Groups and Amenability

M2 AAG, 08-07-2025

Contents

1 Prologue — The Banach–Tarski Paradox

Question. Is there a unique (up to scalar) measure that is invariant under the action by $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$?

- For $n \leqslant 2$, yes.
- FOr $n \geqslant 3$, no.

$\Longrightarrow$ What is the difference between $\mathrm{Isom}(\mathbb{R}^2)$ and $\mathrm{Isom}(\mathbb{R}^3)$?

1.1 Free groups

Denote the free group generated by S by $\mathbb{F}(S)$. The elements of $\mathbb{F}(S)$ are reduced words, that is, words that do not contain xx^{-1}-form.

Example 1.1

Let X be a bonquet of n circles $\{c_1, \ldots, c_n\}$. Then $\pi_1(X) = \mathbb{F}(c_1, \ldots, c_n)$. The *rank* of a free group is exactly the cardinal of its generating set.

For $a \in S$, define the

- *attractive set* of a being $\{w \in \mathbb{F}(S) : w = a^n x : n > 0\}$.
- *repulsive set* of a being $\{w \in \mathbb{F}(S) : w = a^{-n} x : n > 0\}$.

Lemma 1.2: Universal Property

Let S be a set and $\mathbb{F}(S)$ be the free group generated by S. For any group G and $\varphi : S \to G$, it extends to a unique homomorphism $\bar{\varphi} : \mathbb{F}(S) \to G$.

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d] & G \\ \mathbb{F}(S) \arrow[ur, "\bar{\varphi}"'] & \end{tikzcd}$$

Proof. The only possible choice is

$$\bar{\varphi}(s_1 \cdots s_n) = \varphi(s_1) \cdots \varphi(s_n),$$

where $w = s_1 \cdots s_n$ with $s_i \in S \cup S^{-1}$.

Proposition 1.3

Any free group $\mathbb{F}_k$, $k \geqslant 2$ contains a suibgroup isomorphic to $\mathbb{F}_n$ for each $n \in \mathbb{N}$.

Proof. By induction, it suffices to prove $\mathbb{F}_3 \leqslant \mathbb{F}_2$. Denote $\mathbb{F}_2 = \langle a, b \rangle$ and let $x = b$, $y = aba^{-1}$, $z = a^2ba^{-2}$. Then $\langle x, y, z \rangle = \mathbb{F}_3$ is a free group that can be considered as a subgroup of $\mathbb{F}_2$. $\square$

Theorem 1.4: Nielsen–Schreiner

A group G acting freely without inversion on a tree T is free. Moreover, any subgroup of a free group is free.

Proof. Let $X = T/G$ be the quotient set. Since the action $G \curvearrowright T$ is free, and without inversion, *i.e.*, $\forall x \in T\,(\mathrm{Stab}(x) = 1)$ and no element flips an edge. So X is a graph without loops. Note that T is simply connected and $\pi_1(T) = G$, so G is free.

For the « moreover » part, let $G = \mathbb{F}(S)$ and $G \curvearrowright \mathrm{Cay}(G,S)$ acts freely without inversion. Then for any subgroup $H \leqslant G$, note that $H \curvearrowright \mathrm{Cay}(G,S)$ without inversion, then H is free since $H = \pi_1(\mathrm{Cay}(G,S)/H)$. $\square$

Question. How can we determine if a group is free or not?

Lemma 1.5: Ping-pong lemma

Suppose G acts on X. If there are $a, b \in G$ and 4 disjoint sets $A^{\pm},\ B^{\pm} \subset X$, such that

- $a(A^+ \cup B^+ \cup B^-) \subset A^+$;
- $a^{-1}(A^- \cup B^+ \cup B^-) \subset A^-$;
- $b(B^+ \cup A^+ \cup A^-) \subset B^+$;
- $b^{-1}(B^- \cup A^+ \cup A^-) \subset B^-$.

Then the subgroup $\langle a, b\rangle$ generated by a and b is exactly $\mathbb{F}(a,b)$.

Proof. Let $S = \{a, b\}$ and define $\varphi|_S = \mathrm{id}_S$. By the universal property of free group, one can extend φ to $\bar{\varphi}$ and $\mathrm{im}\,\bar{\varphi} = \mathbb{F}(S)$. So it suffices to prove that $\bar{\varphi}$ is injective. This is equivalent to for any non-empty reduced word w, we have $\varphi(w) \neq 1$.

Define A^+ the attractive set of a and A^- the repulsive set of a. And similarly define B^+ and B^-. For $w = a_1 \cdots a_n$, $x \in A^+ \sqcup A^- \sqcup B^+ \sqcup B^-$ such that x is not in the attractive set of a_1, x is not in the repulsive set of a_n. Then

$$\begin{aligned} a_n x \in \text{ the attractive set of } a_n &\implies a_{n-1}a_n x \in \text{the attractive set of } a_{n-1} \\ &\implies \text{(Induction)}\, a_{n-k}\cdots a_n \in \text{the attractive set of } a_{n-k} \\ &\implies wx = a_1 \cdots a_n x \in \text{the attractive set of } a_1. \end{aligned}$$

Hence $wx \neq x$, so $\varphi(w) \neq 1$. $\square$

Proposition 1.6

SO(3) contains a free subgroup generated by

$$a = \frac{1}{3}\begin{bmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad b = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{bmatrix}.$$

Proof. Let w be a non-empty word in a, b. We claim that if w is reduced and ends with a, then $w[1,0,0]^\top = 3^{-k}[x, y\sqrt{2}, z]^\top$ with $x, y, z \in \mathbb{Z}$ and $y \notin 3\mathbb{Z}$. The claim proves that $w[1,0,0]^\top \neq [1,0,0]^\top$ because any non-trivial word can be conjugated to a word that ends with a.

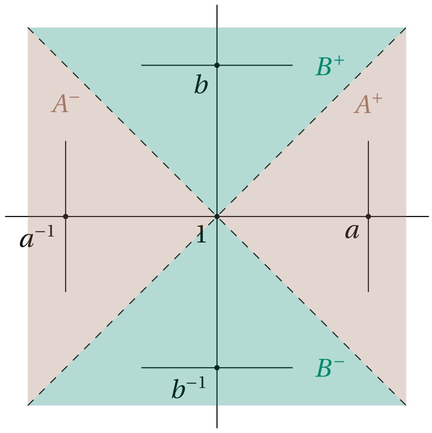
Do induction:

$$a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 2\sqrt{2} \\ 0 \end{bmatrix}.$$

So it holds for $k = 1$. Now suppose it holds for $\leqslant k-1$. If $w = a_k a_{k-1} \cdots a_1 = a_k x$, and $x[1,0,0]^\top = 3^{-(k-1)}[x, y\sqrt{2}, z]^\top$ with $y \notin 3\mathbb{Z}$. Then it suffices to calculate $a,\ a^{-1},\ b,\ b^{-1}$ acting on $[x, y\sqrt{2}, z]^\top$ and check $y' \notin 3\mathbb{Z}$. $\square$

Example 1.7

Denote $X = \text{Cay}(\mathbb{F}_2)$, the Caylay graph of $\mathbb{F}_2$. And $\mathbb{F}_2 \curvearrowright X$ by left-multiplication. One can define $A^{\pm}, B^{\pm} \subset X$ as in the graph, and ping-pong lemma applies.



Note that $\text{SO}(2) \cong (\mathbb{R}/\mathbb{Z}, +)$ is Abelian so $\text{SO}(2)$ does not contain $\mathbb{F}_2$. Because the subgroup of Abelian group must be Abelian.

1.2 Paradoxes and paradoxial decompositions

Definition 1.8: Equidecomposibility

Let $G \curvearrowright X$. Say $A, B \subset X$ are G-*equidecomposible* if there are finite partitions $A = \coprod_{i\in I} A_i$ and $B = \coprod_{i\in I} B_i$, with $g_i \in G$ such that $g_i A_i = B_i$. Denoted as $A \equiv_G B$.

Example 1.9

Let $G = \text{Isom}(\mathbb{R}^2)$ and $X = \mathbb{R}^2$. Then $G \curvearrowright X$ preserves area. Let A, B be polygons with same area, then $A \equiv_G B$.

This is no more true in $\mathbb{R}^3$. Because Hilbert's 3^{rd} problem has been solved negative by Max Delm with half-spaces splitting.

Lemma 1.10

Equidecomposibility is an equivalence relation on 2^X.

Proof. Since $A \equiv_G B$, one has $A = \coprod_{i\in I} A_i$, $B = \coprod_{i\in I} B_i$ with $g_i A_i = B_i$. Since $B \equiv_G C$, one has $B = \coprod_{j\in J} B'_j$, $C = \coprod_{j\in J} C'_j$ with $h_j B'_j = C'_j$. Therefore, let $A_{ij} = A_i \cap g_i^{-1}(B'_j)$, then $g_i A_{ij} = B_i \cap B'_j$, and hence

$$h_j g_i A_{ij} = h_j (B_i \cap B'_j) = h_j(B_i) \cap C'_j =: C_{ij}.$$

So $A \equiv_G C$. $\square$

Definition 1.11: Paradoxial

Say $G \curvearrowright X$ is *paradoxial* if there exist disjoint subsets $A \subset B \subset X$ such that $A \equiv_G X$ and $B = \equiv_G X$.

Proposition 1.12

$\mathbb{F}_2 \curvearrowright \mathbb{F}_2$ by multiplication is paradoxial.

Proof. Denote A^+ the attractive set of a, A^- the repulsive set of a and similarly B^+ and B^-. Then

$$\mathbb{F}_2 = \{1\} \sqcup A^+ \sqcup A^- \sqcup B^+ \sqcup B^-.$$

Define $A = A^+ \sqcup A^-$, $B = B^+ \sqcup B^-$. Then $\mathbb{F}_2 \equiv_{\mathbb{F}_2} A$ because $a(\mathbb{F}_2 \setminus A^-) \subset A^+$. Similarly $B \equiv_{\mathbb{F}_2} \mathbb{F}_2$. □

Corollary 1.13

Let $G \curvearrowright X$ such that there exists $\mathbb{F}_2 \leqslant G$, and the action restricted to $\mathbb{F}_2$ is free. Then the action $G \curvearrowright X$ is paradoxial.

Proof. Since X is a disjoint union of $\mathbb{F}_2$-orbits, there exists $(x_i)_{i\in I} \subset X$ such that $(\mathbb{F}_2 x_i)_{i\in I}$ are mutually disjoint. Then

$$\mathbb{F}_2 \to X = \coprod_{i\in I} \mathbb{F}_2 x_i, \qquad g \mapsto g x_i$$

is a bijection. Thus we can define

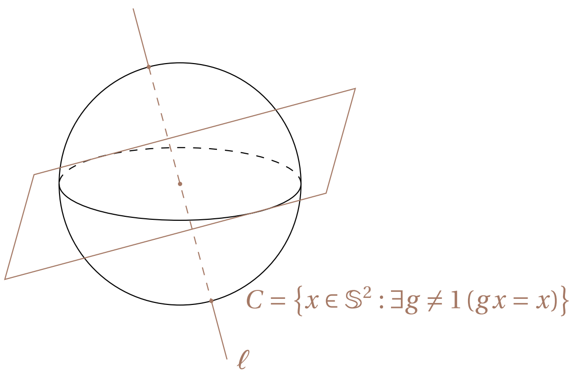
$$X_A := \coprod_{i\in I} A^{\pm} x_i \equiv_{\mathbb{F}_2} \left(\coprod_{i\in I} A^- x_i \right) \sqcup \left(\coprod_{i\in I} (\mathbb{F}_2 \setminus A^-) x_i \right) = X.$$

And similarly $X_B = \coprod_{i\in I} B^{\pm} x_i \equiv_{\mathbb{F}_2} X$. □

Theorem 1.14: Hausdorff paradox

There exists a countable $C \subset \mathbb{S}^2$ and a subgroup $G \leqslant \mathrm{SO}(3)$ generating by 2 rotations preserving C such that $G \curvearrowright \mathbb{S}^2 \setminus C$ is paradoxial.

Proof. Let $G = \langle a, b \rangle$ and C the collection of all points of the sphere that lies on the axis of $g \in G$. Note that $\mathbb{F}_2$ acts on C freely.



Then C is countable because G is countable. Also, g preserves C by construction. Thus we can apply Corollary 1.13 with $F = G$ and $X = \mathbb{S}^2 \setminus C$. □

Corollary 1.15

$\mathrm{SO}(3) \curvearrowright \mathbb{S}^2$ is paradoxial.

Proof. Choose ℓ such that $\ell \cap C = \varnothing$. Denote R_α the rotation by α. Since C is countable, one can find α such that $R_\alpha^n(C)$ are mutually disjoint for $n \geqslant 0$. Then define $C^* = \bigcup_{n\in\mathbb{N}} R_\alpha^n(C)$ and C^* is also countable. Then

$$\mathbb{S}^2 \equiv_{\mathrm{SO}(3)} C^* \sqcup (\mathbb{S}^2 \setminus C^*) \equiv R_\alpha(C^*) \sqcup (S^2 \setminus C^*) \equiv (C^* \setminus C) \sqcup (\mathbb{S}^2 \setminus C^*) \equiv \mathbb{S}^2 \setminus C.$$

Hence $\left\{C, \mathbb{S}^2 \setminus C\right\}$ is an SO(3)-paradoxial decomposition of $\mathbb{S}^2$.

Theorem 1.16: Banach–Tarski paradox

The unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$ has a paradoxial decomposition under $\mathrm{Isom}(\mathbb{R}^3)$.

Proof. By Corollary 1.13, one can radically extending the SO(3)-paradoxial decomposition of $\mathbb{S}^2$ to $r\mathbb{S}^2$ for $r \in (0,1]$. Then $\mathbb{B}^3 = \{0\} \sqcup \coprod_{r\in(0,1]} r\mathbb{S}^2$, done. □

2 Characterisations of Amenability for Discrete Groups

In this chapter, we always endow a group G with discrete topology. Unless stated, we assume G is a countable group.

2.1 von Neumann's criterion – via invariant means

Let X be a set. We define some spaces first:

- Denote $\mathrm{Prob}_f(X)$ the space of finitely supported probabilities on X, *i.e.*, the space of finite convex conbinations of Dirac measures.
- Denote $\mathrm{Prob}(X)$ the space of countable supported probabilities on X, *i.e.*,

$$\left\{\mu = \sum_{i\in I}\lambda_i\delta_{x_i} : x_i \in X,\ \lambda_i \geqslant 0,\ \sum_{i\in I}\lambda_i = 1\right\} \subset \ell^1(X).$$

  If X is countable, then $\mathrm{Prob}(X)$ is exactly the space of probability measures on X.
- Denote $\mathscr{M}(X)$ the space of means on X. A *mean* on X is a functional $m \in \ell^\infty(X)^*$ such that m is positive, *i.e.*, $\forall f \in \ell^\infty(X)$, $f \geqslant 0 \implies m(f) \geqslant 0$, and normalised, *i.e.*, $m(\mathbb{1}_X) = 1$.
- Denote $\mathscr{M}_f(X)$ the space of finitely additive probability measures on X, *i.e.*, a function

$$\mu : 2^X \to [0,1], \qquad A \mapsto \mu(A)$$

  such that $\mu(X) = 1$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$.

Also, recall some results in functional analysis,

- For a set X, $\ell^1(X)^* = \ell^\infty(X)$.
- For a locally convex topological vector space E, it is dense in (E^{**}, wk^*). In particular, $\ell^1(X)^*$ is dense in $\ell^\infty(X)^*$.
- Let C be a convex subset of a normed space E, C is norm-closed if and only if C is wk-closed.
- One has $E \subset E^{**}$, then $(E, \mathrm{wk}) = (E, \mathrm{wk}^*)$ when E is locally convex.

Proposition 2.1

There are one-to-one correspondences

(1) $\mathrm{Prob}_f(X) \to \mathscr{M}(X)$ given by $\sum_{i\in I}\lambda_i\delta_{x_i} \mapsto \left[f \mapsto \sum_{i\in I}\lambda_i f(x_i)\right]$.

(2) $\mathscr{M}(X) \to \mathscr{M}_f(X)$ given by $m \mapsto [A \mapsto m(\mathbb{1}_A)]$.

Proof. (1) Obvious, The interesting part is that $\mathrm{Prob}_f(X)$ is wk*-dense in $\mathscr{M}(X)$. If not, there exists $m_0 \in \mathscr{M}(X) \setminus \overline{\mathrm{Prob}_f(X)}$. By Hahn–Banach theorem, there exists $f \in \ell^\infty(X)$ such that

$$\max_{m\in\mathrm{Prob}_f(X)} m(f) \leqslant \mu < m_0(f).$$

In particular, $\delta_x(f) = f(x) \leqslant \mu < m_0(f)$, hence $\|f\|_\infty \leqslant m_0(f)$. This contradicts with the fact $\|m\| = 1$.

(2) *Injectivity.* Denote $\hat{m}(A) = m(\mathbb{1}_A)$. If $\hat{m}_1 = \hat{m}_2$ on simple functions $f = \sum_{i=1}^n \lambda_i \mathbb{1}_{X_i}$, $X = \coprod_{i=1}^n X_i$, then

$$m_1(f) = \sum_{i=1}^n \lambda_i \hat{m}_1(X_i) = \sum_{i=1}^n \lambda_i \hat{m}_2(X_i) = m_2(f).$$

By continuity $m_1 = m_2$.

Surjectivity. If $\mu \in \mathrm{Prob}_f(X)$, define

$$\hat{\mu}(f) := \sum_{i=1}^{n} \lambda_i \mu(X_i), \qquad f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{X_i}.$$

Then $|\hat{\mu}(f)| \leqslant \|f\|_\infty$. So $\bar{\mu}$ can be continuously extended to $\ell^\infty(X)$ by density of simple functions. And $\bar{\mu}(f) \geqslant 0$ and $\bar{\mu}(\mathbb{1}_X) = 1$ can be simply verified. □

Assume $G \curvearrowright X$ by $x \mapsto g \cdot x$. Then define an action $G \curvearrowright \ell^\infty(X)$ by

$$(g \cdot f)(x) = f(g^{-1}x), \qquad \forall f \in \ell^\infty(X),\ \forall x \in X,\ \forall g \in G.$$

Then $g \cdot \mathbb{1}_A = \mathbb{1}_{gA}$, and the action is norm-preserving. Then define $G \curvearrowright \ell^\infty(X)^*$ by duality

$$(g \cdot m)(f) = m(g^{-1} \cdot f), \qquad \forall m \in \ell^\infty(X)^*,\ \forall f \in \ell^\infty(X),\ \forall g \in G.$$

The action is still norm-preserving. In particular,

$$\widehat{g \cdot m}(A) = (g \cdot m)(\mathbb{1}_A) = m(g^{-1} \cdot \mathbb{1}_A) = m(\mathbb{1}_{g^{-1}A}) = \hat{m}(g^{-1}A).$$

Thus we defined an action $G \curvearrowright \mathscr{M}_f(X)$.

Definition 2.2: One-side amenability

A mean m is said to be *left-invariant* if $\forall g \in G$, $g \cdot m = m$. Say a group G is *left-amenable* if it admits a left-invariant mean.

Similarly, one can define right-invariance by right group actions, and right-amenability of G.

Definition–Proposition 2.3: Amenability

A mean m is said to be *invariant* if it is both left and right-invariant. And a group G is *amenable* if it admits an invariant mean. For groups, left-amenability is equivalent to amenability.

Proof. Let $m_L \in \ell^\infty(G)^*$ be a left-invariant mean. For $f \in \ell^\infty(G)$, define $\tilde{f} \in \ell^\infty(G)$ by $\tilde{f}(x) := f(x^{-1})$. Then define

$$m_R : \ell^\infty(G) \to \mathbb{C}, \qquad f \mapsto m_L(\tilde{f}).$$

It is linear because m_L is linear. And note that

$$(f \cdot g)(x) = f(xg) = \tilde{f}(g^{-1}x^{-1}) = g \cdot \tilde{f}(x).$$

Hence

$$m_R(f \cdot g) = m_L(g \cdot \tilde{f}) = m_L(\tilde{f}) = m_R(f).$$

Note that $\tilde{\tilde{f}} = f$, so the inverse still holds. □

2.2 Følner criterion

Definition 2.4: Følner set

Let G be a group. Say it satisfies *Følner's condition* if for any $\varepsilon > 0$, for any finite subset $K \subset G$, there exists a finite subset $F \subset G$ such that

$$\frac{|Fg \triangle F|}{|F|} \leqslant \varepsilon, \qquad \forall g \in K.$$

The subset F is called a *right* (ε, Q)*-Følner set* for G.

Definition–Proposition 2.5: Følner sequence

If G is countable, then G satisfies Følner's condition if and only if it admits a sequence of finite subsets $(F_n)_{n\geqslant 1} \subset G$, such that

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0, \qquad \forall g \in G.$$

Say $(F_n)_{n\geqslant 1}$ is a *right Følner sequence.*

Proof. $\Rightarrow$: Denote $G = \{g_1, g_2, \dots\}$ since it is countable. Let $K_n = \{g_1, \dots, g_n\}$, by Følner's condition, there exists a finite $F_n \subset G$ such that

$$\frac{|F_n g \triangle F_n|}{|F_n|} \leqslant \frac{1}{n}, \qquad \forall g \in K_n.$$

Then $(F_n)_{n\geqslant 1}$ is a Følner sequence.

$\Leftarrow$: For any finite subset $K \subset G$, there exists $n \in \mathbb{N}$ large enough such that

$$\frac{|F_n g \triangle F_n|}{|F_n|} \leqslant \varepsilon, \qquad \forall g \in K.$$

So G satisfies Følner's condition. $\square$

Remark. *One can similarly define left Følner set / sequence. If* $(F_n)_{n\geqslant 1}$ *is a right Følner sequence, then* $(F_n^{-1})_{n\geqslant 1}$ *is a left Følner sequence.*

Proposition 2.6

Let G be a group generated by a finite symmetric subset S. Then $(F_n)_{n\geqslant 1}$ is a Følner sequence if and only if

$$\lim_{n\to\infty} \frac{|F_n s \triangle F_n|}{|F_n|} = 0, \qquad \forall s \in S.$$

Proof. $\Rightarrow$: Trivial.

$\Leftarrow$: Since $G = \langle S \rangle$, for any $g \in G$, it is of the form $g = s_1 \cdots s_k$ with $s_i \in S$. Note that $|F_n g^{-1} \triangle F_n| = \|\mathbb{1}_{F_n}(\cdot g) - \mathbb{1}_{F_n}(\cdot)\|_1$. Hence for $g \in G$,

$$\begin{aligned}
\|\mathbb{1}_{F_n}(\cdot g) - \mathbb{1}_{F_n}(\cdot)\|_1 &= \|\mathbb{1}_{F_n}(\cdot s_1 \cdots s_k) - \mathbb{1}_{F_n}(\cdot)\|_1 \\
&\leqslant \sum_{i=1}^{k-1} \|\mathbb{1}_{F_n}(\cdot s_1 \cdots s_{i+1}) - \mathbb{1}_{F_n}(\cdot s_1 \cdots s_i)\|_1 \\
&= \sum_{i=1}^{k-1} \|\mathbb{1}_{F_n}(\cdot s_{i+1}) - \mathbb{1}_{F_n}(\cdot)\|_1 \\
&= \sum_{i=1}^{k-1} |F_n s_{i+1}^{-1} \triangle F_n|.
\end{aligned}$$

Here we use the fact that $\|f(\cdot g)\|_1 = \|f\|_1$ for $f \in \ell^1(G)$. Let $n \to \infty$, by assumption, $\|\mathbb{1}_{F_n}(\cdot g) - \mathbb{1}_{F_n}(\cdot)\|_1 = |F_n g^{-1} \triangle F_n| \to 0$. $\square$

Now if $G = \langle S \rangle$ as in Proposition 2.6, one can obtain a (*right*) Cayley graph $\mathrm{Cay}(G, S)$, whose vertices are G and the edges are (g, gs) for $g \in G$ and $s \in S$.

Let X be a connected graph, A be a subset of vertices. Denote the 1-neighbourhood of A by $[A]_1$. The *external boundary* of A is defined as

$$\partial A := [A]_1 \setminus A.$$

Proposition 2.7

Let $G = \langle S \rangle$ with S finite and symmetric. Let $X = \mathrm{Cay}(G, S)$. Then $(F_n)_{n \geqslant 1}$ is a Følner sequence if and only if $\lim_{n\to\infty} |\partial F_n| / |F_n| = 0$.

Proof. $\Rightarrow$: Note that $F_n \subset X$ is finite. By definition,

$$\partial F_n = [F_n]_1 \setminus F_n = \bigcup_{s \in S} F_n s \setminus F_n.$$

Hence $\partial F_n \subset \bigcup_{s\in S} F_n s \triangle F_n$, so $|\partial F_n| \leqslant |F_n s \triangle F_n|$ and the conclusion follows.

$\Leftarrow$: This is because

$$|F_n s \triangle F_n| \leqslant |F_n s \setminus F_n| + |F_n \setminus F_n s| = |F_n s \setminus F_n| + \left|F_n s^{-1} \setminus F_n\right| \leqslant 2 |\partial F_n|.$$

So we conclude the proof. $\square$

Theorem 2.8: Følner's criterion

For a discrete group G, it is amenable if and only if it satisfies Følner's condition. Moreover, if G is countable, it is equivalent to G admitting a Følner's sequence.

Proof. $\Rightarrow$: Let m be an invariant mean on $\ell^\infty(G)$. We prove by contradiction. Assume that there is no Følner sequence exists. Then there is some finite subset $K \subset G$ and $\varepsilon > 0$ such that for any finite $F \subset G$, one can find $g \in K$ such that

$$|F \triangle Fg| \geqslant \varepsilon |F|.$$

Now fix a particular $F \subset G$, define a function

$$h_F(x) := \frac{1}{|K|} \sum_{g \in K} |\mathbb{1}_F(x) - \mathbb{1}_F(xg)|.$$

Then h_F is bounded below because by assumption on F, $\exists g \in K$ such that $|x : \mathbb{1}_F(x) \neq \mathbb{1}_F(xg)| \geqslant \varepsilon |F|$. So $h_F(x) \geqslant \frac{1}{|K|}$ on at least $\varepsilon |F|$ points with $h_F(x) \geqslant 0$.

Since m is invariant,

$$m(\mathbb{1}_F) = m(g \cdot \mathbb{1}_F) \implies m(\mathbb{1}_F - g \cdot \mathbb{1}_F) = 0.$$

But positivity of m gives

$$m(h_F) = \frac{1}{E} \sum_{g \in K} m(\mathbb{1}_F - g \cdot \mathbb{1}_F) = 0,$$

which forces $m(h_F) = 0$. While

$$0 = m(h_F) \geqslant \frac{1}{|E|} m(\mathbb{1}_{\{x : h_F(x) \geqslant 1/|K|\}}) = \frac{1}{|E|} m(\mathbb{1}_{K'})$$

for some subset $K' \subset G$ satifsying $|K'| \geqslant \varepsilon |F|$. Since $\varepsilon > 0$, $m(\mathbb{1}_{K'}) > 0$, which leads to a contradiction.

Therefore, for every finite $K \subset G$ and $\varepsilon > 0$, there exists a finite F with

$$\max_{g \in K} \frac{\left|F \triangle F g\right|}{|F|} < \varepsilon.$$

Let $\varepsilon = 1/n$ and one can choose $(F_n)_{n\geqslant 1}$ to form a Følner sequence.

$\Leftarrow$: For $(\varepsilon_n)_{n\geqslant 1} \to 0$ decreasingly, one can choose a family of finite sets $(F_n)_{n\geqslant 1}$. Note that $\psi_n := \mathbb{1}_{F_n}/|F_n| \in \mathscr{M}_f(G)$. By compactness of $\mathscr{M}(G)$, there exists a cluster point $\psi \in \mathscr{M}(G)$ of $(\psi_n)_{n\geqslant 1}$. It suffices to prove that ψ is left-invariant.

For any $\varepsilon > 0$, $K \subset G$ and $g \in G$, there exists a $n \in \mathbb{N}$ large enough such that $\left|\psi_n(A) - \psi(A)\right| \leqslant \varepsilon/3$, $\left|\psi_n(gA) - \psi(gA)\right| \leqslant \varepsilon/3$. On the other hand,

$$\left|\psi_n(gA) - \psi(gA)\right| \leqslant \left\|\psi_n(g\cdot) - \psi_n\right\|_1 = \frac{\left|g^{-1}F_n \triangle F_n\right|}{|F_n|} \leqslant \frac{\varepsilon}{3}$$

for large enough n. Hence $\left|\psi(A) - \psi(gA)\right| \leqslant \varepsilon$. Let $\varepsilon \to 0$, done. □

2.3 Reiter's condition – functional analysis characterisation

To establish Reiter's condition, we need a technical lemma.

Lemma 2.9

Let X be a set, $(A_t)_{t>0}$ and $(B_t)_{t>0}$ be non-increasing families of fin X. Then

$$\left|\int_0^\infty (\mathbb{1}_{A_t} - \mathbb{1}_{B_t})\,\mathrm{d}t\right| = \int_0^\infty \left|\mathbb{1}_{A_t} - \mathbb{1}_{B_t}\right|\mathrm{d}t.$$

Proof. For any fixed $x \in X$, assume $\exists t_0 > 0$ such that $(\mathbb{1}_{A_{t_0}} - \mathbb{1}_{B_{t_0}})(x) > 0$, which is $x \in A_{t_0} \setminus B_{t_0}$. So for any $t > t_0$, $x \notin B_t$, hence $(1_{A_t} - \mathbb{1}_{B_t})(x) > 0$. This means that for any fixed $x \in X$, $\mathbb{1}_{A_t} - \mathbb{1}_{B_t}$ has constant sign. □

Lemma 2.10: Co-area formula

Let $f \in \ell^1(G)$ and $f \geqslant 0$. For $t \geqslant 0$, define $F_t = \{f \geqslant t\}$.

(1) $\|f\|_1 = \int_0^\infty |F_t|\,\mathrm{d}t$.

(2) For any $g \in G$, we have the *co-area formula*

$$\left\|f(\cdot) - f(\cdot g^{-1})\right\|_1 = \int_0^\infty \left|F_t \triangle F_t g\right|\mathrm{d}t.$$

Proof. (1) By Fubini theorem,

$$\|f\|_1 = \int_G f(x)\,\mathrm{d}x = \int_G \int_0^\infty \mathbb{1}_{\{f\geqslant t\}}(x)\,\mathrm{d}t\,\mathrm{d}x = \int_0^\infty \int_G \mathbb{1}_{\{f\geqslant t\}}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty |F_t|\,\mathrm{d}t.$$

(2) Let $A_t = F_t$ and $B_t = F_t g$. Then $(A_t)_{t>0}$, $(B_t)_{t>0}$ are non-increasing families. By Lemma 2.9,

$$\begin{aligned}\left\|f(\cdot) - f(\cdot g^{-1})\right\|_1 = \int_G \left|\int_0^\infty (\mathbb{1}_{A_t} - \mathbb{1}_{B_t})\,\mathrm{d}t\right|\mathrm{d}x &= \int_G \int_0^\infty \left|\mathbb{1}_{A_t} - \mathbb{1}_{B_t}\right|\mathrm{d}t\,\mathrm{d}x \\ &= \int_0^\infty \int_G \left|\mathbb{1}_{A_t} - \mathbb{1}_{B_t}\right|\mathrm{d}x\,\mathrm{d}t = \int_0^\infty |A_t \triangle B_t|\,\mathrm{d}t.\end{aligned}$$

Then we proved the co-area formula. □

Theorem 2.11: Reiter's criterion

Let G be a group. The followings are equivalent.

(1) G is amenable.

(2) For any $\varepsilon > 0$ and any $K \in \operatorname{fin} G$, there exists $f \in \ell^1(G)$ with $\|f\|_1 = 1$ such that

$$\|f(\cdot) - f(\cdot g^{-1})\|_1 < \varepsilon, \qquad \forall g \in K.$$

This is called the *Reiter's condition.*

(3) For any $1 \leqslant p < \infty$, $\varepsilon > 0$ and $K \in \operatorname{fin} G$, there exists $f \in \ell^p(G)$ with $\|f\|_p = 1$ such that

$$\|f(\cdot) - f(\cdot g^{-1})\|_p < \varepsilon, \qquad \forall g \in K.$$

This is called the *Reiter p-condition.*

Proof. By Theorem 2.8, one can replace (1) with Følner's condition.

(1) ⇒ (2): For any finite subset $K \subset G$ and $\varepsilon > 0$, the Følner condition gives a non-empty finite set $F \subset G$ such that $|F \triangle gF|/|F| < \varepsilon$, $\forall g \in K$. Let $f := \mathbb{1}_F/|F| \in \ell^1(G)$, $\|f\|_1 = 1$ and $\operatorname{supp} f = F$ is finite. So for any $g \in K$,

$$\frac{|Fg \triangle F|}{|F|} = \|f(\cdot g^{-1}) - f(\cdot)\|_1.$$

(2) ⇒ (3): By Reiter's condition, for given $F \subset G$ and $\varepsilon > 0$, there exists $f \in \ell^1(G)$ with $\|f\|_1 = 1$, f finitely supported, such that

$$\|f - g \cdot f\|_1 < \varepsilon, \qquad \forall g \in F.$$

Let $\tilde f(x) := f(x)^{1/p}$, then $\tilde f \in \ell^p(G)$ with $\|\tilde f\|_p = 1$, and $\operatorname{supp} \tilde f = \operatorname{supp} f$. While for any $g \in F$, since $|b - a|^p \leqslant |a^p - b^p|$,

$$\begin{aligned}\|\tilde f - \tilde f(\cdot g^{-1})\|_p^p &= \sum_{x \in G} |\tilde f(x) - \tilde f(xg^{-1})|^p = \sum_{x \in G} |f(x)^{1/p} - f(xg^{-1})^{1/p}|^p \\ &\leqslant \sum_{x \in G} |f(x) - f(xg^{-1})| = \|f - f(\cdot g^{-1})\|_1 < \varepsilon.\end{aligned}$$

(3) ⇒ (1): Define an order on $\Lambda := \{(E, \varepsilon) : E \subset G \text{ is a finite subset}, \ \varepsilon > 0\}$:

$$(E, \varepsilon) \leqslant (E', \varepsilon') :\Longleftrightarrow E \subset E', \ \varepsilon \geqslant \varepsilon'.$$

Then Λ is directed. Let $p = 1$.

For any $(E, \varepsilon) \in \Lambda$, one can fix an almost invariant vector $f_{(E,\varepsilon)} \in \ell^1(G)$ with norm 1 and positive. Now $\ell^1(G)$ can be seen as a subspace of $\ell^\infty(G)$ with the unit ball of $\ell^1(G)$ mapped into the unit ball of $\ell^\infty(G)$. By Banach–Alaoglu's theorem, there exists a subnet $(f_i)_{i \in I}$ that converges in the wk*-topology. Denote its limit as M. Then $M(\varphi) \geqslant 0$ for any $\varphi \geqslant 0$, and $M(\mathbb{1}_G) = 1$. Now fix $g \in G$ and $\varphi \in \ell^\infty(G)$,

$$|M(\varphi(\cdot g^{-1})) - M(\varphi)| = \left|\lim_{i \in I} f_i(\varphi(\cdot g^{-1})) - \lim_{i \in I} f_i(\varphi)\right| \leqslant \limsup_{i \in I} |f_i(\varphi(\cdot g^{-1})) - f_i(\varphi)|.$$

Note that

$$f_i(\varphi(\cdot g^{-1})) = \sum_{x \in G} f_i(x) \varphi(xg^{-1}) = \sum_{y \in G} f_i(yg) \varphi(y) = \varphi(f_i(\cdot g)).$$

Thus

$$\limsup_{i \in I} |f_i(\varphi(\cdot g^{-1})) - f_i(\varphi)| = \limsup_{i \in I} |\varphi(f_i(\cdot g) - f_i)| \leqslant \|\varphi\|_\infty \limsup_{i \in I} \|f_i(\cdot g) - f_i\| = 0.$$

So M is an right-invariant mean on G, which implies that G is amenable. □

2.4 Fell's criterion – via unitary representations

Definition 2.12: Relations of unitary representations

Let G be a group and (π_1, H_1), (π_2, H_2) be two unitary representations. We say

(1) π_1 and π_2 are *isomorphic*, denoted as $\pi_1 \cong \pi_2$, if there exists $\phi : H_1 \to H_2$ interwining π_1 and π_2.

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & H_2 \\ \big\downarrow{\scriptstyle \pi_1(g)} & & \big\downarrow{\scriptstyle \pi_2(g)} \\ H_1 & \xrightarrow{\phi} & H_2 \end{array}$$

(2) π_1 is *contained* in π_2, denoted as $\pi_1 < \pi_2$, if there exists a subrepresentation π_2' of π_2, such that $\pi_1 \cong \pi_2'$. In other words, there exists a $\pi_2(G)$-invariant subspace $V_2 \subset H_2$ and $\phi : H_1 \to V_2$ such that the diagram commutes.

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & V_2 \\ \big\downarrow{\scriptstyle \pi_1(g)} & & \big\downarrow{\scriptstyle \pi_2'(g)} \\ H_1 & \xrightarrow{\phi} & V_2 \end{array}$$

(3) π_1 is *weakly contained* in π_2, denoted as $\pi_1 \prec \pi_2$, if for any $\varepsilon > 0$, $K \Subset G$ and $v \in H_1$, there exist $\{w_1, \ldots, w_\ell\} \subset H_2$ such that

$$\left| \langle \pi_1(g) v, v \rangle - \sum_{i=1}^{\ell} \langle \pi_2(g) w_i, w_i \rangle \right| < \varepsilon, \qquad \forall g \in K.$$

It is obvious that

$$\pi_1 \cong \pi_2 \quad \Longrightarrow \quad \pi_1 < \pi_2 \quad \Longrightarrow \quad \pi_1 \prec \pi_2.$$

Denote the trivial representation $1 : G \to \{\mathrm{id}\} \subset \mathrm{U}(H)$. Then

(1) If $\pi_1 \prec \bigoplus_{i \in I} \pi_2$, then $\pi_1 \prec \pi_2$. Hence we do not care the multiplicity of π_2 in weakly containing relation.

(2) $1 \prec \bigoplus_{i \in I} \pi_2$ if and only if $1 \prec \pi_2$. It is obvious by (1) in one direction, and by taking $\{w_1, \ldots, w_\ell\}$ with $\sum_{i=1}^{\ell} \| w_i \|^2 = 1$ in the other direction.

(3) $1 \prec \pi$ if and only if for any $\varepsilon > 0$, $K \Subset G$, there exists $w \in H$ with $\| w \| = 1$ such that

$$\left| 1 - \langle \pi(g) w, w \rangle \right| \leq \varepsilon.$$

Definition 2.13: Almost invariant vector

Let G be a group and (π, H) be a representation. Given $\varepsilon > 0$ and $K \Subset G$, say $w \in H$, $\| w \| = 1$ is an (ε, K)*-almost invariant vector*, if

$$\left\| \pi(g) w - w \right\| \leq \varepsilon, \qquad \forall g \in K.$$

By the definition of containing, $1 < \pi$ if and only if π admits a non-trivial invariant vector.

Proposition 2.14

Let G be a group, (π, H) be a unitary representation. Then $1 \prec \pi$ if and only if for any $\varepsilon > 0$, $K \in \operatorname{fin} G$, there exists an (ε, K) almost invariant vector w.

Proof. Because

$$\|w - \pi(g)w\|^2 = 2\left|1 - \langle \pi(g)w, w\rangle\right|.$$

Then done by (3) before Definition 2.13. □

Theorem 2.15: Fell's criterion

Denote λ the left regular representation of G on $H = \ell^2(G)$. Then G is amenable if and only if $1 \prec \lambda$.

Proof. By Proposition 2.14, $1 \prec \lambda$ is equivalent to $\forall \varepsilon > 0$, $\forall K \in \operatorname{fin} G$, $\exists f \in \ell^2(G)$ with $\|f\|_2 = 1$ such that

$$\|f - \lambda(g)f\|_2 = \|f - f(g^{-1}\cdot)\|_2 \leqslant \varepsilon.$$

This is Reiter 2-condition, done by Theorem 2.11. □

2.5 The growth function of a finitely generated group

In this section, we assume G is generated by a finite set S. On $\operatorname{Cay}(G,S)$, one can define the *length metric* by

$$\ell_S(g) := \min\left\{n \in \mathbb{N} : g = s_{i_i}^{\pm 1} \cdots s_{i_n}^{\pm 1},\ s_{i_k} \in S\right\}, \qquad d_S(g,h) := \ell_S(g^{-1}h).$$

Then d_S is left-invariant by definition, *i.e.*, $\forall g, x, y \in G$, $d_S(x,y) = d_S(gx, gy)$.

Definition 2.16: Growth function

The *growth function* of G associated to S is defined as

$$f : \mathbb{N} \to \mathbb{N}, \qquad n \mapsto |B_S(n)|,$$

where $B_S(n) := \left\{g \in G : d_S(1,g) \leqslant n\right\}$ denotes the ball of radius n centred at the origin. Moreover, denote $S_S(n)$ its sphere.

One can immediately obtain $f(n) \leqslant (2|S|)^n$ for $G = \langle S \rangle$.

Example 2.17

(1) For free groups $G = \mathbb{F}_k$, $k \geqslant 2$, one has

$$|B_S(0)| = 1, \qquad |B_S(1)| = 2k+1.$$

And inductively, all elements on the sphere $S_S(n)$ generates $2k$ elements, except for the one stepping back to the origin. All of them lie in $B_S(n+1)$. Hence $|B_S(n+1) \setminus B_S(n)| = (2k-1)|S_S(n)|$. Therefore

$$|B_S(n)| = 1 + 2k \sum_{\ell=1}^{n} (2k-1)^{\ell-1} = 1 + 2k \frac{(2k-1)^n - 1}{(2k+1) - 1} \sim (2k^n), \qquad n \to \infty.$$

(2) For free Abelian groups $G = \mathbb{Z}^k$, denote $\{e_i\}$ the canonical basis. Then by counting principles,

$$|B_S(n)| = \left| \left\{ (x_1, \dots, x_k) \in \mathbb{Z}^k : \sum_{i=1}^{k} |x_i| \leqslant n \right\} \right| \leqslant (2n+1)^k \sim n^k, \qquad n \to \infty.$$

Definition 2.18: Quasi-isometry

Let (X, d_X) and (Y, d_Y) be two metric spaces, $\lambda,\ c > 0$. Let $f : X \to Y$ be a map. Say

(1) f is a (λ, c)*-quasi-isometric embedding*, if

$$\frac{1}{\lambda} d_Y(f(x), f(x')) - c \leqslant d_X(x, x') \leqslant \lambda d_Y(f(x), f(x')) + c, \qquad \forall x, x' \in X.$$

(2) f is a *quasi-isometric embedding* if such $(\lambda, c) \in \mathbb{R}^2_{>0}$ in (1) exists.

(3) f is *bornological* if there exists $M \geqslant 0$ such that $\forall y \in Y\, \exists x \in X\, (d_Y(y, f(x))) \leqslant M$.

(4) f is a *quasi-isometry* if f is both a quasi-isometric embedding and bornological.

Proposition 2.19

Let G be a group with two finite generating set $S,\ S'$. Then $\mathrm{id} : G \to G$ is quasi-isometry between vertices of $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S')$.

Proof. Let $c = \max\{||s||_{S'} : s \in S\}$ and $c' = \max\left\{|t|_S : t \in S'\right\}$. Since both S and S' are finite, c and c' are finite. Then

$$\frac{1}{c'} d_{S'} \leqslant d_S \leqslant c d_{S'},$$

which gives the quasi-isometry. □

Definition 2.20: Growing type

Let $G = \langle S \rangle$ and f be the growth function. Say G has

(1) *polynomial growth* of degree $\leqslant k$, if $f(n) = O(n^k)$.

(2) *exponential growth*, if $\limsup_{n\to\infty} f(n)^{1/n} > 1$.

(3) *sub-exponential growth*, if $\limsup_{n\to\infty} f(n)^{1/n} = 1/$

(4) *intermediate growth*, if it has sub-exponential growth but does not have polynomial growth.

The basic examples in Example 2.17 gives that $\mathbb{Z}^k$ has polynomial growth, while $\mathbb{F}_k$ has exponential growth.

Theorem 2.21

Let $G = \langle S \rangle$ with S being a finite group.

(1) If G has sub-exponential growth, then it is amenable.

(2) If G is Abelian, then it has polynomial growth, and hence amenable.

Proof. (1) Since $\limsup_{n\to\infty} f(n)^{1/n} = 1$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, $|B(n+1)| / |B(n)| < 1 + \varepsilon$. Then for any $s \in S$, one has

$$\frac{|sB(n)| \setminus B(n)}{|B(n)|} \leqslant \frac{|B(n+1) \setminus B(n)|}{|B(n)|} < \varepsilon \quad \Longrightarrow \quad \frac{|sB(n) \triangle B(n)|}{|B(n)|} < 2\varepsilon.$$

Then G satisfies Følner's condition for S, so G is amenable.

(2) Each finite generated Abelian group is a quotient of $\mathbb{Z}^k$. While $\mathbb{Z}^k$ has polynomial growth of degree $\leqslant k$, G has polynomial growth of degree $\leqslant k$. Therefore G has sub-exponential and one can conclude by (1). $\square$

Now we consider the growing type of nilpotent group. Recall that the *decreasing central series* $\mathscr{C}^k G$ is defined as

$$\mathscr{C}^0 G := G, \qquad \mathscr{C}^k G := [\mathscr{C}^{k-1} G, G].$$

And a group G is *nilpotent* if the decreasing central series is of finite length.

Lemma 2.22

Let G be a finitely generated nilpotent group. $H = [G,G]$. Let S_H, S_G be finite generating sets of H and G respectively. Then there exists $p \in \mathbb{R}[x]$ such that

$$|h|_{S_H} \leqslant p\big(|h|_{S_G}\big), \qquad \forall h \in H.$$

We say that H has *polynomial distortion* in G.

Proof. Since G is finitely generated, $H = [G,G]$ is also finitely generated. Denote G/H the quotient group and $\pi : G \to G/H$ is the canonical projection. Then G/H is an Abelian finitely generated group. Let $\{e_1, \ldots, e_k\}$ be a generating set of G/H, $\{t_1, \ldots, t_k\}$ be a chocie of their pre-images for π. Let S be a finite symmetric generating set of H, and

$$S_H := S \cup \left\{[\cdots[[s, t_{i_1}^{\pm 1}], t_{i_2}^{\pm 1}]\cdots, t_{i_\ell}^{\pm 1}] : \ell \in \mathbb{N}\right\} \cup \left\{[\cdots[[t_{i_1}, t_{i_2}^{\pm 1}], t_{i_3}^{\pm 1}]\cdots, t_{i_\ell}^{\pm 1}] : \ell \in \mathbb{N}\right\}.$$

Then S_H is finite because G is nilpotent, hence length of iterated commutators is bounded.

Let $S_G = S_H \cup \{t_1, \ldots, t_k\}$ be the generating set of G. Now let $h \in H$ and write h as a word of minimal length for S_G, *i.e.*,

$$h = h_0 t_{i_1}^{\pm 1} h_1 \cdots t_{i_\ell}^{\pm 1} h_\ell, \qquad h_i = x_{j_1,i} \cdots x_{j_i,i}, \qquad x_{j_k,i} \in S_H.$$

Observe that the sum of process of t_i in h is 0 because $\forall h \in H$, $\pi(h) = 0$. Also, $x t_i^{\pm 1} = t_i^{\pm 1} x [x, t_i^{\mp 1}]$.

We move t_{i_ℓ} to the left thanks to the previous observation up to meet the opposite power of t_{i_ℓ}. We get a new word w_1 and continue with the right most t_i, then we get a word w_2. And then $w_3, w_4, \ldots, w_\ell$ to cancel all t_i with their inverses. Let $\ell_k(w_i)$ be the number of k-fold commutators in w_i. Then

$$\ell_0(w_{i+1}) = \ell_0(w_i) - 2, \qquad \ell_k(w_{i+1}) \leqslant \ell_k(w_i) + \ell_{k-1}(w_i).$$

We claim that for $k,\ i \in \mathbb{N}$, $\ell_k(w_i) \leqslant i \cdot n^k$ and $\ell_k(w_i) \leqslant n^{k+1}$, where $n = |h|_{S_G}$.

For $i = 1$, it is obvious. And

$$\ell_k(w_{i+1}) \leqslant \ell_k(w_0) + \ell_k(w_{i-1}) \leqslant i \cdot n^k + n^{(k-1)+1} \leqslant (i+1) n^k.$$

And then $\ell_k(w_{i+1}) \leqslant n^{k+1}$ because i does not go above n. Then we proved the claim.

At the end, we get a word in S_H that represents h with the length of at most $\sum_{k=0}^{d} n^{k+1}$, which is a polynomial in $n = |h|_{S_G}$. And d is the degree of nilpotency. $\square$

Theorem 2.23

Let G be a finitely generated nilpotent group. Then G has polynomial growth.

Proof. Denote $H = [G, G]$ and use the same notations as in Lemma 2.22. Let S_H be the generating set of H, $S_G = S_H \cup \{t_1, \ldots, t_k\}$. Let $g \in G$, which can be written as $g = t_1^{\alpha_1} \cdots t_k^{\alpha_k}$. Then $\sum_{i=1}^{k} |\alpha_i| \leqslant |g|_{S_G}$. Hence

$$|h|_{S_G} \leqslant |g|_{S_G} + \sum_{i=1}^{k} |\alpha_i| \leqslant 2\,|g|_{S_G}.$$

Using Lemma 2.22, there exists a polynomial $p \in \mathbb{R}[x]$ such that $|h|_{S_H} \leqslant p\big(|h|_{S_G}\big)$. So there exists $r,\ c > 0$ such that $p(x) \leqslant cx^r$. Thus

$$|h|_{S_G} \leqslant p\Big(|g|_{S_G}\Big) \leqslant c\Big(2\,|g|_{S_G}\Big)^r.$$

In the ball of radius r for S_G, we have at most $(2r+1)^k$ possibilities for α_i's, and $q\Big(c\Big(2\,|g|_{S_G}\Big)^r\Big)$ for h, where q is an increasing polynomial such that $\big|B_{S_H}(r)\big| \leqslant q(r)$. So G has polynomial growth. $\square$

Example 2.24

(1) Denote the *discrete Heisenburg group*

$$H_3 = \left\{ \begin{bmatrix} 1 & n & \ell \\ & 1 & m \\ & & 1 \end{bmatrix} : m,\ n,\ \ell \in \mathbb{Z} \right\},$$

and

$$a = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Then $H_3 = \langle a, b, c \rangle$. H_3 is not Abelian because $ab \neq ba$. While $[H_3, H_3] = \langle c \rangle = Z(H_3)$. The Abeliansation $H_3/[H_3, H_3] = \mathbb{Z}^2$. So H_3 is nilpotent of degree 2.

(2) Denote $H = [H_3, H_3]$. $S_{H_3} = \{a, b, c\}$ and $S_H = \{c\}$. Now

$$g = \left[\begin{bmatrix} 1 & 2^n & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & 2^n \\ & & 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & & 2^n \\ & 1 & \\ & & 1 \end{bmatrix}.$$

So $|g|_{S_H} = 2^{2n}$. However, $|g|_{S_{H_3}} \leqslant 4^{2n}$ since $g = [a^{2n}, b^{2n}]$. Thus $|g|_{S_H} \sim |g|_{S_{H_3}}^2$. The distortion is quadratic.

DM I – Gromov's Polynomial Growth Theorem

The aim of this homework is to prove the following theorem:

Theorem 2.25: Gromov

If G is a finitely generated group of polynomial growth, then G has a finite index subgroup nilpotent group.

We say that a group with a finite index nilpotent group is *virtually nilpotent*, so the theorem says that a group with polynomial growth is virtually nilpotent. This is a converse to the fact that any finitely generated nilpotent group has polynomial growth.

Part I — The other direction

Question 1

Let G be a group and H be a subgroup of finite index. Prove that G is finitely generated if and only if H is finitely generated.

$\Longleftarrow$: If H is finitely generated with generating set $S_H = \{t_1, \dots, t_m\}$, since $[G:H] = n < +\infty$, one has the coset decomposition

$$G = \coprod_{k=1}^{n} g_k H.$$

Then $S_G = \left\{t_1, \dots, t_m, g_1, \dots, g_n\right\}$ is a generating set of G. Thus G is also finitely generated.

$\Longrightarrow$: Now suppose G is finitely generated with $S_G = \{s_1, \dots, s_n\}$. Then by Schreier's lemma, let $\{r_1, \dots, r_m\} \subset G$ be a left coset representatives for H,

$$H = \left\langle t_i s_j (t_i s_j)^{-1} : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n \right\rangle,$$

which is finitely generated. □

Question 2

Prove that a finitely generated group G and a finite index subgroup H are quasi-isometric.

It suffices to prove that the inclusion $\iota : H \hookrightarrow G$ is a quasi-isometry. We set some notations.

- Denote S_G as the generating set of G and S_H as that of H. (*By (1), S_H is finite.*)
- Let ℓ_G be the length function on G w.r.t. S_G and ℓ_H be that on H w.r.t. S_H. They induce word metrics d_G and d_H respectively.
- The coset decomposition $G = \coprod_{i=1}^{n} H g_i$, where $g_i \in G$. (*Since* $[G:H] < +\infty$.)

Let $c = \max \ell_G(t) : t \in S_H$, $c' = \max\{\ell_H(s) : s \in S_G\}$. Since both S_G and S_H are finite, $c,\ c' > 0$ are finite. Then

$$d_G(h_1, h_2) \leqslant c \cdot d_H(h_1, h_2), \qquad \forall h_1, h_2 \in H$$

because each generator of H is length at most c in H. On the other hand, for $h = h_1^{-1} h_2 \in H$, if $\ell_G(h) = m$, then $h = s_1 s_2 \cdots s_m$ with each $s_i \in S_G$, then $s_i = h_i g_j$ for some $g_j \in G$. Thus

$$d_H(h_1, h_2) \leqslant c' m = c' \cdot d_G(h_1, h_2).$$

Then
$$\frac{1}{c'}d_H(h_1,h_2) \leqslant d_G(h_1,h_2) \leqslant c d_H(h_1,h_2), \qquad \forall h_1,h_2 \in H.$$
So ι is a quasi-isometric embedding.

Then we prove that ι is bornological. For any $g \in G$, there exist $h \in H$ and $g_i \in G$ such that $g = hg_i$. So $d_G(g,h) = d_G(g_i,1_G) = \ell_G(g_i)$. Let $M = \max\{\ell_G(g_i) : 1 \leqslant i \leqslant n\}$, then for any $g \in G$, $\exists h \in H$ such that $d_G(g,h) \leqslant M$. □

Question 3

Let G be a finitely generated group and H a finite index subgroup. Prove that G has polynomial growth if and only if H has polynomial growth.

This is immediate by (2).

$\Longrightarrow$: $|B_H(k)| \leqslant |B_G(c'k)| \sim c_1 k^d$ for some $c_1, d > 0$.

$\Longleftarrow$: For $g \in B_G(k)$, $g = hg_i$. Then by (2), $d_H(1_H,h) \leqslant c'(k+M)$. So
$$|B_G(k)| \leqslant n\,|B_H(c(k+M))| \sim c_2 k^{d'}$$
for some $c_2, d' > 0$. □

Question 4

Let G be a finitely generated nilpotent group. The goal of this question is to prove that its commutator subgroup $G' = [G,G]$ is finitely generated. Let S be a finite generating set of G.

(a) Prove that any commutator can be written as a product of iterated commutators $[\cdots[[s_1,s_2],s_3],\ldots,s_n]$ with $s_i \in S$.

(b) Prove that there is a finite quantity of such iterated commutators of elements of S.

(c) Conclude that G' is finitely generated.

(4a) Let $g = s_1\cdots s_k$, $g' = s_1'\cdots s_\ell'$ with $s_i,\ s_j' \in S$. And note that
$$[ab,c] = a[b,c]a^{-1}[a,c], \qquad [a,bc] = [a,c]c^{-1}[a,b]c.$$
Then the proof is done by applying the above identities repeatly to $[g,g']$.

(4b) Let S_k be the set of k-fold commutators. We have $|S_k| \leqslant |S|^{k+1}$. Since G is nilpotent, there exists $m > 0$ such that $G_m = \{1\}$, where $G_{i+1} = [G,G_i]$ is defined inductively with $G_0 = G$. So if $k > m$, any k-fold commutator is trivial. Therefore, the quantity of iterated commutators are bounded by $\sum_{k=1}^m |S|^{k+1}$, which is finite.

(4c) Immediate by (4a) and (4b). □

Question 5

Let G be the lamplighter group $(\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})) \rtimes \mathbb{Z}$, where $\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ is the set of $(\mathbb{Z}/2\mathbb{Z})$-valued finite supported bi-infinite sequences $(a_i)_{i\in\mathbb{Z}}$. The group $\mathbb{Z}$ acts on $\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ via $n\cdot(a_i) = (a_{i-n})$.

(a) Prove that the lamplighter group is a finitely generated solvable group.

(b) Prove that its derived subgroup G' is not finitely generated.

(c) Explain why the proof that finitely generated nilpotent group have polynomial growth does not work for solvable groups.

(5a) Define t as the generator that corresponds to the lamplighter moving a step to the right, and a as the generator that change the state of the bulb where he is. Then

$$G = \langle t, a : a^2 = 1,\ [t^k a t^{-k}, t^\ell a t^{-\ell}] = 1 \rangle.$$

Hence G is finitely generated. Also, there is a normal series

$$\{1\} \lhd \bigoplus_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \lhd G$$

such that each subquotient is Abelian. Hence G is solvable by definition.

(5b) For any $n \in \mathbb{Z}$,

$$[a, t^n] = a t^n a^{-1} t^{-n} = a - t^n a,$$

which corresponds to the action that change the state of bulb in position 0 and n simultaneously. So G' can be seen as a translation-invariant subgroup of $\bigoplus_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$, which cannot be finitely generated. This is because if there exists $S = \{n_1, \ldots, n_k\} \subset \mathbb{Z}$, $\left\{\sum_{i=1}^n s_i : n \in \mathbb{N},\ s_i \in S\right\}$ is strictly contained in $\mathbb{Z}$.

(5c) In a nilpotent group, the commutators cannot iterate infinitely, which limiting the possible combinations of elements. This ensures the length of the product of elements grows polynomially. While a solvable group can has exponential growth. $\square$

Question 6

Give an example of a group of polynomial growth that is not nilpotent.

Consider the dihedral group $D_\infty = \langle t, a : a^2 = 1,\ ata = t^{-1} \rangle$. Thus the elements of D_∞ is of form either t^k or at^k. Thus $|B(n)| = 2n + 1$. But D_∞ is not nilpotent. $\square$

Part II — The space of Lipschitz harmonic functions

In this part, let G be a finitely generated group with finite symmetric generating set S. For $f : G \to \mathbb{R}$, the function is called *harmonic* if $\forall g \in G$,

$$f(g) = \frac{1}{|S|} \sum_{s \in S} f(gs).$$

Question 7

If $G = \mathbb{Z}$, what are the harmonic functions for $S = \{\pm 1\}$?

It suffices to solve the equation

$$f(n) = \frac{1}{2}(f(n+1) + f(n-1)) \Longleftrightarrow f(n+1) - 2f(n) + f(n-1) = 0.$$

Hence $f(n) = an + b$, where $a, b \in \mathbb{R}$. Therefore, the harmonic functions on $\mathbb{Z}$ are constants and linear functions. $\square$

Question 8

Prove that a harmonic function that has a maximum or a minimum is constant.

By replacing f with $-f$, we may assume that f has a maximum at $g_0 \in G$. Then

$$f(g_0) = \frac{1}{|S|}\sum_{s\in S} f(g_0 s) \leqslant \frac{1}{|S|}\sum_{s\in S} f(g_0) = f(g_0)$$

forces that $f(g_0 s) = f(g_0)$ holds for every $s \in S$. So f is a constant on $g_0 S$. Iterate the above argument, one has f is a constant on $g_0 \langle S\rangle = G$. $\square$

Question 9

Prove that a function $f : G \to \mathbb{R}$ is Lipschitz (*for the word distance on G associated to S*) if there is $C > 0$ such that $\forall g \in G,\ \forall s \in S$,

$$\left|f(gs) - f(g)\right| \leqslant C.$$

For all $g, g' \in G$, let $d_S(g, g') = n$, *i.e.*, $g' = g s_1 \cdots s_n$. Then

$$\left|f(g) - f(g')\right| = \left|f(g) - f(g s_1 \cdots s_n)\right| \leqslant \sum_{i=1}^{n} \left|f(g s_1 \cdots s_{i-1}) - f(g s_1 \cdots s_i)\right| \leqslant nC = C d_S(g, g').$$

Hence f is Lipschitz with Lipschitz constant C. $\square$

A crucial ingredient in the proof of Gromov's theorem that we follow is the following difficult result due to Kleiner. We will assume it.

Theorem 2.26: Kleiner

If G has polynomial growth then the space of Lipschitz harmonic functions $\mathrm{HL}_s(G)$ has finite dimension and contains at least one non-constant function.

For a Lipschitz function, we define its Lipschitz norm $\|F\|_{\mathrm{Lip}}$ to be the least possible Lipschitz constant, *i.e.*,

$$\|F\|_{\mathrm{Lip}} := \inf\left\{C > 0 : \forall g, h \in G\left(\left|f(g) - f(h)\right| \leqslant C d_S(g, h)\right)\right\}.$$

Question 10

(a) Prove that the Lipschitz norm is a seminorm on $\mathrm{HL}_s(G)$ and that G acts by precomposition on $\mathrm{HL}_S(G)$ preserving this seminorm.

(b) What are the functions $f \in \mathrm{HL}_S(G)$ such that $\left\|f\right\|_{\mathrm{Lip}} = 0$? We denote N this space.

(c) Prove that the Lipschitz norm induces a norm on $E = \mathrm{HL}_S(G)/N$ on G acts by linear isometries on this space.

(d) Prove that there is a G-invariant scalar product on E.

(10a) Non-negativity and homogeneity is obvious. The triangle inequality results from

$$\left|(f_1 + f_2)(g) - (f_1 + f_2)(g')\right| \leqslant \left|f_1(g) - f_1(g')\right| + \left|f_2(g) - f_2(g')\right| \leqslant (\left\|f_1\right\|_{\mathrm{Lip}} + \left\|f_2\right\|_{\mathrm{Lip}}) d_S(g, g').$$

Thus $\|\cdot\|_{\mathrm{Lip}}$ is a seminorm. The group G acts on $\mathrm{HL}_s(G)$ by $(g \cdot f)(x) = f(g^{-1}x)$. Since d_S is G-invariant,

$$\left\|g \cdot f\right\|_{\mathrm{Lip}} = \inf\left\{C > 0 : \left|f(g^{-1}x) - f(g^{-1}y)\right| \leqslant C d_S(x, y)\right\} = \left\|f\right\|_{\mathrm{Lip}}.$$

Therefore the G-action preserves $\|\cdot\|_{\mathrm{Lip}}$.

(10b) If $\|f\|_{\mathrm{Lip}} = 0$, then $\forall g, g' \in G$,

$$\|f(g) - f(g')\| \leqslant 0 \cdot d_S(g, g') = 0.$$

So f must be a constant function.

(10c) This is immediate by (10a) and the fact that the constant functions are G-invariant.

(10d) Let $\langle \cdot, \cdot \rangle$ be the inner product induced by $\|\cdot\|_{\mathrm{Lip}}$ on E, *i.e.*,

$$\langle f_1, f_2 \rangle = \frac{1}{2}(\|f_1 + f_2\|_{\mathrm{Lip}}^2 - \|f_1\|_{\mathrm{Lip}}^2 - \|f_2\|_{\mathrm{Lip}}^2).$$

It is G-invariant because G acts by linear isometries on E by (10c). □

Part III — The compact linear case

In this part, we consider a subgroup G of $\mathrm{U}_n(\mathbb{C})$. For $g \in \mathrm{U}_n(\mathbb{C})$, we denote by $\|g\|$ its operator norm.

Question 11

Prove that for $g, h \in \mathrm{U}_n(\mathbb{C})$,

$$\|[g,h] - 1\| \leqslant 2\|g - 1\| \|h - 1\|.$$

For $g, h \in \mathrm{U}_n(\mathbb{C})$ we have $\|g\| = \|g^{-1}\| = \|h\| = \|h^{-1}\| = 1$. Observe that

$$[g,h] - 1 = ghg^{-1}h^{-1} - 1 = (gh - hg)g^{-1}h^{-1}$$

and

$$gh - hg = gh - g - h + 1 - hg + g + h - 1 = (g-1)(h-1) - (h-1)(g-1).$$

Therefore

$$\|[g,h] - 1\| = \|gh - hg\| \leqslant 2\|g - 1\| \|h - 1\|.$$

□

Our aim in this part is to prove the following theorem:

Theorem 2.27

Let G be a finitely generated subgroup of $\mathrm{U}_n(\mathbb{C})$ that has polynomial growth, then G is virtually Abelian.

Let G fix as in the theorem. For $\varepsilon \in \left(0, \frac{1}{10}\right)$, define

$$G' = \langle g \in G : \|g - 1\| < \varepsilon \rangle \leqslant G.$$

Question 12

Prove that there is a constant C depending on n and ε such that G' has index at most C in G.

Observe that $\mathrm{U}_n(\mathbb{C})$ is a compact group. Hence we consider a maximal ε-net K of G, which by compactness, is finite. If there are more than $|K|$ elements in G/G', there exists $g_1,\ g_2 \in G/G'$, $g_1 \neq g_2$ such that $g_1 \cdot B(1,\varepsilon) \cap g_2 \cdot B(1,\varepsilon) \neq \varnothing$. So there exists $h_1,\ h_2 \in G'$ such that

$$\|g_1 h_1 - g_2 h_2\| < \varepsilon \implies \|g_1^{-1} g_2 - h_1 h_2^{-1}\| < \varepsilon \implies g_1^{-1} g_2 \in G'.$$

This contradicts the fact that $g_1 \neq g_2$. Thus

$$\left|[G:G']\right| \leqslant |K| = C(n,\varepsilon)$$

by the properties of covering number. $\square$

Question 13

Assume G' has a central element g that is not a homothety, *i.e.*, not $\lambda 1$ for some $\lambda \in \mathbb{C}$.

(a) Using the spectral theorem prove that the centraliser of g can be identified with $\mathrm{U}_{n_1}(\mathbb{C}) \times \cdots \times \mathrm{U}_{n_k}(\mathbb{C})$ where $n = \sum_{i=1}^k n_i$.

(b) Prove the theorem under this assumption and by assuming that the theorem is true for lower dimensions.

(13a) By the spectral theorem, since $g \in \mathrm{U}_n(\mathbb{C})$, it has eigenvalues $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ with multiplicity $n_1, \ldots, n_k$. Let $u \in \mathrm{U}_n(\mathbb{C})$ such that

$$ugu^{-1} = \mathrm{diag}\left\{\lambda_1 \cdot \mathrm{id}_{n_1}, \ldots, \lambda_k \cdot \mathrm{id}_{n_k}\right\},$$

then for any $x \in Z(g)$, $[ugu^{-1}, x] = u[g,x]u^{-1} = 0$. Hence

$$Z(g) = \mathrm{U}_{n_1}(\mathbb{C}) \times \ldots \mathrm{U}_{n_k}(\mathbb{C}).$$

Here $\sum_{i=1}^k n_i = n$ by the dimension argument.

(13b) We do induction on n. For the case $n = 1$, it is obvious.

Now assume G' has a central element that is not a homothety and the theorem holds for all dimensions $< n$. By (13a), we can decompose G' as a direct product of some $\mathrm{U}_{n_i}(\mathbb{C})$, with each $n_i < n$. And by assumption, the theorem holds in lower dimensions, thus holds in dimension n. $\square$

From now on, we assume that elements in $Z(G')$ are homotheties.

Question 14

Prove that G' is generated by a finite set S of elements g such that $\left\|g-1\right\| < \varepsilon$.

From (12) we know that G' is of finite index, then by (1), G' is also finitely generated. And by the definiton of G', we can find a finite subset generating all the elements in $\left\{g \in G : \left\|g-1\right\| < \varepsilon\right\}$, hence one can choose S a finite subset of $\left\{g \in G : \left\|g-1\right\| < \varepsilon\right\}$. $\square$

Let $h_1 \in S$ that is not a homothety and with $\|h_1 - 1\| = \delta_1 < \varepsilon$.

Question 15

Prove that for any $g \in S$, $\left\|[g,h_1]-1\right\| < 2\delta_1\varepsilon$ and $\det([g,h_1]) = 1$.

Using (11),

$$\left\|[g,h_1]-1\right\| \leqslant 2\left\|g-1\right\| \|h_1 - 1\| < 2\delta_1\varepsilon.$$

Since $g,\ h_1 \in \mathrm{U}_n(\mathbb{C})$, $\det([g,h_1]) = \det(gh_1g^{-1}h^{-1}) = 1$. $\square$

Question 16

Prove that if $[g, h_1]$ is a homothety for all $g \in S$ then h_1 is central in G' and we have a contradiction.

By (15), $[g, h_1] \in G'$ and has determinant 1. Now S is finite, if ε is small enough,

$$\left\| [g, h_1] - 1 \right\| < 2\delta_1 \varepsilon < 2\varepsilon^2 \implies [g, h_1] = 1.$$

In other words, $h_1 \in Z(G')$. This contradicts the fact that $Z(G')$ consists of homotheties. □

Question 17

Construct by induction a sequence $(h_k)_{k \geqslant 1}$ of elements of G' such that $h_k = [g_{k-1}, h_{k-1}]$ for some g_{k-1}, h_k is not a homothety and $\|h_k - 1\| = \delta_n < 2\delta_{n-1}\varepsilon$.

Let $h_2 := [g_1, h_1]$ for some $g_1 \in S$, by (16),

$$\delta_2 = \|h_2 - 1\| \leqslant 2 \left\| g_1 - 1 \right\| \|h_1 - 1\| < 2\delta_1 \varepsilon.$$

Now by (15), $\det([g_1, h_1]) = 1$. Repeat the argument in (16), we know that h_2 is not a homothety. Then iterate the above reasoning. □

Question 18

(a) Prove that there is $c > 0$ such that all products $h_1^{i_1} \cdots h_m^{i_m}$ are distinct for $i_1, \ldots, i_m \in (0, c/\varepsilon)$.

(b) Prove that there is $L > 0$ such that all products $h_1^{i_1} \cdots h_m^{i_m}$ for $i_1, \ldots, i_m \in (0, c/\varepsilon)$ lie in some ball of radius $Lm2^m/\varepsilon$ for the word metric associated to S.

(c) Prove this is a contradiction with polynomial growth.

(18a) By our construction in (17), all h_k's are not homotheties and $\delta_k = \|h_k - 1\| < 2\varepsilon \cdot \delta_{k-1}$. Thus for sufficiently small $c > 0$, for example, $c < \min_{1 \leqslant k \leqslant m} \frac{\varepsilon \log \delta_{k-1}}{\log 2\varepsilon + \log \delta_{k-1}}$, when $0 < i_k,\ i'_k < c\varepsilon^{-1}$,

$$\left\| h_1^{i_1} \cdots h_k^{i_k} \cdots h_m^{i_m} - h_1^{i_1} \cdots h_k^{i'_k} \cdots h_m^{i_m} \right\| = \left\| h_k^{i_k - i'_k} - 1 \right\| \leqslant \delta_k^{c/\varepsilon} < \delta_{k-1} = \|h_{k-1} - 1\|$$

holds for all $k \in \{2, 3, \ldots, m\}$. Thus each $h_1^{i_1} \cdots h_m^{i_m}$ are distinct.

(18b) Let ℓ_k be the word length of h_k. By the construction of (h_k),

$$h_{k+1} = [g_{k+1}, h_k] \implies \ell_{k+1} \leqslant 2\ell_k + 2 \implies \ell_k \leqslant 3 \cdot 2^{k-1} \leqslant 2^{k+1}.$$

Hence

$$\ell(h_1^{i_1} \cdots h_m^{i_m}) \leqslant \sum_{k=1}^{m} i_k \cdot 2^{k+1} \leqslant \frac{c \cdot 2^{m+2}}{\varepsilon}$$

Let $L = 4c$ then $\ell(h_1^{i_1} \cdots h_m^{i_m}) \leqslant L2^m/\varepsilon \leqslant Lm2^m/\varepsilon$.

(18c) If G' has polynomial growth, there exists $d > 0$ such that $|B(r)| \leqslant Cr^d$. While by (18a) and (18b), there are at least $(c\varepsilon^{-1})^m$ elements in $B(Lm\varepsilon^{-1}2^m)$. Hence

$$C\left(\frac{Lm2^m}{\varepsilon}\right)^d \geqslant \left| B\left(\frac{Lm2^m}{\varepsilon}\right) \right| \geqslant \left(\frac{c}{\varepsilon}\right)^m \implies d \geqslant \frac{m \log(c\varepsilon^{-1}) - \log C}{m \log 2 + \log(Lm\varepsilon^{-1})}.$$

Let $m \to \infty$, $d \geqslant \log(c\varepsilon^{-1})/\log 2$ for all $\varepsilon \in \left(0, \frac{1}{10}\right)$. While ε can be chosen arbitarily small, hence this contradicts the fact that G' has polynomial growth. □

Question 19

Gather all what has been proved in this section to prove Theorem 0.3.

Let G and G' be defined as before. If $Z(G')$ contains an element that is not a homothety, then done by (13). Otherwise, by the discussion in (14)–(18), each $s \in S$ needs to be a homothety, which means that G' is an Abelian subgroup with finite index. □

Part IV — Groups with infinite Abelian quotient

Let G be a finitely generated group. It is said to have *polynomial growth of exponent at most* $d \in \mathbb{N}$ if there exists $C > 0$ such that $|B_S(n)| \leqslant Cn^d$ for some finite generating set S.

In this part, we aim to prove the following theorem.

Theorem 2.28

Let $d \in \mathbb{N}$. Suppose that any finitely generated group of polynomial growth of exponent at most $d-1$ is virtually nilpotent. Let G be a finitely generated group of polynomial growth of exponent at most d. Assume that G' is a finite index group of G with a normal subgroup $N \leqslant G'$ such that G'/N is infinite cyclic, then G is virtually nilpotent.

We shall fix G and G' as in the theorem.

Question 20

(*Might need corrections, see Remark below.*) Let G' be a finitely generated group with surjective homomorphism $\phi : G' \to \mathbb{Z}$. Let $\{s_1, \dots, s_m\}$ be a finite generating set. Use Bézout's identity to prove that we can find a generating set $\{e_1, \dots, e_m\}$ such that $\{e_1, \dots, e_{m-1}\}$ and their conjugates $e_m^k e_i e_m^{-k}$ by powers of e_m generates $\ker\phi$.

Remark. *There might be some problems if we keep e_m instead introducing a new element. Because if we let $e_m = s_1^{k_1} \cdots s_m^{k_m} \in G$ and define $e_i = s_i e_m^{-a_i}$ for $i \in \{1, \dots, m-1\}$, then we have*

$$\begin{cases} s_i = e_i e_m^{-a_i} & , i \in \{1, \dots, m-1\}, \\ s_m = e_m^{-1/k_m} \prod_{j=1}^{m-1} e_j^{-k_j} & , i = m \end{cases}$$

However, we cannot define e_m^{-1/k_m} if $k_m \neq 1$. So I would like to introduce a new element e here and replace e_m with e, which affacts all other questions in this part.

Or, there is another way to solve this problem. We can assume à priori there is an element, for example, s_m such that $\phi(s_m) = 1$.

Denote $a_i = \phi(s_i) \in \mathbb{Z}$. Since ϕ is surjective, $\gcd(a_1, \dots, a_m) = 1$. Therefore by Bézout's identity, $\exists k_i \in \mathbb{Z}$ such that

$$a_1 k_1 + \cdots + a_m k_m = 1.$$

Let $e = s_1^{k_1} \cdots s_m^{k_m} \in G$, then $\phi(e) = 1$. Now let $e_i = s_i e^{-a_i}$, we have $\phi(e_i) = a_i - a_i = 0$, *i.e.*, $e_i \in \ker\phi$. And for each i, $s_i = e_i e^{-a_i}$ is a product of $\{e_1, \dots, e_m, e\}$, thus $\{e_1, \dots, e_m, e\}$ generates G'.

For simplicity we denote $e_{i,k} := e^k e_i e^{-k}$ for $i \in \{1, \dots, k\}$ and $k \in \mathbb{Z}$. If $g \in \ker\phi$, since $\{e_1, \dots, e_m, e\}$ generates G', g can be written as a word consisting $\{e_1, \dots, e_m, e\}$ with the powers of e sum to zero. Note that for all $k \in \mathbb{Z}$, $g \mapsto e^{-k} g e^k$ maps $\ker\phi$ to $\ker\phi$, it can be written as a word consisting $\{e_{i,k} : 1 \leqslant i \leqslant m,\ k \in \mathbb{Z}\}$. Hence $\ker\phi$ is generated by $\{e_{i,k} : 1 \leqslant i \leqslant m,\ k \in \mathbb{Z}\}$. $\square$

Question 21

(*Changed a little bit, see Remark above.*) Let $S_k = \{e_{i,k'} : |k'| \leqslant k,\ 1 \leqslant i \leqslant m\}$, and B_k be the elements in the subgroup generated by S_k of length at most k for the word length given by S_k.

(a) Prove that if $S_{k+1} \not\subset B_k B_k^{-1}$, then $|B_{k+1}| \geqslant 2|B_k|$.

(b) Deduce that for some k large enough, $S_{k+1} \subset B_k B_k^{-1}$.

(c) Prove that for some k large enough, S_k generates $\ker\phi$.

(21a) If $S_{k+1} \not\subset B_k B_k^{-1}$, choose $g \in S_{k+1} \setminus B_k B_k^{-1}$, then for any $g_1,\ g_2 \in B_k$, $g \neq g_1 g_2^{-1}$. This is equivalent to

$$\forall g_1,\ g_2 \in B_k\,(g g_2 \neq g_1) \implies B_k \cap g B_k = \varnothing.$$

While $B_k \subset B_{k+1}$, $gB_k \subset B_{k+1}$, so

$$|B_{k+1}| \geqslant |B_k| + \left|gB_k\right| = 2|B_k|.$$

(21b) Prove by contradiction. If $\forall k \in \mathbb{N}$, $S_{k+1} \not\subset B_k B_k^{-1}$, then by (21a), $|B_k|$ grows exponentially. This contradicts the fact that G' is of polynomial growth.

(21c) By (21b), there exists $k \in \mathbb{N}$ such that $S_{k+1} \subset B_k B_k^{-1}$, with the right hand side generated by S_k. Thus S_k generates $\ker\phi$. $\square$

We fix k such that S_k generates $\ker\phi$.

Question 22

(a) Prove that the ball of radius R generated by S_k and e is at least $R/2$ times as large as the ball of radius $R/2$ generated by S_k.

(b) Prove that $\ker\phi$ is finitely generated with polynomial growth at most $d-1$.

(c) Conclude that $\ker\phi$ is virtually nilpotent.

(22a) For any $g \in B_{\ker\phi, S_k}(R)$ and $1 \leqslant i \leqslant R$, $ge^i \in B_{G', S_k \cup \{e\}}$. So

$$R\left|B_{\ker\phi, S_k}(R)\right| \leqslant \left|B_{G', S_k \cup \{e\}}(2R)\right|.$$

Replace R with $R/2$ then we obatin the desired result.

(22b) We know that G' is of polynomial growth of exponent at most d, so $\exists c > 0$ such that

$$R\left|B_{\ker\phi, S_k}(R)\right| \leqslant \left|B_{G', S_k \cup \{e\}}(2R)\right| \leqslant c(2R)^d \quad \implies \quad \left|B_{\ker\phi, S_k}(R)\right| \leqslant c' R^{d-1}.$$

(22c) Immediate by the assumption. $\square$

Question 23

(a) Prove that $\ker\phi$ contains a normal finite index nilpotent subgroup N and there is some power M such that for any element $g \in \ker\phi$, $g^M \in N$.

(b) Let N' be the subgroup of $\ker\phi$ generated by these powers g^M. Prove this is a nilpotent normal subgroup of $\ker\phi$.

(23a) We know from (22c) that $\ker\phi$ is virtually nilpotent, so there exists a nilpotent subgroup H of finite index. Let

$$N = \bigcap_{g\in\ker\phi} gHg^{-1} \lhd \ker\phi,$$

this is actually a finite intersection because $[\ker\phi : H] < +\infty$, and hence $[\ker\phi : N] < +\infty$. The intersection of nilpotent groups is still nilpotent, so N is a nilpotent normal subgroup of finite index.

Now assume c is the degree of nilpotency of N, for any $g \in \ker\phi$, and consider $\bar{g} \in \ker\phi/N$ the image of g in $\ker\phi/N$. Because $[\ker\phi : N] < +\infty$, $\ker\phi/N$ is a finite group, and denotes its order as m. Then $(g^m)^{c!} \in N$ by the nilpotency of N. Let $M = m \cdot c!$ then we are done.

(23b) By Lagrange's theorem, $\bar{g}^M = 1_{\ker\varphi/N}$, so $N' \leqslant N$. The subgroup of a nilpotent group is still nilpotent, so N' is a nilpotent subgroup. For all $h \in \ker\phi$,

$$hg^M h^{-1} = (hgh^{-1})^M \in N'$$

since $hgh^{-1} \in \ker\phi$. Thus $N' \lhd \ker\phi$. $\square$

Question 24

Let $M \in \mathbb{N}$. Let Q be a finitely generated group that is nilpotent and such that any element satisfies $g^M = 1$. Prove that Q is finite and deduce that N' has finite index in N and thus has finite index in $\ker\phi$.

We do induction on the degree of nilpotency c. If $c = 1$, Q is Abelian. Then Q is a finitely generated torsion Abelian group, by the structure theorem of Abelian group, Q is finite.

Now assume the result holds for Q with degree of nilpotency $\leqslant c-1$. If Q has degree of nilpotency c, consider the extension

$$1 \longrightarrow Z(Q) \longrightarrow Q \longrightarrow Q/Z(Q) \longrightarrow 1$$

Now $Z(Q)$ is finitely generated Abelian and $\forall z \in Z(Q)$, $z^M = 1$. By assumption $Z(Q)$ is finite. The quotient $Q/Z(Q)$ is of degree of nilpotency $c-1$, with each element $(gZ(Q))^M = Z(Q)$, thus by assumption $Q/Z(Q)$ is finite. The extension of finite groups is still finite, so Q is finite.

Come back to N'. Consider the quotient N/N', for any xN', $(xN')^M = N'$, thus N/N' is finitely generated group that is nilpotent. So N/N' is finite and thus $[N : N'] = |N/N'| < +\infty$. By (23a),

$$[\ker\phi : N'] = [\ker\phi : N] \cdot [N : N'] < +\infty,$$

thus N' is a normal subgroup of $\ker\phi$ with finite index. $\square$

Question 25

The infinite cyclic group generated by e acts on $\ker\phi$ by conjugations.

(a) Prove that N' is invariant for this action and we can construct the semi-direct product $G'' = \mathbb{Z} \ltimes_e N'$.

(b) Prove that G'' has finite index in G and thus has polynomial growth.

(25a) Let $g \mapsto ege^{-1}$ be an automorphism of $\ker\phi$. If $\operatorname{ord} g = M$, then $\operatorname{ord} ege^{-1} = M$ as well. So N' is actually a characteristic subgroup of $\ker\phi$, and thus invariant for the conjugation $g \mapsto ege^{-1}$. Thus, we can construct $G'' = \mathbb{Z} \ltimes_e N'$ by

$$(k_1, x_1) \cdot (k_2, x_2) = (k_1 + k_2, \mathrm{e}^{k_2} x_1 \mathrm{e}^{-k_2} x_2).$$

(25b) Denote $E = \langle e \rangle \cong \mathbb{Z}$, then as sets we have $G'' = EN'$ and $G' = E\ker\phi$. Decompose $\ker\phi := \coprod_{i\in I} N' g_i$, where $g_i \in G'$, then

$$\coprod_{i\in I} G'' g_i = \coprod_{i\in I} EN' g_i = E \coprod_{i\in I} N' g_i = E\ker\phi = G'.$$

By (24), $[G' : G''] = |I| = [\ker\phi : N'] < +\infty$. And by our assumption

$$[G : G''] = [G : G'][G' : G''] < +\infty.$$

So G'' has polynomial growth because G has polynomial growth. □

Let G be a group, $g \in G$ and H a subgroup of G normalised by g. We say that g *acts unipotently* on H if $\exists k \in \mathbb{N}$ such that for any iterated commutators $[x_1, [x_2, [\dots [x_{n-1}, x_n]]]]$ is trivial when $x_i = g$ or $x_i \in H$ and $\left|\{i : x_i = g\}\right| \geqslant k$.

Question 26

Prove that if there is some power e^a that acts unipotently on N', then G'' is virtually nilpotent.

Assume that N' has degree of nilpotency c. If e^a acts unipotently on N', then the n-fold commutator

$$[x_1, [x_2, [\dots [x_{n-1}, x_n]]]], \qquad x_i \in N' \cup \{e^a\}$$

would be trivial if $\left|\{i : x_i \in N'\}\right| \geqslant c+1$ or $|\{i : x_i = e^a\}| \geqslant k$. Now let n large enough, for example, $n = k(c+1)$. Then one of the following two cases must happen:

- $\left|\{i : x_i \in N'\}\right| \geqslant c + 1$.
- There exists a form of $[e^a, [e^a, [\dots, [e^a, x]]]]$ with k-copies of e^a and $x \in N'$.

Therefore, in any of the above cases, $\mathscr{C}^{k(c+1)} \langle N' \cup \{e^a\} \rangle = 1$. Therefore, N' is a nilpotent subgroup of G'' with finite index, and this gives that G'' is virtually nilpotent. □

Question 27

Let $Z(N')$ be the centre of N'.

(a) Prove that $Z(N')$ is a finitely generated Abelian group.

(b) Assume that N' is nilpotent in m steps. Prove that $N'/Z(N')$ is nilpotent in $m-1$ steps.

(c) Prove there is a finite Abelian group H and some ℓ such that $Z(N') \cong \mathbb{Z}^\ell \times H$.

(d) Prove that H is a characteristic subgroup of $Z(N')$.

(e) Prove that any automorphism of $Z(N')$ induces an automorphism on H.

(f) Prove that there is a $a \in \mathbb{N}$ large enough such that the action induced by conjugation by e^a is trivial on H.

(g) Prove that e^a induces an action on $\mathbb{Z}^\ell \cong Z(N')/H$.

(27a) By (26), N' is a finitely generated nilpotent group, hence $Z(N')$, as its centre, is finitely generated and Abelian.

(27b) This comes from the definition using upper central series. Note that N' has an upper central series of length m with $Z(N')$ being the first term.

(27c) Immediate from (27a) and the structure theorem of finitely generated Abelian group.

(27d) By (27c), H is a torsion subgroup of $Z(N')$ thus any automorphism preserves the order of an element $h \in H$. Hence H is a characteristic subgroup.

(27e) For any $\alpha \in \operatorname{Aut} Z(N')$, simply restricting it then we obtain $\alpha|_H \in \operatorname{Aut} H$ due to (27d).

(27f) Since H is finite, $\operatorname{Aut} H$ is also finite. We take $a \in \mathbb{N}$ being a multiple of $|\operatorname{Aut} H|$, then by Lagrange's theorem, $h \mapsto e^a h e^{-a}$ acts trivially on H.

(27g) The action is well-defined due to (27f). Concretely,

$$e^a (xH) e^{-a} = (e^a x e^{-a}) H$$

since $e^a H e^{-a} = H$. □

Question 28

Prove that it suffices (*arguing by induction on the number of steps of nilpotency*) to prove there is some power e^a that acts unipotently on $Z(N')$.

We do induction on the degree of nilpotency c of N'. If $c = 1$, N' is Abelian, then it is obvious because any $a \in \mathbb{N}$ satisfies the requirement.

Now assume that the proposition is true for $\leqslant c-1$. Using (27b), we know that $N'/Z(N')$ has degree of nilpotency $c-1$, thus by assumption there is some $a_0 \in \mathbb{N}$ such that e^{a_0} acts unipotently on $N'/Z(N')$. If $g \in N'$,

$$[e^{a_0}, [e^{a_0}, \dots, [e^{a_0}, g]]] \in Z(N')$$

when thre are at least k_0-copies of e^{a_0}. Thus if there is e^a that acts unipotently on $Z(N')$, let $a' = \operatorname{lcm}(a, a_0)$ then we are done. □

Question 29

Prove that the group of automorphisms of the group $(\mathbb{Z}^\ell, +)$ is $\mathrm{SL}_\ell(\mathbb{Z})$. (*maybe another typo?*)

For any $f \in \operatorname{Aut} \mathbb{Z}^\ell$, let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the canonical basis of $\mathbb{Z}^\ell$, we have

$$f(\varepsilon_j) = \sum_{i=1}^{\ell} a_{ij} \varepsilon_i, \qquad a_{ij} \in \mathbb{Z}.$$

Thus $A = [a_{ij}] \in \operatorname{Mat}_\ell(\mathbb{Z})$. Also, f is invertible, so $\det A \neq 0$, hence $A \in \mathrm{GL}_\ell(\mathbb{Z})$. And $f^{-1} \in \operatorname{Aut} \mathbb{Z}^\ell$ so similarly $A^{-1} \in \operatorname{Mat}_\ell(\mathbb{Z})$. This implies $\det A = \pm 1$, *i.e.*, $A \in \mathrm{SL}_\ell(\mathbb{Z})$.

On the other hand, if $A \in \mathrm{SL}_\ell(\mathbb{Z})$, one can simply define

$$f_A : \mathbb{Z}^\ell \to \mathbb{Z}^\ell, \qquad v \mapsto Av$$

and it is easy to check $f_A \in \operatorname{Aut} \mathbb{Z}^\ell$. □

Question 30

Let $A \in \mathrm{SL}_\ell(\mathbb{Z})$ be the matrix corresponding to the action e^a on $\mathbb{Z}^\ell$.

(a) Prove that $\|A^n\|$ cannot grow exponentially and thus A has unit complex eigenvalues.

(b) By Kronecker theorem about root of unity, all eigenvalues of A are roots of unity.

(c) Prove that some power of A is unipotent.

(d) Prove that the action of some power of e^a is unipotent.

(30a) We prove by contradiction. If there is an eigenvalue of A whose norm is not 1, then $\exists \lambda \in \operatorname{Sp} A$ such that $|\lambda| > 1$ since $\det A = \pm 1$. Let E_λ be the eigenspace of λ.

Let $v \in \{\varepsilon_1, \ldots, \varepsilon_n\}$ such that $P_{E_\lambda} v \neq 0$. And by choosing a large enough $k \in \mathbb{N}$, one can assume $B = A^k$ has an eigenvalue $|\lambda'| > 3$, which gives us that

$$\left\|B^{n+1} v\right\| > \|B^n v\|, \qquad \forall n \in \mathbb{N}.$$

Therefore

$$\left\|B^{n+1} v\right\| > \|B^n v\| + \|B^n v\| > \cdots > \sum_{i=0}^{n} \left\|B^i v\right\| + \|v\| > \sum_{i=0}^{n} \left\|B^i v\right\|.$$

This implies that for each $I \subset \{0, \ldots, n\}$, $\sum_{i \in I} B^i v$ are different because the norm of difference must be non-zero. While

$$\left\{\sum_{i\in I} B^i v : I \subset \{0,\ldots,n\}\right\} \subset B_G(nk) \implies \left|\left\{\sum_{i\in I} B^i v : I \subset \{0,\ldots,n\}\right\}\right| \leqslant |B_G(nk)|.$$

As $n \to \infty$, the left hand side grows exponentially, and the right hand side by our assumption grows polynomially. This leads to a contradiction. Therefore each eigenvalue λ should have norm 1.

(30b) Since $A \in \mathrm{SL}_\ell(\mathbb{Z})$, the characteristic polynomial $p_A \in \mathbb{Z}[x]$ and is monic. By (30a), all roots of p_A have absolute value at most 1. So by Kronecker's theorem, f is a product of cyclotomic polynomials and(or) a power of x, in other words, each eigenvalue is a root of unity.

(30c) Assume that A has eigenvalues $\{\lambda_1, \ldots, \lambda_\ell\}$ and there exists $k_i \in \mathbb{Z}$ such that $\lambda_i^{k_i} = 1$ by (30b). Take $k = \operatorname{lcm}(k_1, \ldots, k_\ell)$, then $A^k - \mathrm{id}_\ell$ is nilpotent. Therefore A^k is unipotent.

(30d) It is immediate from (30c) that $g \mapsto e^{ka} g e^{-ka}$ is unipotent. □

Question 31

Explain why all what has been done in this part prove Theorem 0.4.

We assumed that any finitely generated group of polynomial growth of exponent at most $d-1$ is virtually nilpotent, and want to do induction on d.

- So in (25), we constructed a group $G'' = \mathbb{Z} \ltimes_e N'$ that has polynomial growth of exponent at most $d-1$. Also G'' has finite index in G. So if we can prove G'' is virtually nilpotent, we can pass the nilpotency to G.
- Then in (26), we showed that it suffices to find some $a \in \mathbb{N}$ such that e^a acts unipotently on N'.
- By considering the centre $Z(N')$, we proved in (27)–(28) that it suffices to find some $a \in \mathbb{N}$ such that e^a acts unipotently on $Z(N')$.
- Through (29)–(30), we proveed that there does exist an $a \in \mathbb{N}$ such that e^a acts unipotently on $Z(N')$.

Then all have been done. □

Part V — Putting all together

Let G be a finitely generated group with polynomial growth of exponent at most d. We prove by induction on d that G is virtually nilpotent.

Question 32

What is the base case? Prove it.

If $d = 0$, then G is actually a finite group. In this case, 1 is nilpotent and has finite index in G, thus G is virtually nilpotent. $\square$

Question 33

Let us consider the action of G on the quotient space E from Part II. Prove that the image A of G in $\mathrm{GL}(E)$ is virtually Abelian.

Recall that $E = \mathrm{HL}(G)/\left\{f \in \mathrm{HL}(G) : \|f\|_{\mathrm{Lip}} = 0\right\}$. Kleiner's theorem implies that $\dim E < +\infty$. Also (10c) said that the action preserves $\|\cdot\|_{\mathrm{Lip}}$, hence the image A of G can be seen as a subgroup of $\mathrm{U}_n(\mathbb{C})$, where $n = \dim E$. Then by Theorem 0.3, A is virtually Abelian. $\square$

Question 34

(a) Prove that if A is infinite, then G has a finite index subgroup with a homomorphism to $\mathbb{Z}$ with infinite image.

(b) Using Theorem 0.4, prove we are done.

(34a) To simplify the notations, we shall denote $\rho : G \to \mathrm{U}_n(\mathbb{C})$ the map in (33). Now $A = \rho(G)$ is finitely generated because G is finitely generated. By (33), A is virtually Abelian, so there exists $B \leqslant A$ an Abelian subgroup with finite index. The subgroup B must be infinite because $|B|\cdot[A : B] = |A|$ as cardinals. So B is a finitely generated infinite Abelian group hence is of the form $B = \mathbb{Z}^\ell \times H$, where H is a finite group. Let

$$\phi : B \to \mathbb{Z}, \qquad (x_1, \ldots, x_\ell, h) \mapsto x_1.$$

It is a homomorphism with image $\mathbb{Z}$.

(34b) Observe that $\phi : B \to \mathbb{Z}$ is surjective. Then $\rho^{-1}(H) = G'$ is a subgroup of G with finite index and $\phi \circ \rho : G' \to \mathbb{Z}$ is surjective. Let $N = \ker(\phi \circ \rho)$, then $N \lhd G'$ and G'/N is infinite cyclic. Then we can apply Theorem 0.4 to G. $\square$

Question 35

(a) Prove that if A is finite, then G has a finite index subgroup G' that acts trivially on E.

(b) Prove in that case, that for any $g \in G'$ and $f \in \mathrm{HL}(G)$, $g \cdot f = f + \lambda_g(f)$, where $\lambda_g(f) \in \mathbb{R}$.

(c) Prove that for any $g \in G'$, λ_g is a linear function on $\mathrm{HL}(G)$ that yields a homomorphism $G' \to (\mathrm{HL}(G)^*, +)$.

(35a) Let $G' = \ker \rho$, then $[G : G'] = \left|G/\ker \rho\right| = |A| < +\infty$ by our assumption. And G' acts trivially on E is immediate by construction.

(35b) Recall the action $(g\cdot f)(x) := f(g^{-1}x)$. Since G' acts trivially on E, for $f\in \mathrm{HL}(G)$, $\|g\cdot f - f\|_{\mathrm{Lip}} = 0$. By (10b), $\lambda_g(f) = g\cdot f - f$ is constant.

(35c) The linearity is trivial. Let $g, h\in G'$,

$$\begin{aligned}\lambda_{gh}(f) = gh\cdot f - f = g\cdot(h\cdot f) - f \\ = g\cdot(f+\lambda_h(f)) - f = g\cdot f + \lambda_h(f) - f = f + \lambda_g(f) + \lambda_h(f) - f = \lambda_g(f) + \lambda_h(f).\end{aligned}$$

So $g\mapsto \lambda_g$ is a group homomorphism. □

Question 36

Prove that if $g\mapsto\lambda_g$ has infinite image, we can use Theorem 0.4 to conclude.

Again, to simplify the notations, denote $\lambda : g\mapsto\lambda_g$. If $\lambda(G')$ is infinite, then λ is a group homomorphism from G to an infinite Abelian group. Then it is a similar argument as (34b). □

Question 37

(a) Prove that if $g\mapsto\lambda_g$ has finite image, then there is another finite index subgroup G'' that acts trivially on $\mathrm{HL}(G)$.

(b) Prove in that case, that all Lipschitz harmonic functions take only finitely many values.

(c) Prove that this implies that all harmonic Lipschitz harmonic functions are constant and this is a contradiction.

(37a) If $\lambda(G')$ is finite, let $G'' := \ker\lambda$, then similar as (35a), G'' is an infinite subgroup of G' with finite index.

(37b) We consider the action of G'' on $\mathrm{HL}(G)$, it acts trivially. For $f\in\mathrm{HL}(G)$, $g\in G''$,

$$f(g^{-1}x) = g\cdot f(x) = f(x) + \lambda_g(f) = f(x), \qquad x\in G.$$

Therefore, f is constant on each $G''x$. Since G'' has finite index, there are only finitely many cosets of G'', so f take only finitely many values.

(37c) Using the maximum value principle, f should be constant. But Kleiner's theorem claims that there exists at least one non-constant function, which leads to a contradiction. □

3 Fixed Point Theorem and Amenable Topological Groups

3.1 Day's fixed point theorem

In this section, we always assume the sets we mentioned are endowed in a locally convex topological vector space E.

Definition–Proposition 3.1: Barycentre

Let (K,μ) be a probability measure space with K compact and convex, μ Radon. Then there exists a unique $x_\mu \in K$ such that

$$\varphi(x_\mu) = \int_K \varphi(x)\,\mathrm{d}\mu(x), \qquad \forall \varphi \in E^*,$$

called the *μ-barycentre* of K.

Proof. If $\dim K < \infty$, then take $x_\mu = \int_K x\,\mathrm{d}\mu(x)$, which is well-defined because K is bounded.

Now assume $\dim K = \infty$. For any $\varphi \in E^*$, consider the set

$$C_\varphi = \left\{ x \in K : \varphi(x) = \int_K \varphi(x)\,\mathrm{d}\mu(x) \right\}.$$

Thus the set of μ-barycentres is $\bigcap_{\varphi\in E^*} C_\varphi$. To show it is non-empty, we consider

$$\Phi_F : K \to \mathbb{R}^F, \qquad x \longmapsto (\varphi(x))_{x\in F}$$

for each finite subset $F \subset E^*$. This is because each C_φ is closed convex, hence finite intersection property can apply. By the finite-dimensional case, $(\Phi(K), \Phi(\mu))$ has a $\Phi(\mu)$-barycentre. Moreover,

$$\Phi^{-1}(\{\Phi(\mu)\}\text{-barycentres}) = \bigcap_{\varphi\in F} C_\varphi.$$

Therefore $\bigcap_{\varphi\in E^*} C_\varphi$ is non-empty. □

Theorem 3.2: Day

Let G be a group. The follows are equivalent:

(1) G is amenable.

(2) For any action $G \curvearrowright X$, there is a G-invariant mean on X.

(3) For any action $G \curvearrowright X$ by homeomorphisms on compact convex X, there is a G-invariant Radon probability measure on X.

(4) Any action $G \curvearrowright X$ by continuous affine transformation on compact convex X admits a fixed point.

Proof. (1) ⇒ (2): Let $m : \ell^\infty(G) \to \mathbb{C}$ be a left-invariant mean. For $x \in X$, consider the map $G \xrightarrow{\cdot x} G\cdot x \longrightarrow X$, which gives a map

$$\ell^\infty(X) \longrightarrow \ell^\infty(G\cdot x) \longrightarrow \ell^\infty(G) \xrightarrow{m} \mathbb{C}$$

Then we have a G-invariant mean on $\ell^\infty(X)$.

(2) ⇒ (1): Let $X = G$ then it is okay.

(2) $\Rightarrow$ (3): By duality $C(X)^* \cong M(X)$, the restriction map $\mathscr{M}(X) \to 2^X$ gives a correspondence between means and Radon probability measures since $\psi \in \mathscr{M}(X) \implies \|\psi\| = 1$. And the G-invariance of ψ corresponds to G-equivariance of μ_ψ.

(3) $\Rightarrow$ (4): Since X is compact and convex, there exists a unique μ-barycentre x_μ of X by Definition-Proposition 3.1. Note that the action $G \curvearrowright X$ is affine and preserves μ, then it also preserves μ-barycentres. Hence it fixes x_μ.

(4) $\Rightarrow$ (1): Apply to the action $G \curvearrowright \mathscr{M}(G)$ by left multiplication. □

Corollary 3.3

For any action of an amenable group by homeomorphisms on a compact convex set X, there exists an invariant probability measure.

Proof. Let G be such a group. $\mathrm{Prob}(X) \subset C(X)^*$ is a compact and convex set. By amenability, there exists a fixed point in $\mathrm{Prob}(X)$ implies the existence of invariant probability measure. □

3.2 Amenable topological group

Now we consider *topological groups* instead of discrete groups.

Definition 3.4: Amenability

A topological group G is *amenable* if for any continuous affine actions on a compact convex topological vector space K, there is a fixed point.

Here continuous actions means that the map $G \times K \to K$, $(g, x) \mapsto g \cdot x$ is continuous. So we need the topology on G. Also, Day's theorem implies that for discrete groups, the definition of amenability is equivalent to our previous definition.

Corollary 3.5

Let τ_1, τ_2 be two topologies on G, with τ_1 finer thatn τ_2. If (G, τ_1) is amenable, then (G, τ_2) is amenable.

Theorem 3.6: Kakutani

Let C be a convex and compact set, $f : C \to C$ be a continuous affine map. Then f has a fixed point.

Proof. For $x \in C$, define $x_n = \frac{1}{n+1} \sum_{i=0}^n f^i(x) \in C$. Let y be an accumulation point of $(x_n)_{n \geqslant 0}$. If $f(y) \neq y$, by Hahn–Banach theorem, there exists $\varphi \in E^*$ such that $\varphi(f(y)) \neq \varphi(y)$. However,

$$\begin{aligned}
\left|\varphi(x_n) - \varphi(f(x_n))\right| &= \left|\varphi\left(\frac{1}{n+1}\sum_{i=0}^n f^i(x)\right) - \varphi\left(\frac{1}{n+1}\sum_{i=0}^n f^{i+1}(x)\right)\right| \\
&= \frac{1}{n+1}\left|\varphi(x) - \varphi(f^{n+1}(x))\right| \leqslant \frac{2\|\varphi\|_\infty}{n+1} \to 0.
\end{aligned}$$

Thus $\left|\varphi(y) - \varphi(f(y))\right| = 0$. Contradiction. Therefore y is a fixed point of f. □

Theorem 3.7: Markov–Kakutani

Let C be a convex and compact set, S be a collection of commuting continuous affine maps $C \to C$. Then there exists $x \in C$ such that $f(x) = x$ holds for all $f \in S$.

Proof. We want $\bigcap_{f \in S} \mathrm{Fix}(f)$. Note that each $f \in S$, $\mathrm{Fix}(f)$ is compact and convex. It suffices to prove that for any $\{f_1, \ldots, f_n\} \subset S$,

$$\bigcap_{i=1}^{n} \mathrm{Fix}(f_i) \neq \varnothing.$$

We prove by induction on n.

When $n = 1$, this is Kakutani's fixed point theorem. Done.

Now assume $K = \bigcap_{i=1}^{n-1} \mathrm{Fix}(f_i) \neq \varnothing$. It is compact and convex, and for all $x \in \mathrm{Fix}(f_i)$,

$$f_n(f_i(x)) = f_i(f_n(x)) = f_n(x) \quad \Longrightarrow \quad f_i(x) \text{ is a fixed point of } f_n.$$

So K is f_n-invariant and then done by Kakutani's theorem. □

Corollary 3.8

Any Abelian group is amenable for discrete topology.

Proof. Let G be Abelian. Then each action $\alpha_g \curvearrowright E$ commutes. Let $S = \{\alpha_g\}$ and apply Markov–Kakutani theorem. For any compact G-space, there is a fixed point. Apply Day's theorem then we are done. □

Theorem 3.9

Compact groups are amenable.

Proof. By definition, it suffices to prove that every compact group G has an invariant measure. If G is finite, take $\mu(A) := |A| / |G|$. So henceforth we assume G is infinite.

Let μ be a diffuse (*i.e., there are no atoms*) probability measure on G with full support. Let λ be some μ-stationary (*i.e.*, $\mu * \lambda = \lambda$) probability measure. Such measure exists because $\mathrm{Prob}(G)$ is a compact and convex subspace of $M(X)$ for wk*-topology and

$$\mathrm{Prob}(G) \to \mathrm{Prob}(G), \qquad \lambda \mapsto \mu * \lambda \left[f \mapsto \int_G \int_G f(xy) \, \mathrm{d}\mu(x) \, \mathrm{d}\lambda(y) \right]$$

is a continuous affine action. Then the existence is ensured by Markov–Kakutani theorem.

For $f \in C(G)$, define $\varphi(g) = \int_G f(xy) \, \mathrm{d}\lambda(y)$, then $\varphi \in C(G)$. The compactness of G implies φ has a maximum at $g_0 \in G$. Replacing f by $f'(x) = f(g_0 x)$, we may assume $g_0 = 1$. Thus

$$\int_G \varphi(x) \, \mathrm{d}\mu(x) = \int_G \int_G f(xy) \, \mathrm{d}\mu(x) \, \mathrm{d}\lambda(y) = \int_G f(x) \, \mathrm{d}\mu * \lambda(x) = \int_G f(x) \, \mathrm{d}\lambda(x) = \varphi(1).$$

Since $\forall g \in G$, $\varphi(g) \leqslant \varphi(1)$, we have $\varphi(g) = \varphi(1)$ μ-a.e.. Since μ has full support and φ continuous, $\varphi(g) = \varphi(1)$ holds for all $g \in G$. Then φ is constant and λ is invariant by

$$\forall g \in G \left(\int_G f(xy) \, \mathrm{d}\lambda(y) = \int_G f(y) \, \mathrm{d}\lambda(y) \right) \quad \Longrightarrow \quad g * \lambda = \lambda.$$

Now let K be some convex compact space in E. Assume G acts by continuous affine maps on K. We want to prove there exists a fixed point in K. Let $x \in K$ and define

$$o : G \to K, \qquad g \mapsto g \cdot x$$

the orbit map. If λ is an invariant probability measure on G, $o * \lambda$ is invariant on K. Thus the $(o * \lambda)$-barycentre is a G-fixed point. □

Remark. *Now we see the same group can have different amenability with different topologies. For example,* $\mathrm{SO}(3)$ *is not amenable as a discrete group because it contains* $\mathbb{F}_2$ *as a subgroup. However, when endowed with the Euclidean topology,* $\mathrm{SO}(3)$ *is compact hence amenable.*

3.3 Permanence properties of amenability

For topological groups, we want a criterion using means because it is usually easier to handle. Instead of $\ell^\infty(G)$, we require continuity since now we have topologies on G.

Definition 3.10: Left uniformly continuous functionals

Let G be a topological group.

(1) A function $f : G \to \mathbb{C}$ is *left uniformly continuous* if

$$\forall \varepsilon > 0 \, \exists U \in \mathcal{N}(1) \left(gh^{-1} \in U \implies \left| f(g) - f(h) \right| < \varepsilon \right).$$

The space of such functions is denoted as $\mathrm{UCB}(G)$. Then $(\mathrm{UCB}(G), \|\cdot\|_\infty)$ is a Banach space.

(2) A *mean* on $\mathrm{UCB}(G)$ is a linear functional $m : \mathrm{UCB}(G) \to \mathbb{C}$ such that m is positive and $m(\mathbb{1}_G) = 1$.

Lemma 3.11

For $f \in \ell^\infty(G)$, the map $g \mapsto g \cdot f$ is continuous if and only if $f \in \mathrm{UCB}(G)$. In other words, G acts on $\mathrm{UCB}(G)$ continuously.

Proof. Let $f \in \mathrm{UCB}(G)$, we want $g \cdot f \in \mathrm{UCB}(G)$. Let $\varepsilon > 0$ and U a neighbourhood, then

$$\forall g, h \in G \left(gh^{-1} \in U \implies \left| f(g) - f(h) \right| < \varepsilon \right).$$

Now for $h_1,\ h_2 \in G$, $g \in G$,

$$\left| (g \cdot f)(h_1) - (g \cdot f)(h_2) \right| = \left| f(g^{-1} h_1) - f(g^{-1} h_2) \right|$$

if $h_1 h_2^{-1} \in gUg^{-1}$, then $g^{-1} h_1 h_2^{-1} g \in U$. Therefore the right hand side is less than ε. □

Theorem 3.12

Let G be a topological group. Then G is amenable if and only if there exists a G-invariant mean on $\mathrm{UCB}(G)$.

Proof. $\Rightarrow$: Let K be the set of means on $\mathrm{UCB}(G)$. It is a compact subspace of $\mathrm{UCB}(G)^*$ since $\mu(f) \leqslant \|f\|_\infty$ implies K is bounded. And G acts continuously by affine maps on K. By amenability, there exists a fixed point, *i.e.*, an invariant mean on $\mathrm{UCB}(G)$.

$\Leftarrow$: Let μ be a G-invariant mean on UCB(G), and K be some compact and convex space in E, which G acts on by continuous affine maps. Let $x \in K$, define the orbit map

$$f : G \to K, \qquad g \mapsto g \cdot x$$

and denote $\bar{f} : G \to \mathbb{C}$, $g \mapsto f(g \cdot x)$. Define $m(f) = \mu(\bar{f})$, where μ is the G-invariant mean on UCB(G). Then m is linear, positive and $m(\mathbb{1}_G) = 1$. Thus $m \in \mathrm{Prob}(K)$ and G-invariant. Therefore, the m-barycentre is a G-fixed point. □

Theorem 3.13: Permanence properties

We have the following permanence properties of amenability:

(1) If $h : G \to H$ is an epimorphism between topological groups, then G amenable implies H amenable.

(2) If G has a dense amenable subgroup for induced topology, then G is amenable.

(3) If G contains a dense increasing unions of amenable groups, then G is amenable.

(4) The topological group G is amenable if the closure of any finitely generated subgroup is amenable.

(5) If H is an open subgroup of an amenable topological group G, then H is amenable for the subspace topology.

(6) Let $N \lhd G$, then G is amenable if both N (*with subspace topology*) and G/N (*with quotient topology*) are amenable. Moreover, if N is open, then this is an equivalence.

Proof. (1) Let C be a compact convex H-space with action $\alpha : H \times C \to C$. Define

$$\beta : G \times C \to C, \qquad (g, x) \mapsto \alpha(h(g), x).$$

Then β is a continuous affine G-action. So Kakutani's theorem gives a G-fixed point $x \in C$. But for any $\ell \in H$, there exists $g \in G$ such that $h(g) = \ell$. Then

$$\ell \cdot x = h(g) \cdot x = \beta(g, x) = x.$$

So α has an H-fixed point.

(2) Let $H \leqslant G$ be dense and amenable. Let C be a compact convex G-space. By amenability of H, $\exists x \in C$ such that $h \cdot x = x$ holds for all $h \in H$. The action is continuous and H is dense, so it can be extended by continuity and $g \cdot x = x$ for all $g \in G$. Therefore G is amenable.

(3) Let G be a group and $\bigcup_{i \in I} G_i$ be some dense directed union of amenable subgroups. It suffices to prove that $\bigcup_{i \in I} G_i$ is amenable. Let C be a compact convex G-space. We want $\bigcup_{\in I} G_i$ has a fixed point in C, *i.e.*,

$$\bigcap_{i \in I} \mathrm{Fix}(G_i) \neq \varnothing, \qquad \mathrm{Fix}(G_i) = \left\{x \in C : \forall g \in G_i \, (g \cdot x = x)\right\}.$$

This is a convex compact subspace. By finite intersection property, it suffices to prove that $\forall \{i_1, \dots, i_n\} \subset I$, $\bigcap_{k=1}^n \mathrm{Fix}(G_{i_k}) \neq \varnothing$. And since I is directed, there exists $j \in I$ such that $G_j \supset G_{i_k}$ for $k \in \{1, \dots, n\}$. So

$$\varnothing \neq \mathrm{Fix}(G_j) \subset \bigcap_{k=1}^n \mathrm{Fix}(G_{i_k}).$$

(4) Apply (3) with $I = \operatorname{fin} G$ and $G_i = \langle i \rangle$ for $i \in I$.

(5) Let G be an amenable group and $H \leqslant G$ open. For any $k \in H \setminus G$, choose $g_k \in k$. For $f \in \mathrm{UCB}(H)$, define

$$\bar{f} : G \to \mathbb{C}, \qquad g \mapsto f(gg_k^{-1}), \qquad k \in Hg.$$

Then $\bar{f} \in \mathrm{UCB}(G)$ and $\forall h \in H\,(\overline{h \cdot f} = h \cdot \bar{f})$. Let μ be a G-invariant mean on $\mathrm{UCB}(G)$, define $\nu(f) = \mu(\bar{f})$ for $f \in \mathrm{UCB}(H)$. Then

$$\bar{f} \geqslant 0, \qquad \overline{f_1 + \lambda f_2} = \bar{f}_1 + \lambda \bar{f}_2, \qquad \overline{\mathbb{1}_H} = \mathbb{1}_G.$$

So ν is a mean on $\mathrm{UCB}(H)$ with

$$h \cdot \nu(f) = \nu(h^{-1} \cdot f) = \mu(\overline{h^{-1} \cdot f}) = \mu(h^{-1} \cdot \bar{f}) = \mu(\bar{f}) = \nu(f).$$

Hence H is amenable.

(6) Let C be a compact convex G-space. Then $\mathrm{Fix}(N) \subset C$ is compact, convex and G-invariant. Now N acts trivially on $\mathrm{Fix}(N)$ because $n \cdot (g \cdot x) = g \cdot (n' \cdot x)$. □

Example 3.14

Let

$$\mathfrak{S}_\infty := \left\{ f : \mathbb{N} \to \mathbb{N},\ f \text{ bijective} \right\}, \qquad \mathfrak{S}_n = \left\{ g \in \mathfrak{S}_\infty : \forall i \geqslant n+1\,(g(i) = i) \right\}.$$

The topology on $\mathfrak{S}_\infty$ is defined by basic neighbourhoods of $g \in \mathfrak{S}_\infty$ being of the form

$$\left\{ h \in \mathfrak{S}_\infty : \forall i \in F\,(h(i) = g(i)),\ F \subset \mathbb{N} \text{ finite} \right\}.$$

Note that $\mathbb{F}_2 \leqslant \mathfrak{S}_\infty$. We can construct an embedding: let $\varphi : \mathbb{F}_2 \to \mathbb{N}$ be a bijection because both of them are countable. For any $g \in \mathbb{F}_2$, $\mathbb{F}_2 \curvearrowright \mathbb{F}_2 \xrightarrow{\varphi} \mathbb{N}$ by left multiplication is a permutation. So $\mathbb{F}_2$ can be considered as a subgroup of $\mathfrak{S}_\infty$.

Since the topology of $\mathfrak{S}_n$ is discrete, and every $\mathfrak{S}_n$ is finite, so $\mathfrak{S}_n$ is amenable. Thus $\overline{\bigcup_{n \geqslant 1} \mathfrak{S}_n}$ is amenable. For $h \in \mathfrak{S}_\infty$, and $F \subset \mathbb{N}$ a finite set, one can find $g \in \mathfrak{S}_\infty$ with finite support such that $\forall i \in F\,(g(i) = h(i))$. Therefore $\mathfrak{S}_\infty = \overline{\bigcup_{n \geqslant 1} \mathfrak{S}_n}$.

Example 3.15

Let H be a separable real HIlbert space, $\mathrm{O}(H)$ is amenable for the wk-topology. Fix $(e_i)_{i \geqslant 1}$ an ONB of H. And denote

$$\mathrm{O}(n) := \left\{ g \in \mathrm{O}(H) : g|_{\overline{\operatorname{span}}\{e_i : i \geqslant n\}} = \mathrm{id} \right\}.$$

This is exactly the orthogonal group of n dimensional space. We have $\mathrm{O}(H) = \overline{\bigcup_{n \geqslant 1} \mathrm{O}(n)}$. Hence $\mathrm{O}(H)$ is amenable. But $\mathrm{O}(H)$ can be non-compact.

Corollary 3.16

IF G is an amenable discrete group, and $H \leqslant G$, then H is amenable.

Proof. This is because every subgroup of a discrete group is open. □

Corollary 3.17

Any solvable group is amenable for discrete topology, hence for any topology.

Proof. Let G be a group. Consider its derived series $(\mathscr{D}^n G)_{n\geqslant 0}$. Since G is solvable, there exists $n \in \mathbb{N}$ such that $\mathscr{D}^n G = 1$. We prove the result by induction on the solvability index.

If $n = 1$, G is Abelian and thus G is amenable. Assume the result holds for index $n-1$ and G has solvable index n. By induction, G' is amenable. Moreover, G/G' is Abelian. We have the extension

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G/G' \longrightarrow 1$$

Hence G is amenable by the permanence properties. $\square$

For a topological group G, it is locally compact if and only if there exists a unique (*up to scalar*) Haar measure on G. Therefore one can define integrals.

Theorem 3.18

Let G be a locally compact group and H be a closed subgroup. If G is amenable, then H is amenable.

Proof. We want a map $\mathrm{UCB}(H) \to \mathrm{UCB}(G)$, $f \mapsto f'$ that is linear, H-equivariant and $\mathbb{1}'_H = \mathbb{1}_G$. Since G is amenable, there exists a G-invariant mean $m \in \mathrm{UCB}(G)^*$. Define $\mu(f) = m(f')$, one can easily check that $\mu \in \mathrm{UCB}(H)^*$ is an H-invariant mean. $\square$

Remark. The locally compactness is necessary here. It is used to prove the existence of continuous local sections $G/H \to G$.

Example 3.19

The group $\mathrm{SL}(n,\mathbb{R})$ is not amenable if $n \geqslant 2$. One can consider a subgroup

$$F = \left\langle \begin{bmatrix} 1 & 2 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & & & \\ 2 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \right\rangle.$$

Then $F \leqslant \mathrm{SL}(n,\mathbb{Z})$ is a free and discrete subgroup in $\mathrm{SL}(n,\mathbb{R})$. So it is a closed non-amenable subgroup.

3.4 Group C^*-algebras

Let G be a group, and

$$\mathbb{C}G := \left\{ \sum_{g\in G} a_g g : \text{finitely many } a_g \neq 0 \right\} = \{ f : G \to \mathbb{C} : f \text{ finitely supported} \}.$$

Then $\mathbb{C}G$ is a *-algebra. If $\pi : G \to \mathscr{U}(H)$ is a unitary representation of G, one can extend π to $\mathbb{C}G$ via linearity, still denoted as $\pi : \mathbb{C}G \to \mathscr{B}(H)$.

Definition 3.20: Group C^*-algebra

Let G be a discrete group.

(1) Let $\lambda : G \to \mathscr{U}(\ell^2(G))$, $g \mapsto \lambda(g) : [\delta_x \mapsto \delta_{gx}]$ be the left-regular representation. Define $C^*_\lambda(G) = \overline{\lambda(\mathbb{C}G)} \subset \mathscr{B}(H)$, called the *reduced C^*-algebra* of G.

(2) Define a norm on $\mathbb{C}G$: $\|a\|_u := \sup\left\{\|\pi(a)\| : \pi \text{ representation}\right\}$. The completion $\overline{\mathbb{C}G}^{\|\cdot\|_u} =: C^*(G)$ is called the *full C^*-algebra* of G.

Proposition 3.21: Universal property

For any representation (π, H) of G, it can be uniquely extended to $\pi : C^*(G) \to \mathscr{B}(H)$. In other words, any *-morphism $\mathbb{C}G \to C^*_\pi(G)$ extends to $C^*(G) \to C^*_\pi(G)$.

$$\begin{tikzcd} \mathbb{C}G \arrow[r, "\pi"] \arrow[d] & C^*_\pi(G) \\ C^*(G) \arrow[ur, "\tilde{\pi}"'] & \end{tikzcd}$$

Proof. By definition

$$\|a\|_u = \sup\left\{\|\pi(a)\| : \pi \text{ is any representation}\right\} \geqslant \|\pi(a)\| = \|a\|_\pi .$$

Hence $(\mathbb{C}G, \|\cdot\|_u) \to \mathscr{B}(H)$ is a contractive *-morphism. It follows that it extends to the C^*-completion $C^*(G)$. □

It would be convenient to see $C^*(G)$ as a group algebra associated to a specific unitary representation of G. We want $\bigoplus_{\pi \text{ rep}} \pi$, but it does not make sense. To overcome the difficulty, note that the supremum can equivalently be taken over all *cyclic representations*, *i.e.*, $\exists v \in H$ such that $(\mathbb{C}G)v$ is dense in H. So if G is countable,

$$\pi \text{ cyclic} \quad \Longrightarrow \quad H \text{ separable} \quad \Longleftrightarrow \quad H \cong \ell^2(I), \qquad I \subset \mathbb{N}.$$

Thus we can pick $\pi_u = \bigoplus_\pi \pi$ over all representations $\pi : G \to \mathscr{B}(\ell^2(I))$. And the universal property applied, we have $C^*(G) \cong C^*_{\pi_u}(G)$.

We shall consider two case of interest:

- Say $\chi : A \to \mathbb{C}$ is a *character* of a C^*-algebra A. Then there is an one-to-one correspondence between the set of characters of $C^*(G)$ and 1-dimensional unitary representation of G.
- There exists a natural C^*-morphism $\lambda : C^*(G) \to C^*_\lambda(G)$.

Recall some properties of states and GNS construction:

(1) A *state* is a linear map $\phi : A \to \mathbb{C}$ with $\phi(1) = 1$ and $\forall a \in A\,(\phi(aa^*) \geqslant 0)$. Denote the space of states on A as $S(A)$.

(2) States on $\ell^\infty(X)$ are means, $S(\ell^\infty(X)) = \mathscr{M}(X)$.

(3) Let $\pi : A \to \mathscr{B}(H)$ be a representation. For $v \in H$, $\|v\| = 1$, one can define a state $\phi_{\pi,v} : a \mapsto \langle \pi(a)v, v\rangle$, called the *vector state*.

(4) The GNS construction told us all states are vector states.

(5) For any state $\phi : A \to \mathbb{C}$, there exists a representation (π, H) of A and a morphism $V \in \mathscr{B}(\mathbb{C}, H)$ of norm 1, such that $\phi(a) = V^*\pi(a)V$.

(6) Given H_1, H_2 Hilbert spaces, a C^*-algebra A, a $*$-morphism $\pi : A \to \mathscr{B}(H_1)$ and a bounded operator $W : H_2 \to H_1$. Then the linear map $A \to \mathscr{B}(H_1)$, $a \mapsto W^*\pi(a)W$ is bounded of norm less than $\|W\|^2$.

We have functors $\mathsf{Set} \to \mathsf{Hilb}$, $X \mapsto \ell^2(X)$. Then

- The disjoint union $X_1 \sqcup X_2$ corresponds to direct sum $\ell^2(X_1) \oplus \ell^2(X_2)$.
- The direct product $X_1 \times X_2$ corresponds to tensor product $\ell^2(X_1) \otimes \ell^2(X_2)$ by a bilinear map

$$\ell^2(X_1) \otimes \ell^2(X_2) \to \ell^2(X_1 \times X_2), \qquad (f_1, f_2) \mapsto f_1 \otimes f_2[(x_1, x_2) \mapsto f_1(x_1)f_2(x_2)].$$

If $(e_i)_{i\in I}$ is an ONB of $\ell^2(X_1)$ and $(f_j)_{j\in J}$ is an ONB of $\ell^2(X_2)$, then $(e_i \otimes f_j)_{i,j}$ is an ONB of $\ell^2(X_1) \otimes \ell^2(X_2)$. Also, our definition of the tensor product does not depend up to canonical isomorphism on a choice of $H_1 \cong \ell^2(X_1)$ and $\mathbb{H}_2 \cong \ell^2(X_2)$.

Definition 3.22: Tensor product for representations

Let (π_1, H_1) and (π_2, H_2) be unitary representations of a group G. Define the *tensor product*

$$\pi_1 \otimes \pi_2 : G \to \mathscr{B}(H_1 \otimes H_2), \qquad (\pi_1 \otimes \pi_2)(g)(v \otimes w) := \pi_1(g)v \otimes \pi_2(g)w.$$

Moreover, π_1 and π_2 unitary implies $\pi_1 \otimes \pi_2$ unitary.

One can rewrite

$$H_1 \otimes H_2 \cong \ell^2(X_1) \otimes H_2 = \bigoplus_{x\in X_1} H_2$$

using the block representation, *i.e.*, see $A \otimes B = [a_{ij}] \otimes B = [a_{ij}B]$ as a block matrix. Let π be a unitary representation of G on H. Let $1_{\ell^2(I)}$ be the trivial representation. Then $\pi \otimes 1_{\ell^2(I)}$ naturally identifies with $\bigoplus_{i\in I} \pi$.

Lemma 3.23: Fell's absorption

Let (π, H) be a unitary representation of G and λ be the left regular representation of G. Then $\lambda \otimes \pi$ is conjugate to $\lambda \otimes 1_H$. So any representation tensor with λ yields a multiple of λ.

Proof. Using the block matrix, what we want is that there exists an ONB such that the matrix of $\lambda\otimes\pi$ is diagonal by blocks, all of which are copies of λ. Then

$$(\lambda \otimes \pi)(g) = \begin{array}{c} \begin{bmatrix} & \pi(g) & \cdots \\ & \vdots & \end{bmatrix} \\ g \end{array} gx, \qquad (\lambda \otimes 1)(g) = \begin{array}{c} \begin{bmatrix} & \mathrm{id}_H & \cdots \\ & \vdots & \end{bmatrix} \\ x \end{array} gx.$$

So define U be a block diagonal operator with $\pi(x)$ on (x, x). Then

$$U^{-1}((\lambda \otimes \pi)(g))U(\delta_x \otimes w) = U^{-1}((\lambda \otimes \pi)(g))(\delta_x \otimes \pi(x)w) = U^{-1}(\delta_{gx} \otimes \pi(gx)w) = \delta_{gx} \otimes w.$$

Or in elements,

$$\mathrm{id}_H \text{ at } (x,x) \xrightarrow{U} \pi(x) \text{ at } (x,x) \xrightarrow{(\lambda\otimes\pi)(g)} \pi(gx) \text{ at } (gx,x) \xrightarrow{U^{-1}} \pi(g) \text{ at } (x,x).$$

Then we conclude the proof. $\square$

Corollary 3.24

For any unitary representation (π, H) of G, $\lambda \otimes \pi$ extends to a representation of $C^*_\lambda(G)$. Moreover, if λ extends to $C^*_\lambda(G)$, then $\bigoplus_{i \in I} \lambda$ extends as well.

Proof. By Lemma 3.23, for any $a \in \mathbb{C}G$,

$$\|a\|_{\lambda\otimes\pi} = \|a\|_{\lambda\otimes\mathrm{id}} = \|a\|_\lambda .$$

Then we conclude the proof. □

3.5 C^*-characterisation of amenability

For $\phi : G \to \mathbb{C}$ a function, denote the multiplier associated to ϕ by

$$m_\phi : G \to \mathbb{C}G, \qquad g \mapsto \phi(g)g,$$

and by linearity extends to $\mathbb{C}G$.

Lemma 3.25

Let $\phi \in S(C^*(G))$. Then the multiplier associated to ϕ induces norm 1 operators $m^u_\phi : C^*(G) \to C^*(G)$ and $m^\lambda_\phi : C^*_\lambda(G) \to C^*_\lambda(G)$.

Proof. By GNS construction, there exists a representation (π, H) and $v \in H$, $\|v\| = 1$ such that $\phi = \phi_{\pi,v}$. Let π_u be the universal representation, and define

$$W : H^u \to H^u \otimes H, \qquad x \mapsto x \otimes v.$$

Then $\|W\| = 1$ because $\|v\| = 1$. Also, its adjoint $W^* : H^u \otimes H \to H^u$ is given by $y \otimes z \mapsto y\langle z, v\rangle$. By (6) in properties of GNS construction, $C^*(G) \to \mathscr{B}(H^u)$, $a \mapsto W^*(\pi_u \otimes \pi)(a)W$ is bounded and of norm 1. Moreover, when restricted to elements of G,

$$g \mapsto W^*(\pi_u \otimes \pi)(g)W = \phi(g)\pi_u(g).$$

One can identify $\mathbb{C}G$ with its image by π_u, and then deduce that the linear map is m^u_ϕ. As for m^λ_ϕ, it is similar as the case m^u_ϕ by replacing π_u by λ.

- $a \mapsto W^*(\pi_u \otimes \pi)W$ extends to $C^*(G)$ by the argument.
- $a \mapsto W^*(\lambda \otimes \pi)W$ extends to $C^*_\lambda(G)$ by Fell's absorption and Corollary 3.24. □

Theorem 3.26

Let G be a group. The following are equivalent.

(1) $C^*(G)$ and $C^*_\lambda(G)$ are isomorphic.

(1') The canonical morphism $\lambda : C^*(G) \to C^*_\lambda(G)$ is an isomorphism.

(2) $C^*_\lambda(G)$ has a character.

(2') $C^*_\lambda(G)$ admits a character which extends the trivial representation.

(3) G is amenable.

Proof. (1) ⇒ (2): Take $\pi : C^*(G) \cong C^*_\lambda(G) \to \mathscr{B}(H)$ an 1-dimensional unitary representation of G. Then the universal property of $C^*(G)$ applies.

(1') ⇒ (2'): Similarly as (1) ⇒ (2).

(2) ⇒ (2'): Let $\chi : C^*_\lambda(G) \to \mathbb{C}$ be a character. Define a unitary operator $U \in \mathscr{U}(\ell^2(G))$, $\delta_g \mapsto \chi(g)\delta_g$. Note that $U^*(\mathbb{C}G)U = \mathbb{C}G$, $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g \overline{\chi(g)} g$. Indeed,

$$U^*\lambda(g)U\delta_x = \chi(x)U^*\lambda(g)\delta_x = \chi(x)U^*\delta_{gx} = \chi(x)\overline{\chi(gx)}\delta_{gx} = \overline{\chi(g)}\delta_{gx}.$$

Hence it preserves $C^*_\lambda(G)$. Therefore $\chi'(a) = \chi(U^*aU)$ defines a character on $C^*_\lambda(G)$, with $\chi'(g) = \chi(U^*gU) = \overline{\chi(g)}\chi(g) = 1$.

(2') ⇒ (3): We want to show that λ admits almost invariant vectors. Let $1 \in K$ for some finite symmetric subset $K \subset G$. Fix $\varepsilon > 0$, let $\mu_K = \frac{1}{|K|}\sum_{x\in K} x \in \mathbb{C}G$ and consider $\mu = \mu_K \cdot \mu_K$. Then

- μ is a positive element of $C^*_\lambda(G)$,
- $K \subset \operatorname{supp}\mu$.

Let $\delta = \min_{x\in K}\mu^2(x) > 0$. Note that $\|\mu_K\|_\lambda \leqslant \|\mu_K\|_1 = 1$, hence $\chi(\mu_K) = 1 = \chi(\mu)$. Since $\|\chi(\mu)\| \leqslant \|\mu\|_\lambda$, we deduce that $\|\mu\|_\lambda = 1$. Since μ is a positive element of $C^*_\lambda(G)$, there exists $v \in H$, $\|v\| = 1$ such that

$$\langle \mu(x)v, v\rangle \geqslant 1 - \frac{\varepsilon^2}{2\delta}, \qquad \forall x \in K.$$

We can verify that μ is a probability, *i.e.*, $\sum_{g\in G}\mu(g) = 1$, one gets

$$1 - \langle \mu(x)v, v\rangle = \langle v, v\rangle - \langle \mu(x)v, v\rangle = \frac{1}{2}\sum_{g\in G}\|\lambda(g)v - v\|_2^2\,\mu(g).$$

We deduce that $\sum_{g\in G}\|\lambda(g)v - v\|_2^2 \leqslant \varepsilon\delta^{-1}$, and therefore $\forall g \in K$,

$$\|\lambda(g)v - v\|_2^2 \leqslant \frac{1}{\delta}\|\lambda(g)v - v\|_2^2\,\mu(g) \leqslant \varepsilon^2.$$

Hence v is an (ε, Q)-almost invariant vector.

(3) ⇒ (1'): Let $(F_n)_{n\geqslant 1}$ be a Følner sequence. Denote $v_n = \mathbb{1}_{F_n}/\sqrt{|F_n|}$, $\phi_n = \phi_{\lambda,v_n} \in S(G)$.

- For any $g \in G$, $\lim_{n\to\infty}\phi_n(g) = 1$ by almost-invariance of $\mathbb{1}_{F_n}/\sqrt{|F_n|}$. So $\langle \lambda(g)v_n, v_n\rangle = 1$.
- For $n \in \mathbb{N}$, $\operatorname{supp}\phi_n$ is finite because $\phi_n(g) = \frac{1}{|F_n|}\langle \mathbb{1}_{gF_n}, \mathbb{1}_{F_n}\rangle \neq 0$, so $gF_n \cap F_n \neq \varnothing$. This means $g \in F_nF_n^{-1}$, which is a finite set.

By Lemma 3.25, $m^u_{\phi_n}(x) \in \mathbb{C}[F_nF_n^{-1}]$, a finite dimensional subspace of $C^*(G)$. By continuity of $m^u_{\phi_n}$ and the denseness $\mathbb{C}G \subset C^*(G)$, one has $m_{\phi_n}(C^*(G)) \subset \mathbb{C}[F_nF_n^{-1}]$. Therefore

$$\begin{array}{ccc} C^*(G) & \xrightarrow{m^u_{\phi_n}} & \mathbb{C}G \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ C^*_\lambda(G) & \xrightarrow{m^\lambda_{\phi_n}} & \mathbb{C}G \end{array} \qquad \Longrightarrow \qquad \begin{array}{ccc} C^*(G) & \xrightarrow{m^u_{\phi_n}} & C^*(G) \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ C^*_\lambda(G) & \xrightarrow{m^\lambda_{\phi_n}} & C^*_\lambda(G) \end{array}$$

We claim that λ is injective.

Firstly, $\forall y \in C^*(G)$, $\lim_{n\to\infty} m^u_{\phi_n}(y) = y$. This is because $\forall g \in G$, $\phi_n(g) \to 1$. By continuity and denseness $\mathbb{C}G \subset C^*(G)$. If $y \in C^*(G)$, $z \in \mathbb{C}G$ with $\|y - z\| < \varepsilon$, then

$$\left\| m^u_{\phi_n}(y) - m^u_{\phi_n}(z)\right\| < \varepsilon$$

as $m^u_{\phi_n}(z) \to z$.

Secondly, $\forall n \in \mathbb{N}$, $m^u_{\phi_n}(C^*(G)) \subset \mathbb{C}G$. We have proved $m^u_{\phi_n}(C^*(G)) \subset \mathbb{C}[F_n F_n^{-1}] \subset \mathbb{C}G$. Now since our diagram commutes, if $x \in \ker\lambda$, then

$$\lambda(m^u_{\phi_n}(x)) = m^u_{\phi_n}(\lambda(x)) = 0$$

because $m^u_{\phi_n}(x) \in \mathbb{C}G$. By the first claim, let $n \to \infty$ then we have $x = 0$. Hence λ is injective. On the other hand, $\operatorname{im}\lambda \supset \mathbb{C}G$, so $\operatorname{im}\lambda$ is dense in $\mathbb{C}G$. So λ is actually an isomorphism. $\square$

DM II – A Necessary Condition of Amenability for Countable Discrete Group

The main purpose of this homework is to give another characterisation of amenability for countable discrete group G, that if λ_G weakly contains a finite dimensional representation, then G is amenable.

Part I — A sufficient criterion of week containment

Let (π, H), (π', H') be unitary representations of a group G. Let X be a subset of H, such that $\|x\| = 1$ and span X is dense in H. The goal of this part is to prove the following claim:

Claim. $\pi \prec \pi'$ if and only if the coefficients $\langle \pi(g)x, x\rangle$ for $x \in X$ are approximable by finite sums of a diagonal coefficients of π'.

We denote by W the set of $v \in H$ such that for all finite subset Q of G, for all $\varepsilon > 0$, there exist $w_1, \dots, w_k$ vectors of H' such that for all $g \in Q$,

$$\left|\langle \pi(g)v, v\rangle - \sum_i \langle \pi'(g)w_i, w_i\rangle\right| \leqslant \varepsilon.$$

Questions 1 – 5.

(1) Show that W is invariant by scalar multiplication.

(2) Show that W is closed.

(3) Show that for all $v \in W$, $\operatorname{span}(\pi(G)v)$ is contained in W.

(4) Show that W is stable by sum.

(5) Conclude **Claim**.

Part II — C^*-Characterisation of amenability

We shall use the characterisation of amenability according to which the trivial representation is weakly contained in the regular representation. In this part, we propose to give an alternative proof of the fact that:

Fact. An amenable group G satisfies $C^*(G) = C^*_\lambda(G)$.

Questions 6 – 9.

(6) Let π, σ, ρ be unitary representations of G. Show that if $\pi \prec \sigma$, then $\pi \otimes \rho \prec \sigma \otimes \rho$. (*Hint: use* **(5)**)

(7) Deduce from the **(6)** that if the trivial representation is weakly contained in the regular representation, then every representation is weakly contained in the regular representation.

(8) Let $a \in C^*(G)$ be in the kernel of λ. Show using **(7)** that if G is amenable, then $a = 0$.

(9) Conclude **Fact**.

Part III — Case where the regular representation contains a representation of finite dimension

In this part, assume G is a countable group.

Questions 10 – 12.

(10) Show that the regular representation of G contains the trivial representation if and only if G is finite.

(11) Let $f \in \ell^2(G)$, and let $G = \{g_0, g_1, \dots, g_n, \dots\}$ be an enumeration of G. Show that if G is infinite, then $\lim_{n\to\infty} \langle \lambda(g_n) f, f \rangle = 0$.

(12) Show that the regular representation of G contains a finite dimensional representation if and only if G is finite.

Part IV — Tensor product and endomorphisms

Let (π, H) be a unitary representation of G. We denote $\bar{\pi}$ the « dual » representation of π, *i.e.*, defined on H^* by composition $\bar{\pi}(g)\varphi = \varphi \circ \pi(g^{-1})$, for all $\varphi \in H^*$, $g \in G$.

We shall assume H has finite dimension in this part.

Questions 13 – 16.

(13) Using the trace, equip $\mathscr{B}(H)$ with an hermitian scalar product $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$.

(14) We define a linear map $\Psi : H \otimes H^* \to \mathscr{B}(H)$ by $\Psi(v \otimes \varphi) := v\varphi$, *i.e.*, the endomorphism which to $x \in H$ associates $v\varphi(x)$. Show that Ψ is an isometry for $\langle \cdot, \cdot \rangle_{\mathrm{tr}}$.

(15) Show that for all $g \in G$ and $w \in H \otimes H^*$, $\Psi((\pi \otimes \bar{\pi})(g)(w)) = \pi(g)\Psi(w)\pi(g^{-1})$.

(16) Deduce that $\pi \otimes \bar{\pi}$ contains the trivial representation.

Part V — Case where the regular representation weakly contains a representation of finite dimension

We suppose λ, the left regular representation of G, weakly contains a finite dimensional representation.

Question 17.

Show that G is amenable.

4 The Unique Trace Property

4.1 Revisit traces

The group action $G \curvearrowright S(C^*(G))$ by conjugation $g \cdot \phi := \phi(g^{-1} \cdot g)$ gives the intuition that the fixed points corresponds to traces in the sense of states.

Definition 4.1: Trace

A *trace* on a C^*-algebra A is a state ϕ such that for any $a, b \in A$, $\phi(ab) = \phi(ba)$.

If $A = C^*(G)$, it suffices to verify $\phi(ab) = \phi(ba)$ for $b \in G$. This is because by linearity one can extend the previous formula to $b \in \mathbb{C}G$, and by denseness of $\mathbb{C}G$ in $C^*(G)$, one can further extends to the case $b \in C^*(G)$.

We have some basic facts about traces.

(1) For finite dimensional representations $\pi : A \to \mathrm{Mat}_n(\mathbb{C})$, the canonical trace $a \mapsto \frac{1}{n}\mathrm{Tr}\,\pi(a)$ is a trace.

(2) If A admits a character, then characters are traces.

(3) $C^*_\lambda(G)$ admits a character if and only if G is amenable. But $C^*_\lambda(G)$ has a canonical trace $\tau : a \mapsto \langle a\delta_1, \delta_1\rangle$, where δ_1 is the Dirac function at the neutral element.

(4) On G, the canonical trace $\tau(1) = 1$ and $\tau(g) = 0$ for $g \neq 1$.

Question. When the canonical trace on G is the only trace? If G only has the canonical trace τ, we say G has the *unique trace property*.

The answer is no for non-amenable groups. Because the trivial character is different from the canonical trace.

Definition 4.2: Conditional expectation

Let G be a group and $H \leqslant G$. Denote $p_H : \ell^2(G) \to \ell^2(H)$ the orthogonal projection. Then define

$$E_H : \mathbb{C}G \to \mathbb{C}H, \qquad a \mapsto p_H a p_H.$$

Now $\operatorname{im} E_H$ is dense in $\overline{\mathbb{C}H}$, hence is contained in $C^*_\lambda(H)$. On the other hand, $E_H|_{\mathbb{C}H} = \mathrm{id}$, hence is identity on $C^*_\lambda(H)$. Therefore $\operatorname{im} E_H = C^*_\lambda(H)$. Now one can extend E_H to $C^*_\lambda(G)$.
The projection $E_H : C^*_\lambda(G) \to C^*_\lambda(H)$ is called the *conditional expectation*.

Proposition 4.3

If G contains a non-trivial normal amenable subgroup, then $C^*_\lambda(G)$ has at least two distinct traces.

Proof. Let $N \triangleleft G$ be an amenable normal subgroup. Then $E_N : C^*(G) \to C^*(N)$ induces a map $S(C^*(G)) \to S(C^*(N))$. Now N is amenable implies there exists trivial character $\chi : C^*(N) \to \mathbb{C}$. Thus $\chi \circ E_N$ is a trace on G. Indeed, for any $g, k \in G$,

$$\chi \circ E_N(g^{-1}kg) = \begin{cases} 1, & k \in N, \\ 0, & k \notin N. \end{cases}$$

since E_N is just killing all coefficients that are not indexed by H. If N is non-trivial, $\chi \neq \tau_N$, where τ_N is the canonical trace on N. While $\chi \circ E_N|_G = \chi|_G \neq \tau_N$. □

Definition–Proposition 4.4: Amenable radical

et G be a group. There exists a maximal normal amenable subgroup of G, called the *amenable radical* of G, denoted as $\mathrm{Rad}(G)$.

Proof. Observe that if N_1, N_2 are normal subgroups, then $N_0 := N_1 \cap N_2 \lhd G$, $N := N_1 N_2 \lhd G$. One can just consider

$$1 \longrightarrow N_0 \longrightarrow N \longrightarrow (N_1/N_0) \times (N_2/N_0) \longrightarrow 1$$

The amenability is stable under taking quotient, subgroup and extension. So N is amenable. Now $\{N : N \lhd G\}$ is directed with respect to inclusion, so the directed limit is also amenable and normal. $\square$

4.2 Power's property

Definition 4.5: Power's property

A group G is said to satisfy *Power's property* if given any finite subset $F \subset G \setminus \{1\}$, $\forall N \in \mathbb{N}$, there exists a partition $G = C \sqcup D$ and $\gamma_1, \ldots, \gamma_N \in G$, such that

(1) $\forall i, j \in \{1, \ldots, n\}$, $\gamma_i D \cap \gamma_j D = \varnothing$ for $i \neq j$.

(2) $gC \subset D$ for all $g \in F$.

Proposition 4.6

If G satisfies Power's property, then it is non-amenable.

Proof. We prove by contradiction. Let m be a left-invariant finitely additive measure of G. Then $gC \subset D$ implies

$$m(C) = m(gC) \leqslant m(D), \qquad m(G) \leqslant 2m(D).$$

But $\gamma_i D \cap \gamma_j D = \varnothing$ gives $3m(D) \leqslant m(G)$. Hence $m(G) \leqslant 2m(D) \leqslant \frac{2}{3} m(G)$, contradiction. $\square$

Proposition 4.7

Power's property $\Longrightarrow$ Unique trace property.

Proof. Let $\phi : C^*_\lambda(G) \to \mathbb{C}$ be a trace. We want to show that $\phi(g) = 0$ unless $g = 1$, which implies $\phi\_\mathbb{C}G = \tau|_{\mathbb{C}G}$ and hence $\phi = \tau$. Since G has power's property, fix $g \in G \setminus \{1\}$. For $n \in \mathbb{N}$, define $\tilde{g} = \frac{1}{n}\sum_{k=1}^n \gamma_k g \gamma_k^{-1} \in \mathbb{C}G$. Then

$$\phi(\tilde{g}) = \frac{1}{n}\sum_{k=1}^n \phi(\gamma_k g \gamma_k^{-1}) = \frac{1}{n}\sum_{k=1}^n \phi(g) = \phi(g)$$

because traces are invariant under conjugation. Thus by $|\phi(\tilde{g})| \leqslant \|\tilde{g}\|_\lambda$, it suffices to prove $\|\tilde{g}\|_\lambda \to 0$ as $n \to \infty$.

Denote $C_i = \gamma_i C = \gamma_i(G \setminus D) = G \setminus \gamma_i D$. Since $\gamma_i g \gamma_i^{-1} C_i \subset \gamma_i D$, let p_i be the orthogonal projection on functions supported on $D_i = \gamma_i D$. One has

$$(1 - p_i)\lambda(\gamma_i g \gamma_i^{-1})(1 - p_i) = 0$$

because $1-p_i$ is the projection on supp $\subset C$. Denote $g_i = \lambda(\gamma_i g \gamma_i^{-1})$,

$$g_i = g_i p_i + g_i(1-p_i) = g_i p_i + p_i g_i(1-p_i).$$

Decompose $\tilde{g} = \tilde{g}_1 + \tilde{g}_2$ with $\tilde{g}_1 = \frac{1}{n}\sum_{i=1}^n g_i p_i$, $\tilde{g}_2 = \frac{1}{n}\sum_{i=1}^n p_i g_i(1-p_i)$. Then for all $f \in \ell^2(G)$,

$$\|\tilde{g}_1\|_\lambda = \frac{1}{n}\left\|\sum_{i=1}^n g_i p_i(f)\right\|_2 \leqslant \frac{1}{n}\left\|\sum_{i=1}^n p_i(f)\right\|_2 \leqslant \frac{1}{n}\left(\sum_{i=1}^n \|p_i(f)\|_2^2\right)^{1/2} \leqslant \frac{1}{n}\|f\|_2.$$

So $\|\tilde{g}_1\|_\lambda \leqslant 1/\sqrt{n}$. And for $f \in \ell^2(G)$ with $\|f\|_2 = 1$, one has

$$\left\langle \sum_{i=1}^n p_i g_i(1-p_i)f, \sum_{i=1}^n p_i g_i(1-p_i)f \right\rangle = \sum_{i=1}^n \langle p_i g_i(1-p_i)f, p_i g_i(1-p_i)f\rangle \leqslant n.$$

So $\|\tilde{g}_2\|_\lambda \leqslant 1/\sqrt{n}$.

Thus,

$$\|\tilde{g}\|_\lambda \leqslant \|\tilde{g}_1\|_\lambda + \|\tilde{g}_2\|_\lambda \leqslant \frac{2}{\sqrt{n}} \to 0, \qquad n \to \infty.$$

So $\forall g \neq 1$, $\phi(\tilde{g}) = 0$. This implies $\phi = \tau$. □

As an example, we consider free groups.

Proposition 4.8

$\mathbb{F}_2 = \langle a, b\rangle$ satisfies the Power's property, and hence has unique trace property.

Proof. The elements of $\mathbb{F}_2$ are of the form $a^{k_1} b^{\ell_1} \cdots a^{k_n} b^{\ell_n}$, $k_i, \ell_i \neq 0$. Denote A_0 the set of words starting with a non-zero power of a. Then

$$B_0 = \mathbb{F}_2 \setminus A_0 = \{\text{words starting with a non-zero power of } b\} \cup \{1\}.$$

Take $F = \{\text{words of length} \leqslant n\} \setminus \{1\}$.

Let $A = a^n B_0$ and $f = a^{k_1} b^{\ell_1} \cdots a^{k_n} b^{\ell_n} a^{-t}$, the elements in fA are of the form $a^{k_1} b^{\ell_1} \cdots a^{k_n} b^{\ell_n} a^{n-t}\beta$, where $\beta \in B_0$. We have $|k_1| \leqslant n$ by definition, hence $fA \cap A = \varnothing$. Denote $B = \mathbb{F}_2 \setminus A$, then similar reasoning leads to $a^{-n}B = A_0$, *i.e.*, $B = a^n A_0$. Let $g_k = b^k a^{-n}$, then $g_k B = b^k A_0$. By definition of A_0, when $j \neq k$, $b^k A_0 \cap b^j A_0 = \varnothing$. □

One can describe Power's property via its full group C^*-algebra. To do this, we introduce Dixmier's property.

Definition 4.9: Dixmier's property

A unital C^*-algebra A is said to satisfy *Dixmier's property* if for all $a \in A$, the closure of the convex hull of uau^* for $u \in \mathscr{U}(A)$ intersects non-trivially the centre of A.

Proposition 4.10

Let A be a C^*-algebra.

(1) If A satisfies Dixmier's property, and the centre of A is reduced to $\mathbb{C}1$, then A has at most one trace.

(2) If in addition A admits a faithful trace τ, *i.e.*, $\tau(x^*x) = 0 \implies x = 0$, then A is simple, *i.e.*, the only two-sided ideals are $\{0\}$ and A.

Proof. For $x \in A$, denote $C_x = \overline{\text{co}}\{uxu^* : u \in \mathscr{U}(A)\}$. By assumption, $\forall x \in A$, we have $C_x \cap Z(A) \neq \varnothing$.

(1) Let τ be a trace on A. Since $Z(A) = \mathbb{C}1$ in this case, any trace is constant on C_x for each $x \in A$. Therefore $\exists \lambda \in \mathbb{C}$ such that $\lambda 1 \in C_x \cap \mathbb{C}1$, then $\tau(x) = \lambda$.

(2) Now let τ be a faithful trace on A, and I be a non-trivial ideal of A. Let $x \in I$ with $x \neq 0$, by the faithfulness, $\tau(x^* x) > 0$. By the reasoning in (1), $\exists \lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda 1 \in C_x \subset I$. Now $1 = \frac{1}{\lambda}(\lambda 1) \in I$, hence $I = A$. □

Now we introduce a similar result as in Proposition 4.7.

Lemma 4.11

Let $(x_i)_{i=1}^n$ be a finite family of elements of $\mathscr{B}(H)$ of norm at most 1, and $(H_i)_{i=1}^n$ a family of pairwise orthogonal closed subspace of H. We suppose that for all $i \in [\![1, n]\!]$, we have $x_i H_i^\perp \subset H_i$. Prove that

$$\left\| \frac{1}{n} \sum_{i=1}^n x_i \right\| \leqslant \frac{2}{\sqrt{n}}.$$

Proof. By assumption $x_i H_i^\perp \subset H_i$. Denote $E = \left(\bigoplus_{i=1}^n H_i\right)^\perp$, then $H = E \oplus \bigoplus_{i=1}^n H_i$. Then let $p_i : H \to H_i$ be the orthogonal projection on H_i. Note that $(p_i)_{i=1}^n$ satisfies

$$\sum_{i=1}^N p_i \leqslant 1, \qquad \forall i \neq j (p_i p_j = 0), \qquad \forall (1 - p_i) x_i (1 - p_i) = 0.$$

Write $x_i = p_i x_i + (1 - p_i) x_i p_i$, then

$$\left\| \sum_{i=1}^n x_i \right\| \leqslant 2\sqrt{n} \Longleftarrow \left\| \sum_{i=1}^n p_i x_i \right\| \leqslant \sqrt{n}, \text{ and } \left\| \sum_{i=1}^n p_i x_i^* (1 - p_i) \right\| \leqslant \sqrt{n}.$$

For the first term on RHS,

$$\left\| (\sum_{i=1}^n p_i x_i)^* (\sum_{i=1}^n p_i x_i) \right\| = \left\| \sum_{i,j=1}^n x_i^* p_i p_j x_j \right\| = \left\| \sum_{i=1}^n x_i^* p_i x_i \right\| \leqslant n.$$

And the second term on RHS, we have

$$\left\| (\sum_{i=1}^n p_i x_i^* (1 - p_i))^* (\sum_{i=1}^n p_i x_i^* (1 - p_i)) \right\| = \left\| \sum_{i,j=1}^n (1 - p_i) x_i p_i p_j x_j^* (1 - p_j) \right\| = \left\| \sum_{i=1}^n (1 - p_i) x_i p_i x_i^* (1 - p_i) \right\| \leqslant n.$$

Combining the two estimates, done. □

Theorem 4.12

Let Γ be a countable group satisfying Powers' property. Then for all $a \in C_\lambda^*(\Gamma)$, the closed convex hull of uau^* for $u \in \mathscr{U}(A)$ contains $\tau(a)1$. So $C_\lambda^*(\Gamma)$ is simple and has a unique trace.

Proof. One can start treating the case where $\tau(a) = 0$ and $a \in \mathbb{C}\gamma$. Let $a = \sum_{g \in G} a(g)\delta_g \in \mathbb{C}\Gamma$ with $\tau(a) = 0$. Denote $F = \{g \in \Gamma : a(g) \neq 0\}$. Then $\tau(a) = 0$ implies $1 \notin F$. By Powers' property, $\exists A, B$ such that $\Gamma = A \sqcup B$, such that $\forall f \in F (fA \subset B)$, and $\exists g_1, \ldots, g_N \in \Gamma (j \neq k \implies g_j B \cap g_k B = \varnothing)$.

Define $H_i = \ell^2(g_i B)$. Then

$$\lambda(g_i) H_i^\perp \subset \ell^2(B)^\perp = a\ell^2(A) \subset \ell^2(B) \implies \lambda(g_i)\ell^2(B) \subset H_i.$$

So we can apply the previous lemma with $x_i = \lambda(g_i) a \lambda(g_i^{-1})$ to obtain

$$\frac{1}{N} \left\| \sum_{i=1}^N \lambda(g_i) a \lambda(g_i)^{-1} \right\| \leqslant \frac{2}{\sqrt{N}}.$$

Let $a \in C^*_\lambda(\Gamma)$. By density of $\mathbb{C}\Gamma$ in $C^*_\lambda(\Gamma)$, $\forall \varepsilon > 0 \, \exists a_0 \in \mathbb{C}\Gamma \, (\|a - a_0\| < \varepsilon)$ with $\tau(a) = \tau(a_0)$. Then pick $b = a_0 - \tau(a_0)1$, $\exists u_1, \dots, u_N$ such that $\| \frac{1}{N} \sum_{i=1}^N u_i b u_i^* \| \leqslant \varepsilon$. Thus

$$\left\| \frac{1}{N} \sum_{i=1}^{N} u_i (a - \tau(a)1) u_i^* \right\| \leqslant 2\varepsilon \implies \left\| \left(\frac{1}{N} \sum_{i=1}^{N} u_i a u_i^* \right) - \tau(a)1 \right\| \leqslant 2\varepsilon,$$

which concludes the proof. □

4.3 Theory of G-boundary

To establish the Power's property for a group G, the general strategy is to find a compact G-space with «*north / south dynamics*», *i.e.*,

$$\forall g \neq 1 \, \exists g^-, g^+ \in X \, (x \neq g^- \implies g^n x \to g^+).$$

What compact G-spaces do we have now? An important example is $\mathscr{M}(G)$, the means on G.

Lemma 4.13

The point stabilisers of $\mathscr{M}(G)$ are amenable. In particular,

$$\ker[G \curvearrowright \mathscr{M}(G)] \subset \mathrm{Rad}(G).$$

Proof. Assume $m \in \mathscr{M}(G)$ is H-invariant, where $H \leqslant G$ and $G = \coprod_{x \in X} xH$. The action $\psi : G \to H$, $xh \mapsto h$ is H-equivariant for the right action. Thus $m_H := \psi \cdot m$ is an H-invariant mean on H. Hence H is amenbale. □

Let X be a compact G-space, $M(X)_1$ be the space of Radon probability measures on X. It is compact and convex. Let $\iota : X \hookrightarrow M(X)_1$, $x \mapsto \delta_x$ be the inclusion. Then ι is injective, continuous. So $\iota(X)$ is homeomorphic to X.

Actually, $\iota(X) = \operatorname{Ext} M(X)_1$ because Dirac measures are extremal points of $M(X)_1$.

Definition 4.14: G-boundary

Let X be a G-space. It is

(1) *minimal*, if it has no proper closed G-invariant subset.

(2) *strongly proximal*, if for any $\mu \in M(X)_1$, $\exists x \in X$ such that $\delta_x \in \overline{G \cdot \mu}$.

(3) a *G-boundary*, if X is both minimal and strongly proximal.

If X is a G-boundary, the minimality implies the equivalence between being strongly proximal and $\forall \mu \in M(X)_1 \, (X \subset \overline{G \cdot \mu})$.

Proposition 4.15

A compact G-space is a G-boundary if and only if X is the only minimal closed G-invariant subspace of $M(X)_1$.

Proof. $\Rightarrow$: Assume that X is a G-boundary. Let Y be a minimal G-invariant subspace of $M(X)_1$. The strong proximality implies

$$\forall \mu \in Y \, \exists x \in X \, (\delta_x \in \overline{G \cdot \mu}).$$

Hence $\overline{G\cdot x} \subset \overline{G\cdot\mu}$ by G-invariance. Since X is minimal, $X \subset \overline{G\cdot\mu} \subset Y$. And the minimality of Y gives $Y = X \subset \overline{G\cdot\mu}$.

$\Leftarrow$: It suffices to prove strong proximality. Let $\mu \in M(X)_1$, using Zorn's lemma one can obtain a minimal G-space Y with $Y = X \subset \overline{G\cdot\mu}$. $\square$

If G is amenaable, then for any G-spaces X, by the fixed point characterisation of amenability, there exists $\mu \in M(X)_1$ fixed by action of G, *i.e.*, μ is G-invariant. Hence X is a G-boundary if X is a singleton.

Proposition 4.16

If $N \triangleleft G$ is an amenable normal subgroup, and X is a G-boundary. Then N acts trivially on X.

Proof. The amenability of N implies that N fixes some $\mu \in M(X)_1$. The set of fixed points

$$M(X)_1^N := \{\mu \in M(X)_1 : n\cdot\mu = \mu,\ \forall n \in N\}.$$

Since N is normal, $\forall g \in G$, one has

$$n\cdot(g\cdot\mu) = ng\cdot\mu = gg^{-1}ng\cdot\mu = gn'\cdot\mu = g\cdot\mu.$$

Thus $M(X)_1^N$ is a closed G-invariant subset. By Proposition 4.15, $M(X)_1^N = X$. Hence elements of X are fixed by N. $\square$

Definition 4.17: Hyperbolic elements

Given a G-space X, say $g \in G$ is *hyperbolic* if there exists $g^{\pm} \in X$ such that

$$g^n y \to g^+, \qquad \forall y \in X \setminus \{g^-\}.$$

The element $g^+ \in G$ is called a *hyperbolic attractor* of X.

The set of hyperbolic attractors is denoted as X_{hyp}. (*It can be empty!*)

We immediately have:

- X_{hyp} is G-invariant.
- If $g \in G$ is hyperbolic, it has at most 2 fixed points, with g^+ being one of them. If $g^+ = g^-$, then g^+ is the only fixed point.

Thus, we may assume $\{g^+, g^-\}$ the set of fixed point of g. Then

$$\forall x \in X_{\text{hyp}}\, \exists g \in G\, (x = g^+), \qquad \forall k \in G\, (kx = (kgk^{-1})^+).$$

Proposition 4.18

Let X be a G-space. Assume that there exist two hyperbolic elements g, h such that $\{g^{\pm}\} \cap \{h^{\pm}\} = \varnothing$. Then $\overline{X_{\text{hyp}}}$ is a G-boundary.

Proof. (1) We first prove $Z = \overline{X_{\text{hyp}}}$ is strongly proximal. For all $\mu \in \mathscr{M}(Z)_1$, $\forall f \in C(Z)$, pointwisely we have

$$f(g^n(z)) \to \tilde{f}(z) = \begin{cases} f(g^-), & \text{if } z = g^-, \\ f(g^+), & \text{otherwise.} \end{cases}$$

Also, $g^n \cdot \mu(f) = \mu(g^n(f)) = \mu(f(g^n \cdot))$. By DCT,

$$\mu(f(g^n \cdot)) \to f(g^+)\mu(Z \setminus \{g^-\}) + f(g^-)\mu(\{g^-\}).$$

Denote $\alpha = \mu(\{g^-\})$, then $g^n \cdot \mu \to \alpha\delta_{g^+} + (1-\alpha)\delta_{g^-}$.

Since $h^+ \notin \{g^\pm\}$, apply the same argument, one has $h^n \cdot (\lim_{m\to\infty} g^m \cdot \mu) \to \delta_{h^+}$. Hence $h^+ \in \overline{G \cdot \mu}$. So Z is strongly proximal.

(2) Then prove the minimality. For $z \in Z$ and $k^+ \in X_{\text{hyp}}$. We want to show that $k^+ \in \overline{G \cdot z}$. By denseness of X_{hyp} in Z, this implies $Z \subset \overline{G \cdot z}$. If $z \neq k^-$, then $k^n z \to k^+$, which implies $k^+ \in \overline{G \cdot z}$. If $z = k^-$, then if $k^- = g^-$, one can use h to move it out, *i.e.*, $hz \neq k^-$. Else, we use g to move it out, *i.e.*, $gz \neq k^-$. Then we back to the first case. □

Example 4.19

Consider $\mathbb{F}_2 = \langle a, b\rangle$. The boundary $\partial\mathbb{F}_2 = \{a^\pm, b^\pm\}$. Then $a, b \in \mathbb{F}_2$ are hyperbolic because $\overline{(\partial\mathbb{F}_2)_{\text{hyp}}} = \partial\mathbb{F}_2$.

Proposition 4.20

Let G be a group, $(X_i)_{i\in I}$ be a family of strongly proximal G-spaces. Then $X = \prod_{i\in I} X_i$ is strongly proximal.

Proof. We start with $|I| = 2$. Let $X = X_1 \times X_2$ then we consider $\pi_i : M(X)_1 \to M(X_1)_1$. Let $\mu \in M(X)_1$, by strong proximality of X_1,

$$\exists x_1 \in X_1\,(\delta_{x_1} \in \overline{\pi_1(G \cdot \mu)}).$$

Let $\mu' \in \overline{G \cdot \mu}$ such that $\pi_1(\mu') = \delta_{x_1}$, then $\mu' = \delta_{x_1} \otimes \nu$ for $\nu \in M(X_2)_1$. Similarly, $\exists (g_j)_{j\in I} \subset G$ such that $g_j \nu \to \delta_{x_2}$. Then by compactness, there exists a subset(*wlog,; let it be* $(g_j)_{j\in J}$ *itself*), such that $g_j\delta_{x_1} \to y_1 \in X_1$. So $g_j(\delta_{x_1} \otimes \nu) \to \delta_{(y_1, x_2)}$.

By induction, it holds for any finite I.

If I is infinite, denote $X_J \subset \prod_{i\in J} X_i$ for $J \subset I$, and $\pi_J : M(X)_1 \to M(X_J)_1$. Fix $\mu \in M(X)_1$, for all $F \in \operatorname{fin} I$,

$$Z_F := \left\{\nu \in \overline{G \cdot \mu} : \pi_F \nu \text{ is Dirac}\right\}$$

is a non-empty closed set by the finite case. Therefore,

$$Z_F \cap Z_{F'} = Z_{F\cup F'} + \text{compactness} \quad \Longrightarrow \quad Z = \bigcap_{F \in \operatorname{fin} I} Z_F \neq \varnothing.$$

We claim that $Z = \left\{\nu \in \overline{G \cdot \mu} : \nu \text{ is Dirac}\right\}$. This is because $\forall \nu \in Z$, $\forall i \in I$, $\pi_i \mu = \delta_{x_i}$. Hence ν is supported on the singleton $x = (x_i)_{i\in I} \in X$. □

4.4 Fustenburg boundary

Proposition 4.21: Fustenburg's lemma

Let X be a G-boundary, and Y be a minimal G-set. Then every continuous G-equivariant map $\Psi : Y \to M(X)_1$ is usch that $\Psi(Y) \subset X$. Moreover, if Ψ exists, then it is unique.

Proof. By Proposition 4.15, the minimality of X implies $\Psi(Y) = X$ because Y minimal $\Longrightarrow$ $\Psi(Y)$ minimal. If ψ, $\psi' : Y \to X$, then take $\psi'' = \frac{1}{2}(\psi + \psi')$, one has $\psi'' : Y \to X$.

If $y \in Y$ such that $\psi(y) \neq \psi'(y)$, then $\psi''(y) \notin X$ because it is not Dirac. But ψ'' is G-equivariant. So $\psi = \psi'$ must happen. □

Definition–Proposition 4.22: Fustenburg boundary

Let G be a group. There is a unique maximal G-boundary of G, called the *Fustenburg boundary* of G, denoted as $\partial_F G$. Here maximal is in the following sense: for any G-boundary X, there exists a G-equivariant $\Psi : \partial_F G \to X$.

Note that Proposition 4.21 says the map $\partial_F G \to X$ is unique and surjective. Hence surjective only depends on the minimality of X. The uniqueness of $\partial_F G$ also comes from Proposition 4.21.

Proof. *Existence*: The minimality of X implies $X = \prod_{i \in I} X_i$ with each X_i minimal, where

$$I = \{G\text{-equivariant homeomorphic classes}\}.$$

Let X_0 be a minimal closed G-invariant subset of X, then X_0 is a G-boundary. Denote $\pi_i : X \to X_i$, X_i minimal implies $\pi_i(X) = X_i$, which further implies that X_0 is Fustenburg.

Uniqueness: If X_0, X_0' are Fustenburg, there exist G-equivariant maps $\Psi : X_0 \to X_0'$, $\Phi : X_0' \to X_0$. Then Proposition 4.21 implies $\Psi\Phi = \mathrm{id}$, $\Phi\Psi = \mathrm{id}$. So $X_0 = X_0'$. □

Example 4.23

Note that X and Y both minimal does not imply $X \times Y$ minimal. For example, let $X = Y = \mathbb{Z}/2\mathbb{Z}$. Then $\mathbb{Z}/2\mathbb{Z} \curvearrowright X$ is minimal. However, $\mathbb{Z}/2\mathbb{Z} \curvearrowright X \times X$ is not minimal.

Definition 4.24: Convex / Minimal G-space

Let X be a G-space. Say it is

(1) *convex*, if it is equipped a G-action by continuous affine transformations.

(2) *minimal*, if it does not contain any non-empty proper G-invariant closed convex subset.

Recall Milman's theorem: for a convex compact set K, $K = \overline{\mathrm{co}}\,A \Longrightarrow \mathrm{Ext}\,K \subset \bar{A}$.

Lemma 4.25

Let X be a compact space, $x_0 \in X$, $\mu \in M(X)_1$. The following are equivalent:

(1) $\mu = \delta_{x_0}$.

(2) For any $f \in C(X)$ with $f(x_0) = 0$, $\mu(f) = 0$.

(3) For any open $\Omega \ni x_0$, $\mu(X \setminus \Omega) = 0$.

Proof. (1) ⇔ (2): Easy by definition.

(1) ⇒ (3): Trivial.

(3) ⇒ (2): Let $\varepsilon > 0$ and $f \in C(X)$ vanish at x_0. By continuity of f, there exists $\Omega_0 \ni x_0$ such that

$$\forall x \in \Omega_0 \left(\left|f(x)\right| < \varepsilon\right).$$

Since Ω_0 is open, there exists an open neighbourhood $\Omega \ni x_0$ with $\bar{\Omega} \subset \Omega_0$. Pick a continuous $h : X \to [0,1]$ with $h|_\Omega = 0$, $h|_{X\setminus\Omega_0} = 1$, and $f_\varepsilon = f \cdot h$. Then

$$\left\|f - f_\varepsilon\right\|_\infty < \varepsilon, \qquad f_\varepsilon|_\Omega = 0.$$

By (3), $\mu(f_\varepsilon) = 0$, so $\mu(f) < \varepsilon$. Let $\varepsilon \to 0$, done. □

Definition–Proposition 4.26: Barycentre map

For a compact set K, define the *barycentre map*

$$\mathrm{bary} : M(K)_1 \to K, \qquad \mu \mapsto x_\mu,$$

where x_μ satisfies $\varphi(x_\mu) = \int_K \varphi(x)\,\mathrm{d}\mu(x)$. Then bary is wk*-continuous.

Proof. Let $(\mu_i)_{i\in I} \subset M(K)_1$ be a net with $\mu_i \to \mu$. By properly choosing subnets, we may assume $x_{\mu_i} \to z \in K$. It suffices to prove $z = x_\mu$. For $\varphi \in E^*$, by continuity of φ, $\varphi(x_{\mu_i}) \to \varphi(z)$. Besides,

$$\varphi(x_{\mu_i}) = \int_K \varphi(x)\,\mathrm{d}\mu_i(x) \to \int_K \varphi(x)\,\mathrm{d}\mu(x) = \varphi(x_\mu).$$

By uniqueness, $\forall \varphi \in E^* \left(\varphi(z) = \varphi(x_\mu)\right)$. So $z = x_\mu$. □

Proposition 4.27

Let K be a minimal convex G-space. Then $\overline{\mathrm{Ext}\,K}$ is a G-boundary.

Proof. We claim that if $x \in \mathrm{Ext}\,K$, $\mu \in M(K)_1$, then $\mathrm{bary}(\mu) = x$ if and only if $\mu = \delta_x$.

⇒: Obvious.

⇐: If $\mathrm{bary}(\mu) = x$ but $\mu \neq \delta_x$, by Lemma 4.25, there exists an open set $\Omega \ni x$ such that $\mu(X \setminus \Omega) > 0$. Denote for $A \subset K$, $\mu(A) > 0$ the measure $\mu_A(B) = \mu(A \cap B)/\mu(A)$. Then $\exists t = \mu(\Omega) < 1$,

$$x = \mathrm{bary}(\mu) = t\,\mathrm{bary}(\mu_\Omega) + (1-t)\,\mathrm{bary}(\mu_{X\setminus\Omega})$$

since $\mu = t\mu_\Omega + (1-t)\mu_{X\setminus\Omega}$. If $\mathrm{bary}(\mu_{X\setminus\Omega}) \neq x$, then it contradicts with the fact that $x \in \mathrm{Ext}\,K$. But obviously $x \notin \overline{X\setminus\Omega}$. Since x is extremal, by Milman's theorem, $x \notin \overline{\mathrm{co}}(X\setminus\Omega)$. Besides, $\mathrm{supp}\,\mu_{X\setminus\Omega} \subset \overline{X\setminus\Omega}$ implies $\mathrm{bary}(\mu_{X\setminus\Omega}) \neq x$.

Now the claim holds. Let $\mu \in M(\overline{\mathrm{Ext}\,K})_1$. By wk*-continuity of bary,

$$\mathrm{bary}(\mathrm{co}(G\cdot\mu)) = \mathrm{co}(G\cdot x_\mu) = K$$

since K is minimal. In particular, $\mathrm{Ext}\,K \cap \{\text{Dirac measures}\} \subset \overline{\mathrm{co}}(G\cdot\mu)$. While Dirac measures are extremal points of $\overline{\mathrm{co}}(G\cdot\mu)$. Hence by Milman's theorem,

$$\{\text{Dirac measures}\} \subset \overline{G\cdot\mu}.$$

Hence $\mathrm{Ext}\,K$ is strongly proximal. □

Theorem 4.28

Point stabilisers of the action of G on $\partial_F G$ are amenable. In particular, the kernel of the action is $\operatorname{Rad}(G)$.

Proof. It suffices to prove the existence of a G-boundary X, such that point stabilisers are amenable. Let K be a minimal closed convex compact G-invariant subset of $\mathcal{M}(G)$ with the action of G by left multiplication.

By Proposition 4.27, $\overline{\operatorname{Ext} K}$ is a G-boundary. If H is the stabiliser of a point of K, then it fixes a mean on G(*that is H-invariant*). Then let $\pi : G \to H$ denote the projection, we may define $m_H(A) := m(\pi^{-1}(A))$. Then m_H is left-invariant. So H is amenable.

Since $\operatorname{Rad}(G)$ is the maximal normal amenable subgroup of G. By Proposition 4.16, $\operatorname{Rad}(G)$ acts trivially on $\partial_F G$. □

4.5 Crossed products and unique trace property

Let G be a group and A be a unital C^*-algebra. There is a G-action

$$\alpha : G \to \operatorname{Aut} A, \qquad g \mapsto \alpha_g.$$

If X is a G-space, $A = C(X)$, define the *twisted group ring* $A_\alpha[G]$ by

$$A_\alpha[G] := \left\{ \sum_{g \in G} a_g g : a_g \in A \right\},$$

with $ga := \alpha_g(a)g$. Then $A_\alpha[G]$ is a *-algebra with

$$(ag)^* = g^* a^* = g^{-1} a^* = \alpha_{g^{-1}}(a^*) g^{-1}.$$

By definition, $A \subset A_\alpha[G]$, $\mathbb{C}G \subset A_\alpha[G]$.

Definition 4.29: Covariant representation

A *covariant representation* of $A_\alpha[G]$ is a pair (π, σ), such that (π, H) is a representation of A, (σ, H) is a unitary representation of G, such that

$$\sigma(g)\pi(a)\sigma(g)^* = \pi(\alpha_g(a)), \qquad \forall a \in A,\ \forall g \in G.$$

If $f : A_\alpha[G] \to \mathcal{B}(H)$ is a *-morphism, then $(f|_A, f|_G)$ is a covariant representation of $A_\alpha[G]$, for any $\hat{\pi} : A \to \mathcal{B}(H)$, define

$$\begin{aligned} \pi : A \to \mathcal{B}(\ell^2(G) \otimes H), &\qquad a \mapsto [\delta_g \otimes \xi \mapsto \delta_g \otimes \hat{\pi}(\alpha_{g^{-1}}(a))\xi], \\ \sigma : G \to \mathcal{U}(\ell^2(G) \otimes H), &\qquad g \mapsto \lambda_g \otimes \mathrm{id}_H. \end{aligned}$$

Then $\forall f \in \mathcal{B}(\ell^2(G), H)$, $(\pi(a)f)(g) = \hat{\pi}(\alpha_{g^{-1}}(a))(f(g))$. THen we have a representation $f : A_\alpha[G] \to \mathcal{B}(\ell^2(G) \otimes H)$.

Definition 4.30: Reduced crossed product

Let $\rho : A \to \mathscr{B}(H_1)$ be a faithful *-morphism, $H = H_1 \otimes \ell^2(G)$. Let $\pi : A \to \mathscr{B}(H)$ given by

$$\pi(a)(v \otimes \delta_g) = \rho(\alpha_{g^{-1}}(a))\, v \otimes \delta_g$$

and let $\tilde{\lambda} = \mathrm{id} \otimes \lambda$. Then $(\pi, \tilde{\lambda})$ is a covariant representation by previous discussion. Hence it defines a representation of $A_\alpha[G]$, still denoted as f. Define a C^*-algebra

$$A \rtimes_\lambda G := \overline{f(A_\alpha[G])},$$

called the *reduced crossed product.*

By definition, $A \rtimes_\lambda G$ contains a copy of $C^*_\lambda(G) \cong \overline{\tilde{\lambda}(\mathbb{C}G)}$ and a copy of $A \cong \pi(A)$.

Lemma 4.31: Multiplicative lemma

Let A be a unital C^*-algebra, $\psi \in S(A)$. Denote the *multiplicative domain* of ψ as

$$A_\psi := \left\{a \in A : \psi(a^* a) = \psi(a^*)\psi(a)\right\}.$$

Then $a \in A_\psi$ if and only if $\forall b \in A$, $\psi(ab) = \psi(ba) = \psi(a)\psi(b)$. In particular, A_ψ is a C^*-subalgebra of A.

Proof. By GNS construction, there exists a representation (π, H) and $v \in H$, $\|v\| = 1$ such that $\psi = \psi_{\pi,v}$. For all $a \in A$, if $a \in A_\psi$, we have

$$\psi(a)\psi(a^*) = \left|\psi(a)\right|^2 = |\langle \pi(a) v, v\rangle| \leqslant \|\pi(a) v\|^2 = \langle \pi(a^* a) v, v\rangle = \psi(aa^*).$$

The equality is attained when $\pi(a) v$ is colinear with v. By $\psi(a) = \langle \pi(a) v, v\rangle$, one has $\pi(a) v = \psi(a) v$.

Similarly, $\pi(a^*) v = \psi(a^*) v = \overline{\psi(a)}\, v$. So for all $b \in A$,

$$\psi(ba) = \langle \pi(b)\pi(a) v, v\rangle = \psi(a) \langle \pi(b) v, v\rangle = \psi(a)\psi(b).$$

And $\psi(ab) = \overline{\psi(b^* a^*)} = \overline{\psi(a^*)\psi(b^*)} = \psi(a)\psi(b)$. □

Now we can prove the big theorem.

Theorem 4.32

A group G has unique trace property if and only if $\mathrm{Rad}(G) = 1$.

Proof. $\Rightarrow$: By Proposition 4.3.

$\Leftarrow$: Assume $\mathrm{Rad}(G) = 1$, X is a G-boundary such that $G \curvearrowright X$ faithful, and $A = C(X) \rtimes_\lambda G \supset C^*_\lambda(G)$. Let τ be a trace such that $\tau(g_0) \neq 0$ with $g_0 \neq 1$. Faithfulness of $G \curvearrowright X$ implies $\exists x_0 \in X\, (g_0 x_0 \neq x_0)$.

We extend τ to $\varphi \in S(A)$ using Hahn–Banach theorem, satisfying

- $\varphi|_{C^*_\lambda(G)} = \tau$, so $\forall g \in G\, \forall a \in C^*_\lambda(G)\, (\varphi(gag^{-1}) = \varphi(a))$, and hence $g \cdot \varphi|_{C^*_\lambda(G)} = \varphi|_{C^*_\lambda(G)}$.
- $\varphi|_{C(X)} = \mu$ for some $\mu \in M(X)_1$.

Since X is a G-boundary, there exists a net $(g_i)_{i\in I} \subset G$ such that $g_i \cdot \mu \to \delta_{x_0}$ up to extracting a subnet, we may assume $g_i \cdot \varphi \to \psi$.

We know $\psi|_{C(X)} = \delta_{x_0}$, so $g_0 \cdot \psi|_{C(X)} = \delta_{g_0 \cdot x_0}$. Also, $\psi|_{C^*_\lambda(G)} = \tau$ because $g \cdot \varphi|_{C^*_\lambda(G)} = \varphi|_{C^*_\lambda(G)} = \tau$. Let $h \in C(X)$ such that $h(x_0) = 0$ and $h(g_0 x_0) = 1$, by Lemma 4.31,

$$g_0 \in A_\psi \implies \begin{cases} \psi(h \cdot_\lambda g_0) = \psi(h)\psi(g_0) = h(x_0)\tau(g_0) = 0, \\ \psi(g_0 \cdot_\lambda h) = \psi(g_0)\psi(h) = 0. \end{cases}$$

While $\psi(g_0 \cdot_\lambda h) = \psi(g_0 h g_0^{-1} \cdot g_0) = \psi(g_0 \cdot_\lambda h)\psi(g_0)$. By assumption, $\tau(g_0) \neq 0$. Now

$$\psi(h) = h(x_0) = 0, \qquad \psi(g_0 \cdot h) = h(g_0 x_0) = 1,$$

which contradicts $\psi(h) = \psi(g_0 \cdot h)$. Therefore such trace τ cannot exists. In other words, G only has the canonical trace. □

Now we turn to the simplicity of reduced crossed product.

Definition–Proposition 4.33: Conditional expectation

There exists a unique continuous linear map $E : A \rtimes_\lambda \Gamma \to A$ such that for $a \in A$ and $g \in \Gamma$, we have

$$E(au_g) = \begin{cases} a, & \text{if } g = 1_\Gamma, \\ 0, & \text{if } g \neq 1_\Gamma. \end{cases}$$

Moreover, $E(x) \in A_+$ if $x \in (A \rtimes_\lambda \Gamma)_+$.

Proof. For $\sum_{g\in\Gamma} a_g u_g$, $a_g \in A$, $u_g^* = u_g^{-1} = u_{g^{-1}}$, $u_1 = 1$, $u_{gh} = u_g u_h$. The action is given by $\alpha_g(a) = u_g a u_g^{-1}$. Let $\pi : A \to \mathscr{B}(H)$, $U : \Gamma \to \mathscr{U}(H)$ be representations such that (π, U) is a covariant representation of $A \rtimes_\alpha \Gamma$, *i.e.*,

$$\forall g \in \Gamma\, \forall a \in A\, (U(g)\pi(a)U(g)^* = \pi(\alpha_g(a))).$$

Then

$$\theta_{\pi,U}(\sum_{g\in\Gamma} a_g u_g) = \sum_{g\in\Gamma} \pi(a_g)U(g) \implies \left\| \theta_{\pi,U}(\sum_{g\in\Gamma} a_g u_g) \right\| \leqslant \sum_{g\in\Gamma} \|a_g\|.$$

Assume $\pi_0 : A \to \mathscr{B}(H)$ is injective, $H = \ell^2(\Gamma, H_0)$. For $\xi \in H$,

$$U(g)(\xi)(h) = \xi(g^{-1}h), \qquad \pi(a)\xi(h) = \pi_0(\alpha_{h^{-1}}(a))\xi(h).$$

Thus

$$\begin{aligned}(U(g)\pi(a)U(g)^*\xi)(h) &= (\pi(a)U(g^{-1})\xi)(g^{-1}h) \\ &= \pi_0(\alpha_{(g^{-1}h)^{-1}}(a))(U(g^{-1})\xi)(g^{-1}h) = \pi_0(\alpha_{h^{-1}}\alpha_g(a))\xi(h) = \pi(\alpha_g(a))\xi(h).\end{aligned}$$

Now define $u : H_0 \to H = \bigoplus_{g\in\Gamma} H_0$ by $u(\xi)(h) = \xi$ if $h = 1_\Gamma$, otherwise $u(\xi)(h) = 0$. Then one can check that $\pi(a)u = u\pi_0(a)$. The adjoint is given by $u^* : H \to H_0$, $\xi \mapsto \xi(1)$. Then

$$u^*\pi(au_g)u = \begin{cases} \pi_0(a), & \text{if } g = 1_\Gamma, \\ 0, & \text{if } g \neq 1_\Gamma \end{cases} \implies u^*\pi(au_g)u\xi(h) = \begin{cases} \pi_0(\alpha_{h^{-1}}(a))\xi(h), & \text{if } g = h, \\ 0, & \text{if } g \neq h. \end{cases}$$

By density of $\{au_g : a \in A,\ g \in \Gamma\}$ in $A \rtimes_\lambda \Gamma$, for $T \in A \rtimes_{\alpha,r} \Gamma$, define

$$E_0 : A \rtimes_\lambda \Gamma \to \mathscr{B}(H_0), \qquad T \mapsto u^*\pi(T)u.$$

It is continuous, and $\operatorname{im} E_0 \subset \pi_0(A) \cong A$, since $E_0(\sum_{g\in\Gamma} a_g u_g) = \pi_0(a_1)$, and π_0 faithful. Then take E such that $E_0 = \pi_0 E$, which is the morphism we desired.

If $T = S^*S \geqslant 0$, then $E_0(T) = u^*\pi(S^*S)u = (\pi(S)u)^*(\pi(S)u) \geqslant 0$. Hence the claim about positivity holds. □

Lemma 4.34

Let J be a two-sided ideal of $A \rtimes_\lambda \Gamma$. Then $A \cap J$ and $E(J)$ are two-sided Γ-invariant ideals of A.

Proof. For $a \in A \cap J$, $E(a) = a \in E(J)$, so $A \cap J \subset E(J)$. For $x \in J$, $u_g x u_g^* = \alpha_g(x) \in J$, and $E(u_g x u_g^*) = \alpha_g(E(x))$. Hence $E(J)$ is Γ-invariant. □

We give an example to show that in $A \cap J \subset E(J)$, the equality may not be attained. Let $A = \mathbb{C}$ and $\Gamma = \mathbb{Z}/2\mathbb{Z}$, then $A \rtimes_\lambda \Gamma = \mathbb{C} \oplus \mathbb{C}$. Let $\alpha_g(a) = \mathrm{i}a$, $U_\varepsilon = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$. Denote $J_+ = \mathbb{C} \oplus 0$, $J_- = 0 \oplus \mathbb{C}$ be the non-trivial ideals of $A \cong \left\{ \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix} : \lambda \in \mathbb{C} \right\}$. Thus $A \cap J_+ = A \cap J_- = \left\{ \begin{bmatrix} 0 & \\ & 0 \end{bmatrix} \right\}$. But for $\lambda, \mu \in \mathbb{C}$,

$$E\left(\begin{bmatrix} \lambda & \\ & \mu \end{bmatrix} \right) = \frac{\lambda + \mu}{2} \implies E(J_+) = E(J_-) = A.$$

So $A \cap J_+ \subset E(J_+) = A$, the inclusion in strict.

Lemma 4.35

Let A be a C^*-algebra and Γ be a discrete group. Let $\alpha : g \mapsto \alpha_g$ be an action of Γ on A by *-automorphisms. Let φ be a state on A. Then $\varphi \circ E$ is a trace on $A \rtimes_\lambda \Gamma$ if and only if

$$\varphi(ab) = \varphi \circ E(ab) = \varphi \circ E(ba) = \varphi(ba),$$

where E is the conditional expectation.

Proof. ote that if $a, b \in A$, we have $E(ax) = aE(x)$, $E(xb) = E(x)b$ for all $x \in A \rtimes_\lambda \Gamma$.

$\Rightarrow$: Let $\varphi : A \to \mathbb{C}$ be a state, $a, b \in A$, then

$$\varphi(ab) = \varphi \circ E(ab) = \varphi \circ E(ba) = \varphi(ba)$$

implies that φ is a trace on A. For $h \in \Gamma$,

$$\varphi \circ E(a\alpha_h(b)) = \varphi(au_h b u_{h^{-1}}) = \varphi \circ E(bu_{h^{-1}} a u_h) = \varphi \circ E(b\alpha_{h^{-1}}(a)).$$

Note that $\alpha_h(\alpha_{h^{-1}}(b)) = u_h u_{h^{-1}} a u_{h^{-1}} b u_h^{-1} = a u_h b u_{h^{-1}}$. If we denote $c = \alpha_{h^{-1}(a)b}$, then LHS $= \varphi(\alpha_h(c)) = \varphi(c) =$ RHS. This implies $\varphi \circ \alpha_h = \varphi$, so φ is a trace on $A \rtimes_\lambda \Gamma$.

$\Leftarrow$: We reverse the reasoning. □

Moreover, $A = C_0(X)$ is commutative, then φ is a trace on $A \rtimes_\lambda \Gamma$ if and only if φ is a Γ-invariant probability measure on X.

From now on, we assume that a discrete group Γ acts by homeomorphisms on a locally compact space X. We shall set

$$\alpha_g(f)(x) = f(g^{-1}x), \qquad \forall g \in \Gamma,\ \forall f \in C_0(X),\ \forall x \in X.$$

And fix a point $x \in X$.

Proposition 4.36

There exists a representation π_x of the reduced crossed product $C_0(X) \rtimes_\lambda \Gamma$ in the Hilbert space $\ell^2(\Gamma)$, such that

- $\pi_x(u_g) = \lambda_g$, where $\lambda_g(\xi)(h) = \xi(g^{-1}h)$ for $g, h \in \Gamma$, $\xi \in \ell^2(\Gamma)$.

- $((\pi_x(f))(\xi))(h) = f(hx)\xi(h)$ for $f \in C_0(X)$, $\xi \in \ell^2(\Gamma)$, $h \in \Gamma$.

Moreover,

(1) Let $x, y \in X$. Suppose x, y are in the same Γ-orbit. Then representations π_x and π_y are equivalent.

(2) Suppose that the orbit of x is dense. Then π_x is a faithful representation of $C_0(X) \rtimes_\lambda \Gamma$.

(3) If the stabiliser of x is reduced to the identity element 1_Γ, the representation π_x is irreducible.

Proof. One can define the left-regular representation λ on Γ and the multiplication representation

$$M_x(f) : \ell^2(\Gamma) \to \ell^2(\Gamma), \qquad \xi(g) \mapsto f(gx)\xi(g), \qquad \forall \xi \in \ell^2(\Gamma), \forall g \in \Gamma.$$

Then M_x satisfies $\|M_x f\| \leqslant \|f\|_\infty$, being a non-degenerate *-representation of $C_0(X)$. One can simply check M_x and λ satisfies the covariant condition because

$$\lambda_g M_x(f) \lambda_g^* = M_x(\alpha_g(f)),$$

where $\alpha_g(f)(y) = f(g^{-1}y)$. So the existence of such π_x follows.

(1) $\ell^2(\Gamma) = \ell^2(\Gamma, \mathbb{C})$. We want something « interwine » $\pi_x : C_0(X) \rtimes_\lambda \Gamma \to \mathscr{B}(\ell^2(\Gamma))$. For $f \in C_0(X)$,

$$\rho_g \pi_x(f) \rho_{g^{-1}}(\xi)(h) = \pi_x(f)\rho_{g^{-1}}(\xi)(hg) = f(hgx)\rho_{g^{-1}}(\xi)(hg) = f(hgx)\xi(h) = \pi_{gx}(f)\xi(h).$$

Hence $\rho_g \pi_x(f) \rho_{g^{-1}} = \pi_{gx}(f)$.

(2) Let $H_0 = \ell^2(\Gamma)$. The representation $\pi_0 : C_0(X) \to \mathscr{B}(H_0)$ is given by $\pi_0(f)\xi(g) = f(gx)\xi(g)$. Then if $\pi_0(f) = 0$, $\forall g \in G(f(gx) = 0)$. Moreover, Γx is dense, $f = 0$, so π_0 is faithful. Note that $\pi_0 = \bigoplus_{g\in\Gamma} \pi_{gx}$, by (2), $\pi_0 = \bigoplus_{g\in\Gamma} \pi_x$. Thus π_x is faithful in this case.

(3) By assumption $\Gamma^x = \{g \in \Gamma : gx = x\} = 1_\Gamma$. If $T \in \mathscr{B}(\ell^2(\Gamma))$ such that $T\pi_x(S) = \pi_x(S)T$ for all S, then take $g \neq h$, $gx \neq hx$. Take $f \in C_0(X)$ with $f(gx) = 1$, $f(hx) = 0$. We have

$$\pi_x(f)\delta_g = f(gx)\delta_g = \delta_g, \qquad \pi_x(f)\delta_h = f(hx)\delta_h = 0.$$

But

$$\begin{aligned} \langle \delta_h, T\delta_g \rangle &= \langle \delta_h, T\pi_x(f)\delta_g \rangle = \langle \delta_h, \pi_x(f) T\delta_g \rangle = \langle \pi_x(\bar{f}\delta_h), T\delta_g \rangle = 0. \\ \langle \delta_h, T\delta_h \rangle &= \langle \lambda_h \delta_1, T\lambda_h \delta_1 \rangle = \langle \delta_1, T\delta_1 \rangle = \alpha. \end{aligned}$$

The second identity used the fact that λ_h is unitary. Hence $T = \alpha \cdot \mathrm{id}_{\ell^2(\Gamma)}$, which is irreducible. □

Theorem 4.37: Simplicity of crossed product

If $C_0(X) \rtimes_\lambda \Gamma$ is simple, then the action of Γ in X is minimal, *i.e.*, all Γ-orbits are dense.

Conversely, if the action of Γ in X is minimal and that there exists $x \in X$ whose stabiliser is reduced to the identity element of Γ, then $C_0(X) \rtimes_\lambda \Gamma$ is simple.

Proof. (1) If U is an open Γ-invariant subset,

$$C_0(X) \subset C_0(U) \rtimes_\lambda \Gamma \subset C_0(X) \rtimes_\lambda \Gamma.$$

But $C_0(X) \rtimes_\lambda \Gamma$ is simple, either $U = \varnothing$, or $U = X$.

(2) Let $x \in X$ such that $\Gamma^x = \{1_\Gamma\}$. By the minimality of G-action, let $J \neq C_0(X) \rtimes_\lambda \Gamma$ be an ideal in $C_0(X) \rtimes_\lambda \Gamma$, $B = C_0(X) \rtimes_\lambda \Gamma / J$, the following diagram commutes:

$$\begin{tikzcd} C_0(X) \rtimes_\lambda \Gamma \arrow[d] \arrow[dr] & \\ C_0(X) \arrow[r, "\varphi"] & B \end{tikzcd}$$

Here $\varphi : x \mapsto f(x)$. By Hahn–Banach theorem , one can extend φ to $C_0(X) \rtimes_\lambda \Gamma$, such that

$$\varphi(f u_g) = f(x)\varphi(u_g) = \begin{cases} 1, & \text{if } g = 1_\Gamma, \\ 0, & \text{if } g \neq 1_\Gamma. \end{cases}$$

Thus $B = \{1\}$, contradict the fact that $J \neq C_0(X) \rtimes_\lambda \Gamma$. □

5 C^*-simplicity

5.1 Characterisation using amenable radical

Definition–Proposition 5.1: Simplicity

Let A be a C^*-algebra. Say A is *simple* if one of the following equivalent condition holds:

(1) Every quotient is an isomorphism.

(2) Any ideal is either 0 or A.

(3) Any non-trivial *-morphism $A \to B$ is a monomorphism.

(4) Every non-trivial *-representation is faithful.

Proof. (1) ⇒ (2): For any ideal $I \lhd A$, consider the canonical quotient $q : A \to A/I$. By (1), q is an isomorphism, so $I = \ker q = 0$. Therefore either $I = 0$. or $I = A$.

(2) ⇒ (1): Now either $A/I \cong A$, or $A/I = 0$, so every non-zero quotient is an isomorphism.

(2) ⇔ (3): For any *-homomorphism $\varphi : A \to B$, its kernel $\ker\varphi$ is a closed two-sided ideal in A, and φ is a monomorphism if and only if $\ker\varphi = 0$. Thus

- (2) ⇒ $\ker\varphi = 0$ or A. When $\ker\varphi = A$, $\varphi = 0$, contradict the fact that φ is non-trivial. So (3) holds.
- (3) ⇒ the quotient $q : A \to A/I$ with $I = \ker\varphi = 0$. So (2) holds.

(2) ⇔ (4): A *-representation is a *-morphism, and its kernel is a closed ideal. The faithfulness of the representation is equivalent to say $\ker\pi = 0$. □

Definition 5.2: C^*-simplicity

Say a group G is *C^*-simple* if $C^*_\lambda(G)$ is simple.

Remark. The full group C^*-algebra $C^*(G)$ is simple if and only if $G = \{1\}$, so we use the reduced group C^*-algebra to define C^*-simplicity. Also, it implies that if G is C^*-simple, then G is non-amenable.

Question. How $\mathrm{Rad}(G)$ affects C^*-simplicity?

Lemma 5.3

Given two unitary representation π, σ of G, the followings are equivalent:

(1) $\pi \prec \sigma$.

(2) $S_\pi(G) \subset S_\sigma(G)$, where $S_\pi(G) = \overline{\mathrm{co}}\{\varphi_{\pi,v} : v \in H_\pi\}$.

Note that here we «*forget*» the multiplicity of π because $\bigoplus_{i\in I} \pi \prec \pi$.

Proof. We first claim that $S_\pi(G) = S(C^*_\pi(G))$.

Since $\varphi_{\pi,v} \in C^*_\pi(G)$ with convexity and closeness of $S(C^*_\pi(G))$, so $S_\pi(G) \subset S(C_{\pi^*(G)})$ is quite obvious. We want $S_\pi(G)$ is sufficiently large to detext positivity of elements of $C^*_\pi(G)$. For $a \in C^*_\pi(G)$ with $a = \pi(x)$ for some $x \in C^*(G)$, one has

$$a \in \mathscr{B}(H_\pi) \text{ is positive} \quad \Longleftrightarrow \quad \forall v \in H_\pi\,(\langle av, v\rangle = \varphi_{\pi,v}(a) \geqslant 0). \tag{5.1}$$

Then secondly we claim that if A is unital, then E *detects positive, i.e.*, $\forall a \in A\,\varphi \in E\,(\varphi(a) \geqslant 0)$ implies $a \geqslant 0$. Then $S(A) = \overline{\mathrm{co}}E$.

Since $S(A) = \{\varphi \in A^*_{\mathrm{sa}} : \varphi \text{ positive, } \varphi(1) = 1\}$, where A_{sa} is the R-vector space of self-adjoint operators. Hence φ is determined by its restriction to A_{sa}. Assume that $\overline{\mathrm{co}}E \not\subset S(A)$. By Hahn–Banach theorem, for $(A^*_{\mathrm{sa}}, \mathrm{wk}^*)$, there is an $x \in A_{\mathrm{sa}}$ such that $\forall \varphi \in E\,(\varphi(x) \geqslant 0)$. While $\exists \psi \in S(A)\,(\psi(x) < 0)$. This contradicts the fact that $\forall \varphi \in E\,(\varphi(x) \geqslant 0) \implies x \geqslant 0$. Therefore the second claim holds.

Then let $E = S_\pi(G)$ and $A = C^*_\pi(G)$. Using (5.1), $S_\pi(G)$ detects positive. Hence $S_\pi(G) \supset S(C^*_\pi(G))$. Therefore the first claim also holds.

(2) ⇒ (1): Trivial.

(1) ⇒ (2): We know that $\sigma \prec \sigma$ if and only if $\forall K \in \operatorname{fin} G$, $\exists \varepsilon > 0$, $\forall v \in H_\pi$ and $\|v\| = 1$, there exist $w_1, \ldots, w_n$ such that

$$\left\langle \pi(g)v, v - \sum_{i=1}^{n} \langle \sigma(g)w_i, w_i\rangle \right\rangle \leqslant \varepsilon.$$

Denote $\operatorname{sgn} w_i = w_i / \|w_i\|$, we have

$$\left|\varphi_{\pi,v}(g) - \sum_{i=1}^{n} \|w_i\|^2 \varphi_{\sigma,\operatorname{sgn} w_i}(g)\right| \leqslant \varepsilon.$$

Apply to $g = 1$, then $\left|1 - \sum_{i=1}^n \|w_i\|^2\right| \leqslant \varepsilon$. Thus take $t_i = \|w_i\|^2 \left(\sum_{j=1}^n \|w_j\|^2\right)^{-1}$, one has

$$\left|\varphi_{\pi,v}(g) - \sum_{i=1}^{n} t_i \varphi_{\sigma \operatorname{sgn} w_i}(g)\right| \leqslant \varepsilon$$

with $\sum_{i=1}^n t_i \varphi_{\sigma \operatorname{sgn} w_i}(g) \in S_\sigma(G)$, done. □

Proposition 5.4

Let π, σ be two unitary representations, the followings are equivalent.

(1) $\pi \prec \sigma$.

(2) For all $a \in C^*(G)$, $\|a\|_\pi \leqslant \|a\|_\sigma$.

(3) $\ker\sigma \subset \ker\pi$.

Proof. (3) ⇒ (2): The inclusion $\ker\sigma \subset \ker\pi$ gives a *-morphism

$$\begin{tikzcd} C^*(G) \arrow[r, "\sigma"] \arrow[dr, "\pi"'] & C^*_\sigma(G) \arrow[d] \\ & C^*_\pi(G) \end{tikzcd}$$

And all *-morphism of C^*-algebras does not increase norm, so

$$\|a\|_\pi = \|\pi(a)\| \leqslant \|\sigma(a)\| = \|a\|_\sigma.$$

(2) $\Rightarrow$ (1): By Lemma 5.3, it suffices to prove $S(C^*_\pi(G)) \subset S(C^*_\sigma(G))$. But

$$S(C^*_\pi(G)) = \{\text{states on } \mathbb{C}G \text{ which are Lipschitz w.r.t. } \|\cdot\|_\pi\},$$

and Lipschitz w.r.t. $\|\cdot\|_\pi$ implies Lipschitz w.r.t. $\|\cdot\|_\sigma$ because $\|\cdot\|_\pi \leqslant \|\cdot\|_\sigma$.

(1) $\Rightarrow$ (3): By Lemma 5.3, if $\pi \prec \sigma$, then $S_\pi(G) \subset S_\sigma(G)$. If $x \in \ker\sigma$, then for all $v \in H_\sigma$ with $\|v\| = 1$, $\langle \sigma(x)v, v\rangle = \varphi_{\sigma,v}(x) = 0$. Hence $\forall \varphi \in S_\sigma(G)\,(\varphi(x) = 0)$. Thus for all $\psi \in S_\pi(G) \subset S_\sigma(G)$, $\varphi(\pi(x)) = 0$. Hence $x \in \ker\pi$. □

Proposition 5.5

If H is an amenable group of G, then $\lambda_{G/H} \prec \lambda_G$.

Proof. Let $f \in \ell^2(G/H)$ with $\|f\|_2 = 1$. If H is finite, define $\tilde{f}(g) = \frac{1}{\sqrt{|H|}} f(gH)$. If H is infinite, since H is amenable, there exists a series $(v_n)_{n\geqslant 1} \subset \ell^2(H)$ with $\|v_n\| = 1$ such that

$$\|\lambda(h)v_n - v_n\| \to 0, \qquad n \to \infty.$$

For example, $(F_n)_{n\geqslant 1}$ is the Følner sequence(*or net, if G is not countable*). Let $v_n = \frac{1}{\sqrt{|F_n|}} \mathbb{1}_{F_n}$. Fix a section $X \subset G$ of G/H and define

$$\tilde{f}_n(xh) = f(xH)v_n(h).$$

For any $g_0 \in G$, note that for all $g_0, x \in X$, there exists a unique $h_{g_0,x} \in H$ and $x' \in X$ such that $g_0^{-1}x = x'h_{g_0}$. One has

$$\begin{aligned}
\langle \lambda(g_0)\tilde{f}_n, \tilde{f}_n\rangle &= \sum_{xH\in\operatorname{supp} f}\sum_{h\in H} \tilde{f}_n(g_0^{-1}xh)\tilde{f}_n(xh) \\
&= \sum_{xH\in\operatorname{supp} f}\sum_{h\in H} f(g_0^{-1}xH)f(xH)v_n(h)v_n(h_{g_0,x}^{-1}) \\
&= \sum_{xH\in\operatorname{supp} f}\sum_{h\in H} f(g_0^{-1}xH)f(xH)\underbrace{\langle v_n, \lambda(h_{g_0,x})v_n\rangle}_{\to 1 \text{ as } n\to\infty} \\
&\to \sum_{xH\in\operatorname{supp} f} f(g_0^{-1}xH)f(xH) \\
&= \langle \lambda_{G/H}(g_0)f, f\rangle.
\end{aligned}$$

Done. □

Now we can give the characterisation of C^*-simplicity using $\operatorname{Rad}(G)$.

Theorem 5.6

If G is C^*-simple, then $\operatorname{Rad}(G) = 1$.

Proof. Let $H = \operatorname{Rad}(G) \leqslant G$ and assume $H \neq 1$. By Proposition 5.5, $\lambda_{G/H} \prec \lambda_G$ since H is amenable. But the normality of H implies $\lambda_G \nprec \lambda_{G/H}$. This is because $1 - h \in \mathbb{C}G$ such that

$$1 - h \notin \ker\lambda_G,\ 1 - h \in \ker\lambda_{G/H} \implies \ker\lambda_{G/H} \not\subset \ker\lambda_G \implies \lambda_G \nprec \lambda_{G/H}$$

by Proposition 5.4. Also, we have $\ker\lambda_G \subset \ker\lambda_{G/H}$. Hence there exists a homomorphism $C^*_{\lambda_G}(G) \to C^*_{\lambda_{G/H}}(G)$ whose kernel is not trivial (*contains* $1 - h \in \ker\lambda_{G/H}$.) □

Corollary 5.7

If G is C^*-simple, then G has unique trace property.

5.2 Characterisation using Fustenburg boundary

Question. Does the inverse of Corollary 5.7 holds?

Answer. No. We need a group G with $\mathrm{Rad}(G) = 1$, but G is not C^*-simple.

The strategy is to find G with *sufficiently large* amenable subgroup H, by Theorem 5.6, there exists a homomorphism $C^*_\lambda(G) \to C^*_{\lambda_{G/H}}(G)$. We want H to be such that this homomorphism is non-injective (*even at the level of group algebra*).

Definition 5.8: Free / Topological free

Let X be a compact G-space. The action $G \curvearrowright X$ is called:

(1) *free*, if for all $x \in X$, $G_x := \{g \in G : gx = x\} = \{1\}$.

(2) *topologically free*, if for all $x \in X$, $G_x^0 := \{g \in G : \exists \text{open } U \ni x\, (g|_U = \mathrm{id})\} = \{1\}$.

Since $G_x^0 \leqslant G_x$, the action is free $\implies$ topologically free.

Denote $X^0 = \{x \in X : G_x^0 \neq \{1\}\}$, this is an open set. In particular, if X is minimal, then either $X^0 = X$, or $X^0 = \varnothing$ (*which implies topological freeness*).

We admit the following theorem, particularly the equivalence between (1) and (2).

Theorem 5.9

Let G be a group. The followings are equivalent:

(1) G is C^*-simple.

(2) $G \curvearrowright \partial_F G$ is topologically free.

(3) Every G-boundary with amenable stabilisers is topologically free.

(4) There is a G-boundary with amenable stabilisers that is topologically free.

Proof. We only prove (4) $\Rightarrow$ (2). Assume $U \subset \partial_F G$ is open, then $\partial_F G$ is covered by finitely many translation of U. By minimality, GU is a G-invariant open subset of X, so $GU = X$. Since $\pi : \partial_F G \to X$ is surjective, this means that $\pi(U)$ has non-empty interior because $X \subset \bigcup_{i=1}^n g_i \pi(U)$ implies $\exists i\, (g_i \pi(U) \neq \varnothing)$.

Hence if $g \in G \setminus \{1\}$, $g|_U = \mathrm{id}$, then $g|_{\pi(U)} = \mathrm{id}$. Therefore there is an open non-empty subset $V \subset X$ such that $g|_V = \mathrm{id}$. So for all $x \in V$, $g \in G_x^0$, one has $G_x^0 = \{1\}$. So we proved that $G \curvearrowright \partial_F$ *not* topologically free $\implies$ $G \curvearrowright X$ *not* topologically free. □

Example 5.10

Let $\mathbb{F}_2 = \langle a, b \rangle$ be the free group and $X = \partial_F \mathbb{F}_2$. We have seen that X is a G-boundary, since we have proved that we can find a pair $g, h \in \mathbb{F}_2$ with $\left|\{g^\pm, h^\pm\}\right| = 4$. Moreover, the stabilisers of points are either trivial, or of the form g^+ for some $g \in G$. This implies $G_x^0 = \{1\}$ for all $x \in X$.
In the case where $x = g^+$, one can choose(*this is not trivial, but true!*) g to be a generator of G_x. Since g moves non-trivially on $G \setminus \{g^\pm\}$, on any neighbourhood of g^+, $G_x = \langle g \rangle$.

Remark. *Let Y, Z be G-spaces. If $\pi : Y \to Z$ is G-equivariant and $y \in Y$, it is obvious that $G_y < G_{\pi(y)}$. Hence Z free $\implies$ Y free.*

Corollary 5.11

If G is not C^*-simple, then there exists an amenable subgroup $H \leqslant G$ such that for all finite subset $F \subset G$,

$$\bigcap_{f \in F} f H f^{-1} \neq \{1\}.$$

Proof. By Theorem 5.9, $G_x \neq \{1\}$ for some $x \in \partial_F G$. Moreover, since G_x is amenable because there is a mean that is invariant under left-translation of G_x. We claim that $H = G_x$ satisfies the conclusion of the corollary.

Let $s \in G \setminus \{1\}$ which acts as the identity on a non-empty open set $U \subset \partial_F G$. By strong proximality, for every finite subset $F \subset G$, one can find $r \in G$ such that $rFx \subset U$. Therefore,

$$\forall f \in F \,(srfx = rfx) \quad \Longrightarrow \quad \forall f \in F \,(r^{-1}sr \in G_{fx} = fG_x f^{-1}).$$

Then $r^{-1}sr \in \bigcap_{f \in F} fHf^{-1}$. □

Now for a G-space X, we associate two families of subgroups of G: $\{G_x : x \in X\}$ and $\left\{G_x^0 : x \in X\right\}$. There is a natural question: what is the connection between the properties of actions and the properties of these subgroups?

Start from a baby case: if $G \curvearrowright X$ is transive, and $x \in X$, $H = G_x$. Then

$$G \curvearrowright X \cong G \curvearrowright G/H.$$

And the stabiliser of gH is gHg^{-1}. So $\{G_x : x \in X\}$ are the conjugacy classes of H.

Denote

$$\mathrm{Sub}(G) = \{H : H \leqslant G\} \subset 2^G \cong \{0,1\}^G.$$

This is a closed and compact space, with G acting on it by conjugation. So $\mathrm{Sub}(G)$ is a compact G-space. For $H \in \mathrm{Sub}(G)$,

$$\overline{GH} = \left\{\overline{g^{-1}Hg} : g \in G\right\}$$

is a G-subspace of $\mathrm{Sub}(G)$.

Then we consider permanence properties for group operations of C^*-simplicity.

(1) By taking quotients: No! $\mathbb{F}_2$ is C^*-simple and has amenable quotients.

(2) By taking subgroups: No! Every non-trivial group has a non-trivial amenable subgroup.

(3) By taking normal subgroups: Actually yes. But the proof is not trivial.

(4) By taking extensions: Yes! We shall prove it from now on.

Theorem 5.12

Let G be a group, $N \lhd G$. Then G is C^*-simple if and only if N and $C_G(N)$ are C^*-simple. Here

$$C_G(N) = \left\{g \in G : \forall n \in N \,(gn = ng)\right\}$$

is the centraliser of N in G.

Special case. If $G = N \times Q$, then G is C^*-simple if and only if both N and Q are C^*-simple.

Proof of Theorem 5.12.

By normaility, $\partial_F N$ will come with a G-action which extends the N-action. We will construct a G-boundary of G by considering

$$\partial_F N \times \partial_F K \times \partial_F (G/L), \qquad K = C_G(N),\ L = NK.$$

Lemma 5.13

Assume N is a normal subgroup of G. Then the N-action on $\partial_F N$ uniquely extends to a G-action. In particular, $\partial_F N$ is a G-boundary.

Proof. Let $\tau \in \operatorname{Aut} N$. Then $\partial_F N$ admits two actions: the original one, and the «*twisted*» one $n \cdot_\tau x := \tau(n)x$.

The universality of $\partial_F N$ implies there is a unique N-equivariant map

$$(\partial_F N, \cdot) \xrightarrow{\bar{\tau}} (\partial_F N, \cdot_\tau), \qquad \bar{\tau}(n \cdot x) := \tau(n) \cdot \bar{\tau}(x), \forall n \in N.$$

Now let $\sigma_g(h) = ghg^{-1}$ for some $g \in G$. The previous remark implies that $\bar{\sigma}_g$ gives rise to an action of G on $\partial_F N$. So

$$\bar{\sigma}_g(h \cdot x) = ghg^{-1} \cdot \bar{\sigma}_g(x), \qquad \forall g \in G,\ \forall h \in N,\ \forall x \in \partial_F N.$$

We want to check that $\bar{\sigma} : G \curvearrowright \partial_N G,\ g \mapsto [x \mapsto \bar{\sigma}_g(x)]$ extends the action $N \curvearrowright \partial_N G$. Hence we need to show that $\partial_F N \to \partial_F N,\ x \mapsto h^{-1}\bar{\sigma}_h(x)$ is the identity. By minimal property, we just need to prove the map is N-equivariant. This is done by

$$h^{-1}\bar{\sigma}_h(kx) = h^{-1}(hkh^{-1})\bar{\sigma}_h(x) = kh^{-1}\bar{\sigma}_h(x), \qquad \forall k \in N.$$

So we conclude the proof of the lemma.

Now recall $K = C_G(N)$, $L = NK$. We have $N \leqslant L \leqslant G$. By Lemma 5.13, there exists an action

$$G \curvearrowright \partial_F N \times \partial_F K \times \partial_F (G/L) =: X.$$

We claim that X is a G-boundary. The strong proximality is stable under product, so it suffices to prove X is minimal. Since N and K commute, they act trivially on each other's Fustenburg boundaries. Then let $x = (y, z, u) \in X$, $x' = (y', z', u') \in X$, one has

$$\begin{aligned} G \curvearrowright \partial_F (G/L) \text{ minimal } &\implies \exists (g_i)_{i \in I} \subset G\,(g_i u \to u,\ g_i(x) \to (y'', z'', u')). \\ K \curvearrowright \partial_F K \text{ minimal } &\implies \exists (k_j)_{j \in J} \subset K\,(k_j z'' \to z',\ k_j(y'', z'', u) \to (y'', z', u)). \\ N \curvearrowright \partial_F N \text{ minimal } &\implies \exists (n_\ell)_{\ell \in L} \subset N\,(n_\ell y'' \to y',\ n_\ell(y'', z', u') \to (y', z', u')). \end{aligned}$$

where we fix $\partial_F K$ in the second row because $N \curvearrowright \partial_F K$ trivially, and we fix $\partial_F N$ in the third row because $K \curvearrowright \partial_F N$ trivially. So X is a G-boundary. Moreover, the stabilisers are amenable.

Now

$$G_x = \operatorname{Stab}(y, z, u) = \operatorname{Stab}(y) \cap \operatorname{Stab}(z) \cap \operatorname{Stab}(u) = N_y K_z (G/L)_u,$$

and $G_x \cap L = N_y K_z$, $G_x/(L \cap G_x) = (G/L)_u$. By stability of amenability under extension, G_x is amenable.

$\Rightarrow$: If G is C^*-simple, then $G \curvearrowright X$ is topologically free. So $N \curvearrowright \partial_F N$ and $K \curvearrowright \partial_F K$ is topologically free because open sets in G are of the form $\prod \{\text{open sets in components}\}$.

$\Leftarrow$: If both N and K are C^*-simple, pick $x \in X$ and $g \in G_x^0$. In particular, $g \in G_y^0$. If we can prove that $y \in K$, then the proof is done because $g \in K_y^0 = \{1\}$. To do this, we need a lemma.

Lemma 5.14

Assume $N \lhd G$, $C_G(N)$ are C^*-simple, then $g \in C_G(N)$ if and only if its action on $\partial_F N$ is *not* topologically free.

Proof. $\Rightarrow$: If $g \in C_G(N)$, then

$$g \in \ker[G \curvearrowright N] \implies g \in \ker[G \curvearrowright C_G(N)] \implies g \in G_y^0,\ \forall y \in \partial_F N.$$

$\Leftarrow$: Assume $g \in G$ acts trivially on a non-empty $U \subset \partial_F N$. We claim that if $\forall n \in N (U \cap n^{-1}U) \neq \varnothing$, then n commutes with g. This is because $N \lhd G \implies [g,n] = gng^{-1}n^{-1} \in N$. So

$$g|_U = \mathrm{id},\ n^{-1}gn|_{n^{-1}U} = \mathrm{id},\ g^{-1}|_U = \mathrm{id} \quad \implies \quad g^{-1}n^{-1}gn|_{U\cap n^{-1}U} = \mathrm{id}.$$

Hence by C^*-simplicity of N, $g^{-1}n^{-1}gn = 1$.

In order to conclude, we need to prove that $\{n \in N : U \cap n^{-1}U \neq \varnothing\}$ generates N. To do this, let $H = \operatorname{span}\{n \in N : U \cap n^{-1}U \neq \varnothing\}$. Then HU is open, non-empty and satisfies $\forall t \in N \setminus H$, $tHU \cap HU = \varnothing$. And NU is open and N-invariant. By minimality of $\partial_F N$, $NU = \partial_F N$.

Let Σ be a section of N/H in N, $\partial_F N = \coprod_{\sigma\in\Sigma} \sigma HU$. By compactness Σ is finite. So one can consider $\mu := \frac{1}{|\Sigma|}\sum_{\sigma\in\Sigma}\delta_\sigma$. Observe that $\psi : \partial_F N \to N/H$, $x \in \sigma HU \mapsto \bar{\sigma}$ is N-invariant. Thus $\psi * \mu$ is an N-invariant measure on N/H. This contradicts strong proximality unless Σ is a singleton, which is equivalent to $N = H$ because

$$\exists (n_i)_{i\in I} \subset N (n_i \cdot \mu \to \delta_x) \quad \implies n_i(\psi * \mu) \to \delta_{\bar{x}'}$$

for some $x' \in N/H$. Taking $\psi * \mu = n_i(\psi * \mu)$ being uniform into consideration, $N = H$. Therefore $g \in C_G(N)$. So we conclude the proof of this lemma.

Now by Lemma 5.14, $g \in G_y^0$ implies g does not act topologically freely on G, so it is not topologically free on $\partial_F N$. So $g \in C_G(N) = K$. Then we conclude the proof of Theorem 5.12. □

Corollary 5.15

C^*-simplicity is stable under extension, and passes to normal subgroup of G.

Proof. Consider the extension

$$1 \longrightarrow N \longrightarrow G \xrightarrow{p} G/N \longrightarrow 1$$

with N and G/N are C^*-simple. We want to prove that $C_G(N)$ is C^*-simple. Note that $p|_{C_G(N)}$ is injective if and only if $C_G(N) \cap N = \{1\}$. Indeed, $C_G(N) \cap N = Z(N)$, the centre of N, which is an Abelian normal subgroup. Hence trivial. But by the previous theorem, $p(C_G(N)) \cong C_G(N)$ is a normal subgroup of G/N, so it is C^*-simple. □

6 Non-positive Curvature

In this chapter, the goal is to determine which Lie groups are amenable using geometric methods. It is a fact that spaces with non-positive can be defined using only distances.

Definition 6.1: Geodesic segment

Let (X, d) be a metric space. A *geodesic segment* is an image of an isometric $f : I \to X$, where $I \subset \mathbb{R}$ is a compact interval. Here isometric means that

$$\forall s, t \in I \, (d(f(s), f(t)) = |s - t|).$$

For $x, y \in X$, a *midpoint* of x and y is a point m such that $d(m, x) = d(m, y) = \frac{1}{2} d(x, y)$.

Note that there can be infinitely many midpoints. For example, in a sphere $\mathbb{S}^2$, the midpoints of the north pole n and the south pole s are the equator.

6.1 CAT(0) spaces

Definition 6.2: CAT(0) space

A CAT(0) *space* is a complete space (X, d) such that it is

(1) *geodesic*, *i.e.*, any two points are extremal points of a geodesic segment.

(2) For all $x, y, z \in X$ and m a midpoint of a geodesic between x and z,

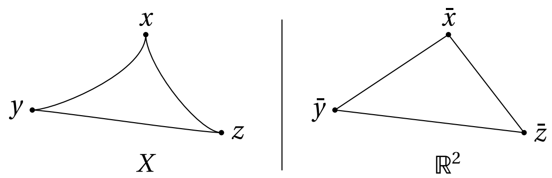
$$d(z, m)^2 \leqslant \frac{1}{2}\left(d(x, z)^2 + d(y, z)^2\right) - \frac{1}{4} d(x, y)^2. \tag{6.1}$$

If $\leqslant$ is replaced by $=$ in (6.1), we obtain the parallelogram identity in $\mathbb{R}^2$. The notation CAT(κ) means «*curvature bounded above by* κ». So CAT(0) is a name for non-positive curvature.

Definition 6.3: Comparison triangle

Denote the geodesic segment between x and y as $[x, y]$, and $\triangle(x, y, z)$ the geodesic triangle. Define the *comparison triangle* $\triangle(\bar{x}, \bar{y}, \bar{z})$ as the triangle in $\mathbb{R}^2$ such that

$$d(x, y) = d(\bar{x}, \bar{y}), \qquad d(y, z) = d(\bar{y}, \bar{z}), \qquad d(z, x) = d(\bar{z}, \bar{x}).$$



For a point p on a side of $\triangle(x, y, z)$, denote $\bar{p}$ the comparison point in $\triangle(\bar{x}, \bar{y}, \bar{z})$.

Proposition 6.4

Let (X, d) be a complete geodesic space. The following are equivalent:

(1) (X, d) is CAT(0).

(2) For any geodesic triangle $\triangle(x, y, z)$ with comparison triangle $\triangle(\bar{x}, \bar{y}, \bar{z})$, for p, q on sides of $\triangle(x, y, z)$, one has $d(p, q) \leqslant d(\bar{p}, \bar{q})$. In other words, *triangles are not fatter than those in the Euclidean space.*

Proof. (1) $\Rightarrow$ (2): Let X be a CAT(0) space. Let $x, y, z \in X$ and m be the midpoint of x and y. The

CAT(0) inequality tells exactly that $d(z,m) \leqslant d(\bar{z},\bar{m})$. Repeating the argument, $\forall p \in [x,y]$ with $d(x,p) = 2^{-n}k \cdot d(x,y)$, $d(z,p) \leqslant d(\bar{z},\bar{p})$. So by the denseness, it holds for all $p \in [x,y]$.

Now for $q \in [z,x]$, we apply the previous step to triangle $\triangle(z,p,x)$, done.

(2) $\Rightarrow$ (1): Let $x,y,z \in X$ and m be the midpoint of x and y.

Consider $\triangle(\bar{x},\bar{y},\bar{z})$. Let $p = m$, $q = z$, one has

$$d(m,z) = d(p,q) \leqslant d(\bar{p},\bar{q}).$$

Apply the parallelogram identity,

$$d(\bar{z},\bar{m})^2 = \frac{1}{2}\big(d(\bar{x},\bar{z})^2 + d(\bar{y},\bar{z})^2\big) - \frac{1}{4}d(\bar{x},\bar{y})^2 = \frac{1}{2}\big(d(x,z)^2 + d(y,z)^2\big) - \frac{1}{4}d(x,y)^2.$$

Combing the two identites, we have the CAT(0) identity. □

Lemma 6.5

In a CAT(0) space, geodesic segments are unique.

Proof. Let s_1, s_2 be two such segments. Take m_i the midpoint of s_i. The comparison condition gives

$$d(m_1,m_2) \leqslant d(\bar{m}_1,\bar{m}_2) = 0,$$

so $m_1 = m_2$. Apply the argument inductively on half-segments $[y,m_1]$ and $[y,m_2]$. This gives that s_1 and s_2 have all points in common at distance $2^{-n}k \cdot d(x,y)$ from y. By denseness, $s_1 = s_2$. □

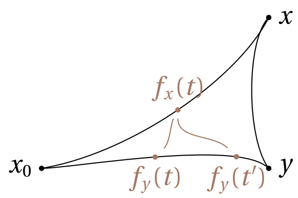
Henceforth we can use the notation $[x,y]$ to denote the *unique* geodesic segment between x and y.

Lemma 6.6

Any CAT(0) space is contractible.

Proof. Let X be a CAT(0) space and $x_0 \in X$. For $x \in X$, let $f_x : [0,1] \to X$ be the map such that $f_x(t)$ is the point on $[x_0,x]$ such that

$$d(f_x(t),x_0) = (1-t)d(x_0,x).$$



Now let $f : [0,1] \times X \to X$, $(x,t) \mapsto f_x(t)$. This is a continuous retraction from X to $\{x_0\}$ because

$$\begin{aligned} d(f_x(t), f_y(t')) &\leqslant d(f_x(t), f_y(t)) + d(f_y(t), f_y(t')) \\ &\leqslant d(\overline{f_x(t)}, \overline{f_y(t)}) + d(\overline{f_y(t)} + \overline{f_y(t')}) \leqslant c_1\left|x-y\right| + c_2\left|t-t'\right|. \end{aligned}$$

Therefore f gives the required retraction. □

Example 6.7

Here are some examples of CAT(0) spaces.

(1) Euclidean spaces $(\mathbb{R}^n, |\cdot|)$ or $(\mathbb{C}^n, |\cdot|)$. Also, Hilbert spaces are CAT(0) spaces.

(2) Trees with geodesic distances.

(3) Any simply connected Riemannian manifold of non-positive sectional curvature.

(4) The space of positive definite matrices with Euclidean topology.

Definition 6.8: Circumcentre, circumradius

Let (X,d) be a metric space and $Y \subset X$ be a subset. The *circumradius* of Y is defined as

$$r_0 := \inf\{r > 0 : \exists x \in X\,(Y \subset B(x,r))\}.$$

A *circumcentre* of Y is a point $x \in X$ such that $Y \subset \bar{B}(x, r_0)$.

Lemma 6.9

If (X,d) is CAT(0), $Y \subset X$ is bounded, then there exists a unique $x \in X$ such that $Y \subset \bar{B}(x,r)$, where

$$r = \inf\{s > 0 : \exists z \in X\,(B(z,s) \supset Y)\}.$$

i.e., any bounded subset of a CAT(0) space has a unique circumcentre.

Proof. Let $Y \subset X$ be a bounded subset and r_0 be the circumradius. Let (x_n, r_n) be a sequence such that $Y \subset B(x_n, r_n)$ and $r_n \to r_0$. For $n, m \in \mathbb{N}$, let $\mu_{n,m}$ be the midpoint of x_n and x_m. By definition of r_0, $\exists y \in Y$ such that

$$d(\mu_{m,n}, y) > r_0 - \varepsilon$$

for some $\varepsilon > 0$. By the CAT(0) inequality,

$$d(x_n, x_m)^2 \leqslant 4\left(\frac{1}{2}\left(d(x_n,y)^2 + d(x_m,y)^2\right) - d(y, \mu_{m,n})^2\right) \leqslant 8\varepsilon, \qquad n, m \to \infty.$$

So $(x_n)_{n\geqslant 1}$ is a Cauchy sequence. Let x be the limit, for $y \in Y$,

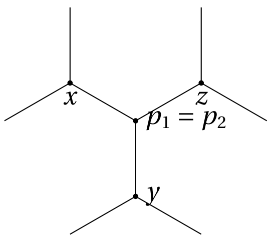
$$d(y, x_n) \leqslant r_n \quad \Longrightarrow \quad d(y,x) \leqslant r_0.$$

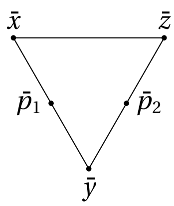
So x is the circumcentre. The uniqueness follows from one more use of the CAT(0) inequality. □

Remark. Please note that $d(p,q) < d(\bar{p}, \bar{q})$ can happen. For example, consider the tree below, we have

$$d(x,y) = d(y,z) = d(z,y) = 2.$$

Let p_1 be the midpoint of $[x,y]$ and p_2 be that of $[y,z]$. But $d(p_1,p_2) = 0 < d(\bar{p}_1, \bar{p}_2)$.





Theorem 6.10: Cartan fixed point theorem

Let X be a CAT(0) space and G be a group acting by isometries on X. If G has a bounded orbit($\Longleftrightarrow$*all orbits are bounded*), then G has a fixed point.

Remark. *The* CAT(0) *condition is necessary:* $\mathrm{SO}(n) \curvearrowright \mathbb{S}^{n-1}$ *by isometries with bounded orbits, but has no fixed points.*

Proof. Let Y be some orbit of G. By assumption Y is bounded, and thus has a circumcentre c. Since G acts by isometries, $g\cdot c$ is also a circumcentre. By uniqueness, $g\cdot c = c$ for any $g\in G$. Hence c is a fixed point. □

Corollary 6.11

Let G be a bounded subgroup of $\mathrm{GL}(n,\mathbb{R})$, then G is conjugated to a subgroup of $\mathrm{O}(n)$.

Proof. Let $\mathrm{SDP}(n,\mathbb{R})$ be the space of symmetric positive-definite matrices of dimension n. It has a Riemannian metric of non-positive curvature and it is simply connected. So $\mathrm{SDP}(n,\mathbb{R})$ is CAT(0). Also, $\mathrm{GL}(n,\mathbb{R})$ acts by isometries on $\mathrm{SDP}(n,\mathbb{R})$ via

$$g\cdot M := (g^{\mathsf{T}})^{-1}Mg^{-1}, \qquad \forall g\in \mathrm{GL}(n,\mathbb{R}),\ \forall M\in \mathrm{SDP}(n,\mathbb{R}).$$

If $H\leqslant \mathrm{GL}(n,\mathbb{R})$ is bounded, there is a fixed point $M\in\mathrm{SDP}(n,\mathbb{R})$ such that $\forall h\in H\,((h^{\mathsf{T}})^{-1}Mh^{-1}=M)$. So M has a square root, $S\in\mathrm{SDP}(n,\mathbb{R})$ such that $M=S^2$. Therefore,

$$\begin{aligned}(h^{\mathsf{T}})^{-1}SSh^{-1}=SS \quad &\Longrightarrow \quad (S^{-1}(h^{\mathsf{T}})^{-1}S)(Sh^{-1}S^{-1})=\mathrm{id}_n\\ &\Longrightarrow \quad (Sh^{-1}S^{-1})^{\mathsf{T}}(Sh^{-1}S^{-1})=\mathrm{id}_n.\end{aligned}$$

Thus $Sh^{-1}S^{-1}\in\mathrm{O}(n,\mathbb{R})$ and $h\in S\cdot\mathrm{O}(n)\cdot S^{-1}$. □

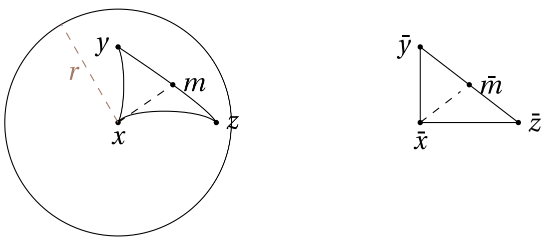
6.2 Convex subsets of a CAT(0) space

Definition 6.12: Convex sets

Let X be a CAT(0) space. A subset C is called *convex* if for any $x,y\in C$, $[x,y]\subset C$.

Any ball in a CAT(0) space is convex. For any $y,z\in B(x,r)$ and $u\in[y,z]$, thanks to a comparison triangle,

$$d(\bar{x},\bar{u})\leqslant \max\{d(x,z),d(x,y)\}\leqslant r.$$



Proposition 6.13

Let X be a CAT(0) space. The distance function is convex, *i.e.*, for $c,c':[0,1]\to X$ two parametrisations of geodesic segments with constant speed, for all $t \in [0,1]$, one has

$$d(c(t), c'(t)) \leqslant (1-t)d(c(0), c'(0)) + td(c(1), c'(1)).$$

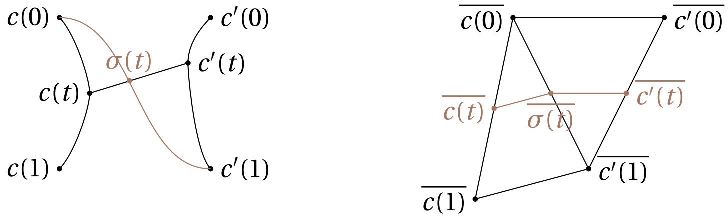
Proof. Take two comparison triangles $\triangle(\overline{c(1)}, \overline{c(0)}, \overline{(c'(1)}))$ and $\triangle(\overline{c(0)}, \overline{c'(1)}, \overline{c'(0)})$. Let $\sigma(t) \in [c(0), c'(1)]$ such that $d(c(0), \sigma(t)) = td(c(0), c'(1))$. Then

$$d(c(t), c'(t)) = d(c(t), \sigma(t)) + d(\sigma(t), c'(t)) \leqslant d(\overline{c(t)}, \overline{\sigma(t)}) + d(\overline{\sigma(t)}, \overline{c'(t)}).$$

However, we have

$$d(\overline{c(t)}, \overline{\sigma(t)}) = td(c(1), c'(1)), \qquad d(\overline{\sigma(t)}, \overline{c'(t)}) = (1-t)d(c(0), c'(0)),$$

combining the inequalities then we obtain the result. □



Proposition 6.14

Let C be a closed convex subspace of a CAT(0) space (X, d). Then for $x \in X$, there is a unique $c \in C$ such that

$$d(x, c) = \inf\{d(x, y) : y \in C\}.$$

The point $c \in C$ is called the *projection* of x to C.

Proof. Let $y_n \in C$ be a minimising sequence, $d(x, y_n) \to \ell$, where $\ell = \inf\{d(x, y) : y \in C\}$. For $m, n \in \mathbb{N}$, let $\mu_{n,m}$ be the midpoint of $[y_n, y_m]$, then by CAT(0) inequality,

$$\begin{aligned} d(y_n, y_m)^2 &\leqslant 4\left(\frac{1}{2}\left(d(x, y_n)^2 + d(x, y_m)^2\right) - d(x, \mu_{n,m})^2\right) \\ &\leqslant 4\left(\frac{1}{2}\left(d(x, y_n)^2 + d(x, y_m)^2\right) - \ell^2\right). \end{aligned}$$

The right hand side tends to 0 as $n, m \to \infty$. Thus $(y_n)_{n\geqslant 1}$ is Cauchy. While C is closed, hence it is complete. So $\exists y \in C\,(y_n \to y)$. By continuity of the distance function to a point, $d(x, y) = \ell$.

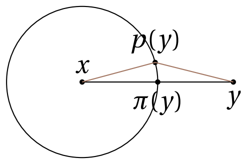
For uniqueness, take $y_1, y_2 \in C$ with $d(x, y_i) = c$. Then use the same inequality to get that $d(y_1, y_2) = 0$. □

Similarly, we have $x \in C$ if and only if its projection on C is itself.

Proposition 6.15

The projection map $p : X \to C$ is a Lipschitz map.

Proof. We only prove the case for balls, *i.e.*, $C = \bar{B}(x_0, r)$. We claim that $y \notin C$, the projection of y is the point on $[x, y]$ at distance r from x. Let $\pi(y)$ be this point, and $p(y)$ is the projection of y on C.



We have

$$d(x,y) \leqslant d(x,p(y)) + d(p(y),y)$$

so

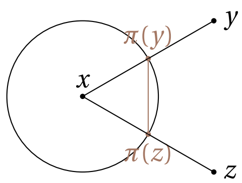
$$d(p(y),y) \geqslant d(x,y) - d(x,p(y)) \geqslant d(x,y) - r \geqslant d(x,\pi(y)) + d(\pi(y),y) - r = d(\pi(y),y).$$

By the uniqueness of projection, $\pi(y) = p(y)$.

Back to the triangle, for $y,z \in X$, in $\triangle(\bar{x},\bar{y},\bar{z})$,

$$d(\pi(y),\pi(z)) \leqslant d(\overline{\pi(y)},\overline{\pi(z)}) \leqslant d(\bar{y},\bar{z}) = d(y,z).$$

Then we conclude the proof. $\square$



6.3 The boundary at infinity

Definition 6.16: Geodesic ray, asymptotic rays

Let X be a CAT(0) space.

(1) A *geodesic ray* is (the image of) an isometric embedding $c : \mathbb{R}_+ \to X$.

(2) Two geodesic rays $c_1,\ c_2$ are *asymptotic* if there are bounded Hausdorff distance one from another, *i.e.*, $\exists A > 0$ such that

$$\forall x \in c_1(\mathbb{R}_+)\, \exists y \in c_2(\mathbb{R}_+)\, (d(x,y) \leqslant A).$$

and vice versa. ($\Longleftrightarrow$ *They have the same direction*)

(3) The *boundary at infinity* of X is the set of equivalence classes of geodesic rays, denoted as ∂X. A point in ∂X is often denoted as $c(\infty)$ or ξ.

A class of rays is given by a unit vector, which indicates the direction of the half-line.

So there is an one-to-one correspondence

$$\partial X \longleftrightarrow \mathbb{S}^1, \qquad \xi = c(\infty) \in \partial X \longleftrightarrow \{\text{direction of } c\}.$$

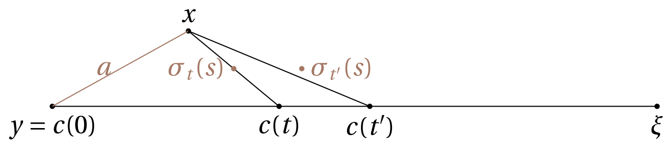
For example, $X = \mathbb{R}^n$ with Euclidean metric is a CAT(0) space, $\partial\mathbb{R}^n$ is a set of parallel classes of half-lines. This is a sphere at infinity.

Proposition 6.17

Let X be a CAT(0) space and $x \in X$, $\xi \in \partial X$. There exists a unique geodesic ray $c : \mathbb{R}_+ \to X$ such that $c(\infty) = \xi$, $c(0) = x$.

Proof. (1) The uniqueness is from convexity of the metric.

(2) Now we prove the existence. Let $y = c(0)$ and $a = d(x, y)$. Denote $\sigma_t(s)$ the point on $[x, c(t)]$ at distance s from x. We claim that for fixed s, $\lim_{t\to\infty} \sigma_t(s) = \sigma(s) \in X$ and $\sigma : \mathbb{R}_+ \to X$ is a geodesic ray asymptotic to c.



The triangle inequality gives

$$t - a \leqslant d(x, c(t)) \leqslant t + a.$$

Consider the comparison triangle $\triangle(\bar{x}, \overline{c(t)}, \overline{c(t')})$. Let α be the angle at $\bar{x}$. Then the law of cosines gives

$$\cos \alpha = \frac{d(x, c(t))^2 + d(x, c(t'))^2 - d(c(t), c(t'))^2}{2d(x, c(t))d(x, c(t'))} \geqslant \frac{(t-a)^2 + (t'-a)^2 - (t-t')^2}{2(t+a)(t'+a)}.$$

Let $t, t' \to \infty$, the right hand side tends to 1. So $\alpha \to 0$. Thus $d(\overline{\sigma_t(s)}, \overline{\sigma_{t'}(s)}) \to 0$ and hence $\sigma_t(s)$ converges for fixed s. By completeness of X, denote $\sigma(s)$ as its limit. Then

$$\forall t \in \mathbb{R}_+ \, (d(\sigma_t(s), \sigma_t(s')) = \left|s - s'\right|) \implies d(\sigma(s), \sigma(s')) = \left|s - s'\right|$$

and $d(\sigma(s), c(R_+)) < \infty$. Therefore c and σ are asymptotic. $\square$

We give a spectific construction of the closure $X \cup \partial X$ using projective limit.

Definition 6.18: Projective limit

Let $(X_t)_{t>0}$ be a family of topological spaces with continuous maps $\pi_{s,r} : X_s \to X_r$ for $r > s$, such that for all $r > s > t$, $\pi_{r,t} = \pi_{r,s} \circ \pi_{s,t}$. Define the *projective limit* as

$$\varprojlim X_t := \left\{ (x_t)_{t>0} \in \prod_{t \in \mathbb{R}_+} X_t : \forall s > t \, (\pi_{s,t}(x_s) = x_t) \right\}$$

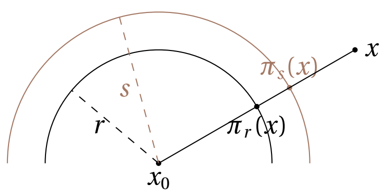
endowed with the induced topology from the product topology on $\prod_{t\in\mathbb{R}_+} X_t$.

Example 6.19

Let $x_0 \in X$ in a CAT(0) space, $X_t = \bar{B}(x_0, t)$. Define

$$\pi_{r,s} : X_s \to X_r, \qquad \pi_s(x) \mapsto \pi_r(s).$$

Then we have a bijection between $\bar{X} = X \cup \partial X$ and $\varprojlim X_t$.



Proof. (1) For $x \in X$, let $x_t = \pi_t(x)$. For $\xi \in \partial X$ we associate a geodesic ray c from x_0 to ξ. Then

$$\bar{X} \to \varprojlim \bar{B}(x_0, r), \qquad X \ni x \mapsto (x_r)_{r>0}, \quad \partial X \ni \xi \mapsto (c(r))_{r>0}$$

is injective.

(2) Let $x = (x_r)_{r>0} \in \varprojlim \bar{B}(x_0, r)$. Then $\forall s > r$, $\pi_{s,r}(x_s) = x_r$. Assume that for some $s > r$, $x_s \neq x_r$. Then for all $r < t$, $x_r = \pi_r(x_s)$ is the projection on the r-ball. Thus if we denote $t_0 := \sup\{r > 0 : d(x_0, x_r) = r\}$:

- when $t_0 < \infty$, $[0, t_0] \to X$, $r \mapsto x_r$ is a geodesic segment because for all $s, r > t_0$, $x_s = x_r = x_{t_0}$. This implies $x \in \varprojlim \bar{B}(x_0, r)$.
- when $t_0 = \infty$, $\mathbb{R}_+ \to X$, $r \mapsto x_r$ is a geodesic ray, thus $x \in \varprojlim \bar{B}(x_0, r)$ as well. □

This construction $\bar{X} = \varprojlim \bar{B}(x_0, r)$ does not depend on x_0.

Definition 6.20: Cone topology

The *cone topology* on $\bar{X}$ is the topology identified with $\varprojlim \bar{B}(x_0, r)$.

The cone topology coincides with the metric topology on X. Additionally, $[x_0, x_n]$ converges to $[x_0, \xi]$ uniformly on bounded subsets if and only if $(x_n)_{n\geqslant 1}$ is a sequence of X that converges to $\xi \in \partial X$.

Recall that a metric space is *proper* if any closed ball is compact.

Proposition 6.21

Let X be a proper CAT(0) space, then $\bar{X}$ is a compacitification of X. In other words, $\bar{X}$ is compact and $X \subset \bar{X}$ is open and dense.

Proof. Since X is proper, $\bar{B}(x_0, r)$ is compact, by Tychonoff's theorem, $\prod_{r\geqslant 0} \bar{B}(x_0, r)$ is compact. Using the fact that the projective limit of closed sets are closed if and only if $\bar{X}$ is Hausdorff, we obtain $\varprojlim \bar{B}(x_0, r)$ closed in $\bar{X}$, thus is compact.

Since any geodesic ray is a limit of geodesic segments $[x_0, x_n]$, so X is dense in $\bar{X}$. For any closed ball, the inclusion $i : \bar{B}(x, r) \to i(\bar{B}(x, r)) \subset X$ is a continuous bijection since $\bar{B}(x_0, r)$ is comapct. This is an isomorphism. So $i(X)$ is open in $\bar{X}$. □

Example 6.22

This is a non-example. Take a tree X with infinite degree at any vertex, $\bar{X} = X \cup \partial X$ is no more compact. Take a sequence of distinct neighbourhood of the origin.

Proposition 6.23

Any isometry of a CAT(0) space X extends uniquely as a homeomorphism of $\bar{X}$.

Proof. Let $g \in \mathrm{Isom}(X)$. For any $r > 0$ and $x_0 \in X$,

$$g(\pi_{\bar{B}(x_0,r)}(x)) = \pi_{\bar{B}(gx_0,r)}(gx).$$

Therefore g gives the homeomorphism

$$\varprojlim \bar{B}(x_0, r) \to \varprojlim \bar{B}(gx_0, r), \qquad (x_r)_{r>0} \mapsto (gx_r)_{r>0}.$$

Since $\bar{X}$ does not depend on the choice of x_0, the isomorphism g then gives a homeomorphism of X. □

6.4 Horofunctions and Busemann functions

Let X be a CAT(0) space, and $C(X)$ be the algebra of continous functions on X, endowed with the topology of uniform convergence on bounded subsets of X. This is a locally convex topological vector space with seminorms

$$P_{x,r}(f) := \sup_{y \in B(x,r)} \left|f(y)\right|, \qquad x \in X,\ r > 0.$$

Then let $C_*(X) := C(X)/\{\text{constants}\}$. Then

$$C_*(X) \cong \left\{f \in C(X) : f(x_0) = 0\right\} = C_0(X).$$

We have an embedding

$$i : X \to C_*(X), \qquad x \mapsto d(x,\cdot) - d(x,x_0).$$

It is injective and continuous, but not necessary the same topology, *i.e.*, $\bar{X} \neq \hat{X}$ in general, where $\hat{X}$ is the closure of $i(X)$ in $C_*(X)$.

Definition 6.24: Horofunction, Busemann function

Let X be a CAT(0) space and $\xi \in \partial X$. A point $h \in \hat{X} \setminus X$ is called a *horofunction*. A sublevel set is called a *horoball* and a level set is called a *horosphere*.
Denote c the geodesic ray from x_0 to ξ. The *Busemann function* is defined as

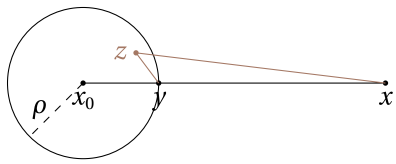
$$\beta_\xi(x,x_0) = \lim_{t\to\infty} d(x,c(t)) - t.$$

The limit exists, the convergence is uniform on bounded sets and $\rho_\xi(\cdot,x_0)$ is Lipschitz. We shall admit the following two lemmata. The proofs are easy.

Lemma 6.25

Let X be a CAT(0) space, $x_0,\ x,\ z \in X$ such that $z \in B(x_0,\rho)$ for some $\rho > 0$. If for any $\varepsilon > 0$, there exists $r > 0$ such that $d(x,x_0) > r$ and $y \in [x_0,x]$ with $d(x_0,y) = \rho$, then

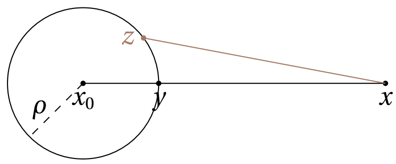
$$d(z,y) + d(y,x) - d(z,x) < \varepsilon.$$



Lemma 6.26

Let $x_0,\ x,\ y \in X$ with $d(x_0,y) = \rho$ and $y \in [x_0,x]$. If $z \in X$ with $d(x_0,z) = \rho$, then

$$d(x,z) - d(x,y) > \frac{d(y,z)^2}{2\rho}.$$



Theorem 6.27

In a CAT(0) space, any horofunction is a Basemann function.

Proof. Let $h \in \hat{X} \setminus X$ vanishing at x_0, then h is a limit of a sequence $(d(x_n, \cdot) - d(x_n, x_0))_{n\geqslant 1}$. We want to prove that x_n converges to $\xi \in \partial X$.

(1) We claim that $(x_n)_{n\geqslant 1}$ is unbounded. Otherwise, $\exists R > 0$ such that $\forall n \in \mathbb{N}(x_n \in \bar{B}(x_0, R))$. In that case,

$$\inf_{y \in \bar{B}(x_0,R)} (d(x_n, y) - d(x_n, x_0)) = -d(x_n, x_0).$$

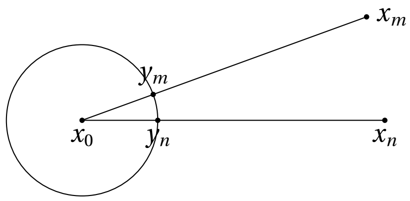
By uniform convergence, there exists a point $x \in \bar{B}(x_0, R)$ such that

$$h(x) = \inf_{y \in \bar{B}(x_0,R)} h(y) = -\liminf_{n\to\infty} d(x_n, x_0).$$

By triangle inequality, $x_n \to x$. So $(x_n)_{n\geqslant 1}$ is unbounded.

(2) It suffices to prove that $\forall \rho > 0$, if $y_n \in [x_0, x_n]$ with $d(y_n, x_0) = \rho$, $(y_n)_{n\geqslant 1}$ is convergent. Now the sequence $(d(x_n, \cdot) - d(x_n, x_0))_{n\geqslant 1}$ is uniformly convergent on $\bar{B}(x_0, \rho)$. So $\forall \varepsilon > 0$ and n, m large enough, $\forall z \in \bar{B}(x_0, \rho)$,

$$|d(x_n, z) - d(x_n, x_0) - d(x_m, z) + d(x_m, x_0)| < \varepsilon.$$



Take $z = y_n$, we have

$$\left| d(x_n, y_n) - d(x_n, y_n) - d(y_n, x_0) - d(x_m, y_n) - d(x_m, y_m) + d(y_m, x_0) \right| < \varepsilon,$$

which implies $\left| d(x_m, y_m) - d(x_m, y_n) \right| < \varepsilon$. By Lemma 6.25, we have

$$d(y_n, y_m) + d(y_m, x_m) - d(y_n, x_m) < \varepsilon,$$

so $d(y_n, y_m) < 2\varepsilon$. Hence $(y_n)_{n\geqslant 1}$ is Cauchy, $y_n \to y_\rho$. Now $(y_\rho)_{\rho>0}$ is a geodesic ray, so $x_n \to \xi = [(x_\rho)_{\rho>0}]$.

(3) Then we identify h and $\beta_\xi(\cdot, x_0)$. First, $h(x_0) = 0 = \beta_\xi(x_0, x_0)$ by definition. And by (2),

$$h(x) = \lim_{n\to\infty} d(x_n, x) - d(x_n, x_0) = \lim_{t\to\infty} d(x, c(t)) - d(c(t), c(0)) = \beta_\xi(x, x_0)$$

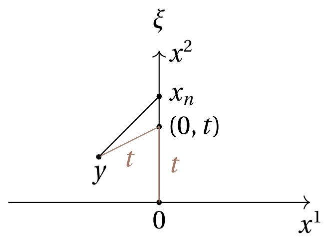
for any $x \in [x_0, \xi]$. $\square$

Example 6.28

Let $X = \mathbb{R}^2$ and $x_n = (0, n)$. Then $x_n \to \xi \in \partial X$. We have

$$\beta_\xi(y, 0) = \lim_{t\to\infty} d(y, (0, t)) - t =: h(y).$$

And $h(y) - h(z) = \left\langle y - z, \lim_{n\to\infty} \frac{x_n}{\|x_n\|} \right\rangle$.



6.5 Behaviour of individual isometries

Definition 6.29: Classification of isometries

Let X be a CAT(0) space and g be an isometry of X. Define its *translation length*

$$\ell(g) := \inf_{x\in X} d(gx, x).$$

The isometry g is said to be

- *elliptic*, if $\ell(g)$ is a minimum and $\ell(g) = 0$.
- *hyperbolic*, if $\ell(g)$ is a minimum and $\ell(g) > 0$.
- *parabolic*, if $\ell(g)$ is not a minimum.

Example 6.30

(1) Let $X = \mathbb{R}^2$, $g \in \mathrm{Isom}(\mathbb{R}^2)$. Then

- If g is a rotation or a symmetry, then g is elliptic.
- If g is a translation or a gliding symmetry, then g is hyperbolic.

Recall that a gliding symmetry is an isometry of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

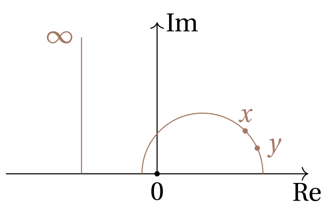
Additionally, any $g \in \mathrm{Isom}(\mathbb{R}^n)$ is not parabolic.

(2) Let $X = \ell^2(\mathbb{Z})$ and $\sigma : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the shift map $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$. Then $g(x) = \sigma(x) + \delta_0$ is parabolic.

(3) Let X be a tree, then any isometry is either elliptic or hyperbolic. To see this, it suffices to prove that $\exists x_0 \in X$ such that $d(gx_0, x_0)$ is the minimum.

(4) Let $\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ be the hyperbolic plane with $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. Then $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ and $\mathrm{PSL}(2,\mathbb{R})$ acts by isometries on $\mathbb{H}^2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}.$$



- The elliptic isometries are of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. This forms a group and i is a fixed point.

- The hyperbolic isometries are of the form $\begin{bmatrix} \lambda & \\ & 1/\lambda \end{bmatrix}$ for $\lambda \in \mathbb{R}$. The line $\{z \in \mathbb{H}^2 : \operatorname{Re} z = 0\}$ is invariant and the action is by translation.
- The parabolic isometries are of the form $\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$ for $t \in \mathbb{R}$. The action is by horizontal translation.

Each isometry here has vanishing translation length.

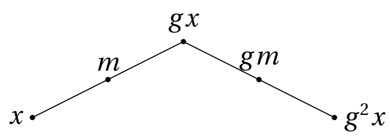
Definition–Proposition 6.31: Axis of a hyperbolic isometry

Let g be a hyperbolic isometry, then there is a g-invariant geodesic line L on which g acts by translations, called the *axis* of g.

Proof. Let $\min(g) := \{x \in X : d(gx, x) = \ell(g)\}$. Choose $x \in \min(g)$ and m the midpoint of $[x, gx]$, then gm is the midpoint of $[gx, g^2x]$. If $[x, gx] \cup [gx, g^2x]$ is not a segment, then by a comparison triangle:

$$d(m, gm) \leqslant \frac{1}{2} d(x, g^2x) \leqslant \frac{1}{2}\big(d(x, gx) + d(gx, g^2x)\big) = d(x, gx) = \ell(g).$$

Also, $d(m, gm) = d(m, gx) + d(gx, gm)$ since $d(m, gx) = \frac{1}{2} d(x, gx)$. We have equality in the triangle inequality, so $[m, gx] \cup [gx, gm] = [m, gm]$. And we have a contradiction.



By induction, $\bigcup_{n\in\mathbb{N}} [g^n x, g^{n+1} x]$ is a geodesic line and the action is by translations of length $\ell(g)$. □

Then we consider parabolic isometries.

Lemma 6.32

Let X be a CAT(0) space, and $(X_n)_{n\geqslant 1}$ is a nested sequence(*i.e.*, $m \geqslant n \implies X_m \subset X_n$) of bounded closed convex subspaces. Then $\bigcap_{n\in\mathbb{N}} X_n \neq \varnothing$.

Proof. Let r_n be the circumradius and c_n be the circumcentre of X_n. Since the projection on X_n is 1-Lipschitz, $c_n \in X_n$. Then let $\mu_{n,m}$ be the midpoint of $[c_n, c_m]$ and let $x \in X_m$. The CAT(0) inequality gives

$$d(x, \mu_{n,m})^2 \leqslant \frac{1}{2}\big(d(x, c_n)^2 + d(x, c_m)^2\big) - \frac{1}{4} d(c_n, c_m)^2 \leqslant \frac{1}{2}(r_n^2 + r_m^2) - \frac{1}{4} d(c_n, c_m)^2.$$

Since $X_m \subset X_n$, $r_m \leqslant r_n$, $(r_n)_{n\geqslant 1}$ converges to some $r \in \mathbb{R}$. We may find $x \in X_m$ such that $d(x, \mu_{n,m}) \geqslant r_m$. Thus

$$d(c_n, c_m)^2 \leqslant 4\left(\frac{1}{2}(r_n^2 + r_m^2) - r_m^2\right) \to 0, \qquad n, m \to \infty.$$

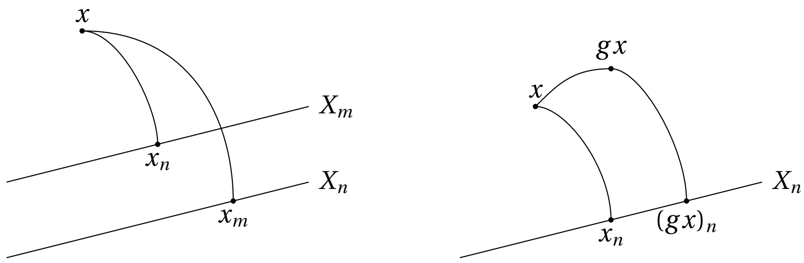
Then $(c_n)_{n\geqslant 1} \subset X$ is Cauchy, and the completeness of X gives the existence of the limit $c \in X$. While $\forall m \geqslant n\, (c_m \in X_n)$, so $c \in \bigcap_{n\in\mathbb{N}} X_n$. □

Lemma 6.33

Let X be a proper CAT(0) space. Let $(X_n)_{n\in\mathbb{N}}$ be a nested sequence of closed convex subspaces such that $\bigcap_{n\in\mathbb{N}} X_n = \varnothing$. Then $\exists \xi \in \bigcap_{n\in\mathbb{N}} \partial X_n$ and for all $g \in \operatorname{Isom}(X)$ such that $\forall n \in \mathbb{N}\, (gX_n = X_n)$, we have

$g\xi = \xi$.

Proof. Let $x \in X$ and x_n the projection of x on X_n. Since X is proper, up to extracting a subsequence, $[x, x_n]$ converges to some geodesic ray $[x, \xi)$. Here $d(x, x_n) \to \infty$ because of Lemma 6.32. Then for all $m \geq n$, $x_m \in X_m \subset X_n$. Therefore $\xi \in \partial X_n$ implies $\xi \in \bigcap_{n\in\mathbb{N}} \partial X_n$.



Assume $g \in \mathrm{Isom}(X)$ with $\forall n \in \mathbb{N} (gX_n = X_n)$. The projection is 1-Lipschitz, thus

$$d(x_n, gx_n) \leq d(x, gx).$$

So $d_H([x, \xi), [gx, g\xi)) \leq d(x, gx)$. The two geodesic rays are asymptotic, *i.e.*, $\xi = g\xi$. □

Proposition 6.34

Let $g \in \mathrm{Isom}(X)$ be a parabolic isometry, X is a proper CAT(0) space. Then g has a fixed point at infinity.

Proof. Set $X_n = \left\{x \in X : d(x, gx) \leq \ell(g) + \frac{1}{n}\right\}$. Since g is parabolic, $\bigcap_{n\in\mathbb{N}} X_n = \varnothing$. By Lemma 6.33, $\exists \xi \in \partial X$ such that $\xi = g\xi$. □

Definition 6.35: Splitting

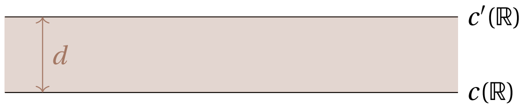
Let X be a CAT(0) space. A *splitting* of X is a pair (Y, Z) of CAT(0) spaces such that $X \cong Y \times Z$ with distance

$$d((y_1, z_1), (y_2, z_2))^2 = d(y_1, y_2)^2 + d(z_1, z_2)^2.$$

To deal with the splitting, we shall introduce several results. Some of them are not proved in this course.

Proposition 6.36: Flat strip

Let $c, c' : \mathbb{R} \to X$ be two geodesic rays in the CAT(0) space X. IF $d_H(c(\mathbb{R}), c'(\mathbb{R})) < \infty$, then $\overline{\mathrm{co}}(c(\mathbb{R}) \cup c'(\mathbb{R}))$ is isomorphic to $[0, d] \times \mathbb{R}$ for some $d \geq 0$.



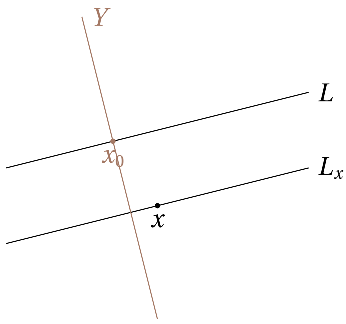
Consider a map

$$\mathbb{R} \to \mathbb{R}, \qquad t \mapsto d(c(t), c'(\mathbb{R})).$$

The map is continuous and convex, bounded from $\mathbb{R}$ to $\mathbb{R}$, so it is a constant $d \geq 0$.

Proposition 6.37

Assume that X is the union of all geodesic lines at bounded Hausdorff distance from some geodesic L. Then $X \cong \mathbb{R} \times Y$, where Y is the preimage of L.



Definition 6.38: Clifford translation

An isometry g of a CAT(0) space X is called a *Clifford translation* if $X = \min(g)$.

Theorem 6.39

Let g be a Clifford translation with $\ell(g) > 0$, then X splits as $Y \times \mathbb{R}$ and g acts on $Y \times \mathbb{R}$ via $g \cdot (y,t) = (y, t + \ell(g))$.

Definition 6.40: Euclidean de Rham factor

Let X be a space that splits as $H \times Z$, where H is a Hilbert space and Z is a CAT(0) space, and any Clifford translation comes from a translation in H. Moreover, this splitting is invariant under any isometry, *i.e.*, any $g \in \mathrm{Isom}(X)$ is of the form $g_1 \times g_2$ with $g_1 \in \mathrm{Isom}(H)$ and $g_2 \in \mathrm{Isom}(Z)$.
The space H is called the *Euclidean de Rham factor* of X.

Definition 6.41: Flatness

Let X be a CAT(0) space. A map $f : X \to \mathbb{R}$ is called *affine* if it is affine in restriction to any geodesic segment $c : I \to X$, *i.e.*, $f \circ c : I \to \mathbb{R}$ is affine.
Let $\xi \in \partial X$. It is called *flat* if the Busemann function $\beta_\xi(\cdot, x_0) : X \to \mathbb{R}$ is affine.

A basic example of flat boundary points comes from $\mathbb{R}^n$. Each $\xi \in \partial\mathbb{R}^n$ is flat.

Lemma 6.42

Let X be a CAT(0) space and $\xi \in \partial X$. For any $x,\ y,\ z \in X$,

$$\beta_\xi(x,z) = \beta_\xi(x,y) + \beta_\xi(y,z).$$

In other words, changing the base point, the Busemann function change by a constant.

Proof. Recall that $\beta_\xi(x,z) = \lim_{t\to\infty} d(x,c(t)) - t$. We have seen that if $(x_n)_{n\geqslant 1} \subset X$ with $x_n \to \xi$, then

$$\begin{aligned}\beta_\xi(x,z) &= \lim_{n\to\infty} d(x,x_n) - d(z,x_n) \\ &= \lim_{n\to\infty} (d(x,x_n) - d(y,x_n)) + (d(y,x_n) - d(z,x_n)) \\ &= \beta_\xi(x,y) + \beta_\xi(y,z).\end{aligned}$$

Then we conclude the proof. □

Proposition 6.43

Let X be a CAT(0) space such that $\mathrm{Isom}(X) \curvearrowright X$ is minimal, *i.e.*, there is no invariant strict closed convex subspace. If X has trivial Euclidean de Rham factor, then there is no flat point at infinity. In other words, if ∂X has a flat point, then $X \,\mathrm{Cont}\, Y \times \mathbb{R}$.

Lemma 6.44

Let X be a CAT(0) space. $G \leqslant \mathrm{Isom}(X)$ without fixed point at infinity. Then there is a closed convex G-invariant subspace $Y \subset X$ that is minimal for these properties.

Proof. By Zorn's lemma, it suffices to prove that for any chain $\mathscr{Y}$ of closed convex subspaces, $\bigcap \mathscr{Y} \neq \varnothing$. If $\bigcap \mathscr{Y} = \varnothing$, then $\forall n \in \mathbb{N}$, $\exists Y_n \in \mathscr{Y}$ such that $d(x_0, Y_n) \to \infty$. So $\bigcap_{n \in \mathbb{N}} Y_n = \varnothing$. This implies the existence of $\xi \in \partial Y_n$ such that ξ is G-invariant, then we have a contradiction. □

Definition 6.45: Flatness of subspace

A closed convex subspace of a CAT(0) space is *flat* if it is isometric to some Hilbert space.

6.6 Adams–Ballman theorem

Theorem 6.46: Adams–Ballman

If X is a proper CAT(0) space, and G is an amenable topological group, acting continuously by isometries on X. Then

- either G fixes a point in ∂X,
- or G stablises a flat subspace(*i.e.*, a subspace isometric to $\mathbb{R}^n$ for $n \geqslant 0$).

Proof. Assume that G has no fixed point at infinity. By Lemma 6.44, we may assume that $G \curvearrowright X$ is minimal. The space X is proper, so $\bar{X}$ is compact and $G \curvearrowright \bar{X}$ continuously. By amenability, there exists $\mu \in \mathrm{Prob}(\bar{X})$ such that μ is G-invariant.

(1) If $\mu(X) > 0$, then there exists a bounded ball B large enough, such that $\mu(B) > \mu(X)/2$. This implies that for all $g \in G$, $gB \cap B \neq \varnothing$. Otherwise

$$\mu(gB \cup B) = \mu(gB) + \mu(B) = 2\mu(B) > \mu(X),$$

while $\mu(gB \cup B) \leqslant \mu(X)$, contradiction. So any point in B has a bounded orbit. By Cartan fixed point theorem, there is a global fixed point. So there is an invariant flat subspace of dimension 0.

(2) If $\mu(X) = 0$, fix $x_0 \in X$ and define

$$f : X \to \mathbb{R}, \qquad x \mapsto \int_{\partial X} \beta_\xi(x, x_0)\, \mathrm{d}\mu(\xi).$$

It is well-defined because $\xi \mapsto \beta_\xi(x, x_0)$ is continuous and bounded by $d(x_0, x)$.

$$\begin{aligned} f(gx) &= \int_{\partial X} \beta_\xi(gx, x_0)\, \mathrm{d}\mu(\xi) = \int_{\partial X} \beta_{g^{-1}\xi}(x, g^{-1}x_0)\, \mathrm{d}\mu(\xi) \\ &= \int_{\partial X} \beta_\xi(x, g^{-1}x_0)\, \mathrm{d}\mu(\xi) = \int_{\partial X} (\beta_\xi(x, x_0) + \beta_\xi(x_0, g^{-1}x_0))\, \mathrm{d}\mu(\xi) \\ &= f(x) - \underbrace{f(g^{-1}x_0)}_{\text{constant w.r.t } x} . \end{aligned}$$

Let $r = \inf_{x\in X} f(x)$ and take $(r_n)_{n\geqslant 1} \to r$ decreasing. Then define $X_n = f^{-1}(-\infty, r_n]$. Then X_n is closed since $\beta_\xi(\cdot, x_0)$ is 1-Lipschitz; X_n is convex since f is convex. If $\bigcap_{n\in\mathbb{N}} X_n = \varnothing$, then there is $\xi \in \bigcap_{n\in\mathbb{N}} \partial X_n$ such that ξ is G-invariant. If $\bigcap_{n\in\mathbb{N}} X_n \neq \varnothing$, then f has a minimum, and its minimum level set is G-invariant. It is closed and convex, by minimality, it is X. So f is a constant. But

$$f(x) = \int_{\partial X} \beta_\xi(x, x_0)\,\mathrm{d}\mu(\xi)$$

with each β_ξ is convex. Thus for almost all $\xi \in \partial X$, β_ξ is affine, *i.e.*, almost all points are flat.

But X has de Rham decomposition $X \cong Y \times H$, where H is a Hilbert space and Y has no Euclidean factor. Up to consider Y, we may assume that X has no Euclidean factor and thus no flat points. □

Example 6.47

$\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ is amenable, and $\mathrm{Isom}(\mathbb{R}^n) \curvearrowright \mathbb{R}^n$. We have seen that

$$G = \left\{ \begin{bmatrix} \lambda & t \\ & 1/\lambda \end{bmatrix} : t \in \mathbb{R},\ \lambda > 0 \right\} \leqslant \mathrm{SL}(2, \mathbb{R}).$$

And it fixes $\infty \in \partial\mathbb{H}^2$. The group is solvable thus amenable. The group G acts transitively on $\mathbb{H}^2$.

7 Epilogue — Amenable Lie Groups

In this chapter, we shall give the main result: a connected Lie group is amenable if and only if it is a compact extension of a solvable group.

Any countable discrete group is a Lie group of dimension 0. That is why the connectedness is required.

Definition 7.1: Sovability by compact

A topological group G is a *compact extension of a solvable group* or *solvable by compact*, if there exists $S \lhd G$ such that $K := G/S$ is compact. Using terms of extension, we have the short exact sequence

$$1 \longrightarrow S \longrightarrow G \longrightarrow K \longrightarrow 1$$

7.1 Geometry of the space of positive-definite matrices

Let $\mathrm{SDP}_n(\mathbb{R}) \subset \mathrm{Mat}_n(\mathbb{R})$ be the subspace of positive-definite matrices with topology in $\mathbb{R}^{n^2}$.

(1) It is a dense and open subspace of $\mathrm{Sym}_n(\mathbb{R})$, the symmetric matrices. So $\mathrm{SDP}_n(\mathbb{R})$ is a manifold of dimension $n(n+1)/2$.

(2) It is a Riemannian manifold with metric at S being

$$\langle X, Y\rangle_S := \mathrm{Tr}\,(S^{-1} X S^{-1} Y).$$

It can be identified with the space of scalar products on $\mathbb{R}^n$ and with the space of ellipsoids in $\mathbb{R}^n$.

Proposition 7.2

The sectional curvature is non-positive at any point. The exponential map at id_n is

$$\mathrm{Sym}_n(\mathbb{R}) \to \mathrm{SDP}_n(\mathbb{R}), \qquad X \mapsto \exp X := \sum_{k\geqslant 0} \frac{X^k}{k!}.$$

The norm at id_n is $\|X\|^2 = \mathrm{Tr}\, X^2$, the sum of the squares of coefficients of X.

A geodesic through id_n is $t \mapsto \exp(tX)$, where X is symmetric and $\mathrm{Tr}\, X^2 = 1$.

Proof. See the lecture notes of previous course. □

Corollary 7.3

The space $\mathrm{SDP}_n(\mathbb{R})$ is CAT(0), and the distance is given by

$$d(\mathrm{id}_n, A) := \sqrt{\sum_{i\in I} \log \lambda_i^2}, \qquad A \in \mathrm{SDP}_n(\mathbb{R}),$$

where $(\lambda_i)_{i\in I}$ are eigenvalues(with multiplicity) of A.

Remark. We can give an explicit formula for the distance between $A, B \in \mathrm{SDP}_n(\mathbb{R})$ by

$$d(A,B) = \sqrt{\sum_{i\in I} \log\left(\frac{\lambda_i}{\mu_i}\right)^2}, \qquad A = \mathrm{diag}\{\lambda_i\},\ B = \mathrm{diag}\{\mu_i\}.$$

But there is no way to prove the CAT(0) inequality using the above formula.

Proof. The space $\mathrm{SDP}_n(\mathbb{R})$ is CAT(0) because the section curvature, by Proposition 7.2, is non-positive. And $\exp : \mathrm{Sym}_n(\mathbb{R}) \to \mathrm{SDP}_n(\mathbb{R})$ is a diffeomorphism. So $\mathrm{SDP}_n(\mathbb{R})$ is simply connected.

For the formula, it suffices to prove the case when

$$A = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\} = \exp(\mathrm{diag}\{\log\lambda_1, \dots, \log\lambda_n\}) = \exp\left(t \cdot \mathrm{diag}\left\{\frac{\log\lambda_1}{t}, \dots, \frac{\log\lambda_n}{t}\right\}\right),$$

where $t = \sqrt{\sum_{i\in I} \log\lambda_i^2}$ to ensure $\mathrm{diag}\left\{\frac{\log\lambda_1}{t}, \dots, \frac{\log\lambda_n}{t}\right\}$ is of norm 1. So A is time t of the geodesic starting at id_n, then $d(\mathrm{id}_n, A) = t$. □

The group $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathrm{SDP}_n(\mathbb{R})$ via

$$g \cdot S := gSg^{\mathsf{T}}, \qquad g \in \mathrm{GL}_n(\mathbb{R}), \quad S \in \mathrm{SDP}_n(\mathbb{R}).$$

Lemma 7.4

This is a transitive action by isometries.

Proof. Since $S \in \mathrm{SDP}_n(\mathbb{R})$, there is $R \in \mathrm{SDP}_n(\mathbb{R})$ such that $S = R^2 = RR^{\mathsf{T}} = R \cdot \mathrm{id}_n$. So the orbit of id_n is exactly $\mathrm{SDP}_n(\mathbb{R})$, thus the action is transitive.

To show it acts by isometries, for $X,\ Y \in \mathrm{Sym}_n(\mathbb{R})$,

$$\begin{aligned}\langle d_S g X, d_S g Y \rangle_{g\cdot S} &= \mathrm{Tr}\left((gSg^{\mathsf{T}})^{-1} gXg^{\mathsf{T}} (gSg^{\mathsf{T}})^{-1} gYg^{\mathsf{T}}\right) \\ &= \mathrm{Tr}\left((g^{\mathsf{T}})^{-1} S^{-1} X S^{-1} Y g^{\mathsf{T}}\right) = \mathrm{Tr}\left(S^{-1} X S^{-1} Y\right) = \langle X, Y \rangle_S.\end{aligned}$$

So it is an isometry. □

Definition 7.5: Symmetric space

A connected Riemannian manifold M is said to be *symmetric* if $\forall p \in M$, $\exists \sigma_p \in \mathrm{Isom}(M)$ such that

$$\sigma_p(p) = p, \qquad d_p\sigma(p) = -\mathrm{id}_{T_pM}.$$

Lemma 7.6

$\mathrm{SDP}_n(\mathbb{R})$ is a symmetric space.

Proof. Since the space is homogeneous, it suffices to prove the existence of a symmetry at id_n. Let

$$\sigma : \mathrm{SDP}_n(\mathbb{R}) \to \mathrm{SDP}_n(\mathbb{R}), \qquad S \mapsto S^{-1}.$$

Then σ is a smooth involution with $\sigma(\mathrm{id}_n) = \mathrm{id}_n$. Let $S \in \mathrm{SDP}_n(\mathbb{R})$, $H \in \mathrm{Sym}_n(\mathbb{R})$,

$$\begin{aligned}(S+H)^{-1} &= (\mathrm{id} + S^{-1}H)^{-1}S^{-1} \\ &= (\mathrm{id}_n - S^{-1}H + o(H))S^{-1} = S^{-1} - S^{-1}HS^{-1} + o(H).\end{aligned}$$

So $d_S\sigma(H) = S^{-1}HS^{-1}$. For $X,\ Y \in \mathrm{Sym}_n(\mathbb{R})$,

$$\begin{aligned}\langle d_S\sigma(X), d_S\sigma(Y)\rangle_{S^{-1}} &= \mathrm{Tr}\,(S(-S^{-1}XS^{-1})S(-S^{-1}YS^{-1})) \\ &= \mathrm{Tr}\,(XS^{-1}YS^{-1}) = \mathrm{Tr}\,(S^{-1}XS^{-1}Y) = \langle X, Y\rangle_S.\end{aligned}$$

This implies the symmetry of $\mathrm{SDP}_n(\mathbb{R})$. □

Lemma 7.7

A flat subsapce of $\mathrm{SDP}_n(\mathbb{R})$ containing id_n is of the form $\exp V$, where V is a linear subspace of $\mathrm{Sym}_n(\mathbb{R})$ consisting matrices.

Proof. By Riemannian geometry. □

Corollary 7.8

A maximal flat subspace of $\mathrm{SDP}_n(\mathbb{R})$ is of the form

$$S\left\{\mathrm{diag}\{\lambda_1,\ldots,\lambda_n\} : \lambda_i \geqslant 0\right\}S$$

for some $S \in \mathrm{SDP}_n(\mathbb{R})$.

Proof. Commuting symmetric matrices are simultineously diagonalisable. □

Remark. The rank of $\mathrm{SDP}_n(\mathbb{R})$*, in other words, the maximal dimension of a flat subspace, is n.*

Let $\mathrm{SDP}_n^1(\mathbb{R})$ be the space of positive-definite matrices with determinant 1. This is a convex subspace of $\mathrm{SDP}_n(\mathbb{R})$.

Proposition 7.9

The map

$$\mathbb{R} \times \mathrm{SDP}_n^1(\mathbb{R}) \to \mathrm{SDP}_n(\mathbb{R}), \qquad (t, S) \mapsto \mathrm{e}^{t/\sqrt{n}}S$$

is an isometry.

Remark. *This shows that the Euclidean de Rham factor of* $\mathrm{SDP}_n(\mathbb{R})$ *is* $\{\lambda \mathrm{id}_n : \lambda > 0\}$*, the space of positive homotheties. The action of* $\mathrm{GL}_n(\mathbb{R})$ *on* $\mathrm{SDP}_n^1(\mathbb{R})$ *is*

$$g \cdot S := (\det g^2)^{-1/n} g S g^\top, \qquad g \in \mathrm{GL}_n(\mathbb{R}), \quad S \in \mathrm{SDP}_n^1(\mathbb{R}).$$

Thus homotheties act on $\mathrm{SDP}_n^1(\mathbb{R})$ *trivially. And the action of* $\mathrm{SL}(n,\mathbb{R})$ *is formally simplier.*

Proposition 7.10

The stabiliser of a point at infinity of $\mathrm{SDP}_n^1(\mathbb{R})$ is conjugated to a subgroup of blockwise upper triangular matrices.

Proof. Let $\xi \in \partial \mathrm{SDP}_n^1(\mathbb{R})$. ξ is the class of some $t \mapsto \exp(tX)$, in which $X \in \mathrm{Sym}_n(\mathbb{R})$ satisfies $\mathrm{Tr}\, X^2 = 1$ and $\mathrm{Tr}\, X = 0$. So up to conjugating X, we may assume

$$X = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}, \qquad \forall i \leqslant j\, (\lambda_i \geqslant \lambda_j),\ \lambda_1 \neq \lambda_n.$$

Let $g \in \mathrm{Stab}(\xi)$, this means that $g \exp(tX) g^\top$ and $\exp(tX)$ are at uniformly bounded distance. So

$$\exp\left(-\frac{tX}{2}\right) g \exp(tX) g^\top \exp\left(-\frac{tX}{2}\right) \text{ and } \mathrm{id}_n = \exp\left(-\frac{tX}{2}\right) \exp(tX) \exp\left(-\frac{tX}{2}\right)$$

are at uniformly bounded distance. So $\exp\left(-\frac{tX}{2}\right) g \exp(tX) g^\top \exp\left(-\frac{tX}{2}\right)$ is uniformly bounded.

Now for $j > i$ such that $\lambda_j < \lambda_i$, the (i,j)-coefficient of this matrix is $\exp\left(\frac{t(\lambda_j - \lambda_i)}{2}\right) g_{ij}$. When $t \to \infty$, it goes to infinity if and only if $g_{ij} \neq 0$. So $g_{ij} = 0$ for $\lambda_j > \lambda_i$. Therefore g is blockwise upper triangular. □

Lemma 7.11

The kernel of this action is given by homotheties.

Proof. If g is in the kernel, it is upper triangular in any basis. So it must be a homothety. □

Remark. *The group that is really acting is* $\mathrm{PGL}_n(\mathbb{R})$.

7.2 Amenable Lie groups

Definition 7.12: Decomposition of vector spaces

A decomposition of vector space $\mathbb{R}^n = \bigoplus_{i=1}^k E_i$ is *not trivial* if $k \geqslant 2$. An element $g \in \mathrm{GL}_n(\mathbb{R})$ *preserves* this decomposition if $\forall i \in \{1, \dots, k\}$, $gE_i = E_i$.

Lemma 7.13

If $g \in \mathrm{GL}_n(\mathbb{R})$ stabilising a flat subspace with positive dimension in $\mathrm{SDP}_n^1(\mathbb{R})$, then g preserves a factor of a non-trivial decomposition.

Proof. Let A be a flat subspace invariant under g. We may assume that A contains id_n and elements of A are diagonal. Let $M \in A$ be an element with maximal number of eigenspaces. So $\mathbb{R}^n = \bigoplus_{i=1}^k E_i$ with E_i being the eigenspaces of M.

If $N \in A$, N is an homothety on each E_i. Now

$$gg^\top = g \cdot \mathrm{id}_n \in A \quad \Longrightarrow \quad gg^\top = \sum_{i=1}^k \lambda_i P_i,$$

where $P_i : \mathbb{R}^n \to E_i$ is the orthogonal projection. If $s = \sum_{i=1}^k \sqrt{\lambda_i} P_i$ and $h = sg$, then $hh^\top = \mathrm{id}_n$. So $h \in \mathrm{O}_n(\mathbb{R})$ and s preserves the decomposition.

$$g \cdot M = s^{-1} h \cdot M = s^{-1}(hMh^\top)(s^{-1})^\top = s^{-1}(hMh^{-1})(s^{-1})^\top \in A.$$

Therefore $g \cdot M$ is a homothety in restriction to E_i.

If M has k distinct eigenvalues, then $h \cdot M$ also has k distinct eigenvalues, and the eigenspaces of $h \cdot M$ are exactly the eigenspaces of M. Hence $\exists \sigma \in \mathfrak{S}_k$ such that $h \cdot E_i = E_{\sigma(i)}$. Now

$$g^{-1} E_i = h^{-1} s E_i = h^{-1} E_i = E_{\sigma^{-1}(i)},$$

and by dimension arguments, $gE_i = E_{\sigma(i)}$. □

Theorem 7.14

Let $G \leqslant \mathrm{GL}_n(\mathbb{R})$ be a closed subgroup. If G is amenable, then G is solvable by compact.

Proof. Do induction on dimension. For $n = 1$, $G \leqslant \mathbb{R}$, trivial.

Now we assume the result holds for $k < n$. If G preserves a decomposition $\mathbb{R}^n = \bigoplus_{i=1}^k E_i$, we set G_i to be the image of G in $\mathrm{GL}(E_i)$. Then

$$G \subset \mathrm{diag}\{\mathrm{GL}(E_1), \ldots, \mathrm{GL}(E_k)\}.$$

Each G_i is amenable, so by induction hypothesis, there exist $S_i \lhd G_i$ closed and solvable such that G_i/S_i are compact. Let $S = \bigcap_{i=1}^k \ker[G \to G_i/S_i]$. It is closed in $S_1 \times \cdots \times S_k$ thus S is solvable. Hence

$$G/S \leqslant (G_1/S_1) \times \cdots \times (G_k/S_k)$$

is a closed subgroup of a compact group. Therefore G/S is compact. □

Lemma 7.15

Let G be a topological group with a normal closed subgroup S such that G/S is solvable by compact, then G is solvable by compact.

Proof. Let $\pi : G \to G/S$ be the quotient map. Since G/S is solvable by compact, there exists a solvable normal subgroup $R \lhd G/S$ such that $(G/S)/R$ is compact. Let $N = \pi^{-1}(R)$. We have $N \lhd G$ because $R \lhd G/S$, and N is the preimage of a closed group hence is closed. The solvability pulls back along quotient, thus $G/N \cong (G/S)/R$ is compact. □

Now $G \curvearrowright \mathrm{SDP}_n^1(\mathbb{R})$ continuously. By Adams–Bullmann theorem, either G stabilises a flat subspace, or G fixes a point at infinity.

(1) If G fixes a point, then G itself is compact.

(2) If G fixes a flat subspace of positive dimension, all its elements permute the factor of the decomposition. So G has a finite index subsgroup G' preserving a non-trivial decomposition. So G' is solvable by compact. By Lemma 7.15, G is solvable by compact.

(3) IF G fixes a point at infinity, there is a non-trivial decomposition $\mathbb{R}^n = \bigoplus_{i=1}^k E_i$ such that $\forall g \in G\big(g(\oplus_{i=1}^k E_i) \subset (\oplus_{i=1}^k E_i)\big)$. We use the map $G \to \mathrm{GL}(E_i)$ as above. The image of G in $\prod_{i=1}^k \mathrm{GL}(E_i)$ is

amenable and thus solvable by compact. The kernel is given by upper triangular with 1's on the diagonal. So the kernel is solvable. Then apply Lemma 7.15 again, we deduce that G is solvable by compact.

Now we can prove our goal.

Theorem 7.16

A connected Lie group is amenable if and only if it is a compact extension of a solvable group.

Proof. G is a connected amenable Lie group, so G has a solvable radical R, *i.e.*, the unique maximal normal solvable subgroup and $G' = G/R$ is semisimple. So the adjoint representation $\mathrm{ad} : G' \to \mathrm{GL}_n(\mathbb{R})$ has kernel being the centre of G', which is discrete. G' is an extension of Abelian group by solvable by compact group. So G is solvable by compact. □

7.3 Tits Alternative, the statement

Theorem 7.17: Tits

Let Γ be a finitely generated subgroup of $\mathrm{GL}_n(\mathbb{k})$, where $k \in \mathbb{N}$ and $\mathbb{k}$ is a field. Then either Γ is virtually solvable, or Γ contains $\mathbb{F}_2$.

Remark. *The theorem is called « alternative » because if Γ is finitely generated and linear, then Γ is either amenable (because virtually solvable), or Γ is not amenable(because $\mathbb{F}_2 \leqslant \Gamma$).*

Moreover, if $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$, the « finitely generated » condition can be removed.

von Neumann–Day's Question. If Γ is a non-amenable group, is it true that Γ contains $\mathbb{F}_2$?

In general, the answer is no. But it is true when Γ is finitely generated.

Definition 7.18: Algebraic set

Let $X \subset \mathrm{Mat}_n(\mathbb{R})$. Say X is *algebraic* or *Zariski closed* if X is the zeros of a family of polynomial maps $\{p_i : \mathrm{Mat}_n(\mathbb{R}) \to \mathbb{R}\}$.

The *Zariski closure* of $G \leqslant \mathrm{GL}_n(\mathbb{R})$ is the smallest algebraic set that contains G, denoted as $\overline{G}^Z$, which is still a subgroup.

We say a Zariski closed subspace of $\mathrm{Mat}_n(\mathbb{R})$ is *connected* if it is not a finite union of at least 2 disjoint Zariski closed subspaces.

Example 7.19

(1) $\mathrm{SL}(n, \mathbb{R}) = \mathrm{zero}(\det - 1)$, so it is algebraic.

(2) The groups $\mathrm{SO}(p,q)$ and $\mathtt{SU}(p,q) = \{M \in \mathrm{Mat}_n(\mathbb{R}) : M^* J_{p,q} M = J_{p,q}\}$ are algebraic, where $J_{p,q} = \mathrm{diag}\{\mathrm{id}_p, -\mathrm{id}_q\}$.

(3) For the stabiliser of subspaces $V \subset \mathbb{R}^n$, one can find a basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that $V = \mathrm{span}\{e_1, \dots, e_k\}$. Then

$$G = \{g \in \mathrm{GL}_n(\mathbb{R}) : gV = V\} = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}.$$

Theorem 7.20

Let $G \leqslant \mathrm{GL}_n(\mathbb{R})$ be an unbounded subgroup. If $\overline{G}^Z$ is connected, and $G \curvearrowright \mathbb{R}^n$ is irreducible, then $\mathbb{F}_2 \leqslant G$.

We shall give the strategy of the proof: Play ping-pong on some projective space $\mathbb{P}(V)$ where $V = \mathbb{R}^n$ or $V = \wedge^d\mathbb{R}^n$. We want to find elements in G with contracting properties.

7.4 Cartan decomposition

Theorem 7.21: Cartan

Let $g \in \mathrm{SL}(n,\mathbb{R})$, then $\exists k_1,\ k_2 \in \mathrm{SO}(n)$ and $a = \mathrm{diag}\{a_1,\dots,a_n\}$ with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$, and $g = k_1 a k_2$.

Proof. Since $g^\top g$ is positive-definite so $\exists k \in \mathrm{SO}(n)$ such that

$$k^\top g^\top g k = \mathrm{diag}\{b_1,\dots,b_n\}$$

with $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$. Set $a_i = \sqrt{b_i}$ and $a = \mathrm{diag}\{a_1,\dots,a_n\}$. Then

$$(a^{-1})^\top k^\top g^\top g k a^{-1} = \mathrm{id}.$$

So $gka^{-1} =: k_1 \in \mathrm{SO}(n)$. Now we have $g = k_1 a k_1^{-1}$ and set $k_2 = k_1^{-1} \in \mathrm{SO}(n)$, done. $\square$

Remark. This decomposition is not necessary unique. But the a_i's are unique because they are eigenvalues of $g^\top g$. If $g_n \to \infty$, then $a_1(g_n) \to \infty$ because $a_1(g_n) = \|g_n\|$.

Let $d \in \mathbb{N}$. The exterior product $\wedge^d\mathbb{R}^n$ is the dual space of the space of alternating d-linear forms on $\mathbb{R}^n$. If $\{e_1,\dots,e_n\}$ is a baiss of $\mathbb{R}^n$, then $\left\{e_{i_1} \wedge \cdots \wedge e_{i_d} : i_1 < \cdots < i_d\right\}$ is a basis of $\wedge^d\mathbb{R}^n$. If $\langle\cdot,\cdot\rangle$ is a scalar product on $\mathbb{R}^n$, one defines

$$\langle u_1 \wedge \cdots \wedge u_d, v_1 \wedge \cdots \wedge v_d\rangle := \det[\langle u_i, v_j\rangle].$$

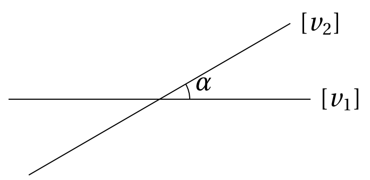
If $g \in \mathrm{SL}(n,\mathbb{R})$, we denote $\bar{g} \in \mathrm{SL}(\wedge^d\mathbb{R}^n)$ defined by

$$g(v_1 \wedge \cdots \wedge v_d) := gv_1 \wedge \cdots \wedge gv_d.$$

Let V be a Euclidean space. On $\mathbb{P}(V)$, we put the following distance

$$d([v_1],[v_2]) := \sin\alpha = \frac{\|v_1 \wedge v_2\|}{\|v_1\|\,\|v_2\|}$$

where $[v] = \mathbb{R}v \in \mathbb{P}(V)$ with $v \in V$.



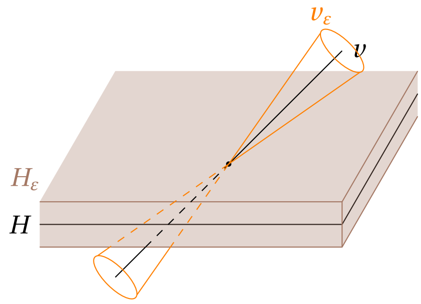
The group of projective transformation $\mathrm{PGL}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})/\{\lambda\mathrm{id}\}$. If we denote $\bar{g}$ the image of $g \in \mathrm{GL}_n(\mathbb{R})$, and for $\bar{g}_1 = \bar{g}_2$, $g_1,\ g_2 \in \mathrm{SL}(n,\mathbb{R})$, then for all $i \in \{1,\dots,n\}$, $a_i(g_1) = a_i(g_2)$.

Definition 7.22: ε-contracting

Let $\varepsilon > 0$. An element $g \in \mathrm{PSL}(n,\mathbb{R})$ is called *ε-contracting* if $\exists v \in \mathbb{P}(\mathbb{R}^n)$ and H an image of a hyperplane, such that

$$g(\mathbb{P}(\mathbb{R}^n) \setminus H_\varepsilon) \subset v_\varepsilon,$$

where v_ε, H_ε are the ε-neighbourhood of v and H respectively.



Remark. $g \in \mathrm{SL}(n,\mathbb{R})$ *such that* $\bar{g} \in \mathrm{PSL}(n,\mathbb{R})$ *is ε-contracting if and only if for all* $k_1,\ k_2 \in \mathrm{SO}(n)$, $k_1 g k_2$ *is ε-contracting because the metric on* $\mathbb{P}(\mathbb{R}^n)$ *is* $\mathrm{SO}(n)$*-invariant.*

Proposition 7.23

For any $\varepsilon > 0$, $g \in \mathrm{SL}(n,\mathbb{R})$, there exists $M > 0$ such that if $a_1(g)/a_2(g) > M$, then g is ε-contracting.

Proof. It suffices to consider the case where $g = \mathrm{diag}\{a_1(g), \dots, a_n(g)\}$. Set $v = e_1$ and $H = \mathrm{span}\{e_2, \dots, e_n\}$. We have

$$d([u], H) = \frac{|u_1|}{\|u\|}, \qquad u = u_1 e_1 + u',\ u' \in H$$

because $\forall [w] \in H$, $d([u],[w]) = \frac{\|u \wedge w\|}{\|u\|\|w\|}$. Let $w = u'$, then

$$u \wedge w = (u_1 e_1 \wedge u') \wedge u' = u_1 e_1 \wedge u'.$$

If $[u] \notin H_\varepsilon$, and $|u_1| / \|u\| > \varepsilon$, then $d([gu],[v]) = \frac{\|gu \wedge e_1\|}{\|gu\|}$. We denote $u = \sum_{i=1}^n u_i e_i$ and then $gu = \sum_{i=1}^n a_i(g) u_i e_i$. Therefore

$$\|gu \wedge e_1\| = \left\| \left(\sum_{i=1}^n a_i(g) u_i e_i \right) \wedge e_1 \right\| = \left\| \sum_{i=2}^n a_i(g) u_i e_i \wedge e_1 \right\| \leqslant \left(\sum_{i=2}^n a_i(g)^2 u_i^2 \right)^{1/2} \leqslant a_2(g)\, \|u\|.$$

While $\|gu\| \geqslant \|a_1(g) u_1\| \geqslant a_1(g)\varepsilon \|u\|$. THus

$$d([gu],[v]) < \frac{a_2(g)\,\|u\|}{a_1(g)\varepsilon\,\|u\|} = \frac{a_2(g)}{a_1(g)} \cdot \frac{1}{\varepsilon}.$$

Let $M > \varepsilon^{-2}$, then $d([gu],[v]) < \varepsilon^2 \cdot \frac{1}{\varepsilon} = \varepsilon$. $\square$

If $G \leqslant \mathrm{SL}(n,\mathbb{R})$ is unbounded, one can find a sequence $(g_m) \subset G$ such that $\|g_m\| \to \infty$. Then $a_1(g_m) \to \infty$. But $g_m \in \mathrm{SL}(n,\mathbb{R})$, $\det g_m = \prod_{i=1}^n a_i(g_m) = 1$. So if $\frac{a_k(g_m)}{a_{k+1}(g_m)}$ is bounded for all $k \in \{1, \dots, n-1\}$, then we obtain a contradiction with $a_1(g_m) \to \infty$. Denote

$$d = \min \left\{ k \in \{1, \dots, n-1\} : \frac{a_k(g_m)}{a_{k+1}(g_m)} \to \infty \right\}.$$

And consider the action on $\mathbb{P}(\wedge^d \mathbb{R}^n)$. If $g = \mathrm{diag}\{a_1(g), \dots, a_n(g)\}$ and $\{e_1, \dots, e_n\}$ are the eigenvectors of g, then $e_{i_1} \wedge \cdots \wedge e_{i_d}$ are eigenvectors of $\bar{g}$ for the eigenvalur $a_{i_1}(g) \cdots a_{i_d}(g)$. So

$$a_1(\bar{g}) = a_1(g) \cdots a_d(g), \qquad a_2(\bar{g}) = a_1(g) \cdots a_{d-1}(g) a_{d+1}(g).$$

And $a_1(\bar{g})/a_2(\bar{g}) = a_d(g)/a_{d+1}(g) \to \infty$.

Therefore, we conclude that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $m \geqslant N$, $\bar{g}_m$ is ε-contracting on $\mathbb{P}(\wedge^d \mathbb{R}^n)$. To conclude the Tits alternative, we need to find $\gamma_1 \in G$ such that γ_1 is ε-contracting for (H_1^+, v_1^+) and (γ_1^{-1}) is ε-contracting for (H_1^-, v_1^-). And $\gamma_2 \in G$ such that γ_2 is ε-contracting for (H_2^+, v_2^+) and (γ_2^{-1}) is ε-contracting for (H_2^-, v_2^-). Moreover, we need

$$d(v_1^\pm, H_2^- \cup H_2^+) > \varepsilon, \qquad d(v_2^\pm, H_1^- \cup H_1^+) > \varepsilon.$$

Then we can apply ping-pong lemma by letting

$$A^+ = (v_1^+)\varepsilon, \qquad A^- = (v_1^-)\varepsilon, \qquad B^+ = (v_2^+)\varepsilon, \qquad B^- = (v_2^-)\varepsilon.$$

Then $\mathbb{F}_2 = \langle \gamma_1, \gamma_2 \rangle \leqslant G$.