$ is as follows:

- $\det\left(\mathbf{B}\right) = -\det\left(\mathbf{A}\right)$ for Type I operations.
- $\det\left(\mathbf{B}\right) = \alpha \det\left(\mathbf{A}\right)$ for Type II operations.
- $\det\left(\mathbf{B}\right) = \det\left(\mathbf{A}\right)$ for Type III operations.

- It is now possible to evaluate the determinant of an elementary matrix associated with any of the three types of elementary operations.
- Let $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$ be elementary matrices of Types I, II, and III, respectively.
- $\det(\mathbf{I}) = 1$, $\det(\mathbf{E}) = -1$, $\det(\mathbf{F}) = \alpha$ and $\det(\mathbf{G}) = 1$.

$$\begin{aligned}
\det(\mathbf{EA}) &= -\det(\mathbf{A}) &&= \det(\mathbf{E})\det(\mathbf{A}),\\
\det(\mathbf{FA}) &= \alpha\det(\mathbf{A}) &&= \det(\mathbf{F})\det(\mathbf{A}),\\
\det(\mathbf{GA}) &= \det(\mathbf{A}) &&= \det(\mathbf{G})\det(\mathbf{A}).
\end{aligned}$$

- In other words, $\det(\mathbf{PA}) = \det(\mathbf{P})\det(\mathbf{A})$ whenever $\mathbf{P}$ is an elementary matrix of Type I, II, or III. It' s easy to extend this observation to any number of these elementary matrices, $\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_k$, by writing

$$\begin{aligned}
\det(\mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A}) &= \det(\mathbf{P}_1)\det(\mathbf{P}_2\cdots\mathbf{P}_k\mathbf{A})\\
&= \det(\mathbf{P}_1)\det(\mathbf{P}_2)\det(\mathbf{P}_3\cdots\mathbf{P}_k\mathbf{A})\\
&\ \ \vdots\\
&= \det(\mathbf{P}_1)\det(\mathbf{P}_2)\cdots\det(\mathbf{P}_k)\det(\mathbf{A}).
\end{aligned}$$

Invertibility and Determinants

- $\mathbf{A}_{n\times n}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$

  or, equivalently,
- $\mathbf{A}_{n\times n}$ is singular if and only if $\det(\mathbf{A}) = 0$.

- It might be easy to get idea that det($\mathbf{A}$) is somehow a measure of how close $\mathbf{A}$ is to being singular, but this is not necessarily the case.
- **Small Determinants$\not\Leftrightarrow$ Near Singularity.**
- A minor determinant (or simply a minor) of $\mathbf{A}_{m\times n}$ is defined to be the determinant of any $k \times k$ submatrix of $\mathbf{A}$. For example,

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \text{ and } \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -6 \text{ are } 2\times 2 \text{ minors of } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

An individual entry of $\mathbf{A}$ can be regarded as a 1×1 minor, and $\det(\mathbf{A})$ itself is considered to be a 3×3 minor of $\mathbf{A}$.

- $rank(\mathbf{A})$ = the size of the largest nonzero minor of $\mathbf{A}$.

Problem: Use determinants to compute the rank of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{pmatrix}$.

Solution: Clearly, there are 1×1 and 2×2 minors that are nonzero, so $rank(\mathbf{A}) \geq 2$. In order to decide if the rank is three, we must see if there are any 3×3 nonzero minors. There are exactly four 3×3 minors, and they are

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 1 \\ 7 & 8 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 3 & 1 \\ 4 & 6 & 1 \\ 7 & 9 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 3 & 1 \\ 5 & 6 & 1 \\ 8 & 9 & 1 \end{vmatrix} = 0.$$

Since all 3×3 minors are 0, we conclude that $rank(\mathbf{A}) = 2$.

Product Rules

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for all $n \times n$ matrices.
- $\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} = \det(\mathbf{A})\det(\mathbf{D})$ if $\mathbf{A}$ and $\mathbf{D}$ are square.

- The product rule provides a practical way to compute determinants.
- The definition of a determinant is purely algebraic, but there is a concrete geometrical interpretation.
- A solid in $\mathfrak{R}^m$ with parallel opposing faces whose adjacent sides are defined by vectors from a linearly independent set $\{x_1, x_2, \cdots, x_n\}$ is called an n-dimensional parallelepiped.
- A two-dimensional parallelepiped is a parallelogram, and a three-dimensional parallelepiped is a skewed rectangular box.
- When $\mathbf{A} \in \mathfrak{R}^{m\times n}$ has linearly independent columns, the volume of the n-dimensional parallelepiped generated by the columns of $\mathbf{A}$ is $V_n = \left[\det(\mathbf{A}^T\mathbf{A})\right]^1/2$. In particular, if $\mathbf{A}$ is square, then $V_n = |\det(\mathbf{A})|$.

- For every nonsingular matrix $\mathbf{A}$, there is a permutation matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{LU}$ in which $\mathbf{L}$ is lower triangular with 1's on its diagonal, and $\mathbf{U}$ is upper triangular with the pivots on its diagonal.
- The product rule guarantees that $\det(\mathbf{P})\det(\mathbf{A}) = \det(\mathbf{L})\det(\mathbf{U})$.

Computing a Determinant

If $\mathbf{PA}_{n\times n} = \mathbf{LU}$ is an LU factorization obtained with row interchanges (use partial pivoting for numerical stability), then

$$\det(\mathbf{A}) = \sigma u_{11} u_{22} \cdots u_{nn}.$$

The u_{ii} 's are the pivots, and σ is the sign of the permutation. That is,

$$\sigma = \begin{cases} +1 & \text{if an } \textit{even} \text{ number of row interchanges are used,} \\ -1 & \text{if an } \textit{odd} \text{ number of row interchanges are used.} \end{cases}$$

If a zero pivot emerges that cannot be removed (because all entries below the pivot are zero), then $\mathbf{A}$ is singular and $\det(\mathbf{A}) = 0$.

- It's sometimes necessary to compute the derivative of a determinant whose entries are differentiable functions.

Derivative of a Determinant

If the entries in $\mathbf{A}_{n\times n} = [a_{ij}(t)]$ are differentiable functions of t, then

$$\frac{d\left(\det\left(\mathbf{A}\right)\right)}{dt} = \det\left(\mathbf{D}_1\right) + \det\left(\mathbf{D}_2\right) + \cdots + \det\left(\mathbf{D}_n\right),$$

where $\mathbf{D}_i$ is identical to $\mathbf{A}$ except that the entries in the i^{th} row are replaced by their derivatives—i.e., $\left[\mathbf{D}_i\right]_{k*} = \begin{cases} \mathbf{A}_{k*} & \text{if } i \neq k, \\ d\,\mathbf{A}_{k*}/dt & \text{if } i = k. \end{cases}$

- Evaluate the derivative $d(\det(\mathbf{A}))/dt$ for $\mathbf{A} = \begin{pmatrix} e^t & e^{-t} \\ \cos t & \sin t \end{pmatrix}$.

$$\frac{d\left(\det\left(\mathbf{A}\right)\right)}{dt} = \begin{vmatrix} \mathrm{e}^t & -\mathrm{e}^{-t} \\ \cos t & \sin t \end{vmatrix} + \begin{vmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ -\sin t & \cos t \end{vmatrix} = \left(\mathrm{e}^t + \mathrm{e}^{-t}\right)\left(\cos t + \sin t\right).$$

Proof. This follows directly from the definition of a determinant by writing

$$\frac{d\big(\det(\mathbf{A})\big)}{dt} = \frac{d}{dt}\sum_p \sigma(p) a_{1p_1} a_{2p_2} \cdots a_{np_n} = \sum_p \sigma(p) \frac{d\big(a_{1p_1} a_{2p_2} \cdots a_{np_n}\big)}{dt}$$

$$= \sum_p \sigma(p) \Big(a'_{1p_1} a_{2p_2} \cdots a_{np_n} + a_{1p_1} a'_{2p_2} \cdots a_{np_n} + \cdots + a_{1p_1} a_{2p_2} \cdots a'_{np_n} \Big)$$

$$= \sum_p \sigma(p) a'_{1p_1} a_{2p_2} \cdots a_{np_n} + \sum_p \sigma(p) a_{1p_1} a'_{2p_2} \cdots a_{np_n} + \cdots + \sum_p \sigma(p) a_{1p_1} a_{2p_2} \cdots a'_{np_n}$$

$$= \det(\mathbf{D}_1) + \det(\mathbf{D}_2) + \cdots + \det(\mathbf{D}_n). \quad \blacksquare$$

Additional Properties of Determinants

- The purpose of this section is to present some additional properties of determinants that will be helpful in later developments.

Block Determinants

If $\mathbf{A}$ and $\mathbf{D}$ are square matrices, then

$$\det\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{cases} \det(\mathbf{A})\det\left(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) & \text{when } \mathbf{A}^{-1} \text{ exists,} \\ \det(\mathbf{D})\det\left(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right) & \text{when } \mathbf{D}^{-1} \text{ exists.} \end{cases}$$

The matrices $\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are called the ***Schur complements*** of $\mathbf{A}$ and $\mathbf{D}$, respectively

Proof. If $\mathbf{A}^{-1}$ exists, then $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C}\mathbf{A}^{-1} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} \end{pmatrix}$

- Since the determinant of a product is equal to the product of the determinants, it's only natural to inquire if a similar result holds for sums.
- In other words, is $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$? Almost never!

- Nevertheless, there are still some statements that can be made regarding the determinant of certain types of sums.

Rank-One Updates

If $\mathbf{A}_{n\times n}$ is nonsingular, and if $\mathbf{c}$ and $\mathbf{d}$ are $n \times 1$ columns, then

- $\det\left(\mathbf{I} + \mathbf{c}\mathbf{d}^T\right) = 1 + \mathbf{d}^T\mathbf{c}$,
- $\det\left(\mathbf{A} + \mathbf{c}\mathbf{d}^T\right) = \det\left(\mathbf{A}\right)\left(1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\right)$.

- The proof follows by applying the product rules

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{d}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbf{I} + \mathbf{c}\mathbf{d}^T & \mathbf{c} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{d}^T & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{c} \\ \mathbf{0} & 1 + \mathbf{d}^T\mathbf{c} \end{pmatrix}$$

- Write $\mathbf{A} + \mathbf{c}\mathbf{d}^T = \mathbf{A}(\mathbf{I} + \mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T)$.

Problem: For $\mathbf{A} = \begin{pmatrix} 1+\lambda_1 & 1 & \cdots & 1 \\ 1 & 1+\lambda_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+\lambda_n \end{pmatrix}$, $\lambda_i \neq 0$, find $\det(\mathbf{A})$.

Solution: Express $\mathbf{A}$ as a rank-one updated matrix $\mathbf{A} = \mathbf{D} + \mathbf{e}\mathbf{e}^T$, where $\mathbf{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{e}^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$.

$$\det(\mathbf{D} + \mathbf{e}\mathbf{e}^T) = \det(\mathbf{D})\left(1 + \mathbf{e}^T\mathbf{D}^{-1}\mathbf{e}\right) = \left(\prod_{i=1}^{n} \lambda_i\right)\left(1 + \sum_{i=1}^{n} \frac{1}{\lambda_i}\right).$$

Cramer's Rule

In a nonsingular system $\mathbf{A}_{n\times n}\mathbf{x} = \mathbf{b}$, the i^{th} unknown is

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i = \left[\mathbf{A}_{*1} \,|\, \cdots \,|\, \mathbf{A}_{*i-1} \,|\, \mathbf{b} \,|\, \mathbf{A}_{*i+1} \,|\, \cdots \,|\, \mathbf{A}_{*n}\right]$. That is, $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that column $\mathbf{A}_{*i}$ has been replaced by $\mathbf{b}$.

Proof. Since $\mathbf{A}_i = \mathbf{A} + (\mathbf{b} - \mathbf{A}_{*i})\,\mathbf{e}_i^T$, where $\mathbf{e}_i$ is the i^{th} unit vector,

$$\det(\mathbf{A}_i) = \det(\mathbf{A})\Big(1 + \mathbf{e}_i^T \mathbf{A}^{-1}(\mathbf{b} - \mathbf{A}_{*i})\Big) = \det(\mathbf{A})\Big(1 + \mathbf{e}_i^T(\mathbf{x} - \mathbf{e}_i)\Big)$$

$$= \det(\mathbf{A})\,(1 + x_i - 1) = \det(\mathbf{A})\,x_i.$$

Thus $x_i = \det(\mathbf{A}_i)/\det(\mathbf{A})$ because $\mathbf{A}$ being nonsingular insures $\det(\mathbf{A}) \neq 0$

Problem: Determine the value of t for which $x_3(t)$ is minimized in

$$\begin{pmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1/t \\ 1/t^2 \end{pmatrix}.$$

Solution: Only one component of the solution is required, so it's wasted effort to solve the entire system. Use Cramer's rule to obtain

$$x_3(t) = \frac{\begin{vmatrix} t & 0 & 1 \\ 0 & t & 1/t \\ 1 & t^2 & 1/t^2 \end{vmatrix}}{\begin{vmatrix} t & 0 & 1/t \\ 0 & t & t^2 \\ 1 & t^2 & t^3 \end{vmatrix}} = \frac{1 - t - t^2}{-1} = t^2 + t - 1, \quad \text{and set} \quad \frac{d\,x_3(t)}{dt} = 0$$

- Recall that minor determinants of $\mathbf{A}$ are simply determinants of submatrices of $\mathbf{A}$.
- We are now in a position to see that in an $n \times n$ matrix the $n-1 \times n-1$ minor determinants have a special significance.

Cofactors

The ***cofactor*** of $\mathbf{A}_{n\times n}$ associated with the (i,j)-position is defined as

$$\mathring{A}_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the $n-1 \times n-1$ minor obtained by deleting the i^{th} row and j^{th} column of $\mathbf{A}$. The matrix of cofactors is denoted by $\mathring{\mathbf{A}}$.

Problem: For $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 6 \\ -3 & 9 & 1 \end{pmatrix}$, determine the cofactors $\mathring{A}_{21}$ and $\mathring{A}_{13}$.

Solution:

$\mathring{A}_{21} = (-1)^{2+1} M_{21} = (-1)(-19) = 19$ and $\mathring{A}_{13} = (-1)^{1+3} M_{13} = (+1)(18) = 18$.

The entire matrix of cofactors is $\mathring{\mathbf{A}} = \begin{pmatrix} -54 & -20 & 18 \\ 19 & 7 & -6 \\ -6 & -2 & 2 \end{pmatrix}$.

- The cofactors of a square matrix $\mathbf{A}$ appear naturally in the expansion of det $(\mathbf{A})$. For example,

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\qquad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right) \\ &\qquad + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) \\ &= a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + a_{13}\mathring{A}_{13}.\end{aligned}$$

- This expansion is called the cofactor expansion of det$(\mathbf{A})$ in terms of the first row. It can also be written as any other row or column.

Cofactor Expansions

- $\det(\mathbf{A}) = a_{i1}\mathring{A}_{i1} + a_{i2}\mathring{A}_{i2} + \cdots + a_{in}\mathring{A}_{in}$ (about row i).
- $\det(\mathbf{A}) = a_{1j}\mathring{A}_{1j} + a_{2j}\mathring{A}_{2j} + \cdots + a_{nj}\mathring{A}_{nj}$ (about column j).

Problem: Use cofactor expansions to evaluate $\det(\mathbf{A})$ for

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 7 & 1 & 6 & 5 \\ 3 & 7 & 2 & 0 \\ 0 & 3 & -1 & 4 \end{pmatrix}.$$

Solution: To minimize the effort, expand $\det(\mathbf{A})$ in terms of the row or column that contains a maximal number of zeros. For this example, the expansion in terms of the first row is most efficient because

$$\det(\mathbf{A}) = a_{11}\mathring{A}_{11} + a_{12}\mathring{A}_{12} + a_{13}\mathring{A}_{13} + a_{14}\mathring{A}_{14} = a_{14}\mathring{A}_{14} = (2)(-1)\begin{vmatrix} 7 & 1 & 6 \\ 3 & 7 & 2 \\ 0 & 3 & -1 \end{vmatrix}.$$

Now expand this remaining 3×3 determinant either in terms of the first column or the third row. Using the first column produces

$$\begin{vmatrix} 7 & 1 & 6 \\ 3 & 7 & 2 \\ 0 & 3 & -1 \end{vmatrix} = (7)(+1)\begin{vmatrix} 7 & 2 \\ 3 & -1 \end{vmatrix} + (3)(-1)\begin{vmatrix} 1 & 6 \\ 3 & -1 \end{vmatrix} = -91 + 57 = -34,$$

so $\det(\mathbf{A}) = (2)(-1)(-34) = 68$. You may wish to try an expansion using different rows or columns, and verify that the final result is the same.

- In the previous example, we were able to take advantage of the fact that there were zeros in convenient positions.
- However, for a general matrix $\mathbf{A}_{n\times n}$ with no zero entries, it's not difficult to verify that successive application of cofactor expansions requires $n!\left(1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{(n-1)!}\right)$ multiplications to evaluate det $(\mathbf{A})$.
- Even for moderate values of n, this number is too large for the cofactor expansion to be practical for computational purposes.
- Nevertheless, cofactors can be useful for theoretical developments such as the following determinant formula for $\mathbf{A}^{-1}$.

Determinant Formula for $\mathbf{A}^{-1}$

The ***adjugate*** of $\mathbf{A}_{n\times n}$ is defined to be $\operatorname{adj}(\mathbf{A}) = \mathring{\mathbf{A}}^T$, the transpose of the matrix of cofactors—some older texts call this the ***adjoint*** matrix. If $\mathbf{A}$ is nonsingular, then

$$\mathbf{A}^{-1} = \frac{\mathring{\mathbf{A}}^T}{\det(\mathbf{A})} = \frac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})}.$$

Proof. $\left[\mathbf{A}^{-1}\right]_{ij}$ is the i^{th} component in the solution to $\mathbf{A}\mathbf{x} = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j^{th} unit vector. By Cramer's rule, this is

$$\left[\mathbf{A}^{-1}\right]_{ij} = x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$

where $\mathbf{A}_i$ is identical to $\mathbf{A}$ except that the i^{th} column has been replaced by $\mathbf{e}_j$, and the cofactor expansion in terms of the i^{th} column implies that

$$\det(\mathbf{A}_i) = \begin{array}{c} \begin{array}{ccccc} \phantom{a_{11}} & & i^{th} & & \phantom{a_{1n}} \\ & & \downarrow & & \end{array} \\ \begin{vmatrix} a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{j1} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} \end{array} = \mathring{A}_{ji}. \quad \blacksquare$$

Problem: For $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, determine a general formula for $\mathbf{A}^{-1}$.

Solution: $\operatorname{adj}(\mathbf{A}) = \mathring{\mathbf{A}}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\det(\mathbf{A}) = ad - bc$, so

$$\mathbf{A}^{-1} = \frac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercises

1. What is the volume of the parallelepiped generated by the three vectors $x_1 = (3, 0, -4, 0)^T$, $x_2 = (0, 2, 0, -2)^T$ and $x_3 = (0, 1, 0, 1)^T$?
2. If $\mathbf{A}$ is $n \times n$, explain why det$(\alpha\mathbf{A})$=α^ndet$(\mathbf{A})$ for all scalars α.
3. Use a cofactor expansion to evaluate each of the following determinants.

   (a) $\begin{vmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ -2 & 2 & 1 \end{vmatrix}$, (b) $\begin{vmatrix} 0 & 0 & -2 & 3 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{vmatrix}$.
4. By example, show that det$(\mathbf{A} + \mathbf{B}) \neq$ det$(\mathbf{A}) + det(\mathbf{B})$.
5. Using square matrices, construct an example that shows that

$$\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \neq \det(\mathbf{A})det(\mathbf{D}) - \det(\mathbf{B})det(\mathbf{C}).$$