Index Theory, Lecture Notes

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Contents

1 Remind: What you need to know first

Let us assume that all manifolds M appeared in this course have empty boundary.

1.1 Vector bundles

We start with vector bundles.

Definition 1.1: Vector bundle

Let $(V_i, \psi_i)_{i\in I}$ be an atlas of M, and define an equivalent relation on $\coprod_{i\in I} V_i \times \mathbb{K}^n$ by

$$(x, v) \sim (y, w) :\Longleftrightarrow x = y,\ w = \psi_{ij}(x)v,$$

where $(x, v) \in V_i \times \mathbb{K}^n$ and $(y, w) \in V_j \times \mathbb{K}^n$. Denote $E = \left(\coprod_{i\in I} V_i \times \mathbb{K}^n\right)/\sim$ and the natural projection $\pi : E \to M$, $[(x, v)] \mapsto x$ such that

(1) The diagram below commutes.

$$\begin{tikzcd} \pi^{-1}(V_i) \arrow[r, "\psi_i"] \arrow[dr, "\pi"'] & V_i \times \mathbb{K}^n \arrow[d, "\mathrm{pr}_1"] \\ & V_i \end{tikzcd}$$

(2) For all $i, j \in I$ such that $V_i \cap V_j \neq \varnothing$, there exists $\psi_{ij} : V_i \cap V_j : \mathrm{GL}_n(\mathbb{K})$ smooth and such that the diagram below commutes.

$$\begin{tikzcd} \pi^{-1}(V_i \cap V_j) \arrow[r, "\psi_i"] \arrow[dr, "\psi_j"'] & (V_i \cap V_j) \times \mathbb{K}^n \arrow[d, "\psi_j \circ \psi_i^{-1}|_{V_i \cap V_j \times \mathbb{K}^n}"] \\ & (V_i \cap V_j) \times \mathbb{K}^n \end{tikzcd}$$

In particular, each fibre $E_x := \pi^{-1}(x)$, there exists a well-defined K-vector space structure.

The maps $\{\psi_{ij}\}$ is called the *cocycle* of E, which satisfies

$$\psi_{ij}^{-1} = \psi_{ji}, \qquad \psi_{ij}\psi_{jk} = \psi_{ik}. \tag{1.1}$$

Conversely, if we have a family of maps $\{\psi_{ij} : V_i \cap V_j \to \mathrm{GL}_n(\mathbb{K})\}$ such that (1.1) holds, then $\exists \pi : E \to M$ a vector bundle such that $\{\psi_{ij}\}$ is its cocycle.

Example 1.2

Let $\pi : E \to M$ be a vector bundle and $\{\psi_{ij}\}$ be its cocycle.

(1) The *dual bundle* $E^* = \coprod_{x\in M} E_x^*$, where E_x^* is the linear 1-forms on E_x. The cocycle of E^* is $\left\{(\psi_{ij}^{\top})^{-1}\right\}$.

(2) If $\mathbb{K} = \mathbb{C}$, $\bar{E} = \coprod_{x\in M} \bar{E}_x$ is a vector bundle with cocycle $\{\bar{\psi}_{ij}\}$.

(3) Consider $E^{\otimes k} = \coprod_{x\in M} E_x^{\otimes k}$ all k-linear forms on E_x, *i.e.*, $\forall v_1, \dots, v_k \in E_x$,

$$v_1 \otimes \cdots \otimes v_k : E_x^* \times \cdots \times E_x^* \to \mathbb{K}, \qquad (\alpha_1, \dots, \alpha_k) \mapsto \prod_{i=1}^{k} (v_i, \alpha_i).$$

(4) Denote $S^k(E)$ the k-symmetric forms on E, $\Lambda^k(E)$ the k-alternating forms on E. Then $S^k(E) \subset E^{\otimes k}$, $\Lambda^k(E) \subset E^{\otimes k}$. And $S^k(E)$ is isomorphic to the space of homogeneous polynomials of

degree k on E. In particular,

$$S^\bullet(E) := \bigoplus_{k\geqslant 0} S^k(E), \qquad \Lambda^\bullet(E) := \bigoplus_{k\geqslant 0} \Lambda^k(E)$$

are called the *symmetric algebra* of E and the *exterior algebra* of E.

(5) Let $F \to M$ be another vector bundle. We can define their *tensor product* $E \otimes F := \coprod_{x\in M} E_x \otimes F_x$. In particular,

$$E^* \otimes F = \mathrm{Hom}(E,F), \qquad e^* \otimes f \mapsto [v \mapsto (e,v) f].$$

(6) Let N be a manifold and $f : N \to M$ be a smooth map. The *pullback* of $\pi : E \to M$ is defined as

$$f^* E := \coprod_{y\in N} E_{f(y)}, \qquad \left\{(f^{-1}(U_i)), \{\psi_{ij} \circ f\}_{i,j}\right\}.$$

Definition 1.3: C^∞-sections

The C^∞*-sections* of $E \to M$ is defined as

$$C^\infty(M,E) := \{s : M \to E : s \text{ is smooth, } s(x) \in E_x\}.$$

i.e., those smooth maps such that $\pi \circ s = \mathrm{id}_M$.

If $s \in C^\infty(M,E)$, the diagram below commutes.

$$\begin{tikzcd}
\pi^{-1}(V_i) \arrow[d, shift left, "\pi"] \arrow[dr, "\psi_i"] & \\
V_i \arrow[u, shift left, "s"] \arrow[r, "(\mathrm{id}, s_i)"'] & V_i \times \mathbb{K}
\end{tikzcd}$$

We obatin $\{s_i : U_i \to \mathbb{K}^n\}$ smooth and such that

$$\forall i,j \in I, (\psi_{ij} s_i = s_j). \tag{1.2}$$

Conversely, if $\{s_i\}$ satisfies (1.2), then $\exists s \in C^\infty(M,E)$ such that $\{s_i\}$ comes from s.

Proposition 1.4

The smooth sections $C^\infty(M,E)$ is non-empty.

Example 1.5

(1) When $\mathbb{K} = \mathbb{R}$, g^E is an *Euclidean metric* on E if $g^E \in C^\infty(M, E^* \otimes E^*)$ such that $\forall x \in M$, g^E_x is a scalar product.

(2) When $\mathbb{K} = \mathbb{C}$, h^E is an *Hermetian metric* on E if $h^E \in C^\infty(M, E^* \otimes \bar{E}^*)$ such that $\forall x \in M$, h^E_x is an Hermetian product.

Proposition 1.6: Tensoriality

Let E, F be vector bundles on M, $A : C^\infty(M,E) \to C^\infty(M,F)$, then the following are equivalent:

(1) $\forall f \in C^\infty(M,\mathbb{K})$, $\forall s \in C^\infty(M,E)$, $A(fs) = f \cdot A(s)$, *i.e.*, $[A,f] = 0$.

(2) $\exists \tilde{A} \in C^\infty(M, \mathrm{Hom}(E,F))$ such that $\forall s \in C^\infty(M,E)$, $A(s)(x) = \tilde{A}(x)s(x)$.

1.2 Differention and Integration

Let $TM \to M$ be the tangent bundle of M, *i.e.*, for an atlas (V_α, ψ_α), TM is defined by

$$\psi_{\alpha\beta}(x) := d_x(\varphi_\beta \circ \varphi_\alpha^{-1}).$$

So $\Omega^\bullet(M) = \bigoplus_{k\geqslant 0} \Omega^k(M) = \bigoplus_{k\geqslant 0} C^\infty(M, \Lambda^k(T^*M))$. Here $\Omega^k(M)$ denotes the set of k-forms on M.

Proposition 1.7: Exterior differential

There exists a unique map $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ such that

(1) For all $f \in C^\infty(M, \mathbb{K}) = \Omega^0(M)$, $df \in \Omega^1(M)$ is the usual differential.

(2) $d \circ d = 0$.

(3) $\forall \omega \in \Omega^k(M)$, $\forall \eta \in \Omega^\bullet(M)$, one has $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

The map d is called the *(exterior) differential*.

For $I = (i_1, \ldots, i_k)$, denote $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. If $\omega = f\,dx_I$, then

$$(d\omega)_x(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k} (df)_x dx_I(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}).$$

Definition 1.8: de Rham cohomology

As $d \circ d = 0$, one can define a complex $(\Omega^\bullet(M), d)$ and

$$H^k(M) := \frac{\ker(d|_{\Omega^k})}{\operatorname{im}(d|_{\Omega^{k-1}})}$$

the *de Rham cohomology*.

Assume $\dim M = n$, $\mathbb{K} = \mathbb{R}$. For $\omega \in \Omega^n(M)$, one can associate $(\omega_\alpha)_{\alpha\in A}$ with $\omega_\alpha : V_\alpha \to \Lambda^n(\mathbb{R}^n)$. While for each $\alpha \in A$, ω_α can be identified with a signed measure $\underline{\omega}_\alpha$ on V_α via

$$\omega_\alpha = f_\alpha dx_1 \wedge \cdots \wedge dx_n \longleftrightarrow \underline{\omega}_\alpha = f_\alpha dx_1 \cdots dx_n.$$

Then for $\varphi \in C_c^\infty(V_\alpha)$, one can define

$$\underline{\omega}_\alpha(\varphi) = \int_{V_\alpha} \varphi f_\alpha dx_1 \cdots dx_n.$$

Remark. *The signed measures $(\underline{\omega}_\alpha)_{\alpha\in A}$ do not patch together into a signed measure on M because*

$$(\psi_{\alpha\beta})_* \underline{\omega}_\alpha|_{V_\alpha\cap V_\beta} = \det\psi_{\alpha\beta} \cdot \underline{\omega}_\beta|_{V_\alpha\cap V_\beta}.$$

However, in the change of variable formula for Lebesgue measure, we should have $\left|\det\psi_{\alpha\beta}\right|$ instead of $\det\psi_{\alpha\beta}$.

Definition 1.9: Orientable bundle

Say a $\mathbb{R}$-vector bundle $\pi : E \to M$ is *orientable* if it can be defined by a cocycle $\{\psi_{ij}\}$ with $\psi_{ij}(x) \in \mathrm{GL}_r^+(\mathbb{R})$, *i.e.*, $\det\psi_{ij} > 0$.

Then M is orientable if TM is. So if M is orientable, $\omega \in \Omega^n(M)$ associates with a signed measure on M, denoted as $\underline{\omega}$, defined by

$$\underline{\omega}(\varphi) := \int_M \varphi\omega, \qquad \forall \varphi \in C_c^\infty(M).$$

More generally, if $\omega \in \Omega^\bullet(M)$, let $\omega^{[n]}$ be its component in $\Omega^n(M)$ then

$$\int_M \omega := \int_M \omega^{[n]}, \qquad \text{if } 1 \in L^1(\omega^{[n]}).$$

Theorem 1.10: Stokes' theorem

Let M be compact and orientable, $\forall s \in \Omega^\bullet(M)$,

$$\int_M ds = 0.$$

Thus, we obtain that if M is compact and orientable, the pairing

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}, \qquad ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$$

is well-defined.

1.3 Differential operator on manifolds

We assume M to be a manifold of dimension n, and $E,\ F \to M$ to be vector bundles in this section. The operators from E to F is denoted as

$$\mathrm{Op}(E,F) := \mathscr{B}(\mathbb{C}^\infty(M,E), C^\infty(M,F)).$$

Definition 1.11: Differential operator

A *differential operator* of order $\leqslant k$ from E to F is $P \in \mathrm{Op}(E,F)$ such that

(1) There exists an open covering $M = \bigcup_\alpha U_\alpha$ such that $U_\alpha \cong V_\alpha \subset \mathbb{R}^n$, $E|_{U_\alpha} \cong V_\alpha \in \mathbb{K}^{r_1}$, $F|_{U_\alpha} \cong V_\alpha \times \mathbb{K}^{r_2}$.

(2) There exists an operator

$$P_\alpha : C^\infty(V_\alpha, \mathbb{K}^{r_1}) \to C^\infty(V_\alpha, \mathbb{K}^{r_2}), \qquad \varphi \mapsto \sum_{|I| \leqslant k} a_\alpha^I \frac{\partial^{|I|}}{\partial x^I} \varphi,$$

where each $a_\alpha^I \in C^\infty(V_\alpha, \mathrm{Mat}_{r_1,r_2}(\mathbb{K}))$, such that if $s \in C^\infty(M,E)$ (*associates to* $\{s_\alpha\}$), $Ps \in C^\infty(M,F)$ (*associates to* $\{\sigma_\alpha\}$) with $\sigma_\alpha = P_\alpha s_\alpha$.

Denote by $\mathscr{D}\mathrm{iff}^{\leqslant k}(E,F)$ the space of such operators, and

$$\mathscr{D}\mathrm{iff}^k(E,F) = \mathscr{D}\mathrm{iff}^{\leqslant k}(E,F) \setminus \mathscr{D}\mathrm{iff}^{\leqslant k-1}(E,F).$$

Example 1.12

(1) If $A \in C^\infty(M, \mathrm{Hom}(E,F))$, then $A \in \mathscr{D}\mathrm{iff}^0(E,F)$.

(2) The differential $d : \Omega^i(M) \to \Omega^{i+1}(M) \in \mathscr{D}\mathrm{iff}^1(\Lambda^i T^*M, \Lambda^{i+1} T^*M)$ because when $\omega_\alpha = \sum_I \omega_{\alpha,I}$,

$$d_\alpha \omega_\alpha = \sum_I \sum_j \frac{\partial \omega_{\alpha,I}}{\partial x_j} dx_j \wedge dx_I = \sum_I \sum_j a_\alpha^{(j)} dx_I = d^{V_\alpha} \omega_\alpha,$$

here d^{V_α} denotes the usual differential on $\Omega^\bullet(V_\alpha)$.

Proposition 1.13: Inductive characterisation

Let $P \in \mathrm{Op}(E,F)$. Then $P \in \mathscr{D}\mathrm{iff}^{\leqslant k}(E,F)$ if and only if $\forall f \in C^\infty(M,\mathbb{R})$, $[P,f] \in \mathscr{D}\mathrm{iff}^{\leqslant k-1}(E,F)$.

Proof. $\Longrightarrow$: On $\mathbb{R}^n$, $[\partial_{x_i}, f]$ is just multiplication by $\frac{\partial f}{\partial x_i}$, then it is immediate for ∂_{x_i}. We do induction: if $I = (i_1, \dots, i_{k-1}, i_k)$, and denote $I' = (i_1, \dots, i_{k-1})$.

$$\begin{aligned}\frac{\partial^{|I|}}{\partial x_I} f = \frac{\partial^{|I'|}}{\partial x_{I'}} \frac{\partial}{\partial x_{i_k}} f &= \frac{\partial^{|I'|}}{\partial x_{I'}} f \frac{\partial}{\partial x_{i_k}} + \text{diff.op. of order} \leqslant k-1 \\ &= f \frac{\partial^{|I'|}}{\partial x_{I'}} \frac{\partial}{\partial x_{i_k}} + \text{diff.op. of order} \leqslant k-1.\end{aligned}$$

Hence $[\frac{\partial^{|I|}}{\partial x_I}, f]$ is a differential operator with order $\leqslant k-1$. And for $Q = [P, f]$, one can consider $Q_\alpha = [P_\alpha, f_\alpha]$ for each $\alpha \in A$, which is an operator on $\mathbb{R}^n$, and patch them together.

$\Longleftarrow$: **Step 1.** The operator P is locally well-defined. $\forall x \in M$, if $s_1 = s_2$ on a neighbourhood $U_x \ni x$, then $\forall W_x \ni x$ with $\bar{W}_x \subset U_x$, one has $Ps_1|_{W_x} = Ps_2|_{W_x}$.

Now assume $k = 0$ and $f \in C_c^\infty(U_x)$ with $f|_{\bar{W}_x} = 1$. Then on W_x,

$$Ps_1 = fPs_1 = Pfs_1 + [P, f]s_1 = Pfs_2 + [P, f]s_2 = fPs_2 = Ps_2.$$

In particular, the definition $P|_U$ is independent from the open neighbourhood U. We can take $s \in C^\infty(U, E|_U)$ and $x \in U$, then define

$$P|_U s(x) := Pfs(x)$$

with $f = 1$ on a neighbourhood W_x of x with $\bar{W}_x \subset U_x$ and $f = 0$ outside U_x.

Step 2. Now let $P_\alpha = P|_{U_\alpha}$ is a classical differential operator, $P_\alpha : C^\infty(V_\alpha, \mathbb{K}^{r_1}) \to C^\infty(V_\alpha, \mathbb{K}^{r_2})$. If $x_0 \in V_\alpha$ and $s \in C^\infty(V_\alpha, \mathbb{K}^{r_1})$, we write

$$s = \sum_{|I| \leqslant k} (x - x_0)^I \left(\frac{\partial^{|I|}}{\partial x_I} s \right)(x_0) + \sum_{|J| = k+1} (x - x_0)^J s_J(x)$$

and we want to calculate $(P_\alpha s)(x_0)$.

If $|J| = k+1$, denote $J = (i_1, \dots, i_n)$ and with for example $i_i \geqslant 1$. Let $I = (i_1 - 1, i_2, \dots, i_n)$, $|I| = k$,

$$\begin{aligned}P_\alpha (x - x_0)^J s_J &= P_\alpha (x_1 - x_{0,1})(x - x_0)^I s_J \\ &= (x_1 - x_{0,1}) P_\alpha (x - x_0)^I s_J + \underbrace{[P_\alpha, (x_1 - x_{0,1})]}_{\text{diff.op. of order} \leqslant k-1} (x - x_0)^I s_J.\end{aligned}$$

Note that the first term equals zero on $x = x_0$, so we can continue this reasoning by doing induction on k that $(P_\alpha (x - x_0)^J s_J)(x_0) = 0$.

If $|I| = k$,

$$P_\alpha \left((x - x_0)^I \left(\frac{\partial^{|I|}}{\partial x_I} s(x_0) \right) \right) = (P_\alpha)(x - x_0)^I \frac{\partial^{|I|}}{\partial x_I} s(x_0)$$

so by setting $a_\alpha^I(x_0) = (P_\alpha (x - x_0)^I)(x_0)$,

$$(P_\alpha s)(x_0) = \sum_{|I| \leqslant k} a_\alpha^I(x_0) \frac{\partial^{|I|}}{\partial x_I} s(x_0).$$

It remains to prove that a_α^I is smooth. This because

$$(x - x_0)^I = \sum_{I' \subset I} c_{I'} x_0^{I'} x^{I \setminus I'}$$

so

$$(P_\alpha (x - x_0)^I)(x_0) = \sum_{I' \subset I} c_{I'} x_0^{I'} (P_\alpha x^{I \setminus I'})(x_0)$$

with both $x_0^{I'}$ and $P_\alpha x^{I \setminus I'}$ smooth at x_0. $\square$

Definition 1.14: Total symbol

If P is a differential operator on $V \subset \mathbb{R}^n$, $P = \sum_{|I|\leqslant k} a_I(x) \frac{\partial^{|I|}}{\partial x_I}$, we can define the *total symbol* of P by

$$\sigma_{\mathrm{tot}}(P)(x,\xi) = \sum_{|I|\leqslant k} a_I(x)(\mathrm{i}\xi)^I, \qquad x \in V,\ \xi \in T_x^* V \cong \mathbb{R}^n.$$

Then $\sigma_{\mathrm{tot}}(P)$ is a smooth map from V to polynomial of degree $\leqslant k$ on T^V with values in $\mathrm{Hom}(E,F)$. Hence

$$\sigma_{\mathrm{tot}}(P) \in C^\infty(V, \bigoplus_{i\leqslant k} S^i(T^*V)\otimes E^* \otimes F).$$

If $P \in \mathscr{D}\mathrm{iff}^{\leqslant k}(E,F)$, then $(\sigma_{\mathrm{tot}}(P_\alpha))_{\alpha\in A}$ do not patch together. But we have:

Proposition 1.15: Principal symbol

The local sections

$$\sigma(P_\alpha)(x,\xi) = \sum_{|I|=k} a_\alpha^I(x)(\mathrm{i}\xi)^I$$

do patch together and define $\sigma(P) \in C^\infty(M, S^k(TM)\otimes E^*\otimes F)$, called the *principal symbol* of P.

Proof. For $f \in C^\infty(M,\mathbb{R})$, $\mathrm{e}^{-\mathrm{i}tf} P \mathrm{e}^{\mathrm{i}tf}$ is a differential operator and

$$\mathrm{e}^{-\mathrm{i}tf} P \mathrm{e}^{\mathrm{i}tf}|_{U_\alpha} = \mathrm{e}^{-\mathrm{i}tf} P_\alpha \mathrm{e}^{\mathrm{i}tf} = \mathrm{e}^{-\mathrm{i}tf}\left(\sum_{|I|\leqslant k} \frac{\partial^{|I|}}{\partial x_I}\right)\mathrm{e}^{\mathrm{i}tf}$$

$$\frac{\partial}{\partial x_j}(\mathrm{e}^{\mathrm{i}tf} s) = \left(\mathrm{i}t\frac{\partial f}{\partial x_j}\mathrm{e}^{\mathrm{i}tf}\right)s + \mathrm{e}^{\mathrm{i}tf}\frac{\partial s}{\partial x_j} = \mathrm{e}^{\mathrm{i}tf}\left(\frac{\partial}{\partial x_j} + \mathrm{i}t\cdot df(e_j)\right).$$

Denote $\left(\frac{\partial}{\partial x} + \mathrm{i}t\cdot df\right)^I = \prod_{j=1}^n \left(\frac{\partial}{\partial x_j} + \mathrm{i}t\cdot df(x_j)\right)^{i_j}$, then

$$\mathrm{e}^{-\mathrm{i}tf} P \mathrm{e}^{\mathrm{i}tf}|_{U_\alpha} = \sum_{|I|\leqslant k} a_\alpha^I \left(\frac{\partial}{\partial x} + \mathrm{i}t\cdot df\right)^I,$$

which is a polynomial w.r.t. t. The leading term is $t^k \sum_{|I=k|} a_\alpha^I (\mathrm{i}df)^I$.

Set $A \in C^\infty(M, S^k(TM)\otimes E^*\otimes F)$ by defining $A(x,\xi)$ to be the coefficient in t^k of $\mathrm{e}^{-\mathrm{i}tf} P \mathrm{e}^{\mathrm{i}tf}$, if $x \in M$ and $\xi \in T_x^* M = (df)_x$ for some smooth f. We have proved that it does not depend on f, and A is a global section with $A_\alpha = \sigma(P_\alpha)$. □

Remark. Let $\mathrm{Ad}_g P = gPg^{-1}$, $\mathrm{ad}(X)P = [X,P]$. *Then* $\mathrm{Ad}_{\mathrm{e}^X} = \mathrm{e}^{\mathrm{ad}(X)}$. *So*

$$\mathrm{e}^{-\mathrm{i}tf} P \mathrm{e}^{\mathrm{i}tf} = \mathrm{Ad}_{\mathrm{e}^{-\mathrm{i}tf}} P = \mathrm{e}^{\mathrm{ad}(-\mathrm{i}tf)} P$$

as $\sigma(P)(x,(df)_x) = \frac{(-\mathrm{i})^k}{k!}(\mathrm{ad}(f))^k P$.

Proposition 1.16

The following sequence is exact.

$$0 \longrightarrow \mathscr{D}\mathrm{iff}^{\leqslant k-1}(E,F) \xrightarrow{\iota} \mathscr{D}\mathrm{iff}^{\leqslant k}(E,F) \xrightarrow{\sigma} C^\infty(M, S^k(TM)\otimes E^*\otimes F) \longrightarrow 0$$

Here ι is the canonical inclusion and σ is the map defined by principal symbol.

Proof. It is obvious that ι is injective and $\sigma\circ\iota = 0$. So the exactness at $\mathscr{D}\mathrm{iff}^{\leqslant k-1}$ is done. If $\sigma(P) = 0$, then $P \in \mathscr{D}\mathrm{iff}^{\leqslant k-1}$, so it suffices to prove σ is surjective.

Let $A \in C^\infty(M, S^k(TM) \otimes E^* \otimes F)$ and denote $(A_\alpha)_{\alpha \in A}$ its local sections. On V_α, there exists P_α such that $\sigma_{\text{tot}}(P_\alpha) = A_\alpha$. Let P be defined by

$$Ps = \sum_{\alpha \in A} P_\alpha \varphi_\alpha s$$

with $(\varphi_\alpha)_{\alpha \in A}$ being a locally finite partition of unity subordinate to $(V_\alpha)_{\alpha \in A}$. Now $P_\alpha \varphi_\alpha s \in C_c^\infty(V_\alpha, F) \subset C^\infty(M, F)$. Thus we can calculate

$$\begin{aligned}
\sigma(P)(x, (df)_x) &= \text{coefficient of } t^k \text{ in } \mathrm{e}^{-\mathrm{i}tf} P \mathrm{e}^{\mathrm{i}tf} \\
&= \text{coefficient of } t^k \text{ in } \mathrm{e}^{-\mathrm{i}tf} \sum_{\alpha \in A} P_\alpha \varphi_\alpha \mathrm{e}^{\mathrm{i}tf} \\
&= \sum_{\alpha \in A} \sigma(P_\alpha)(x, (df)_x) \varphi_\alpha(x) \\
&= \sum_{\alpha \in A} A_\alpha \varphi_\alpha = \sum_{\alpha \in A} A \varphi_\alpha = A.
\end{aligned}$$

We conclude the proof. □

Definition 1.17: Elliptic operator

Say $P \in \mathscr{D}\text{iff}(E, F)$ is *elliptic* if $\forall x \in M$, $\forall \xi \in T_x^* M \setminus \{0\}$,

$$\sigma(P)(x, \xi) : E_x \to F_x$$

is invertible. In particular, $E_x \cong F_x$.

1.4 Statement of Atiyah–Singer index theorem

Let M be a compact manifold and $E, F \to M$ be vector bundles.

Definition 1.18: Index (of a diff. op.)

If $P \in \mathscr{D}\text{iff}(E, F)$ is elliptic, then both $\ker P$ and $\operatorname{coker} P$ are of finite dimension. And we set

$$\operatorname{Ind} P := \dim \ker P - \dim \operatorname{coker} P$$

its *index*.

Note that $\dim \ker P$ is the dimension of the space of solutions to $Ps = u$, and $\dim \operatorname{coker} P$ is the number of constriants on u to be able to solve $Ps = u$.

Example 1.19

We consider a baby case: if $M = \{\text{pt}\}$, $P : E \to F$ linear. Then $\operatorname{Ind} P = \dim E - \dim F$ by basic linear algebra.

Question. Why consider $\operatorname{Ind} P$?

The first reason is the rank theorem on infinite dimensional spaces gives something interesting. The second reason is that $\dim \ker P$ and $\dim \operatorname{coker} P$ are not stable, for example,

$$\dim \ker(\varepsilon \mathrm{id}_{\mathbb{R}^n}) = n\delta_{\varepsilon \neq 0}, \quad \dim \operatorname{coker}(\varepsilon \mathrm{id}_{\mathbb{R}^n}) = n\delta_{\varepsilon = 0}.$$

But $\operatorname{Ind} P$ is stable. We admit the following proposition.

Proposition 1.20

$\operatorname{Ind} P$ only depends on the homotopy class of P, *i.e.*, if $t \mapsto P_t$ is a C^0 family of elliptic operators, then $\operatorname{Ind} P_t = \text{const}$.

Corollary 1.21

$\operatorname{Ind} P$ only depends on $\sigma(P)$.

Proof. If $\sigma(P_0) = \sigma(P_1)$, define $P_t = (1-t)P_0 + tP_1$, then $\sigma(P_t) = \sigma(P_0)$ as P_t is elliptic. So $\operatorname{Ind} P_0 = \operatorname{Ind} P_1$. □

Moreover, $\sigma_0, \sigma_1 \in C^\infty(M, S^k(TM) \otimes E^* \otimes F)$ are said to be *regularly homotopic* if there exists a C^0 path of elliptic symbols σ_t linking them.

Proposition 1.22

$\operatorname{Ind} P$ only depends on the regular homotopy class of $\sigma(P)$.

Proof. See **[Lawson, Michelsohn, Spin Geometry. Chap 3, Sec 7.]** □

Question. How to express $\operatorname{Ind} P$ with $\sigma(P)$ in a *topological* way?

Theorem 1.23: Atiyah–Singer

Let P be an elliptic operator.

(1) If $\dim M$ is odd, $\operatorname{Ind} P = 0$.

(2) If $\dim M$ is even, then $\operatorname{Ind} P = \int_{TM} \operatorname{ch}(\sigma(P))\pi^*(\widehat{A}(TM)^2)$.

where $\pi : TM \to M$ is the projection. $\operatorname{ch}(\cdot)$ and $\widehat{A}(\cdot)$ are characteristic classes.

To prove (2), there are roughly two steps.

Step 1. Prove that there exists a Dirac operator D on another manifold(*or vector bundle*) such that $\operatorname{Ind} P = \operatorname{Ind} D$. We have completed this part last year by introducing tangent groupoids.

Step 2. Prove that for Dirac operators,

$$\operatorname{Ind} D = \int_M \widehat{A}(TM)\operatorname{ch}(E/S).$$

The formula above is the aim of this course.

There are other proofs: topology, noncommutative geometry, for example. This theorem is one of the great results of the 20th century. It unifies in particular the following 3 fundamental theorems, from different branches of mathematics.

- Gauss–Bonnet–Chern theorem (*in differential geometry*):

$$\chi(M) = \int_M e(TM).$$

- Hirzebruch's signature theorem (*in topology*):

$$\sigma(M) = \int_M L(TM).$$

- Riemann–Roch–Hirzebruch theorem (*in algebraic geometry*):

$$\sum_{j=0}^{\dim M}(-1)^j \dim H^{0,j}(M,E) = \int_M \mathrm{Td}(TM)\mathrm{ch}(E)$$

The objective of this course is to present step 2 of the proof and to show how the 3 previous theorems can be deduced from the Atiyah–Singer theorem.

From now on, we assume that $\dim M$ is even. For simplicity, we will also assume that M is orientable.

2 Chern–Weil Theory

2.1 Connections and curvatures of vector bundles

Let M be a manifold of dimension n, $E \to M$ be a vector bundle of rank r. Set $\Omega^\bullet(M,E) := C^\infty(M, \Lambda^\bullet T^*M \otimes E)$.

Definition 2.1: Connection

A *connection* on E is an operator

$$\nabla^E : C^\infty(M,E) \to \Omega^1(M,E)$$

such that

(1) ∇^E is $\mathbb{K}$-linear.

(2) $\forall f \in C^\infty(M)$, $\forall s \in C^\infty(M,E)$, $\nabla^E(fs) = df \cdot s + f \cdot \nabla^E s$. (*which is equivalent to* $[\nabla^E, f] = df$.)

We denote $(\nabla^E s)(X) =: \nabla^E_X s$ for $X \in TM$, the derivative of s in the direction X. Note that (1) and (2) in the definition above is equivalent to $\nabla^E \in \mathscr{D}\mathrm{iff}^1(E, T^*M \otimes E)$ such that $\sigma(\nabla^E)(x,\xi) = \mathrm{i}\xi\lambda$. Hence ∇^E is elliptic.

Example 2.2: Trivial connection

If $E = \mathbb{K}^r$ is the trivial bundle,

$$s \in C^\infty(M,E) \quad \Longleftrightarrow \quad \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix} \in (C^\infty(M))^r.$$

And $ds = [ds_1 \ \cdots \ ds_n]^\top$ is a connection, called the *trivial connection*.

Proposition 2.3

(1) There exists a connection.

(2) If ∇^E_0 is a connection, then ∇^E is a connection if and only if $\nabla^E - \nabla^E_0 \in \Omega^1(M, \mathrm{End}\, E)$.

Proof. (1) This is because $\mathscr{D}\mathrm{iff}^{\leq 1}(E, T^*M \otimes E) \to C^\infty(M, S^1(T^*M) \otimes E^* \otimes T^*M \otimes E)$ is surjective.

(2) ∇^E is a connection if and only if $\sigma(\nabla^E - \nabla^E_0) = 0$, which is equivalent to $\nabla^E - \nabla^E_0 \in \mathscr{D}\mathrm{iff}^0(E, T^*M \otimes E)$. □

Locally on U_α, $E \cong \mathbb{K}^r$. Any connection ∇^E can be written as

$$\nabla^E|_{U_\alpha} = d + \Gamma^E_\alpha,$$

where d is the trivial connection and $\Gamma^E_\alpha \in \Omega^1(U_\alpha, \mathrm{Mat}_r(\mathbb{K}))$ is the *Christoffel tensor*. Please note that $\{\Gamma^E_\alpha\}_{\alpha\in A}$ do not patch together.

To make the notations simpler, we shall denote the C^∞-sections $C^\infty(M,E) =: \Gamma(E)$.

Example 2.4: Induced connections

Let (E,∇^E), (F,∇^F) be two bundles with connections.

(1) One can define ∇^{E^*} on E^* by

$$d(\varphi,u) := (\nabla^{E^*}\varphi, u) + (\varphi, \nabla^E u), \qquad \forall \varphi \in \Gamma(E^*), \forall u \in \Gamma(E).$$

And locally, $\Gamma^{E^*}_\alpha = -(\Gamma^E_\alpha)^\top$.

(2) If E is complex, $\nabla^{\bar{E}}$ defined by $\nabla^{\bar{E}}\bar{u} := \overline{\nabla^E u}$ is a connection on $\bar{E}$. Locally we have $\Gamma^{\bar{E}}_\alpha = \overline{\Gamma^E_\alpha}$.

(3) On $E \otimes F$,

$$\nabla^{E\otimes F}(u,v) := \nabla^E u \otimes v + u \otimes \nabla^F v$$

determines a connection, with $\Gamma^{E\otimes F}_\alpha = \Gamma^E_\alpha \otimes 1 + 1 \otimes \Gamma^F_\alpha$ locally.

(4) On $\mathrm{Hom}(E,F)$, $\nabla^{\mathrm{Hom}(E,F)} := \nabla^F f - f\nabla^E$ for all $f: E \to F$.

(5) Let $f: X \to M$ be a smooth map, there exists a unique ∇^{f^*E} such that

$$(\nabla^{f^*E}_U (s\circ f))(x) := (\nabla^E_{f_*U} s)(f(x)), \qquad \forall s \in \Gamma(E),\ \forall U \in T_x X.$$

Here $\nabla^{f^*E}_U(s\circ f) \in \Gamma(f^*E)$. Locally, $\Gamma^{f^*E}_\alpha = f^*\Gamma^E_\alpha$.

If $\iota: X \to M$ is a submanifold, there does not exist a natural restriction of differential operator. So we cannot define $\nabla^E|_X : C^\infty(X, E|_X) \to C^\infty(X, T^*M|_X \otimes E)$. But by (5) in the previous example,

$$\nabla^{\iota^*E} : C^\infty(X, E|_X) \to C^\infty(X, T^*X \otimes E|_X)$$

is well-defined.

Definition 2.5: Adjoint connection

Denote $\langle\cdot,\cdot\rangle$ the metric on E. The *adjoint connection* of ∇^E is $\nabla^{E,*}$ (*note the difference between* ∇^{E^*}*!*) defined by

$$\left\langle \nabla^{E,*}u, v\right\rangle = d(u,v) - \left\langle u, \nabla^E v\right\rangle.$$

Note that $\nabla^{E,*,*} = \nabla^{E,*}$. If $\nabla^E = \nabla^{E,*}$, we say that ∇^E is a *metric connection.*

Metric connections do exist. For any connection ∇^E, simply taking

$$\nabla := \frac{1}{2}(\nabla^E + \nabla^{E,*})$$

and we can obtain a metric connection.

The connection $\nabla^E : \Gamma(E) \to \Omega^1(M,E)$ can be extended to

$$\nabla^E : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$$

by setting

$$\nabla^E(\alpha\otimes s) := d\alpha\otimes s + (-1)^k\alpha\wedge\nabla^E s, \qquad \forall\alpha\in\Omega^k(M,E),\ \forall s\in\Gamma(E).$$

Proposition 2.6

Let $X, Y\in\Gamma(TM)$, $s\in\Gamma(E)$. Then

$$(\nabla^E)^2 s(X,Y) = (\nabla^E_X\nabla^E_Y - \nabla^E_Y\nabla^E_X - \nabla^E_{[X,Y]})s = ([\nabla^E_X,\nabla^E_Y] - \nabla^E_{[X,Y]})s.$$

Proof. Let $\{e_k\}$ be an ONB of E on $U\subset M$, $\nabla^E s|_U = \sum_k \alpha_k\otimes e_k$, where $\alpha_k\in\Omega^1(U)$. So

$$\nabla^E(\nabla^E s) = \sum_k \nabla^E(\alpha_k\otimes e_k) = \sum_k (d\alpha_k\otimes e_k - \alpha_k\otimes\nabla^E e_k).$$

Thus

$$\begin{aligned}(\nabla^E)^2 s(X,Y) &= \sum_k (X\alpha_k(Y) - Y\alpha_k(X) - \alpha_k[X,Y])e_k - \big(\alpha_k(X)\nabla^E_Y e_k - \alpha_k(Y)\nabla^E_X e_k\big)\\ &= \sum_k \big(\nabla^E_X(\alpha_k(Y)) - \nabla^E_Y(\alpha_k(X))\big) - \nabla^E_{[X,Y]}s\\ &= \nabla^E_X\nabla^E_Y s - \nabla^E_Y\nabla^E_X s - \nabla^E_{[X,Y]}s.\end{aligned}$$

Then we conclude the proof. □

Now ∇^E is a differential operator of order 1 because $\sigma(\nabla^E)(x,\xi) = \mathrm{i}\xi\wedge$. So

$$\sigma((\nabla^E)^2) = (\sigma(\nabla^E))^2(x,\xi) = -\xi\wedge\xi = 0.$$

Thus $(\nabla^E)^2$ is of order $\leqslant 1$.

Definition–Proposition 2.7: Curvature

$(\nabla^E)^2$ is of order 0 as it defines an element $R^E\in\Omega^2(R,\operatorname{End}E)$ by

$$(\nabla^E)^2 s(X,Y) =: R^E(X,Y)s.$$

Then R^E is called the *curvature* of ∇^E.

Proof. For any $\psi\in C^\infty(M)$, by the expression of $(\nabla^E)^2$,

$$(\nabla^E)(\psi s)(X,Y) = (\nabla^E)^2 s(\psi X, Y) = (\nabla^E)^2 s(X,\psi Y) = \psi(\nabla^E)^2 s(X,Y).$$

Thus the map

$$\Gamma(TM)\times\Gamma(TM)\times\Gamma(E)\to\Gamma(E), \qquad (X,Y,s)\mapsto(\nabla^E)^2 s(X,Y)$$

is $C^\infty(M)$-bulinear. And by tensoriality lemma, it defines $R^E\in\Gamma(T^*M\otimes T^*M\otimes E)$. But it is also antisymmetric in (X,Y), so $R^E\in\Omega^2(M,\operatorname{End}E)$. □

Example 2.8

(1) If $E=\mathbb{C}$, $\nabla^E = d$, then $R^E = 0$.

(2) If $\operatorname{rank}E = 1$, locally $\{e\}$ is the basis of E, and $\nabla^E e = \alpha\otimes e$ for $\alpha\in\Omega^1(M)$. Then

$$(\nabla^E)^2 e = d\alpha\otimes e - \alpha\wedge\nabla^E e = d\alpha\otimes e.$$

So $R^E = d\alpha$. ($\operatorname{End}E = \mathbb{K}$ *as* $\operatorname{rank}E = 1$.)

Remark. *There is something interesting: α is only locally defined but $R^E = d\alpha$ is globally defined.*

Moreover, if $\tilde{\nabla}^E$ is another connection, $\tilde{\nabla}^E = \nabla^E + \Gamma$ for $\Gamma \in \Omega^1(M, \mathrm{End}\, E) = \Omega^1(M)$, then

$$\tilde{\alpha} = \alpha + \Gamma, \qquad \tilde{R}^E = R^E + d\Gamma.$$

In particular, $[R^E] = [\tilde{R}^E] \in H^2(M)$. As a result, if $[R^E] \neq 0$, then E is not trivial.

Proposition 2.9: Bianchi identity

$[\nabla^E, R^E] = 0.$

Proof. Just simple calculation. □

Proposition 2.10

If ∇^E is a metric connection, then the curvature R^E is a 2-form on M that take values in anti self-adjoint endomorphisms of E.

Proof. For any $s_1,\ s_2$, let $\{e_k\}$ be a basis of E and $\nabla^E s_1 = \sum_k \alpha_k \otimes e_k$ for $\alpha_k \in \Omega^1(M)$, then

$$\begin{aligned}
d\left\langle \nabla^E s_1, s_2 \right\rangle &= \sum_k d(\alpha_k \otimes \langle e_k, s_2\rangle) \\
&= \sum_k d\alpha_k \otimes \langle e_k, s_2\rangle - \alpha_k \wedge d\,\langle s_1, s_2\rangle \\
&= d\alpha_k \langle e_k, s_2\rangle - \alpha_k \wedge (\left\langle \nabla^E e_k, s_2\right\rangle + \left\langle e_k, \nabla^E s_2\right\rangle) \\
&= \sum_k \left\langle \nabla^E(\alpha_k \otimes e_k), s_2\right\rangle - \left\langle \nabla^E s_1, \nabla^E s_2\right\rangle \\
&= \left\langle R^E s_1, s_2\right\rangle - \left\langle \nabla^E s_1, \nabla^E s_2\right\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= d\circ d\,\langle s_1, s_2\rangle \\
&= d\big(\left\langle \nabla^E s_1, s_2\right\rangle + \left\langle s_1, \nabla^E s_2\right\rangle\big) \\
&= \left\langle R^E s_1, s_2\right\rangle - \left\langle \nabla^E s_1, \nabla^E s_2\right\rangle + \left\langle \nabla^E s_1, \nabla^E s_2\right\rangle + \left\langle s_1, R^E s_2\right\rangle.
\end{aligned}$$

Hence $\left\langle R^E s_1, s_2\right\rangle = -\left\langle s_1, R^E s_2\right\rangle$. □

2.2 Chern–Weil Theory

Assume that M is a manifold, $E \to M$ is a vector bundle with ∇^E being a connection on it.

Definition 2.11: $\mathbb{Z}/2\mathbb{Z}$-grading, superthing

Say V is a *superspace* if V is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V^+ \oplus V^-$. We say V^+ is the *even part* of V and V^- is the *odd part*.

Similarly, say $\mathscr{A}$ is a *superalgebra* aif $\mathscr{A}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded algebra(*with identity*) and a superspace $\mathscr{A} = \mathscr{A}^+ \oplus \mathscr{A}^-$ such that

$$\mathscr{A}^+\cdot\mathscr{A}^+ \subset \mathscr{A}^+, \quad \mathscr{A}^+\cdot\mathscr{A}^- \subset \mathscr{A}^-, \quad \mathscr{A}^-\cdot\mathscr{A}^+ \subset \mathscr{A}^-, \quad \mathscr{A}^-\cdot\mathscr{A}^- \subset \mathscr{A}^+.$$

If V or $\mathscr{A}$ is a super thing, define the *degree* of u, denoted as $\deg u \in \mathbb{Z}/2\mathbb{Z}$, with

$$\begin{cases} \deg u = 0, & \text{if } u \in V^+ \text{ or } \mathscr{A}^+, \\ \deg u = 1, & \text{if } u \in V^- \text{ or } \mathscr{A}^-. \end{cases}$$

Remark. *Basically the terminology « super » means that there is a $\mathbb{Z}/2\mathbb{Z}$-grading on the object. If V or $\mathscr{A}$ is a normal thing, we see it as a super thing by considering $V = V \oplus 0$ or $\mathscr{A} = \mathscr{A} \oplus 0$.*

From now on, the vector space V would be finite dimensional.

Definition 2.12: Super-commutator

For a superalgebra $\mathscr{A}$, define the *super-commutator* on $\mathscr{A}$ by

$$[\cdot,\cdot] : \mathscr{A} \times \mathscr{A} \to \mathscr{A}, \qquad (a,b) \mapsto (-1)^{\deg a \cdot \deg b} ba.$$

Say $\mathscr{A}$ is *super-commutative* if $[\cdot,\cdot] = 0$.

A simple calculation verifies that on a superalgebra, we have

$$[a,[b,c]] = [[a,b],c] + (-1)^{\deg a \cdot \deg b}[b,[a,c]].$$

Example 2.13

(1) $\Omega^\bullet(M) = \Omega^{2\bullet}(M) \oplus \Omega^{2\bullet+1}(M)$ is a superalgebra. It is also super-commutative.

(2) If V is a superspace, then $\operatorname{End} V$ is a superalgebra with

$$(\operatorname{End} V)^+ = \left\{f \in EndV : f(V^\pm) \subset V^\pm\right\}, \qquad (\operatorname{End} V)^- = \left\{f \in EndV : f(V^\pm) \subset V^\mp\right\}.$$

(3) If $F \to M$ is super-vector bundle, then $\Omega^\bullet(M,F)$ is a superspace with

$$\Omega^\bullet(M,F)^\pm = \Omega^+(M,F^\pm) \oplus \Omega^-(M,F^\mp).$$

Definition 2.14: Supertrace

Let $\alpha : \mathscr{A} \to \mathbb{K}$ be a trace on $\mathscr{A}$. Say α is a *supertrace* if $\forall a,b \in \mathscr{A}$, $\alpha[a,b] = 0$.

Proposition 2.15

Let V be a superspace and $f = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \operatorname{End} V$. Define

$$\operatorname{Tr}_s f := \operatorname{Tr} A - \operatorname{Tr} D.$$

Then Tr_s is a supertrace on V.

Proof. This is because

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & \\ & D \end{bmatrix} + \begin{bmatrix} & B \\ C & \end{bmatrix} \in (\operatorname{End} V)^+ + (\operatorname{End} V)^-.$$

Then it is done by definition. □

Remark. *If $V = V \oplus 0$ is a normal vector space, then $\operatorname{Tr}_s = \operatorname{Tr}$ on V.*

Definition 2.16: Super-tensor product

Let $\mathscr{A}$, $\mathscr{B}$ be superalgebras, define their *super-tensor product* by

- $\mathscr{A} \hat{\otimes} \mathscr{B} = \mathscr{A} \otimes \mathscr{B}$ as vector spaces.

- The even part and odd part are defined as

$$(\mathscr{A}\hat{\otimes}\mathscr{B})^+ := (\mathscr{A}^+\otimes\mathscr{B}^+)\oplus(\mathscr{A}^-\otimes\mathscr{B}^-), \qquad (\mathscr{A}\hat{\otimes}\mathscr{B})^- := (\mathscr{A}^+\otimes\mathscr{B}^-)\oplus(\mathscr{A}^-\otimes\mathscr{B}^+).$$

- If $a, a' \in \mathscr{A}^\pm$, $b, b' \in \mathscr{B}^\pm$, define

$$(a\otimes b)(a'\otimes b') = (-1)^{\deg a'\cdot\deg b} aa'\otimes bb'.$$

Question. Why we introduce super-tensor product?

This is because on superalgbras, the commutator becomes super-commutator. By defining super-tensor product, $\mathscr{A}\otimes 1$ and $1\otimes\mathscr{B}$ are sub-superalgebras of $\mathscr{A}\hat{\otimes}\mathscr{B}$ which are super-commute.

$$(a\otimes 1)(1\otimes b) = a\otimes b, \qquad (1\otimes b)(a\otimes 1) = (-1)^{\deg a\cdot\deg b} a\otimes b.$$

Let $\mathscr{A}$ be a super-commutative superalgebra, V be a superspace. Then $\mathrm{Tr}_s : \mathrm{End}\, V \to \mathbb{K}$ can be extended to

$$\mathrm{Tr}_s : \mathscr{A}\hat{\otimes}\mathrm{End}\, V \to \mathscr{A}, \qquad a\otimes M \mapsto a\cdot\mathrm{Tr}_s M.$$

The fact that this defines a supertrace follows from the following lemma.

Lemma 2.17

Let $\mathscr{A}$ be a super-commutative algebra and $\mathscr{B}$ be a superalgebra. For all $a, b \in \mathscr{A}$ and $M, N \in \mathscr{B}$,

$$[a\otimes M, b\otimes N] = (-1)^{\deg M \deg b} ab\otimes[M,N].$$

Proof. First, if $1 = 1^+ + 1^-$, by $1\otimes a^+ = a^+$, we obtain that $1^- = 0$. Hence $\deg 1 = 0$. Denote $A = a\otimes M$, $B = b\otimes N$, then $\deg A = \deg a + \deg M$, $\deg B = \deg b + \deg N$. Thus

$$\begin{aligned} AB &= (-1)^{\deg M \deg b} ab\otimes MN \\ BA &= (-1)^{\deg N \deg a} ba\otimes NM = (-1)^{\deg N \deg a + \deg a \deg b} ab\otimes NM. \end{aligned}$$

Hence

$$\begin{aligned} [A,B] &= AB - (-1)^{\deg A \deg B} BA \\ &= AB - (-1)^{(\deg a + \deg M)(\deg b + \deg N)} BA \\ &= (-1)^{\deg M \deg b} ab\otimes MN - (-1)^{\deg M \deg b} ab\otimes(-1)^{\deg M \deg N} NM \\ &= (-1)^{\deg M \deg b} ab\otimes[M,N]. \end{aligned}$$

Then we conclude the proof. □

We apply this formalism:

- On each fibre, $\Lambda^\bullet T_x^* M$ is a super-commutative superalgebra, and $E_x = E_x\oplus 0$ is a superspace. Then $\mathrm{Tr}_s = \mathrm{Tr}$ on E_x. One can define

$$\mathrm{Tr} : \Lambda^\bullet T_x^* M\otimes\mathrm{End}\, E_x \to \Lambda^\bullet T_x^* M, \qquad \alpha\otimes f \mapsto \alpha\mathrm{Tr}\, f.$$

  Here we use $\otimes$ instead of $\hat{\otimes}$ because $\mathrm{End}\, E_x$ is purely even.

- Globally speaking, $\Omega^\bullet(M,E)$ is a superspace, $\Omega^\bullet(M)$ is a super-commutative superalgebra. We have the canonical trace

$$\mathrm{Tr} : \Omega^\bullet(M, \mathrm{End}\, E) \to \Omega^\bullet(M).$$

Note that $\Omega^\bullet(M, \operatorname{End} E) \cong \Omega^\bullet(M) \hat{\otimes} \operatorname{End} E$ is a superalgebra, then

$$[\alpha \otimes A, \beta \otimes B] = \alpha \wedge \beta \otimes [A, B].$$

The connection $\nabla^E : \Omega^\bullet(M, E)^\pm \to \Omega^\bullet(M, E)^\mp$ is an odd operator, and

$$R^E = (\nabla^E)^2 = \frac{1}{2}[\nabla^E, \nabla^E].$$

Remark. *Note that* Tr *is a trace on* $\Omega^\bullet(M, \operatorname{End} E)$ *but not on* $\operatorname{End}(\Omega^\bullet(M, E))$.

Proposition 2.18

For all $X \in \Omega^\bullet(M, \operatorname{End} E)$, $d\operatorname{Tr}(X) = \operatorname{Tr}([\nabla^E, X])$.

Proof. (1) If $\tilde{\nabla}^E$ is another connection, $\tilde{\nabla}^E = \nabla^E + \Gamma$ for some $\Gamma \in \Omega^1(M, \operatorname{End} E)$. Thus

$$\operatorname{Tr}([\tilde{\nabla}^E - \nabla^E, X]) = \operatorname{Tr}([\Gamma, X]) = 0.$$

So $\operatorname{Tr}([\nabla^E, X])$ is independent on the choice of ∇^E.

(2) The formula is local, so it is enough to prove it on small open sets. For a small open set U, we can trivialise E and denote $\nabla^E = d$. For $\alpha \in \mathbb{C}^\infty(\mathbb{K}^n, \Lambda^k T^*M)$, $A \in C^\infty(\mathbb{K}^n, \operatorname{Mat}_n(\mathbb{K}))$,

$$d\operatorname{Tr} A = \sum_i dA_{ii} = \operatorname{Tr}(dA).$$

And

$$\begin{aligned} d\operatorname{Tr}(\alpha \otimes A) = d(\alpha \operatorname{Tr} A) &= d\alpha \operatorname{Tr} A + (-1)^{\deg \alpha} \alpha \wedge d\operatorname{Tr} A \\ &= \operatorname{Tr}((d\alpha) \otimes A + (-1)^{\deg \alpha} \alpha \wedge dA). \end{aligned}$$

Moreover, since $d(Au) = dA \cdot u + A \cdot du = dA \cdot u$, hence

$$\begin{aligned} d[\alpha \otimes A] &= d(\alpha \otimes A) - (-1)^{\deg \alpha}(\alpha \otimes A)d \\ &= d\alpha \otimes A + (-1)^{\deg \alpha} \alpha \wedge (dA - Ad) = d\alpha \otimes A + (-1)^{\deg \alpha} \alpha \wedge dA. \end{aligned}$$

Therefore, $d\operatorname{Tr}(\alpha \otimes A) = \operatorname{Tr}([d, \alpha \otimes A])$. ☐

Definition 2.19: Characteristic form

For $f \in \mathbb{R}[[z]]$, $f = \sum_{k \geqslant 0} a_k z^k$, set

$$f(E, \nabla^E) := \operatorname{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi} R^E\right)\right) \in \Omega^\bullet(M, \mathbb{C}),$$

called the *characteristic form* of E associated to ∇^E and f.

Now $f(E, \nabla^E)$ is well-defined since $(R^E)^{(\dim M + 1)/2} = 0$, the series is actually a finite sum. And $f(E, \nabla^E) = \sum_{k \geqslant 0} a_k \operatorname{Tr}\left(\left(\frac{\mathrm{i}}{2\pi} R^E\right)^k\right)$.

If $\alpha = \alpha^{(0)} + \alpha^{(\geqslant 1)} \in \Omega^\bullet(M, \mathbb{C})$, where $\alpha^{(0)} \in \Omega^0(M, \mathbb{C})$ and $\alpha^{(\geqslant 1)} \in \Omega^{\geqslant 1}(M, \mathbb{C})$. For $x \in M$, assume $\alpha_x^{(0)} \neq 0$, then we can define

$$\alpha_x^{-1} = (\alpha_x^{(0)})^{-1} \left(1 + \sum_{k \geqslant 1} \left(-\frac{\alpha_x^{(\geqslant 1)}}{\alpha_x^{(0)}}\right)^k\right) \in \Omega^\bullet(M, \mathbb{C}).$$

In particular, if $f(0) \neq 0$, $f(E, \nabla^E)$ is invertible on $\Omega^\bullet(M, \mathbb{C})$.

Definition–Theorem 2.20: Characteristic class, Chern-Weil Theorem

(1) The characteristic form $f(E,\nabla^E)$ is closed.

(2) Its cohomology class $[f(E,\nabla^E)] \in H^\bullet(M,\mathbb{C})$ is independent on ∇^E.

We denote $f(E) := [f(E,\nabla^E)]$, called the *characteristic class* of E associated to f.

Proof. (1) By Bianchi's identity,

$$d f(E,\nabla^E) = d\mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R^E\right)\right) = \mathrm{Tr}\left(\left[\nabla^E, f\left(\frac{\mathrm{i}}{2\pi}R^E\right)\right]\right) = 0.$$

(2) Let ∇_0^E, ∇_1^E be two connections. Define a family of connections linking them by $\nabla_t^E := (1-t)\nabla_0^E + t\nabla_1^E$. Consider the canonical projection $\pi : M\times\mathbb{R} \to M$, $(x,t)\mapsto x$, which induces a map between $\pi^*E \to M\times\mathbb{R}$ to $E\to M$. Note that $C^\infty(M\times\mathbb{R},\pi^*E) = C^\infty(M,E)\otimes C^\infty(\mathbb{R})$. Let

$$\nabla^{\pi^*E}s(x,t) = \nabla_t^E + \underbrace{dt\cdot \mathrm{i}\frac{\partial}{\partial t}}_{\text{trivial connection for the « } t \text{ part »}}.$$

This is a connection on π^*E, and the curvature

$$R^{\pi^*E} = R_t^E + dt\wedge \underbrace{\alpha_t}_{\text{decompose according to the degree in } dt}.$$

Thus

$$\mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R^{\pi^*E}\right)\right) = \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\right) + dt\wedge Q_t.$$

Moreover, $d^{M\times\mathbb{R}} = d^M + dt\wedge \mathrm{i}\frac{\partial}{\partial t}$, and

$$0 = d^{M\times\mathbb{R}}\mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R^{\pi^*E}\right)\right) = d^M f(E,\nabla^E) + dt\wedge\frac{\partial f(E,\nabla_t^E)}{\partial t} - dt\wedge d^M Q_t.$$

Thus $\frac{\partial}{\partial t}(f(E,\nabla_t^E)) = d^M Q_t$, which implies

$$f(E,\nabla_1^E) - f(E,\nabla_0^E) = \int_0^1 d^M Q_t dt = d^M\left(\int_0^1 Q_t dt\right).$$

Hence $[f(E,\nabla_1^E)] = [f(E,\nabla_0^E)]$ on $H^\bullet(M,\mathbb{C})$. $\square$

Proposition 2.21

Assume E is a complex vector bundle. If ∇^E is metric with respect to some Hermitian metric, then $f(E,\nabla^E)\in\Omega^\bullet(M,\mathbb{R})$. In particular, $f(E)\in H^\bullet(M,\mathbb{R})$.

Proof. Recall that $R^E\in\Omega^2(M,\mathrm{End}\,E)$. Let

$$* : \Omega^\bullet(M,\mathrm{End}\,E)\to\Omega^\bullet(M,\mathrm{End}\,E),\quad \alpha_1\wedge\cdots\wedge\alpha_k\otimes f \mapsto (-\alpha_k)\wedge\cdots\wedge(-\alpha_1)\otimes f^*,$$

where f^* denotes the adjoint w.r.t. h^E. Then $*^2 = 1$, $(AB)^* = B^*A^*$ for $A,B\in\Omega^\bullet(M,\mathrm{End}\,E)$, and

$$\mathrm{Tr}\,A^* = (-1)^k(-1)^{(k(k-1))/2}\overline{\mathrm{Tr}\,A},\qquad A\in\Omega^k(M,\mathrm{End}\,E).$$

Thus $(R^E)^* = R^E$. So

$$\begin{aligned}\mathrm{Tr}\left(\left(\frac{\mathrm{i}}{2\pi}R^E\right)^k\right) &= \left(\frac{\mathrm{i}}{2\pi}\right)^k\mathrm{Tr}\,((R^E)^k) = \left(\frac{\mathrm{i}}{2\pi}\right)^k\mathrm{Tr}\,(((R^E)^k)^*)\\ &= \left(\frac{\mathrm{i}}{2\pi}\right)^k(-1)^{2k+k(2k-1)}\overline{\mathrm{Tr}\,((R^E)^k)} = \overline{\mathrm{Tr}\left(\left(\frac{\mathrm{i}}{2\pi}R^E\right)^k\right)}.\end{aligned}$$

Therefore $\mathrm{Tr}\left(\left(\frac{\mathrm{i}}{2\pi}R^E\right)^k\right)\in\mathbb{R}$. $\square$

Remark. *If E is a real bundle, in pratice, we take $f \in \mathbb{R}[[z]]$ to be even so that*

$$f\left(\frac{\mathrm{i}}{2\pi}R^E\right) = \sum_{k\geqslant 0} a_{2k}(-1)^k\left(\frac{\mathrm{i}}{2\pi}\right)^k (R^E)^k \in \Omega^{4\bullet}(M,\mathbb{R}).$$

Example 2.22

Let $E \to M$ be a complex vector bundle.

(1) Take $f(z) = \mathrm{e}^z$,

$$\mathrm{ch}(E,\nabla^E) := \mathrm{Tr}\left(\exp\left(\frac{\mathrm{i}}{2\pi}R^E\right)\right),$$

and $\mathrm{ch}(E)$ is called the *Chern character* of E.

(2) Take $f(z) = \log\frac{z}{1-\exp(z)}$, note that $\exp(\mathrm{Tr}(\log A)) = \det A$,

$$\mathrm{Td}(E,\nabla^E) = \exp(f(E,\nabla^E)) = \det\left(\frac{\frac{\mathrm{i}}{2\pi}R^E}{1-\exp\left(\frac{\mathrm{i}}{2\pi}R^E\right)}\right)$$

is called the *Todd form* and $\mathrm{Td}(E)$ is called the *Todd class* of E.

(3) Take $f(z) = \log(1+z)$,

$$c(E,\nabla^E) = \exp(f(E,\nabla^E)) = \det\left(1+\frac{\mathrm{i}}{2\pi}R^E\right) = 1+\sum_{j\geqslant 0} c_j(E,\nabla^E),$$

where $c_j(E,\nabla^E) \in \Omega^{2j}(M,\mathbb{C})$. Then $c(E,\nabla^E)$ is called the *total Chern form* of E and $c(E)$ the *total Chern class* of E. Similarly, $c_j(E,\nabla^E)$ is called the *j-th Chern form* of E and $c_j(E)$ the *j-th Chern class* of E.

Remark. *Let $j = 1$ in (3), we have $c_1(E,\nabla^E) = \mathrm{Tr}\left(\frac{\mathrm{i}}{2\pi}R^E\right) \in \Omega^2(M,\mathbb{C})$. In particular, if E is a line bundle, then $c_1(E,\nabla^E) = \frac{\mathrm{i}}{2\pi}R^E$.*

Example 2.23

Let $E \to M$ be a real vector bundle, ∇^E be a metric connection w.r.t. g^E. Then ∇^E can be extended by $\mathbb{C}$-linearity to $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$. And we know $\nabla^{E_{\mathbb{C}}}$ such that $R^{E_{\mathbb{C}}} = R^E$. In this case, $(R^E)^{\mathsf{T}}(X,Y) = -R^E(X,Y)$.

(1) The total Chern form becomes

$$\begin{aligned} 1+\sum_{j\geqslant 1} c_j(E_{\mathbb{C}},\nabla^{E_{\mathbb{C}}}) &= \det\left(\left(1+\frac{\mathrm{i}}{2\pi}R^E\right)^{\mathsf{T}}\right) \\ &= \det\left(1-\frac{\mathrm{i}}{2\pi}R^E\right) = 1+\sum_{j\geqslant 1}(-1)^j c_j(E_{\mathbb{C}},\nabla^{E_{\mathbb{C}}}). \end{aligned}$$

Thus $c_{2j+1}(E_{\mathbb{C}},\nabla^{E_{\mathbb{C}}}) = 0$. Define

$$p_j(E) := (-1)^j c_{2j}(E_{\mathbb{C}}) \in H^{4j}(M,\mathbb{R})$$

the *j-th Pontryagin class* of E.

(2) Take $f(z) = \frac{1}{2}\log\left(\frac{z/2}{\sinh(z/2)}\right)$. This is an even function,

$$\hat{A}(E,\nabla^E) = \exp(f(E,\nabla^E)) = \left(\det\left(\frac{\frac{\mathrm{i}}{4\pi}R^E}{\sinh\left(\frac{\mathrm{i}}{4\pi}R^E\right)}\right)\right)^{1/2}.$$

Then $\hat{A}(E)$ is called the *$\hat{A}$-genus* of E.

(3) Take $f(z)=\frac{1}{2}\log(\frac{z/2}{\tanh(z/2)})$.

$$\hat{L}(E,\nabla^E)=\exp(f(E,\nabla^E))=\left(\det\left(\frac{\frac{\mathrm{i}}{4\pi}R^E}{\tanh\left(\frac{\mathrm{i}}{4\pi}R^E\right)}\right)\right)^{1/2}.$$

Then $\hat{L}(E)$ is called the *$\hat{L}$-genus* of E.

Remark. For $p(E,\nabla^E)=1+\sum_{j\geqslant 1}p_j(E,\nabla^E)$, then

$$p(E,\nabla^E)=\det\left(\left(1-\left(\frac{R^E}{2\pi}\right)^2\right)^{1/2}\right).$$

2.3 Euler class

Let $(V,\langle\cdot,\cdot\rangle,\mathrm{vol}_V)$ be an Euclidean oriented vector space with dimension n. Then $\Lambda^n V^*\cong\mathbb{R}$ because we fixed an orientation $\mathrm{vol}_V\in\Lambda^n V^*\setminus\{0\}$. We can assume that vol_V is of unit norm. Fix an oriented ONB $\{e_i\}$ of V, then

$$\mathrm{vol}_V=e^1\wedge\cdots\wedge e^n,$$

where $\{e^i\}$ is the dual basis of V^*. Recall the notation that for $\alpha\in\Lambda^\bullet V^*$, $\alpha^{[k]}$ denotes its component in $\Lambda^k V^*$.

Definition 2.24: Pfaffian

Let $A\in\operatorname{End}V$ be anti-symmetric. Denote

$$\omega_A:=\langle\cdot,A\cdot\rangle=\frac{1}{2}\sum_{i,j}\langle e_i,Ae_j\rangle e_i\wedge e_j=\sum_{i<j}\langle e_i,Ae_j\rangle e^i\wedge e^j\in\Lambda^2V^*.$$

Then define the *Pfaffian* of A by

$$\exp(\omega_A)^{[n]}=\mathrm{Pf}(A)\cdot\mathrm{vol}_V.$$

By definition, $\mathrm{Pf}(A)$ is a polynomial in the entries of A.

Example 2.25

If n is odd, then $\mathrm{Pf}(A)=0$.
If n is even, take $n=2$ as an example, in this case $A=\begin{bmatrix} & \theta\\ -\theta & \end{bmatrix}$. So

$$\omega_A=\frac{1}{2}(\theta e^1\wedge e^2-\theta e^2\wedge e^1)=\theta e^1\wedge e^2.$$

Thus

$$\exp(\omega_A)=1+\omega_A=1+\theta e^1\wedge e^2,$$

then $\mathrm{Pf}(A)=\theta$ by definition.

Proposition 2.26

$\mathrm{Pf}(A)^2=\det A$.

Proof. True if n is odd. Now assume $n = 2k$ is even. There exists an ONB $\{e_i\}$ such that

$$A = \operatorname{diag}\left\{\begin{bmatrix} & \theta_1 \\ -\theta_1 & \end{bmatrix}, \dots, \begin{bmatrix} & \theta_n \\ -\theta_n & \end{bmatrix}\right\} =: \operatorname{diag}\{A_1, \dots, A_k\}.$$

Thus $\omega_A = \sum_{i=1}^k \omega_{A_i}$ with ω_{A_i} mutually commutes. Thus

$$\operatorname{Pf}(A)\mathrm{vol}_V = \exp(\omega_A)^{[n]} = \left(\prod_{i=1}^k \exp(\omega_{A_i})\right)^{[n]} = \prod_{i=1}^n \operatorname{Pf}(A_i)\cdot \mathrm{vol}_V = \prod_{i=1}^k \theta_i \cdot \mathrm{vol}_V.$$

And $\det A = \prod_{i=1}^k \theta_i^2$. □

Definition 2.27: Euler form

Let $E \to M$ be a real oriented vector bundle, g^E the Euclidean metric and ∇^E the metric connection. Define the *Euler form*

$$e(E, g^E, \nabla^E) := \operatorname{Pf}\left(\frac{R^E}{2\pi}\right).$$

Here Pf is w.r.t. g^E and ∇^E.

By our previous discussions, if $\operatorname{rank} E$ is odd, then $e(E, g^E, \nabla^E) = 0$.

Definition–Proposition 2.28: Euler class

(1) The Euler form is closed, *i.e.*, $de(E, g^E, \nabla^E) = 0$.

(2) $e(E) := [e(E, g^E, \nabla^E)] \in H^\bullet(M)$ is independent on g^E and ∇^E, called the *Euler class* of E.

Proof. (1) Let R^E be the curvature.

$$\omega_{R^E} = \frac{1}{2}\left\langle e_i, R^E e_j\right\rangle e^i \wedge e^j \in \Omega^2(M, \Lambda^2 E^*).$$

Now $[\nabla^E, g^E] = 0$ and $[\nabla^E, R^E] = 0$, thus for $e, f \in E$,

$$\begin{aligned}\nabla^{\Lambda^\bullet E^*}\omega_{R^E}(e,f) &= d\omega_{R^E}(e,f) - \omega_{R^E}(\nabla^E e, f) - \omega_{R^E}(e, \nabla^E f) \\ &= d\left\langle e, R^E f\right\rangle - \left\langle \nabla^E, R^E f\right\rangle - \left\langle e, R^E \nabla^E f\right\rangle = 0.\end{aligned}$$

Hence $\nabla^{\Lambda^\bullet E^*}\omega_{R^E} = 0$, and therefore $\nabla^{\Lambda^\bullet E^*}\exp(\omega_{R^E}) = 0$. This gives

$$\nabla^{\Lambda^\bullet E^*}(\operatorname{Pf}(R^E)\mathrm{vol}_E) = d\operatorname{Pf}(R^E)\mathrm{vol}_E + \operatorname{Pf}(R^E)\lambda^{\Lambda^\bullet E^*}\mathrm{vol}_E = 0.$$

But locally, we have

$$\begin{aligned}\nabla^{\Lambda^\bullet E^*}\mathrm{vol}_E &= \nabla^{\Lambda^\bullet E^*}(e^1 \wedge \cdots \wedge e^n) \\ &= \sum_{i=1}^n e^1 \wedge \cdots \wedge \nabla^{E^*} e^i \wedge \cdots \wedge e^n \\ &= \sum_{i=1}^n e^1 \wedge \cdots \wedge \sum_{j=1}^n \left\langle \nabla^{E^*} e^i, e_j\right\rangle e^j \wedge \cdots \wedge e^n \\ &= \sum_{i=1} \left\langle \nabla^{E^*} e^i, e_i\right\rangle e^1 \wedge \cdots \wedge e^n \\ &= \sum_{i=1}^n \left\langle \nabla^{E^*} e^i, e_i\right\rangle \mathrm{vol}_V.\end{aligned}$$

It remains to compute

$$\left\langle \nabla^{E^*} e^i, e_i\right\rangle = d\left\langle e^i, e_i\right\rangle - \left\langle e^i, \nabla^E e_i\right\rangle = -\left\langle e_i, \nabla^E e_i\right\rangle = 0.$$

This is because $d\langle e_i, e_i\rangle = 2\left\langle e_i, \nabla^E e_i\right\rangle = 0$. Hence (1) is proved.

(2) If there are (∇_0^E, g_0^E) and (∇_1^E, g_1^E) on E, then there exists a family (∇_t^E, g_t^E) linking them with ∇_t^E being the metric connection w.r.t. g_t^E. We define

$$g_t^E := t g_1^E + (1-t) g_0^E, \qquad \tilde{\nabla}_t^E := t\nabla_1^E + (1-t)\nabla_0^E, \qquad \nabla_t^E := \frac{1}{2}(\tilde{\nabla}_t^E + (\tilde{\nabla}_t^E)^*),$$

where the dual in ∇_t^E is w.r.t. g_t^E. For $\pi : M \times [0,1] \to M$, the pullback is

$$g^{\pi^* E}(x,t) = g_t^E(x), \qquad \nabla^{\pi^* E} := \nabla_t^E + \frac{1}{2}\left(dt\frac{\partial}{\partial t} + \left(dt\frac{\partial}{\partial t}\right)^*\right) = \nabla_t^E + dt\left(\frac{\partial}{\partial t} + A_t\right).$$

So $g^{\pi^* E}$ is a metric and $\nabla^{\pi^* E}$ is a metric connection w.r.t. $g^{\pi^* E}$. Denote $R^{\pi^* E} = R_t^E + dt \wedge B_t$. Hence

$$\begin{aligned} e(\pi^* E, g^{\pi^* E}, \nabla^{\pi^* E}) &= \mathrm{Pf}\left(\frac{R_t^E + dt\wedge Q_t}{2\pi}\right) \\ &= \mathrm{Pf}\left(\frac{R_t^E}{2\pi}\right) + dt\wedge\beta_t = e(E, g_t^E, \nabla_t^E) + dt \wedge \beta_t. \end{aligned}$$

While by (1), $d^{M\times[0,1]} e(\pi^* E, g^{\pi^* E}, \nabla^{\pi^* E}) = 0$, which is,

$$(d^M + dt\wedge \frac{\partial}{\partial t}) e(\pi^* E, g^{\pi^* E}, \nabla^{\pi^* E}) = dt \wedge \frac{\partial e(E, g_t^E, \nabla_t^E)}{\partial t} - dt\wedge d^M\beta_t + (\text{terms without } dt) = 0.$$

So $\frac{\partial}{\partial t} e(E, g_t^E, \nabla_t^E) = d^M \beta_t$ and hence

$$e(E, g_1^E, \nabla_1^E) - e(E, g_0^E, \nabla_0^E) = d\int_0^1 \beta_t dt.$$

Then we conclude the proof. □

2.* Additional Exercises

Exercise 1: *K*-Theory

Let M be a compact manifold. Let $\mathrm{Vect}(M)$ be the set on complex vector bundles on M. We define the K^0-group of M as follows:

$$K^0(M) = \mathrm{Vect}(M) \times \mathrm{Vect}(M)/\sim,$$

where

$$(V, V') \sim (W, W') \Longleftrightarrow \exists F \in \mathrm{Vect}(M)\,(V \oplus W' \oplus F \simeq V' \oplus W \oplus F).$$

(1) Prove that $\oplus$ and $\otimes$ on $\mathrm{Vect}(M)$ define operations on $K^0(M)$, denoted by $+$ and $\times$.

(2) We denote $[(V,0)]$ by $[V]$ for $V \in \mathrm{Vect}(M)$. Prove that $[(V,V')] + [V'] = [V]$.

(3) Deduce that $(K^0(M), +)$ is an Abelian group and that $[(V,V')] = [V] - [V']$. Prove that $(K^0(M), +, \times)$ is a ring.

(4) Prove that

$$K^0(\mathrm{pt}) \to \mathbb{Z}, \qquad [V] - [V'] \mapsto \dim V - \dim V'$$

is an isomorphism.

(5) Prove that the Chern character $\mathrm{ch} : \mathrm{Vect}(M) \to H^{2\bullet}(M, \mathbb{R})$ induces a ring morphism $\mathrm{ch} : K^0(M) \to H^{2\bullet}(M, \mathbb{R})$.

(1) If $(V_1, V_1') \sim (V_2, V_2')$, then

$$V_1 \oplus V_2' \oplus F \simeq V_1' \oplus V_2 \oplus F \implies (V_1 \oplus W) \oplus (V_2' \oplus W') \oplus F \simeq (V_1' \oplus W') \oplus (V_2 \oplus W) \oplus F,$$

hence $(V_1, V_1') \oplus (W, W') \sim (V_2, V_2') \oplus (W, W')$. Thus

$$+ : K^0(M) \times \mathrm{Vect}(M) \to K^0(M), \qquad ((V, V'), (W, W')) \mapsto (V \oplus W, V' \oplus W')$$

is well-defined. And similarly for $(W_1, W_1') \sim (W_2, W_2')$.

For tensor products, $V_1 \oplus V_2' \oplus F \simeq V_1' \oplus V_2 \oplus F$ implies that

$$(V_1 \oplus V_2' \oplus F) \otimes W \simeq (V_1' \oplus V_2 \oplus F) \otimes W, \qquad (V_1 \oplus V_2' \oplus F) \otimes W' \simeq (V_1' \oplus V_2 \oplus F) \otimes W'.$$

Thus

$$(V_1 \otimes W) \oplus (V_1' \otimes W') \oplus (V_2' \otimes W) \oplus (V_2 \otimes W') \oplus F' \simeq (V_1' \otimes W) \oplus (V_1 \otimes W') \oplus (V_2 \otimes W) \oplus (V_2' \otimes W') \oplus F'$$

with $F' = F \otimes (W \oplus W')$. Thus

$$\times : K^0(M) \times K^0(M), \qquad ((V, V'), (W, W')) \mapsto ((V \otimes W) \oplus (V' \otimes W'), (V \otimes W') \oplus (V' \otimes W))$$

is well-defined.

(2) By (1) we have $[(V, V')] + [V'] = [(V \oplus V', V')]$. And it is obvious that

$$V \oplus V' \oplus 0 \simeq V' \oplus V \oplus 0,$$

hence $[(V, V')] + [V'] = [V]$. From now on, we denote $[(V, V')] := [V] - [V']$.

(3) It is easy to verify that $[(V, V)]$ for any $V \in \mathrm{Vect}(M)$ is the additive identity (*and we denote it as* 0) and the reverse of $[(V, V')]$ is $[(V', V)]$. Hence $(K^0(M), +)$ is a group. It is Abelian because the direct sum of vector bundles is commutative.

Similarly, it is easy to verify that 0 is exactly the zero element of $\times$ and $[\mathbb{C}]$ is the multiplicative identity. The distributivity over + is a verification by simple calculation. And $(K^0(M), \times)$ is commutative because the tensor product of vector bundles is commutative.

(4) A vector bundle on a point is simply a finite dimensional vector space. Hence there is a morphism

$$K^0(\mathrm{pt}) \to \mathbb{Z}, \qquad [V] - [W] \mapsto \dim V - \dim W.$$

It is injective because $\dim V_1 - \dim W_1 = \dim V_2 - \dim W_2$ implies that $(V_1, W_1) \sim (V_2, W_2)$. It is surjective by taking $V = \mathbb{C}^n$ and $W = 0$ for $n \geqslant 0$, and $V = 0$ and $W = \mathbb{C}^n$ vice versa.

(5) Recall that $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$, $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \wedge \mathrm{ch}(W)$. For $[V] - [W] \in K^0(M)$, the Chern character induces

$$\mathrm{ch} : K^0(M) \to H^{2\bullet}(M, \mathbb{R}), \qquad [V] - [W] \mapsto \mathrm{ch}(V) - \mathrm{ch}(W).$$

It is well-defined because $[V_1] - [W_1] = [V_2] - [W_2]$ gives $V_1 \oplus W_2 \oplus F \simeq V_2 \oplus W_1 \oplus F$, hence $\mathrm{ch}(V_1) + \mathrm{ch}(W_2) + \mathrm{ch}(F) = \mathrm{ch}(V_2) + \mathrm{ch}(W_1) + \mathrm{ch}(F)$.

Then it is a simple (*but complicated*) calculation that ch is a ring homomorphism between $(K^0(M), +, \times)$ and $(H^{2\bullet}(M, \mathbb{R}), +, \wedge)$. $\square$

Exercise 2: Transgression formula and Chern-Simons form

Let $E \to M$ be a vector bundle on a compact manifold. Let ∇_0^E and ∇_1^E be two connections on E and let $f \in \mathbb{R}[[X]]$. We know that $[f(E,\nabla_0^E)] = [f(E,\nabla_1^E)]$ in $H^\bullet(M,\mathbb{R})$.

(1) Let ∇_t^E be a smooth path of connection linking ∇_0^E and ∇_1^E, and let R_t^E be the curvature of ∇_t^E. We define

$$\mathscr{T} := \frac{\mathrm{i}}{2\pi}\int_0^1 \mathrm{Tr}\left(\frac{\partial \nabla_t^E}{\partial t} f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\right) dt.$$

Prove that

$$\mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_1^E\right)\right) - \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_0^E\right)\right) = d\mathscr{T}.$$

This formula is called the *transgression formula* and the form $\mathscr{T}$ is called the *transgression term.*

(2) Prove that if we construct $\mathscr{T}$ from another path $\nabla_t^{E'}$ linking ∇_0^E and ∇_1^E, then $\mathscr{T}(\nabla_\bullet^E)$ and $\mathscr{T}(\nabla_\bullet^{E'})$ differ by an exact form.

We thus get a form $\tilde{f}(\nabla_0^E,\nabla_1^E) \in \Omega^\bullet(M,\mathbb{C})/d\Omega^\bullet(M,\mathbb{C})$, depending only on ∇_0^E and ∇_1^E, such that

$$[f(E,\nabla_1^E)] - [f(E,\nabla_0^E)] = d\tilde{f}(\nabla_0^E,\nabla_1^E).$$

This form is called the *Chern–Simons form* associated with f, ∇_0^E and ∇_1^E.

(3) Prove that if ∇_i^E, $i = 0,1$ are Hermitian connections w.r.t. some metrics h_i^E, then the imaginary part of $\tilde{f}(\nabla_0^E,\nabla_1^E)$ is exact and thus $\tilde{f}(\nabla_0^E,\nabla_1^E) \in \Omega^\bullet(M,\mathbb{R})/d\Omega^\bullet(M,\mathbb{R})$.

(1) Denote $\omega_t := \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\right)$, then

$$\begin{aligned}\frac{\mathrm{d}\omega_t}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\right) = \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\left(f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\frac{\mathrm{d}R_t^E}{\mathrm{d}t}\right)\\ &= \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\left(f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)d^{\nabla_t^E}\frac{\partial \nabla_t^E}{\partial t}\right) = d\left(\frac{\mathrm{i}}{2\pi}\mathrm{Tr}\left(f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\frac{\partial\nabla_t^E}{\partial t}\right)\right).\end{aligned}$$

Integrate over $[0,1]$, the left hand side is

$$\int_0^1 \frac{\mathrm{d}\omega_t}{\mathrm{d}t}\,\mathrm{d}t = \omega_1 - \omega_0 = \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_1^E\right)\right) - \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R_0^E\right)\right).$$

While the right hand side is

$$\int_0^1 d\left(\frac{\mathrm{i}}{2\pi}\mathrm{Tr}\left(f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\frac{\partial\nabla_t^E}{\partial t}\right)\right)\mathrm{d}t = d\left(\int_0^1 \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\left(f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\frac{\partial\nabla_t^E}{\partial t}\right)\mathrm{d}t\right) = d\mathscr{T}.$$

(2) For $E \to M$, denote $\tilde{E} := E \times \mathbb{R}$ and $\tilde{M} = M \times \mathbb{R}$. Let $\nabla_{0,t}^E$ and $\nabla_{1,t}$ be two paths from ∇^E to $\nabla^{E'}$, and $\mathscr{T}(\nabla_{0,t}^E)$, $\mathscr{T}(\nabla_{1,t}^E)$ be the associated transgression terms. Note that ∇_t^E is a path from ∇^E to $\nabla^{E'}$, we have

$$\nabla^{\tilde{E}} = \nabla_t^E + dt \wedge \frac{\partial}{\partial t} \qquad \Longrightarrow \qquad R^{\tilde{E}} = R_t^E + dt \wedge \frac{\partial \nabla_t^E}{\partial t}.$$

Therefore

$$f(\nabla^{\tilde{E}}) = f(\nabla_t^E) + \frac{\mathrm{i}}{2\pi}\mathrm{Tr}\left(\frac{\partial\nabla_t^E}{\partial t}f'\left(\frac{\mathrm{i}}{2\pi}R_t^E\right)\right)dt.$$

Hence

$$\mathscr{T} = \int_0^1 \left(dt \text{ part of } f(\nabla^{\tilde{E}})\right) = \int_0^1 \underbrace{\iota_{\partial_t} f(\nabla^{\tilde{E}})}_{\in\Omega^\bullet(M)}\,dt.$$

As a consequence,

$$\mathscr{T}(\nabla^E_{1,t}) - \mathscr{T}(\nabla^E_{0,t}) = \int_0^1 \iota_{\partial_t}(f(\nabla^{\tilde E}_1) - f(\nabla^{\tilde E}_0))dt.$$

But if $\nabla^{\tilde E}_s$ is a path from $\nabla^{\tilde E}_0$ to $\nabla^{\tilde E}_1$, we have

$$f(\nabla^{\tilde E}_1) - f(\nabla^{\tilde E}_0) = d^{\tilde M}\mathscr{T}(\nabla^{\tilde E}_s) = \left(d^M + dt\wedge\frac{\partial}{\partial t}\right)\mathscr{T}(\nabla^{\tilde E}_0).$$

Thus

$$\iota_{\partial_t}(f(\nabla^{\tilde E}_1) - f(\nabla^{\tilde E}_0)) = d^M\iota_{\partial_t}\mathscr{T}(\nabla^{\tilde E}_s) + \frac{\partial}{\partial t}(\mathscr{T}(\nabla^{\tilde E}_s)^{[0]}),$$

where $\alpha\in\Omega(\tilde M)$ decomposes as $\alpha^{[0]} + \alpha^{[1]}dt$ with $\alpha^{[i]}\in\Omega(M)$. Then

$$\mathscr{T}(\nabla^E_{1,t}) - \mathscr{T}(\nabla^E_{0,t}) = d^M\left(\int_0^1 \iota_{\partial_s}\mathscr{T}(\nabla^{\tilde E}_s)\right) + \mathscr{T}(\nabla^{\tilde E}_s)^{[0]}|_{t=1} - \mathscr{T}(\nabla^{\tilde E}_s)^{[0]}|_{t=0}.$$

Now, for $t=0$ or 1, $\nabla^{\tilde E}_s = \text{const} = \nabla^E$ or $\nabla^{E'}$. Hence

$$\left.\frac{\partial\nabla^{\tilde E}_s}{\partial s}\right|_{M\times\{t\}} = 0 \qquad\Longrightarrow\qquad \mathscr{T}(\nabla^{\tilde E}_s)|_{t=0\text{ or }1} = 0.$$

(3) From (1) and (2) we know that the transgression form $\mathscr{T} = \tilde f(\nabla^E_0,\nabla^E_1)$. In this part, ∇^E_i are metric, hence R^E_i are anti self-adjoint. Since $f\in\mathbb{R}[[X]]$,

$$\overline{f'\left(\frac{\mathrm{i}}{2\pi}R^E_i\right)} = f'\left(\overline{\frac{\mathrm{i}}{2\pi}R^E_i}\right) = f'\left(-\frac{\mathrm{i}}{2\pi}\overline{R^E_i}\right) = f'\left(\frac{\mathrm{i}}{2\pi}R^E_i\right).$$

And (1) gives us

$$d\tilde f(\nabla^E_0,\nabla^E_1) = \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R^E_1\right)\right) - \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R^E_0\right)\right),$$

which by our argument before, is a real form. Thus $d(\operatorname{Im}\tilde f) = 0$ as a difference of cohomology class, which implies $\operatorname{Im}\tilde f$ is exact. □

Remark. *There is also a double transgression method in* ***[MM07, Thm B.5.4]****.*

Exercise 3: Characteristic class of flat vector bundles

Let $F\to M$ be a complex vector bundle on a compact connected manifold. We assume that F is flat, that is, there exists a connection ∇^F on it such that its curvature vanishes.

(1) Prove that for any $f\in\mathbb{R}[[X]]$, $f(F) = \operatorname{rank} F$.

(2) Let h^F be an Hermitian metric on F, and let $\nabla^{F,*}$ be the dual of ∇^F w.r.t. h^F. We define a from $f^\circ(F,h^F)\in\Omega^\bullet(M,\mathbb{C})/d\Omega^\bullet(M,\mathbb{C})$ by

$$f^\circ(F,h^F) = \mathrm{i}\pi\tilde f(F,\nabla^F,\nabla^{F,*}).$$

Prove that $f^\circ(F,h^F)$ is a real, odd and closed form.

(3) Prove that if h^F_0 and h^F_1 are two metrics on F, then

$$[f^\circ(F,h^F_1)] = [f^\circ(F,h^F_0)]\in H^{2\bullet+1}(M,\mathbb{R}).$$

This class is simply denoted by $f^\circ(F)$.

(4) Prove that if F is unitarily flat, that is, there is a flat connection ∇^F and an Hermitian metric h^F on F such that ∇^F is metric for h^F, then $f^\circ(F) = 0$.

(1) Now ∇^F is flat, hence $R^F = 0$. By definition, if $f = \sum_{k\geqslant 0} a_k X^k$, then

$$f(F,\nabla^F) = \mathrm{Tr}\left(f\left(\frac{\mathrm{i}}{2\pi}R^F\right)\right) = \mathrm{Tr}\,(f(0)) = \mathrm{Tr}\,(a_0\mathrm{id}) = a_0\cdot\mathrm{rank}\,F.$$

Since M is connected, $H^0(M,\mathbb{C}) = \mathbb{C}$. Thus $f(F) = [f(F,\nabla^F)] = a_0\cdot\mathrm{rank}\,F$. (*We may need to assume $a_0 = 1$ for $f \in \mathbb{R}[[X]]$ to get $f(F) = \mathrm{rank}\,F$.*)

(2) By the definition of dual connection, $R^{F,*} = -\overline{(R^F)}^{\top} = 0$, thus $\nabla^{F,*}$ such that the curvature vanishes. By Exercise 2, the Chern–Simons form $\tilde{f}(F,\nabla^F,\nabla^{F,*})$ such that

$$d\tilde{f}(F,\nabla^F,\nabla^{F,*}) = [f(F,\nabla^{F,*})] - [f(F,\nabla^F)] = a_0(\mathrm{rank}\,F - \mathrm{rank}\,F) = 0.$$

Hence $\tilde{f}(F,\nabla^F,\nabla^{F,*})$ is closed. So $f^\circ(F,h^F)$ is well-defined.

- $f^\circ(F,h^F)$ is closed because $\tilde{f}(F,\nabla^F,\nabla^{F,*})$ is closed.
- $f^\circ(F,h^F)$ is real since

$$\begin{aligned}\overline{f^\circ(F,h^F)} = \overline{\mathrm{i}\pi\tilde{f}(F,\nabla^F,\nabla^{F,*})} &= -\mathrm{i}\pi\overline{\tilde{f}(F,\nabla^F,\nabla^{F,*})}\\ &= -\mathrm{i}\pi\tilde{f}(F,\nabla^{F,*},\nabla^F) = \mathrm{i}\pi\tilde{f}(F,\nabla^F,\nabla^{F,*}) = f^\circ(F,h^F).\end{aligned}$$

- By Exercise 2, $d\tilde{f}(F,\nabla^F,\nabla^{F,*}) = [f(F,\nabla^{F,*})] - [f(F,\nabla^F)]$, where the right hand side is even. Then $f^\circ = \mathrm{i}\pi\tilde{f}$ is odd.

(3) Choose a path of Hermitian connection $\left\{h_t^F : 0 < t < 1\right\}$ linking h_0^F and h_1^F, with ∇_t^F be the corresponding flat connection and $\nabla_t^{F,*}$ the dual w.r.t. h_t^F. Then the Chern–Simons form

$$\tilde{f}(F,\nabla_t^F,\nabla_t^{F,*}),$$

by (2), is closed. Therefore

$$f^\circ(F,h_1^F) - f^\circ(F,h_0^F) = \mathrm{i}\pi\int_0^1 \frac{\mathrm{d}\tilde{f}_t}{\mathrm{d}t}dt = -\frac{1}{2}\int_0^1 d\left(\mathrm{Tr}\left(\frac{\partial\nabla_t^{F,*}}{\partial t}f'(0)\right)\right)$$

since $R_t^F = 0$ for each $t \in [0,1]$. Let $\eta = -\frac{1}{2}\int_0^1 \mathrm{Tr}\left(\frac{\partial\nabla_t^{F,*}}{\partial t}f'(0)\right)dt$, then

$$f^\circ(F,h_1^F) - f^\circ(F,h_0^F) = d\eta \implies [f^\circ(F,h_1^F)] - [f^\circ(F,h_0^F)] = 0 \in H^{2\bullet+1}(M,\mathbb{R}).$$

(4) Let ∇^F be the metric connection w.r.t. h^F, which means $\nabla^F = \nabla^{F,*}$. Thus $\tilde{f}(F,\nabla^F,\nabla^{F,*}) = 0$, which implies $f^\circ(F,h^F) = 0$. □

3 Analysis of Elliptic Operators

In this chapter, we fix (M,g^{TM}) an oriented compact Riemannian manifold of dimension n. E, $F \to M$ are two vector bundled endowed with Hermitian metrics h^E, h^F and connections ∇^E, ∇^F.

3.1 Sobolev spaces

Let $\nabla^{T^*M^{\otimes k}\otimes E}$ be the connection induced on $T^*M^{\otimes k}\otimes E$,

$$\nabla^{(k)} : \Gamma(E) \to \Gamma(T^*M^{\otimes k}\otimes E), \qquad s \mapsto \nabla^{T^*M^{\otimes(k-1)}}(\cdots(\nabla^{T^*M^{\otimes 2}}(\nabla^{T^*M\otimes E}(\nabla^E s)))\cdots).$$

Definition 3.1: Sobolev spaces

For $u \in \Gamma(E)$, define its *k-th Sobolev norm* by

$$\|u\|_{k,2} := \left(\sum_{j=0}^{k} \int_M \left|\nabla^{(j)} u(x)\right| d\mathrm{vol}_M(x) \right)^{1/2}.$$

And the *k-th Sobolev space* $W^{k,2}(M,E)$ is the completion of $\Gamma(E)$ w.r.t. the norm $\|\cdot\|_{k,2}$. This is equivalent to

$$W^{k,2}(M,E) := \left\{ u \in L^2(M,E) : \nabla^{(j)} u \text{ is weakly defined and is in } L^2 \text{ for } 0 \leqslant j \leqslant k \right\}.$$

Here $L^2(M,E)$ denotes the set of L^2-sections.

The standard notation for k-th Sobolev space is actually H^k. To avoid possible confusion (*for example, the de Rham cohomology* $H^k(M)$), we use the notation $W^{k,2}(M,E)$ instead of $H^k(M,E)$.

Proposition 3.2

For $P \in \mathscr{D}\mathrm{iff}^{\leqslant m}(E,F)$ and for all $k \geqslant m$, P extends to a C^0 linear map $W^{k,2}(M,E) \to W^{k-m,2}(M,E)$, which is equivalent to $\exists c > 0$,

$$\|Pu\|_{k-m,2} \leqslant c\,\|u\|_{k,2}.$$

Proof. We first do induction on m.

If $m = 0$, $P = A \in \mathrm{Hom}(E,F)$, then do induction on k.

- If $k = 0$, $\|\cdot\|_{0,2} = \|\cdot\|_2$. Then $\|Au\|_2 \leqslant c\,\|u\|_2$ since M is compact.
- For $k \geqslant 1$, if $1 \leqslant \ell \leqslant k$, we have

$$\nabla^{(\ell)} As = \nabla^{(\ell-1)}(A\nabla^E + (\nabla^F A - \nabla^E A)s),$$

and $B = \nabla^F A - \nabla^E A$ is of order 0, so

$$\begin{aligned}
\|As\|_{k,2}^2 &\leqslant \sum_{\ell=1}^{k} c_\ell \int_M \left(\left|\nabla^{(\ell-1)} A\nabla^E s\right|^2 + \left|\nabla^{(\ell-1)} Bs\right|^2 \right) d\mathrm{vol}_M + \int_M |As|^2\, d\mathrm{vol}_M \\
&\leqslant c \left\| A\nabla^E s \right\|_{k-1,2}^2 + c' \|Bs\|_{k,2}^2 + \|As\|_2^2 \\
&\leqslant c \left\| \nabla^E s \right\|_{k-1,2}^2 + c' \|s\|_{k-1,2}^2 \\
&\leqslant c\,\|s\|_{k,2}^2.
\end{aligned}$$

The third inequality comes from induction on k.

For $m \geqslant 1$, for $s \in C^\infty(M,E)$ and locally on U_α, $P = P_\alpha = \sum_{|I| \leqslant M} a_I^\alpha \frac{\partial^{|I|}}{\partial x_I}$ and $\frac{\partial}{\partial x_i} = \nabla_{e_i} + \Gamma(e_i)$ for $\{e_i\}$ a basis of $\mathbb{R}^n$. Denote $I = (j_1, \ldots, j_n)$, we have

$$\frac{\partial^{|I|}}{\partial x_I} = \prod_{i=1}^{n} (\nabla_{e_i} + \Gamma(e_i))^{j_i} = (\nabla_e)^I + R_I^\alpha$$

with R_I^α of order $\leqslant m-1$. Let $\{\varphi_\alpha\}$ be a partition of unity and $s_\alpha = \varphi_\alpha s$. Then

$$\|Ps\|_{k-m,2} = \left\| \sum_\alpha P_\alpha s_\alpha \right\|_{k-m,2} \leqslant \sum_\alpha \|P_\alpha s_\alpha\|_{k-m}.$$

Now we estimate $\|P_\alpha s_\alpha\|_{k-m,2}^2$.

$$\begin{aligned}\|P_\alpha s_\alpha\|_{k-m,2}^2 &\leqslant c\sum_{\ell=0}^{k-m}\int_{U_\alpha}\left|\nabla^{(\ell)}((\nabla_e)^I + R_I^\alpha)s_\alpha\right|^2 d\mathrm{vol}\\ &\leqslant c\sum_{\ell=0}^{k-m}\int_{U_\alpha}\left(\left|\nabla^{(\ell)}(\nabla_e)^I s_\alpha\right|^2 + \left|\nabla^{(\ell)}R_I^\alpha s_\alpha\right|^2\right)d\mathrm{vol},\end{aligned}$$

by an elementary inequality $(a_1+\cdots+a_n)^2 \leqslant n(a_1^2+\cdots+a_n^2)$. Note that $\left|\nabla^{(\ell)}(\nabla_e)^I s_\alpha\right|^2 \leqslant c\left|\nabla^{(\ell+m)}s_\alpha\right|^2$, and left multiplication by φ_α is of order 0, we have

$$\begin{aligned}\|P_\alpha s_\alpha\|_{k-m,2} &\leqslant c\left(\|s_\alpha\|_{k,2} + \left\|R_I^\alpha s_\alpha\right\|_{k-m,2}\right)\\ &\leqslant c\left(\|s\|_{k,2} + \|s\|_{k-1,2}\right) \leqslant c\,\|s\|_{k,2}.\end{aligned}$$

The second term comes from the induction. So we conclude that $\|Ps\|_{k-m,2} \leqslant c\,\|s\|_{k,2}$. □

Definition 3.3: Space C^ℓ

Let $C^\ell(M,E)$ be the space of ℓ-th continuous differentable sections, endowed with the norm

$$\|u\|_{C^\ell} := \sup_{x\in M}\left|\sum_{j=0}^{\ell}\left|\nabla^{(j)}u(x)\right|\right|.$$

There are some results from the PDE course, we would not prove them in this course. But we shall use them in the next section.

Theorem 3.4: Sobolev embedding

If $k > -\frac{n}{2}+\ell$, then there is an inclusion $W^{k,2}(M,E) \hookrightarrow C^\ell(M,E)$. In other words,

$$u \in W^{k,2}(M,E) \implies u \in C^\ell(M,E), \qquad \|u\|_{C^\ell} \leqslant c_{\ell,k}\,\|u\|_{k,2}.$$

Theorem 3.5: Rellich lemma

If $k < k'$, then the inclusion $W^{k',2}(M,E) \hookrightarrow W^{k,2}(M,E)$ is compact. In other words, if $\{u_j\}$ is a bounded sequence for $\|\cdot\|_{k',2}$, then there exists a subsequence $(j_k)_{k\geqslant 1} \subset \mathbb{N}$ such that $\{u_{j_k}\}$ is convergent for $\|\cdot\|_{k,2}$.

Also, the topological dual of $W^{k,2}(M,E)$ is denoted as $W^{-k,2}(M,E)$ with norm

$$\|\varphi\|_{-k,2} := \sup_{u\in W^{k,2}(M,E)}\frac{(\varphi,u)}{\|u\|_{k,2}}.$$

We have a C^0 inclusion

$$C^0(M,E^*) \hookrightarrow W^{-k,2}(M,E), \qquad s \mapsto \left[(\varphi_s,\cdot) := \int_M (s,\cdot)d\mathrm{vol}_M\right].$$

3.2 Analytic properties of elliptic operators

(α) **Formal adjoint**

Definition 3.6: Formal adjoint

Let $P \in \mathscr{D}\mathrm{iff}(E,F)$. A *formal adjoint* of P is a differential operator $P^* \in \mathscr{D}\mathrm{iff}(F,E)$ such that $\forall u \in \Gamma(E),\ v \in \Gamma(F)$,

$$\langle Pu, v\rangle = \langle u, P^* v\rangle .$$

A formal adjoint is not an adjoint because for unbounded operators, there may be problems with the domain.

Proposition 3.7

(1) If P has a formal adjoint P^*, it is unique and P is the formal adjoint of P^*.

(2) If $P_1,\ P_2$ have formal adjoints, then P_1+P_2, P_1P_2 also have formal adjoints. And

$$(P_1+P_2)^* = P_1^* + P_2^*, \qquad (P_1P_2)^* = P_2^* P_1^*.$$

(3) If P is of order 0, then P has a formal adjoint.

Proof. Check the definition. □

Example 3.8

Assume ∇^E is Hermitian, $X \in C^\infty(M, TN)$, then $\nabla_X \in \mathscr{D}\mathrm{iff}(E,E)$. Recall that $\operatorname{div} X$ is defined by $\mathscr{L}_X d\mathrm{vol}_M = \operatorname{div} X \cdot d\mathrm{vol}_M$, and the divergence formula

$$\int_M X(\varphi) d\mathrm{vol}_M = -\int_M \varphi \cdot \operatorname{div} X d\mathrm{vol}_M, \qquad \forall \varphi \in C^\infty(M).$$

We get

$$\begin{aligned}\int_M (\langle \nabla^E_X u, v\rangle + \langle u, \nabla^E_X v\rangle) d\mathrm{vol}_M &= \int_M \iota_X (d\langle u, v\rangle) \\ &= \int_M X(\langle u, v\rangle) d\mathrm{vol}_M = -\int_M \operatorname{div} X \cdot \langle u, v\rangle\, d\mathrm{vol}_M.\end{aligned}$$

Thus $(\nabla^E_X)^* = -\nabla^E_X - \operatorname{div} X$.

Example 3.9

Let ∇^E be an Hermitian connection, $\nabla^E \in \mathscr{D}\mathrm{iff}(E, T^*M \otimes E)$. Locally,

$$\nabla^E = \sum_{i=1}^n \nabla^E_{\frac{\partial}{\partial x_i}} dx^i = \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} + \Gamma_i\right) dx^i.$$

Hence $(\nabla^E)^* = \sum_{i=1}^n (dx^i)^* \left(\frac{\partial}{\partial x_i} + \Gamma_i\right)^*$. We know that $(dx^i)^* = \iota_{e_i}$, where $\{e_i\}$ is the metric dual of $\{dx^i\}$.

Proposition 3.10

Every differential operator has a formal adjoint.

Definition 3.11: Formally self-adjoint

A differential operator P is said to be *formally self-adjoint* if $P = P^*$.

Thus PP^* and P^*P are formally self-adjoint for any differential operator P.

(β) **Fundamental results on elliptic operators**

The following theorems are from the course of PDE. We do not prove them here.

Theorem 3.12: Elliptic estimate

Let $L \in \mathscr{D}\mathrm{iff}^m(E,F)$ be elliptic, then $\exists c > 0$ such that $\forall u \in W^{k+m,2}(M,E)$,

$$\|u\|_{k+m,2} \leqslant c(\|Lu\|_{k,2} + \|u\|_2).$$

Remark. *Recall that* $\|Lu\|_{k,2} \leqslant c\,\|u\|_{k+m,2}$.

Theorem 3.13: Regularity

For $u \in L^2(M,E)$ and $v \in W^{k,2}(M,F)$, if $Lu = v$ weakly, *i.e.*, $\forall \varphi \in \mathbb{C}^\infty(M,F)$, $\langle u, L^*\varphi\rangle = \langle v, \varphi\rangle$, then $u \in W^{k+m,2}(M,E)$. Lu is defined and $Lu = v$.

Corollary 3.14

If $u \in L^2(M,E)$ and Lu is weakly smooth, then u is smooth. In particular, $Lu = 0$ weakly implies u is smooth.

Proof. If $Lu = v$ weakly, v is smooth, then $v \in \bigcup_{k\in\mathbb{R}} W^{k,2}(M,E)$ and hence $u \in \bigcap_{k\in\mathbb{R}} W^{k+m,2}(M,E) = C^\infty(M,E)$ by Sobolev embedding. □

(γ) **Fredholm index of L**

If $L \in \mathscr{D}\mathrm{iff}^m(E,F)$ is elliptic, it defines an operator

$$L_k : W^{k+m,2}(M,E) \to W^{k,2}(M,F).$$

By the corollary above, for all $k, k' \in \mathbb{R}$, $\ker L_k = \ker L_{k'} \subset C^\infty(M,E)$. We denote the kernel by $\ker L$ because it is actually independent of $k \in \mathbb{R}$.

Theorem 3.15

$\ker L$ is of finite dimension.

Proof. (1) First, we prove that $\ker L$ is closed in $L^2(E)$. Set $(u_i) \subset \ker L$ and $u_i \to u$ in $L^2(M,E)$ w.r.t. $\|\cdot\|_2$. Thus $\forall \varphi \in C^\infty(M,E)$,

$$\langle u_i, L^*\varphi\rangle = 0 = \int_M \langle u_i(x), L^*\varphi(x)\rangle\, dx \implies \langle u, L^*\varphi\rangle = 0.$$

So $Lu = 0$ weakly and hence $u \in \ker L$.

(2) If $B \subset \ker L$ is a closed ball for $\|\cdot\|_2$, then B is compact. Let $(u_i) \subset B$, *i.e.*, a $\|\cdot\|_2$-bounded sequence. By elliptic estimate,

$$\|u_i\|_{m,2} \leqslant c(\|Lu_i\|_2 + \|u_i\|_2) \leqslant c'.$$

As $W^{k,2} \hookrightarrow W^{0,2} = L^2$ is compact, (u_i) has a converging subsequence in L^2. So we conclude that B is compact and hence $\ker L$ is of finite dimension by Riesz theorem.

Now our aim is to deal with $\operatorname{coker} L$.

Lemma 3.16: Poincaré inequality

There exists $c > 0$ such that $\forall u \in W^{k+m,2}(M,E) \cap (\ker L)^\perp$,

$$\|u\|_{k+m,2} \leqslant c\,\|Lu\|_{k,2}.$$

Proof. We prove by contradiction. If $\exists (u_i)_{i\geqslant 1} \subset W^{k+m,2}(M,E) \cap (\ker L)^\perp$ such that

$$\|u_i\|_{k,2} = 1, \qquad \|u_i\|_{m+k,2} \geqslant i\,\|Lu_i\|_{k,2}.$$

Then the elliptic estimate gives

$$i\,\|Lu_i\|_{k,2} \leqslant \|u_i\|_{k+m,2} \leqslant c(\|Lu_i\|_{k,2} + \|u_i\|_2) \leqslant c(\|Lu_i\|_{k,2} + 1).$$

So $\|Lu_i\|_{k,2} \to 0$ as $i \to \infty$ and hence $\|u_i\|_{k+m,2} \leqslant c$ for some $c > 0$. The inclusion $W^{m+k,2} \hookrightarrow W^{k,2}$ is compact so we can assume that $u_i \to u_\infty$ in $W^{k,2}$. But then $\|u_\infty\|_{k,2} = 1$ with $u_\infty \in (\ker L)^\perp$. Let $v_i = Lu_i \to 0$ in $W^{k,2}$, for any $\varphi \in C^\infty(M,F)$,

$$\int_M \langle u_i, L^*\varphi\rangle = \int_M \langle v_i, \varphi\rangle \implies \int_M \langle u_\infty, L^*\varphi\rangle = 0.$$

Thus $Lu_\infty = 0$ weakly, so $u_\infty \in \ker L \setminus \{0\}$. contradiction. □

Theorem 3.17

We shall denote L instead of L_k because the following holds for all $k \in \mathbb{R}$.

(1) $\operatorname{im} L$ is closed in $W^{k,2}(M,F)$.

(2) $\operatorname{im} L = (\ker L^*)^\perp$, $\operatorname{im} L^* = (\ker L)^\perp$.

Proof. (1) Denote $v_n = Lu_n$ with $u_n \in W^{k+m,2}$. Assume that $v_n \to v_\infty$ in $W^{k,2}$ and since $\ker L$ is of finite dimension, one can decompose

$$u_n = u_n^0 + u_n^\perp, \qquad u_n^0 \in \ker L, \qquad u_n^\perp \in (\ker L)^\perp.$$

Thus $v_n = Lu_n^\perp$. By Poincaré inequality,

$$\left\| u_n^\perp \right\|_{k+m,2} \leqslant c\,\|v_n\|_{k,2} \leqslant c'.$$

As $W^{k_m,2} \hookrightarrow W^{k,2}$ compactly, up to contracting a subsequence, we can assume $u_n^\perp \to u_\infty^\perp$ in $W^{k,2}$. Then $\forall \varphi \in C^\infty$,

$$\int_M \langle u_n^\perp, L^*\varphi\rangle = \int_M \langle v_n, \varphi\rangle \implies \int_M \langle u_\infty^\perp, L^*\varphi\rangle = \int_M \langle v_\infty, \varphi\rangle.$$

Thus $v_\infty = Lu_\infty^\perp$ weakly. By the regularity theorem, $u_\infty^\perp \in W^{m+k,2}$ and $v_\infty = Lu_\infty^\perp \in \operatorname{im} L$.

(2) We only prove $\operatorname{im} L = (\ker L^*)^\perp$, the other claim can be proved similarly.

If $v = Lu$, $u \in W^{k+m,2}$, and $w \in \ker L^*$, then $w \in C^\infty$ since L^* is elliptic. Thus

$$\langle v, w\rangle = \langle u, L^* w\rangle = 0 \implies v \in (\ker L^*)^\perp,$$

this gives us $\operatorname{im} L \subset (\ker L^*)^\perp$.

Now if $v \in (\ker L^*)^\perp$, assume $v \notin \operatorname{im} L$. Because $\operatorname{im} L$ is closed, by Hahn–Banach theorem, $\exists w \in W^{k,2}$ such that $\langle w, v\rangle \neq 0$ and $w \in (\operatorname{im} L)^\perp$. Then $\forall u \in W^{k+m,2}$,

$$\langle w, (L^*)^* u\rangle = \langle w, Lu\rangle = 0,$$

the left hand side weakly define $\langle L^* w, u\rangle$. In particular, this is true for u smooth, hence $L^* w = 0$ weakly. So $w \in C^\infty$ and $L^* w = 0$ implies $w \in \ker L^*$. Therefore $\langle v, w\rangle = 0$ since $v \in (\ker L^*)^\perp$, which leads to a contradiction. $\square$

Corollary 3.18

The operator $L_k : W^{m+k,2} \to W^{k,2}$ is a Fredholm operator. And its index is independent of $k \in \mathbb{R}$. We shall denote it as $\operatorname{Ind} L$.

We also obtain that

Definition–Proposition 3.19: Green operator

Let $L \in \mathscr{D}\mathrm{iff}^m(E,E)$ be elliptic, then

$$L_k : W^{k+m,2}(M,E) \cap (\ker L)^\perp \to \operatorname{im} L_k$$

has a bounded inverse G_k by Poincaré inequality, called the *Green operator*.

We extend G_k by zero on $(\operatorname{im} L_k)^\perp = \ker L_k^*$. Then

$$\begin{cases} L_k G_k = \mathrm{id} - P^\perp_{\ker L^*}, & P^\perp_{\ker L^*} G_k = 0, \\ G_k L_k = \mathrm{id} - P^\perp_{\ker L}, & G_k P^\perp_{\ker L} = 0. \end{cases}$$

Moreover, $u \in C^\infty(M,E) \implies G_k u \in C^\infty(M,E)$.

3.3 Hodge Theory

Definition 3.20: Elliptic complex

Let $E^0, \dots, E^N$ be Hermitian vector bundles on M. Then a complex

$$0 \longrightarrow C^\infty(M,E^0) \xrightarrow{\partial_0} C^\infty(M,E^1) \xrightarrow{\partial_1} \cdots \longrightarrow C^\infty(M,E^N) \longrightarrow 0$$

is called an *elliptic complex* if and only if

- $\partial^2 = 0$, *i.e.*, it is a complex.
- $\partial_i \in \mathscr{D}\mathrm{iff}^1(E^i, E^{i+1})$.
- $D = \partial + \partial^* \in \mathscr{D}\mathrm{iff}^1(E^\bullet, E^\bullet)$, where $E^\bullet = \bigoplus_{i=0}^N E^i$.

For such a complex, we can define

$$H^i(E^\bullet, \partial) := \frac{\ker \partial_i}{\operatorname{im} \partial_{i-1}}.$$

Set $\Delta = D^2 = (\partial\partial^* + \partial^*\partial)$. Then Δ preserves each $C^\infty(M,E^i)$. Denote $\Delta_i := \Delta|_{C^\infty(M,E^i)} \in \mathscr{D}\mathrm{iff}^2(E^i,E^i)$, then each Δ_i is elliptic.

Lemma 3.21

$\ker \Delta = \ker D = \ker \partial \cap \ker \partial^*$.

Proof. (1) Since $\Delta = D^2 = D^* D$, $\ker \Delta = \ker D$.

(2) For $u \in C^\infty(M, E^\bullet)$,

$$\langle \Delta u, u\rangle = \langle \partial\partial^* u, u\rangle + \langle \partial^*\partial u, u\rangle = \|\partial^* u\|^2 + \|\partial u\|^2.$$

Thus $\ker\Delta = \ker\partial \cap \ker\partial^*$. □

Theorem 3.22: Hodge

For each $i \in \{0, \dots, n\}$, we have

(1) Hodge decomposition: $L^2(M, E^i) = \ker\Delta_i \oplus \operatorname{im}\partial_{i-1} \oplus \operatorname{im}\partial_i^*$. Here ∂_{i-1}, ∂_i^* are seen as bounded operators $W^{1,2} \to L^2$.

(2) $H^i(E^\bullet, \partial) = \ker\Delta_i$.

(3) $H^i(E^\bullet, \partial)$ is finite dimensional.

Proof. (1) By the closed range theorem,

$$L^2(M, E^i) = \ker D \oplus \operatorname{im} D^* = \ker D \oplus \operatorname{im} D.$$

(2) Denote P_i the orthogonal projection $L^2(M, E^i) \to \ker\Delta_i$, and define

$$Z^i(M, E^\bullet) = \left\{u \in C^\infty(M, E^i) : \partial_i u = 0\right\}, \qquad B^i(M, E^\bullet) = \partial_{i-1}(C^\infty(M, E^{i-1})).$$

So $H^i(M, E^\bullet) = Z^i(M, E^\bullet)/B^i(M, E^\bullet)$. Define $\pi_i : \ker\Delta_i \hookrightarrow Z^i \twoheadrightarrow H^i$. We want to prove that π_i is an isomorphism.

Injectivity. Let $u \in \ker\Delta_i$ such that $\pi_i u = 0$. Then $u = \partial_{i-1} v$ for some v. But $u \in \ker\Delta_i \subset \ker\partial_i^*$, so

$$\partial_i^*\partial_{i-1} v = 0 \implies \langle \partial^*\partial v, v\rangle = \|\partial v\| = 0.$$

So $\partial v = 0 = u$.

Surjectivity. Let $[u] \in H^i(E^\bullet, \partial)$. By (1), $u = P_i u + Dv$ with $v \in W^{1,2}(M, E^\bullet)$. As $Dv = u - P_i u \in C^\infty$, in fact v is smooth, we write $v = v_{i-1} \oplus v_{i+1}$, such that

$$u = P_i u + \partial_{i-1} v_{i-1} + \partial_i^* v_{i+1}.$$

Thus

$$0 = \partial_i u = \partial_i \partial_i^* v_{i+1} \implies \left\|\partial_i^* v_{i+1}\right\|^2 = 0 \implies \partial_i^* v_{i+1} = 0.$$

Hence $u = P_i u + \partial_{i-1} v_{i-1}$, which is, $[u] = [P_i u] \in H^i(E^\bullet, \partial)$ and $[P_i u] = \pi_i(P_i u)$. So π_i is surjective.

(3) It follows from (2) and ellipticity of Δ_i. □

Additionally, using L^2 version of Hodge decomposition and regularity theorem for D, we obtain the smooth Hodge decomposition:

Corollary 3.23: Smooth Hodge decomposition

$C^\infty(M, E^i) = \ker\Delta_i \oplus \partial_{i-1}(C^\infty(M, E^{i-1})) \oplus \partial_i^*(C^\infty(M, E^{i+1}))$.

For $(x, \xi) \in T^*M$, we have a finite dimensional complex

$$0 \longrightarrow E_x^0 \xrightarrow{\sigma(\partial_0)(x,\xi)} E_x^1 \xrightarrow{\sigma(\partial_1)(x,\xi)} \cdots \longrightarrow E_x^N \longrightarrow 0$$

Then

$$D \leftrightarrow \sigma(\partial)(x,\xi) + \sigma(\partial)^*(x,\xi) = \sigma(\partial - \partial^*)(x,\xi), \qquad \Delta \leftrightarrow \sigma(-\Delta)(x,\xi).$$

We have finite dimensional Hodge theorem

$$H^\bullet(E_x^\bullet, \sigma(\partial)(x,\xi)) \cong \ker(\sigma(-\Delta)(x,\xi)).$$

So $(C^\infty(M,E),\partial)$ is elliptic if and only if $(E_x^\bullet, \sigma(\partial)(x,\xi))$ is acyclic, *i.e.*, exact, which means $H^\bullet = 0$ for $\xi \neq 0$.

3.4 Heat kernel

In this section we consider the operator $\mathrm{e}^{-t\Delta}$. Let $t \to 0$, we obtain local properties. Let $t \to +\infty$, we obtain the projection on Harmonic forms, which indicates global properties. So it is like an interpolation between local and global properties.

(α) **Spectral theory of symmetric elliptic operators**

Assume that $L \in \mathscr{D}\mathrm{iff}^m(E,E)$ is a symmetric elliptic operator, $m > 0$. Denote $\operatorname{Sp} L$ the eigenvalues of L (*not the spectrum!*). For $\lambda \in \operatorname{Sp} L$, denote $E_\lambda = \ker(L - \lambda)$ the eigenspace.

Lemma 3.24

$\operatorname{Sp} L \neq \mathbb{R}$, *i.e.*, $\exists r > 0$ such that $E_r = 0$.

Proof. Prove by contradiction. If $\exists u_r$ with $\|u_r\|_2 = 1$, such that $Lu_r = ru_r$ for any $r \in \mathbb{R}$, then $u_r = L(\frac{1}{r}u_r)$ for $r \neq 0$, which implies $u_r \in \operatorname{im} L = (\ker L)^\perp$. By Poincaré inequality,

$$\|u_r\|_{m,2} \leqslant c\|ru_r\|_2 = cr.$$

Thus $\|u_r\|_{m,2} \to 0$ as $r \to 0$, contradiction. □

Theorem 3.25

(1) $\operatorname{Sp} L$ is a discrete on-empty subset of $\mathbb{R}$.

(2) $L^2(M,E) = \oplus_{\lambda \in \operatorname{Sp} L} E_\lambda$. If P_λ denotes the orthogonal projection on E_λ, then for $u \in L^2(M,E)$,

$$u = \sum_{\lambda \in \operatorname{Sp} L} P_\lambda u, \qquad \|u\|_2^2 = \sum_{\lambda \in \operatorname{Sp} L} \|P_\lambda u\|_2^2.$$

(3) If $u \in W^{m,2}(M,E)$, then

$$\sum_{\lambda \in \operatorname{Sp} L} |\lambda|^2 \|P_\lambda u\|^2 < +\infty, \qquad Lu = \sum_{\lambda \in \operatorname{Sp} L} \lambda P_\lambda u.$$

Proof. Replacing L by $L - r$, we may assume that $0 \notin \operatorname{Sp} L$, which is equivalent to $\ker L = 0$. In that case, the Green operator $G : L^2(M,E) \to W^{m,2}(M,E)$ is invertible with its inverse being $L_m : W^{m,2}(M,E) \to L^2(M,E)$.

As $W^{m,2} \hookrightarrow L^2$ is compact, the composition $L^2 \xrightarrow{G} W^{m,2} \hookrightarrow L^2$ is a compact operator, still denoted by G. For $u, v \in L^2$, denote $\tilde{u} = Gu$, $\tilde{v} = Gv \in W^{m,2}(M,E)$,

$$\langle Gu, v\rangle = \langle \tilde{u}, L\tilde{v}\rangle = \langle L\tilde{u}, \tilde{v}\rangle = \langle u, Gv\rangle$$

since L is symmetric. Hence G is a compact bounded symmetric operator from $L^2(M,E)$ to $L^2(M,E)$, and $\ker G = 0$. Then everything follows from the spectral theorem for such operators. □

Theorem 3.26

Set $d(\Lambda) = \dim\big(\bigoplus_{|\lambda|<\Lambda} E_\lambda\big) =: \dim E(\Lambda)$. Then $\exists c > 0$ such that

$$d(\Lambda) \leqslant c\Lambda^{n(n+2m+2)/2m}.$$

Proof. For $k \in \mathbb{N}$, $u \in E(\Lambda)$, $u = \sum_{|\lambda|<\Lambda} a_\lambda u_\lambda$ with $\|u_\lambda\|_2 = 1$. Then $L^k u = \sum_{|\lambda|<\Lambda} a_\lambda \lambda^k u_\lambda$ and

$$\left\|L^k u\right\|_2 = \sum_{|\lambda|<\Lambda} |a_\lambda|^2 |\lambda|^{2k} \leqslant \sum_{|\lambda|<\Lambda} |a_\lambda|^2 \Lambda^{2k} = \Lambda^{2k} \|u\|_2^2.$$

By elliptic estimates for L^k,

$$\|u\|_{mk,2} \leqslant c(1+\Lambda^k)\,\|u\|_2\,.$$

If $mk > \frac{n}{2}+1$,. by Sobolev embedding,

$$\sup|\nabla u| \leqslant \|u\|_{C^1} \leqslant c(1+\Lambda^k)\,\|u\|_2\,.$$

For $\varepsilon > 0$, we say $X \subset M$ is ε-dense if $M = \bigcup_{x\in X} B(x,\varepsilon)$. Define

$$N(\varepsilon) := \min\{|X| : X \text{ is } \varepsilon\text{-dense in } M\}\,.$$

Then $N(\varepsilon)$ is finite because M is compact. Moreover, as $\dim M = n$, there exist a constant K such that $N(\varepsilon) \leqslant K\varepsilon^{-m}$.

Assume $d(\Lambda) > N(\varepsilon)$ for some $\varepsilon > 0$, and set X an ε-dense set with $|X| = N(\varepsilon)$. Then the subspace $V = \{u \in E(\Lambda) : \forall x \in X\,(u(x) = 0)\}$ is of codimension $N(\varepsilon)$. In particular, $V \neq 0$. For $u \in V$ with $\|u\|_2 = 1$,

$$\begin{cases} \sup|\nabla u| \leqslant c(1+\Lambda^k), \\ \forall x \in X\,(u(x)=0). \end{cases} \xRightarrow{\text{mean value theorem}} \|u\|_{C^0} \leqslant c\varepsilon(1+\Lambda^k).$$

But $1 = \int_M |u|^2 \leqslant \mathrm{vol}(M)\cdot\|u\|_{C^0} \leqslant c\mathrm{vol}(M)\varepsilon^2(1+\Lambda^k)^2$. Hence

$$\varepsilon \geqslant \frac{1}{c(1+\Lambda^k)\sqrt{\mathrm{vol}(M)}} =: 2\varepsilon_\Lambda.$$

Then we have

$$d(\Lambda) \leqslant N(\varepsilon_\Lambda) \leqslant K\varepsilon_\Lambda^{-m} \leqslant c\Lambda^{nk}.$$

Choosing the smallest k such that $mk > \frac{n}{2}+1$, *i.e.*, $k = \lfloor\frac{n+2m+2}{2m}\rfloor$, we obtain the desired result. □

(β) **Heat kernel**

In this part, we assume that $L \geqslant 0$, *i.e.*,

$$\langle Lu, u\rangle \geqslant 0, \qquad \forall u \in W^{m,2}(M,E).$$

Fix $\{u_k\}$ a Hilbert basis of L^2 such that $Lu_k = \lambda_k u_k$, $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. And for $v \in E_x$, denote the metric dual w.r.t. $\langle v,\cdot\rangle_x$ as $v^* \in E_x^*$.

Definition–Proposition 3.27: Heat kernel

For $t > 0$ and $x, y \in M$, set

$$K_t(x,y) := \sum_{k\geqslant 1} \mathrm{e}^{-\lambda_k t} u_k(x) \otimes u_k(y)^* \in E_x \otimes E_y^*.$$

For all $\ell \in \mathbb{N}$, this converges uniformly in the C^ℓ-topology in $[a,b]\times M\times M$ for any $0<a<b$. Thus for $u \in L^2(M,E)$, set

$$(\mathrm{e}^{-tL}u)(x) := \int_M K_t(x,y)u(y)dy.$$

And e^{-tL} is called the *heat kernel* of L.

Proof. Fix $s\in\mathbb{N}$ such that $ms > \frac{n}{2}+\ell$. By Sobolev embedding and elliptic estimate,

$$\|u_k\|_{C^0} \leqslant c\|u_k\|_{s,2} \leqslant c(\|u_k\|_2 + \|L^* u_k\|_2) \leqslant c(1+\lambda_s).$$

Now, as $d(lambda a_k) = k$, so $\lambda_k \geqslant ck^\gamma$, where $\gamma = \frac{2m}{n(n+2m+2)}$. So

$$\mathrm{e}^{-t\lambda_k}\lambda_k^s \leqslant c\mathrm{e}^{-ck^\gamma} + k^{s\gamma}.$$

And the convergence for the C^ℓ-topology follows from the convergence of $\int_1^\infty \mathrm{e}^{-tx}x^{-s}dx$. □

Theorem 3.28

For $t>0$, $v\in L^2(M,E)$, $u(t,x) = \mathrm{e}^{-tL}v(x)$. Then u is smooth on $(0,+\infty)\times M$ and

$$\frac{\partial u}{\partial t} + Lu = 0.$$

Finally, if $v\in W^{m,2}(M,E)$, then $\left\|e^{-tL}v - v\right\|_2 = 0$ as $t\to 0$.

Proof. Since $K_t(\cdot,\cdot)$ is smooth, u is also smooth. And

$$\frac{\partial}{\partial t}K_t(x,y) = -\sum_{k\geqslant 1}\lambda_k \mathrm{e}^{-t\lambda_k}u_k(x)\otimes u_k(y)^* = -L_x K_t(x,y).$$

Hence $\frac{\partial}{\partial t}\mathrm{e}^{-tL}v = -L\mathrm{e}^{-tL}v$.

$$\mathrm{e}^{-tL}v(x) = \sum_{k\geqslant 1}\mathrm{e}^{-t\lambda_k}u_k(x)\langle v,u_k\rangle, \qquad \mathrm{e}^{-tL}v - v = \sum_{k\geqslant 1}a_k(\mathrm{e}^{-t\lambda_k}-1)u_k,$$

where a_k is the coefficient of v. Thus

$$\left\|\mathrm{e}^{-tL}v - v\right\|_2^2 = \sum_{k\geqslant 1}|a_k|^2(\mathrm{e}^{-t\lambda_k}-1)^2 =: \sum_{k\geqslant 1}\varphi_k(t),$$

and $\varphi_k(t)\to 0$ as $t\to 0$ with $|\varphi_k(t)| \leqslant |\lambda_k|^2|a_k|^2$. Hence $(\varphi_k(t))_{k\geqslant 1}\in \ell^1(\mathbb{N})$. Therefore, we get the result from the DCT. □

Definition 3.29: Trace of the heat kernel

Note that $K_t(x,x)\in E_x\otimes E_x^* = \mathrm{End}\,E_x$, the *trace* of the heat kernel is

$$\mathrm{Tr}\,\mathrm{e}^{-tL} := \int_M \mathrm{Tr}\,(K_t(x,x))dx.$$

As

$$\mathrm{Tr}\,(u_k(x)\otimes u_k(x)^*) = \left|u_k(x)^2\right|, \qquad \int_M |u_k|^2 = 1,$$

we have $\mathrm{Tr}\,\mathrm{e}^{-tL} = \sum_{k\geqslant 1}\mathrm{e}^{-t\lambda_k}$. For any Hilbert basis $\{e_i\}$ of $L^2(M,E)$, we have

$$\mathrm{Tr}\,\mathrm{e}^{-tL} = \sum_i \left\langle \mathrm{e}^{-tL}e_i, e_i\right\rangle.$$

3.* Additional Exercises

Exercise 1: The closedness of elliptic operators

Let M be a compact oriented Riemannian manifold, and $E,\ F \to M$ be complex Hermitian vector bundles. Let $L \in \mathscr{D}\mathrm{iff}^m(E,F)$ be an elliptic differential operator of order m. Suppose that we have sequences $u_n \in W^{m,2}(M,E)$ and $f_n \in L^2(M,F)$ such that

- For all $n \in \mathbb{N}$, $Lu_n = f_n$.
- There are $u \in L^2(M,E)$ and $f \in L^2(M,F)$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} f_n = f$ for $\|\cdot\|_2$.

Prove that $u \in W^{m,2}(M,E)$ and that $Lu = f$ and that $\lim_{n\to\infty} u_n = u$ for $\|\cdot\|_{m,2}$.

Apply the elliptic estimate for u_n,

$$\|u_n\|_{m,2} \leqslant c(\|Lu_n\|_2 + \|u_n\|_2).$$

We know that $(u_n)_{n\geqslant 1}$ and $(f_n)_{n\geqslant 1}$ are convergent in L^2-norm, so they are bounded in L^2-norm. Thus $(u_n)_{n\geqslant 1}$ is bounded in $W^{m,2}$-norm. By Rellich lemma, there exists a subsequence $(u_{n_k})_{k\geqslant 1}$ that converge to u_∞ in $W^{m,2}$-norm. While $W^{m,2} \hookrightarrow L^2$ is compact, hence $(u_{n_k})_{k\geqslant 1}$ converges to u_∞ in L^2-norm. This gives $u = u_\infty \in W^{m,2}$.

Now $L : W^{m,2}(M,E) \to L^2(M,E)$ is bounded, thus the convergence is preserved, $Lu_n = f_n \to Lu$. By the uniqueness of limit again, $Lu = f$. We can apply elliptic estimate for $u - u_n$,

$$\|u - u_n\|_{m,2} \leqslant c(\left\|f - f_n\right\|_2 + \|u - u_n\|_2) \to 0.$$

Then $u_n \to u$ in $W^{m,2}$-norm. □

Exercise 2: de Rham complex

Let M be a compact oriented Riemannian manifold.

(1) Let $d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ be the differential. Prove that d is a differential operator of order 1, with $\sigma(d)(x,\xi) = \mathrm{i}\xi\wedge \in \mathrm{End}(\Lambda^\bullet T_x^* M)$.

(2) For $\xi \in T_x^*M$, prove that the metric dual of $\xi\wedge$ is $\iota_{\xi^\sharp}$, where $\xi^\sharp \in T_xM$ is the metric dual of ξ.

(3) Prove the Cartan formula: for $\xi \in T_x^*M$, we have $\xi \wedge \iota_{\xi^\sharp} + \iota_{\xi^\sharp}\xi\wedge = |\xi|^2\, \mathrm{id}_{\Lambda^\bullet T_x^*M}$.

(4) Prove that $(\Omega^\bullet(M), d)$ is an elliptic complex.

(1) For a k-form $\omega = \sum_{|I|=k} a_I dx^I$,

$$d\omega = \sum_{|I|=k}\sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I,$$

thus d is an operator with order 1. And by definition,

$$\sigma(d)(x,\xi) = \sum_{j=1}^n \mathrm{i}\xi_j \otimes (dx^j \wedge \cdot) = \mathrm{i}\xi \wedge \cdot.$$

(2) Assume $\alpha \in \Lambda^k T_x^*M,\ \beta \in \Lambda^{k+1} T_x^*M$. Let $* : \Lambda^\bullet T_x^*M \to \Lambda^{n-\bullet} T_x^*M$ be the Hodge star operator. A basic identity of Hodge theory gives that

$$\iota_{\xi^\sharp} * \omega = *(\omega \wedge \xi).$$

Thus

$$\left\langle \xi \wedge \omega, \eta \right\rangle = \xi \wedge \omega \wedge *\eta = (-1)^k \omega \wedge \xi \wedge *\eta = (-1)^k \omega \wedge (-1)^{k(n-k)} **(\xi \wedge *\eta)$$
$$= (-1)^{k+k(n-k)} \omega \wedge *((-1)^{k(n-k)} * \eta \wedge \xi) = (-1)^{2k+2k(n-k)} \omega \wedge *(\iota_{\xi^\sharp}\eta) = \left\langle \omega, \iota_{\xi^\sharp}\eta \right\rangle.$$

Which gives the result that $(\xi\wedge)^* = \iota_{\xi^\sharp}$, where $\sharp$ is the musical isomorphism.

(3) For $\alpha \in \Lambda^\bullet T_x^* M$,

$$(\xi \wedge \iota_{\xi^\sharp} + \iota_{\xi^\sharp} \xi \wedge)\alpha = \xi \wedge \iota_{\xi^\sharp} \alpha + \iota_\xi (\xi \wedge \alpha)$$
$$= \xi \wedge \iota_{\xi^\sharp} \alpha + \xi(\xi^\sharp)\alpha - \xi \wedge \iota_{\xi^\sharp} \alpha = |\xi|^2 \alpha.$$

Thus $\xi \wedge \iota_{\xi^\sharp} + \iota_{\xi^\sharp} \xi \wedge = |\xi|^2 \mathrm{id}_{\Lambda^\bullet T_x^* M}$.

(4) It is obvious that $d^2 = 0$, and $d_i \in \mathscr{D}\mathrm{iff}^1(\Omega^i(M), \Omega^{i+1}(M))$ by (1). Then d_i and d_{i+1}^* are differential operators of order 1, thus $D = d_i + d_{i+1}^* \in \mathscr{D}\mathrm{iff}^1(\Omega^\bullet(M), \Omega^\bullet(M))$. Hence the de Rham complex is an elliptic complex. □

Remark. The coefficient is different from the original exercise. I cannot prove the original version but I believe there must be some different choice of coefficient that caused this problem.

Exercise 3: Differential operators and connections

Let M be a compact oriented Riemannian manifold, let $E,\ F \to M$ be two vector bundles and ∇^E be a connection on E. For $k \in \mathbb{N}^*$, recall that $\nabla^{(k)} : C^\infty(M, E) \to C^\infty(M, T^*M^{\otimes k} \otimes E)$ is the differential operator defined by composing the k-connections $\nabla^{T^*M^{\otimes j} \otimes E} \to C^\infty(M, T^*M^{\otimes j+1} \otimes E)$ induced by ∇^E and the Levi-Civita connection of M.

(1a) Prove that $\sigma(\nabla^{(k)})(x,\xi) = (\xi \otimes \xi \otimes \cdots \otimes \xi) \otimes \mathrm{id}_{E_x} : E_x \to T_x^* M^{\otimes k} \otimes E_x$.

(1b) Let $\mathfrak{s}_k : T_x^* M^{\otimes k} \to S^k(T^*M)$ be the natural symmetrisation morphism. We denote by $\mathfrak{s}_k^E$ the morphism $\mathfrak{s}_k \otimes \mathrm{id}_E : T^*M^{\otimes k} \otimes E \to S^k(T^*M) \otimes E$. Set $P_k^E = \mathfrak{s}_k^E \circ \nabla^{(k)}$, then the principal symbol of P_k^E is an element of $S^k(TM) \otimes E^* \otimes S^k(T^*M) \otimes E$. Prove that if we see $\sigma(P_k^E)$ as an element of $\mathrm{End}\,(S^k(T^*M) \otimes E)$, it equals the identity.

(1c) Deduce a new proof of the fact that $\sigma : \mathscr{D}\mathrm{iff}^{\leqslant k}(E,F) \to C^\infty(M, \mathrm{Hom}(S^k(T^*M) \otimes E, F))$ is surjective.

(2a) Let $P \in \mathscr{D}\mathrm{iff}^1(E,F)$. Prove that there exists $A \in C^\infty(M, \mathrm{Hom}(E,F))$ such that $P = \sigma(P) \circ \nabla^E + A$, where $\sigma(P) \circ \nabla^E$ is defined as the composition

$$C^\infty(M,E) \xrightarrow{\nabla^E} C^\infty(M, T^*M \otimes E) \xrightarrow{\sigma(P)} C^\infty(M,F).$$

(2b) Prove that any differential operator is a sum of operators obtained by composing bundle morphism with connections.

(2c) Deduce that every differential operator has a formal adjoint.

Proof. (1a)

4 Clifford Algebra and Dirac Operators

In this chapter, we aim to solve the equation $D^2 = \Delta$ in some sense.

Definition 4.1: (Generalised) Laplacian, Dirac operator

A *generalised Laplacian* on an Hermitian vector field E over a Riemannian oriented compact manifold M is $L \in \mathscr{D}\mathrm{iff}^2(M, E)$ such that

- L is symmetric.
- $\forall (x, \xi) \in T^*M$, $\sigma(L)(x, \xi) = |\xi|^2 \operatorname{id}_{E_x}$.

And a *Dirac operator* is a self-adjoint operator $D \in \mathscr{D}\mathrm{iff}^1(E, E)$ such that $D^2 = L$.

The operator L is called a Laplacian because locally $\xi = [x_1, \dots, x_n]^\top$, we have

$$L = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \text{lower order terms.}$$

If $E = E^+ \oplus E^-$ is a superspace, then $D \in \mathscr{D}\mathrm{iff}^1(E^+, E^-)$ is a Dirac operator if and only if $\mathscr{D} = \begin{bmatrix} & D^* \\ D & \end{bmatrix} \in \mathscr{D}\mathrm{iff}^1(E, E)$ is.

4.1 Clifford algebra and Spin groups

If D is a Dirac operator, for $x \in M$, set $c(\xi) = \mathrm{i}\sigma(D)(x, \xi)$, then $c(\xi)^2 = -|\xi|^2$. By polarisation,

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2\langle \xi, \eta \rangle.$$

We can use these relation to define an algebra C, which we will study, along with its representations $c : C \to \operatorname{End} E$.

In this section, we shall fix $(V, \langle \cdot, \cdot \rangle) = (V, q)$ an n-dimensional Euclidean space.

(α) **Clifford algebra**

Definition 4.2: Clifford algebra

The *Clifford algebra* of (V, q) is the $\mathbb{R}$-algebra generated by $\{1, v \in V\}$ and relations

$$vw + wv = -2\langle v, w \rangle, \qquad \forall v, w \in V,$$

denoted as $\mathrm{Cl}(V, q)$ or just $\mathrm{Cl}(V)$.

We have a concrete realisation: let $\mathscr{I} \lhd T(V)$ generated by $X = \{v \otimes w + w \otimes v - 2\langle v, w \rangle : v, w \in V\}$, then

$$\mathrm{Cl}(V) = T(V)/\mathscr{I}.$$

Note that $T(V)$ is a superalgebra and $X \subset T^+(V)$, hence $\mathscr{I} = \mathscr{I}^+ \oplus \mathscr{I}^-$ with $\mathscr{I}^\pm = \mathscr{I} \cap T^\pm(V)$. So $\mathrm{Cl}(V)$ is also a superalgebra with

$$\mathrm{Cl}^\pm(V) = \operatorname{im}[T^\pm(V) \to T(V)/\mathscr{I}].$$

As a baby example, if $V = \mathbb{R}$, e_1 is the canonical basis of $\mathbb{R}$. Then in $\mathrm{Cl}(\mathbb{R})$, $e_1^2 = -1$, so

$$\mathrm{Cl}(\mathbb{R}) = \langle 1, e_1 : e_1^2 = -1 \rangle = \mathbb{C}.$$

Proposition 4.3: Universal property

$\mathrm{Cl}(V)$ is the unique algebra satisfying the following universal property:

(1) There exists an inclusion $V \hookrightarrow C(V)$.

(2) For all $\mathbb{R}$-algebra $\mathscr{A}$ and all morphism $\varphi : V \to \mathscr{A}$, such that

$$\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = -2\langle v, w\rangle, \qquad \forall v, w \in V.$$

There exists a unique algebra morphism $\tilde{\varphi}$ such that the diagram commutes.

$$\begin{tikzcd} \mathrm{Cl}(V) \arrow[dr, dashed, "\tilde{\varphi}"] & \\ V \arrow[u, hook] \arrow[r, "\varphi"] & \mathscr{A} \end{tikzcd}$$

Proposition 4.4

Let (V, q_V) and (W, q_W) be two Euclidean spaces,

$$f : V \otimes W \to \mathrm{Cl}(V) \hat{\otimes} \mathrm{Cl}(W), \qquad (v, w) \mapsto v \otimes 1 + 1 \otimes w$$

induces an isomorphism $\tilde{f} : \mathrm{Cl}(V \oplus W) \to \mathrm{Cl}(V) \hat{\otimes} \mathrm{Cl}(W)$.

Proof. First,

$$\begin{aligned} f(v, w)^2 &= (v \otimes 1 + 1 \otimes w)^2 \\ &= v^2 \otimes 1 + 1 \otimes w^2 + v \otimes w - v \otimes w = -|v|^2 - |w|^2 = -|(v, w)|^2. \end{aligned}$$

So $\tilde{f}$ such that the following graph commutes exists

$$\begin{tikzcd} \mathrm{Cl}(V \oplus W) \arrow[dr, "\tilde{f}"] & \\ V \oplus W \arrow[u, hook] \arrow[r, "f"] & \mathrm{Cl}(V) \hat{\otimes} \mathrm{Cl}(W) \end{tikzcd}$$

In the same way,

$$\begin{cases} \psi_1 : V \to V \oplus W \to \mathrm{Cl}(V \oplus W), \\ \psi_2 : W \to V \oplus W \to \mathrm{Cl}(V \oplus W). \end{cases} \overset{\text{induces}}{\Longrightarrow} \begin{cases} \tilde{\psi}_1 : \mathrm{Cl}(V) \to \mathrm{Cl}(V \oplus W), \\ \tilde{\psi}_2 : \mathrm{Cl}(W) \to \mathrm{Cl}(V \oplus W). \end{cases}$$

Then

$$(\tilde{\psi}_1 \otimes \tilde{\psi}_2) \circ \tilde{f} : \mathrm{Cl}(V \oplus W) \xrightarrow{\tilde{f}} \mathrm{Cl}(V) \hat{\otimes} \mathrm{Cl}(W) \xrightarrow{\tilde{\psi}_1 \otimes \tilde{\psi}_2} \mathrm{Cl}(V \oplus W)$$

satisfies that $(\tilde{\psi}_1 \otimes \tilde{\psi}_2) \circ f = \mathrm{id}$ because the following diagram commutes

$$\begin{tikzcd} \mathrm{Cl}(V \oplus W) \arrow[dr, "(\tilde{\psi}_1 \otimes \tilde{\psi}_2) \circ \tilde{f}"] & \\ V \oplus W \arrow[u, hook] \arrow[r] & \mathrm{Cl}(V \oplus W) \end{tikzcd}$$

and then done by the uniqueness. □

Corollary 4.5

Let (V, q) be an Euclidean space of dimension n, $\{e_i\}$ be an orthonormal basis. Then

$$\mathrm{Cl}(V) = \mathrm{Cl}\left(\bigoplus_{i=1}^{n} \mathbb{R} e_i\right) = \widehat{\bigotimes_{i=1}^{n}} \mathrm{Cl}(\mathbb{R} e_i) \cong (\mathbb{R} \otimes \mathbb{R} e_i)^n.$$

Hence $\dim \mathrm{Cl}(V) = 2^n$ and we have an isomorphism of vector space

$$\sigma : \mathrm{Cl}(V) \to \Lambda^\bullet V^*, \qquad e_{i_1}\cdots e_{i_k} \mapsto e^{i_1}\wedge\cdots\wedge e^{i_k}, \qquad i_1 < \cdots < i_k,$$

where $\{e^i\}$ the dual basis. The map σ is called the *symbol map.*

The inverse of σ is given by

$$\mathbf{c} : \Lambda^\bullet V^* \to \mathrm{Cl}(V), \qquad e^{i_1}\wedge\cdots\wedge e^{i_k} \mapsto e_{i_1}\cdots e_{i_k}, \qquad i_1 < \cdots < i_k,$$

called the *quantisation map.*

Denote $\mathrm{Cl}_i(V) = \mathrm{span}\{v_1\cdots v_k : k \leqslant i,\ v_i \in V\} \subset \mathrm{Cl}(V)$. Then $\mathrm{Cl}_i(V)\cdot\mathrm{Cl}_j(V) \subset \mathrm{Cl}_{i+j}(V)$. So $\{\mathrm{Cl}_i(V)\}$ is a filtration of $\mathrm{Cl}(V)$. Let $\mathrm{gr}_\bullet(\mathrm{Cl})(V) = \bigoplus_{i\geqslant 0}(\mathrm{Cl}_i(V)/\mathrm{Cl}_{i+1}(V))$ be the graded algebra associated. Then

$$v : \Lambda^\bullet V^* \to \mathrm{gr}_\bullet(\mathrm{Cl}(V)), \qquad e^{i_1}\wedge\cdots\wedge e^{i_k} \mapsto \mathrm{gr}_k(e_{i_1},\ldots,e_{i_k})$$

is an algebra isomorphism. We have

$$\begin{tikzcd}
\mathrm{Cl}_i(V) \arrow[r, "\mathrm{gr}_i"] \arrow[dr, "\sigma(\cdot)^{[i]}"'] & \mathrm{gr}_i(\mathrm{Cl}(V)) \arrow[d, "\cong"] \\
& \Lambda^i V^*
\end{tikzcd}$$

The map $-\mathrm{id} : V \to V$ induces a map $T(V) \to T(V)$ which perserves $\mathscr{I}$, thus induces a map $\varepsilon : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ with

$$\mathrm{Cl}^\pm(V) = \ker(\varepsilon \mp 1).$$

(β) **The Spin group**

Definition 4.6: Spin group

The *spin group* of (V, q) is

$$\mathrm{Spin}(V) := \{v_1\cdots v_k : v_i \in V,\ |v_i| = 1,\ k \text{ is even}\} \subset \mathrm{Cl}^+(V).$$

This is a group because $v_1\cdots v_k v_k\cdots v_1 = (-1)^k = 1$.

We denote $\mathrm{Cl}(n)$ the Clifford algebra of Euclidean space $\mathbb{R}^n$, and $\mathrm{Spin}(n)$ its spin group. If $x = v_1\cdots v_k \in \mathrm{Spin}(V)$, $x^{-1} = v_k\cdots v_1$, so if $v \in V \subset \mathrm{Cl}(V)$,

$$xvx^{-1} = v_1\cdots v_k v v_k\cdots v_1.$$

Note that $v_i v = -vv_i - 2q(v, v_i)$, hence $xvx^{-1} \in V$ is of length 1. Let

$$\rho : \mathrm{Spin}(V) \to \mathrm{GL}(V), \qquad x \mapsto [v \mapsto xvx^{-1}],$$

be a representation of $\mathrm{Spin}(V)$.

Proposition 4.7

For all $x \in \mathrm{Spin}(V)$, $\rho(x) \in \mathrm{SO}(V)$.

Proof. If $x = v_1 \in V$ with $|v_1| = 1$, then

$$xvx^-1 = v_1 v v_1^{-1} = -v_1 v v_1 = (vv_1 + 2q(v, v_1))v_1 = -(v - 2q(v, v_1)v_1).$$

This is the orthogonal reflexion of v with respect to $\langle v_1 \rangle^{\perp}$. If $x = v_1 \cdots v_k$, we repeat the above argument and get that $\rho(x)$ is a product of $k = 2\ell$-copies of orthogonal reflexions, hence $\rho(x) \in \mathrm{SO}(V)$. □

Theorem 4.8

If $n = \dim V \geqslant 2$, then $\rho : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is a non-trivial double covering. In particular, if $\dim V \geqslant 3$, as $\pi_1(\mathrm{SO}(V)) = \mathbb{Z}/2\mathbb{Z}$, ρ is the universal cover of $\mathrm{SO}(V)$.

Proof. (1) ρ is surjective because each $A \in \mathrm{SO}(V)$ can be decomposed as a product of orthogonal reflexions.

(2) We claim that $\ker \rho = \{\pm 1\}$. Since $-1 = v_1 v_1 \in \mathrm{Spin}(V)$, $|v_1| = 1$, it is obvious that

$$\rho(-1)v = (-1)v(-1)^{-1} = v,$$

thus $\ker \rho \supset \{\pm 1\}$.

For the other direction, we first claim that $\forall a \in \mathrm{Cl}^+(V)$, $\forall v \in V$, we have

$$\sigma([a, v]) = 2\iota_v \sigma(a) \in \Lambda^\bullet V^*.$$

If $v = e_i$, and for $I = \{i_1, \dots, i_k\}$ with $i_1 < \cdots < i_k$, denote $e_I = e_{i_1} \cdots e_{i_k}$, $a = e_I$. Then

$$[a, v] = e_I e_i - e_i e_I.$$

But $\forall i \neq j\ (e_i e_j + e_j e_i = 0)$, if $i \notin I$, $[a, v] = 0$ and of course $\iota_v \sigma(a) = \iota_{e_i} e_I = 0$. If $i \in I$, for example $i = i_m$, then

$$e_i e_I = (-1)^{m-1} e_{i_1} \cdots e_{i_m} e_{i_m} e_{i_{m+1}} \cdots e_{i_k} = (-1)^{m-1}(-1)^{k-m} e_{i_1} \cdots e_{i_k} e_{i_m} = -e_I e_i.$$

Thus $[a, v] = -2e_i e_I = -2(-1)^m e_{I \setminus \{i_m\}}$ and $\iota_{e_{i_m}} e^I = (-1)^{m-1} e^{I \setminus \{i_m\}}$.

Back to $x \in \ker \rho$, by the claim above, $\forall v \in V$, $\iota_v \sigma(x) = 0$ implies that $\sigma(x) \in \Lambda^0 V^*$. So $x \in \mathrm{Cl}_0(V) = \mathbb{R}$. But $|x| = 1$, so $x = \pm 1$, this implies $\ker \rho \subset \{\pm 1\}$.

(3) To prove $\mathrm{Spin}(V) \to \mathrm{SO}(V)$ is non-trivial, we need $\mathrm{Spin}(V) \ncong \mathrm{SO}(V) \times (\mathbb{Z}/2\mathbb{Z})$. As $\dim V \geqslant 2$, we can define $\gamma(t) = \exp(2te_1 e_2) \in \mathrm{Cl}^+(V)$.

$$\begin{aligned}\gamma(t) &= \sum_{j \geqslant 0} \frac{(2te_1 e_2)^{2j}}{(2j)!} + \frac{(2te_1 e_2)^{2j+1}}{(2j+1)!} = \sum_{j \geqslant 0} (-1)^j \frac{(2t)^{2j}}{(2j)!} + (-1)^j \frac{(2t)^{2j+1}}{(2j+1)!} e_1 e_2 \\ &= \cos 2t + \sin 2t \cdot e_1 e_2 = (\cos t \cdot e_1 + \sin t \cdot e_2)(-\cos t \cdot e_1 + \sin t \cdot e_2).\end{aligned}$$

Thus $\gamma(0) = e_1(-e_1) = 1$, $\gamma(\pi/2) = e_2 e_2 = -1$. So $\mathrm{Spin}(V) \ncong \mathrm{SO}(V) \times \{\pm 1\}$. □

For $k \in \{0, \dots, n\}$, set $\mathrm{Cl}^k(V) = \sigma^{-1}(\Lambda^k V^*)$.

Lemma 4.9

If $\dim V \geqslant 2$, we have $\mathfrak{spin}(V) = \mathrm{Cl}^2(V)$, where $\mathfrak{spin}(V)$ is the Lie algebra of $\mathrm{Spin}(V)$. Moreover, if ρ still denotes $d_*\rho : \mathfrak{spin}(V) \to \mathfrak{so}(V)$, then

$$\rho(a) \cdot v = [a, v] = av - va, \qquad \forall a \in \mathfrak{spin}(V),\ \forall v \in V,$$

where the product is in $\mathrm{Cl}(V)$.

Proof. Using $\gamma(t)$, we have that $\exp(te_1e_2) \in \mathrm{Spin}(V)$ and hence $e_1e_2 \in \mathfrak{spin}(V)$. But

$$\dim(\mathfrak{spin}(V)) = \dim(\mathfrak{so}(V)) = \frac{n(n-1)}{2} = \dim \mathrm{Cl}^2(V),$$

so $\mathfrak{spin}(V) = \mathrm{Cl}^2(V)$. And

$$\rho(a) \cdot v = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{ta} v \mathrm{e}^{-ta})\bigg|_{t=0} = av - va$$

gives the formula. □

As $\rho : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is a double cover, $\rho : \mathfrak{spin}(V) \to \mathfrak{so}(V)$ is an isomorphism. If $A \in \mathfrak{so}(V)$, set

$$c(A) = \frac{1}{2} \sum_{i<j} \langle Ae_i, e_j \rangle e_i e_j \in \mathrm{Cl}^2(V),$$

note that this is independent on the choice of $\{e_i\}$, and $\frac{1}{2}\sum_{i<j} = \frac{1}{4}\sum_{i,j}$.

Now, to distinguish between $e_i \in V$ and $e_i \in \mathrm{Cl}(V)$, we often denote the latter by $c(e_i) = \mathbf{c}(e^i) \in \mathrm{Cl}(V)$, where $e^i \in \Lambda^1 V^*$. If $\{e_i\}$ such that

$$A = \mathrm{diag}\left\{ \begin{bmatrix} & -\theta_1 \\ \theta_1 & \end{bmatrix}, \dots, \begin{bmatrix} & -\theta_k \\ \theta_k & \end{bmatrix} \right\},$$

then $c(A) = \frac{1}{2}\sum_{j=1}^k \theta_j c(e_{2j-1})c(e_{2j})$, and for $v = \sum_{i=1}^n v_i e_i$,

$$\rho(e_1e_2) \cdot v = [e_1e_2, v] = \sum_{i=1}^n v_i(e_1e_2e_i - e_ie_1e_2) = 2v_1e_2 - 2v_2e_1.$$

In matrix form, this is $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} & -2 \\ 2 & \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Thus we conclude $\rho(c(A)) = A$, *i.e.*, $\rho = c^{-1}$.

In summary,

$$\begin{tikzcd}
\mathfrak{so}(V) \arrow[r, "\omega_*"] \arrow[dr, "c"'] & \Lambda^2 V^* \arrow[d, "\frac{1}{2}\mathbf{c}"] & \\
& \mathrm{Cl}^2(V) & \supset \quad \mathfrak{spin}(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
A = \sum_{i,j} \langle Ae_i, e_j \rangle e^i \otimes e_j \arrow[r, mapsto] \arrow[dr, mapsto] & \omega_A = \langle A\cdot, \cdot \rangle = \frac{1}{2}\sum_{i,j} \langle Ae_i, e_j \rangle e^i \wedge e^j \arrow[d, mapsto] \\
& c(A) = \frac{1}{4}\sum_{i,j} \langle Ae_i, e_j \rangle c(e_i)c(e_j)
\end{tikzcd}$$

4.2 Clifford modules and spinors

(α) **Clifford modules**

Definition 4.10: Clifford modules

A *Clifford module* is a vector space E which is a $\mathrm{Cl}(V)$-module and which is $(\mathbb{Z}/2\mathbb{Z})$-graded such that the $\mathrm{Cl}(V)$-action is conpatible with the grading.

Example 4.11

(1) $\mathrm{Cl}(V)$ itself is a Clifford module.

(2) $\Lambda^\bullet V^*$ is a Clifford module. If $v \in V \subset \mathrm{Cl}(V)$, set

$$c(v) \cdot \alpha := v^\flat \wedge \alpha - \iota_v \alpha,$$

where $\flat : TV \to T^*V$ is the musical isomorphism. Then

$$c(v)^2 = (v^\flat \wedge - \iota_v)^2 = -v^\flat \wedge \iota_v - \iota_v v^\flat = -|v|^2$$

because

$$\iota_v(v^\flat \wedge \alpha) = \iota_v v^\flat \wedge \alpha - v^\flat \wedge \iota_v \alpha = |v|^2 \wedge \alpha - v^\flat - \iota_v \alpha.$$

Hence this extend to $c : \mathrm{Cl}(V) \to \mathrm{End}\,(\Lambda^\bullet V^*)$.

Let $\{e_i\}$ be an orthonormal basis of V. Define

$$G = \left\langle -1, e_1, \ldots, e_n : (-1)^2 = -e_i^2 = 1,\ \forall i \neq j\,(e_i e_j + e_j e_i = 0) \right\rangle \subset \mathrm{Cl}(V).$$

There is an one-to-one correspondence between

$$\left\{\text{complex Cl}(V)\text{-modules of finite dim}\right\} \longleftrightarrow \left\{\text{complex finite dim'l rep. } \pi \text{ of } G \text{ s.t. } \pi(-1) = -\mathrm{id}\right\},$$

here G, as we defined above, is a finite group.

Theorem 4.12

(1) Every finite dimensional complex $\mathrm{Cl}(V)$-module is a direct sum of irreducible $\mathrm{Cl}(V)$-module.

(2) Every finite dimensional complex $\mathrm{Cl}(V)$-module admits an Hermitian metric such that

$$\langle c(v)\cdot, \cdot\rangle = -\langle \cdot, c(v)\cdot\rangle, \qquad \forall v \in V.$$

(3) Such a metric is unique up to scalar if the module is irreducible.

Definition 4.13: Self-adjoint Clifford algebra

If $(E, \langle\cdot,\cdot\rangle)$ is a Clifford module such that $\forall v \in V\,(c(v)^* = -c(v))$, we say that E is a *self-adjoint Clifford algebra.*

Remark. In this case,

$$c(a)^* = -c(a), \qquad \forall a \in \mathrm{Cl}^2(V) = \mathfrak{spin}(V).$$

So the representation of $\mathrm{Spin}(V) \subset \mathrm{Cl}^+(V)$ *induced by the module structure is unitary on* $(E, \langle\cdot,\cdot\rangle)$.

Definition 4.14: Chinality operator

Denote $\mathbb{C}\mathrm{l}(V) := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$. Assume that $\dim V$ is even and that V is oriented, $\{e_i\}$ is a directed orthonormal basis of V. Set

$$\Gamma := \mathrm{i}^{n/2} e_1 \cdots e_n \in \mathbb{C}\mathrm{l}(V),$$

the *chinality operator.* It is independent of the choice of $\{e_i\}$.

Lemma 4.15

$\Gamma^2 = 1$ and $\forall v \in V\,(\Gamma v + v\Gamma = 0)$.

Proof. Since V is of even dimension, hence

$$\Gamma^2 = \mathrm{i}^n e_1 \cdots e_n e_1 \cdots e_n = (-1)^{n/2}(-1)^n(-1)^{(n-1)(n-2)/2} = 1.$$

By linearity it suffices to prove the case $v = e_j$. Thus

$$\Gamma e_j = \mathrm{i}^{n/2} e_1 \cdots e_m e_j = \mathrm{i}^{n/2}(-1)^{n-1} e_j e_1 \cdots e_m = -e_j \Gamma.$$

Done. □

Thus, if E is a complex Clifford module, we can define $E^{\pm} := \{c(\Gamma)e = \pm e\}$. If the orientation is reserved, Γ is replaced by $-\Gamma$ and $E^{\pm}$ becomes $E^{\mp}$.

(β) **Spinors**

Here we assume that $\dim V$ is even and that V is oriented. Let J be a complex structure on V that compatible with $\langle\cdot,\cdot\rangle$, *i.e.*,

$$J \in \operatorname{End} V, \qquad J^2 = -1, \qquad \langle J\cdot, J\cdot\rangle = \langle\cdot,\cdot\rangle .$$

Then

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \ker(J - \mathrm{i}) \oplus \ker(J + \mathrm{i}) =: W \oplus W'.$$

Note that $W' = \bar{W}$. We extend $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ by $\mathbb{C}$-bilinearity to $\langle\cdot,\cdot\rangle : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$.

- $\langle\cdot,\bar{\cdot}\rangle$ is an Hermitian metric.
- $\langle W, W\rangle = 0$, $\langle W', W'\rangle = 0$.

Theorem 4.16

If V is an even dimensional oriented Euclidean space, then

(1) There exists a unique (up to isomorphism) complex $(\mathbb{Z}/2\mathbb{Z})$-graded Clifford module $S = S^+ \oplus S^-$, called the *spinor module*, such that $\mathbb{C}\mathrm{l}(V) \cong \operatorname{End} S$.

(2) If E is a $(\mathbb{Z}/2\mathbb{Z})$-graded complex Clifford module, then $E \cong F \otimes S$, where $F = \operatorname{Hom}_{\mathbb{C}\mathrm{l}(V)}(S, E) = \left\{f : S \to E : f \circ c(a)|_S = c(a)|_E \circ f\right\}$. Moreover,

$$c(a)|_E = c(a)|_S \otimes \mathrm{id}_F, \qquad \operatorname{End} F \cong \operatorname{End}_{\mathbb{C}\mathrm{l}(V)}(E).$$

The $(\mathbb{Z}/2\mathbb{Z})$-graded space F is called the *twisting space* of E.

Proof. (1) The explicit construction of S is given by $S := \Lambda^\bullet \bar{W}^*$. If $s \in S$, $w \in W$, set

$$c(w) \cdot s := \sqrt{2} w^\flat \wedge s, \qquad w^\flat(u) = \langle w, \bar{u}\rangle .$$

Set also $c(\bar{w}) \cdot s = -\sqrt{2}\iota_{\bar{w}} s$. If $x \in V \subset V_{\mathbb{C}} = W \oplus \bar{W}$, then let $v = w + \bar{w}'$. But as $v \in V$, $v = w + \bar{w}$, we define

$$c(v) = c(w) + c(\bar{w}) = \sqrt{2}(w^\flat - \iota_{\bar{w}}).$$

Then $c(v)^2 = -2|w|^2 = -|v|^2$. Hence $c : \mathrm{Cl}(V) \to \operatorname{End} V$ is a Clifford module. Note that

$$\dim_{\mathbb{C}} \mathbb{C}\mathrm{l}(V) = \dim_{\mathbb{R}} \Lambda^\bullet V^* = 2^n = \dim_{\mathbb{C}}(\Lambda^\bullet \bar{W}^*).$$

So we just need to prove that $c : \mathbb{C}\mathrm{l}(V) \to \operatorname{End} S$, the extension of c, is injective.

Let $\{w_j\}$ is an oriented orthonormal basis of W and $\{\bar{w}^j\} = \left\{w_j^\flat\right\}$ be a basis of $\bar{W}^*$. Define

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j), \qquad e_{2j} = \frac{1}{\sqrt{2}}(w_j - \bar{w}_j).$$

Then $\{e_j\}$ is an oriented orthonormal basis of V. If $a = \sum_{i_1<\cdots<i_k} a_I e_{i_1}\cdots e_{i_k} \in C(V)$, then

$$\begin{aligned} c(a) &= \sum_{J\sqcup K=I} a_{JK} c(w_{j_1})\cdots c(w_{j_p}) c(\bar{w}_{k_1})\cdots c(\bar{w}_{k_q}) \\ &= \sum_{J\sqcup K=I} b_{JK} \bar{w}^{j_1}\wedge\cdots\wedge\bar{w}^{j_p}\iota_{\bar{w}_{k_1}}\cdots\iota_{\bar{w}_{k_q}}, \end{aligned}$$

where $b_{JK} = (-1)^q 2^{(p+1)/2} a_{JK}$. Assume that $c(a)|_S = 0$ and that $b_{JK} \neq 0$ for some J, K. Let $q_0 = \inf\{|K| : \exists J\,(b_{JK}\neq 0)\}$, then $\forall K^0$ such that $|K^0| = q_0$, we have

$$0 = c(a)w^{k_1^0}\wedge\cdots\wedge w^{k_q^0} = \sum_J (-1)^{\varepsilon_k} b_{JK^0}\bar{w}^{j_1}\wedge\cdots\wedge\bar{w}^{j_k}$$

because

$$\begin{cases} \iota_{\bar{w}_K}\bar{w}^{K^0} = 0, & |K| > |K^0|, \\ \iota_{\bar{w}_K}\bar{w}^{K^0} = (-1)^{\varepsilon_k}\delta_{K=K_0}, & |K| = |K^0|, \\ b_{JK} = 0, & |K| < |K^0|, \end{cases} \implies \forall J\,(b_{JK} = 0).$$

This contradicts the definition of q_0.

(2) The unicity of S and the second point comes from the fact that for any vector space X, $\operatorname{End} X$ is a simple algebra. So it has a unique irreducible representation which is $\operatorname{End} X \curvearrowright X$. And for any finite dimensional $\operatorname{End} X$-module E,

$$\operatorname{Hom}_{\operatorname{End} X}(X,E)\otimes X \xrightarrow{\sim} E, \qquad A\otimes x \mapsto Ax$$

and $\operatorname{End}(\operatorname{Hom}_{\operatorname{End} X}(X,E)) \cong \operatorname{End}_{\operatorname{End} X} X$. □

Proposition 4.17

We have $\mathrm{Spin}(V) \subset \mathbb{C}\mathrm{l}(V)$. So $\mathrm{Spin}(V)$ acts on S and preserves $S^\pm$. Moreover, S^+ and S^- are two irreducible and non-isomorphism representation of $\mathrm{Spin}(V)$.

Proof. (1) For $v \in V$, $v = w + \bar{w}$, we have

$$c(v) = \sqrt{2}(w^\flat - \iota_{\bar{w}})$$

exchange S^+ and S^-. So $\mathrm{Spin}(V) \subset \mathrm{Cl}^+(V)$ preserves them.

(2) For this part, see **[LM16, Chap I, Prop 5.15]**. □

Let $E = F\otimes S$ be a Clifford module. As $\Gamma \curvearrowright S^\pm$ as $\pm\mathrm{id}$, we have $\mathrm{Tr}_s^S(\Gamma) = 2^{n/2}$. Moreover, Γ acts as $1\otimes\Gamma|_S$ on E. So if $A \in \operatorname{End} F \cong \operatorname{End}_{\mathrm{Cl}(V)} E$, we have

$$\mathrm{Tr}_s^S(\Gamma A) = \mathrm{Tr}_s^{F\otimes S}(\Gamma|_S\otimes A|_F) = 2^{n/2}\mathrm{Tr}_s^F(A).$$

As the object F may not actually exists, we can define:

Definition 4.18: Relative supertrace

The map

$$\mathrm{Tr}_s^{E/S} : \operatorname{End}_{\mathrm{Cl}(V)} E \to \mathbb{C}, \qquad A \mapsto 2^{-n/2}\mathrm{Tr}_s^E(\Gamma A)$$

is called the *relative supertrace*.

4.3 Dirac operators

(α) **Reminders on principal bundles**

Let M be a manifold and K be a Lie group.

Definition 4.19: Principal bundle

A K-principal bundle on M is a vector bundle $\pi : P \to M$ such that

(1) There exists an open covering $M = \bigcup_\alpha U_\alpha$ and ψ_α diffeomorphisms such that the diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\psi_\alpha", "\sim"'] \arrow[dr] & U_\alpha \times K \arrow[d, "\mathrm{pr}_1"] \\ & U_\alpha \end{tikzcd}$$

(2) If $U_\alpha \cap U_\beta \neq \varnothing$, $\exists \psi_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, K)$ such that the diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U_\alpha \cap U_\beta) \arrow[r, "\psi_\alpha"] \arrow[dr, "\psi_\beta"'] & (U_\alpha \cap U_\beta) \times K \arrow[d, "\psi_\beta \circ \psi_\alpha^{-1}"] & (x, k) \arrow[d, mapsto] \\ & (U_\alpha \cap U_\beta) \times K & (\psi_{\alpha\beta}(x), k) \end{tikzcd}$$

In this case, $P \circlearrowleft K$ and on $\pi^{-1}(U_\alpha) = U_\alpha \times K$, the action is given by

$$(x, k) \cdot k' := (x, kk').$$

Moreover, $\forall x \in M$, $\pi^{-1}(x)$ is diffeomorphic to K and $P/K = M$.

Example 4.20

(1) Let $E \to M$ be a vector bundle of rank r, and let

$$\mathrm{GL}(E) := \coprod_{x \in M} \left\{u : \mathbb{K}^r \to E_x : u \text{ is invertible}\right\}.$$

Then $\mathrm{GL}(E)$ is a $\mathrm{GL}_r(\mathbb{K})$-principal bundle, called the *frame bundle* of E.

(2) If g^E is an Euclidean metric on E, let

$$\mathrm{O}(E) := \coprod_{x \in M} \mathrm{Isom}((\mathbb{R}^r, (\mathbb{R}^r)^E), (E_x, g_x^E)).$$

Then $\mathrm{O}(E)$ is a $\mathrm{O}(n)$-principal bundle.

(3) If E is moreover oriented, we can define $\mathrm{SO}(E)$.

Let $P \to M$ be a K-principal bundle, $\rho : K \to \mathrm{GL}(V)$ be a representation. Define $P \times_\rho V = (P \times V)/K$ with $P \times V \circlearrowleft K$ by

$$(p, v) \cdot k := (pk, \rho(k^{-1})v).$$

Then $P \times_\rho V \to M$, $[p, v] \mapsto \pi(p)$ makes $P \times_\rho V$ a vector bundle on M with each fibre isomorphic to V. Indeed, if $\varphi_{\alpha\beta}$ is in the definition of P, let E be the vector bundle defined by $\left\{(U_\alpha),\ \rho \circ \varphi_{\alpha\beta}\right\}$, then

$$P \times_\rho V \xrightarrow{\sim} E = \coprod_{x \in M} U_\alpha \times V / \sim, \qquad [p_\rho \cdot v] \mapsto [x, \rho(k)v],$$

where $p_\rho = [x, k]$.

Remark. *We have $C^\infty(M, P\times_\rho V) = C^\infty(P,V)^K$ as functions (not sections).*

Example 4.21

Let $P \to M$ be a G-principal bundle, $\rho : G \to \mathrm{GL}(V)$ be a representation and the fibre product $P\times_\rho V \to M$.

(1) Let $E \to M$ be a vector bundle, then $E = \mathrm{GL}(E)\times_{\mathrm{GL}_n(\mathbb{K})} \mathbb{K}^n$.

(2) If there exists a structure on V preserved by G, then we have a structure on $P\times_\rho V$. For example, let $\langle\cdot,\cdot\rangle$ be an inner product on V and $\rho : G \to \mathrm{O}(V)$. Let $E = P\times_\rho V$ on P, then we have a constant $\langle\cdot,\cdot\rangle \in C^\infty(P, V^*\otimes V^*)$ which is G-invariant,

$$\langle\cdot,\cdot\rangle \in C^\infty(P, V^*\otimes V^*)^G = C^\infty(P, E^*\otimes E^*).$$

Then we get an inner product on E.

(3) $\mathrm{O}(n) \circlearrowright T(\mathbb{R}^n)$ and preserves $\mathscr{I} = \{u\otimes v + v\otimes u - 2\langle u,v\rangle : u,v\in\mathbb{R}^n\}$. This induces $\mathrm{O}(n) \circlearrowright \mathrm{Cl}(\mathbb{R}^n) = \mathrm{Cl}(n)$. Now if g^{TM} is a Riemannian metric on M, we can define

$$\mathrm{Cl}(TM) := \mathrm{O}(TM)\times_{\mathrm{O}(n)} \mathrm{Cl}(n),$$

called the *Clifford bundle* of (M, g^{TM}). And $\mathrm{Cl}(TM)_x = \mathrm{Cl}(T_xM, g_x^{TM})$.

(β) **Spin manifolds**

Let (M, g^{TM}) be an oriented Riemannian manifold, $\mathrm{Cl}(TM) = \mathrm{SO}(TM)_{\times \mathrm{SO}(n)}\mathrm{Cl}(n)$. We want to define $S^{TM} = \coprod_{x\in TM} S^{T_xM}$. But

- S^V is not canonically defined from $(V, \langle\cdot,\cdot\rangle)$.
- $\mathrm{SO}(TM)\times_{\mathrm{SO}(n)} S^{T_xM}$ has no meaning as $\mathrm{SO}(n)$ does not act on $S^{\mathbb{R}^n}$.

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r] \arrow[d] & \mathrm{GL}(S^{\mathbb{R}^n}) \\ \mathrm{SO}(n) \arrow[ur, dashed, "\times"] & \end{tikzcd} \qquad \begin{tikzcd} -1 \arrow[r, mapsto] \arrow[d, mapsto] & -1 \\ 1 \arrow[ur, dashed, mapsto, "\times"] & \end{tikzcd}$$

Definition 4.22: Spin manifold, spin bundle

(1) An oriented Riemannian manifold is called *spin* if there exists a $\mathrm{Spin}(n)$-principal bundle $\mathrm{Spin}(M) \to M$ and a map $\xi : \mathrm{Spin}(M) \to \mathrm{SO}(TM)$ such that

$$\begin{tikzcd} \mathrm{Spin}(M) \arrow[r, "\xi"] \arrow[dr] & \mathrm{SO}(TM) \arrow[d] \\ & M \end{tikzcd}$$

commutes, and $\forall p \in \mathrm{Spin}(M)$, $\forall k \in \mathrm{Spin}(n)$, we have $\xi(p\cdot k) = \xi(p)\cdot\rho(k)$, where $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. A chocie of such a bundle is called a *spin structure* on M. It gives a coherent way of taking the universal cover of $\mathrm{SO}(T_xM)$ for $x\in M$.

(2) If $(M, g^{TM}, \mathrm{Spin}(M))$ is a spin manifold of even dimension, we define

$$S^{TM} := \mathrm{Spin}(M)\times_{\mathrm{Spin}(n)} S^{\mathbb{R}^n}$$

the *spin bundle*. It has a unique (up to scalar) metric such that

$$\forall u \in C^\infty(M,TM)\,(c(u)^* = -c(u)), \qquad S^{TM} = S^{TM,+} \oplus S^{TM,-}.$$

Remark. *If M is spin, then $TM = \mathrm{Spin}(M) \times_\rho \mathbb{R}^n$. The existence of spin structure on M is equivalent to $w_2 = 0 \in H^2(M,\mathbb{Z}/2\mathbb{Z})$, where w_2 is the second Stiefel–Whitney class of M. In this case,*

$$\{\text{spin structure}\}/\sim \xleftrightarrow{1-1} H^1(M,\mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z}/2\mathbb{Z}).$$

(γ) **Dirac operator**

Let (M, g^{TM}) be an oriented Riemannian manifold. The metric g^{TM} induces g^{Cl} on $\mathrm{Cl}(M)$, with $\{e_i\}$ the ONB of TM corresponding to $\{e_I\}$ the ONB of $\mathrm{Cl}(M)$. Let ∇^{TM} be the Levi–Civita connection on TM, which induces a connection ∇ on $T(TM)$ such that ∇ preserves $\mathscr{I}$, and

$$\nabla(a\otimes b) = \nabla a \otimes b + a \otimes \nabla b.$$

So it induces a connection ∇^{Cl} such that

$$\nabla^{\mathrm{Cl}}_v(v_1,\dots,v_k) = \sum_i v_1\cdots v_{i-1}(\nabla^{\mathrm{Cl}}_v v_i)v_{i+1}\cdots v_k.$$

Proposition 4.23

The connection ∇^{Cl} preserves g^{Cl}.

Proof. For $\{e_i\}$ an ONB of TM, we know that

$$g^{\mathrm{Cl}}(e_i,e_j) = g^{TM}(e_i,e_j) = \delta_{ij}.$$

And extend g^{Cl} by multilinearity. Then for any vector field v,

$$v(g^{TM}(e_i,e_j)) = g^{TM}(\nabla^{TM}_v e_i, e_j) + g^{TM}(e_i, \nabla^{TM}_v e_j) = 0.$$

And by the bilinearity of g^{Cl}, it suffices to prove that for $a = e_I$, $b = e_J$, $g^{\mathrm{Cl}}(e_I,e_J) = 0$ if $I\cap J \neq \varnothing$. By induction, observe that

$$v(g^{\mathrm{Cl}}(e_I,e_J)) = \sum_{m=1}^k \left(g^{\mathrm{Cl}}(e_{i_1}\cdots(\nabla^{\mathrm{Cl}}_v e_{i_m})\cdots e_{i_k}, e_J) + g^{\mathrm{Cl}}(e_I, b_{j_1}\cdots(\nabla^{\mathrm{Cl}}_v b_{j_m})\cdots e_{j_\ell})\right).$$

Then done by induction. □

Definition 4.24: Clifford module, Clifford connection

(1) A $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $E\to M$ is a *Clifford module* if there is a $\mathrm{Cl}(M)$-action

$$C^\infty(M,\mathrm{Cl}(M))\times C^\infty(M,E)\to C^\infty(M,E), \qquad (a,s)\mapsto c(a)\cdot s,$$

and such that $c(v): E^\pm \to E^\mp$ for $v\in TM$.

(2) If $h^E = h^{E,+}\oplus h^{E,-}$ is a metric, (E,h^E) is *self-adjoint* if $c(v)^* = -c(v)$, $v\in TM$.

(3) A connection ∇^E on E, preserving $E^\pm$ is called a *Clifford connection* if

$$\nabla^{\mathrm{End}\,E}c(a) = c(\nabla^{\mathrm{Cl}}a), (\Longleftrightarrow [\nabla^E, c(a)] = c(\nabla^{\mathrm{Cl}}(a)))$$

for $a \in C^\infty(M, \mathrm{Cl}(M))$.

Assume that M is spin, then ∇^{TM} induces $\nabla^{S^{TM}}$, still called the Levi–Civita connection. Indeed,

$$\begin{aligned} TM &= \mathrm{Spin}(M) \times_\rho \mathbb{R}^n, & \rho &: \mathrm{Spin}(n) \to \mathrm{SO}(n), \\ S^{TM} &= \mathrm{Spin}(M) \times_\lambda S^{\mathbb{R}^n}, & \lambda &: \mathrm{Spin}(n) \to \mathrm{GL}(S^{\mathbb{R}^n}). \end{aligned}$$

Locally, $\nabla^{TM} = d + \Gamma^{TM}_\alpha$, where $\Gamma^{TM}_\alpha \in \Omega^1(U_\alpha, \mathfrak{so}(n))$. Recall also

$$d_1\rho : \mathfrak{spin}(n) \xrightarrow{\sim} \mathfrak{so}(n), \qquad \mathrm{Cl}(A) \leftarrow A.$$

(here 1 denotes the identity of the group.) Thus Γ^{TM}_α gives $c(\Gamma^{TM}_\alpha) \in \Omega^1(U_\alpha, \mathfrak{spin}(n))$ and $\nabla^{S^{TM}} := d + d_1\lambda(c(\Gamma^{TM}_\alpha))$ locally, where $d_1\lambda : \mathfrak{spin}(n) \to \mathfrak{gl}(S^{\mathbb{R}^n})$.

If $\mathrm{Spin}(M)$ is defined by $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(n)$, then TM is defined by $\rho \circ \psi_{\beta\alpha}$ and S^{TM} is defined by $\lambda \circ \psi_{\beta\alpha}$. Hence $\{d + \Gamma^{TM}_\alpha\}$ glues into ∇^{TM} if and only if

$$(\rho \circ \psi_{\beta\alpha})(d + \Gamma^{TM}_\alpha)(\rho \circ \psi_{\beta\alpha})^{-1} = d + \Gamma^{TM}_\beta.$$

This implies

$$(\lambda \circ \psi_{\beta\alpha})(d + d_1\lambda(c(\Gamma^{TM}_\alpha)))(\lambda \circ \psi_{beta\alpha})^{-1} = d + d_1\lambda(c(\Gamma^{TM}_\beta)).$$

So $\nabla^{S^{TM}}$ is globally defined.

Proposition 4.25

If M is a spin manifold,

(1) $\nabla^{S^{TM}}$ is an Hermitian Clifford connection.

(2) Its curvature is

$$R^{S^{TM}}(\cdot,\cdot) := c(R^{TM}(\cdot,\cdot)) = \frac{1}{4}\sum_{i,j} \left\langle R^{TM}(\cdot,\cdot)e_i, e_j \right\rangle c(e_i)c(e_j).$$

Proof. (1) $\nabla^{S^{TM}}$ is Hermitian if and only if $c(\Gamma^{TM}_\alpha)^* = -c(\Gamma^{TM}_\alpha)$. This is done because

$$(c(e_i)c(e_j))^* = -c(e_i)c(e_j).$$

Since $\{e_i\}$ is an ONB of $\mathbb{R}^n$ independent of $x \in U_\alpha$,

$$\begin{aligned} [\nabla^{S^{TM}}, c(u)] &= \nabla^{S^{TM}}\left(\sum_i \langle u, e_i\rangle\, c(e_i)\right) - \sum_i \langle u, e_i\rangle\, c(e_i)\nabla^{S^{TM}} \\ &= \sum_i (d\,\langle u, e_i\rangle)c(e_i) + \sum_i \left\langle u(e_i), [\nabla^{S^{TM}}, c(e_i)]\right\rangle, \end{aligned}$$

where

$$[\nabla^{S^{TM}}, c(e_i)] = [d + c(\Gamma^{TM}_\alpha), c(e_i)] = [c(\Gamma^{TM}_\alpha), c(e_i)] = c(\Gamma^{TM}_\alpha e_i).$$

The last equality results from the fact that $c(A)$ acts on $\mathbb{R}^n = \mathrm{Cl}^1(\mathbb{R}^n)$ as A since $c = \rho^{-1}$. Thus,

$$\begin{aligned} [\nabla^{S^{TM}}, c(u)] &= \sum_i (d\,\langle u(e_i)\rangle)c(e_i) + \sum_i u(e_i)c(\gamma^{TM}_\alpha e_i) \\ &= c\left(\sum_i (d\,\langle u, e_i\rangle)e_i + \langle u, e_i\rangle\, \Gamma^{TM}_\alpha e_i\right) = c(\nabla^{TM} u). \end{aligned}$$

(2) Since $\nabla = d + \Gamma$, $R = d\Gamma + \Gamma \wedge \Gamma$. So

$$R^{S^{TM}} = dc(\Gamma_\alpha^{TM}) + c(\Gamma_\alpha^{TM}) \wedge c(\Gamma_\alpha^{TM}) =: A + B.$$

Note that $c(e_i)c(e_j)$ is constant,

$$A = \frac{1}{4} d(\langle \Gamma_\alpha^{TM} e_i, e_j \rangle c(e_i)c(e_j)) = \frac{1}{4} d \langle \Gamma_\alpha^{TM} e_i, e_j \rangle c(e_i)c(e_j).$$

And since c is a Lie algebra homomorphism,

$$\begin{aligned}
B &= \sum_{i,j} e^i c(\Gamma_\alpha^{TM} e_i) \wedge e^j c(\Gamma_\alpha^{TM} e_j) = \sum_{i,j} e^i \wedge e^j [c(\Gamma_\alpha^{TM} e_i), c(\Gamma_\alpha^{TM} e_j)] \\
&= \sum_{i<j} e^i \wedge e^j c([\Gamma_\alpha^{TM} e_i, \Gamma_\alpha^{TM} e_j]) = \sum_{i,j} c\big(e^i \Gamma_\alpha^{TM} e_i \wedge e^j \Gamma_\alpha^{TM} e_j\big) \\
&= c(\Gamma_\alpha^{TM} \wedge \Gamma_\alpha^{TM}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
R^{S^{TM}} &= \frac{1}{4} d \langle \Gamma_\alpha^{TM} e_i, e_j \rangle c(e_i)c(e_j) + c(\Gamma_\alpha^{TM} \wedge \Gamma_\alpha^{TM}) \\
&= c(d\Gamma_\alpha^{TM}) + c(\Gamma_\alpha^{TM} \wedge \Gamma_\alpha^{TM}) = c(R^{TM}).
\end{aligned}$$

Then we conclude the proof. $\square$

Theorem 4.26

If M is spin, every Clifford module E can be written as $E = F \otimes S^{TM}$ with $F = \mathrm{Hom}_{\mathrm{Cl}(M)}(S^{TM}, E)$ and $\mathrm{Cl}(M) \circlearrowright F$ trivially. Moreover, if ∇^E is a Clifford connection, there exists a unique ∇^F such that

$$\nabla^E = \nabla^F \otimes 1 + 1 \otimes \nabla^{S^{TM}}.$$

The vector bundle F is called the *twisting bundle* of E.

Proof. Set $F_x = \mathrm{Hom}_{\mathrm{Cl}(M)_x}(S_x^{TM}, E_x)$, thus $E = F \otimes S^{TM}$. Define ∇^F by

$$(\nabla_u^F f)(s) := \nabla_u^E(f(s)) - f(\nabla_u^{S^{TM}} s), \qquad \forall s \in C^\infty(M, S^{TM}).$$

And for any $\varphi \in C^\infty(M)$, we have

$$(\nabla_u^F f)(\varphi s) = \varphi(\nabla_u^F f)(s) \implies \nabla_u^F \in C^\infty(M, \mathrm{Hom}(S^{TM}, E)).$$

Now

$$(\nabla_u^F f)(c(a)\cdot s) = \nabla_u^E(\rho(c(a)\cdot s)) - f(\nabla_u^{S^{TM}}(c(a)\cdot s)) = c(a)\cdot(\nabla_u^E f)(s).$$

Thus $\nabla_u^F f \in C^\infty(M, F)$. $\square$

If M is not spin, S^{TM} exists locally but not globally. However, $R^{S^{TM}} = c(\mathbb{R}^{TM})$ exists globally.

Proposition 4.27

For each manifold M and any Clifford module E, there exists an isomorphism $\mathrm{End}\, E \xrightarrow{\sim} \mathrm{Cl}(M) \otimes \mathrm{End}_{\mathrm{Cl}(M)} E$ under which

$$R^E \in \Omega^2(M, \mathrm{End}\, E) \cong \Omega^2(M, \mathrm{Cl}(M) \otimes_{\mathrm{End}\,\mathrm{Cl}(M)} E)$$

decomposes as $R^E = c(R^{TM}) \otimes 1 + 1 \otimes R^{E/S}$, where $R^{E/S} \in \Omega^2(M, \mathrm{End}_{\mathrm{Cl}(M)} E)$ is called the *twisting curvature.*

Note that if M is spin, $E = F \otimes S^{TM}$, we have

$$\nabla^E = \nabla^F \otimes 1 + 1 \otimes \nabla^{S^{TM}}, \qquad R^E = R^F \otimes 1 + 1 \otimes R^S,$$

so $R^F = R^{E/S}$, with $\mathrm{Cl}(M) \otimes \mathbb{C} = \operatorname{End} S$ and $\operatorname{End} F = \operatorname{End}_{\mathrm{Cl}(M)} E$.

Proof. Define $F^{E/S} = R^E - c(R^{TM}) \otimes 1 \in \Omega^2(M, \operatorname{End} E)$. Let us prove that $F^{E/S} \wedge : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$ commutes with $c(v)$ for all $v \in TM$. If this is proved, then $F^{E/S} \in \Omega^2(M, \operatorname{End}_{\mathrm{Cl}(M)} E)$ so it is of the form $1 \otimes R^{E/S}$.

We calculate

$$[R^E, c(v)] = [\nabla^E, [\nabla^E, c(v)]] = [\nabla^E, c(\nabla^{TM} v)] = c(R^{TM} v).$$

Recall that $\rho : \mathfrak{spin}(V) = \mathrm{Cl}^2(V) \to \mathfrak{so}(V)$ defines the action of a on V, and is given by

$$\rho(a) \cdot v := [a, v] \in \mathrm{Cl}^1(V) = V.$$

So for $A \in \mathfrak{so}(V)$, $c(A) \in \mathrm{Cl}^2(V)$ acts on V as A, $[c(A), v] = Av \in \mathrm{Cl}^1(V)$ because $c = \rho^{-1}$. Applied here in $\mathrm{Cl}^1(TM)$, and making $\mathrm{Cl}(TM)$ acts on E,

$$[c(R^{TM}), \underbrace{c(v)}_{\text{acts on } E}\ ] = c(\underbrace{R^{TM} v}_{\text{acts on } E}).$$

Then we conclude $[F^{E/S}, c(v)] = 0$. $\square$

Definition 4.28: Dirac operator

Let ∇^E be an Hermitian Clifford connection on a $\mathbb{Z}/2\mathbb{Z}$-graded self-adjoint Clifford module E. The *Dirac operator* associated with ∇^E is

$$D^E : \Gamma(E) \xrightarrow{\nabla^E} \Gamma(T^*M \otimes E) \xrightarrow{g^{TM}} \Gamma(TM \otimes E) \xrightarrow{u \otimes e \mapsto c(u) \cdot e} \Gamma(E).$$

Locally we have $D^E = \sum_{i=1}^n c(e_i) \nabla^E_{e_i}$ if $\{e_i\}$ is an ONB of TM.

Proposition 4.29

D^E is a formally self-adjoint Dirac type operator.

Proof. Locally $\nabla^E = d + \Gamma^E$. So for $\xi = \sum_j \xi_j e^j \in T^*_x M$ and $f \in C^\infty(M)$ such that $d_x f = \xi$,

$$\begin{aligned}\sigma(D^E)(x,\xi) = \mathrm{i}[D^E, f] &= \mathrm{i} \sum_j (c(e_j) \nabla^E_{e_j} f - f c(e_j) \nabla^E_{e_j}) \\ &= \mathrm{i} \sum_j c(e_j) d_x f(e_j) \xi_j = \mathrm{i} \sum_j \xi_j c(e_j) = \mathrm{i} c(\xi^\sharp).\end{aligned}$$

So $\sigma(D^E)(x,\xi) = \mathrm{i}c(\xi^\sharp)$ and $\sigma(D^E)(x,\xi)^2 = -c(\xi^\sharp)^2 = |\xi|^2$.

For $s_1, s_2 \in \Gamma(E)$,

$$\begin{aligned}\langle D^E s_1, s_2 \rangle_x &= \sum_j \left\langle c(e_j) \nabla^E_{e_j} s_1, s_2 \right\rangle_x = -\sum_j \left\langle \nabla^E_{e_j} s_1, c(e_j) s_2 \right\rangle_x \\ &= -\left(\sum_j (d_x \langle s_1, c(e_j) s_2 \rangle) e_i - \left\langle s_1, \nabla^E_{e_j} c(e_j) s_2 \right\rangle_x \right) \qquad (4.1) \\ &= -\sum_j (d_x \langle s_1, c(e_j) s_2 \rangle) e_i + \sum_j \left\langle s_1, c(\nabla^{TM}_{e_j} e_j) + c(e_j) \nabla^E_{e_j} s_2 \right\rangle.\end{aligned}$$

If V is Euclidean, $V^*\otimes V^* \xrightarrow{\sim} V^*\otimes V = \operatorname{End} V$, $\varphi\mapsto M_\varphi$. We have a trace Tr defined on $V^*\otimes V^*$. If $\{v_j\}$ is an ONB and $\varphi\in V^*\otimes V^*$,

$$\operatorname{Tr}\varphi := \sum_j \varphi(e_j,e_j).$$

For $\beta\in\Omega^1(M)$, $\nabla^{T^*M}\beta\in\Gamma(T^*M\otimes T^*M)$, so we can define

$$\operatorname{Tr}(\nabla^{T^*M}\beta) := \sum_j (\nabla^{T^*M}\beta)(e_j,e_j) = \sum_j d(\beta(e_j)e_j) - \beta(\nabla^{TM}_{e_j}e_j).$$

In (4.1), set $\alpha(u) = \langle s_1, c(u)s_2\rangle\in\Omega^1(M)$. Then

$$\left\langle D^E s_1, s_2\right\rangle_x = \left\langle s_1, D^E s_2\right\rangle_x - \operatorname{Tr}(\nabla^{T^*M}\alpha)_x.$$

So it suffices to prove $\int_M \operatorname{Tr}(\nabla^{T^*M}\alpha)\,\mathrm{dvol}_M = 0$. This is proved by the following lemma. □

Lemma 4.30

For any $\beta\in\Omega^1(M)$, $\int_M \operatorname{Tr}(\nabla^{T^*M}\beta)\,\mathrm{dvol}_M = 0$.

Proof. Let $w\in\Gamma(TM)$, $w=\beta^\sharp$. Then

$$\begin{aligned}
\mathscr{L}_w\,\mathrm{dvol}_M &= \mathscr{L}_w(e^1\wedge\cdots\wedge e^n) = \sum_i e^1\wedge\cdots\wedge \overbrace{\mathscr{L}_w e^i}^{=\sum_j(\mathscr{L}_w e^i,e_j)e^j}\wedge\cdots\wedge e^n\\
&= \sum_i \overbrace{(\mathscr{L}_w e^i, e_i)}^{=-(e^i,\mathscr{L}_w e_i)=-\langle e_i,[w,e_i]\rangle}\,\mathrm{dvol}_M \overbrace{=}^{\nabla^{TM}\text{ is torsion free}} -\sum_i \left\langle e^i, \nabla^{TM}_w e_i - \nabla^{TM}_{e_i}w\right\rangle \mathrm{dvol}_M\\
&= \sum_i \left\langle \nabla^{TM}_{e_i}w, e_i\right\rangle \mathrm{dvol}_M = \sum_i \left((d\langle w,e_i\rangle)e_i - \left\langle w, \nabla^{TM}_{e_i}e_i\right\rangle\right)\mathrm{dvol}_M\\
&= \operatorname{Tr}(\nabla^{T^*M}\beta)\,\mathrm{dvol}_M.
\end{aligned}$$

Thus

$$\int_M \operatorname{Tr}(\nabla^{T^*M}\beta)\,\mathrm{dvol}_M = \int_M \mathscr{L}_w\,\mathrm{dvol}_M = \int_M (d\iota_w + \iota_w d)\mathrm{vol}_M = 0$$

by Cartan's formula and Stokes' formula. □

Remark. *We have* $\Gamma(E) = \Gamma(E^+)\oplus\Gamma(E^-)$ *and* $D^E = \begin{bmatrix} & D^E_- \\ D^E_+ & \end{bmatrix}$ *with* $D^E_- = (D^E_+)^*$.

Consider $F\to M$ a vector bundle, ∇^F its connection. Let

$$\Delta^F : \Gamma(F) \xrightarrow{\nabla^F} \Gamma(T^*M\otimes F) \xrightarrow{\nabla^{T^*M\otimes F}} \Gamma(T^*M\times T^*M\otimes F) \xrightarrow{-\operatorname{Tr}} \Gamma(F)$$

be the *Bochner Laplacian.*

Observe that

$$(\nabla^{T^*M\otimes F}\nabla^F s)(X,Y) = (\nabla^F_X\nabla^F_Y - \nabla^F_{\nabla^{TM}_X Y})s.$$

So if $\{e_j\}$ is an ONB,

$$\Delta^F = -\operatorname{Tr}(\nabla^{T^*M\otimes F}\nabla^F) = -\sum_j (\nabla^F_{e_j}\nabla^F_{e_j} - \nabla^F_{\nabla^{TM}_{e_j}e_j}).$$

Proposition 4.31

(1) Δ^F is a generalised Laplacian.

(2) $\left\langle\Delta^F s_1, s_2\right\rangle = \left\langle\nabla^F s_1, \nabla^F s_2\right\rangle$. So Δ^F is positive and formally self-adjoint.

Proof. (1) Calculate the principal symbol, for $x \in M$ and $\xi = \sum_j \xi_j e^j \in T_x M$,

$$\sigma(\Delta^F)(x,\xi) = -\sum_j (\mathrm{i}\xi_j)^2 = |\xi|^2 .$$

Hence Δ^F is a generalised Laplacian.

(2) Let $\beta(u) = \langle \nabla^F_u s_1, s_2 \rangle$. To calculate $\mathrm{Tr}(\nabla^{TM}\beta)$, we first calculate

$$(\nabla^{TM}_{e_j}\beta)(e_j) = e_j\beta(e_j) - \beta(\nabla^{TM}_{e_j} e_j) = e_j \left\langle \nabla^F_{e_j} s_1, s_2 \right\rangle_x - \left\langle \nabla^F_{\nabla^{TM}_{e_j}} e_j s_1, s_2 \right\rangle_x .$$

Thus

$$\mathrm{Tr}(\nabla^{TM}\beta) = \sum_j e_j \left\langle \nabla^F_{e_j} s_1, s_2 \right\rangle_x - \left\langle \nabla^F_{\nabla^{TM}_{e_j}} e_j s_1, s_2 \right\rangle_x .$$

While by Leibniz rule, we have

$$\begin{aligned}
\langle \Delta^F s_1, s_2 \rangle_x &= -\sum_j \left\langle \nabla^F_{e_j} \nabla^F_{e_j} s_1, s_2 \right\rangle_x + \left\langle \nabla^F_{\nabla^{TM}_{e_j}} s_1, s_2 \right\rangle_x \\
&= -\sum_j e_j + \left\langle \nabla^F_{e_j} s_1, s_2 \right\rangle_x + \left\langle \nabla^F_{e_j} s_1, \nabla^F_{e_j} s_2 \right\rangle_x + \left\langle \nabla^F_{\nabla^{TM}_{e_j}} e_j s_1, s_2 \right\rangle_x \\
&= \sum_j \left\langle \nabla^F_{e_j} s_1, \nabla^F_{e_j} s_2 \right\rangle_x - \mathrm{Tr}(\nabla^{TM}\beta)_x .
\end{aligned}$$

By integrating on M for both sides,

$$\langle \Delta^F s_1, s_2 \rangle = \langle \nabla^F s_1, \nabla^F s_2 \rangle$$

since M is compact and has no boundary, the integral of the divergence vanishes.

Thus Δ^F is positive because $\langle \Delta^F s, s \rangle = \left\| \nabla^F s \right\|^2 \geqslant 0$. It is formally self-adjoint since

$$\langle \Delta^F s_1, s_2 \rangle = \overline{\langle \nabla^F s_2, \nabla^F s_1 \rangle} = \overline{\langle \Delta^F s_2, s_1 \rangle} = \langle s_1, \Delta^F s_2 \rangle .$$

Then we conclude the proof. □

Theorem 4.32: Lichnerowicz formula

Let D^E be the Dirac operator on E, a $\mathbb{Z}/2\mathbb{Z}$-graded self-adjoint Clifford module and ∇^E be an Hermitian Clifford connection. Then

$$(D^E)^2 = \Delta^E + \frac{r^M}{4} + \mathbf{c}(R^{E/S})$$

where r^M is the scalar curvature, $\mathbf{c}$ is the composition

$$\mathbf{c} : \Lambda^2 TM \xrightarrow{\mathbf{c}} \mathrm{Cl}^2(TM) \xrightarrow{c} \mathrm{End}\, E.$$

So that $\mathbf{c}(R^{E/S}) = \frac{1}{2}\sum_{i,j} R^{E/S}(e_i, e_j) c(e_i) c(e_j)$.

Proof. Omitted because I am lazy. □

4.4 Restate the Atiyah–Singer theorem

Let (M, g^{TM}) be a compact oriented even dimensional Riemannian manifold. Let $(E = E^+ \oplus E^-, h^E)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded self-adjoint complex Clifford module. Let ∇^E be an Hermitian Clifford connection on (E, h^E) and D^E the associated Dirac operator.

Recall that

$$\hat{A}(TM,\nabla^{TM}) = \det{}^{1/2}\left(\frac{\frac{\mathrm{i}}{4\pi}R^{TM}}{\sinh\left(\frac{\mathrm{i}}{4\pi}R^{TM}\right)}\right), \qquad \hat{A}(TM) = [\hat{A}(TM,\nabla^{TM})].$$

We extend $\mathrm{Tr}_s{}^{E/S} : \Gamma(\mathrm{End}_{\mathrm{Cl}(M)}E) \to C^\infty(M,\mathbb{C})$ to $\Omega^2(M,\mathrm{End}_{\mathrm{Cl}(M)}E) \to \Omega^2(M,\mathbb{C})$ as in Chern–Weil theory.

Definition 4.33: Relative Chern character

he *relative Chern character* of E is defined as

$$\mathrm{ch}(E/S) := [\mathrm{ch}(E/S),\nabla^E]$$

with $\mathrm{ch}(E/S,\nabla^E) = \mathrm{Tr}_s{}^{E/S}(\exp(\frac{\mathrm{i}}{2\pi}R^{E/S}))$.

Remark. *Locally, M is spin and $E = F\otimes S^{TM}$, $\nabla^E = \nabla^F\otimes 1 + 1\otimes\nabla^S$. Moreover, ∇^F preserves F^+ and F^-,*

$$\mathrm{ch}(E/S,\nabla^E) = \mathrm{Tr}_s{}^F\left(\exp\left(\frac{\mathrm{i}}{2\pi}R^F\right)\right) = \mathrm{ch}(F^+,\nabla^{F^+}) - \mathrm{ch}(F^-,\nabla^{F^-}).$$

$\mathrm{ch}(E/S,\nabla^E)$ *is thus closed and its cohomology class is independent of* ∇^E.

Theorem 4.34: Atiyah–Singer, restated

Use the same notation as before,

$$\mathrm{Ind}\, D^E_+ = \int_M \hat{A}(TM)\mathrm{ch}(E/S).$$

Remark. *Here* $\mathrm{Ind}\, D^E_+ = \dim\ker D^E_+ - \dim\ker D^E_-$, *the « super-dimension » of* $\ker D^E$.

5 Three special cases of the Atiyah–Singer theorem

In this chapter, we discuss the three special cases listed at the end of Chapter 1.

5.1 The de Rham operator

Let (M,g^{TM}) be an oriented Riemannian manifold. The metric g^{TM} induces $g^{\Lambda^\bullet T^*M}$ on $\Lambda^\bullet T^*M$, which is a Clifford module with Clifford action defined by

$$c(v)\cdot\alpha := v^\flat\wedge\alpha - \iota_v\alpha, \qquad \forall v\in TM.$$

Moreover,

$$\langle c(v)\cdot\alpha,\beta\rangle = \langle v^\flat\wedge\alpha,\beta\rangle - \langle\iota_v\alpha,\beta\rangle = \langle\alpha,\iota_v\beta\rangle - \langle\alpha,v^\flat\wedge\beta\rangle = -\langle\alpha,c(v)\cdot\beta\rangle.$$

Hence $\Lambda^\bullet T^*M$ is self-adjoint as a Clifford module.

Denote $\nabla^{\Lambda^\bullet T^*M}$ the Levi–Civita connection induced by ∇^{TM}.

Proposition 5.1

(1) $\nabla^{\Lambda^1 T^*M} = \nabla^{T^*M}$,

(2) For each $v\in TM$, $\nabla_v^{\Lambda^\bullet T^*M}(\alpha\wedge\beta) = \nabla_v^{\Lambda^\bullet T^*M}\alpha\wedge\beta + \alpha\wedge\nabla_v^{\Lambda^\bullet T^*M}\beta$,

(3) If ∇^{TM} preserves g^{TM}, then $\nabla^{\Lambda^\bullet T^*M}$ preserves $g^{\Lambda^\bullet T^*M}$.

Proof. (1) and (2) are immediate since this is how $\nabla^{\Lambda^\bullet T^*M}$ defined. To prove (3), since ∇^{TM} preserves g^{TM}, then for all $v \in TM$ and $X,\ Y \in \Gamma(TM)$,

$$v \cdot g^{TM}(X,Y) = g^{TM}(\nabla_v^{TM} X, Y) + g^{TM}(X, \nabla_v^{TM} Y).$$

Thus ∇^{T^*M} preserves g^{T^*M}. Thus for $\alpha, \beta \in \Lambda^1 T^*M$,

$$v \cdot \langle \alpha, \beta \rangle = \langle \nabla_v^{T^*M} \alpha, \beta \rangle + \langle \alpha, \nabla_v^{T^*M} \beta \rangle.$$

And one can extend to $\Lambda^\bullet T^*M$ by (2) and obtain the desired conclusion. □

Lemma 5.2

$\nabla^{\Lambda^\bullet T^*M}$ is a Clifford connection.

Proof. The connection $\nabla^{\Lambda^\bullet T^*M}$ preserves the $\mathbb{Z}/2\mathbb{Z}$-grading. Observe that:

(1) $(\nabla_w^{TM} v)^* = \nabla_w^{T^*M} v^\flat$ since

$$\nabla_w^{T^*M} v^\flat \cdot u = w(v^\flat(u)) - v^\flat(\nabla_w^{TM} u) = w\langle v, u\rangle - v^\flat(\nabla_w^{TM} u) = \langle \nabla_w^{TM} v, u \rangle.$$

(2) $\nabla_w^{\Lambda^\bullet T^*M}(\iota_v \alpha) = \iota_{\nabla_w^{TM} v}\alpha + \iota_v(\nabla_w^{\Lambda^\bullet T^*M}\alpha)$. Indeed, $\forall \beta \in \Lambda^\bullet T^*M$,

$$\begin{aligned}
\langle \nabla_w^{\Lambda^\bullet T^*M}(v^\flat \wedge \beta), \alpha \rangle &= w\langle v^\flat \wedge \beta, \alpha \rangle - \langle v^\flat \wedge \beta, \nabla_w^{\Lambda^\bullet T^*M} \alpha \rangle \\
&= w\langle \beta, \iota_v \alpha \rangle - \langle \beta, \iota_v \nabla_w^{\Lambda^\bullet T^*M} \alpha \rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \nabla_w^{\Lambda^\bullet T^*M}(v^\flat \wedge \beta), \alpha \rangle &= \langle (\nabla_w^{TM} v)^* \wedge \beta + v^\flat \wedge \nabla_w^{\Lambda^\bullet T^*M} \beta, \alpha \rangle \\
&= \langle \beta, \iota_{\nabla_w^{TM} v} \alpha \rangle + \langle \nabla_w^{\Lambda^\bullet T^*M} \beta, \iota_v \alpha \rangle \\
&= \langle \beta, \iota_{\nabla_w^{TM} v} \alpha \rangle + w\langle \beta, \iota_v \alpha \rangle - \langle \beta, \nabla_w^{\Lambda^\bullet T^*M} \iota_v \alpha \rangle.
\end{aligned}$$

So $\forall \beta \in \Lambda^\bullet T^*M$,

$$0 = \langle \beta, \iota_v \nabla_w^{\Lambda^\bullet T^*M} \alpha + \iota_{\nabla_w^{TM} v} \alpha - \nabla_w^{\Lambda^\bullet T^*M} \iota_v \alpha \rangle.$$

Therefore, using (1) and (2),

$$\begin{aligned}
[\nabla_v^{\Lambda^\bullet T^*M}, c(v)]\alpha &= \nabla_v^{\Lambda^\bullet T^*M}(v^\flat \wedge \alpha - \iota_v \alpha) - c(v)\nabla_v^{T^*M}\alpha \\
&= \nabla_v^{T^*M} v^\flat \wedge \alpha + v^\flat \wedge \nabla_v^{\Lambda^\bullet T^*M} \alpha - \iota_{\nabla_v^{TM} v}\alpha - \iota_v \nabla_v^{\Lambda^\bullet T^*M}\alpha - c(v)\nabla_v^{\Lambda^\bullet T^*M}\alpha \\
&= c(\nabla_v^{TM} v)\alpha.
\end{aligned}$$

Hence $\nabla^{\Lambda^\bullet T^*M}$ is a Clifford connection. □

Proposition 5.3

Let $\{e_i\}$ be a local ONB of TM with dual $\{e^i\}$. Then locally,

$$d = \sum_i e^i \wedge \nabla_{e_i}, \qquad d^* = -\sum_i \iota_{e_i} \nabla_{e_i},$$

and thus the Dirac operator associated with $\nabla := \nabla^{\Lambda^\bullet T^*M}$ is $D = d + d^*$.

Proof. Set $\tilde{d} = \sum_i e^i \wedge \nabla_{e_i}$. By the uniqueness, it suffices to prove that $\tilde{d}$ satisfies the properties of exterior differential.

(1) For $\alpha, \beta \in \Lambda^\bullet T^* M$,

$$\tilde{d}(\alpha \wedge \beta) = \sum_i e^i \nabla_{e_i}(\alpha \wedge \beta) = \sum_i e^i (\nabla_{e_i} \alpha \wedge \beta + \alpha \wedge \nabla_{e_i} \beta) = \tilde{d}\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \tilde{d}\beta.$$

(2) Let $f \in C^\infty(M) = \Lambda^0 T^* M$,

$$\tilde{d} f = \sum_i e^i \wedge \nabla_{e_i} f = \sum_i e^i df(e_i) = df.$$

(3) Let $f \in C^\infty(M)$,

$$\begin{aligned}\tilde{d}^2 f = \tilde{d} df &= \sum_i e^i \wedge \nabla_{e_i} df = \sum_i \sum_j e^i \wedge e^j (\nabla_{e_i} df)(e_j) \\ &= \sum_i \sum_j e^i \wedge e^j (e_i(df(e_j)) - df(\nabla_{e_i} e_j)) = \sum_i \sum_j a_{ij} e^i \wedge e^j.\end{aligned}$$

Then $a_{ij} = e_j(e_i(f)) - df(\nabla_{e_j} e_i)$. Since ∇^{TM} is torsion-free,

$$a_{ij} - a_{ji} = [e_i, e_j] f - df(\nabla_{e_j} e_i - \nabla_{e_i} e_j) = 0.$$

Thus $\tilde{d}^2 f = 0$. So $\tilde{d} = d$ by uniqueness.

Then for $\alpha, \beta \in \Lambda^\bullet T^* M$,

$$\begin{aligned}\langle d\alpha, \beta \rangle_x &= \sum_i \langle \nabla_{e_i} \alpha, \iota_{e_i} \beta \rangle_x = \sum_i e_i \langle \alpha, \iota_{e_i} \beta \rangle_x - \langle \alpha, \nabla_{e_i}(\iota_{e_i} \beta) \rangle_x e_i \\ &= \sum_i e_i \langle \alpha, \iota_{e_i} \beta \rangle_x - \left\langle \alpha, \iota_{\nabla_{e_i} e_i} \beta \right\rangle_x - \langle \alpha, \iota_{e_i} \nabla_{e_i} \beta \rangle_x \\ &= \operatorname{Tr}(\nabla^{TM} \gamma)_x + \left\langle \alpha, -\sum_i \iota_{e_i} \nabla_{e_i} \beta \right\rangle_x,\end{aligned}$$

where $\gamma(u) = \langle \alpha, \iota_u \beta \rangle$. Integrate on M then we obtain $d^* = -\sum_i \iota_{e_i} \nabla_{e_i}$. □

Definition 5.4: Hodge Laplacian

The Laplacian Δ of the elliptic complex $(\Omega^\bullet(M), d)$ is called the *Hodge Laplacian*, satisfying $\Delta = D^2$. Also, $\Delta|_{\Omega^0(M)}$ is exactly the Laplace–Beltrami operator.

By Hodge theory, we have

$$H^i_{\mathrm{dR}}(M, \mathbb{R}) = \ker \Delta|_{\Omega^i(M)} = \ker D|_{\Omega^i(M)}.$$

Also

$$\operatorname{Ind} D^+ = \dim \ker D^+ - \dim \ker D^- = \sum_k \dim H^{2k}(M) - \sum_k \dim H^{2k+1}(M) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Hence Atiyah–Singer theorem implies that

$$\chi(M) = \int_M \hat{A}(TM) \operatorname{ch}(\Lambda^\bullet T^* M / S).$$

If $\dim M = 2\ell + 1$, then $\chi(M) = 0$ and $\int_M (\text{even form}) = 0$. We now assume that $\dim M = 2\ell$ and compute $\int_M \hat{A}(TM) \operatorname{ch}(\Lambda^\bullet T^* M / S)$.

Since $R^{TM} \in \Omega^2(M, \mathfrak{so}(TM))$, locally we can decompose it with respect to a positive oriented ONB such that

$$\frac{1}{2\pi} R^{TM} = \operatorname{diag}\left\{ \begin{bmatrix} & -x_1 \\ x_1 & \end{bmatrix}, \ldots, \begin{bmatrix} & -x_\ell \\ x_\ell & \end{bmatrix} \right\}, \qquad x_i \in \Omega^2(M).$$

In particular, $\{\pm x_j\}$ are called the *Chern roots*.

We have

$$\hat{A}(TM,\nabla^{TM}) = \det{}^{1/2}\left(\frac{\frac{\mathrm{i}}{4\pi}R^{TM}}{\sinh\left(\frac{\mathrm{i}}{4\pi}R^{TM}\right)}\right) = \prod_{j=1}^{\ell}\det{}^{1/2}\left(\frac{\frac{\mathrm{i}}{2}\begin{bmatrix} & -x_j \\ x_j & \end{bmatrix}}{\sinh\left(\frac{\mathrm{i}}{2}\begin{bmatrix} & -x_j \\ x_j & \end{bmatrix}\right)}\right).$$

Observe that

$$\left(\frac{\mathrm{i}}{2}\begin{bmatrix} & -x \\ x & \end{bmatrix}\right)^{2k} = (-1)^k\left(\frac{x}{2}\right)^{2k}(-\mathrm{id})^k = \left(\frac{x}{2}\right)^{2k}\mathrm{id}.$$

So if $f(z) = \sum_{k\geqslant 0} a_{2k}z^{2k}$,

$$\det{}^{1/2}\left(f\left(\frac{\mathrm{i}}{2}\begin{bmatrix} & -x \\ x & \end{bmatrix}\right)\right) = \det{}^{1/2}\left(f\left(\frac{x}{2}\right)\mathrm{id}\right) = f\left(\frac{x}{2}\right).$$

Thus

$$\hat{A}(TM,\nabla^{TM}) = \prod_{j=1}^{\ell}\frac{x_j/2}{\sinh(x_j/2)}.$$

Now we compute $\mathrm{ch}(\Lambda^\bullet T^*M/S)$. Locally M is spin and we have an isomorphism of Clifford modules

$$\Lambda^\bullet T^*M\otimes C \cong \mathrm{Cl}(TM)\otimes\mathbb{C} \cong \mathrm{End}\,S = S^*\otimes S.$$

Thus $\Lambda^\bullet T^*M\otimes\mathbb{C} = F\otimes S$ with $F = S^*$. The $\mathbb{Z}/2\mathbb{Z}$-grading on F is given by $F^\pm = (S^*)^\pm = (S^\pm)^*$ because in this case

$$(\Lambda^\bullet T^*M)^+\otimes\mathbb{C} \cong \mathrm{Cl}(TM)^+\otimes\mathbb{C} \cong (\mathrm{End}\,S)^+ = (S^+)^*\otimes S^+ \oplus (S^-)^*\otimes S^- = (F\otimes S)^+.$$

So $F = \mathbb{S}^*$ is a $\mathbb{Z}/2\mathbb{Z}$-graded space.

Recall that

$$R^S = c(R^{TM}) = \frac{1}{4}\sum_{i,j}\langle R^{TM}e_i, e_j\rangle c(e_i)c(e_j), \qquad R^{\Lambda^\bullet T^*M} = R^F = R^{S^*} = -(R^S)^* = R^S$$

under $\mathrm{End}\,S^*\cong\mathrm{End}\,S$. Hence locally,

$$\begin{aligned}\mathrm{ch}(\Lambda^\bullet T^*M/S) &= \mathrm{ch}(F^+,\nabla^{F,+}) - \mathrm{ch}(F^-,\nabla^{F,-})\\ &= \mathrm{Tr}^{S^+}\left(\exp\left(\frac{\mathrm{i}}{2\pi}R^S\right)\right) - \mathrm{Tr}^{S^-}\left(\exp(\frac{1}{2\pi}R^S)\right) = \mathrm{Tr}_s{}^S\left(\exp\left(\frac{\mathrm{i}}{2\pi}R^S\right)\right) = \mathrm{Tr}^S\left(\Gamma\exp\left(\frac{\mathrm{i}}{2\pi}R^S\right)\right),\end{aligned}$$

where $\Gamma = \mathrm{i}^\ell e_1\cdots e_{2\ell}$ is the grading operator. For $n = 2\ell$, if $A = \mathrm{diag}\left\{\begin{bmatrix} & -\theta_1 \\ \theta_1 & \end{bmatrix},\dots,\begin{bmatrix} & -\theta_\ell \\ \theta_\ell & \end{bmatrix}\right\}\in\mathfrak{so}(V)$, we have seen that

$$c(A) = \frac{1}{2}\sum_j\theta_j c(e_{2j-1})c(e_{2j}),$$

where $c(e_{2j-1})c(e_{2j})$ commutes with each other. So

$$\exp(c(A)) = \prod_{j=1}^{\ell}\exp\left(\frac{1}{2}\theta_j c(e_{2j-1})c(e_{2j})\right) = \prod_{j=1}^{\ell}\left(\cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2}c(e_{2j-1})c(e_{2j})\right).$$

Lemma 5.5

(1) If $a\in\mathrm{Cl}^{n-1}(V)$, then $\mathrm{Tr}_s{}^S(c(a)) = 0$.

(2) If $a = \lambda e_1\cdots e_{2\ell}$, then $\mathrm{Tr}_s{}^S(c(a)) = (-2\mathrm{i})^\ell\lambda$, $\lambda\in\mathbb{R}$.

Proof. (1) Let $I\subset\{1,\dots,2k\}$ and $j\notin I$. Then $e_je_I = (-1)^{|I|}e_Ie_j$, so $[e_j,e_I] = 0$. As $[a,bc] = [a,b]c + a[b,c]$,

$$[e_j,e_je_I] = [e_j,e_j]e_I + 0 = -2e_I.$$

So $e_I = -\frac{1}{2}[e_j, e_j e_I]$, and $\mathrm{Tr}_s{}^S(c(e_I)) = 0$.

(2) Let $a = \lambda e_1 \cdots e_{2\ell} = \frac{\lambda}{\mathrm{i}^\ell}\Gamma$, then

$$\mathrm{Tr}_s{}^S(c(a)) = \mathrm{Tr}^S(\Gamma a) = \frac{\lambda}{\mathrm{i}^\ell}\mathrm{Tr}^S(\Gamma^2) = \frac{\lambda}{\mathrm{i}^\ell}\dim S = \frac{\lambda}{\mathrm{i}^\ell}2^\ell = (-2\mathrm{i})^\ell\lambda.$$

Done. □

By this lemma,

$$\begin{aligned}\mathrm{Tr}_s{}^S(\exp(c(a))) &= \mathrm{Tr}_s{}^S\left(\prod_{j=1}^{\ell}\sin\frac{\theta_j}{2}c(e_{2j-1})c(e_{2j})\right)\\ &= (-2\mathrm{i})^\ell\prod_{j=1}^{\ell}\sin\frac{\theta_j}{2} = \prod_{j=1}^{\ell}(-2\mathrm{i})\sin\frac{\theta_j}{2} = \prod_{j=1}^{\ell}2\sinh\left(-\frac{\mathrm{i}\theta_j}{2}\right).\end{aligned}$$

If we apply this to $\mathrm{ch}(\Lambda^\bullet T^*M/S, \nabla^{\Lambda^\bullet T^*M}) = \mathrm{Tr}_s{}^S\big(\exp\big(\frac{\mathrm{i}}{2\pi}c(R^{TM})\big)\big)$, by replacing θ_j by $\mathrm{i}x_j$, we get

$$\mathrm{ch}(\Lambda^\bullet T^*M/S, \nabla^{\Lambda^\bullet T^*M}) = \prod_{j=1}^{\ell}2\sinh\frac{x_j}{2}.$$

Therefore, in conclusion,

$$\hat{A}(TM, \nabla^{TM})\mathrm{ch}(\Lambda^\bullet T^*M/S, \nabla^{\Lambda^\bullet T^*M}) = \prod_{j=1}^{\ell}x_j.$$

Recall the definition of $\mathrm{Pf}(A)$ for $A \in \mathfrak{so}(V)$,

$$\mathrm{Pf}(A) = \exp(\omega_A)^{[\dim V]}/\mathrm{vol}_V.$$

When $A = \mathrm{diag}\left\{\begin{bmatrix} & -\theta_1\\ \theta_1 & \end{bmatrix}, \dots, \begin{bmatrix} & -\theta_\ell\\ \theta_\ell & \end{bmatrix}\right\}$, $\mathrm{Pf}(A) = \prod_{j=1}^{\ell}\theta_j$. Thus, the Euler form

$$e(TM, \nabla^{TM}) = \mathrm{Pf}\left(\frac{R^{TM}}{2\pi}\right) = \prod_{j=1}^{\ell}x_j.$$

And hence

$$\chi(M) = \int_M e(TM).$$

This is *Gauss–Bonnet–Chern theorem.*

5.2 Signature operator

We still consider the vector bundle $E = \Lambda^\bullet T^*M$, but with a different grading.

Definition 5.6: Hodge star operator

he *Hodge star operator* is defined as

$$*: \Lambda^j V^* \to \Lambda^{n-j}V^*, \qquad \alpha \mapsto *\alpha,$$

such that $\forall \alpha, \beta \in \Lambda^j V^*$, $\alpha \wedge *\beta = \langle\alpha, \beta\rangle \mathrm{vol}_V$.

Recall that $\Gamma = \mathrm{i}^{n/2}e_1\cdots e_n \in \mathrm{Cl}(V)$ if $\dim V = n$ is even. We can also define $\Gamma = \mathrm{i}^{(n+2)/2}e_1\cdots e_n \in \mathrm{Cl}(V)$ when n is odd. In this case,

$$\Gamma^2 = 1, \qquad \forall v \in V\,(\Gamma v + v\Gamma = 0).$$

In both cases, $\Gamma = \mathrm{i}^{\lfloor (n+1)/2\rfloor}e_1\cdots e_n$.

On $\Lambda_{\mathbb{C}} V^* = \Lambda^\bullet V^* \otimes \mathbb{C}$, we define a $\mathbb{Z}/2\mathbb{Z}$-grading by

$$(\Lambda^\bullet_{\mathbb{C}} V^*)^\pm = \ker(\Gamma \mp \mathrm{id}),$$

where Γ denotes the action of Γ on $\Lambda^\bullet_{\mathbb{C}} V^*$.

Proposition 5.7: O

$\Lambda^j_{\mathbb{C}} V*$, we have

(1) $\Gamma = \mathrm{i}^{\lfloor (n+1)/2 \rfloor} (-1)^{nj + j(j+1)/2} *$.

(2) $*^2 = (-1)^{(n-j)j} \mathrm{id}$.

Proof. We only need to check these on $e^1 \wedge \cdots \wedge e^j$ because $\{e_{i_1}, \dots, e_{i_j}\}$ can always extend to a positive oriented ONB of V.

$$\begin{aligned} c(e_k) e^i \wedge \alpha &= e^k \wedge e^i \wedge \alpha - \iota_{e_k}(e^i \wedge \alpha) \\ &= e^k \wedge e^i \wedge \alpha - \delta_{ik} \alpha + e^i \wedge \iota_{e_k} \alpha = \begin{cases} -e^i \wedge c(e_k)\alpha, & i \neq k, \\ -\alpha + e^i \wedge \alpha, & i = k. \end{cases} \end{aligned}$$

In particular, if $\alpha = e^J$ with $i \notin J$, $c(e_i) e^i \wedge \alpha = -\alpha$. Thus

$$\begin{aligned} \Gamma e^1 \wedge \cdots \wedge e^j &= \mathrm{i}^{\lfloor (n+1)/2 \rfloor} c(e_1) \cdots c(e_n) e^1 \wedge \cdots \wedge e^j \\ &= \mathrm{i}^{\lfloor (n+1)/2 \rfloor} (-1)^{j(n-j)} \underbrace{c(e_1) \cdots c(e_j)(e^1 \wedge \cdots \wedge e^j)}_{=(-1)^{j(j+1)/2} c(e_1) e^1 \wedge \cdots \wedge c(e_j) e^j} \underbrace{c(e_{j+1}) \cdots c(e_n)}_{=\sigma(e_{j+1} \cdots e_n) = e^{j+1} \wedge \cdots \wedge e^n} \cdot 1 \\ &= \mathrm{i}^{\lfloor (n+1)/2 \rfloor} (-1)^{j(n-j) + j + j(j+1)/2} * (e^1 \wedge \cdots \wedge e^j) \\ &= \mathrm{i}^{\lfloor (n+1)/2 \rfloor} (-1)^{nj + j(j+1)/2} * (e^1 \wedge \cdots \wedge e^j). \end{aligned}$$

And

$$*^2 (e^1 \wedge \cdots \wedge e^j) = (-1)^{j(n-j)} e^1 \wedge \cdots \wedge e^j$$

because $(e_{j+1}, \dots, e_n, e_1, \dots, e_j)$ can be obtained by $j(n-j)$ permutations of $(e_1, \dots, e_n)$. □

Proposition 5.8

If (M, g^{TM}) is oriented, we can define $*$ on $\Lambda^\bullet T^* M$ and then on $\Omega^j(M)$, $d^* = (-1)^{n+1+nj} * d *$.

Proof. For any $\alpha \in \Omega^{j-1}(M)$, $\beta \in \Omega^j(M)$,

$$\begin{aligned} \langle \alpha, d^* \beta \rangle = \langle d\alpha, \beta \rangle &= \int_M d\alpha \wedge * \beta = \int_M d(\alpha \wedge * \beta) - (-1)^{j-1} \alpha \wedge d(*\beta) \\ &= (-1)^j \int_M \alpha \wedge d(*\beta) = (-1)^{j + (n-j+1)(j-1)} \int_M \alpha \wedge ** d * \beta = (-1)^{nj+n+1} \langle \alpha, * d * \beta \rangle \end{aligned}$$

using Stokes' theorem. □

Since $\nabla^E = \nabla^{\Lambda^\bullet T^* M}$ as earlier, $[\nabla^E, c(a)] = c(\nabla^{TM} a)$ for $a \in TM$. But does ∇^{TM} preserve the $\mathbb{Z}/2\mathbb{Z}$-grading on E?

Lemma 5.9

$\nabla^{\mathrm{Cl}}_v \Gamma = 0$ for any $v \in TM$.

Proof. Denote c_{jk} the element $e_1 \cdots e_n$ with e_j replaced by e_k. Then

$$\nabla^{\mathrm{Cl}}_v \Gamma = \mathrm{i}^p \nabla^{\mathrm{Cl}}_v (e_1 \cdots e_n) = \mathrm{i}^p \sum_j e_1 \cdots e_{j-1} (\nabla^{TM}_v e_j) e_{j+1} \cdots e_n = \mathrm{i}^p \sum_{j \neq k} \langle \nabla^{TM}_v e_j, e_k \rangle c_{jk}.$$

Hence $c_{jk} = (-1)^{|k-j|} e_{\{1,\dots,n\}\setminus\{j,k\}}$ is symmetric in j and k. And $\langle \nabla_v^{TM} e_j, e_k \rangle$ is anti-symmetric in j,k. So $\nabla_v^{\mathrm{Cl}}\Gamma = 0$. □

As a consequence,

$$[\nabla^E, \Gamma] = c(\nabla^{\mathrm{Cl}}\Gamma) = 0.$$

So ∇^E preserves $E^\pm$. Therefore, if $\dim M$ is even, $\Lambda^\bullet_{\mathbb{C}} T^*M = E_{\mathbb{C}}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Clifford module and $\nabla^{E_{\mathbb{C}}}$ is a Clifford connection. If furthermore, $\dim M = 4k$ for some $k \in \mathbb{N}$, then Γ is real and we have the same thing for E instead of $E_{\mathbb{C}}$.

Theorem 5.10: Poincaré duality

If M is compact and oriented, then

$$Q : H^j(M,\mathbb{R}) \times H^{n-j}(M,\mathbb{R}) \to \mathbb{R}, \qquad ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$$

is a non-degenerate pairing.

Proof. By Hodge theory, for $[\alpha] \in H^j(M,\mathbb{R})$, we may assume $\alpha \in \ker\Delta$. So

$$d * \alpha = (-1)^{\varepsilon} * d^* \alpha = 0.$$

And thus

$$Q([\alpha],[*\alpha]) = \int_M \alpha \wedge *\alpha = \|\alpha\|^2 \neq 0$$

since $\alpha \neq 0$. □

Recall that if Q is non-degenerate quadratic form on V, there exists an ONB $\{e_i\}$ such that Q has matrix form

$$Q = \operatorname{diag}\{\underbrace{1,\dots,1}_{p \text{ copies}}, \underbrace{-1,\dots,-1}_{q \text{ copies}}\}.$$

Moreover, p and q does not depend on the choice of $\{e_i\}$. The *signature* of Q is $\sigma(Q) = p - q$.

If $\dim M$ is even, $\alpha,\ \beta \in H^{n/2}(M,\mathbb{R})$,

$$\int_M \alpha \wedge \beta = (-1)^{n/2} \int_M \beta \wedge \alpha.$$

So from now on, we assume $n = 4k$.

Definition 5.11: Signature

The *signature* of M is defined as the signature of

$$H^{n/2}(M,\mathbb{R}) \times H^{n/2}(M,\mathbb{R}) \to \mathbb{R}, \qquad ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta,$$

denoted as $\sigma(M)$.

Proposition 5.12

Let $D = \begin{bmatrix} & D^- \\ D^+ & \end{bmatrix}$ be the decomposition of D w.r.t. $E^\pm$. Then $\operatorname{Ind} D^+ = \sigma(M)$. (*Note that D^+ here is not the same as D^+ in section 5.1*).

Proof. Let $\{\alpha_i\}$ be a basis of $H^j(M) := \ker \Delta|_{\Omega^j(M)}$. As $[\nabla^E, \Gamma] = 0$ and $\forall v \in TM$, $\Gamma c(v) + c(v)\Gamma = 0$, we have $D\Gamma = -\Gamma D$. Moreover, $\Gamma^2 = 1$, so Γ induces an isomorphism $H^j(M) \xrightarrow{\sim} H^{m-j}(M)$. Let $\alpha_i^{\pm} := \alpha_i \pm \Gamma\alpha_i$ be the decomposition of $\alpha_i \in E^+ \oplus E^-$. Then if $j \neq n-j$, *i.e.*, $j \neq \frac{n}{2}$, $\{\alpha_i^{\pm}\}$ form a basis of $H^j(M) \oplus H^{n-j}(M)$. Moreover,

$$(H^j(M) \oplus H^{n-j}(M))^+ = \operatorname{span}\{\alpha_i^+\}, \qquad (H^j(M) \oplus H^{n-j}(M))^- = \operatorname{span}\{\alpha_i^-\}.$$

Thus

$$\begin{aligned}
\operatorname{Ind} D^+ &= \dim\ker D^+ - \dim\ker D^- \\
&= \dim(H^\bullet(M))^+ - \dim(H^\bullet(M))^- \\
&= \sum_{j\neq n/2} (\dim(H^j(M) \oplus H^{n-j}(M))^+ - \dim(H^j(M) \oplus H^{n-j}(M))^-) + \dim H^{n/2}(M)^+ - \dim H^{n/2}(M)^- \\
&= \dim H^{n/2}(M)^+ - \dim H^{n/2}(M)^-.
\end{aligned}$$

As $n = 4k$, $\Gamma = *$ on $\Lambda^{n/2} T^* M$. The Poincaré pairing gives that

$$(\alpha, \beta) = \int_M \alpha \wedge \beta = \langle \alpha, *\beta \rangle.$$

Let $\{e_i\}$ be a basis of $H^{n/2}(M)$ with

$$\|\alpha_i\| = 1, \qquad \Gamma\alpha_i = \begin{cases} \alpha_i, & 1 \leqslant i \leqslant p, \\ -\alpha_i, & p+1 \leqslant i \leqslant p+q. \end{cases}$$

In this basis, $Q = \operatorname{diag}\{1, \dots, 1, -1, \dots, -1\}$ and $\alpha_i \in H^+$ for $1 \leqslant i \leqslant p$, $\alpha_i \in H^-$ for $p+1 \leqslant i \leqslant p+q$. Thus

$$\sigma(M) = p - q = \dim H^{n/2}(M)^+ - \dim H^{n/2}(M)^- = \operatorname{Ind} D^+,$$

then we conclude the proof. □

Thus

$$\sigma(M) = \int_M \hat{A}(TM)\operatorname{ch}(E/S).$$

Recall that locally we have

$$\frac{1}{2\pi} R^{TM} = \operatorname{diag}\left\{ \begin{bmatrix} & -x_1 \\ x_1 & \end{bmatrix}, \dots, \begin{bmatrix} & -x_\ell \\ x_\ell & \end{bmatrix} \right\}, \qquad \hat{A}(TM, \nabla^{TM}) = \prod_{j=1}^{\ell} \frac{x_j/2}{\sinh(x_j/2)}.$$

Let us compute $\operatorname{ch}(E/S)$, where $E_{\mathbb{C}} = F \otimes S$ with $F \cong S^*$. Hence $\operatorname{ch}(E/S) = \operatorname{ch}(F^+) - \operatorname{ch}(F^-)$. Note that the grading on F is not the same as that in section 5.1.

We have Γ acts on $F \otimes S$ as $1 \otimes \Gamma|_S$, so

$$\mathbb{E}_{\mathbb{C}}^{\pm} = F \otimes S^{\pm}, \qquad F^+ = F = S^*, \qquad F^- = 0.$$

Hence

$$\operatorname{ch}(E/S) = \operatorname{ch}(F^+) - 0 = \operatorname{Tr}^F\left(\exp\left(\frac{\mathrm{i}}{2\pi} R^F\right)\right) = \operatorname{Tr}^S\left(\exp\left(\frac{\mathrm{i}}{2\pi} R^S\right)\right) = \operatorname{Tr}^S\left(\exp\left(\frac{1}{2\pi} c(R^{TM})\right)\right).$$

Now recall that

- When $A = \operatorname{diag}\left\{ \begin{bmatrix} & -\theta_1 \\ \theta_1 & \end{bmatrix}, \dots, \begin{bmatrix} & -\theta_\ell \\ \theta_\ell & \end{bmatrix} \right\}$,

$$\exp(c(A)) = \prod_{j=1}^{\ell} \exp\left(\frac{1}{2}\theta_j c(e_{2j-1})c(e_{2j})\right) = \prod_{j=1}^{\ell} \left(\cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2} c(e_{2j-1})c(e_{2j})\right).$$

- For $a = \lambda e_1 \cdots e_{2\ell} + a'$, $a' \in \mathrm{Cl}^{n-1}(V)$, $\lambda \in \mathbb{R}$, we have $\mathrm{Tr}^S(a) = \mathrm{Tr_s}^S(\Gamma a) = (-2\mathrm{i})^{2\ell}\lambda$. Thus

$$\begin{aligned}\mathrm{Tr}^S(\exp(c(A))) &= \mathrm{Tr_s}^S(\Gamma \exp(c(A))) \\ &= \mathrm{Tr_s}^S\left(\mathrm{i}^{2\ell} e_1 \cdots e_{2\ell} \prod_{j=1}^{\ell}\left(\cos\frac{\theta_j}{2} + \sin\frac{\theta_j}{2} e_{2j-1}e_{2j}\right)\right) = 2^{2\ell}\prod_{j=1}^{\ell}\cos\frac{\theta_j}{2} = \prod_{j=1}^{\ell} 2\cosh\left(\frac{\mathrm{i}\theta_j}{2}\right)\end{aligned}$$

Apply with $\theta_j = -\mathrm{i}x_j$, then $\mathrm{ch}(E/S) = \prod_{j=1}^{\ell} 2\cosh(x_j/2)$ and finally,

$$\hat{A}(TM, \nabla^{TM})\mathrm{ch}(E/S, \nabla^E) = \prod_{j=1}^{\ell}\frac{x_j}{\tanh x_j}.$$

Now

$$\begin{aligned}(TM, \nabla^{TM}) &= \det{}^{1/2}\left(\frac{\frac{\mathrm{i}}{4\pi}R^{TM}}{\tanh\left(\frac{\mathrm{i}}{4\pi}R^{TM}\right)}\right) \\ &= L\det{}^{1/2}\left(f\left(\frac{\mathrm{i}}{2}\mathrm{diag}\left\{\begin{bmatrix} & -x_1 \\ x_1 & \end{bmatrix}, \dots, \begin{bmatrix} & -x_\ell \\ x_\ell & \end{bmatrix}\right\}\right)\right) = \prod_{j=1}^{\ell} f\left(\frac{x_j}{2}\right),\end{aligned}$$

where $f(z) = \frac{z}{\tanh z}$. Therefore we conclude

$$\sigma(M) = 2^{n/2}\int_M L(M),$$

this is *Hirzebruch signature theorem.*

5.3 Dulbeault operator

Let X be a complex manifold with $\dim_{\mathbb{C}} X = n$, $J \in C^\infty(X, \mathrm{End}\, T_{\mathbb{R}}X)$ be a complex structure, *i.e.*, $J^2 = 1$. The tangent $T_{\mathbb{R}}X$, as a real manifold, has real dimension $2n$. Let $T_{\mathbb{C}}X = T_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$, then $\dim_{\mathbb{C}} T_{\mathbb{C}}X = 2n$. Define

$$T^{1,0}(X) = \{u \in T_{\mathbb{C}}X : Ju = \mathrm{i}u\}, \qquad T^{0,1}X = \{u \in T_{\mathbb{C}}X : Ju = -\mathrm{i}u\}.$$

Both of them have complex dimension n and $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$.

Remark. $T^{1,0}X = \overline{T^{0,1}X}$ *(the complex conjugate), and $T^{1,0}X$ is a holomorphic vector bundle on X.*

Let $\Lambda^{p,q}T^*_{\mathbb{C}}X$ be the space of (p,q)-forms, *i.e.*, $\alpha \in \Lambda^{p,q}T^*_{\mathbb{C}}X$ if and only if α is an alternative $(p+q)$-form on $T^*_{\mathbb{C}}X$ that being $\mathbb{C}$-linear in p variables and $\mathbb{C}$-antilinear in q variables. Then

$$\Lambda^k T^*_{\mathbb{C}}X = \bigoplus_{p+q=k}\Lambda^{p,q}T^*_{\mathbb{C}}X$$

as $\mathbb{R}$-vector spaces.

As X is complex, we can find a holomorphic local chart $\{z_1, \dots, z_n\}$ with $z_j = x_j + \mathrm{i}y_j$ and $J\begin{bmatrix} x_j \\ y_j\end{bmatrix} = \begin{bmatrix} -y_j \\ x_j\end{bmatrix}$. Set

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \mathrm{i}\frac{\partial}{\partial y_j}\right) \in T_{\mathbb{C}}X, \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}\right) \in T_{\mathbb{C}}X.$$

Then $T^{1,0}X = \mathrm{span}\left\{\frac{\partial}{\partial z_j}\right\}$, $T^{0,1}X = \mathrm{span}\left\{\frac{\partial}{\partial \bar{z}_j}\right\}$. Let $\{dz_j\}$ and $\{d\bar{z}_j\}$ be the corresponding dual basis, then

$$dz_j = dx_j + \mathrm{i}dy_j, \qquad d\bar{z}_j = dx_j - \mathrm{i}dy_j.$$

And $\forall \alpha \in \Lambda^{p,q}T_{\mathbb{C}}X$,

$$\alpha = \sum_{|I|=p, |K|=q}\alpha_{IK}\, dz_I \wedge d\bar{z}_K.$$

In particular, $f \in C^\infty(X,\mathbb{C})$, $df \in \Omega^1(X,\mathbb{C})$ can be decomposed as

$$df = \sum_{j=1}^{\ell} \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j = \sum_{j=1}^{\ell} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j =: \partial f + \bar{\partial} f,$$

where $\partial f \in \Omega^{1,0}(X,\mathbb{C})$, $\bar{\partial} f \in \Omega^{0,1}(X,\mathbb{C})$. Using Leibniz rule, we extend

$$\begin{cases} \partial : \Omega^{0,0}(X,\mathbb{C}) \to \Omega^{1,0}(X,\mathbb{C}) \\ \bar{\partial} : \Omega^{0,0}(X,\mathbb{C}) \to \Omega^{0,1}(X,\mathbb{C}) \end{cases} \quad \overset{\text{extend}}{\Longrightarrow} \quad \begin{cases} \partial : \Omega^{p,q}(X,\mathbb{C}) \to \Omega^{p+1,q}(X,\mathbb{C}) \\ \bar{\partial} : \Omega^{p,q}(X,\mathbb{C}) \to \Omega^{p,q+1}(X,\mathbb{C}) \end{cases}$$

Then $d = \partial + \bar{\partial}$. Thus $d^2 = 0$ implies $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.

Now take $E \to X$ a holomorphic vector bundle.

Definition 5.13: Dulbeault operator

The *Dulbeault operator* of E

$$\bar{\partial}^E : \Omega^{p,q}(X,E) \to \Omega^{p,q+1}(X,E)$$

is defined as follows:

(1) For a local holomorphic basis $\{e_k\}$ of $E \cong \mathbb{C}^r$, $s = \sum_{k=1}^r s_k e_k$, define

$$\bar{\partial}^E s := \sum_{k=1}^r (\bar{\partial} s_k) \otimes e_k, \qquad i.e., \bar{\partial}^E s = \bar{\partial} \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix}.$$

As transition maps from $\{e_k\}$ to $\{e_k'\}$ are holomorphic, *i.e.*, satisfy $\bar{\partial}\varphi = 0$, this definition is independent on the choice of the basis.

(2) For $\alpha \otimes s \in \Omega^\bullet(X,E)$, where $\alpha \in \Omega^\bullet(X,E)$, $s \in C^\infty(X,E)$,

$$\bar{\partial}^E(\alpha \otimes s) := (\bar{\partial}\alpha) \otimes s + (-1)^{\deg \alpha} \alpha \wedge \bar{\partial}^E s.$$

And it extends to $\bar{\partial}^E : \Omega^{p,q}(X,E) \to \Omega^{p,q+1}(X,E)$ by linearity. Moreover,

$$(\bar{\partial}^E)^2 s = \sum_{k=1}^r \bar{\partial}^2 s_k \otimes e_k + \bar{\partial} s_k \otimes \bar{\partial}^E s_k = 0.$$

So $(\bar{\partial}^E)^2 = 0$.

Definition 5.14: Dulbeault complex

The *Dulbeault complex* is defined as

$$0 \longrightarrow \Omega^{p,0}(X,E) \xrightarrow{\bar{\partial}^E} \Omega^{p,1}(X,E) \xrightarrow{\bar{\partial}^E} \cdots \xrightarrow{\bar{\partial}^E} \Omega^{p,n}(X,E) \xrightarrow{\bar{\partial}^E} 0$$

Its cohomology is called *Dulbeault cohomology*, denoted as $H^{p,\bullet}(X,E)$.

If X is compact, by Hodge theory $\dim H^{p,q}(X,E) < +\infty$.

Proposition 5.15

The Dulbeault complex is elliptic.

Proof. To show the Dulbeault complex is elliptic, it suffices to calculate $\sigma(\bar{\partial}^E)$. For any $x \in X$ and $\xi \in \Omega^{p,1}(X,E)$,

$$\sigma(\bar{\partial}^E)(x,\xi) = \lambda \xi^{0,1} \wedge,$$

where $\xi^{0,1}$ is the $\Omega^{0,1}(X,E)$ component of ξ. Thus the complex

$$0 \longrightarrow \Omega^{p,0}(X,E)\otimes\Omega^{0,0}(X,E) \xrightarrow{\mathrm{id}\otimes(-1)^p\xi^{0,1}\wedge} \Omega^{p,0}(X,E)\otimes\Omega^{0,0}(X,E)\otimes\Omega^{0,1}(X,E) \xrightarrow{\mathrm{id}\otimes(-1)^p\xi^{0,1}\wedge} \cdots$$

is determined by $\sigma(\bar\partial^E)$. This is the $\mathbb{Z}$-graded tensor product of the trivial complex $0\to\Omega^{p,0}(X,E)\xrightarrow{\mathrm{id}}\Omega^{p,0}(X,E)\to 0$ and the Koszul complex

$$0 \longrightarrow \Omega^{0,0}(X,E) \xrightarrow{\mathrm{id}\otimes(-1)^p\xi^{0,1}\wedge} \Omega^{0,1}(X,E) \xrightarrow{\mathrm{id}\otimes(-1)^p\xi^{0,1}\wedge} \cdots$$

Since $\xi^{0,1}\neq 0$ for any $\xi\neq 0$, thus the Koszul complex is exact and hence the Dolbeault complex is elliptic.

□

We set

$$\chi(X,E) := \sum_{j=0}^{n}(-1)^j \dim H^{0,j}(X,E).$$

Theorem 5.16: Riemann–Roch–Hirzebruch

For any compact complex manifold X, for any holomorphic vector bundle $E\to X$,

$$\chi(X,E) = \int_X \mathrm{Td}(T^{1,0}X)\mathrm{ch}(E).$$

To show this theorem, we set a Riemannian metric $g^{T_{\mathbb{R}}X}$ such that $J\in \mathrm{O}(g^{T_{\mathbb{R}}X})$. To simply the notations, we denote

$$g_{ij} = g^{T_{\mathbb{R}}X}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right), \qquad g_{i\bar j} = g^{T_{\mathbb{R}}X}(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar z_j}).$$

Then

$$g_{ij} = g\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right) = g\left(J\frac{\partial}{\partial z_i},J\frac{\partial}{\partial z_j}\right) = -g_{ij},$$

this implies $g_{ij}=0$. Similarly, $g_{i\bar j} = g_{\bar j i} = \overline{g_{\bar i j}}$, and $g_{\bar i\bar j}=0$. Therefore $\omega = g^{T_{\mathbb{R}}X}(\cdot, J\cdot)\in\Omega^{1,1}(X,\mathbb{C})$.

Definition 5.17: Kähler manifold

We say that (X,J,ω) is *Kähler* if $d\omega=0$.

Lemma 5.18

Using the same notations as above,

(1) $d\omega=0$ if and only if $\frac{\partial g_{i\bar j}}{\partial z_k} = \frac{\partial g_{k\bar j}}{\partial z_i}$ for any $i,j,k\in\{1,\dots,n\}$.

(2) For any $i,j\in\{1,\dots,n\}$, we have

$$\nabla_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_j}\in T^{1,0}X, \qquad \nabla_{\frac{\partial}{\partial \bar z_i}}\frac{\partial}{\partial z_j} = 0.$$

And thus there exists $\left\{\Gamma_{ij}^k\right\}$, the Christoffel symbols, such that

$$\nabla_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_j} = \sum_k \Gamma_{ij}^k\frac{\partial}{\partial z_k}, \qquad \nabla_{\frac{\partial}{\partial \bar z_i}}\frac{\partial}{\partial \bar z_j} = \sum_k \Gamma_{\bar i\bar j}^{\bar k}\frac{\partial}{\partial \bar z_k}, \qquad \overline{\Gamma_{ij}^k} = \Gamma_{\bar i\bar j}^{\bar k}.$$

Proof. (1) By definition

$$-\mathrm{i}d\omega = \sum_{i,j} dg_{i\bar j}\wedge dz_i\wedge d\bar z_j = \sum_{i,j}\left(\sum_k \frac{\partial g_{i\bar j}}{\partial z_k}dz_k + \frac{\partial g_{i\bar j}}{\partial \bar z_k}d\bar z_k\right)\wedge dz_i\wedge d\bar z_j.$$

Therefore

$$d\omega = 0 \quad \Longleftrightarrow \quad \begin{cases} \frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i} \\ \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}_j} \end{cases} \quad \Longleftrightarrow \quad \frac{\partial g_{j\bar{i}}}{\partial z_k} = \frac{\partial g_{k\bar{i}}}{\partial z_j}.$$

The second equivalence comes from $\frac{\partial g_{\bar{i}j}}{\partial z_k} = \overline{\left(\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k}\right)} = \overline{\left(\frac{\partial g_{j\bar{i}}}{\partial \bar{z}_k}\right)}$.

(2) The Koszul formula gives that for all $u, v, w \in T_{\mathbb{C}}X$,

$$2g(\nabla_u v, w) = ug(v,w) + vg(w,u) + wg(u,v) + g([u,v]w) - g([v,w]u) + g([w,u],v).$$

Then by (1) and the above formula,

$$g\left(\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = 0, \quad g\left(\nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \frac{\partial g_{k\bar{i}}}{\partial z_j} - \frac{\partial g_{\bar{i}j}}{\partial z_k} = 0, \quad g\left(\nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k}\right) = \frac{\partial g_{j\bar{k}}}{\partial \bar{z}_i} - \frac{\partial g_{\bar{i}j}}{\partial \bar{z}_k} = 0.$$

Therefore $g\left(\nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j}, T^{1,0}X\right) = g\left(\nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j}, T_{\mathbb{C}}X\right) = 0.$ □

Proposition 5.19

Denote $\nabla^{T_{\mathbb{R}}X}$ and $\nabla^{T_{\mathbb{C}}X}$ the Levi–Civita connection on $T_{\mathbb{R}}X$ and $T_{\mathbb{C}}X$. The following are equivalent.

(1) X is Kähler.

(2) $\nabla^{T_{\mathbb{C}}X}$ preserves $T^{1,0}X$ and $T^{0,1}X$.

(3) $\nabla^{T_{\mathbb{R}}X} J = 0$.

Proof. (1) $\Longrightarrow$ (2): By the previous lemma, since $T^{1,0}X = \operatorname{span}\left\{\frac{\partial}{\partial z_i}\right\}$, then for all $u \in T_{\mathbb{C}}X$, $v \in T^{1,0}X$ and $w \in T^{0,1}X$,

$$\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} \in T^{1,0}X,\ \nabla_{\frac{\partial}{\partial \bar{z}_i}} \frac{\partial}{\partial z_j} = 0 \quad \Longrightarrow \quad \nabla_u^{T_{\mathbb{C}}X} v \in T^{1,0}X,\ \nabla_u^{T_{\mathbb{C}}X} w \in T^{0,1}X.$$

This means $\nabla^{T_{\mathbb{C}}X}$ preserves $T^{1,0}X$ and $T^{0,1}X$.

(2) $\Longrightarrow$ (3): This is because $\forall u \in T_{\mathbb{C}}X$, $\forall v \in T^{1,0}X$,

$$\mathrm{i}\nabla_u^{T_{\mathbb{C}}X} v = \nabla_u^{T_{\mathbb{C}}X}(Jv) = (\nabla_u^{T_{\mathbb{R}}X} J)v + J(\nabla_u^{T_{\mathbb{C}}X} v) = (\nabla_u^{T_{\mathbb{R}}X} J)v + \mathrm{i}\nabla_u^{T_{\mathbb{C}}X} v,$$

hence $(\nabla_u^{T_{\mathbb{R}}X} J)v = 0$ for all $u \in T_{\mathbb{C}}X$ and $v \in T^{1,0}X$. Ane therefore $\nabla^{T_{\mathbb{R}}X} J = 0$.

(3) $\Longrightarrow$ (2): $\forall v \in T^{1,0}X$,

$$\nabla_u(\mathrm{i}v) = J(\nabla_u v) \Longleftrightarrow \nabla_u v \in T^{1,0}X.$$

(2), (3) $\Longrightarrow$ (1): For any $u, v \in T^{1,0}X$, by (2), $\nabla_u v$, $\nabla_v u \in T^{1,0}X$. So $[u,v] = \nabla_u v - \nabla_v u \in T^{1,0}X$. By (3), $\nabla\omega = \nabla(g(\cdot, J\cdot)) = 0$, and therefore

$$d\omega = \sum_i dz_i \wedge \nabla_{\frac{\partial}{\partial z_i}} \omega = 0.$$

Then we conclude the proof. □

From now on, we assume that X is Kähler and $E \to X$ is a holomorphic vector bundle. h^E is an Hermitian metric on E. Then there exists a unique connection ∇^E, called the *Chern connection*, such that

(1) ∇^E is holomorphic, *i.e.*, $(\nabla^E)^{0,1} = \bar{\partial}^E$. Specifically,

$$\forall s \in C^\infty(X,E)\ \forall \bar{u} \in C^\infty(X, T^{0,1}X)\ (\nabla_{\bar{u}}^E s = (\bar{\partial}^E s)\bar{u}).$$

(2) ∇^E is metric with respect to h^E.

Thus $\Lambda(T^{0,1}X)^*$ is a Clifford module.

Proposition 5.20

Let ∇ be the Levi–Civita connection on $T_{\mathbb{R}}X$ and $E \to X$ be a holomorphic vector bundle with Chern connection ∇^E.

(1) The connection $\nabla^{\mathscr{E}} = \nabla \otimes 1 + 1 \otimes \nabla^E$ on $\mathscr{E} = \Lambda(T^{0,1}X)^* \otimes E$ is a Clifford connection.

(2) The associated Dirac operator $D^E = \sqrt{2}(\bar{\partial}^E + (\bar{\partial}^E)^*)$.

Proof. (1) The Levi–Civita connection itself satisfies the Leibniz rule by definition

$$\nabla_u(c(v)\alpha) = c(\nabla_u v) + c(v)\nabla_u \alpha.$$

where $c(v) = \sqrt{2}(v^{1,0} \wedge + \iota_{v^{0,1}})$, the coefficient $\sqrt{2}$ comes from normalisation. And for any $w \in E$,

$$(1 \otimes \nabla_u^E)(c(v)(\alpha \otimes w)) = c(v)(\alpha \otimes \nabla_u^E w) = c(v)((1 \otimes \nabla_u^E)(\alpha \otimes w)).$$

So $\nabla^{\mathscr{E}} = \nabla \otimes 1 + 1 \otimes \nabla^E$ is a Clifford connection.

(2) Let $\{e_i\}$ be a local frame of $T^{0,1}X$ and denote $\{\bar{e}_i\}$ be the complex conjugate. Let $\{e^i\}$ be the dual frame on $(T^{0,1}X)^*$. Then $T_{\mathbb{C}}X$ has local frame $\{e_i, \bar{e}_i\}$. Hence

$$d = \sum_{i=1}^n (e^i \wedge \nabla_{e_i} + \bar{e}^i \wedge \nabla_{\bar{e}_i}).$$

By Proposition 5.19, ∇ preserves $T^{1,0}X$ and $T^{0,1}X$, so on a trivialisation U,

$$\bar{\partial}^E = \sum_{i=1}^n \bar{e}^i \wedge \nabla_{\bar{e}_i}, \qquad (\bar{\partial}^E)^* = -\sum_{i=1}^n \iota_{e_i} \nabla_{e_i}.$$

Then by definition $c(\bar{e}_i) = \sqrt{2}\bar{e}^i \wedge$, $c(e_i) = -\sqrt{2}\iota_{e_i}$. And therefore

$$D^E = \sum_{i=1}^n c(e_i)\nabla_{e_i} + c(\bar{e}_i)\nabla_{\bar{e}_i} = \sqrt{2}\sum_{i=1}^n \bar{e}^i \wedge \nabla_{\bar{e}_i} - \iota_{e_i}\nabla_{e_i} = \sqrt{2}(\bar{\partial}^E + (\bar{\partial}^E)^*)$$

by definition of Dirac operator. □

By Hodge theory, we have

$$\operatorname{Ind} D_+^E = \chi(X, E).$$

So one can apply Atiyah–Singer index theorem.

Proposition 5.21

The twisting curvature

$$R^{E/S} = \frac{1}{2}\operatorname{Tr}^{T^{1,0}X}(R^+) + R^E,$$

where R^+ is the curvature of $T^{1,0}X$ for the connection induced by $\nabla^{T_{\mathbb{R}}X}$.

Proof. To simplify the notation, denote $\Lambda = \Lambda^\bullet(T^{0,1}X)^*$. Let $\{w_i\}$ be a local frame of $T^{1,0}X$, $\{\bar{w}_i\}$ the complex conjugate, $\{w^i\}$ the dual and $R \in \Omega^2(X, \operatorname{End} T_{\mathbb{C}}X)$ be the Riemann curvature. By previous computation, on Λ, since $c(w^k) = -\sqrt{2}\iota_{w_k}$ and $c(\bar{w}^\ell) = \sqrt{2}\bar{w}^\ell \wedge$,

$$(\nabla^\Lambda)^2 = \sum_{k,\ell=1}^n (Rw_k, \bar{w}_\ell)\bar{w}^\ell \wedge \iota_{w_k} = -\frac{1}{2}\sum_{k,\ell=1}^n (Rw_k, \bar{w}_\ell)c(\bar{w}^\ell)c(w^k),$$

and

$$R^\Lambda = \frac{1}{4}\sum_{k,\ell=1}^n ((Rw_k, \bar{w}_\ell)c(w^k)c(\bar{w}^\ell) + (R\bar{w}^\ell, w_k)c(\bar{w}^\ell)c(w^k)).$$

Since R is anti-selfadjoint, one has

$$\begin{aligned}
(\nabla^\Lambda)^2 - R^\Lambda &= -\frac{1}{2}\sum_{k,\ell=1}^n (Rw_k,\bar w_\ell)c(\bar w^\ell)c(w^k) - \frac{1}{4}\sum_{k,\ell=1}^n ((Rw_k,\bar w_\ell)c(w^k)c(\bar w^\ell) + (R\bar w^\ell, w_k)c(\bar w^\ell)c(w^k)) \\
&= -\frac{1}{4}\sum_{k,\ell=1}^n (Rw_k,\bar w_\ell)(c(\bar w^\ell)c(w^k) + c(w^k)c(\bar w^\ell)) \\
&= -\frac{1}{4}\sum_{k,\ell=1}^n (Rw_k,\bar w_\ell)\cdot -2\langle w_k,\bar w_\ell\rangle \\
&= \frac{1}{2}\sum_{k=1}^n (Rw_k,\bar w_k) = \frac{1}{2}\mathrm{Tr}^{T^{1,0}X}(R^+).
\end{aligned}$$

So if $E \to X$ is a holomorphic Hermitian vector bundle, and $\nabla^{\mathscr{E}}$ as before, we have

$$R^{E/S} = R^{\mathscr{E}} - R^\Lambda = (\nabla^\Lambda)^2 + R^E - R^\Lambda = R^E + \frac{1}{2}\mathrm{Tr}^{T^{1,0}X}(R^+),$$

then we conclude the proof. □

Proposition 5.22

We have

$$\hat{A}(X,\nabla^{T_{\mathbb{R}}X})\mathrm{Tr}_s{}^{E/S}(\exp(-R^{E/S})) = \mathrm{Td}(X,\nabla^{T_{\mathbb{R}}X})\mathrm{Tr}(\exp(-R^E)),$$

where $\mathrm{Td}(X,\nabla^{T_{\mathbb{R}}X}) = \det\left(\frac{R^+}{\exp(R^+)-1}\right)$.

Proof. Here we modify the definition of Todd class and Chern class by replacing $-\frac{\mathrm{i}}{2\pi}R$ with R, so each of the modification introduce a coefficient $(-2\pi\mathrm{i})^{-n/2}$ in the integral. Note that this do not change the additivity and multiplicity of ch.

In this context, we have

$$\begin{aligned}
\widehat{A}(X,\nabla^{T_{\mathbb{R}}X}) &= \widehat{A}(T_RX)^2 = \det\left(\frac{R^+/2}{\sinh(R^+/2)}\right) \\
&= \det\left(\frac{R^+}{\exp(-R^+/2)(\exp R^+ - 1)}\right) = \det\left(\exp\frac{R^+}{2}\right)\det\left(\frac{R^+}{\exp R^+ - 1}\right) = \det\left(\exp\frac{R^+}{2}\right)\mathrm{Td}(X,\nabla^{T_{\mathbb{R}}X}).
\end{aligned}$$

And by Proposition 5.21, we have

$$\begin{aligned}
R^{E/S} = R^E + \frac{1}{2}\mathrm{Tr}^{T^{1,0}X}(R^+) \quad &\Longrightarrow \quad \exp(R^{E/S}) = \exp(R^E)\exp\frac{R^+}{2} \\
&\Longrightarrow \quad \mathrm{Tr}_s^{E/S}\exp(-R^{E/S}) = \det\left(\exp\left(-\frac{R^+}{2}\right)\right)\mathrm{Tr}^E(\exp(-R^E)).
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{A}(X,\nabla^{T_{\mathbb{R}}X})\mathrm{Tr}_s^{E/S}\exp(-R^{E/S}) &= \det\left(\exp\frac{R^+}{2}\right)\mathrm{Td}(X,\nabla^{T_RX})\det\left(\exp\left(-\frac{R^+}{2}\right)\right)\mathrm{Tr}^E(\exp(-R^E)) \\
&= \mathrm{Td}(X,\nabla^{T_RX})\mathrm{Tr}^E(\exp(-R^E)).
\end{aligned}$$

Then we conclude the proof. □

By the three propositions above, we conclude that

$$\chi(X,E) = \frac{1}{(2\pi\mathrm{i})^n}\int_X \mathrm{Td}(X)\mathrm{ch}(E).$$

The coefficient, as we said before, comes from the modification of Td and ch.

Moreover, even if X is not Kähler, one can prove that D^E has the same symbol as some Dirac operators. For the reference, see **[MM07, Sec 1.3, 1.4.1]** So even for non-Kähler manifolds, Atiyah–Singer theorem still implies Riemann–Roch–Hirzebruch theorem.

6 The Atiyah–Singer index theorem — a heat kernel approach

In this chapter, we assume that:

- (M, g^{TM}) is a compact oriented even-dimensional Riemannian manifold with $n = \dim M$.
- $E = E^+ \oplus E^-$ equipped with h^E is a complex $\mathbb{Z}/2\mathbb{Z}$-graded self-adjoint Clifford module on M.
- ∇^E is a Clifford connection on (E, h^E).
- D is the associated Dirac operator.

6.1 McKean–Singer formula and the local index theorem

The vector space $C^\infty(M, E)$ is $\mathbb{Z}/2\mathbb{Z}$-graded and D^2 preserves this grading. In Chapter 3, we defined

$$\operatorname{Tr} \mathrm{e}^{-tD^2} := \int_M \operatorname{Tr}^{E_x} K_t(x,x)\, \mathrm{dvol}_M(x),$$

where $K_t(\cdot,\cdot)$ is the Schwartz kernel of e^{-tD^2}. If $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ are the eigenvalues of D^2, then

$$\operatorname{Tr} \mathrm{e}^{-tD^2} = \sum_k \mathrm{e}^{-t\lambda_k}.$$

We can also define $\operatorname{Tr}_s \mathrm{e}^{-tD^2}$ in the same way.

Remark. *If an operator A has a Schwartz kernel, we will also denote it by A. In other words,*

$$(As)(x) = \int_M A(x,y) s(y)\, dv(y).$$

Theorem 6.1: McKean–Singer

For any $t > 0$, we have

$$\operatorname{Ind} D_+ = \operatorname{Tr}_s \mathrm{e}^{-tD^2}.$$

Proof. Set $\mathscr{E}_\lambda^\pm = \left\{s \in C^\infty(M, E^\pm) : D^2 s = \lambda s\right\}$. Then $\mathscr{E}_\lambda = \mathscr{E}_\lambda^+ \oplus \mathscr{E}_\lambda^-$, $n_\lambda^\pm = \dim \mathscr{E}_\lambda^\pm$. Then

$$\operatorname{Tr}_s \mathrm{e}^{-tD^2} = \sum_{\lambda \in \operatorname{Sp} D^2} (n_\lambda^+ - n_\lambda^-) \mathrm{e}^{-\lambda t}.$$

For $\lambda > 0$ and $s \in \mathscr{E}_\lambda^\pm$, we have $D^2 D s = D D^2 s = \lambda D s$, thus $Ds = \mathscr{E}_\lambda^\mp$. So we have

$$\begin{tikzcd} \mathscr{E}_\lambda^+ \arrow[r, "D"] \arrow[dr, "\mathrm{id}"'] & \mathscr{E}_\lambda^- \arrow[d, "\frac{1}{\lambda}D"] \\ & \mathscr{E}_\lambda^+ \end{tikzcd}$$

Hence $\mathscr{E}_\lambda^+ \cong \mathscr{E}_\lambda^-$ and $\operatorname{Tr}_s \mathrm{e}^{-tD^2} = n_0^+ - n_0^- = \operatorname{Ind} D_+$. □

Alternative Proof. Let $t \to +\infty$, then e^{-tD^2} converges to $P_{\ker D^2}$ in strong operator topology, where P_V denotes the orthogonal projection to a subspace V. This is because $\forall x, y$,

$$\lim_{t\to+\infty} \mathrm{e}^{-tD^2}(x,y) = \lim_{t\to+\infty} \sum_k \mathrm{e}^{-t\lambda_k} u_k(x) \otimes u_k(y)^* = \sum_{\{k:\lambda_k=0\}} u_k(x) \otimes u_k(y)^* = P_{\ker D^2}(x,y).$$

Then note that for X of degree 1, $X^2 = \frac{1}{2}[X, X]$, we have

$$\frac{\partial}{\partial t} \operatorname{Tr}_s \mathrm{e}^{-tD^2} = \operatorname{Tr}_s \left(\frac{\partial}{\partial t} \mathrm{e}^{-tD^2}\right) = -\operatorname{Tr}_s (D^2 \mathrm{e}^{-tD^2}) = -\frac{1}{2} \operatorname{Tr}_s ([D\mathrm{e}^{-tD^2/2}, D\mathrm{e}^{-tD^2/2}]) = 0.$$

Then we conclude the proof. □

To prove the Atiyah–Singer theorem, we shall compute $\lim_{t\to 0} \operatorname{Tr} \mathrm{e}^{-tD^2}$. Indeed, we have the classical theorem.

Theorem 6.2

There exist $a_j \in C^\infty(M, \operatorname{End} E)$ such that $\forall \ell \in \mathbb{N}$, as $t \to 0$, we have

$$e^{-tD^2}(x,x) = \sum_{j=-k}^{\ell} a_j(x) t^j + O(t^{k+1})$$

uniformly on M. Moreover, a_j is local in the sense that $a_j(x)$ only depends on $D^2|_{B(x,\varepsilon)}$ for any $\varepsilon > 0$.

Proof. Omitted here, see **[BGV03, Thm 2.30]** if you are intrested. □

Thus

$$\operatorname{Ind} D_+ = \sum_{j=-k}^{\ell} t^j \Big(\int_M \operatorname{Tr}_s{}^{E_x} a_j(x)\, \mathrm{dvol}_M(x)\Big) + O(t^{k+1}),$$

which implies

$$\int_M \operatorname{Tr}_s{}^{E_x} a_j(x)\, \mathrm{dvol}_M(x) = \begin{cases} 0, & \text{if } j \neq 0, \\ \operatorname{Ind} D_+, & \text{if } j = 0. \end{cases}$$

Theorem 6.3: Local index theorem

Using the same notations as above,

$$\operatorname{Tr}_s a_j(x)\, \mathrm{dvol}_M(x) = \begin{cases} 0, & \text{if } j < 0, \\ [\hat{A}(TM, \nabla^{TM}) \mathrm{ch}(E/S, \nabla^E)]^n, & \text{if j=0} \end{cases}$$

where $[\alpha]^n$ denotes the part of α in $\Lambda^n T^* M$.

This is equivalent to

$$\operatorname{Tr}_s(\mathrm{e}^{-tD^2}(x,x))\, \mathrm{dvol}_M(x) \to [\hat{A}(TM, \nabla^{TM}) \mathrm{ch}(E/S, \nabla^E)]^n$$

as $t \to 0$, and we get the index theorem after integrating the above equation over M.

Remark. *(1) Local index theorem implies Atiyah–Singer theorem, but the converse is not true. This is because $\int f = \int g \not\Longrightarrow f = g$. The local index theorem was conjectured by McKean who called it the miraculous cancelation conjecture.*

(2) The local index theorem is only valid for Dirac operators associated with a Clifford connection.

6.2 Proof of the local index theorem

To make things clear, we divided the proof into 4 step in this section.

(1) Showing that computing $\lim_{t\to 0} \operatorname{Tr}_s(\mathrm{e}^{-tD^2}(x,x))$ is actually a local problem.

(2) Replace M by a local model $\mathbb{R}^n$.

(3) Getzler rescaling of D^2, then discuss the deformed Laplacian L_u and prove $L_u \to L_0$ as $u \to 0$.

(4) Prove that $L_u \to L_0$ implies $\mathrm{e}^{-L_u} \to \mathrm{e}^{-L_0}$ and then compute $\mathrm{e}^{-L_0}(x,x)$ in a explicit way.

(α) **The problem is local**

Denote r the injectivity radius of M, and fix $\varepsilon \in (0, r/4)$. Let f be a smooth function with

$$f: \mathbb{R} \to [0,1], \qquad \operatorname{supp} f \subset [-\varepsilon, \varepsilon], \qquad f|_{[-\varepsilon/2, \varepsilon/2]} = 1.$$

For $u>0$, we define two functions $F_u,\ G_u:\mathbb{R}\to\mathbb{R}$ by

$$F_u(a):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\cos(va)f(\sqrt{u}v)\mathrm{e}^{-v^2/2}\,\mathrm{d}v,\qquad G_u(a)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\cos(va)(1-f(\sqrt{u}v))\mathrm{e}^{-v^2/2}\,\mathrm{d}v.$$

Then we have:

- For any $a\in\mathbb{R}$, $F_u(a)+G_u(a)=\mathscr{F}(\mathrm{e}^{-v^2/2})(a)=\mathrm{e}^{-a^2/2}$, where $\mathscr{F}(\cdot)$ denotes the Fourier transform.
- F_u and G_u are even functions and are in the Schwartz space of $\mathbb{R}$.

The second property ensures that we can define $F_u(tD)$ and $G_u(tD)$ with $F_u(tD)|_{\mathscr{E}_\lambda}=F_u(t\sqrt{\lambda})\mathrm{id}_{\mathscr{E}_\lambda}$. And we have

$$F_u(\sqrt{u}D)+G_u(\sqrt{u}D)=\mathrm{e}^{-uD^2/2}.$$

Consider the wave equation $(\partial_t^2+D_x^2)w(t,x)=0$. We have the wave operator $w_t=\cos(t|D|)$, such that $w(t,x)=(w_t\cdot f_0)(t,x)$ is the unique solution of the wave euqation with initial condition

$$\begin{cases} w(0,x)=f_0(x),\\ \frac{\partial w}{\partial t}(0,x)=0.\end{cases}$$

Moreover,

$$\operatorname{supp} w(t,\cdot)\subset\left\{x\in M: d(x,\operatorname{supp} f_0)\leqslant t\right\}.$$

This is called the *finite propagation speed property* of the wave equation. (see **[MM07, Appendix D2]**)

Now $w_t(x,y)$ is a Schwartz kernel and

$$(w_t\cdot f_0)(x)=\int_{\mathbb{R}}f(y)w_t(y,x)\,\mathrm{d}y.$$

So formally, $w_t(x_0,x)=(w_t\cdot\delta_{x_0})(x)$. Thus $\operatorname{supp} w_t(x_0,\cdot)\subset B(x_0,t)$ and $w_t(x_0,\cdot)$ only depends on $D|_{B(x_0,t)}$. Therefore, as

$$F_u(\sqrt{u}D)=\frac{1}{\sqrt{2\pi}}\int_{-\varepsilon/\sqrt{u}}^{\varepsilon/\sqrt{u}}\cos(\sqrt{u}v|D|)\mathrm{e}^{-v^2/2}f(\sqrt{u}v)\,\mathrm{d}v,$$

we find that

$$\operatorname{supp} F_u(\sqrt{u}D)(x_0,\cdot)\subset B_M(x_0,\varepsilon).$$

Thus $\operatorname{supp}(F_u(\sqrt{u}D)(x_0,\cdot))\subset D|_{B_M(x_0,\varepsilon)}$.

Proposition 6.4

There exists $c_1,\ c_2>0$ constants, such that $\forall u\in(0,1]$, $\forall x,x'\in M$,

$$\left|G_u(\sqrt{u}D)(x,x')\right|\leqslant c_1\mathrm{e}^{-c_2/u}.$$

Proof. By definition

$$G_u(\sqrt{u}a)=\frac{1}{\sqrt{2\pi u}}\int_{|s|\geqslant\varepsilon/2}\cos(sa)\mathrm{e}^{-s^2/2u}(1-f(s))\,\mathrm{d}s.$$

Using the fact that $a^{2n}\cos(sa)=\left(\frac{\partial}{\partial s}\right)^{2m}\cos(sa)$, integrate by part,

$$\begin{aligned} a^{2m}G_u(\sqrt{u}a)&=\int_{|s|\geqslant 2}\cos(av)\left(\frac{\partial^0}{}\right)^{2m}(\mathrm{e}^{-s^2/2u}(1-f(s)))\,\mathrm{d}s\\ &=\int_{|s|\geqslant\varepsilon/2}\cos(av)\mathrm{e}^{-s^2/2u}\sum_{\ell=0}^{2m}p_\ell\left(\frac{1}{n},0\right)\left(\frac{\partial}{\partial s}\right)^{\ell}(1-f(s))\,\mathrm{d}s,\end{aligned}$$

where p_ℓ is a polynomial of degree ℓ. Therefore

$$a^{2m} G_u(\sqrt{u}a) \leqslant c\mathrm{e}^{-\varepsilon^2/16u}.$$

Replacing a with D, for any $s \in C^\infty(M,E)$,

$$\left\| D^{2m} G_u(\sqrt{u}D)s \right\|_2 \leqslant c\mathrm{e}^{-\varepsilon^2/16} \|s\|_2,$$

where $D^{2n} G_u(\sqrt{u}D)$ has Schwartz kernel $D_x^{2m} G_u(\sqrt{u}D)(x,x')$. Then by ellipltic estimates for D and Sobolev inequalities for $m \geqslant 1$,

$$\left| G_u(\sqrt{u}D)(x,x') \right| \leqslant c_1 \mathrm{e}^{-c_2/u}.$$

Then we conclude the proof. □

To sum up, in this step, we proved

- $\mathrm{e}^{-uD^2/2}(x,x) = F_u(\sqrt{u}D)(x,x) + O(\mathrm{e}^{-c_2/u})$.
- $F_u(\sqrt{u}D)(x,x)$ is local (*only depends on* $D|_{B(x,\varepsilon)}$).

(β) **Replace M by $\mathbb{R}^n$**

For $x_0 \in M$, then exponential map $\exp_{x_0} : B^{T_{x_0}M}(0,\varepsilon) \xrightarrow{\sim} B^M(x_0,\varepsilon)$. Fix an ONB $\{e_i\}$ of $T_{x_0}M$ so that $M_0 := T_{x_0}M \cong \mathbb{R}^n$. On $B^M(x_0,\varepsilon)$, the vector bundle $E = S^{TM} \otimes F$ and we trivialise F and S^{TM} on $B^{M_0}(0,4\varepsilon) \subset \mathbb{R}^n$ by $z \in \mathbb{R}^n$, $|z| \leqslant 4\varepsilon$,

$$S_z^{TM} \cong S_0^{TM}, \qquad F_z \cong F_0$$

via the parallel transport along $\gamma : t \mapsto tz$ with respect to $\nabla^{S^{TM}}$ (*induced by Levi–Civita*) and ∇^F (*induced by* ∇^E). These 2 connections are Hermitian, so we have

$$(S^{TM}, h^{S^{TM}})|_{B^M(0,4\varepsilon)} \cong (S_{x_0}^{TM} \times B^M(x_0,4\varepsilon), h_{x_0}^{S^{TM}}), \qquad (F, h^F)|_{B^M(0,4\varepsilon)} \cong (F_{x_0} \times B^M(x_0,4\varepsilon), h_{x_0}^F).$$

Let $\tilde{e}_i(z)$ be the parallel transport of $e_i \in M_0$ along γ w.r.t. ∇^{TM}. So $\{\tilde{e}_i\}$ is an orthonormal fram e of $TM|_{B^{M_0}(0,4\varepsilon)}$. We have

$$D^2 = \Delta + \frac{r^M}{4} + \frac{1}{2} R^F(\tilde{e}_i, \tilde{e}_j) c(\tilde{e}_i) c(\tilde{e}_j)$$

by Lichnerowicz formula. In our trivialisation,

$$\begin{cases} \nabla^{S^{TM}} = d + \Gamma^S, \\ \nabla^F = d + \Gamma^F, \end{cases} \qquad \Longrightarrow \qquad \nabla^E = d + \Gamma^S + \Gamma^F.$$

On M_0, choose a metric g^{TM_0} such that

$$g^{TM_0}|_{B^{M_0}(0,2\varepsilon)} \cong g^{TM}|_{B^M(0,2\varepsilon)}, \qquad g^{TM_0}|_{M_0 \setminus B^{M_0}(0,4\varepsilon)} = \text{const} = g^{T_{x_0}M}.$$

Denote dvol_{M_0} the volume form of (M_0, g^{TM_0}) and $\mathrm{dvol}_{T_{x_0}M}$ the volume form of $(T_{x_0}M, g^{T_{x_0}M})$. Set $\kappa(z)$ such that

$$\mathrm{dvol}_{M_0}(z) = \kappa(z)\, \mathrm{dvol}_{T_{x_0}M}(z).$$

Then $\kappa(z) = 1$ and $\kappa|_{M_0 \setminus B^{M_0}(0,4\varepsilon)} = 1$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth function with $\operatorname{supp} \rho \subset [-4\varepsilon, 4\varepsilon]$ and $\rho|_{[-2\varepsilon,2\varepsilon]} = 1$. We extend $\nabla^E|_{B^M(0,2\varepsilon)}$ to ∇^{E_0} on $E_0 = E_{x_0} \times M_0 \to M_0$. Then $\nabla^{E_0} = d + \rho(|z|)(\Gamma_z^F + \Gamma_z^S)$ is a Hermitian connection on $(E_0, h^{E_0} = h^{E_{x_0}})$. Then we define

$$L_{x_0} := \Delta^{E_0} + \rho(|z|)\left(\frac{r^M}{4} + \frac{1}{2} R^F(\tilde{e}_i, \tilde{e}_j) c(\tilde{e}_i) c(\tilde{e}_j) \right) \in \mathscr{D}\mathrm{iff}^2(E_0, E_0),$$

where Δ^{E_0} is the Laplacian w.r.t. ∇^{E_0} and g^{E_0}. Then

$$\begin{cases} L_{x_0}|_{B^{M_0}(0,2\varepsilon)} = D^2|_{B^M(x_0,2\varepsilon)}, \\ L_{x_0}|_{M\setminus B^{M_0}(0,3\varepsilon)} = -\sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}. \end{cases} \tag{6.1}$$

What we have done in step (α) is also valid for D^2 replaced by L_{x_0}. Therefore,

$$\mathrm{e}^{-uL_{x_0}/2}(0,0) = F_u(\sqrt{uL_{x_0}})(0,0) + O(\mathrm{e}^{-c/u}).$$

But $F_u(\sqrt{uL_{x_0}})(0,0) = F_u(\sqrt{u}D)(x_0,x_0)$ because of (6.1). Hence

$$\mathrm{e}^{-uD^2/2}(x_0,x_0) = \mathrm{e}^{-uL_{x_0}/2}(0,0) + O(\mathrm{e}^{-c/u}).$$

Define the rescaling on $\mathbb{R}^n$ as follows: for $s \in C^\infty(M_0,E_0)$, define

$$S_u s(z) := s\left(\frac{z}{\sqrt{u}}\right).$$

Set $\nabla_u = S_u^{-1}(\sqrt{u}\nabla^{E_0})S_u$, $L_u = S_u^{-1}(uL_{x_0})S_u$, and $\mathrm{e}^{-L_u}(z,z')$ the Schwartz kernel w.r.t. $\mathrm{dvol}_{T_{x_0}M}(z)$. Then

$$\begin{aligned} \mathrm{e}^{-L_u}s(z) &= (S_u^{-1}\mathrm{e}^{-uL_{x_0}}S_u s)(z) \\ &= \int_{M_0} \mathrm{e}^{-uL_{x_0}}(\sqrt{u}z,z')(S_u s)(z')\,\mathrm{dvol}_{M_0}(z') \\ &= u^{n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-uL_{x_0}}(\sqrt{z},\sqrt{z''})s(z'')\kappa(\sqrt{u}z'')\,\mathrm{dvol}_{M_0}(z''). \end{aligned}$$

So $\mathrm{e}^{-L_u}(z,z') = u^{n/2}\mathrm{e}^{-uL_{x_0}}(\sqrt{u}z,\sqrt{u}z')\kappa(\sqrt{u}z')$. At $z=z'=0$,

$$\mathrm{e}^{-tL_u}(0,0) = u^{n/2}\mathrm{e}^{-uL_{x_0}}(0,0)\kappa(0,0) = u^{n/2}\mathrm{e}^{-L_u}(0,0) + O(\mathrm{e}^{-c/n}).$$

Note that $\forall A \in C^\infty(M_0,E_0)$,

$$S_u^{-1}\frac{\partial}{\partial z_j}S_u = \frac{1}{\sqrt{u}}\frac{\partial}{\partial z_j}, \qquad S_u^{-1}AS_u(z) = A(\sqrt{u}z).$$

So one can check that $L_u \to \Delta^{\mathbb{R}^n} = -\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2}$ as $u \to 0$.

(γ) **Getzler rescaling**

For $u \in (0,1]$, set $c_u(e_i) = \frac{1}{\sqrt{u}}e^i \wedge -\sqrt{u}\iota_{e_i} \in \mathrm{End}(\Lambda^\bullet T_{x_0}^*M)$. Observe the fact that $\{e^i\wedge,\iota_{e_i}\}$ generate $\mathrm{End}(\Lambda^\bullet T_{x_0}^*M)$ as an algebra. More precisely, $\{e^J \wedge \iota_{e_K} : J,\, K \subset \{1,\dots,n\}\}$ is a basis of $\mathrm{End}(\Lambda^\bullet T_{x_0}^*M)$. For $\alpha \in \mathrm{End}(\Lambda^\bullet T_{x_0}^*M)$,

$$\alpha = \sum_{J,K}\alpha_{J,K}e^J \wedge \iota_{e_K}.$$

And we set $\{\alpha\}^{\max} := \alpha_{\{1,\dots,n\},\varnothing}$, the coefficient of $e^1 \wedge \cdots \wedge e^n$ in α.

Set $\tilde{L}_u$ the operator with $c(e_i)$ in L_u replaced by $c_u(e_i)$, and $\tilde{\nabla}_u$ the connection with $c(e_i)$ in ∇_u replaced by $c_u(e_i)$.

Lemma 6.5

We have

$$\mathrm{Tr}_s^{S_{x_0}\otimes F_{x_0}}(\mathrm{e}^{-L_u}(0,0)) = (-2\mathrm{i})^{n/2}u^{n/2}\mathrm{Tr}_s^{F_{x_0}}\left(\left\{\mathrm{e}^{-\tilde{L}_u}(0,0)\right\}^{\max}\right).$$

Proof. We have $c_u(e_i)^2 = -1$, so c_u extends to $c_u : \mathrm{Cl}(T_{x_0}M) \to \mathrm{End}(\Lambda^\bullet T_{x_0}^*M)$. By the uniqueness of the heat kernel,

$$c_u(\mathrm{e}^{-L_u})(z,z') = \mathrm{e}^{-\tilde{L}_u}(z,z').$$

We have seen that if $\alpha = \sum_I \alpha_I c(e_I)$, then

$$\mathrm{Tr}_s^{S_{x_0}}(\alpha) = (-2\mathrm{i})^{n/2}\alpha_{\{1,\ldots,n\}} = (-2\mathrm{i})^{n/2}u^{n/2}\{c_u(\alpha)\}^{\max}$$

in step (β). ☐

As a consequence,

$$\mathrm{Tr}_s(\mathrm{e}^{-uD^2/2}(x_0,x_0)) = (-2\mathrm{i})^{n/2}\mathrm{Tr}_s(\left\{\mathrm{e}^{-\tilde{L}_u}(0,0)\right\}^{max}) + O(\mathrm{e}^{-c/u}).$$

Then we compute $\lim_{u\to 0}\tilde{L}_u$ (*in the sense of taking limit of coefficients*).

Recall that $\Gamma_z^S = \frac{1}{4}\sum_{k,\ell}\left\langle \Gamma_z^{TM}\tilde{e}_k, \tilde{e}_\ell\right\rangle c(\tilde{e}_k)_z c(\tilde{e}_\ell)_z$. But $\tilde{e}_j$ is parallel w.r.t. ∇^{TM}, *i.e.*, $\nabla_{\dot{\gamma}}^{TM}\tilde{e}_j = 0$. So

$$\nabla_{\dot{\gamma}}^S c(\tilde{e}_j) = c(\nabla_{\dot{\gamma}}^{TM}\tilde{e}_j) = 0.$$

This means that in out trivialisation, $c(\tilde{e}_j)_z = \mathrm{const} = c(e_j)_{x_0}$ acts on S_{x_0}.

On the other hand, for any vector bundle $W \to X$ on a manifold with a connection ∇^W and the curvature R^W, if $(U,(z_1,\ldots,z_n))$ is a local chart near $x_0 \in X$, and if $W_z \cong W_{x_0}$, thanks to the parallel transport w.r.t. ∇^W along $\gamma : t \mapsto tz$, we can write $\nabla^W = d + \Gamma^W$ in this trivialisation, where Γ^W is seen as an element of $C^\infty(U, \mathbb{R}^n \otimes \mathrm{End}\,\mathbb{C}^k)$ thanks to the basis $\{e_j\} = \left\{\frac{\partial}{\partial z_j}\right\}$.

Lemma 6.6

Near 0, we have

$$\Gamma_z^W = \frac{1}{2}R_{x_0}^W(R,\cdot) + O(|z|^2),$$

where $R = \sum_{j=1}^n z_j\frac{\partial}{\partial z_j}$ is the radial vector field.

Proof. We have $R^W = d\Gamma^W + \Gamma^W \wedge \Gamma^W$ and $\iota_R\Gamma^W = 0$ because of the parallel transport along γ. Then Cartan's formula gives

$$\mathscr{L}_R\Gamma^W = [\iota_R, d]\Gamma^W = \iota_R d\Gamma^W = \iota_R R^W \tag{6.2}$$

Using $\mathscr{L}_R dz^j = dz^j$ and Taylor expansion of (6.2) at $z = 0$, we obtain

$$\sum_\alpha(|\alpha|+1)(\partial^\alpha\Gamma^W)_{x_0}(e_j)\frac{z^\alpha}{\alpha!} = \sum_\alpha(\partial^\alpha R^W)_{x_0}(R_z, e_j)\frac{z^\alpha}{\alpha!}.$$

Looking the coefficient of z^α for $|\alpha| = 1$, we obtain

$$\sum_{i=1}^n 2(\partial_{e_i}\Gamma^W)_{x_0}e_j z_i = R_{x_0}(e_i, e_j)z_i.$$

So $\sum_{i=1}^n \Gamma^\Gamma(\cdot)_{x_0}z_i = \frac{1}{2}R_{x_0}^W(R_z,\cdot)$. ☐

Remark. *On $M_0 = \mathbb{R}^n$, as $R_z \in T_x\mathbb{R}^n \cong \mathbb{R}^n$ can be identified via $\{e_i\}$ with $(z_1,\ldots,z_n) = "Z$. So we may write $\Gamma_z^F = \frac{1}{2}R^F(Z,\cdot) + O(|z|^2)$.*

Let ∇_V the usual derivative along V on $\mathbb{R}^n$, we have

$$\begin{aligned}(\nabla_u)_V s &= \sqrt{u}S_u^{-1}\nabla_V^{E_0}S_u s\\ &= \sqrt{u}S_u^{-1}(\nabla_V + \rho(|z|))(\Gamma^S(V) + \Gamma^F(V))S_u s\\ &= (\nabla_V + \sqrt{u}\rho(\sqrt{u}|z|))(\Gamma^S_{\sqrt{u}z}(V) + \Gamma^F_{\sqrt{u}z}(V)).\end{aligned}$$

Therefore, as $u \to 0$,

$$
\begin{aligned}
(\tilde{\nabla}_u)_V &= \nabla_V + \frac{\sqrt{u}}{4}\sum_{k,\ell}\left\langle \Gamma^{TM}_{\sqrt{u}z}(V)\tilde{e}_k, \tilde{e}_\ell\right\rangle c_u(e_k)_{x_0} c_u(e_\ell)_{x_0} + \overbrace{\sqrt{u}\Gamma^F_{\sqrt{u}z}(V)}^{=O(\sqrt{u})\text{ by Lem above}} + O(\sqrt{u}) \\
&= \nabla_V + \frac{1}{8}\sum_{k,\ell}\langle R^{TM}_{x_0}(Z,V)e_k, e_\ell\rangle e^k \wedge e^\ell + O(\sqrt{u}) \\
&= \nabla_V + \frac{1}{4}\langle R^{TM}_{x_0}(\cdot,\cdot)Z, V\rangle + O(\sqrt{u}).
\end{aligned}
$$

Recall that for (W, ∇^W) a vector bundle with connection, locally

$$
\Delta^W = -\sum_{i,j} g^{ij}(z)\left(\nabla^W_{\frac{\partial}{\partial z_i}}\nabla^W_{\frac{\partial}{\partial z_j}} - \nabla^W_{\nabla^{TM}_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_j}}\right),
$$

where $[g^{ij}]$ is the inverse of $G := \left[\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right\rangle\right]$. Here in our trivialisation, $e_i = \frac{\partial}{\partial z_i}$. So

$$
\tilde{L}_u = -\sum_{i,j} g_0^{ij}(\sqrt{u}z)(\tilde{\nabla}_{u,e_i}\tilde{\nabla}_{u,e_j} - \tilde{\nabla}_{u,\nabla^{TM}_{e_i}e_j}) + \frac{u}{4}r^M(\sqrt{u}, z) + \frac{u}{2}\sum_{k,\ell} R^F_{\sqrt{u}z}(\tilde{e}_k, \tilde{e}_\ell)c_u(e_k)_{x_0}c_u(e_\ell)_{x_0}.
$$

Here $g_0 = g^{TM_0}$ on M_0 is identity because $G_0(0) = \mathrm{id}$. Then let $u \to 0$, we have

$$
\tilde{L}_0 = -\sum_{i,j}\left(\frac{\partial}{\partial z_j} + \frac{1}{4}\left\langle R^{TM}_{x_0}z, \frac{\partial}{\partial z_j}\right\rangle\right)^2 + R^F_{x_0}.
$$

(δ) **Convergence of the heat kernel**

Theorem 6.7

As $u \to 0$, we have $\mathrm{e}^{-\tilde{L}_u}(0,0) \to \mathrm{e}^{-\tilde{L}_0}(0,0)$.

To prove this theorem, recall that $(E_0 = S^{TM}_{x_0} \otimes F_{x_0}, h^{E_0}) \to M_0 = \mathbb{R}^n$ is the trivial bundle with trivial metric. $\tilde{L}_u$ is a differential operator with coefficients in $\mathrm{End}(\Lambda^\bullet T^*_{x_0}M \otimes F_{x_0})$. We shall use $(\lambda - \tilde{L}_u)^{-1}$, the resolvent of $\tilde{L}_u$. If $a \in \mathbb{C}$ and C is a well-chosen contour,

$$
\mathrm{e}^{-a} = \frac{1}{2\pi\mathrm{i}}\int_C \frac{\mathrm{e}^{-\lambda}}{\lambda - a}\,\mathrm{d}\lambda.
$$

Then by holomorphic functional calculus,

$$
\mathrm{e}^{-\tilde{L}_u} = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{e}^{-\lambda}(\lambda - \tilde{L}_u)^{-1}\,\mathrm{d}\lambda.
$$

We need estimates on $(\lambda - \tilde{L}_u)^{-1}$ and $(\lambda - \tilde{L}_u)^{-1} - (\lambda - \tilde{L}_0)^{-1}$. Note that $\tilde{L}_u$ is not self-adjoint, and M_0 is not compact.

Definition 6.8: Sobolev norm

For $s \in C^\infty_c(M_0, \Lambda^q T^*_{x_0}M \otimes F_{x_0})$, $u \in (0,1]$, define

$$
\|s\|_{u,0} = \int_{\mathbb{R}^n} \|s(t)\|^2_{g^{T_{x_0}M}\otimes h^{F_{x_0}}}\left(1 + |z|\,\rho\left(\frac{\sqrt{u}z}{2}\right)\right)^{2(n-q)}\mathrm{dvol}_{T_xM_0}(z),
$$

where ρ is a smooth function from $\mathbb{R}$ to $[0,1]$ with $\operatorname{supp}\rho \subset [-4\varepsilon, 4\varepsilon]$, $\rho|_{[-2\varepsilon,2\varepsilon]} = 1$, and $\mathrm{dvol}_{T_{x_0}M}$ is the volume form of $(T_{x_0}M, g^{T_{x_0}M})$.

For $k \geqslant 0$, define

$$
\|s\|_{u,k} = \sum_{\ell=0}^{k}\sum_{i_1,\ldots,i_\ell}\left\|\frac{\partial}{\partial z_{i_1}}\cdots\frac{\partial}{\partial z_{i_\ell}}s\right\|^2_{u,0}.
$$

And define S_u^k the completion of C_c^∞ w.r.t. $\|\cdot\|_{u,k}$. Denote S_u^{-k} the dual of S_u^k.

If $A \in \mathscr{B}(S_u^m, S_u^{m'})$ for $m, m' \geqslant 0$, let $\|A\|_{u,m,m'}$ be the operator norm.

Proposition 6.9

There exist constants $c_1,\ c_2,\ c_3,\ c_4 > 0$ such that $\forall u \in (0,1]$, $\forall s, s' \in S^0(M_0, \Lambda^\bullet T^*_{x_0} M \otimes F_{x_0})$,

(1) The symmetric part of $\tilde{L}_u$ satisfies « *uniform elliptic estimates* ».

$$\operatorname{Re}\left\langle \tilde{L}_u s, s\right\rangle \geqslant c_1 \|s\|_{u,1}^2 - c_2 \|s\|_{u,0}^2 .$$

(2) The anti-symmetric part of $\tilde{L}_u$ is uniformly bounded from S_u^1 to S_u^0.

$$\left|\operatorname{Im}\left\langle \tilde{L}_u s, s'\right\rangle\right| \leqslant c_3 \|s\|_{u,1} \left\|s'\right\|_{u,0} .$$

(3) $\tilde{L}_u$ is uniformly bounded from S_u^1 to S_u^{-1}.

$$\left|\left\langle \tilde{L}_u s, s'\right\rangle\right| \leqslant c_4 \|s\|_{u,1} \left\|s'\right\|_{u,1} .$$

Proof. To simplify the notations, denote

$$\mathit{\Phi}_u := \frac{u}{4} r^m(\sqrt{u}z) + \frac{u}{2} \sum_{k,\ell} R^F_{\sqrt{u}z}(\tilde{e}_k, \tilde{e}_\ell) c_u(e_k) c_u(e_\ell).$$

And recall that

$$\tilde{L}_u = \sum_{i,j} g_0^{ij}(\sqrt{u}z)(\tilde{\nabla}_{u,e_i} \tilde{\nabla}_{u,e_j} - \sqrt{u}\tilde{\nabla}_{u,\nabla^{TM}_{e_i} e_j}) + \mathit{\Phi}_u$$

with

$$\begin{aligned}\tilde{\nabla}_u &= \nabla + \sqrt{u}\rho(\sqrt{u}z)\left(\frac{1}{4}\sum_{k,\ell}\left\langle \Gamma^{TM}_{\sqrt{u}z}(\cdot)\tilde{e}_k, \tilde{e}_\ell\right\rangle c_u(e_k)c_u(e_\ell) + \Gamma^F_{\sqrt{u}z}(\cdot)\right) \\ &= \nabla + \rho(\sqrt{u}z)\left(\frac{1}{4}\sum_{k,\ell}\left\langle \frac{1}{\sqrt{u}}\Gamma^{TM}_{\sqrt{u}z}(\cdot)\tilde{e}_k, \tilde{e}_\ell\right\rangle \sqrt{u}c_u(e_k)\sqrt{u}c_u(e_\ell) + \sqrt{u}\Gamma^F_{\sqrt{u}z}(\cdot)\right).\end{aligned}$$

Recall also

$$1_{\{\sqrt{u}z \leqslant 4\varepsilon\}} \frac{1}{\sqrt{u}} \Gamma^W_{\sqrt{u}z}(\cdot) = R^W(Z, \cdot) + O(|z|)$$

uniformly on u. By the definition of $\|\cdot\|_{u,0}$, $1_{\{\sqrt{u}z \leqslant 4\varepsilon\}}\sqrt{u}c_u(e_i)$ is uniformly bounded from S_u^0 to S_u^0. Thus

$$\tilde{\nabla}_u = \nabla + O_u^{0,0}(1),$$

where $O_u^{0,0}(1)$ denotes a uniformly bounded operator from S_u^0 to S_u^0. And $\mathit{\Phi}_u = O_u^{0,0}(1)$. Moreover, since $\operatorname{supp}\rho$ is compact, there exists $c > 0$ such that $\forall u \in (0,1]$,

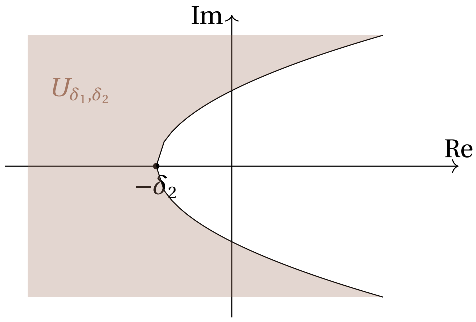
$$\sup_{\mathbb{R}^n}\left|\nabla_\bullet\left(1 + \rho\left(\frac{\sqrt{u}z}{2}\right)\right)\right| \leqslant c.$$

Hence $\tilde{\nabla}_{u,e_i}\tilde{\nabla}_{u,e_j} = \nabla_{e_i}\nabla_{e_j} + O_u^{0,0}(1)\nabla_{e_j} + O_u^{0,0}(1)$.

Therefore, the symmetric part of $\tilde{L}_u$ is of order 2 and the anti-symmetric part is of order 1. Thus we obtain (1), (2) and (3). $\square$

For $\delta_1,\ \delta_2 > 0$, let U_{δ_1,δ_2} be a domain of $\mathbb{C}$ defined by

$$U_{\delta_1,\delta_2} := \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leqslant \delta_1(\operatorname{Im}\lambda)^2 - \delta_2\right\}.$$



Proposition 6.10

There exist $\delta_1,\ \delta_2 > 0$ such that $\forall u \in (0,1]$, $\forall \lambda \in U_{\delta_1,\delta_2}$, the resolvent $(\lambda - \tilde{L}_u)^{-1}$ exists, and is a bounded operator from S_u^{-1} to S_u^1. Moreover, there exists $c > 0$ such that

$$\left\|(\lambda - \tilde{L}_u)^{-1}\right\|_{u,0,0} \leqslant c, \qquad \left\|(\lambda - \tilde{L}_u)^{-1}\right\|_{u,-1,1} \leqslant c(1+|\lambda|)^2.$$

Proof. (1) Let $\lambda \in \mathbb{R}$ and $\lambda < -c_2$, $s \in S^2$ (*actually* S_u^2, *but there is no difference in this case*) with compact support. By Proposition 6.9,

$$\operatorname{Re}\left\langle (\tilde{L}_u - \lambda)s, s\right\rangle \geqslant c_1 \|s\|_{u,1}^2 \geqslant c_1 \|s\|_{u,0}^2 . \tag{6.3}$$

So

$$\|s\|_{u,0} \leqslant \frac{1}{c_1}\left\|(\tilde{L}_u - \lambda)s\right\|_{u,0} . \tag{6.4}$$

Moreover, $\tilde{L}_u - \lambda$ is elliptic of order 2 near 0, and equals $\Delta - \lambda$ for $|z|$ large enough. Thus we obtain

$$\|s\|_2^2 \leqslant c(\left\|(\lambda - \tilde{L}_u)s\right\|_{u,0}^2 + \|s\|_{u,0}^2)$$

by elliptic estimates in Proposition 6.9 in the first case, and Fourier transform in the second case.

Using a cut-off function φ and write $s = \varphi s + (1-\varphi)s$, for any $s \in S^2$,

$$\|s\|_2^2 \leqslant c(u)\Big(\left\|(\lambda - \tilde{L}_u)s\right\|_{u,0}^2 + \|s\|_{u,0}^2\Big).$$

Thus we have elliptic estimates on S^2 and (6.4), which implies (*as in Chapter 3*) the existence of $(\lambda - \tilde{L}_u)^{-1}$.

(2) Let $\lambda = a + \mathrm{i}b$, $s \in S^2$ with compact support. We have

$$\begin{aligned}\left|\left\langle(\lambda - \tilde{L}_u)s, s\right\rangle\right| &\geqslant \max\left\{\left|\operatorname{Re}\left\langle\tilde{L}_u s, s\right\rangle - a\|s\|_{u,0}^2\right|, \left|\operatorname{Im}\left\langle\tilde{L}_u s, s\right\rangle - b\|s\|_{u,0}^2\right|\right\}\\ &\geqslant \max\left\{c_1\|s\|_{u,1}^2 - (c_2 + a)\|s\|_{u,0}^2, -c_3\|s\|_{u,1}\|s\|_{u,0} + |b|\|s\|_{u,0}^2\right\}.\end{aligned}$$

We set $c(\lambda) = \inf_{v \in [1,\infty)} \max\left\{c_1 v^2 - (c_2 + a), -c_3 v + |b|\right\}$, *i.e.*, let $v = \|s\|_{u,1} / \|s\|_{u,0}$. So that

$$\left|\left\langle(\lambda - \tilde{L}_u)s, s\right\rangle\right| \geqslant c(\lambda)\|s\|_{u,0}^2 .$$

If δ_2 large enough and δ_1 small enough, then $c_0 := \inf_{\lambda \in U_{\delta_1,\delta_2}} c(\lambda)$ satisfies $c_0 > 0$. In particular, if $(\lambda - \tilde{L}_u)^{-1}$ exists, then

$$\left\|(\lambda - \tilde{L}_u)^{-1}\right\|_{u,0,0} \leqslant \frac{1}{c_0}. \tag{6.5}$$

But this implies $(\lambda' - \tilde{L}_u)^{-1}$ exists when $|\lambda - \lambda'| < c_0/2$. As $(\lambda - \tilde{L}_u)^{-1}$ exists for $\lambda \in (-\infty, -c_2)$, and we get that $(\lambda - \tilde{L}_u)^{-1}$ exists for $\lambda \in U_{\delta_1,\delta_2}$, and (6.5) holds for any $\lambda \in U_{\delta_1,\delta_2}$. Therefore we obtain the estimate of $\left\|(\lambda - \tilde{L}_u)^{-1}\right\|_{u,0,0}$.

(3) If $\lambda_0 \in \mathbb{R}$, $\lambda_0 < -c_2$, (6.3) implies that for $s \in S^1$ with compact support,

$$\|s\|_{u,1} \leqslant \frac{1}{c_1} \left\| (\lambda_0 - \tilde{L}_u) s \right\|_{u,-1}.$$

In other words,

$$\left\| (\lambda_0 - \tilde{L}_u)^{-1} \right\|_{u,-1,1} \leqslant \frac{1}{c_1}. \tag{6.6}$$

Now for $\lambda \in U_{\delta_1,\delta_2}$,

$$(\lambda - \tilde{L}_u)^{-1} = (\lambda_0 - \tilde{L}_u)^{-1} + \underbrace{(\lambda - \tilde{L}_u)^{-1}}_{S^0 \to S^0} (\lambda_0 - \lambda) \underbrace{(\lambda_0 - \tilde{L}_u)^{-1}}_{S^{-1} \to S^0}.$$

Thus the norm estimate of $S^0 \to S^0$ and (6.6) implies that

$$\left\| (\lambda - \tilde{L}_u)^{-1} \right\|_{u,-1,0} \leqslant \frac{1}{c_1} + \frac{|\lambda - \lambda_0|}{c_0 c_1}.$$

We also have

$$(\lambda - \tilde{L}_u)^{-1} = (\lambda_0 - \tilde{L}_u)^{-1} + (\lambda_0 - \lambda) \underbrace{(\lambda_0 - \tilde{L}_u)^{-1}}_{S^0 \to S^1} \underbrace{(\lambda - \tilde{L}_u)^{-1}}_{S^{-1} \to S^0}.$$

Therefore

$$\left\| (\lambda - \tilde{L}_u)^{-1} \right\|_{u,-1,1} \leqslant \frac{1}{c_1} + \frac{|\lambda - \lambda_0|}{c_0 c_1} \left(\frac{1}{c_1} + \frac{|\lambda - \lambda_0|}{c_0 c_1} \right),$$

this implies the norm estimate of $S^{-1} \to S^1$. □

Lemma 6.11

For any $k \in \mathbb{N}$, $\exists c_k > 0$ such that $\forall s, s' \in S^1$ with compact support,

$$\left| \left\langle [\nabla_{e_{i_1}}, [\cdots, [\nabla_{e_{i_k}}, \tilde{L}_u] \cdots]] s, s' \right\rangle \right| \leqslant c_k \|s\|_{u,1} \left\| s' \right\|_{u,1}.$$

Proof. Since $[\nabla_{e_{i_1}}, [\cdots, [\nabla_{e_{i_k}}, \tilde{L}_u] \cdots]]$ has the same structure as $\tilde{L}_u$, so it is proved as (3) in Proposition 6.9. □

Proposition 6.12

For all $k \in \mathbb{N}$, $\exists m_k \in \mathbb{N}$, $c_k > 0$ such that $\forall u \in (0,1]$, $\forall \lambda \in U_{\delta_1,\delta_2}$, $(\lambda - \tilde{L}_u)^{-1}$ maps S_u^k to S_u^{k+1} with

$$\left\| (\lambda - \tilde{L}_u)^{-1} \right\|_{u,k,k+1} \leqslant c_k (1 + |\lambda|)^{m_k}.$$

Proof. We want to prove that there exist m_k, c_k such that $\forall s \in C_c^\infty$, $\forall e_{i_1}, \ldots, e_{i_{k+1}}$,

$$\left\| \nabla_{e_{i_1}} \cdots \nabla_{e_{i_{k+1}}} (\lambda - \tilde{L}_u)^{-1} s \right\|_{u,0} \leqslant c_k (1 + |\lambda|)^{m_k} \|s\|_{u,0}.$$

Note that $\nabla_{e_{i_1}} \cdots \nabla_{e_{i_{k+1}}} (\lambda - \tilde{L}_u)^{-1}$ is a linear combination of operators of the form

$$\overbrace{[\nabla_{e_{i_1}}, [\cdots, [\nabla_{e_{i_{k'}}}, (\lambda - \tilde{L}_u)^{-1}] \cdots]]}^{=:A} \nabla_{e_{i_{k'+1}}} \cdots \nabla_{e_{i_{k+1}}}, \qquad k' \leqslant k,$$

A itself is a linear combination of operators of the form

$$(\lambda - \tilde{L}_u)^{-1} R_1 (\lambda - \tilde{L}_u)^{-1} R_2 \cdots R_\ell (\lambda - \tilde{L}_u)^{-1}$$

with $R_j \in \mathscr{R}_u := \left\{ [\nabla_{e_{i_1}}, [\cdots, [\nabla_{e_{i_p}}, \tilde{L}_u] \cdots]] \right\}$ since

$$[\nabla, X^{-1}] = \nabla X^{-1} - X^{-1} \nabla = X^{-1} (X \nabla - \nabla X) X^{-1}.$$

By the lemma above, the operators in $\mathscr{R}_u$ are uniformly bounded from S^1_u to S^0_u. Then by Proposition 6.10,

$$\left\|(\lambda-\tilde{L}_u)^{-1}R_1(\lambda-\tilde{L}_u)^{-1}R_2\cdots R_\ell(\lambda-\tilde{L}_u)^{-1}\right\|_{u,0,1}\leqslant c(1+|\lambda|)^m.$$

Finally of course $\left\|\nabla_{e_i}\right\|_{u,1,0}\leqslant c$. This concludes the proof. □

Proposition 6.13

There exist $c>0$ and $k,k'\in\mathbb{N}$ such that $\forall s\in C^\infty_c$, $\forall\lambda\in U_{\delta_1,\delta_2}$, $\forall u\in(0,1]$,

$$\left\|((\lambda-\tilde{L}_u)^{-1}-(\lambda-\tilde{L}_0)^{-1})s\right\|_{0,0}\leqslant c\sqrt{u}(1+|\lambda|)^k\sum_{|\alpha|\leqslant k'}\|Z^\alpha s\|_{0,0},$$

where $\|\cdot\|_{0,0}$ is the limit of $\|\cdot\|_{u,0}$ when $u\to 0$.

Proof. First, $\|\cdot\|_{u,0}\leqslant\|\cdot\|_{0,0}$ for any $u\in(0,1]$ by definition, and that

$$(\lambda-\tilde{L}_u)^{-1}-(\lambda-\tilde{L}_0)^{-1}=(\lambda-\tilde{L}_u)^{-1}(\tilde{L}_u-\tilde{L}_0)(\lambda-\tilde{L}_0)^{-1}. \tag{6.7}$$

By Taylor expansion of the formula of $\tilde{L}_u$, we have

$$\left|\left\langle(\tilde{L}_u-\tilde{L}_0)s,s'\right\rangle\right|\leqslant c\sqrt{u}\left(\sum_{|\alpha|\leqslant k'}\|Z^\alpha s\|_{0,1}\right)\left\|s'\right\|_{u,1},$$

which implies

$$\left\|(\tilde{L}_u-\tilde{L}_0)s\right\|_{u,-1}\leqslant c\sqrt{u}\sum_{|\alpha|\leqslant k'}\|Z^\alpha s\|_{0,1}. \tag{6.8}$$

On the other hand, if $|\alpha\leqslant k|$, replace ∇_{e_i} with Z_i in Proposition 6.12, we obtain

$$\left\|Z^\alpha(\lambda-\tilde{L}_0)^{-1}s\right\|_{0,1}\leqslant c(1+|\lambda|)^{m_k}\sum_{|\beta|\leqslant k}\left\|Z^\beta s\right\|_{0,0}. \tag{6.9}$$

We need to replace the lemma before Proposition 6.12 by $\exists c_p>0$ such that $\left\|[Z_{i_1},[\cdots,[Z_{i_p},\tilde{L}_u]\cdots]]\right\|\leqslant c_p$. Using Proposition 6.12, (6.7), (6.8) and (6.9), we obtain the desired conclusion. □

For $z\in\mathbb{R}^n$, set $S_z=L^2(B(z,2),\Lambda^\bullet T^*_{x_0}M\otimes F_{x_0})$, let $\|\cdot\|_z$ be the corresponding L^2-norm on S_z, and if $A\in\mathscr{B}(S_z)$, let $|||A|||_z$ be the operator norm. Let $K_u=\mathrm{e}^{-\tilde{L}_u}-\mathrm{e}^{-\tilde{L}_0}$ and $C=\partial U_{\delta_1,\delta_2}$. Then by Proposition 6.13,

$$\begin{aligned}|||K_u|||_z&=\left|\left|\left|\int_C\mathrm{e}^{-\lambda}((\lambda-\tilde{L}_u)^{-1}-(\lambda-\tilde{L}_0)^{-1})\,\mathrm{d}\lambda\right|\right|\right|_z\\&\leqslant c\sqrt{u}\int_C\mathrm{e}^{-\mathrm{Re}\,\lambda}(1+|\lambda|)^k\,\mathrm{d}\lambda\leqslant c\sqrt{u}.\end{aligned}$$

Now let φ^z_ν be an approximation of δ_z, *i.e.*,

$$\varphi^z_\nu(w)=\frac{1}{\nu^n}\varphi\Big(\frac{w-z}{\nu}\Big),$$

with φ a nonnegative smooth function, $\operatorname{supp}\varphi\subset B(0,1)$ and $\int\varphi=1$. Observe that for any smooth function f,

$$\left|f(z)-\int_{\mathbb{R}^n}f(w)\varphi^z_\nu(w)\,\mathrm{d}w\right|\leqslant\int_{\mathbb{R}^n}\left|f(z)-f(w)\right|\varphi^z_\nu(w)\,\mathrm{d}w\leqslant\sup_{w\in B(z,1),i\in[[1,n]]}\left|\frac{\partial f}{\partial z_i}(w)\right|\cdot\nu.$$

Proposition 6.14

For all $k \in \mathbb{N}$ and $K > 0$, there exists $c > 0$ such that $\forall u \in (0,1]$, $\forall \alpha, \alpha' \in \mathbb{N}^n$ such that $|\alpha| + |\alpha'| \leqslant k$, $\forall z, z'$ such that $\max\{|z|, |z'|\} \leqslant K$,

$$\left| \frac{\partial^{\alpha+\alpha'}}{\partial z^\alpha \partial z'^{\alpha'}} \mathrm{e}^{-\tilde{L}_u}(z,z') \right| \leqslant c.$$

Proof. As $\mathrm{e}^{-a} = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{e}^{-\lambda}(\lambda - a)^{-1}\,\mathrm{d}\lambda$, its k-th derivative and we obtain

$$\mathrm{e}^{-a} = \frac{(-1)^{k-1}}{2\pi \mathrm{i}}(k-1)! \int_C \mathrm{e}^{-\lambda}(\lambda - a)^{-k}\,\mathrm{d}\lambda.$$

Thus by holomorphic functional calculus

$$\mathrm{e}^{-\tilde{L}_u} = \frac{(-1)^{k-1}}{2\pi \mathrm{i}}(k-1)! \int_C \mathrm{e}^{-\lambda}(\lambda - \tilde{L}_u)^{-k}\,\mathrm{d}\lambda.$$

Note that $\Delta K(z,z') = \Delta_z K(z,z')$, $K\Delta(z,z') = \Delta_{z'} K(z,z')$. By Proposition 6.12,

$$\left\| \Delta^{-k'}(\lambda - \tilde{L}_u)^{-2k'} \right\|_{u,0,0} \leqslant c(1+|\lambda|)^{m_{k'}}.$$

Let $\tilde{L}_u^*$ be the formal adjoint of $\tilde{L}_u$ for $\|\cdot\|_0$, then $\tilde{L}_u^*$ has the same structure as $\tilde{L}_u$ except that e^i is replaced by ι_{e_i}. So that $c_u(e_i)$ is replaced by $\frac{1}{\sqrt{u}}\iota_{e_i} - \sqrt{u} e^i \wedge$. Set

$$\|s\|_{u,0}^* := \left(\int |s|^2 \left(1 + |z|\,\rho\left(\frac{\sqrt{u}z}{2}\right)\right)^{-2(n-q)} \mathrm{d}z \right)^{1/2}, \qquad s \in C_c^\infty(\mathbb{R}^n, \Lambda^q T_{x_0}^* M \otimes F_{x_0}).$$

Everything we have proved for $\tilde{L}_u$ with $\|\cdot\|_{u,0}$ is also true for $\tilde{L}_u^*$ with $\|\cdot\|_{u,0}^*$. In particular,

$$\left\| \Delta^{k'}(\lambda - \tilde{L}_u^*)^{-2k'} \right\|_{u,0,0}^* \leqslant c(1+|\lambda|)^{m_{k'}}.$$

By taking adjoint for $\|\cdot\|_0$, we get

$$\left\| (\lambda - \tilde{L}_u^*)^{-2k'} \Delta^{k'} \right\|_{u,0,0}^* \leqslant c(1+|\lambda|)^{m_{k'}}.$$

Thus we obtain $\left|\Delta_z^k \Delta_{z'}^{k'} \mathrm{e}^{-\tilde{L}_u}(z,z')\right| \leqslant c$. Then by Sobolev embedding, the conclusion is proved. □

Now we apply the above proposition, we obtain

$$\left| K_u(z,z') - \iint K_u(w,w') \varphi_\nu^z(w) \varphi_\nu^{z'}(w')\,\mathrm{d}w\,\mathrm{d}w' \right| \leqslant c\nu.$$

But now

$$\begin{aligned}
\left| \iint K_u(w,w')\varphi_\nu^z(w)\varphi_\nu^{z'}(w')\,\mathrm{d}w\,\mathrm{d}w' \right|^2 &\leqslant \int \left| \overbrace{\int K_u(w,w')\varphi_\nu^z(w)\varphi_\nu^{z'}(w')\,\mathrm{d}w'}^{=\varphi_\nu^z(w) K_u(\varphi_\nu^{z'})(w)} \right|^2 \mathrm{d}w \\
&= \int \left| \varphi_\nu^z(w) K_u(\varphi_\nu^{z'})(w) \right| \mathrm{d}w \\
&\leqslant \underbrace{\int \left|\varphi_\nu^z(w)\right|^2 \mathrm{d}w}_{\leqslant c\nu^{-n} \text{ by definition}} \underbrace{\int \left| K_u(\varphi_\nu^{z'})(w) \right|^2 \mathrm{d}w}_{\leqslant cu\left\|\varphi_\nu^{z'}\right\|_0^2 = cu\nu^{-n}} \\
&\leqslant c\nu^{-n} \cdot cu\nu^{-n} = cu\nu^{-2n}.
\end{aligned}$$

Take $\nu = u^{1/(2n+1)}$ so that $u\nu^{-2n} = \nu$, so

$$\left| K_u(z,z') \right| \leqslant c\nu = cu^{1/(2n+1)} \to 0, \qquad u \to 0.$$

Thus $e^{-\tilde{L}_u} \to e^{-\tilde{L}_0}$ as $u \to 0$.

(ε) **Conclusion**

Now we can prove the Atiyah–Singer theorem as follows:

Since $\tilde{L}_0$ is a harmonic oscillation on $\mathbb{R}^n$, we have the Mehler formula

$$
\begin{aligned}
e^{-t\tilde{L}_0}(z,z') = \frac{\exp(-tR^F_{x_0})}{(4\pi)^{n/2}} \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \\
\cdot \exp\left(-\frac{1}{4t}\left\langle \frac{tR/2}{\tanh(tR/2)} z, z\right\rangle - \frac{1}{4t}\left\langle \frac{tR/2}{\tanh(tR/2)} z', z'\right\rangle + \frac{1}{4t}\left\langle \frac{tR/2}{\sinh(tR/2)} e^{-tR/4} z, z'\right\rangle\right),
\end{aligned}
$$

where $R = R^{TM}_{x_0}$. At $(z,z') = (0,0)$ and $t = 1$,

$$
e^{-t\tilde{L}_0}(0,0) = \frac{1}{(4\pi)^{n/2}} \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right) \exp(-R^F_{x_0}).
$$

As a consequence,

$$
\begin{aligned}
\lim_{u\to 0} \mathrm{Tr}_s(e^{-uD^2/2}(x_0,x_0))\, \mathrm{dvol}_M(x_0) &= \lim_{u\to 0} (-2i)^{n/2} \mathrm{Tr}_s^{F_{x_0}}\left(\left\{e^{-\tilde{L}_u}(0,0)\right\}^{\max}\right) \mathrm{dvol}_M(x_0) \\
&= (-2i)^{n/2} \mathrm{Tr}_s^{F_{x_0}}\left(\left\{e^{-\tilde{L}_0}(0,0)\right\}^{\max}\right) \mathrm{dvol}_M(x_0) \\
&= \left(\frac{-2i}{4\pi}\right)^{n/2} \left\{\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\mathrm{Tr}_s(\exp(-R^F_{x_0}))\right)\right\}^{\max} \\
&= \left\{\det{}^{1/2}\left(\frac{R/4\pi i}{\sinh(R/4\pi i)}\right)\mathrm{Tr}_s\left(\exp\left(-\frac{R^F_{x_0}}{2\pi i}\right)\right)\right\}^{\max} \\
&= \left\{\widehat{A}(TM,\nabla^{TM})\mathrm{ch}(F,\nabla^F)\right\}^{\max}.
\end{aligned}
$$

This is the local index theorem.

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