Distributed Constrained Consensus of Multi-Agent Systems with Uncertainties and Disturbances under Switching Directed Graphs

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***Abstract*— This paper provides a distributed leaderless consensus control framework for nonlinear multi-agent systems with time-varying asymmetric state constraints, uncertainties, and disturbances under switching directed graphs. In such a framework, original constrained states of agents are first transformed into free states in a transformed state space. To deal with switching directed graphs, we drive agents towards consensus in the transformed space by leveraging a model reference control scheme, and it is sufficient that the original states reach consensus strictly subject to the time-varying constraints under mild assumptions. A single-layer neural network with weights adapted online is leveraged to approximate the uncertainties in agent dynamics. For external disturbances and reconstruction errors in the approximation, we introduce a robust term with an adaptive gain for compensation. Distributed consensus algorithms are proposed, respectively, for multi-agent systems with first- or second-order dynamics. We prove convergence to consensus via Lyapunov analysis and study the proposed algorithms' performance using numerical simulations.**

***Index Terms*— Multi-agent systems, leaderless consensus, state constraints, switching directed graphs, distributed control.**

I. Introduction

DISTRIBUTED coordination of multi-agent systems (MASs) has caught substantial attention in the recent two decades owing to its great diversity of applications in multi-robot systems, large-scale power grids, sensor networks, biochemical networks, *etc*. [1]–[3]. Among various coordination tasks, the consensus of MASs has long been of significant interest to the multi-agent systems community and the broader system and control community because it serves as the foundation of many other typical coordination tasks such as containment [4], [5], formation [6], [7], flocking [8], resource allocation [9], and even noncooperative tasks like distributed Nash equilibria seeking [10], [11].

There are several essential topics in the study of consensus. One is the agent dynamics, including integrators [12], general linear systems [13], [14], and nonlinear dynamics [15]–[17]. Another is the communication topology embodying information interaction among agents. Archetypal topologies include undirected graphs, directed graphs, and time-varying switching graphs [18]–[20]. Moreover, external disturbances, as well as uncertainties led by parameter identification errors and unmodeled dynamics, are also extensively studied from the control perspective.

Lately motivated by a variety of practical applications, the problem of consensus with constraints emerged. Li *et al.* [21] introduce a distributed consensus algorithm for heterogeneous linear MASs with nonconvex input constraints. The authors of [22] put forward a standard consensus algorithm with output saturation and propose necessary and sufficient initial conditions for the achievement of consensus. Nédic *et al.* [23] first propose a distributed projected algorithm to drive discrete MASs under time-varying directed graphs to reach consensus with agents' states kept in closed convex sets. Lin and Ren [24] extend the projection algorithm to scenarios involved with bounded communication delays. Lin *et al.* [25] further unfurl the method to solve the consensus of second-order systems with position constraints. In [26], Liu *et al.* propose a discarded consensus strategy to deal with state constraints for both discrete-time and continuous-time integrators. Nguyen *et al.* [27], [28] study state-constrained consensus by transforming the state constraints into equivalent consensus problems with input constraints. Hong *et al.* [17] have investigated the finite-time consensus for second-order nonlinear MASs over undirected graphs with a focus on connectivity preservation, which can be conceived as a kind of constraints on agents' states. The consensus problem with both relative state saturation constraints and time-varying input-delay is studied in [29] by using state saturation functions on MASs' edge dynamics. In [30], Meng *et al.* solve the consensus of nonlinear MASs with time-varying constraints under undirected graphs.

Nevertheless, there are still knowledge gaps between state-

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of-the-art consensus algorithms concerning state constraints and their practical applications in the field. First, most of the aforementioned studies concentrate on time-invariant state constraints [17], [23]–[29]. In real-world situations, however, constraints over agents' states vary from time to time and depend on various factors. Second, while the consensus of MASs with state constraints has been studied in multiple settings, many works rely on restrictive assumptions. The interaction topologies discussed in [17], [27]–[30] are all *fixed* and *undirected* ones. Nonetheless, it is quite common in applications that the interaction topology among agents is unilateral and may change dynamically over time. Consensus algorithms in the above works would fail to fit in such scenarios. Third, none of the works in [23]–[29] has considered the uncertainties in agents' dynamics. We argue that this reliance on perfect model information would severely limit the practical applicability of the approaches therein. As such, the integration of time-varying asymmetric state constraints and switching directed topologies into consensus protocols remains an underexplored topic yet would be of great practical importance.

This paper investigates the leaderless consensus problem of nonlinear MASs with uncertainties, external disturbances, and time-varying state constraints under jointly switching directed graphs. The three factors, *i.e.*, nonlinear dynamics with uncertainties and disturbances, time-varying state constraints, and switching directed topologies, combined together, render the consensus problem rather arduous. To tackle the above challenges, we have synthesized several approaches. First, a state transformation is adopted to turn the constrained states into free ones. Second, we use neural networks to approximate agents' model uncertainties. Third, a robust term is proposed in the consensus algorithm to achieve asymptotic convergence. Finally, a distributed framework is presented to design the consensus algorithms for uncertain MASs under switching directed graphs. Our main contribution in this work is a consensus framework that addresses both *time-varying state constraints* and *general switching directed graphs*.

The closest work to this paper is, perhaps, [30], which addresses the consensus with time-varying state constraints over a *fixed undirected graph*. In spite of borrowing some of its ideas, our method differs from that work in a) addressing the consensus problem under *switching directed* topologies rather than a fixed undirected one, and b) extending the proposed approach to second-order agent dynamics. We argue that the controller brought forward in [30] does not apply to MASs under switching directed graphs that we discuss in this work, since the consensus analysis in [30] hinges on the eigendecomposition of a symmetric Laplacian matrix, which is, however, not generally viable for the non-symmetric Laplacian matrices associated with directed graphs. In contrast, our approach, featuring a model reference adaptive style, circumvents such crux and allows for general *switching directed* topologies. We believe that this improvement provides a theoretical foundation for broader applications for MASs in real-world scenarios.

Notations: We refer to $\mathbf{1}_n$ and $\mathbf{0}_n$ as the $n \times 1$ column vector of all ones and all zeros, respectively. We also denote by $\mathbf{0}_{n\times p}$ and $\mathbf{1}_{n\times p}$ some $n \times p$ matrices with all their entries as zero and one, respectively. Let I_p be the $p \times p$ identity matrix, and let diag $(a_1, \ldots, a_n)$ be an $n \times n$ diagonal matrix with diagonal entries from a_1 to a_n. Throughout this paper, we use $\|\cdot\|$ to denote Euclidean norms for vectors and 2–norms for matrices. The operator $\otimes$ stands for the Kronecker product of two matrices. The notiation tr $\{\cdot\}$ represents the trace of a square matrix. For a vector function $f(t) : \mathbb{R}^+ \to \mathbb{R}^m$, we say that $f(t) \in \mathbb{L}_p$ if $\|f(t)\|_p < \infty$ for $p \in [1, \infty]$, wherein the L_p norm $\|\cdot\|_p$ is defined as $\|f(t)\|_p \triangleq \left(\int_0^\infty \|f(\tau)\|^p \,\mathrm{d}\tau\right)^{\frac{1}{p}}$ for $p < \infty$ and $\|f(t)\|_\infty \triangleq \sup_{t\geq 0} \|f(t)\|$.

II. Problem Formulation and Preliminaries

A. Problem Statement

The constrained consensus problem for first-order MASs with uncertainties and disturbances is the focus of this research. The agents' dynamics are described as

$$\dot{x}_i = u_i + f_i(x_i) + d_i(t), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i \in \mathbb{R}^p$ is the state vector standing for agents' positions, $u_i \in \mathbb{R}^p$ is the control input, $f_i(\cdot) \in \mathbb{R}^p$ is a continuous unknown function representing model uncertainties, and $d_i \in \mathbb{R}^p$ is an unknown external disturbance for the ith agent.

Predefined time-varying constraints on agents' states are considered in this paper. Let $x_{i,k}$ be the kth component of the ith agent's state vector, with $\underline{x}_k(t) \in \mathbb{R}$ and $\overline{x}_k(t) \in \mathbb{R}$ for $k = 1, \ldots, p$ being time differentiable functions. Then, the constraints imposed on the states have the following form

$$\underline{x}_k(t) < x_{i,k}(t) < \overline{x}_k(t) \tag{2}$$

for $i = 1, \ldots, n$, $k = 1, \ldots, p$ and $\forall t \geq 0$.

We also have the following assumptions in regard to the dynamics and constraints.

Assumption 1. The function $f_i(\cdot)$ is continuous and is defined over a compact set.

Assumption 2. The disturbance $d_i(t)$ is piecewise continuous and bounded in the sense of $\|d_i(t)\| \leq d_{im}$, $\forall t \geq 0$, where $d_{im} \geq 0$ is an unknown upper-bound.

Assumption 3. The upper and lower bounds of the state constraints are differentiable with respect to time t, *i.e.*, $\dot{\overline{x}}_k(t)$ and $\dot{\underline{x}}_k(t)$ exist for all $t \geq 0$ and $k = 1, \ldots, p$.

The control objective is to design a distributed control algorithm u_i such that all agents in the MAS asymptotically converge to the same state, *i.e.*,

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = \mathbf{0}_p, \quad \forall i, j = 1, \ldots, n, \tag{3}$$

and meanwhile, all states satisfy the constraints (2).

B. Graph Theory

In this paper, we use switching directed graphs to represent the information interaction among the agents. A directed graph is a pair $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} \triangleq \{1, \ldots, n\}$ being the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the edge set. An edge $(j, i) \in \mathcal{E}$ means that agent i is able to obtain information from agent j. Note that this connection is unilateral. Self-edges (i, i) are prohibited in this paper. In a time-varying directed graph $\mathcal{G}(t)$, we assume that the $\mathcal{V}$ is time-invariant while $\mathcal{E}(t)$ is time-varying.

The adjacency matrix, the degree matrix, and the Laplacian matrix associated with graph $\mathcal{G}(t)$ are notated as $\mathcal{A}(t) \in \mathbb{R}^{n\times n}$, $\mathcal{D}(t) \in \mathbb{R}^{n\times n}$, and $\mathcal{L}_A(t) \in \mathbb{R}^{n\times n}$, respectively. The adjacency matrix $\mathcal{A}(t) = [a_{ij}(t)]$ associated with $\mathcal{G}(t)$ is piecewise continuous and bounded with $a_{ij}(t) \in [\underline{a}, \overline{a}]$ for $0 < \underline{a} < \overline{a}$, if an edge $(j,i) \in \mathcal{E}$ and $a_{ij}(t) = 0$ otherwise. The degree matrix is defined as $\mathcal{D}(t) \triangleq \operatorname{diag}(deg_1, \dots, deg_n)$ with $deg_i \triangleq \sum_{j=1}^{n} a_{ij}(t)$. The Laplacian matrix is then defined by $\mathcal{L}_A(t) \triangleq \mathcal{D}(t) - \mathcal{A}(t)$.

Let t_0, t_1, … be the time sequence of the times when $A(t)$ switches and we assume that $t_m - t_{m-1} \geq t_d$ for all $m = 1, \dots$ with $t_d > 0$.

Assumption 4 ([31]). For an infinite sequence of switching graphs $\mathcal{G}(t_m)$ for $m = 0, 1, \dots$ that characterizes the interaction topologies, there exists an infinite sequence of contiguous, nonempty, uniformly bounded time intervals $[t_{m_j}, t_{m_{j+1}}]$ for $j = 1, \cdots$, starting at $t_{m_1} = t_0$, satisfying that the union of the directed graphs across each such interval contains a directed spanning tree.

There have already been results for the consensus of single integrators without uncertainties or disturbances.

Lemma 1 ([31])**.** *Under Assumption 4, the closed-loop system*

$$\dot{x}_i = -\sum_{j=1}^{n} a_{ij}(t)(x_i - x_j), \quad \forall i, j = 1, \dots, n \tag{4}$$

achieves consensus exponentially.

C. Neural-Network Approximation

Neural networks (NNs) are powerful tools for tackling complex nonlinearities and uncertainties, and NN-based adaptive control problems have been extensively investigated [32]–[34]. In this work, we employ a single-layer feedforward neural network to approximate uncertainties appearing in agent dynamics. It has been proven that NNs such as sigmoidal feedforward NNs and RBFNNs have the universal approximation ability [35]–[37]. The definition of an RBFNN can be found in [38], and a sigmoidal NN has the following definition.

Definition (Sigmoidal Feedforward NNs [38])**.** A *sigmoidal feedforward NN* with n_2 neurons is a map from $\mathbb{R}^{n_1}$ to $\mathbb{R}^{n_3}$ in the form

$$NN(x) = \Theta\sigma(Vx + \theta) + b \tag{5}$$

where $\Theta \in \mathbb{R}^{n_3\times n_2}$ is the matrix of outer-layer weights;

$$\sigma(x) = \left[\sigma_1(V_1 x + \theta_1), \cdots, \sigma_{n_2}(V_{n_2} x + \theta_{n_2})\right]^\top \in \mathbb{R}^{n_2}$$

is the vector of n_2 sigmoid; $V \in \mathbb{R}^{n_2\times n_1}$ is the matrix of inner-layer synaptic weights, with its ith row denoted by $V_i \in \mathbb{R}^{n_1}$; $\theta \in \mathbb{R}^{n_2}$ is the vector of thresholds; and $b \in \mathbb{R}^{n_3}$ denotes the NN bias vector.

For succinctness, we put (5) in a more compact form

$$NN(x) = W\phi(x) \tag{6}$$

by defining $W \triangleq \begin{bmatrix}\Theta & b\end{bmatrix} \in \mathbb{R}^{n_3\times(n_2+1)}$ and $\phi(x) \triangleq \begin{bmatrix}\sigma^\top(x) & 1\end{bmatrix}^\top \in \mathbb{R}^{n_2+1}$.

Owing to NNs' universal approximation properties, a continuous function $f(x) : \mathbb{R}^{n_1} \to \mathbb{R}^{n_3}$ over a compact set $\Omega_x \subset \mathbb{R}^{n_1}$ can be approximated as

$$f(x) = W^*\phi(x) + \epsilon \tag{7}$$

with W^* being the ideal weight matrix and ϵ the approximation error. The ideal weight matrix W^* is defined as

$$W^* \triangleq \underset{W\in\mathbb{R}^{n_3\times(n_2+1)}}{\operatorname{argmin}} \left\{\sup_{x\in\Omega_x} \|f(x) - W\phi(x)\|\right\}.$$

It is worth mentioning that we can keep the approximation error ϵ arbitrarily small by merely adopting sufficiently many nodes in the hidden layer without changing the synaptic weights V and the thresholds θ, and therefore the following assumption holds.

Assumption 5. The approximation error ϵ is bounded by $\|\epsilon\| \leq \epsilon_m$ with $\epsilon_m > 0$.

Here we adopt a sigmoidal feedforward NN with its inner-layer weights and thresholds fixed to approximate the uncertainties in the agent dynamics. The output-layer weights and the biases are tuned online according to adaptation laws designed in Section III. In other words, the NN does not require training beforehand.

Remark 1. We choose sigmoidal NNs over RBFNNs to avoid the curse of dimensionality in the approximation rate [38]. Sigmoidal NNs provide $O\left(n_2^{-\frac{1}{2}}\right)$ rate of approximation while that of RBFNNs is $O\left(n_2^{-\frac{1}{2n_1}}\right)$, which decreases exponentially as the dimension of the input increases. Details about the approximation rates of these two types of NNs can be found in [37] and [39].

D. Useful Lemmas

The following lemmas are useful in the subsequent analysis.

Lemma 2 ([40])**.** *Let* $g, f : \mathbb{R}^+ \to \mathbb{R}$ *and* $l \in [1, \infty]$. *If* $g \in \mathbb{L}_l$ *and* $f \in \mathbb{L}_1$, *then* $\|\int_0^t g(\tau)f(t-\tau)\,\mathrm{d}\tau\|_l \leq \|g\|_l\|f\|_1$.

Lemma 3. *If a differentiable function of time* $e(t) \in \mathbb{L}_2 \cap \mathbb{L}_\infty$ *and its time derivative* $\dot{e} \in \mathbb{L}_\infty$, *then* $\lim_{t\to\infty} e(t) = \mathbf{0}$.

III. Control Algorithms Design

A. Control Framework

Our overall control framework consists of two major parts, namely, a state space transformation and the consensus in the transformed space. Fig. 1 depicts the architecture of our proposed framework.

To address the time-varying state constraints, we first transform all the constrained state variables from the original state space into free states in a transformed state space. Motivated by the error transformation in [41], we introduce a nonlinear function $s = S(x, \overline{x}(t), \underline{x}(t))$. Without loss of generality, we adopt a transformation utilized in [30]:

$$s_{i,k} = \tan\left[\frac{\pi}{2}\cdot\frac{2x_{i,k} - \overline{x}_k(t) - \underline{x}_k(t)}{\overline{x}_k(t) - \underline{x}_k(t)}\right]. \tag{8}$$

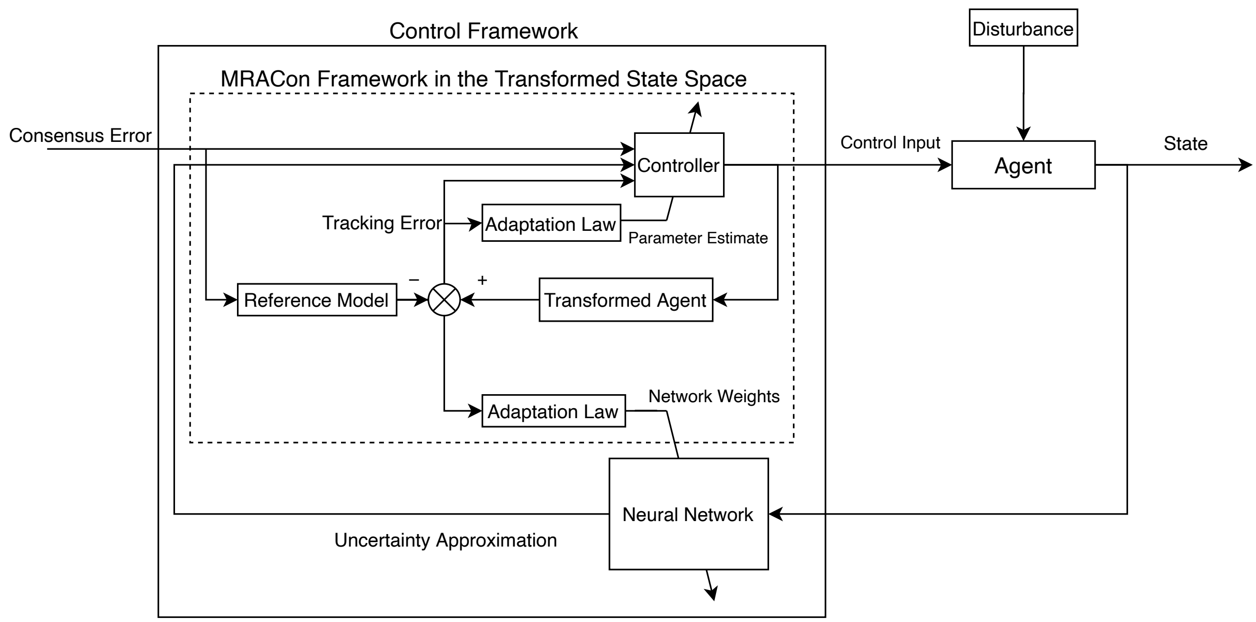

Fig. 1. Block diagram of an agent under our control framework. An actual agent's state is first mapped to another state space by a bijective transformation to tackle the state constraints. In the transformed space, the agent is controlled under the MRACon scheme, where a reference model indicates how the agent should act upon the consensus error so as to achieve consensus, and the transformed agent tracks this reference model under the designed robust adaptive controller. The uncertainties in the agent's dynamics are approximated by a shallow NN whose weights adapt online and are excited by the tracking error between the transformed agent and the reference model. A robust adaptive component in the controller, with the tracking error being the excitation, compensates for the effects of external disturbances and the residual of the NN approximation.

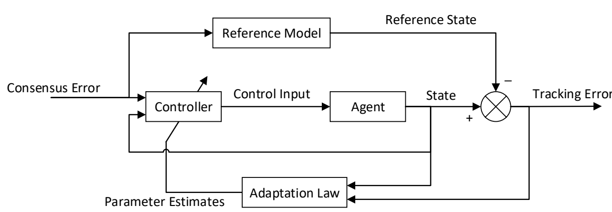

Fig. 2. Block diagram of an agent under the MRACon framework. Under MRACon, an uncertain agent tracks a simpler reference model, which is designed to achieve consensus. The tracking error between the agent's state and that of the reference model is used in the adaptation law that provides parameter estimates for the controller. Once the tracking is achieved, all agents will reach consensus asymptotically.

One can verify that

$$\lim_{s_{i,k}\to+\infty} x_{i,k} = \bar{x}_k(t), \qquad \lim_{s_{i,k}\to-\infty} x_{i,k} = \underline{x}_k(t). \tag{9}$$

From (8) and (9), we see that at any time, the original state $x_{i,k}$ satisfies constraints (2) if and only if its transformed counterpart $s_{i,k}$ remains bounded. Additionally, the original MAS achieves consensus if and only if the transformed MAS achieves consensus thanks to the invertibility of the transformation function S. Therefore, in this way, we have contrived to turn the constrained consensus problem in the original state space into a normal consensus problem in a new transformed state space.

Remark 2. The state transformation is not unique as long as 1) S is a smooth, strictly increasing (and thus invertible) function with respect to x, where each component $x_{i,k}$ of x is defined on $[\underline{x}, \overline{x}]$; 2) S satisfies (9). An alternative for (8) would be

$$s_{i,k} = \ln\left[\frac{x_{i,k} - \underline{x}(t)}{\overline{x}(t) - x_{i,k}}\right].$$

The other part of the framework is the consensus in the transformed space under general switching directed graphs. In order to tackle the asymmetry in the Laplacian matrices of directed graphs, we adopt the Model Reference Adaptive Consensus (MRACon) scheme proposed in [42]. Fig. 2 shows the structure of the MRACon control framework. The general idea of MRACon is that the normal consensus problem could be divided into two parts, videlicet, the tracking to the output of certain reference models for the uncertain agent dynamics, and the consensus of the reference models *per se*. The motivation for utilizing MRACon in our framework is its flexibility and capability in solving leaderless consensus problems for uncertain MASs with various orders under switching directed graphs.

B. Distributed Consensus Algorithm Design

In this subsection, we design a distributed consensus algorithm for MAS (1) under switching directed graphs. Taking the derivative of both sides of (8) with respect to time t, we obtain

$$\dot{s}_{i,k} = \frac{\partial s_{i,k}}{\partial x_{i,k}}\dot{x}_{i,k} + \frac{\partial s_{i,k}}{\partial \bar{x}_k}\dot{\bar{x}}_j + \frac{\partial s_{i,k}}{\partial \underline{x}_k}\dot{\underline{x}}_k. \tag{10}$$

Rewriting (10) in a more compact form, we get a transformed agent with the following dynamics:

$$\dot{s}_i = R_i \left(u_i + f_i(x_i) + d_i(t)\right) + c_i, \tag{11}$$

where $\dot{s}_i = \begin{bmatrix}\dot{s}_{i,1} & \cdots & \dot{s}_{i,p}\end{bmatrix}^\top$, $c_i = \begin{bmatrix}c_{i,1} & \cdots & c_{i,p}\end{bmatrix}^\top$ with $c_{i,k} = \frac{\partial s_{i,k}}{\partial \overline{x}_k}\dot{\overline{x}}_j + \frac{\partial s_{i,k}}{\partial \underline{x}_k}\dot{\underline{x}}_k, k = 1, \ldots, p$, and

$$R_i \triangleq \text{diag}\left(\frac{\partial s_{i,1}}{\partial x_{i,1}}, \ldots, \frac{\partial s_{i,p}}{\partial x_{i,p}}\right). \tag{12}$$

Remark 3. Note that

$$\frac{\partial s_{i,k}}{\partial x_{i,k}} = \frac{\pi}{\overline{x}_k - \underline{x}_k} \cdot \sec^2\left[\frac{\pi}{2} \cdot \frac{2x_{i,k} - \overline{x}_k - \underline{x}_k}{\overline{x}_k - \underline{x}_k}\right] > 0$$

for $j = 1, \ldots, p$. Therefore, R_i is always invertible. Additionally, the partial derivatives $\frac{\partial s_{i,k}}{\partial x_{i,k}}$, $\frac{\partial s_{i,k}}{\partial \overline{x}_k}$ and $\frac{\partial s_{i,k}}{\partial \underline{x}_k}$ are all bounded as long as the transformed state $s_{i,k}$ remains bounded.

Due to the aforementioned universal approximation property, we consider the following NN approximation to approximate the uncertainties

$$f_i(x_i) = W_i\phi(x_i) + \epsilon_i, \tag{13}$$

with W_i being the ideal constant weight matrix and ϵ_i the reconstruction error.

The backbone of our framework is MRACon, where a linear reference model is first proposed for each agent, and then a distributed control law is designed to enable the actual MAS to track that reference model with asymptotic convergence. So the basic steps are 1) design a linear reference model; 2) define the tracking error between the reference states and actual states; 3) design the actual control input so that the tracking error converges to zero asymptotically, and the reference system achieves our control objective.

For the first-order dynamics in (1), taking single integrators as the reference model, we have

$$\dot{z}_i = -\sum_{j=1}^{n} a_{ij}(t)\left(s_i - s_j\right), \quad i = 1, \ldots, n \tag{14}$$

with initial conditions that $z_i(0) = s_i(0)$ for $i = 1, \ldots, n$. In (14), z_i is the reference state, and the single integrator's input is the consensus error of the transformed states s_i defined in (11). Then, by defining the tracking error as

$$e_i \triangleq s_i - z_i, \tag{15}$$

combining (13) we are able to obtain the error dynamics

$$\dot{e}_i = R_i\left[u_i + W_i\phi(x_i) + \epsilon_i + d_i(t)\right] + c_i + \sum_{j=1}^{n} a_{ij}(t)(s_i - s_j). \tag{16}$$

Since the NN's ideal weight W_i is an unknown constant, we need to estimate it. Moreover, there are disturbances and the approximation error of the NN in the error dynamics (16), so we opt for a robust adaptive approach to handle them. Here we further define a robust variable

$$B_i \triangleq \epsilon_{im} + d_{im}, \tag{17}$$

where ϵ_{im} is the upper bound for the reconstruction error of f_i and d_{im} is the bound of $d_i(t)$. The robust gain B_i needs to be estimated as well.

Given the form of the tracking error dynamics (16) and our goal to stabilize the tracking error and drive the reference system toward consensus, we propose the following distributed consensus algorithm:

$$\begin{aligned} u_i = R_i^{-1}\Bigg[&-k_i e_i - \sum_{j=1}^{n} a_{ij}(t)\left(s_i - s_j\right) - \frac{\eta_i e_i}{\|e_i\| + \mu_i(t)} \\ &- \frac{\hat{B}_i \left\|R_i\right\|^2 e_i}{\|R_i\|\,\|e_i\| + \mu_i(t)} - c_i\Bigg] - \hat{W}_i\phi(x_i) \end{aligned} \tag{18}$$

with k_i and η_i being positive constants, $\mu_i(t)$ being some positive function belonging to $\mathbb{L}_1$, and $\hat{W}_i$ and $\hat{B}_i$ the estimate of W_i and B_i, respectively. The adaptation laws for $\hat{W}_i$ and $\hat{B}_i$ are

$$\dot{\hat{W}}_i = \Gamma_i R_i e_i \phi^\top(x_i), \qquad \hat{W}_i(0) = \mathbf{0} \tag{19a}$$

$$\dot{\hat{B}}_i = \delta_i \frac{\|R_i\|^2 \|e_i\|^2}{\|R_i\|\|e_i\| + \mu_i(t)}, \quad \hat{B}_i(0) = 0 \tag{19b}$$

where Γ_i is some positive definite matrix, and δ_i is some positive constant.

Remark 4. As opposed to the classic consensus algorithm (4), agents communicate their transformed states as they appear in (14). This protocol works thanks to the fact that the state transformation S, seen as a function mapping the original states to the transformed states, is one-to-one and onto.

Remark 5. In (19a) and (19b), we use the tracking error e_i as excitation for parameter adaptation. The initial estimate $\hat{W}_i(0)$ is set to be a zero matrix only for convenience. In fact, the estimation of W_i can start with any given bounded matrix with appropriate dimensions, so this NN approximation in our framework is compatible with warm-start setups. However, one would notice that $\dot{\hat{B}}_i$ is non-negative, which means that $\hat{B}_i$ keeps growing as long as perfect tracking is not achieved. Therefore, we always set $\hat{B}_i(0) = 0$, assuming the system does not need "extra" control efforts from the robust term given no *a priori* information about the tracking performance.

Remark 6. The introduction of $\mu_i(t)$ is to ensure that the control algorithm and the adaptation law are continuous.

Before proceeding with the convergence analysis, we introduce an $(n-1)\times n$ matrix Q as defined in [3]:

$$Q = \begin{bmatrix} -1+(n-1)\nu & 1-\nu & -\nu & \cdots & -\nu \\ -1+(n-1)\nu & -\nu & 1-\nu & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\nu \\ -1+(n-1)\nu & -\nu & \cdots & -\nu & 1-\nu \end{bmatrix} \tag{20}$$

with $\nu = \frac{n-\sqrt{n}}{n(n-1)}$. We also point out some useful properties of Q as follows

$$Q\mathbf{1}_n = \mathbf{0}_{n-1}, \tag{21a}$$

$$QQ^\top = I_{n-1}, \tag{21b}$$

$$Q^\top Q = I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top. \tag{21c}$$

The following lemma related to Q also holds.

Lemma 4 ([16]). *For any time-varying vector $x(t) \in \mathbb{R}^n$ and some constant $a \in \mathbb{R}$,*

$$\lim_{t\to\infty} x(t) = a\mathbf{1}_n \quad \Longleftrightarrow \quad \lim_{t\to\infty} Qx(t) = \mathbf{0}_{n-1}.$$

Next comes the main result of this paper.

Theorem 1. *Under Assumption 1–5 and using (18) for (1), if (2) holds at $t = 0$ and $x_i(0)$, $\forall i$ is bounded, then (2) holds for all $t > 0$, and $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$.*

Proof. This proof is two-fold. We first prove that asymptotic tracking to the reference models' output is guaranteed, and then prove that the reference models themselves reach consensus.

Using (18) for (16), we obtain the closed-loop system of error dynamics

$$\begin{aligned}\dot{e}_i =& - k_i e_i - R_i \tilde{W}_i \phi_i(x_i) + R_i\left(\epsilon_i + d_i(t)\right) \\ &- \frac{\hat{B}_i \|R_i\|^2 e_i}{\|R_i\|\|e_i\| + \mu_i(t)} - \frac{\eta_i e_i}{\|e_i\| + \mu_i(t)}\end{aligned} \tag{22}$$

where

$$\tilde{W}_i \triangleq \hat{W}_i - W_i. \tag{23}$$

Consider for each agent the following Lyapunov function candidate

$$V_i = \frac{1}{2} e_i^\top e_i + \frac{1}{2\delta_i}\tilde{B}_i^2 + \frac{1}{2}\operatorname{tr}\left\{\tilde{W}_i^\top \Gamma_i^{-1}\tilde{W}_i\right\}, \tag{24}$$

wherein

$$\tilde{B}_i \triangleq \hat{B}_i - B_i. \tag{25}$$

Taking the derivative of (24) with respect to time t, we have

$$\begin{aligned}\dot{V}_i =& e_i^\top \dot{e}_i + \frac{1}{\delta_i}\tilde{B}_i\dot{\hat{B}}_i + \operatorname{tr}\left\{\tilde{W}_i^\top \Gamma_i^{-1}\dot{\hat{W}}_i\right\} \\ =& - k_i e_i^\top e_i + \frac{1}{\delta_i}\tilde{B}_i\dot{\hat{B}}_i + e_i^\top R_i\left(\epsilon_i + d_i(t)\right) \\ &- \frac{\hat{B}_i\|R_i\|^2\|e_i\|^2}{\|R_i\|\|e_i\| + \mu_i(t)} + \operatorname{tr}\left\{\tilde{W}_i^\top \Gamma_i^{-1}\dot{\hat{W}}_i\right\} \\ &- \operatorname{tr}\left\{e_i^\top R_i \tilde{W}_i \phi(x_i)\right\} - \frac{\eta_i\|e_i\|^2}{\|e_i\| + \mu_i(t)} \\ =& - k_i e_i^\top e_i + \frac{1}{\delta_i}\tilde{B}_i\dot{\hat{B}}_i + e_i^\top R_i(\epsilon_i + d_i(t)) \\ &- \frac{\hat{B}_i\|R_i\|^2\|e_i\|^2}{\|R_i\|\|e_i\| + \mu_i(t)} - \frac{\eta_i\|e_i\|^2}{\|e_i\| + \mu_i(t)} \\ \le& - k_i\|e_i\|^2 + \|R_i\|\|e_i\|B_i - \frac{B_i\|R_i\|^2\|e_i\|^2}{\|R_i\|\|e_i\| + \mu_i(t)} \\ &- \frac{\eta_i\|e_i\|^2}{\|e_i\| + \mu_i(t)} \\ \le& - k_i\|e_i\|^2 + B_i\mu_i(t) - \frac{\eta_i\|e_i\|^2}{\|e_i\| + \mu_i(t)}\end{aligned} \tag{26}$$

where we have used (22) to obtain the second equality, (19a) to get the third equality, and (19b) to gain the first inequality. Integrating both sides of (26) yields

$$\begin{aligned}&V_i(t) + k_i\int_0^t \|e_i(\tau)\|^2 \,\mathrm{d}\tau + \eta_i\int_0^t \frac{\|e_i(\tau)\|^2}{\|e_i(\tau)\| + \mu_i(\tau)}\,\mathrm{d}\tau \\ &\le V_i(0) + B_i\int_0^t \mu_i(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{27}$$

Since $\mu_i(t) \in \mathbb{L}_1$, we can obtain that $V_i \in \mathbb{L}_\infty$, $e_i \in \mathbb{L}_2$ and $\frac{\|e_i\|^2}{\|e_i\|+\mu_i} \in \mathbb{L}_1$. Thus, we can derive e_i, $\tilde{W}_i$, $\tilde{B}_i \in \mathbb{L}_\infty$.

Now we show the boundedness of $\dot{e}_i$. Note that

$$\begin{aligned}&\int_0^t \|e_i(\tau)\|\,\mathrm{d}\tau \qquad (28)\\ =& \int_0^t \frac{\|e_i(\tau)\|^2}{\|e_i(\tau)\| + \mu_i(\tau)}\,\mathrm{d}\tau + \int_0^t \frac{\|e_i(\tau)\|\mu_i(\tau)}{\|e_i(\tau)\| + \mu_i(\tau)}\,\mathrm{d}\tau \\ \le& \int_0^t \frac{\|e_i(\tau)\|^2}{\|e_i(\tau)\| + \mu_i(\tau)}\,\mathrm{d}\tau + \int_0^t \mu_i(\tau)\mathrm{d}\tau \\ <&\infty, \qquad (29)\end{aligned}$$

since $\frac{\|e_i\|^2}{\|e_i\|+\mu_i} \in \mathbb{L}_1$. Therefore, $e_i \in \mathbb{L}_1$. Let z, e be stack vectors of z_i and e_i respectively, so we can rewrite (14) as

$$\begin{aligned}\dot{z} &= -\left(\mathcal{L}_A(t)\otimes I_p\right)s \\ &= -\left(\mathcal{L}_A(t)\otimes I_p\right)z - \left(\mathcal{L}_A(t)\otimes I_p\right)e.\end{aligned} \tag{30}$$

Denote $\Phi(t,\tau)$ as the transition matrix of system (30), then the solution of (30) is

$$z(t) = \Phi(t,0)z(0) - \int_0^t \Phi(t,\tau)\left(\mathcal{L}_A(\tau)\otimes I_p\right)e(\tau)\,\mathrm{d}\tau. \tag{31}$$

As per the connectivity conditions in *Assumption* 4, we can conclude from **Lemma 1** that the system

$$\dot{z} = -\left(\mathcal{L}_A(t)\otimes I_p\right)z \tag{32}$$

is uniformly stable. Hence there must exist some constant $\gamma > 0$ such that $\|\Phi(t,\tau)\| \le \gamma$ [43]. Since $\|\mathcal{L}_A(t)\otimes I_p\| \le c$, the following inequalities hold:

$$\begin{aligned}\|z(t)\| \le& \|\Phi(t,0)z(0)\| + \int_0^t \|\Phi(t,\tau)\|\|\mathcal{L}_A(\tau)\otimes I_p\|\|e(\tau)\|\mathrm{d}\tau \\ \le& \gamma\|z(0)\| + \gamma c\int_0^t \|e(\tau)\|\mathrm{d}\tau.\end{aligned} \tag{33}$$

Because $e_i \in \mathbb{L}_1$, from (33) we know that $z \in \mathbb{L}_\infty$. Thus, we can get from (15) that $s \in \mathbb{L}_\infty$, and this gives us that the constraints (2) are satisfied, which means the transformed states s_i are bounded. Then, as per pure calculation of the partial derivatives of (8), which is given by (10), one can verify that R_i defined in (12) is bounded, as is noted in *Remark* 3. Therefore, from (16) we have $\dot{e}_i \in \mathbb{L}_\infty$.

Since $e_i \in \mathbb{L}_2 \cap \mathbb{L}_\infty$ and $\dot{e}_i \in \mathbb{L}_\infty$, by **Lemma 3**, we will find $\lim_{t\to\infty} e_i(t) = \mathbf{0}_p$ for $i = 1, \ldots, n$. Hence, asymptotic tracking to the reference models is achieved.

Next, we shall show the consensus of (14). Define

$$\hat{z} \triangleq \left(Q\otimes I_p\right)z, \quad \hat{e} \triangleq \left(Q\otimes I_p\right)e. \tag{34}$$

From Lemma 4, we have that $z = \mathbf{1}_n \otimes b$ for some $b \in \mathbb{R}^p$ if and only if $(Q\otimes I_p)z = \mathbf{0}$. This means that the set of consensus states of system (32) is equivalent to the origin of variable $\hat{z}$. In light of this fact, the introduction of the matrix Q is supposed to turn a consensus problem into a stabilization problem. By multiplying both sides of (30) by $Q\otimes I_p$, we get

$$\dot{\hat{z}} = -(Q\mathcal{L}_A(t)\otimes I_p)z - (Q\mathcal{L}_A(t)\otimes I_p)e$$

$$\begin{aligned} &= -(Q\mathcal{L}_A(t)Q^\top Q \otimes I_p)z - (Q\mathcal{L}_A(t)\otimes I_p)e \\ &= -(Q\mathcal{L}_A(t)Q^\top \otimes I_p)\hat{z} - (Q\mathcal{L}_A(t)\otimes I_p)e \end{aligned} \tag{35}$$

where we use the fact that $\mathcal{L}_A Q^\top Q = \mathcal{L}_A(I_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top) = \mathcal{L}_A$ in (21) to obtain the second equality. With **Lemma 1** and **Lemma 4**, combining the relationship between z and $\hat{z}$, we can see that system $\dot{\hat{z}} = -(Q\mathcal{L}_A(t)Q^\top \otimes I_p)\hat{z}$ is uniformly exponentially stable. Thus, system (35) is input-to-state stable with respect to input $-(Q\mathcal{L}_A(t)\otimes I_p)e$ and state $\hat{z}$. Owing to this input-to-state stability, we can infer that $\lim_{t\to\infty}\hat{z} = \mathbf{0}$ since $\lim_{t\to\infty} e_i = \mathbf{0}_p$. Finally, we can conclude that $\lim_{t\to\infty} z = \mathbf{1}_n \otimes b$ where $b \in \mathbb{R}^p$, *i.e.*, the reference system has achieved consensus.

Considering the asymptotic tracking of the transformed system (11) to the reference system (14), along with the consensus of the reference system itself, we can derive that $\lim_{t\to\infty}(s_i - s_j) = \mathbf{0}$, and thus from (8) the final results of this theorem. ■

C. Extension to Second-Order Systems

We have shown in the previous subsection that the proposed consensus framework works for first-order systems with uncertain dynamics and disturbances. In real-world applications, nevertheless, motion equations of physical systems are often described by second-order dynamics, *e.g.*, Euler-Lagrange systems. This motivates us to discuss the same problem for systems with second-order dynamics. However, the extension of the constrained consensus of first-order systems to that of second-order systems is nontrivial. The main difficulty lies in the fact that a trivial extension of the consensus algorithm for first-order MASs under directed graphs to second-order ones only admits necessity but without sufficiency [31]. In this subsection, we show that the proposed consensus framework can be also extended to MASs with second-order agent dynamics, time-varying state constraints, uncertainties, and switching directed interaction topologies.

Now consider a second-order MAS consisting of n agents with the following dynamics

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = u_i + f_i(x_i, v_i) + d_i(t), \quad i = 1, 2, \ldots, n, \end{cases} \tag{36}$$

where $v_i \in \mathbb{R}^p$ denotes the velocity vector, $x_i \in \mathbb{R}^p$, $u_i \in \mathbb{R}^p$, $f_i(\cdot) \in \mathbb{R}^p$, and $d_i \in \mathbb{R}^p$ are consistent with their definitions in Section II.

Assumption 6. For second-order systems, both the upper and lower bounds of the constraints are twice differentiable with respect to time t, *i.e.*, $\ddot{\overline{x}}_k(t)$ and $\ddot{\underline{x}}_k(t)$ exist for all $t \geq 0$ and $k = 1, \ldots, p$.

To approximate the uncertainties, we adopt a similar NN approximation as (7). The only difference is that velocity information is put into the input layer of the NNs, *i.e.*, $f_i(x_i, v_i) = W_i \phi(x_i, v_i) + \epsilon_i$.

Taking the same state transformation (8) and differentiating (10), we have

$$\begin{aligned} \ddot{s}_{i,k} =& \frac{\partial^2 s_{i,k}}{\partial x_{i,k}^2}\dot{x}_{i,k}^2 + 2\frac{\partial^2 s_{i,k}}{\partial x_{i,k}\partial \overline{x}_k}\dot{x}_{i,k}\dot{\overline{x}}_k + 2\frac{\partial^2 s_{i,k}}{\partial x_{i,k}\partial \underline{x}_k}\dot{x}_{i,k}\dot{\underline{x}}_k \\ &+ 2\frac{\partial^2 s_{i,k}}{\partial \overline{x}_k \partial \underline{x}_k}\dot{\overline{x}}_k\dot{\underline{x}}_k + \frac{\partial^2 s_{i,k}}{\partial \overline{x}_k^2}\dot{\overline{x}}_k^2 + \frac{\partial^2 s_{i,k}}{\partial \underline{x}_k^2}\dot{\underline{x}}_k^2 \\ &+ \frac{\partial s_{i,k}}{\partial x_{i,k}}\ddot{x}_{i,k} + \frac{\partial s_{i,k}}{\partial \overline{x}_k}\ddot{\overline{x}}_k + \frac{\partial s_{i,k}}{\partial \underline{x}_k}\ddot{\underline{x}}_k \end{aligned} \tag{37}$$

Representing the transformed agent dynamics, (10) and (37) can be written in a matrix form as

$$\begin{cases} \dot{s}_i = R_i v_i + c_i, \\ \ddot{s}_i = G_i + R_i\left(u_i + W_i\phi(x_i, v_i) + \epsilon_i + d_i\right), \end{cases} \tag{38}$$

where s_i, c_i, and R_i are defined the same as in Section III-B, and $G_i = \begin{bmatrix} g_{i1} & \cdots & g_{ip}\end{bmatrix}^\top$ with

$$\begin{aligned} g_{ij} \triangleq& 2\frac{\partial^2 s_{i,k}}{\partial x_{i,k}\partial \overline{x}_k}\dot{x}_{i,k}\dot{\overline{x}}_k + 2\frac{\partial^2 s_{i,k}}{\partial x_{i,k}\partial \underline{x}_j}\dot{x}_{i,k}\dot{\underline{x}}_k \\ &+ 2\frac{\partial^2 s_{i,k}}{\partial \overline{x}_k \partial \underline{x}_k}\dot{\overline{x}}_k\dot{\underline{x}}_k + \frac{\partial^2 s_{i,k}}{\partial \overline{x}_k^2}\dot{\overline{x}}_k^2 + \frac{\partial^2 s_{i,k}}{\partial \underline{x}_k^2}\dot{\underline{x}}_k^2 \\ &+ \frac{\partial s_{i,k}}{\partial x_{i,k}}\ddot{x}_{i,k} + \frac{\partial s_{i,k}}{\partial \overline{x}_k}\ddot{\overline{x}}_k + \frac{\partial s_{i,k}}{\partial \underline{x}_k}\ddot{\underline{x}}_k. \end{aligned} \tag{39}$$

To tackle the asymmetric Laplacian matrix of a directed graph, we again adopt a model reference adaptive control scheme. Motivated by [15], [20], we employ the following reference model for second-order agents:

$$\ddot{z}_i = -\sum_{j=1}^{n} a_{ij}(t)(s_i - s_j) - \left(\frac{\sum_{j=1}^{n} a_{ij}(t)}{\kappa_i} + \kappa_i\right)\dot{s}_i \tag{40}$$

where $\kappa_i > 0$ for $i = 1, \ldots, n$.

Again, the tracking error e and a robust variable B are defined as (15) and (17), respectively. Referring to the sliding mode control, we introduce a sliding variable

$$q_i \triangleq \dot{e}_i + e_i. \tag{41}$$

Remark 7. The introduction of the sliding variable is merely for the convenience of design and analysis. One can always only use the tracking error e_i and its derivative $\dot{e}_i$ in algorithm implementation.

Then we put forward a distributed algorithm for (36)

$$\begin{aligned} u_i = R_i^{-1}\Bigg[&-G_i - \dot{e}_i - k_i q_i - \left(\frac{\sum_{j=1}^{n} a_{ij}(t)}{\kappa_i} + \kappa_i\right)\dot{s}_i \\ &- \sum_{j=1}^{n} a_{ij}(t)(s_i - s_j) - \frac{\hat{B}_i \left\|R_i\right\|^2 q_i}{\|R_i\|\,\|q_i\| + \mu_i(t)} \\ &- \frac{\eta_i q_i}{\|q_i\| + \mu_i(t)}\Bigg] - \hat{W}_i\phi(x_i, v_i) \end{aligned} \tag{42}$$

with k_i, κ_i, η_i being positive constants, $\mu_i(t) \geq 0$ for all $t \geq 0$, $\mu_i \in \mathbb{L}_1$, and $\hat{W}_i$ and $\hat{B}_i$ being the estimate of W_i and B_i, respectively. The adaptation laws for $\hat{W}_i$ and $\hat{B}_i$ are

$$\dot{\hat{W}}_i = \Gamma_i R_i q_i \phi^\top(x_i, v_i), \qquad \hat{W}_i(0) = \mathbf{0} \tag{43a}$$

$$\dot{\hat{B}}_i = \delta_i \frac{\|R_i\|^2 \|q_i\|^2}{\|R_i\|\|q_i\| + \mu_i(t)}, \quad \hat{B}_i(0) = 0 \tag{43b}$$

where Γ_i is some positive definite matrix, and δ_i is some positive constant.

Theorem 2. *Under Assumption 1–6 and using the control algorithm (42) for (36), if (2) holds at $t = 0$, and $x_i(0)$ and $v_i(0), \forall i$ are bounded, then (2) holds for all $t > 0$, $\lim_{t\to\infty} \|x_i(t)-x_j(t)\| = 0$, and $\lim_{t\to\infty} \|v_i(t)-v_j(t)\| = 0$.*

Proof. The proof also has two parts, namely, the asymptotic tracking to the reference system and the consensus of the reference system.

We first prove the asymptotic tracking. Substituting (15), (41), and (42) into (38), we obtain the closed-loop dynamics of the ith agent with respect to q_i

$$\begin{aligned}\dot{q}_i =& - k_i q_i - R_i \tilde{W}_i \phi(x_i, v_i) + R_i\left(\epsilon_i + d_i\right) \\ & - \frac{\hat{B}_i \left\|R_i\right\|^2 q_i}{\|R_i\| \left\|q_i\right\| + \mu_i(t)} - \frac{\eta_i q_i}{\|q_i\| + \mu_i(t)}\end{aligned} \tag{44}$$

with $\tilde{W}_i$ defined in (23). Consider the following Lyapunov function candidate

$$V_i = \frac{1}{2} q_i^\top q_i + \frac{1}{2}\operatorname{tr}\left\{\tilde{W}_i^\top \Gamma_i^{-1} \tilde{W}_i\right\} + \frac{1}{2\delta_i}\tilde{B}_i^2. \tag{45}$$

The derivative of V_i with respect to time t is

$$\begin{aligned}\dot{V}_i =& q_i^\top \dot{q}_i + \operatorname{tr}\left\{\tilde{W}_i^\top \Gamma_i^{-1} \dot{W}_i\right\} + \frac{1}{\delta_i}\tilde{B}_i \dot{\hat{B}}_i \\ =& - k_i \|q_i\|^2 + q_i^\top R_i(\epsilon_i + d_i) \\ & - \frac{B_i \left\|R_i\right\|^2 \left\|q_i\right\|^2}{\|R_i\| \left\|q_i\right\| + \mu_i(t)} - \frac{\eta_i \left\|q_i\right\|^2}{\|q_i\| + \mu_i(t)}\end{aligned} \tag{46}$$

where we obtain the second equality by taking (43a) and (43b) into (44). Further,

$$\begin{aligned}\dot{V}_i \leq& - k_i \|q_i\|^2 + \|R_i\| \|q_i\| B_i - \frac{B_i \left\|R_i\right\|^2 \left\|q_i\right\|^2}{\|R_i\| \left\|q_i\right\| + \mu_i(t)} \\ & - \frac{\eta_i \left\|q_i\right\|^2}{\|q_i\| + \mu_i(t)} \\ =& - k_i \|q_i\|^2 + \frac{B_i \left\|R_i\right\| \left\|q_i\right\| \mu_i(t)}{\|R_i\| \left\|q_i\right\| + \mu_i(t)} - \frac{\eta_i \left\|q_i\right\|^2}{\|q_i\| + \mu_i(t)} \\ \leq& - k_i \|q_i\|^2 + B_i \mu_i(t) - \frac{\eta_i \left\|q_i\right\|^2}{\|q_i\| + \mu_i(t)}\end{aligned} \tag{47}$$

Integrating both sides of (47) with respect to time t, we get

$$\begin{aligned}& V_i(t) + k_i \int_0^t \|q_i(\tau)\|^2 \, \mathrm{d}\tau + \eta_i \int_0^t \frac{\left\|q_i\right\|^2}{\|q_i\| + \mu_i(t)} \, \mathrm{d}\tau \\ & \leq V_i(0) + B_i \int_0^t \mu_i(\tau) \, \mathrm{d}\tau < \infty,\end{aligned} \tag{48}$$

which implies that $V_i \in \mathbb{L}_\infty$, $q_i \in \mathbb{L}_2$ and $\frac{\|q_i\|^2}{\|R_i\|\|q_i\| + \mu_i} \in \mathbb{L}_1$. Thus, we can derive q_i, $\tilde{W}_i$, $\tilde{B}_i \in \mathbb{L}_\infty$. Similar to (29), we can see $q_i \in \mathbb{L}_1$, since $\frac{\|q_i\|^2}{\|q_i\| + \mu_i} \in \mathbb{L}_1$ and $\mu_i \in \mathbb{L}_1$. Therefore, $q_i \in \mathbb{L}_1$. On the sliding surface, note that (41) is also an input-to-state stable dynamic system with e_i being the state and q_i the input:

$$\dot{e}_i = -e_i + q_i. \tag{49}$$

Examining (49), one can find that its solution is

$$e_i(t) = e^{-t} e_i(0) + \int_0^t e^{-(t-\tau)} q_i(\tau) \, \mathrm{d}\tau. \tag{50}$$

Since the exponential function $e^{-t} \in \mathbb{L}_l$ for $l \in [1, \infty]$, from **Lemma 2** we can get that

$$\left\| \int_0^t e^{-(t-\tau)} q_i(\tau) \, \mathrm{d}\tau \right\|_l \leq \|e^{-t}\|_1 \|q_i\|_l. \tag{51}$$

By letting $l = 1, 2, \infty$, we then have $e_i \in \mathbb{L}_1 \cap \mathbb{L}_2 \cap \mathbb{L}_\infty$. Thus, from (41) we know $\dot{e}_i \in \mathbb{L}_1 \cap \mathbb{L}_\infty$. Invoking Barbalat's lemma, $\lim_{t\to\infty} e_i(t) = \mathbf{0}$. Asymptotic tracking to the reference system is achieved.

Next, we show the consensus of the reference models themselves. Define

$$\begin{aligned} y_i &\triangleq z_i + \frac{1}{\zeta_i}\dot{z}_i, \\ \xi &\triangleq \left[z_1^\top, \cdots, z_n^\top, y_1^\top, \cdots, y_n^\top\right]^\top, \quad (52) \\ e_z &\triangleq \left[e_1^\top, \cdots, e_n^\top, \dot{e}_1^\top, \cdots, \dot{e}_n^\top\right]^\top, \quad (53)\end{aligned}$$

and $\mathrm{Z} \triangleq \operatorname{diag}\left(\zeta_1, \cdots, \zeta_n\right)$. Note that $e_z \in \mathbb{L}_1$ since $\dot{e}_i, e_i \in \mathbb{L}_1$. Then, (40) can be written in a vector form as

$$\dot{\xi} = -\left(\mathcal{L}_1(t) \otimes I_p\right) \xi - \left(\mathcal{L}_2(t) \otimes I_p\right) e_z \tag{54}$$

where

$$\mathcal{L}_1(t) = \begin{bmatrix} \mathrm{Z} & -\mathrm{Z} \\ -\mathrm{Z}^{-1}\mathcal{A}(t) & \mathrm{Z}^{-1}\mathcal{D}(t)\end{bmatrix}$$

$$\mathcal{L}_2(t) = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathrm{Z}^{-1}\mathcal{L}_A(t) & \mathrm{Z}^{-2}\mathcal{D}(t) + I\end{bmatrix}.$$

According to [20], we can consider $\mathcal{L}_1(t)$ as the Laplacian matrix of a directed graph $\mathcal{G}_1(t)$ consisting of $2n$ vertices. Further, it is also proven in [20] that the sequence of switching graphs $\mathcal{G}_1(t_k)$, $k = 0, 1, \ldots$, meets *Assumption* 4 if the sequence of switching graphs $\mathcal{G}(t_k)$, $k = 0, 1, \ldots$, satisfies *Assumption* 4. Let $\Phi(t, \tau)$ be the transition matrix of system (54), then the solution of (54) is

$$\xi(t) = \Phi(t, 0) z(0) - \int_0^t \Phi(t, \tau)\left(\mathcal{L}_2(\tau) \otimes I_p\right) e_z(\tau) \, \mathrm{d}\tau. \tag{55}$$

Hence, we can conclude from **Lemma 1** that the system

$$\dot{\xi} = -\left(\mathcal{L}_1(t) \otimes I_p\right) \xi$$

is uniformly stable. Hence there must exist some constant $\gamma > 0$ such that $\|\Phi(t, \tau)\| \leq \gamma$. On account of the boundedness of $\mathcal{A}(t)$ and Z, $\mathcal{L}_2(t)$ is also bounded, *i.e.*, $\forall t \geq 0$, $\|\mathcal{L}_2(t)\| \leq c$ for some $c \in \mathbb{R}^+$. Hence,

$$\begin{aligned}\xi(t) \leq& \|\Phi(t, 0)\xi(0)\| + \int_0^t \left\|\Phi(t, \tau)\left(\mathcal{L}_2(\tau) \otimes I_p\right) e_z(\tau)\right\| \, \mathrm{d}\tau \\ \leq& \gamma \|\xi(0)\| + \gamma c \int_0^t \|e_z(\tau)\| \, \mathrm{d}\tau.\end{aligned}$$

Combining the fact that $e_z \in \mathbb{L}_1$, we can see $\xi \in \mathbb{L}_\infty$. By the definition of ξ, $\dot{z}_i \in \mathbb{L}_\infty$ and $z_i \in \mathbb{L}_\infty$. Therefore, $\dot{s}_i \in \mathbb{L}_\infty$ and $s_i \in \mathbb{L}_\infty$ since $\dot{e}_i \in \mathbb{L}_\infty$ and $e_i \in \mathbb{L}_\infty$. Then from (40) and (12), $\ddot{z}_i \in \mathbb{L}_\infty$ and $R_i \in \mathbb{L}_\infty$. Thus, we have $\dot{q}_i \in \mathbb{L}_\infty$

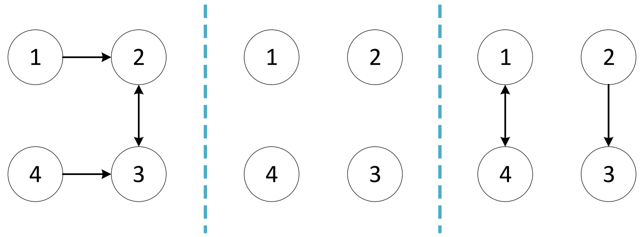

Fig. 3. Communication network

from (44). Since $q_i \in \mathbb{L}_\infty \cap \mathbb{L}_2$ and $\dot{q}_i \in \mathbb{L}_\infty$, by **Lemma 3**, we can conclude $\lim_{t\to\infty} q_i = \mathbf{0}$. Again, due to the input-to-state stability of (49) , $\lim_{t\to\infty} \dot{e}_i = \lim_{t\to\infty} e_i = \mathbf{0}$, *i.e.*, $\lim_{t\to\infty} e_z = \mathbf{0}$.

Then, by letting $\hat{\xi} \triangleq (Q \otimes I_p)\xi$ and following the same steps in the proof of **Theorem 1**, we can obtain that $\lim_{t\to\infty}(z_i - z_j) = \lim_{t\to\infty}(y_i - y_j) = \lim_{t\to\infty}(y_i - z_i) = \mathbf{0}$. Note that

$$\begin{aligned}\lim_{t\to\infty} y_i - z_i &= \lim_{t\to\infty} z_i + \frac{1}{\zeta_i}\dot{z}_i - z_i \\ &= \frac{1}{\zeta_i}\lim_{t\to\infty} \dot{z}_i = \mathbf{0}.\end{aligned}$$

Therefore, $\lim_{t\to\infty} \dot{z}_i = \mathbf{0}$, *i.e.*, the reference models have achieved consensus.

Finally, in consideration of the asymptotic tracking behavior of system (38) to the reference models (40), we have $\lim\limits_{t\to\infty}(s_i - s_j) = \lim\limits_{t\to\infty} \dot{s}_i = \mathbf{0}, \forall i, j = 1, \ldots, n$. From (8), we can derive that $\lim_{t\to\infty} \|x_i - x_j\| = 0$. Then using (10), we can find that $\lim_{t\to\infty}(v_i - v_j) = \mathbf{0}$, and thus the conclusion of the theorem. ∎

Remark 8. From the last step in the proof of **Theorem 2**, one can notice that the derivatives of the original states with respect to time only reach consensus but do not converge to zero. In fact, if the state constraints satisfy certain conditions, the final consensus value will also be stabilized, *i.e.*, $\lim_{t\to\infty} \|v_i(t)\| = 0$, $\forall i$. Without changing the control algorithm, we have the following corollary.

Corollary 3. *Suppose that the constraints satisfy* $\lim_{t\to\infty} \dot{\overline{x}}_k = \lim_{t\to\infty} \dot{\underline{x}}_k = 0, \forall k = 1, \ldots, p$. *Using the control algorithm* (42) *for* (36)*, if* (2) *holds at* $t = 0$*, then* (2) *holds for all* $t > 0$, $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$, *and* $\lim_{t\to\infty} \|v_i(t)\| = 0$.

IV. Numerical Simulations

To illustrate the effectiveness of the proposed control algorithms, we performed simulations on an MAS consisting of four agents with first-order dynamics and another MAS comprising four agents with second-order dynamics.

For both systems, we use the same communication topology represented by a series of switching directed graphs shown in Fig. 3. Note that none of the three graphs contains a directed spanning tree, whereas their union has one. The graphs switch among the three in Fig. 3 every 2 seconds.

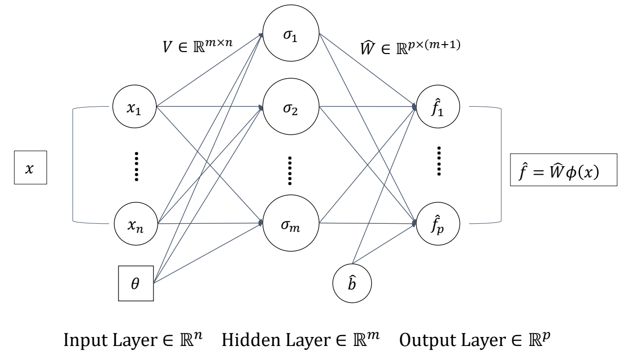

Fig. 4. Single layer neural network structure.

A. First-Order Dynamics

For the first-order scenario, agent dynamics are as (11), with the dimension of states $p = 1$. Predesignated state constraints are

$$\begin{cases}\overline{x} = 3e^{-0.2t} + 1.5 \\ \underline{x} = -5e^{-0.5t} + 1\end{cases}.$$

The initial states are set as $x(0) = [3.2,\ 0.98,\ -1,\ -\pi - 2.1]^\top$, The uncertainties are set as

$$f(x) = \begin{bmatrix} x_1^2 \\ 0.38x_2^3 \\ 2.64\sin\left(100\pi x_3 - \frac{\pi}{3}\right) + 2 \\ e^{0.6x_4}\end{bmatrix}.$$

The bounded external disturbances are

$$d(t) = \begin{bmatrix} 2\sin t \\ 4\cos\frac{\pi}{2000}t \\ \frac{5}{0.01t+1} \\ D_4\end{bmatrix},$$

where D_4 is a random variable such that $D_4 \sim U(-1, 1)$.

The structure of the NNs included in the controller is shown in Fig. 4. To approximate uncertainties, m neurons are used in the hidden layer for each agent, and the activation function is the logistic sigmoid function

$$\sigma(z) = \frac{1}{1 + e^{-z}}. \tag{56}$$

The size of the NN's hidden layer (the number of neurons) m, the fixed synaptic weights $V \in \mathbb{R}^{m\times p}$, along with other control parameters in the control algorithm (18), (19a), and (19b) are all listed in TABLE I.

Fig. 5 shows the states (positions) of the MAS. Clearly, agents' positions converge to the same constant without breaching any state constraints. Fig. 6 shows that the transformed states as well as the reference states achieve consensus in the transformed space, and that the tracking errors all converge to zero asymptotically.

B. Second-Order Dynamics

For the second-order system, agent dynamics are as (36) and the dimension of states $p = 1$. State constraints are set to

TABLE I
CONTROLLER PARAMETERS for the FIRST-ORDER SYSTEM

	Agent 1	Agent 2	Agent 3	Agent 4
k_i	50	30	1	3
δ_i	0.4	0.1	0.05	0.02
η_i	10	1	1	1
Γ_i	$0.1I_1$	$0.1I_1$	$0.07I_1$	$0.05I_1$
m_i	200	100	100	70
V_i	$\mathbf{1}_{200\times 1}$	$\mathbf{1}_{100\times 1}$	$\mathbf{1}_{100\times 1}$	$\mathbf{1}_{70\times 1}$
θ_i	0	0	0	0
$\mu_i(t)$	e^{-t}	e^{-t}	e^{-t}	e^{-t}

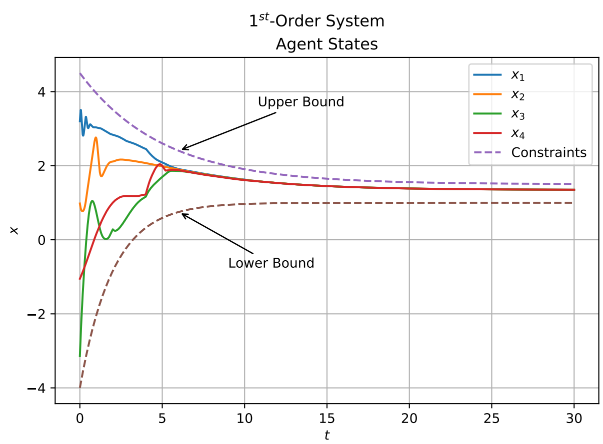

Fig. 5. Position states of the MAS with first-order agent dynamics.

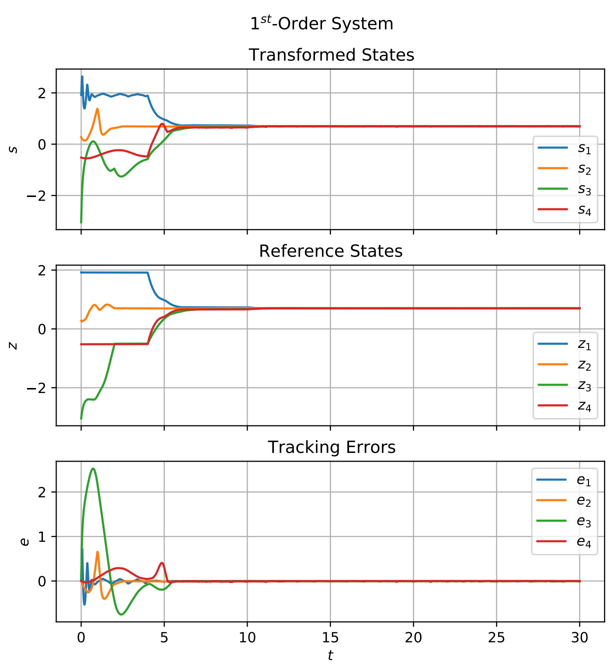

Fig. 6. Transformed states, reference states, and tracking errors of the first-order system.

be

$$\begin{cases} \overline{x} = \dfrac{9+\sin t}{0.7t+1} + 1 \\ \underline{x} = -4e^{-0.2t} + 0.999 \end{cases}.$$

TABLE II
CONTROLLER PARAMETERS for the SECOND-ORDER SYSTEM

	Agent 1	Agent 2	Agent 3	Agent 4
κ_i	1	1	1	1
k_i	50	30	1	3
δ_i	0.4	0.1	0.05	0.02
η_i	10	1	1	1
Γ_i	$0.1I_1$	$0.1I_1$	$0.07I_1$	$0.05I_1$
m_i	100	100	100	100
V_i	$\mathbf{1}_{100\times 2}$	$\mathbf{1}_{100\times 2}$	$\mathbf{1}_{100\times 2}$	$\mathbf{1}_{100\times 2}$
θ_i	0	0	0	0
$\mu_i(t)$	e^{-t}	e^{-t}	e^{-t}	e^{-t}

The initial states are $x(0) = [8,\ 0.98,\ 4.3387,\ -1.957]^\top$, and initial velocities are $v(0) = [0,\ 0,\ 0,\ 0]^\top$. Uncertainties are set as

$$f(x,v) = \begin{bmatrix} 0.0006x_1^2 + 0.0234e^{v_1} \\ x_2 \cdot v_2 + 0.095x_2^3 - 0.39v_2 \\ x_3 + v_3 \\ 0.066x_4^5 + 0.066v_4^3 \end{bmatrix}.$$

Disturbances are

$$d(t) = \begin{bmatrix} -3\cos(2\pi t) \\ -\cos\left(\frac{\pi}{2000}t\right) + 1 \\ \frac{2}{0.01t+1} \\ D_4 \end{bmatrix}$$

where $D_4 \sim U(-1,1)$.

We still use (56) as the activation function of neural networks. Control parameters for (42), (43a), (43b), and NN approximation are shown in TABLE II.

From Fig. 7, one can see that the MAS with second-order dynamics reaches consensus, and the derivatives of states, *i.e.*, agents' velocities, converge to zero asymptotically. Fig. 6 illustrates the consensus of the transformed states and the reference states in transformed space, as well as the asymptotic convergence of tracking errors.

V. Conclusion

This work has focused on the leaderless consensus problem of MASs with time-varying state constraints in the presence of uncertainties and bounded external disturbances under switching directed graphs. To this end, we have proposed distributed consensus algorithms under a feedback control framework incorporating state-space transformation, neural-network approximation, and model reference adaptive consensus. We have further provided Lyapunov-like proofs of the asymptotic convergence of the agents' states by leveraging sliding control and robust adaptive control techniques. Numerical simulation results have validated the efficacy of our framework for MASs with first- or second-order agent dynamics.

Future work could go in a few different directions. An important and natural question is how to extend this framework to the consensus of MASs with higher-order dynamics [44], [45]. For now, we only have extended it to second-order

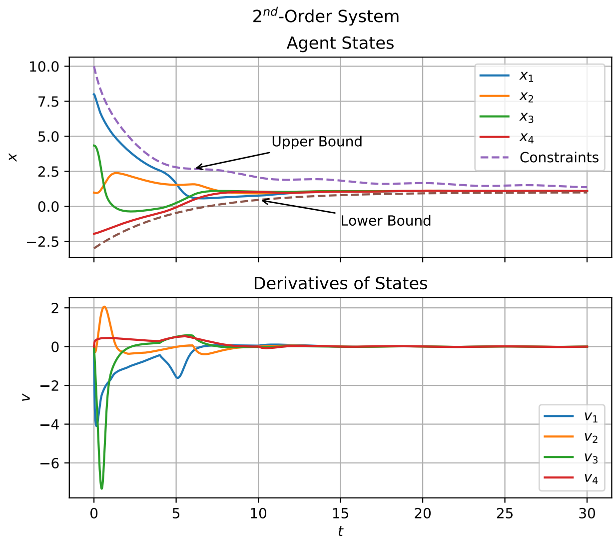

Fig. 7. Positions and velocities of the MAS with second-order agent dynamics.

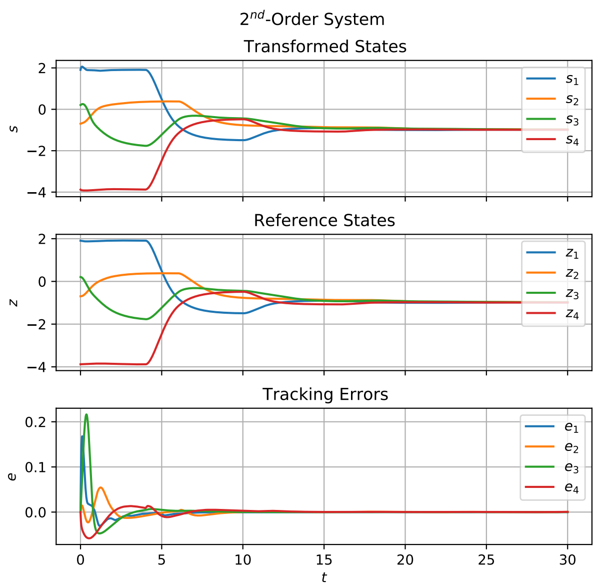

Fig. 8. Transformed states, reference states, and tracking errors of the second-order system.

MASs, while generalization to nth-order systems could significantly enhance the practicability of the framework. Another realistic problem is how we can incorporate input constraints into the framework. In this paper, we assume the agents can apply any desired control input without bounds, whereas there are always input constraints for real-world agents, such as saturation. Previous efforts [21], [27], [46] could be for reference. Addressing this issue remains an interesting yet practical open topic.

References

[1] F. Chen and W. Ren, "On the control of multi-agent systems: A survey," *Found. Trends Syst. Control*, vol. 6, no. 4, pp. 339–499, 2019.

[2] J. Qin, Q. Ma, Y. Shi, and L. Wang, "Recent advances in consensus of multi-agent systems: A brief survey," *IEEE Trans. Ind. Electron.*, vol. 64, no. 6, pp. 4972–4983, Jun. 2017.

[3] L. Scardovi, M. Arcak, and E. D. Sontag, "Synchronization of interconnected systems with applications to biochemical networks: An input-output approach," *IEEE Trans. Autom. Control*, vol. 55, no. 6, pp. 1367–1379, Jun. 2010.

[4] M. Ji, G. Ferrari-Trecate, M. Egerstedt, and A. Buffa, "Containment control in mobile networks," *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1972–1975, Sep. 2008.

[5] F. Zhang, Y.-Y. Chen, and Y. Zhang, "Projection-based containment control of multiple nonlinear systems with switching topologies," *IEEE Trans. Control Netw. Syst.*, vol. 9, no. 4, pp. 1793–1803, 2022.

[6] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, Mar. 2015.

[7] H. Liu, Y. Wang, F. L. Lewis, and K. P. Valavanis, "Robust formation tracking control for multiple quadrotors subject to switching topologies," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 3, pp. 1319–1329, 2020.

[8] S. Ghapani, J. Mei, W. Ren, and Y. Song, "Fully distributed flocking with a moving leader for Lagrange networks with parametric uncertainties," *Automatica*, vol. 67, pp. 67 – 76, 2016.

[9] Z. Deng and T. Chen, "Distributed algorithm design for constrained resource allocation problems with high-order multi-agent systems," *Automatica*, vol. 144, p. 110492, 2022.

[10] A. R. Romano and L. Pavel, "Dynamic NE seeking for multi-integrator networked agents with disturbance rejection," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 1, pp. 129–139, 2020.

[11] Z. Deng, Y. Liu, and T. Chen, "Generalized Nash equilibrium seeking algorithm design for distributed constrained noncooperative games with second-order players," *Automatica*, vol. 141, p. 110317, 2022.

[12] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[13] Z. Li and Z. Duan, *Cooperative Control of Multi-agent Systems: A Consensus Region Approach*, 2nd ed. Boca Raton, FL: CRC Press, Taylor & Francis Group, 2015.

[14] X. Wu, B. Mao, X. Wu, and J. Lu, "Dynamic event-triggered leader-follower consensus control for multiagent systems," *SIAM J. Control Optim.*, vol. 60, no. 1, pp. 189–209, 2022.

[15] K. Tian, Z. Guo, J. Mei, C. Jiang, and G. Ma, "Leaderless consensus for second-order inertia uncertain multi-agent systems under directed graphs without relative velocity information," *IEEE Trans. Netw. Sci. Eng.*, vol. 8, no. 4, pp. 3417–3429, 2021.

[16] J. Mei, "Distributed consensus for multiple Lagrangian systems with parametric uncertainties and external disturbances under directed graphs," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 2, pp. 648–659, 2020.

[17] H. Hong, W. Yu, J. Fu, and X. Yu, "Finite-time connectivity-preserving consensus for second-order nonlinear multiagent systems," *IEEE Trans. Control Netw. Syst.*, vol. 6, no. 1, pp. 236–248, 2019.

[18] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[19] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[20] K. Liu, Z. Ji, and W. Ren, "Necessary and sufficient conditions for consensus of second-order multiagent systems under directed topologies without global gain dependency," *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 2089–2098, Aug. 2017.

[21] X. Li, C. Li, and Y. Yang, "Heterogeneous linear multi-agent consensus with nonconvex input constraints and switching graphs," *Inf. Sci.*, vol. 501, pp. 397–405, Oct. 2019.

[22] Y.-H. Lim and H.-S. Ahn, "Consensus with output saturations," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5388–5395, Oct. 2017.

[23] A. Nédic, A. Ozdaglar, and P. Parrilo, "Constrained consensus and optimization in multi-agent networks," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 922–938, Apr. 2010.

[24] P. Lin and W. Ren, "Constrained consensus in unbalanced networks with communication delays," *IEEE Trans. Autom. Control*, vol. 59, no. 3, pp. 775–781, Mar. 2014.

[25] P. Lin, Y. Liao, H. Dong, D. Xu, and C. Yang, "Consensus for second-order discrete-time agents with position constraints and delays," *IEEE Trans. Cybern.*, vol. 52, no. 9, pp. 9736–9745, 2021.

[26] Z.-X. Liu and Z.-Q. Chen, "Discarded consensus of network of agents with state constraint," *IEEE Trans. Autom. Control*, vol. 57, no. 11, pp. 2869–2874, Nov. 2012.

[27] D. H. Nguyen, T. Narikiyo, and M. Kawanishi, "Multi-agent system consensus under input and state constraints," in *2016 Eur. Control Conf. (ECC)*. IEEE, Jun. 2016, pp. 537–542.

[28] ——, "Robust consensus analysis and design under relative state constraints or uncertainties," *IEEE Trans. Autom. Control*, vol. 63, no. 6, pp. 1784–1790, Jun. 2018.
[29] H. Chu, D. Yue, C. Dou, and L. Chu, "Consensus of multiagent systems with time-varying input delay and relative state saturation constraints," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 51, no. 11, pp. 6938–6944, 2021.
[30] W. Meng, Q. Yang, J. Si, and Y. Sun, "Consensus control of nonlinear multiagent systems with time-varying state constraints," *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 2110–2120, 2017.
[31] W. Ren and R. W. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*. London: Springer London, 2008.
[32] Z.-G. Hou, L. Cheng, and M. Tan, "Decentralized robust adaptive control for the multiagent system consensus problem using neural networks," *IEEE Trans. Syst., Man, Cybern. B*, vol. 39, no. 3, pp. 636–647, Jun. 2009.
[33] J. Mei, W. Ren, B. Li, and G. Ma, "Distributed containment control for multiple unknown second-order nonlinear systems with application to networked Lagrangian systems," *IEEE Trans. Neural Netw. Learning Syst.*, vol. 26, no. 9, pp. 1885–1899, 2015.
[34] D. Li, C. L. P. Chen, Y.-J. Liu, and S. Tong, "Neural network controller design for a class of nonlinear delayed systems with time-varying full-state constraints," *IEEE Trans. Neural Netw. Learning Syst.*, vol. 30, no. 9, pp. 2625–2636, Sep. 2019.
[35] G. Cybenko, "Approximation by superpositions of a sigmoidal function," *Math. Control Signals, Syst.*, vol. 2, no. 4, pp. 303–314, Dec. 1989.
[36] K. Hornik, "Approximation capabilities of multilayer feedforward networks," *Neural Netw.*, vol. 4, no. 2, pp. 251–257, 1991.
[37] J. Park and I. W. Sandberg, "Universal approximation using radial-basis-function networks," *Neural Comput.*, vol. 3, no. 2, pp. 246–257, Jun. 1991.
[38] E. Lavretsky and K. A. Wise, *Robust and Adaptive Control: With Aerospace Applications*. London: Springer London, 2013.
[39] A. Barron, "Universal approximation bounds for superpositions of a sigmoidal function," *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 930–945, May 1993.
[40] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. Philadelphia: Society for Industrial and Applied Mathematics, 2009, no. 55.
[41] C. P. Bechlioulis and G. A. Rovithakis, "Robust adaptive control of feedback linearizable mimo nonlinear systems with prescribed performance," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2090–2099, Oct. 2008.
[42] J. Mei, W. Ren, and Y. Song, "A unified framework for adaptive leaderless consensus of uncertain multi-agent systems under directed graphs," *IEEE Trans. Autom. Control*, vol. 66, no. 12, pp. 6179–6186, 2021.
[43] W. J. Rugh, *Linear System Theory*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1996.
[44] H. Rezaee and F. Abdollahi, "Consensus problem in high-order multiagent systems with Lipschitz nonlinearities and jointly connected topologies," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 47, no. 5, pp. 741–748, 2017.
[45] N. Wang, Y. Wang, G. Wen, M. Lv, and F. Zhang, "Fuzzy adaptive constrained consensus tracking of high-order multi-agent networks: A new event-triggered mechanism," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 52, no. 9, pp. 5468–5480, 2022.
[46] J. Shi, D. Yue, and X. Xie, "Optimal leader-follower consensus for constrained-input multiagent systems with completely unknown dynamics," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 52, no. 2, pp. 1182–1191, 2022.