Harmonic Analysis, Lecture Notes

M2 AAG, EN1722, by Pascal AUSCHER

Contents

1 Covering

Setup. We always consider on $\mathbb{R}^n$ with Euclidean norm $|\cdot|$, the distance $d(x,y) := |x-y|$. The Lebesgue measure is denoted $\mathrm{d}x$. The open ball is defined as

$$B(x,r) := \{y \in \mathbb{R}^n : d(x,y) < r\}.$$

For $B = B(x,r)$ and $\lambda > 0$, let $\lambda B := B(x, \lambda r)$ and $r(B)$ denotes its radius.

For convenience, for $a, b \in \mathbb{Z}$, $a < b$, denote the set $\{a, a+1, \dots, b\} =: [\![a, b]\!]$.

Basic Questions. For a set $\Omega \subset \bigcup_{B \in \mathscr{B}} B = \bigcup \mathscr{B}$, a union of balls. We want to extract a subcovering without too much overlap. Ideaily, we would like to have mutually disjoint balls. But it is not possible in general so as to keep inclusion. There are 2 possible solution.

- Vitali's lemma: not specific to $\mathbb{R}^n$;
- Besicovitch theorem: specific to Euclidean space $\mathbb{R}^n$.

1.1 Vitali's lemma

Lemma 1.1: Vitali covering lemma

Let $\{B_\alpha\}_{\alpha \in I}$ be a collection of balls in $\mathbb{R}^n$ with bounded radius, *i.e.*, $\sup_{\alpha \in I} r(B_\alpha) < +\infty$. Then $\exists I_0 \subset I$ such that

(i) $\{B_\alpha\}_{\alpha \in I_0}$ are mutually disjoint;

(ii) $\bigcup_{\alpha \in I} B_\alpha \subset \bigcup_{\beta \in I_0} 5B_\beta$.

Remark. *(1) Balls can be open or closed.*

(2) There is a possible geometric extension: the statement holds for a metric space (E, d). Hence by changing $(\mathbb{R}^n, d)$ to $(\mathbb{R}^n, d_\infty)$, it is equivalent to replacing balls by cubes in $\mathbb{R}^n$.

(3) The sup condition is necessary. Take $B_i = B(0, i)$ for $i \in \mathbb{N}^*$. Then $\bigcup_{i \in \mathbb{N}^*} B_i = \mathbb{R}^n$ but it is impossible for disjointness.

Proof. Let $M = \sup_{\alpha \in I} r(B_\alpha)$. For $j \in \mathbb{N}$, define

$$I(j) := \{\alpha \in I : 2^{-(j+1)} M < r(B_\alpha) < 2^{-j} M\}.$$

We extract maximal subsets of $I(j)$ by induction. We use Zorn's lemma: any non-empty collection of balls contains a maximal subcollections of mutually disjoint balls.

For $j = 0$: $J(0)$ is the maximal subset of $I(0)$ such that B_α, $\alpha \in J(0)$ are mutually disjoint.

For $j = 1$: $J(1)$ is the maximal subset of $I(1)$ such that B_α, $\alpha \in J(1)$ are mutually disjoint, and disjoint from the balls in generation 0. Take initial family B_α, $\alpha \in I(1)$, it is already disjoint from balls of generation 0.

Induction. If $J(0), \dots, J(k)$ are constructed, take $J(k+1)$ such that B_α, $\alpha \in J(k+1)$ are mutually disjoint, and disjoint from balls selected at generation $0, 1, \dots, k$.

Then set $I_0 = \bigcup_{k \geqslant 0} J(k)$. By construction, B_β, $\beta \in I$ are mutually disjoint. It suffices to show that for fixed $\alpha \in I$, $\exists \beta \in I_0$ such that $B_\alpha \subset 5B_\beta$.

If $\alpha \in I_0$, take $\beta = \alpha$, then $B_\alpha \subset 5B_\alpha = 5B_\beta$. Otherwise, there exists $k \in \mathbb{N}$ such that $\alpha \in I(k)$, *i.e.*, $2^{-(k+1)}M < r(B_\alpha) < 2^{-k}M$. We claim that $\alpha \notin J(k)$. If not, B_α was not selected, hence it meets a ball B_β, $\beta \in J(\ell)$, $0 \leqslant \ell \leqslant k$ by maximality. Thus

$$r(B_\beta) > 2^{-(\ell+1)}M \geqslant 2^{-(k+1)}M \geqslant \frac{1}{2}r(B_\alpha).$$

Then $B_\alpha \cap B_\beta = \varnothing$ implies that $d(x_\alpha, x_\beta) \leqslant r(B_\alpha) + r(B_\beta) < 2r(B_\beta) + r(B_\beta) = 3r(B_\beta)$. Hence

$$B_\alpha = B(x_\alpha, r(B_\alpha)) \subset B(x_\beta, d(x_\alpha, x_\beta) + r(B_\alpha)) \subset B(x_\beta, 5r(B_\beta)),$$

which concludes the proof. □

1.2 Besicovitch Theorem

Definition 1.2: Bounded overlap

A collection of sets $\mathscr{A}$ is said to have *bounded overlap* if $\exists c > 0$ such that $\sum_{A\in\mathscr{A}} \mathbb{1}_A \leqslant c$. In other words, no more than $[c]$ sets A contain x for all $x \in \bigcup \mathscr{A}$.

Remark. *If $c = 1$, then $\mathscr{A}$ is mutually disjoint. Hence $\mathscr{A} \neq \varnothing$ implies $c \geqslant 1$.*

Theorem 1.3: Besicovitch theorem

Let $E \subset \mathbb{R}^n$. $\forall x \in E$, let $B(x) = B(x, r(x))$ is a bounded control of x. Assume that E is bounded or that $\sup_{x\in E} r(B(x)) < +\infty$. Then there exists a subset $E_0 \subset E$ and a constant $c = c(n) \in \mathbb{N}$ such that

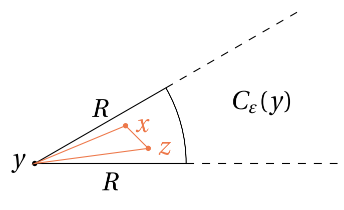
(i) $E \subset \bigcup_{x\in E_0} B(x)$;

(ii) $\sum_{x\in E_0} \mathbb{1}_{B(x)} \leqslant c(n)$, *i.e.*, $\{B(x) : x \in E_0\}$ has bounded overlap.

We need a technical lemma first.

Lemma 1.4

For $y \in \mathbb{R}^n$, $0 \leqslant \varepsilon \leqslant \frac{\pi}{6}$, let $C_\varepsilon(y)$ be the closed sector with vertex y and aparture angle ε. For any $x, z \in C_\varepsilon(y)$ and $R > 0$, if $|x - y| \leqslant R$, $|z - y| \leqslant R$, then $|x - z| \leqslant R$.



Proof. Note that $\cos(\angle(x-y, y-z)) \geqslant \cos 2\varepsilon \geqslant \cos\frac{\pi}{3} = \frac{1}{2}$, we have

$$\begin{aligned}
|x-z|^2 = |(x-y)+(y-z)|^2 &= |x-y|^2 + |y-z|^2 + 2\langle x-y, y-z\rangle \\
&= |x-y|^2 + |y-z|^2 - 2|x-y||y-z|\cos(\angle(x-y, y-z)) \\
&\leqslant |x-y|^2 + |y-z|^2 - |x-y||y-z| \leqslant \max\left\{|x-y|^2, |y-z|^2\right\} \leqslant R^2.
\end{aligned}$$

Note that this lemma depends on the metric on $(\mathbb{R}^n, d)$, because we use the inner product structure. □

Proof of Besicovitch theorem.

We prove only when E is bounded, because when E is unbounded we can use stupid technicalities to simplify the discussion to the bounded case. Define $M = \sup_{x\in E} r(B(x))$.

If $M = +\infty$, then take a ball large enough $B(x_0) \supset E$, then $E_0 = \{x_0\}$, done. Otherwise, for $k \in \mathbb{N}$, let

$$E(k) := \left\{x \in E : 2^{-(k+1)} M < r(B(x)) < 2^{-k} M\right\}.$$

We select centres inductively within each $E(k)$.

<u>*For k = 0:*</u> Since $E(0) \neq \varnothing$, pick $x_{0,0} \in E(0)$ and select $x_{0,i}$, $i = 1, 2, \ldots$ in such a way that $x_{0,i} \in E(0)$, $x_{0,i} \notin \bigcup_{j=0}^{i-1} B(x_0, j)$. The process stops. Indeed,

$$\left|x_{0,i} - x_{0,j}\right| > \frac{M}{2}$$

because E is bounded, $B(x_{0,1}) \subset E + B(0, M)$. Hence $B(x_{0,1}, \frac{M}{4}) \cap B(x_{0,j}, \frac{M}{4} = \varnothing)$. The balls $B(x_{0,i}, \frac{M}{4})$ of the selected points are contained in $B(0, R + M)$ and are mutually disjoint. By volume counting argument, denote A the set of selected $x_{0,i}$,

$$\#A \left|B\left(0, \frac{M}{4}\right)\right| = \sum_A \left|B\left(x_{0,i}, \frac{M}{4}\right)\right| = \left|\bigcup_A B\left(x_{0,i}, \frac{M}{4}\right)\right| \leqslant |B(0, R + M)|.$$

Hence $\#A < +\infty$, and let $E'(0) = A$ be a finite set.

<u>*Induction.*</u> Assume $E'(0), \ldots, E'(k-1)$ have been constructed. Construct $E'(k) \subset E(k)$ by selecting centres $x_{k,i}$ not in the union of all previously selected balls, *i.e.*,

$$x_{k,i} \notin \left(\bigcup_{m=0}^{k-1} \bigcup_{x_{m,i} \in E'(m)} B(x_{m,i})\right) \cup \bigcup_{\ell=0}^{i-1} B(x_{k,\ell}),$$

here $B(x_i)$ refers to the ball centred at x_i. By the same volume counting argument, $\#E'(k) < +\infty$ for all $k \in \mathbb{N}$. Let $E_0 = \bigcup_{k \geqslant 0} E'(k)$ and relabel $E_0 = \{x_1, x_2, \ldots\}$ with the same order. And we check the conditions (i) and (ii):

(i) We prove by contradiction. Suppose $x \in E$ but $x \notin \bigcup_{x_i \in E_0} B(x_i)$. But $\exists k \in \mathbb{N} (x \in E(k))$ and x was not selected. But it should have been selected by our construction. Contradiction.

(ii) Pick $y \in \mathbb{R}^n$, we want to show $\sum_{i \geqslant 1} \mathbb{1}_{B(x_i)}(y) \leqslant c(n)$. We start by counting how many $B(x_i)$ contain y with $x_i \in C_\varepsilon(y)$, where $\varepsilon = \frac{\pi}{6}$. That is, $\#A_y$ with

$$A_y := \left\{x_i \in E_0 : y \in B(x_i),\ x_i \in C_\varepsilon(y)\right\}.$$

If $A_y = \varnothing$, done. If not, let x_i be the first element in A_y and assume that $x_j \in A_y$ comes after, *i.e.*, $j > i$. Then

$$x_i,\ x_j \in C_\varepsilon(y), \qquad \left|x_i - y\right| < r(B(x_i)), \qquad \left|x_j - y\right| < r(B(x_j)).$$

Apply lemma 1.2, we have $\left|x_i - x_j\right| < \max\left\{r(B(x_i)), r(B(x_j))\right\}$. We also know that $x_j \notin B(x_i)$, then $\left|x_j - x_i\right| \geqslant r(B(x_i))$. Thus $r(B(x_i)) < r(B(x_j))$. Let k be the generation of x_i and k' be the generation of x_j, then $k' \geqslant k$, so

$$2^{-k-1} M < r(B(x_i)) \leqslant 2^{-k} M, \qquad 2^{-k'-1} M < r(B(x_j)) \leqslant 2^{-k'} M.$$

The fact that $r(B(x_i)) < r(B(x_j))$ implies $k' = k$. Thus all elements of A_j are of generation k. Therefore, $B(x_j, 2^{-k-2} M)$ are mutually disjoint for $x_j \in A_j$, and are contained in $B(x_i, 2^{-k-1} M)$. Do the volume counting argument,

$$\#A_y \leqslant \frac{\left|B(x_i, 2^{-k-1} M)\right|}{\left|B(0, 2^{-k-2} M)\right|} = 2^{3n} = 8^n.$$

Note that $\mathbb{R}^n$ can be covered by at most $M(n) \in \mathbb{N}$ sectors $C_\varepsilon(y)$, take $c(n) = 8^n M(n)$ then we conclude the proof. □

Remark. (1) *Extension of the proof: there is an organisation* $E_0 = E_1 \cup \cdots \cup E_N$, $N = N(n)$ *is a constant such that* $B(x)$, $x \in E_k$, $k \geqslant 1$ *are mutually disjoint.*

(2) The proof is Euclidean, but this proof works for cubes with sides parallel to the axes.

(3) If E *is unbounded, the condition* $\sup_{x\in E} r(B(x)) < +\infty$ *is necessary.*

(4) Balls must be centred at points in E*. A counterexample is as follows: Let* $E = \{1-2^{-k} : k \geqslant 1\}$*, and let*

$$B_k = [0, 1-2^{-k}] = \bar{B}\left(\frac{1-2^{-k}}{2}, \frac{1-2^{-k}}{2}\right).$$

Then the bounded overlap condition fails.

(5) Besicovitch theorem does not work for arbitrary metric spaces. For example, Heisenburg group.(see ***TD I, Exercise 3****)*

Corollary 1.5: Sard's theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map,

$$A := \left\{ x \in \mathbb{R}^n : \liminf_{r\to 0} \frac{m^*(f(B(x_i, r)))}{m^*(B(x_i, r))} = 0 \right\},$$

then $m^*(f(A)) = 0$, here m^* denotes the exterior Lebesgue measure. Moreover, the images of singular points of a differential map f has Lebesgue measure zero.

Proof. Fix $\varepsilon > 0$. For $x \in A$, $\exists r_x > 0$ such that $m^*(f(B(x, r_x))) \leqslant \varepsilon m^*(B(x, r_x))$. Apply Besicovitch theorem for $B_x = B(x, r_x)$, one can find a countable $A_0 \subset A$, such that $A \subset \bigcup_{x\in A_0} B_x$. Thus $f(A) \subset \bigcup_{x\in A_0} f(B_x)$. Hence

$$\begin{aligned} m^*(f(A)) &\leqslant \sum_{x\in A_0} m^*(f(B_x)) \leqslant \sum_{x\in A_0} \varepsilon m^*(B_x) = \sum_{x\in A_0} \varepsilon \int_{\mathbb{R}^n} \mathbb{1}_{B_x} \,\mathrm{d}x \\ &= \varepsilon \int_{\mathbb{R}^n} \sum_{x\in A_0} \mathbb{1}_{B_x} \,\mathrm{d}x \leqslant \varepsilon \cdot c(n) \cdot m^*\left(\bigcup_{x\in A_0} B_x \right) \leqslant \varepsilon \cdot c(n) \cdot m^*(A + \bar{B}(0,1)). \end{aligned}$$

If we assume A is bounded, $m^*(A+\bar{B}(0,1)) < +\infty$, and let $\varepsilon \to 0$, done. If A is unbounded, apply the above argument to $A \cap B(0,k)$ for each $k \in \mathbb{N}$, and the monotonic convergence theorem implies $m^*(f(A)) = 0$.

For the *moreover* part, let $S = \{x \in \mathbb{R}^n : \mathrm{d}f(x) = 0\}$. For $x \in S$ and $\varepsilon > 0$, $\exists r_x \in (0,1]$ such that $\forall h \in B(0, r_x)$,

$$|f(x+h) - f(x) - h \cdot \mathrm{d}f(x)| < \varepsilon |h|.$$

Hence $f(x+h) \in B(f(x), \varepsilon r_x)$ for all $h \in B(0, r_x)$. Thus $f(B(x, r_x)) \subset B(f(x), \varepsilon r_x)$, which implies

$$\frac{m^*(f(B(x, r_x)))}{m^*(B(x, r_x))} \leqslant \varepsilon^n.$$

Thus $S \subset A$. □

1.3 Dyadic Cubes

Definition 1.6: Dyadic cubes

Let $[0,1)^n$ be the reference cube. For $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, define

$$Q_{j,k} := \{x \in \mathbb{R}^n : 2^j x - k \in [0,1)^n\}.$$

Then $Q_{j,k}$ is called the *dyadic cube* of generation j with lower left corner $k/2^j$.

Set $\mathscr{D}_j := \{Q_{j,k} : k \in \mathbb{Z}^n\}$ the dyadic cubes of generation j, $\mathscr{D} := \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j$, and the sidelength $\ell(Q_{j,k}) = 2^{-j}$ in $\mathbb{R}$.

Remark. *We may start with a different reference cube $R = \prod_{i=1}^n [a_i, a_i + \delta)$ with $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\delta > 0$. We obtain another collection of dyadic cubes, they will share the same properties below:*

- *They can be described as $\varphi(Q_{j,k})$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map such that $\varphi([0,1)^n) = R$.*

Theorem 1.7

The dyadic cubes have the following properties:

(1) $\ell(Q_{j,k}) = 2^{-j}$, $\left|Q_{j,k}\right| = 2^{-jn}$, and $\mathscr{D}_j$ is a partition of $\mathbb{R}^n$.

(2) $\forall j \in \mathbb{Z}$, $\forall k \in \mathbb{Z}^n$, there exists a unique $k \in \mathbb{Z}^n$ such that $Q_{j,k} \subset Q_{j-1,k'}$. We set $Q_{j-1,k'} = \widehat{Q_{j,k}}$ the *parent cube* of $Q_{j,k}$.

(3) $\forall j \in \mathbb{Z}$, $\forall k \in \mathbb{Z}^n$, all $Q_{j+1,k'}$ such that $\widehat{Q_{j+1,k}} = Q_{j,k}$ are called the *children cubes* of $Q_{j,k}$. There are 2^n of them.

(4) $\forall x \in \mathbb{R}^n$, there exists a unique sequence of dyadic cubes $(Q_{j,k_j(x)})_{j\in\mathbb{Z}}$ such that $x \in Q_{j,k_j(x)}$. Moreover, there is a decreasing sequence and $\bigcap_{j\in\mathbb{Z}} Q_{j,k_j(x)} = \{x\}$.

(5) $\forall Q, Q' \in \mathscr{D}$, either $Q \subset Q'$ or $Q' \subset Q$ or $Q \cap Q' = \varnothing$.

(6) Let $\mathscr{E} \subset \mathscr{D}$ non-empty such that $\Omega = \bigcup_{Q\in\mathscr{E}} Q = \bigcup \mathscr{E}$ has finite Lebesgue measure. Let $\mathscr{F} = \{Q \in \mathscr{D} : Q \subset \Omega,\ \widehat{Q} \not\subset \Omega\}$, then $\mathscr{F}$ is composed mutually disjoint dyadic cubes which is a partition of Ω.

Remark. *In (6), it could be that $\mathscr{F} \not\subset \mathscr{E}$. For example, let $\mathscr{E} = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$, $\Omega = [0,1)$ and $\mathscr{F} = \{[0,1)\}$.*

Proof. We only prove (5) and (6).

(5) Suppose $x \in Q \cap Q'$, Q and Q' must be in the family in (4), which is a totally ordered set with respect to inclusion.

(6) Since $\mathscr{E} \neq \varnothing$, let $Q \notin \mathscr{E}$ and $\widehat{Q}^{(k)} := (\widehat{Q})^{\wedge \cdots \wedge}$. Then $\left|\widehat{Q}^{(k)}\right| = 2^{kn}|Q| \to +\infty$ as $k \to +\infty$. Since $|\Omega| < +\infty$, $\exists k_0 \in \mathbb{N}$ such that $\widehat{Q}^{(k_0)} \in \mathscr{F}$. (*one possible k_0 can be chosen as* $\max\{k \in \mathbb{N} : \widehat{Q}^{(k)} \subset \Omega\}$) Note that $\mathscr{F}$ is a partition of Ω, by the argument above, $\Omega \subset \bigcup_{Q'\in\mathscr{F}} Q' = \bigcup \mathscr{F}$.

Any $Q' \in \mathscr{F}$ is contained in Ω, thus $\bigcup \mathscr{F} \subset \Omega$. Let Q', $Q'' \in \mathscr{F}$ with $Q' \neq Q''$, then $Q' \subset Q'' \implies \widehat{Q'} \subset Q''$, which leads to a contradiction. Similarly, $Q'' \not\subset Q'$. Thus $Q' \cap Q'' = \varnothing$. □

Corollary 1.8: Maximal dyadic cubes

Suppose $\Omega \subset \mathbb{R}^n$ is an open set with $|\Omega| < +\infty$. There exists a unique maximal collection of dyadic cubes whose union is Ω. They are called the *maximal dyadic cubes* of Ω.

Proof. Let $\mathscr{E} = \{Q \in \mathscr{D} : Q \subset \Omega\}$. We claim that $\Omega = \bigcup \mathscr{E}$. $\bigcup \mathscr{E} \subset \Omega$ is trivial. For the other direction, since Ω is open, for $x \in \Omega$, one of the $Q_{j,k_j(x)} \subset \Omega$. Hence $\Omega = \bigcup_{x\in\Omega} \{x\} = \bigcup_{Q_{j,k_j(x)}\in\mathscr{E}} Q_{j,k_j(x)}$. Define $\mathscr{F} = \{Q \in \mathscr{D} : Q \subset \Omega,\ \widehat{Q} \not\subset \Omega\} \subset \mathscr{E}$. We have that $\mathscr{F}$ is a partition of Ω made of dyadic cubes which are maximal in Ω.

Then we prove the uniqueness. Let $\{Q_j\}$ and $\{Q'_k\}$ be two maximal collections. Argue by contradiction. For j, Q_j meets at least one Q'_k, thus $Q_j \subset Q'_k$ or $Q'_k \subset Q_j$. If $Q_j \subset Q'_k$ and $Q_j \neq Q'_k$, then $\widehat{Q_j} \subset Q'_k$ by definition. Hence $\widehat{Q_j} \subset \Omega$, it must meet some $Q_{j'}$, contradiction with the maximality of Q_j, thus $Q_j = Q'_k$. This implies that the two collections are the same. □

1.4 The Whitney covering with dyadic cubes

Let (X,d) be a metric space, $E \subset X$, $\operatorname{diam} E := \sup\{d(x,y) : x,y \in E\}$. For closed subsets $E,\ F \subset X$, define the distances

$$d(E,F) := \inf\{d(x,y) : x \in E,\ y \in F\}, \qquad d(x,E) := d(\{x\},E).$$

Theorem 1.9: Whitney dyadic cubes

Let $O \subset \mathbb{R}^n$ be a non-empty open set that is not $\mathbb{R}^n$. There exists a collection $\mathscr{F} = \{Q_i : i \in I\} \subset \mathscr{D}$ such that

(i) $\frac{1}{30} d(Q_i, O^c) \leqslant \operatorname{diam} Q_i \leqslant \frac{1}{10} d(Q_i, O^c)$;

(ii) $O = \bigcup_{i \in I} Q_i$.

(iii) $\{Q_i : i \in I\}$ are mutually disjoint.

The cubes Q_i are called the *Whitney dyadic cubes* for O.

Remark. *(1) We do not need $|O| < +\infty$ here because $d(\widehat{Q}^{(k)}, O^c)$ decreases as $\operatorname{diam}(\widehat{Q}^{(k)})$ increases. (2) We can change the upper bound $\frac{1}{10}$ to $\theta < \frac{1}{2}$. But this will change the lower bound $\frac{1}{30}$ to some μ.*

Proof. Define $\mathscr{E} := \{Q \subset \mathscr{D} : \operatorname{diam} Q \leqslant \frac{1}{10} d(Q_i, O^c)\}$. Note that $Q \in \mathscr{E}$ implies $Q \subset O$, and $\mathscr{E} \neq \varnothing$ has already seen in corollary 1.3. Thus there exists a collection $\mathscr{F} \subset \mathscr{E}$, which forms a partition of O (*maximal dyadic cubes of $\mathscr{E}$ for* $\operatorname{diam} Q \leqslant \frac{1}{10} d(Q, O^c)$.) It remains to check $\frac{1}{30} d(Q_i, O^c) \leqslant \operatorname{diam} Q_i$.

We know that $d(\widehat{Q_i}, O^c) < 10 \operatorname{diam} \widehat{Q_i}$, then

$$d(Q_i, O^c) \leqslant \operatorname{diam} Q_i + d(\widehat{Q_i}, O^c) \leqslant 11 \operatorname{diam} \widehat{Q_i} = 22 \operatorname{diam} Q_i \leqslant 30 \operatorname{diam} Q_i.$$

Then we conclude the proof.

Proposition 1.10: Whitney covering, dyadic cube case

The Whitney dyadic cubes satisfy:

(i) $3Q_i \subset O$.

(ii) If $3Q_i \cap 3Q_j \neq \varnothing$, then $\frac{1}{4} \leqslant \frac{\operatorname{diam} Q_i}{\operatorname{diam} Q_j} \leqslant 4$.

(iii) $\{3Q_i : i \in I\}$ have bounded overlap, *i.e.*, $\sum_{Q_i \in \mathscr{F}} \mathbb{1}_{3Q_i} \leqslant c(n) \in \mathbb{N}$.

Proof. (i) Argue by contradiction. Let $z \in 3Q_i \cap O^c$, then $z \in 3Q_i$ implies $\exists y \in Q_i\ (d(z,y) \leqslant \operatorname{diam} 3Q_i)$. Thus

$$10 \operatorname{diam} Q_i \leqslant d(Q_i, O^c) \leqslant d(y, O^c) \leqslant d(y,z) \leqslant \operatorname{diam} 3Q_i.$$

Contradiction.

(ii) It is enough to prove $\frac{\operatorname{diam} Q_i}{\operatorname{diam} Q_j} \leqslant 4$. Let $y \in 3Q_i \cap 3Q_j$, then $y \in 3Q_i$ implies $d(y, Q_i) \leqslant \operatorname{diam} Q_i$, thus

$$10 \operatorname{diam} Q_i \leqslant d(Q_i, O^c) \leqslant d(y, O^c) + d(Q_i, y) \leqslant d(y, O^c) + \operatorname{diam} Q_i.$$

Hence $d(y,O^c) \geqslant 9 \operatorname{diam} Q_i$. Also, $y \in 3Q_j$ implies $\exists z \in Q_j\,(d(y,z) \leqslant \operatorname{diam} Q_j)$, then

$$9 \operatorname{diam} Q_i \leqslant d(y,O^c) \leqslant d(y,z) + d(z,O^c).$$

For $w \in Q_j$,

$$d(z,O^c) \leqslant d(z,w) + d(w,O^c) \leqslant \operatorname{diam} Q_j + d(w,O^c).$$

Take infimum in w,

$$d(z,O^c) \leqslant \operatorname{diam} Q_j + 30 \operatorname{diam} Q_j = 31 \operatorname{diam} Q_j.$$

Hence $9 \operatorname{diam} Q_i \leqslant \operatorname{diam} Q_j + 31 \operatorname{diam} Q_j$, which implies $\frac{\operatorname{diam} Q_i}{\operatorname{diam} Q_j} \leqslant 4$.

(iii) Fix $i \in I$. Define $A_i := \left\{ j \in I : 3Q_i \cap 3Q_j \neq \varnothing \right\}$. It is enough to show $\# A_i \leqslant c(n)$, where $c(n)$ is a constant uniform with i. Let $j \in A_i$ and $y \in 3Q_i \cap 3Q_j$.

$$\begin{cases} d(y,Q_i) \leqslant \operatorname{diam} Q_i, \\ d(y,Q_j) \leqslant \operatorname{diam} Q_j, \end{cases} \implies d(Q_i,Q_j) \leqslant \operatorname{diam} Q_i + \operatorname{diam} Q_j.$$

For $k \in K = [\![-2,2]\!]$, define $A_i^k := \left\{ j \in A_i : \operatorname{diam} Q_j = 2^k \operatorname{diam} Q_i \right\}$. $A_i = \bigcup_{k=-2}^{2} A_i^k$. For $k \in K$,

$$d(Q_i,Q_j) \leqslant (1+2^k) \operatorname{diam} Q_i \leqslant 5 \operatorname{diam} Q_i \implies Q_j \subset 10Q_i.$$

The Q_j are mutually disjoint, do the volume counting argument again, $\# A_i^k \leqslant \frac{|10Q_i|}{2^{kn}|Q_i|} = (10/2^k)^n \leqslant 40^n$. □

1.5 Whitney covering in metric spaces

Theorem 1.11

Let (E,d) be a metric space and O open, $O \subset E$, $O \neq E$. There exists $\mathscr{E} = \{B_\alpha\}_{\alpha \in I}$ and $c_1 \geqslant 1$, independent of O, such that

(i) $\{B_\alpha\}_{\alpha \in I}$ are mutually disjoint.

(ii) $O = \bigcup_{\alpha \in I} c_1 B_\alpha$.

(iii) $4c_1 B_\alpha \not\subset O$.

Moreover, if (E,d) is separable, then I is at most countable.

Proof. Define $\delta(x) = d(x,O^c)$, then $x \in O$ implies $\delta(x) > 0$. Let $\varepsilon \in (0,\frac{1}{2})$, define

$$\mathscr{B} := \{ B(x,\delta(x)) : x \in O \}.$$

Note that $B(x,\varepsilon\delta(x)) \subset B(x,\delta(x)) \subset O$. Use Zorn's lemma, $\exists \mathscr{E} = \{B_\alpha\}_{\alpha \in I} \subset \mathscr{B}$ the maximal collection of mutually disjoint balls. Let $r_\alpha = \varepsilon\delta(x_\alpha)$, $B_\alpha = B(x_\alpha, r_\alpha)$, and $c_1 = 1/2\varepsilon > 1$. Then

$$4c_1 B_\alpha = B(x_\alpha, 4c_1\varepsilon\delta(x_\alpha)) = B\left(x_\alpha, 4 \cdot \frac{1}{2\varepsilon} \cdot \varepsilon\delta(x_\alpha)\right) = B(x_\alpha, 2\delta(x_\alpha)) \not\subset O.$$

(ii) Argue by contradiction. Suppose $x \in O \setminus \bigcup_{\alpha \in I} c_1 B_\alpha$. By maximality, $\exists \beta \in I$, $B(x,r(x)) \cap B(x_\beta, r(x_\beta)) \neq \varnothing$. Thus

$$\begin{cases} d(x,x_\beta) \leqslant r(x) + r(x_\beta) = \varepsilon(\delta(x) + \delta(x_\beta)), \\ x \notin c_1 B_\beta \implies d(x,x_\beta) \geqslant \frac{1}{2}\delta(x_\beta), \end{cases} \implies \delta(x_\beta) \leqslant \frac{\varepsilon}{\frac{1}{2} - \varepsilon}\delta(x).$$

Note that $B(x_\beta, 2\delta(x_\beta)) \subset B(x, 2\delta(x_\beta) + d(x,x_\beta))$, we have

$$2\delta(x_\beta) + d(x, x\beta) \leqslant \left(\frac{2\varepsilon}{\frac{1}{2} - \varepsilon} + \varepsilon + \frac{\varepsilon^2}{\frac{1}{2} - \varepsilon} \right) \delta(x).$$

Pick $\varepsilon \in (0, \frac{1}{2})$ such that $\eta := \frac{2\varepsilon}{\frac{1}{2}-\varepsilon} + \varepsilon + \frac{\varepsilon^2}{\frac{1}{2}-\varepsilon} < 1$. $B(x_\beta, 2\delta(x_\beta)) \subset B(x, \eta\delta(x)) \subset O$, which is false by (iii).

For the *moreover* part, pick $D \subset E$ a countable dense subset of E. Each ball B_α are open by contradiction, hence $B_\alpha \cap D \neq \varnothing$. Let $d_\alpha \in B_\alpha \cap D$, then

$$\alpha \neq \beta \implies B_\alpha \cap B_\beta = \varnothing \implies d_\alpha \neq d_\beta.$$

Hence $I \to D$, $\alpha \mapsto d_\alpha$ is injective, thus I is at most countable. □

Proposition 1.12

If $(E, d) = (\mathbb{R}^n, d_E)$, the balls $\{c_1 B_\alpha\}_{\alpha \in I}$ has the bounded overlap property. Recall that $c_1 = 1/2\varepsilon$ in the previous theorem.

Proof. Use the volume counting argument, for $\alpha \in I$, define

$$A_\alpha := \left\{\beta \in I : c_1 B_\alpha \cap c_1 B_\beta \neq \varnothing\right\}.$$

We want $\# A_\alpha \leqslant c$ independent of α. We then show that if $\beta \in A_\alpha$, $\frac{1}{3} \leqslant \frac{\delta(x_\beta)}{\delta(x_\alpha)} \leqslant 3$.

Pick $z \in c_1 B_\alpha \cap c_1 B_\beta$, $d(z, x_\beta) \leqslant c_1 r(B_\beta) = \frac{1}{2\varepsilon}\varepsilon\delta(x_\beta) = \frac{1}{2}\delta(x_\beta)$. By triangle inequality, $d(z, O^c) \geqslant \frac{1}{2}\delta(x_\beta)$. Then

$$d(z, O^c) \leqslant d(z, x_\alpha) + d(x_\alpha, O^c) \leqslant \frac{1}{2}\delta(x_\alpha) + \delta(x_\alpha) = \frac{3}{2}\delta(x_\alpha).$$

Thus $\delta(x_\beta) \leqslant 3\delta(x_\alpha)$ and by symmetry $\delta(x_\alpha) \leqslant 3\delta(x_\beta)$.

We have known that $B(x_\beta, \varepsilon\delta(x_\beta))$ disjoint if $\beta \in I$. Thus $B(x_\beta, \frac{\varepsilon}{3}\delta(x_\alpha))$ are disjoint if $\beta \in A_\alpha$. Thus

$$B\left(x_\beta, \frac{\varepsilon}{3}\delta(x_\alpha)\right) \subset B(x_\alpha, \frac{\varepsilon}{3}\delta(x_\alpha) + \frac{1}{2}(\delta(x_\alpha) + \delta(x_\beta))) \subset B(x_\alpha, \left(\frac{\varepsilon}{3} + 2\right)\delta(x_\alpha)).$$

Hence $\# A_\alpha \leqslant \frac{(\varepsilon/3+2)^n}{(\varepsilon/3)^n}$ only depends on the dimension. □

2 Maximal Functions

Setup. We shall use the covering method developed in the previous chapter to solve the problems we met in maximal functions. In summary, we have

Maximal Function	Covering method
centred maximal function M_c	Besicovitch theorem
uncentred maximal function M	Vitali covering lemma
dyadic maximal function M_d	Whitney covering theorem

For convention, $0/0 := 0$.

2.1 Centred maximal functions

Let μ be a positive, Borel, locally finite measure on $\mathbb{R}^n$, ν be a positive Borel measure. Consider the ratio for open balls, for $x \in \mathbb{R}^n$,

$$M_c(\nu)(x) := \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \in [0, +\infty].$$

And if $f \in L^1_{\text{loc}}(\mu)$, denote $M_c(f) := M_c(\nu)$ where $\mathrm{d}\nu = f\,\mathrm{d}\mu$.

Lemma 2.1

The function $x \mapsto M_c(\nu)(x)$ is lower semi-continuous, *i.e.*, $\forall \lambda > 0$, $\{x \in \mathbb{R}^n : M_c(\nu)(x) > \lambda\}$ is open. Hence it is a Borel function.

Remark. M_c *measure the relative size of* ν *with respect to* μ.

Theorem 2.2

There exists a constant $c = c(n) > 0$(*the Besicovitch constant actually*) such that $\forall \lambda > 0$,

$$\mu\{M_c(\nu) > \lambda\} \leqslant \frac{c}{\lambda}\nu(\mathbb{R}^n).$$

In particular, for $f \in L^1_{\text{loc}}(\mu)$,

$$\mu\{M_c(f) > \lambda\} \leqslant \frac{c}{\lambda}\int_{\mathbb{R}^n} |f| \,\mathrm{d}\mu = \frac{c}{\lambda}\|f\|_{L^1(\mu)}.$$

Proof. Let $O_\lambda = \{M_c(\nu) > \lambda\}$, then $x \in O_\lambda$ implies $\exists B_x = B(x, r_x)$ such that $\frac{\nu(B_x)}{\mu(B_x)} > \lambda$. Fix $R > 0$ and apply Besicowitch's theorem to $O_\lambda \cap B(0,R)$, there exists $E_0 \subset O_\lambda \cap B(0,R)$ at most countable, such that $O_\lambda \cap B(0,R) \subset \bigcup_{x\in E_0} B_x$ and $\sum_{x\in E_0} \mathbb{1}_{B_x} \leqslant c(n)$. Thus

$$\begin{aligned}\mu(O_\lambda \cap B(0,R)) \sum_{x\in E_0} \mu(B_x) &\leqslant \frac{1}{\lambda}\sum_{x\in E_0} \nu(B_x) = \frac{1}{\lambda}\sum_{x\in E_0}\int_{\mathbb{R}^n} \mathbb{1}_{B_x}\,\mathrm{d}\nu \\ &= \frac{1}{\lambda}\int_{\mathbb{R}^n}\sum_{x\in E_0}\mathbb{1}_{B_x}\,\mathrm{d}\nu \leqslant \frac{1}{\lambda}\int_{\mathbb{R}^n} c(n)\,\mathrm{d}\nu = \frac{c(n)}{\lambda}\nu(\mathbb{R}^n).\end{aligned}$$

Let $R \nearrow +\infty$, then $\mu(O_\lambda) = \lim_{R\to+\infty} \mu(O_\lambda) \cap B(0,R) \leqslant \frac{c(n)}{\lambda}\nu(\mathbb{R}^n)$. ☐

Corollary 2.3: Lebesgue differentation

Let $f \in L^1_{\text{loc}}(\mu)$, then $\exists L_f \subset \mathbb{R}^n$ such that $\mu(L_f^c) = 0$ and $\forall x \in L_f$,

$$\lim_{r\to 0} \fint_{B(x,r)} |f(x) - f(y)| \,\mathrm{d}\mu(y) = 0.$$

In particular, for μ-a.e. $x \in X$

$$f(x) = \lim_{r\to 0} \fint_{B(x,r)} f(y)\,\mathrm{d}\mu(y).$$

Proof. Note that this is a local statement, hence one can always work on bounded sets on which f is integrable. We assume instead $f \in L^1(\mu)$ by replacing f by $f\mathbb{1}_K$ for some compact K. Moreover, if f is continuous, we may take $L_f = \mathbb{R}^n$.

Since μ is locally finite, $C_c(\mathbb{R}^n)$ is dense in $L^1(\mu)$. So we fix $f \in L^1(\mu)$, for any $\varepsilon > 0$, $\exists g \in C_c(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |f - g|\,\mathrm{d}\mu < \varepsilon$. Set

$$\omega_f(x) = \limsup_{r\to 0} \fint_{B(x,r)} |f(x) - f(y)|\,\mathrm{d}\mu(y), \qquad x \in \mathbb{R}^n.$$

We want $L_f = \{\omega_f = 0\}$, *i.e.*, we need to prove $\{\omega_f > 0\}$ is a μ-null set. It suffices to show that $\forall \lambda > 0$, $\{\omega_f > \lambda\}$ is a μ-null set.

Since $g \in C_c(\mathbb{R}^n)$, $\omega_g = 0$. And for $f \in L^1(\mu)$, $\omega_f \leqslant |f| + M_c(f)$. Then

$$\omega_f \leqslant \omega_{f-g} + \omega_g \leqslant |f - g| + M_c(f - g).$$

Hence

$$\{\omega_f > \lambda\} \subset \left\{|f-g| > \frac{\lambda}{2}\right\} \cup \left\{M_c(f-g) > \frac{\lambda}{2}\right\}.$$

We have

$$\mu\left\{|f-g| > \frac{\lambda}{2}\right\} \leqslant \frac{2}{\lambda}\int_{\mathbb{R}^n} |f-g|\,\mathrm{d}\mu \leqslant \frac{2}{\lambda}\varepsilon \qquad \text{(by Markov)}$$

$$\mu\left\{M_c(f-g) > \frac{\lambda}{2}\right\} \leqslant \frac{2c(n)}{\lambda}\int_{\mathbb{R}^n} |f-g|\,\mathrm{d}\mu \leqslant \frac{2c(n)}{\lambda}\varepsilon \qquad \text{(by maximal thm)}$$

Thus $\mu\{\omega_f > \lambda\} \leqslant \frac{2}{\lambda}(1+c(n))\varepsilon$ is true for all $\varepsilon > 0$, so $\{\omega_f > \lambda\}$ is a μ-null set for all $\lambda > 0$. □

2.2 Uncentred maximal function for doubling measures

Definition 2.4: Doubling measures

Let μ be a Borel, positive, locally finite measure on $\mathbb{R}^n$, μ is said to be a *doubling measure* if

$$\exists c > 0\, \forall x \in \mathbb{R}^n\, \forall r > 0\, (\mu(B(x,2r)) \leqslant c_\mu \mu(B(x,r))),$$

where c_μ is a constant depending on the measure μ.

If μ is the Lebesgue measure on $\mathbb{R}^n$, then one can choose $c_\mu = 2^n$. So Lebesgue measure is a doubling measure.

Let μ, ν be as before, define

$$M\nu(x) := \sup\left\{\frac{\nu(B)}{\mu(B)} : B \ni x \text{ is an open ball}\right\}.$$

Remark. *One has $M_c\nu \leqslant M\nu$ and $M\nu \leqslant c(n)M_c\nu$ when μ is doubling, where $c(n)$ is a constant.*

The doubling condition allows us to use Vitali's covering theorem.

Lemma 2.5

Let B be a collection of open sets and $a_B \in [0,+\infty]$ for $B \in \mathscr{B}$, then

$$g : \mathbb{R}^n \to [0,+\infty], \qquad x \mapsto \sup\{a_B : B \ni x \text{ is an open ball}\}$$

is lower semi-continuous.

Proof. Fix $\lambda > 0$ and $x \in \mathbb{R}^n$ with $g(x) > \lambda$. $\exists B \in \mathscr{B}$, $B \ni x$ such that $a_B > \lambda$. Thus $\forall y \in B$, $g(y) \geqslant a_B > \lambda$. Thus $\{g > \lambda\}$ contains B for all open sets B containing x, hence $\{g > \lambda\}$ is an open set. □

Corollary 2.6

$M\nu$ is lower semi-continuous.

Theorem 2.7

Assume that μ is doubling, there exists a constant c depending on c_μ, such that

$$\mu\{M\nu > \lambda\} \leqslant \frac{c}{\lambda}\nu(\mathbb{R}^n), \qquad \forall \lambda > 0. \tag{2.1}$$

In particular, for $f \in L^1_{\mathrm{loc}}(\mu)$, $Mf = M\nu$ for $\mathrm{d}\nu = |f|\,\mathrm{d}\mu$,

$$\mu\{Mf > \lambda\} \leqslant \frac{c}{\lambda}\int_{\mathbb{R}^n} |f|\,\mathrm{d}\mu.$$

Proof. Fix $k \in \mathbb{N}^*$, define $M_k(\nu) = \sup\left\{\frac{\nu(B)}{\mu(B)} : \text{open ball } B \ni x, r(B) \leqslant k\right\}$. By the previous lemma, $M_k(\nu)$ is lower semi-continuous. Let $k \nearrow +\infty$, then $M_k(\nu) \nearrow M\nu$. So it is enough to prove (2.1) for each $k \in \mathbb{N}^*$.

Let $\lambda > 0$ and $O_\lambda^k = \{M_k(\nu) > \lambda\}$. For all $x \in O_\lambda^k$, $\exists B_x$ open ball with $r(B_x) \leqslant k$ such that $\frac{\nu(B_x)}{\mu(B_x)} > \lambda$. $\{B_x : x \in O_\lambda^k\}$ is a cover of O_λ^k with $r(B_x) \leqslant k$. By Vitali theorem, $\exists E_0 \subset O_\lambda^k$ at most countable(*by volume counting argument for doubling measure for* μ), such that $O_\lambda^k \subset \bigcup_{x\in E_0} 5B_x$. Then

$$\mu(O_\lambda^k) \leqslant \sum_{x\in E_0} \mu(5B_x) \leqslant c_\mu^3 \sum_{x\in E_0} \mu(B_x) \leqslant \frac{c_\mu^3}{\lambda} \sum_{x\in E_0} \nu(B_x) = \frac{c_\mu^3}{\lambda} \nu\left(\bigcup_{x\in E_0} B_x\right) \leqslant \frac{c_\mu^3}{\lambda} \nu(O_\lambda^k) \leqslant \frac{c_\mu^3}{\lambda} \nu(O_\lambda),$$

because $\{B_x\}_{x\in E_0}$ are disjoint and $B_x \subset O_\lambda^k$. Let $k \nearrow +\infty$, we have $\mu(O_\lambda) \leqslant \frac{c_\mu^3}{\lambda} \nu(O_\lambda)$. $\square$

Corollary 2.8: Lebesgue differentation for doubling measures

For $f \in L^1_{\text{loc}}(\mu)$, one has

$$f(x) = \lim_{r(B)\to 0, B\ni x} \fint_B f(y)\, \mathrm{d}\mu(y)$$

for μ-a.e. $x \in X$.

2.3 Dyadic maximal function

Let μ be a Borel, positive, locally finite measure, ν be a positive Borel measure. Let $\mathscr{D}$ be the set of dyadic cubes with reference cube Q_0, $\mathscr{D}(Q_0) \subset \mathscr{D}$ is the collection of dyadic subcubes of Q_0. Define *dyadic maximal functions*

$$M_{d,Q_0}\nu(x) := \sup\left\{\frac{\nu(Q)}{\mu(Q)} : Q \in \mathscr{D}(Q_0),\ Q \ni x\right\}, \quad x \in Q_0,$$
$$M_d\nu(x) := \sup\left\{\frac{\nu(Q)}{\mu(Q)} : Q \in \mathscr{D},\ Q \ni x\right\}, \quad x \in \mathbb{R}^n.$$

Lemma 2.9

If $\lambda > 0$, then $\{M_{d,Q_0}\nu > \lambda\}$ is a union of dyadic subcubes of Q_0; $\{M_d\nu > \lambda\}$ is a union of dyadic cubes in $\mathscr{D}$.

Hence $M_{d,Q_0}\nu$ is Borel measurable on Q_0.

Theorem 2.10

For all $\lambda > 0$, one has

$$\mu\{M_{d,Q_0}\nu > \lambda\} \leqslant \frac{\nu(Q_0)}{\lambda}.$$

(The constant here is exactly 1) Hence for $f \in L^1_{\text{loc}}(Q_0, \mu)$,

$$\mu\{M_{d,Q_0} f > \lambda\} \leqslant \frac{1}{\lambda} \int_{Q_0} |f|\, \mathrm{d}\mu.$$

Proof. Let

$$\Omega_\lambda = \{x \in Q_0 : M_{d,Q_0}\nu(x) > \lambda\}.$$

If $\Omega_\lambda = \varnothing$, done. Otherwise, $\Omega_\lambda \neq \varnothing$, by the previous lemma, Ω_λ is a the union of all dyadic subcubes for Q_0 such that $\nu(Q)/\mu(Q) > \lambda$. Since $|\Omega_\lambda| < +\infty$, let $\mathscr{F}$ be the maximal subcollection. Then $\Omega_\lambda = \bigcup_{Q\in\mathscr{F}} Q$, where $Q \in \mathscr{F}$ are mutually disjoint. Thus

$$\mu(\Omega_\lambda) = \sum_{Q\in\mathscr{F}} \mu(Q) \leqslant \frac{1}{\lambda} \sum_{Q\in\mathscr{F}} \nu(Q) = \frac{1}{\lambda}\nu(\Omega_\lambda)$$

because $\{Q : Q \in \mathscr{F}\}$ is a partition. □

Remark. Now we consider dyadic cubes on $\mathbb{R}$. *Let* $Q_0 = [0,1)$, $Q_k^+ = [0,2^k)$, $Q_k^- = [-2^k,0)$. *Then*

$$M_d\nu(x) = \begin{cases} \sup_{k\geqslant 0} M_{d,Q_k^+}\nu(x), & x \geqslant 0, \\ \sup_{k\geqslant 0} M_{d,Q_k^-}\nu(x), & x < 0. \end{cases}$$

For $x \geqslant 0$, *one has*

$$\mu(\Omega_\lambda^k) \leqslant \frac{1}{\lambda}\nu(\Omega_\lambda^k) \leqslant \nu(\Omega_\lambda^+) \implies \mu(\Omega_\lambda^+) \leqslant \frac{1}{\lambda}\nu(\Omega_\lambda^+).$$

And similar for $x < 0$, $\mu(\Omega_\lambda^-) \leqslant \frac{1}{\lambda}\nu(\Omega_\lambda^-)$. *We then use the fact that* $\Omega_\lambda = \Omega_\lambda^+ \cup \Omega_\lambda^-$.

We provide a new proof of the theorem using stopping time argument. Denote the condition $\frac{\nu(Q)}{\mu(Q)} > \lambda$ as (†).

Proof. If Q_0 satisfies (†), then $\Omega_\lambda = Q_0$, done.

If Q_0 does not satisfy (†), then we divide Q_0 dyadically. Consider two dyadic condition Q of Q_0, and test (†) for Q:

- If (†) holds for Q, we select Q and stop.
- If (†) does not hold for Q, we divide Q dyadically and continue.

Denote $\mathscr{F}'$ the set of selected dyadic subcubes, we claim that

(1) The cubes on $\mathscr{F}'$ are mutually disjoint.

(2) They form a partition of Ω_λ, *i.e.*, $\Omega_\lambda = \bigcup \mathscr{F}'$.

Now $Q \in \mathscr{F}'$ implies $\frac{\nu(Q)}{\mu(Q)} > \lambda$, hence $Q \subset \Omega_\lambda$. If $x \in \Omega_\lambda \setminus \bigcup \mathscr{F}'$, then $x \in \Omega_\lambda$ implies $\exists Q' \in \mathscr{D}(Q_0)$ such that $\frac{\nu(Q)}{\mu(Q)} > \lambda$ and $Q' \ni x$. Then Q' satisfied (†) but was not selected. Hence one of its ancestors must have been selected, hence $Q' \subset \bigcup \mathscr{F}'$, contradiction.

Then by the condition (†),

$$\mu(\Omega_\lambda) = \sum_{Q\in\mathscr{F}'} \mu(Q) \leqslant \frac{1}{\lambda} \sum_{Q\in\mathscr{F}} \nu(Q) = \frac{1}{\lambda}\nu(\Omega_\lambda).$$

So $\mathscr{F}'$ is the collection that we want. □

We obtained two collection of dyadic cubes: $\mathscr{F}$ in the first proof and $\mathscr{F}'$ in the second proof. They are actuallly the same.

Corollary 2.11

For all $f \in L^1_{\text{loc}}(Q_0,\mu)$, one has

$$f(x) = \lim_{x\in Q\in\mathscr{D}(Q_0),\ \operatorname{diam} Q\to 0} \fint_Q f \,\mathrm{d}\mu$$

for μ-a.e. $x \in Q_0$.

Proof. The same as the case M_c, but much easier because the limit is along a sequence. □

2.4 Consequence of Lebesgue differentation

Proposition 2.12

Under the assumption yielding, the Lebesgue differentation (3) results μ-a.e. for $f \in L^1_{\text{loc}}(\mu)$,

$$|f| \leqslant M_c f, \qquad |f| \leqslant M f, \qquad |f| \leqslant M_{d,Q_0} f.$$

Proof. For $|f| \leqslant M_c f$,

$$|f(x)| = \left|\lim_{r\to 0} \fint_{B(x,r)} f(y)\,\mathrm{d}\mu(y)\right| \leqslant \sup_{r>0} \fint_{B(x,r)} |f(y)|\,\mathrm{d}\mu(y) = M_c f(x)$$

for μ-a.e. x. The same proof holds for Mf and $M_{d,Q_0} f$. □

2.5 Maximal function on L^p spaces

Let $\mathscr{M}$ denote either M_c, M or M_{d,Q_0}. We always have for $f \in L^\infty(\mu)$,

$$\mathscr{M} f(x) \leqslant \|f\|_\infty, \qquad \forall x \in \mathbb{R}^n.$$

Hence $\mathscr{M} : L^\infty(\mu) \to L^\infty(\mu)$ is a bounded operator.

Question. Is $\mathscr{M}$ bounded on $L^p(\mu)$ when $1 \leqslant p < +\infty$?

Regretfully, for $p = 1$, $\mathscr{M}$ is never bounded.

Lemma 2.13: Cavalieri's principle

Let μ be a positive measure on set E, $0 < p < +\infty$. $g : E \to [0, +\infty]$ is measurable. Then

$$\int_{\mathbb{R}^n} g^p \,\mathrm{d}\mu = p \cdot \int_0^\infty \mu\{g > \lambda\}\, \lambda^{p-1}\,\mathrm{d}\lambda.$$

Proof. Use Fubini theorem, one has

$$\begin{aligned} p \int_0^\infty \mu\{g > \lambda\}\, \lambda^{p-1}\,\mathrm{d}\lambda &= p \int_0^\infty \int_{\{g>\lambda\}} \mathbb{1}_{\{g>\lambda\}}\,\mathrm{d}\mu \lambda^{p-1}\,\mathrm{d}\lambda \\ &= \int_{\{g>\lambda\}} \int_0^{g(x)} p\lambda^{p-1}\,\mathrm{d}\lambda \mathbb{1}_{\{g>\lambda\}}\,\mathrm{d}\mu = \int_{\{g>\lambda\}} g^p\,\mathrm{d}\mu. \end{aligned}$$

But this only works for σ-finite measures. Either we assume μ is σ-finite, or we prove this for $g_{N,R} = g\mathbb{1}_{\{g\leqslant N\}}\mathbb{1}_{B(0,R)}$, which is a bounded function with bounded support. Then it is enough by monotone convergence theorem as $N \to +\infty$ and $R \to +\infty$. Denote $\mathscr{B}$ the σ-algebra generated by $\{g_{N,R} : N \in \mathbb{N},\ R > 0\}$, and the restriction of μ on the σ-algebra $\mu|_{\mathscr{B}}$ becomes σ-finite. We repeat the above argument. □

Theorem 2.14

Let μ be a Borel, positive, locally finite measure. Let $1 < p < +\infty$, $\exists c_1,\ c_2 > 0$ such that $\forall f \in L^\infty(\mu)$,

$$\|M_c f\|_p \leqslant c_1 \|f\|_p, \qquad \|M_d f\|_p \leqslant \frac{p}{p-1} \|f\|_p,$$

and if μ is doubling, $\|Mf\|_p \leqslant c_2 \|f\|_p$.

Proof. Let c be the best constant in the maximal inequality,

$$\mu\{\mathscr{M} f > \lambda\} \leqslant \frac{c}{\lambda} \|f\|_1, \qquad \forall f \in L^1(\mu),$$

the *best* means that $c=\sup\left\{\frac{\mu\{\mathscr{M}f>\lambda\}}{\|f\|_1}:\lambda>0,\ f\in L^1(\mu),\ f\neq 0\right\}$. Let $\lambda>0$ and $f\in L^1(\mu)\cap L^\infty(\mu)$, set

$$f_\lambda(x)=\begin{cases} f(x), & \text{if } |f(x)|>\lambda/2,\\ 0, & \text{if } |f(x)|\leqslant\lambda/2.\end{cases}$$

We have $|f|\leqslant|f_\lambda|+\frac{\lambda}{2}$. Thus by subadditivity of sup, $\mathscr{M}f\leqslant\mathscr{M}f_\lambda+\frac{\lambda}{2}$. Note that $\{\mathscr{M}f>\lambda\}\subset\{\mathscr{M}f_\lambda>\frac{\lambda}{2}\}$, we have

$$\mu\{\mathscr{M}f>\lambda\}\leqslant\mu\left\{\mathscr{M}f_\lambda>\frac{\lambda}{2}\right\}\leqslant\frac{c}{\lambda/2}\int_{\mathbb{R}^n}|f_\lambda(x)|\,\mathrm{d}\mu(x).$$

The Cavalieri's principle tolds that

$$\begin{aligned}\int_{\mathbb{R}^n}|\mathscr{M}f|^p\,\mathrm{d}\mu\leqslant p\int_0^\infty\frac{c}{\lambda/2}\int_{\mathbb{R}^n}|f_\lambda(x)|\,\mathrm{d}\mu(x)\lambda^{p-1}\,\mathrm{d}\lambda&=\int_{\mathbb{R}^n}2c\int_0^{2|f(x)|}p\lambda^{p-2}\,\mathrm{d}\lambda\,\mathrm{d}\mu(x)\\ =2c\frac{p}{p-1}\int_{\mathbb{R}^n}|2f(x)|^{p-1}|f(x)|\,\mathrm{d}\mu(x)&=2^pc\frac{p}{p-1}\int_{\mathbb{R}^n}|f|^p\,\mathrm{d}\mu.\end{aligned}$$

For $f\in L^p(\mu)$, apply to simple functions with $|f_k|\nearrow|f|$, then

$$\int_{\mathbb{R}^n}|f_k|\,\mathrm{d}\mu\leqslant\int_{\mathbb{R}^n}|f|\,\mathrm{d}\mu,\qquad\mathscr{M}f_k\nearrow\mathscr{M}f,\qquad\forall k\in\mathbb{N}.$$

And we use monotonic convergence theorem, done. □

We also provide a proof for M_{d,Q_0}, and it also works for the uncentred maximal functions.

Proof. We do the proof again but for M_{d,Q_0} with $Q_0\subset\mathbb{R}^n$ the reference cube. By Cavalieri's principle,

$$\begin{aligned}\int_{Q_0}|M_{d,Q_0}f|^p\,\mathrm{d}\mu&=p\int_0^\infty\mu\{M_{d,Q_0}f>\lambda\}\lambda^{p-1}\,\mathrm{d}\lambda\leqslant p\int_0^\infty\frac{1}{\lambda}\int_{\{M_{d,Q_0}f>\lambda\}}|f|\,\mathrm{d}\mu\lambda^{p-1}\,\mathrm{d}\mu\\ &=\int_{Q_0}\int_0^{M_{d,Q_0}f(x)}p\lambda^{p-2}\,\mathrm{d}\lambda\,|f(x)|\,\mathrm{d}\mu(x)=\int_{Q_0}\frac{p}{p-1}(M_{d,Q_0}f(x))^{p-1}|f(x)|\,\mathrm{d}\mu(x).\end{aligned}$$

Apply Hölder inequality, denote $p'=\frac{p}{p-1}$, and note that $\|(\mathscr{M}f)^{p-1}\|_{p'}=\left(\int|\mathscr{M}f|^{p-1}\right)^{(p-1)/p}=\left(\int|\mathscr{M}f|^p\right)$, we have

$$\begin{aligned}\int_{Q_0}|M_{d,Q_0}f|^p\,\mathrm{d}\mu&=\int_{Q_0}\frac{p}{p-1}(M_{d,Q_0}f(x))^{p-1}|f(x)|\,\mathrm{d}\mu(x)\\ &\leqslant\frac{p}{p-1}\left(\int_{Q_0}|M_{d,Q_0}f|^{p-1}\right)^{p/(p-1)}\|f\|_p=\frac{p}{p-1}\|M_{d,Q_0}f\|_p^{p-1}\|f\|_p.\end{aligned}$$

If Q_0 is a cube, apply this to $f_N=f\mathbb{1}_{\{|f|\leqslant N\}}\subset L^\infty(Q_0,\mu)$, then $M_{d,Q_0}f_N\in L^\infty(Q_0,\mu)\subset L^p(Q_0,\mu)$. Simplify and let $N\to+\infty$.

If $Q_0=\mathbb{R}^n$, apply this to $f_{N,k}=f\mathbb{1}_{|f|\leqslant N}\mathbb{1}_{[-2^k,2^k)^n}\in L^\infty(\mathbb{R}^n,\mu)$ with compact support. Simplify and let $N\to+\infty$, $k\to+\infty$. □

2.6 Application to Hardy–Littlewood–Sobolev inequality

For $\lambda>0$, define

$$v_\lambda=\begin{cases}|x|^{-\lambda}, & \text{if } x\in\mathbb{R}^n\setminus\{0\},\\ 0, & \text{if } x=0.\end{cases}$$

Then $v_\lambda\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}x)\iff\lambda<n$. For $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{d}x)$, $f\geqslant 0$ and $\lambda>0$,

$$(v_\lambda*f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^\lambda}\,\mathrm{d}\lambda$$

exists $\mathrm{d}x$-a.e., and $(v_\lambda*f)(x)\in[0,+\infty]$.

Remark. *The function v_λ arises in mathematical physics:*

- *If $\lambda = 2$, $|x|^{-2}$ is the Coulomb potential.*
- *If $\lambda = n-2$, $|x|^{-(n-2)}$ is related to the Laplacian.*

For $f \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\widehat{v_{n-2} * f}(\xi) = c(n)\frac{\hat{f}(\xi)}{|\xi|^2}, \qquad \xi \neq 0,\ n \geqslant 3,$$

and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Also, $\widehat{\Delta f}(\xi) = -|\xi|^2 f(\xi)$. Hence $v_{n-2} * f$ acts like « intergrating twice ».

Theorem 2.15

For $0 < \lambda < n$ and $1 < p < \frac{n}{n-\lambda}$, and q such that $\frac{1}{p} + \frac{\lambda}{n} = 1 + \frac{1}{q}$. Then

(1) For $p = 1$, $\forall u \in L^1(\mathbb{R}^n, \mathrm{d}x)$, $\forall \alpha > 0$, $|\{v_\lambda * u > \alpha\}| \leqslant \frac{c}{\alpha^{n/\lambda}} \|u\|_1^q$.

(2) For $1 < p < \frac{n}{n-\lambda}$, $\forall u \in L^p(\mathbb{R}^n, \mathrm{d}x)$, $\|v_\lambda * u\|_q \leqslant c(n,p,q)\|u\|_p$.

Proof. (1) We first give the *Hedburg's inequality*: For $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{d}x)$ and for all $x \in \mathbb{R}^n$, we have

$$(v_\lambda * |u|)(x) \leqslant K(n,p,q)\|u\|_q^{1-p/q}(M_c u(x))^{p/q}, \tag{H}$$

where $K(n,p,q)$ is a constant. Assume (H) holds, when $p = 1$, $u \in L^1(\mathbb{R}^n, \mathrm{d}x)$ and $\|u\|_1 \neq 0$, denote $q = \frac{n}{\lambda}$, the inequality implies

$$(v_\lambda * |u|)(x) > \alpha \implies M_c u(x) > \frac{\alpha^q}{K(n,p,q)^q} \cdot \frac{1}{\|u\|_1^{q-1}}.$$

Apply maximal inequality,

$$|\{v_\lambda * |u| > \alpha\}| \leqslant \left(\frac{cK(n,p,q)^q}{\alpha^q}\|u\|_1^{q-1}\right)\|u\|_1 = \left(\frac{cK(n,p,q)}{\alpha}\right)^q \|u\|_1^q.$$

Thus $v_\lambda * |u|$ is defined a.e.. We can remove the absolute value in u.

When $1 < p < \frac{n}{n-\lambda}$,

$$\|v_\lambda * |u|\|_q^q \leqslant K(n,p,q)^q \|u\|_q^{q-p}\|M_c u\|_p^p \leqslant K(n,p,q)^q\|u\|_p^{q-p}(c_1\|u\|_p)^p = c_1^p K(n,p,q)^q \|u\|_p^q.$$

Thus $v_\lambda * |u|$ is defined a.e.. We can also remove the absolute value in u.

Now it suffices to prove (H). Assume $u \geqslant 0$,

$$(v_\lambda * u)(x) = \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^\lambda}\mathrm{d}y = \int_{|x-y|\geqslant\delta} \frac{u(y)}{|x-y|^\lambda}\mathrm{d}y + \sum_{k\geqslant 0}\int_{2^{-(k+1)}\delta\leqslant|x-y|\leqslant 2^{-k}\delta} \frac{u(y)}{|x-y|^\lambda}\mathrm{d}y.$$

When $p = 1$,

$$\int_{|x-y|\geqslant\delta} \frac{u(y)}{|x-y|^\lambda}\mathrm{d}y \leqslant \frac{1}{\delta^\lambda}\|u\|_1.$$

When $1 < p < \frac{n}{n-\lambda}$, use Hölder inequality, and denote q, p' such that

$$\frac{1}{p} + \frac{\lambda}{n} = 1 + \frac{1}{q}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

thus $\lambda p' = n + \frac{np'}{q} > n$. Hence

$$\int_{|x-y|\geqslant\delta} \frac{u(y)}{|x-y|^\lambda}\mathrm{d}y \leqslant \left(\int_{|x-y|\geqslant\delta}\frac{\mathrm{d}y}{|x-y|^{\lambda p'}}\right)^{1/p'}\|u\|_p = \left(\int_{|z|\geqslant\delta}\frac{\mathrm{d}z}{|z|^{n+\varepsilon}}\right)^{1/p'}\|u\|_p = \left(\frac{c(n,\varepsilon)}{\delta^\varepsilon}\right)^{1/p'}\|u\|_p = \frac{\|u\|_p}{\delta^{n/q}},$$

here we use spherical coordinate to handle the integral. For $k \geqslant 0$,

$$\int_{2^{-(k+1)}\delta\leqslant|x-y|\leqslant 2^{-k}\delta} \frac{u(y)}{|x-y|^\lambda}\,\mathrm{d}y \leqslant \left(\frac{2^{k+1}}{\delta}\right)^\lambda \int_{|x-y|\leqslant 2^{-k}\delta} u(y)\,\mathrm{d}y \leqslant \left(\frac{2^{k+1}}{\delta}\right)^\lambda \left|B(x,2^{-k}\delta)\right| M_c u(x)$$
$$= \left(\frac{2^k}{\delta}\right)^\lambda 2^\lambda |B(0,1)| \left(\frac{\delta}{2^k}\right)^n M_c u(x) = \frac{\delta^{n-\lambda}}{2^{k(n-\lambda)}} 2^\lambda |B(0,1)| M_c u(x).$$

Therefore,

$$\sum_{k\geqslant 0} \int_{2^{-(k+1)}\delta\leqslant|x-y|\leqslant 2^{-k}\delta} \frac{u(y)}{|x-y|^\lambda}\,\mathrm{d}y \leqslant c(n-\lambda)\delta^{n-\lambda} 2^\lambda |B(0,1)| M_c u(x).$$

We have obtained

$$(v_\lambda * u)(x) = A\delta^{-n/q} + B\delta^{n-\lambda}, \qquad -\frac{n}{q} < 0,\ n-\lambda > 0$$

with $A = c\big(np'/q, n\big)^{1/p'} \|u\|_p$ and $B = c(n-\lambda)2^\lambda |B(0,1)| M_c u(x)$ for all $\delta > 0$. Take δ to minimise (*either take derivative is equal to zero or pick δ with $A\delta^{-n/q} = B\delta^{n-\lambda}$*), then (H) holds. □

3 Introduction to spaces of homogeneous type

Setup. We always consider a *quasi-distance* d on a set E, which is a map $d : E \times E \to [0, +\infty)$ such that

(i) $\forall x, y \in E\,(d(x,y) = 0 \iff x = y)$;

(ii) $d(x,y) = d(y,x)$;

(iii) $\exists A_0 \geqslant 1$ such that d satisfies the triangle-like inequality, $\forall x, y, z \in E$,

$$d(x,z) \leqslant A_0(d(x,y) + d(y,z)).$$

The smallest A_0 is called the *quasi-distance constant.*

So d is a distance if and only if $A_0 = 1$. But the case when $A_0 > 1$ would have interesting different properties. We shall introduce some of them and give several examples.

3.1 Basic definitions and properties

Topology on a quasi-metric space (E,d).

For $x \in E$, $r > 0$, define the ball

$$B(x,r) := \{y \in E : d(x,y) < r\}, \qquad \forall \lambda > 0\,(\lambda B(x,r) := B(x,\lambda r)).$$

These balls form a basis of the topology on E, *i.e.*, $O \subset E$ is an open set when $\forall x \in O \exists r > 0\,(B(x,r) \subset O)$. Denote the Borel σ-algebra generated by the open sets as $\mathscr{B}(E)$.

Remark. The balls are always open when d is actually a distance. But it is not true when d is just a quasi-distance with $A_0 > 1$. Observe that

$$B\left(x, \frac{r}{A_0}\right) \subset \mathrm{Int}(B(x,r)) \subset B(x,r), \qquad \mathrm{Int}(B(x,r)) \subset B(x,r) \subset \mathrm{Int}(B(x, 2A_0, r)).$$

Say $d,\ d'$ are equivalent quasi-distances if

$$\exists k \geqslant 1\,(\frac{d'}{k} \leqslant d \leqslant kd).$$

This measn d and d' induce the same topology. For $\alpha > 0$, d and d^α induces the same topology on E.

Remark. *One can show that there exists $\alpha > 0$ such that d^α is equivalent to a distance. (See* ***TD II, Exercise 6****)*

Doubling measures

Definition 3.1: Doubling measure

Let μ be a positive Borel measure on a quasi-distance space (E, d). It is called *doubling* if

(i) $\exists c \geqslant 1\, \forall x \in E\, \forall r > 0\, (\mu(B(x, 2r)) \leqslant c\mu(B(x, r)) < +\infty)$.

- If μ is the restriction of an exterior measure for which the Borel sets are measurable, then it is OK even if $B(x, r)$ is not a Borel set.
- In the case where μ is a Borel measure, it could be that μ is not the restriction of an exterior measure. If $B(x, r)$ are Borel (*or even open*) sets, then (i) is OK. If not, we replace (i) by the following condition.

(ii) $\exists c' \geqslant 1\, \forall x \in E\, \forall r > 0\, (\mu(\mathrm{Int}(B(x, 2r))) \leqslant c'\mu(\mathrm{Int}(B(x, r))) < +\infty)$.

Note that (i) and (ii) are equivalent if the balls are open. Say $A \subset E$ is *bounded* if $\exists x \in E\, \exists r > 0\, (A \subset B(x, r))$. Then μ is doubling if and only if μ is locally finite.

The measure μ is doubling either $\mu = 0$, which is trivial, or $\forall x \in E$ and $\forall r > 0$,

$$\mu(B(x, r)) > 0 \qquad \text{(in case (i))}, \qquad \mu(\mathrm{Int}(B(x, r))) > 0 \qquad \text{(in case (ii))}.$$

We may assume the non-trivial case in the following discussion.

Definition 3.2: Doubling constant

The constant

$$c_D := \begin{cases} \sup\left\{\dfrac{\mu(B(x, 2r))}{\mu(B(x, r))} : x \in E,\ r > 0\right\}, & \text{in case (i)}, \\ \sup\left\{\dfrac{\mu(\mathrm{Int}(B(x, 2r)))}{\mu(\mathrm{Int}(B(x, r)))} : x \in E,\ r > 0\right\}, & \text{in case (ii)}, \end{cases}$$

is called the *doubling constant*. Moreover, $c_D > 1$ unless $E = \varnothing$ or E is a singleton. So we assume $\#E \geqslant 2$.

Definiton of sht

Definition 3.3: Space of homogeneous type

A triple (E, d, μ) is a *space of homogeneous type* if

- E is a set;
- d is a quasi-metric;
- μ is a positive Borel doubling measure.

We shall use sht for short of space of homogeneous type.

Then we have:

- For $\alpha > 0$, (E, d, μ) is a sht $\Longleftrightarrow$ (E, d^α, μ) is a sht.
- For $f : E \in [0, +\infty)$ a Borel measurable function, f and $1/f$ bounded. Set $\mathrm{d}\nu = f\,\mathrm{d}\mu$, then (E, d, μ) is a sht $\Longleftrightarrow$ (E, d, ν) is a sht.

3.2 Examples of sht

Example 3.4

(1) $(\mathbb{R}^n, d_{\|\cdot\|}, \mathrm{d}x)$ is a sht with $A_0 = 1$ and $c_D = 2^n$.

(2) $(\mathbb{R}^n, d_{\|\cdot\|}, (1 + \|\cdot\|^\alpha)\,\mathrm{d}x)$ is a sht for all $\alpha \in \mathbb{R}$.

(3) $(\mathbb{R}^n, d_{\|\cdot\|}, \|\cdot\|^\alpha\,\mathrm{d}x)$ is a sht if and only if $\alpha > -n$.

(4) $(\mathbb{R}^n, d_{\|\cdot\|}, \mathrm{e}^{-\|\cdot\|^\alpha}\,\mathrm{d}x)$ is not a sht for $\alpha > 0$.

(5) $(\mathbb{Z}^n, d_\infty|_{\mathbb{Z}^n\times\mathbb{Z}^n}$, counting measure) is a sht.

(6) A compact Riemannian manifold with geodesic distance and Riemannian volume is a sht.

(7) Some non-compact Riemannian manifold with geodesic distance and Riemannian volume are shts, for example, when $\mathrm{Ric} \geqslant 0$.

(8) Connected Lie groups with left invariant distance and Haar measure such that balls have polynomial volume growth are shts.

(9) $\Omega \subset \mathbb{R}^n$ is a bounded open subset with $\partial\Omega$ Lipschitz, *i.e.*, local charts by Lipschitz maps $\mathbb{R}^n \to \partial\Omega$. Then $(\Omega, d_{\|\cdot\|}|_{\Omega\times\Omega}, \mathrm{d}x)$ is a sht, $(\partial\Omega, d_{\|\cdot\|}|_{\partial\Omega\times\partial\Omega}, \mathrm{d}\sigma)$ is a sht, where $\mathrm{d}\sigma$ is the surface measure.

Let $\mathscr{D}$ be a collection of dyadic cubes generated by $[0,1)^n$, $E = [0,\infty)^n \subset \mathbb{R}^n$. We define the *dyadic distance* for $x, y \in E$ as

$$d_{\text{dyad}}(x,y) := \inf\left\{\text{length}(Q) : Q \in \mathscr{D},\ x, y \in \mathscr{Q}\right\}.$$

Then $d_{\text{dyad}} \geqslant 0$ and symmetric. $d_{\text{dyad}}(x,y) = 0$ if and only if $x = y$. To show $d = d_{\text{dyad}}$ it is actually a distance, it suffices to prove the triangle inequality. For $x, y, z \in E$, let $Q, R \in \mathscr{D}$ such that $x, y \in Q$, $y, z \in R$. Then $y \in Q \cap R$, hence $Q \subset R$ or $R \subset Q$.

- If $R \subset Q$, then $x, z \in Q$ implies $d(x,z) \leqslant \text{length}(Q) = d(x,y)$;
- If $Q \subset R$, then $x, z \in R$ implies $d(x,z) \leqslant \text{length}(R) = d(y,z)$.

Hence $d(x,z) \leqslant \max\left\{d(x,y), d(y,z)\right\}$, the ultra-metric inequality. Thus d is a metric and

$$\left\{\text{balls for } d_{\text{dyad}} \text{ in } A\right\} = \left\{Q \in \mathscr{D} : Q \subset A\right\}.$$

More precisely, $B(x,r) = Q_{x,r}$, where $Q_{x,r}$ is the unique dyadic cube Q with $x \in Q$ and $\frac{r}{2} \leqslant \text{length}(Q) < r$, which can be taken with $(Q_{j,k_j(x)})_{j\in\mathbb{Z}}$. To show $B(x,r) \subset Q_{x,r}$,

$$\forall y \in B(x,r) \implies d(x,y) < r \implies \exists Q \in \mathscr{D}\,(x, y \in Q,\ \text{length}(Q) < r) \implies Q \subset Q_{x,r}.$$

Hence $y \in Q_{x,r}$. The other side comes from $y \in Q_{x,r} \implies d(x,y) \leqslant \text{length}(Q_{x,r}) < r \implies y \in B(x,r)$.

Let $\mu = \mathrm{d}x$, then

$$\mu(B(x,2r)) = \mu(Q_{x,2r}) = 2^n\mu(Q(x,r)) = 2^n\mu(B(x,r)),$$

hence $\mathrm{d}x$ is doubling.

Example 3.5

Let E defined as above, $d = d_{\text{dyad}}$ be the dyadic distance. $(E, d, \mathrm{d}x)$ is a sht.

Then we give two examples where things can go wrong in the quasi-metric settings.

Example 3.6

Let $F = \mathbb{R}^2$ and $\mu = \mathrm{d}x$. Denote $\mathbb{S}^1 = \left\{x \in \mathbb{R}^2 : \|x\|_2 = 1\right\}$ be the unit Euclidean sphere. Pick $A \subset \mathbb{S}^1$ a symmetric subset, *i.e.*, $x \in A \implies -x \in A$, non-Borel ($A \notin \mathscr{B}(\mathbb{S}^1)$). Define

$$\Sigma := \left\{\frac{x}{2} : x \in \mathbb{S}^1\right\} \cup \{2x : x \in A\}.$$

Then for any $x \in \mathbb{R}^2 \setminus \{0\}$, $\{tx : t \geqslant 0\} \cap \Sigma$ is a singleton. Define $\|x\|_\Sigma = \lambda :\Longleftrightarrow \frac{x}{\lambda} \in \Sigma$ for $x \in \mathbb{R}^2 \{0\}$, and $\|0\|_\Sigma = 0$.

By construction, we have

$$\frac{1}{2}\|x\|_\Sigma \leqslant \|x\|_2 \leqslant 2\|x\|_\Sigma, \qquad \forall x \in \mathbb{R}^2.$$

In fact, for $x \neq 0$, $\|x/\|x\|_\Sigma\|_2 \in \{1/2, 2\}$. Then $x \mapsto \|x\|_\Sigma$ is a quasi-norm with $A_0 = 4$, $(x, y) \mapsto \left\|x - y\right\|_\Sigma$ is a quasi-distance, and equivalent to $d_{\|\cdot\|_2}$. Hence $(\mathbb{R}^2, d_{\|\cdot\|_\Sigma}, \mathrm{d}x)$ is a sht. But

$$B_{\|\cdot\|_\Sigma}(0,1) \cap \mathbb{S}^1 = A \notin \mathscr{B}(\mathbb{S}^1), \qquad \forall x \in \mathbb{S}^1 \left(x \in A \Longleftrightarrow 2x \in \Sigma \Longleftarrow \|x\|_2 = \frac{1}{2}\right),$$

thus $B_{\|\cdot\|_\Sigma}(0,1)$ is not a Borel set in $\mathbb{R}^2$.

Example 3.7

For metric space E, $\forall x \in E$, $y \mapsto d(x, y)$ is Lipschitz continuous, there exist quasi-metric spaces with arbitary $A_0 > 1$ and d is not continuous. (*see* ***TD II, Exercise 7***)

3.3 General properties of balls in sht

Assume (E, d, μ) is a sht and $\#E \geqslant 2$ in this subsection.

Proposition 3.8

There exist $\delta > 1$ and $c > 1$ such that $\forall \lambda > 1$, $\forall B$ ball,

$$\mu(\lambda B) \leqslant c\lambda^\delta \mu(B).$$

The constant δ is called the *homogeneous dimension.*

Proof. If $1 \leqslant \lambda < 2$, then $\mu(\lambda B) \leqslant \mu(2B) \leqslant c_D \mu(B)$. If $2^k \leqslant \lambda < 2^{k+1}$,

$$\mu(\lambda B) \leqslant \mu(2^{k+1}B) \leqslant c_D^{k+1}\mu(B) = c_D 2^{k\log_2 c_D}\mu(B) \leqslant c_D \lambda^{\log_2 c_D}\mu(B).$$

Then we take $c = c_D$, $\delta = \log_2 c_D$. ☐

Remark. Note that $\inf\left\{\delta > 0 : \exists c \text{ (previous proposition holds)}\right\}$ *may not be attained. But for Euclidean spaces* $\mathbb{R}^n$*, it can be attained with* $\delta = n$*.*

Proposition 3.9

There exists $c > 1$ such that $\forall x, y \in E$, $\forall r > 0$,

$$\mu(B(y, r)) \leqslant c\left(1 + \frac{d(x, y)}{r}\right)\mu(B(x, r)).$$

Proof. We only prove when $A_0 = 1$. For $x, y \in E$,

$$B(y,r) \subset B(x, r + d(x,y)) = \lambda B(x,r), \qquad \lambda = 1 + \frac{d(x,y)}{r},$$

and apply Property 1. □

Proposition 3.10

For all $c_0 > 0$, there exists $c_1 > 1$ such that $\forall x, y \in E$, $\forall r > 0$,

$$d(x,y) < c_0 r \implies \mu(B(y,r)) \leqslant c_1 \mu(B(x,r)).$$

It means nearby centres implies comparable mass in a scale invariant way.

Proof. Apply Property 2 with $c_1 = c(1+c_0)^\delta$. □

Proposition 3.11: Geometric doubling

There exists $N \in \mathbb{N}_{\geqslant 1}$ such that $\forall x \in E$, $\forall r > 0$, the ball $B(x, 2r)$ can be covered by at most N balls with radius r.

Proof. We only prove when $A_0 = 1$. For the case $A_0 > 1$, see **TD II, Exercise 6**. Let $\{B(x_i, r/2) : i \in I\}$ be a maximal collection of mutually disjoint balls in E, $\{B(x_i, r) : i \in I\}$ is a covering of E by maximality. Fix $B(x, 2r)$,

$$I_0 := \{i \in I : B(x_i, r) \cap B(x, 2r) = \varnothing\}.$$

For $i \in I_0$, $B(x_i, r) \subset B(x, 3r)$ and $\{B(x_i, r) : i \in I_0\}$ is a covering of $B(x, 2r)$. Then

$$\begin{aligned}
\# I_0 \cdot \mu\Big(B\Big(x, \frac{r}{2}\Big)\Big) = \sum_{i \in I_0} \mu\Big(B\Big(x, \frac{r}{2}\Big)\Big) &\leqslant \sum_{i \in I_0} c_1 \mu\Big(B\Big(x_i, \frac{r}{2}\Big)\Big) && \text{by Property 3 b/c } d(x, x_i) < 3r \\
&\leqslant c_1 \mu(B(x, 3r)) && \text{mutually disjoint and } \subset B(x, 3r) \\
&\leqslant c_1 c \lambda^\delta \mu\Big(B\Big(x, \frac{r}{2}\Big)\Big) && \text{by Property 1.}
\end{aligned}$$

Hence $\# I_0 \leqslant c_1 c \lambda^\delta$. □

Proposition 3.12

E is bounded if and only if $\mu(E) < +\infty$.

Proof. $\Longrightarrow$: Always since μ is locally finite.

$\Longleftarrow$: Assume $\mu(E) < +\infty$. Let $\varepsilon > 0$ to be chosen, and fix $x \in E$, $\exists B = B(x, R)$ such that $\mu(B(x,R)) \geqslant \mu(E)(1-\varepsilon)$. Let us assume $y \notin B(x, 2R)$, set $d = d(x,y) \geqslant 2R$. Then

$$\mu\Big(B\Big(y, \frac{d}{2}\Big)\Big) \geqslant \frac{1}{c_1} \mu\Big(B\Big(x, \frac{d}{2}\Big)\Big), \qquad B\Big(x, \frac{d}{2}\Big) \cap B\Big(y, \frac{d}{2}\Big) = \varnothing.$$

Hence

$$\begin{aligned}
\mu(E) \geqslant \mu\Big(B\Big(x, \frac{d}{2}\Big)\Big) + \mu\Big(B\Big(y, \frac{d}{2}\Big)\Big) &\geqslant \Big(1 + \frac{1}{c_1}\Big) \mu\Big(B\Big(x, \frac{d}{2}\Big)\Big) \\
&\geqslant \Big(1 + \frac{1}{c_1}\Big) \mu(B(x,R)) \geqslant \Big(1 + \frac{1}{c_1}\Big)(1-\varepsilon)\mu(E).
\end{aligned}$$

Contradiction if $(1 + 1/c_1)(1-\varepsilon) > 1$. Then we take $\varepsilon < 1 - 1/(1+1/c_1)$. Thus $E \subset B(x, 2R)$. □

Proposition 3.13: Growth of balls

For all $x \in E$, $r \mapsto \mu(B(x,r))$ is non-decreasing. Then

- either the annulus is empty,
- or there is some mass.

Specifically, when $A_0 = 1$, let $R, \rho > 0$ with $R > 4\rho$, then

- either $B(x, R/2) \setminus B(x, 2\rho) = \varnothing$,
- or $\mu(B(x,R)) \geqslant (1+\varepsilon_{R,\rho})\mu(B(x,\rho))$ with $\varepsilon_{R,\rho} > 0$ independent of x.

Proof. We only prove the case $A_0 = 1$. Assume $z \in B(x,R/2)\setminus B(x,2\rho)$, then $\rho + d(x,z) < R/4 + R/2 < R$ with

$$B(x,\rho) \subset B(x,R), \qquad B(z,\rho) \subset B(x,R), \qquad B(x,\rho)\cap B(z,\rho) = \varnothing.$$

Since $d(x,z) \geqslant 2\rho$,

$$\mu(B(x,R)) \geqslant \mu(B(x,\rho)) + \mu(B(z,\rho)) \geqslant \mu(B(x,\rho))\left(1 + \frac{1}{c}\left(1+\frac{d(x,z)}{\rho}\right)^{-\delta}\right) \geqslant \mu(B(x,\rho))\left(1+\frac{1}{c}\left(1+\frac{R}{2\rho}\right)^{\delta}\right).$$

Let $\varepsilon_{R,\rho} = c^{-1}(1+R/2\rho)^{-\delta}$ independent of x, done. □

Remark. *If $R < \kappa\rho$ with $\kappa > 4$, in particular if ρ and R are comparable, then*

$$\varepsilon_{R,\rho} \geqslant \frac{1}{c}\left(1+\frac{\kappa}{2}\right)^{-\delta},$$

which is independent of ρ, R.

Definition 3.14: Reverse doubling property

Let (E,d,μ) be a sht. E is said to have *reverse doubling property* if $\exists \varepsilon > 0$, for all balls B,

$$\mu(2B) \geqslant (1+\varepsilon)\mu(B).$$

In other words, measure of balls grows at least polynomially w.r.t. radius. If B is not Borel, replace B by $\operatorname{Int} B$.

Remark. *This definition implies that $\forall x \in E\, (\mu(\{x\}) = 0)$. In particular, μ cannot have atoms.*

3.4 Maximal Functions

Let (E,d,μ) be a sht, $f \in L^1_{\text{loc}}(E,\mu)$, $x \in E$. Define

$$M'_\mu f(x) := \sup\left\{ \fint_{\operatorname{Int} B} |f| \,\mathrm{d}\mu : x \in \operatorname{Int} B,\ B \text{ is a ball}\right\},$$

and if B is open, $\operatorname{Int} B = B$.

All results and proofs on $\mathbb{R}^n$ go through up to a little change.

Proposition 3.15: Properties of M'_μ

(1) $M'_\mu f$ is lower semi-continuous. (*This is where* $\operatorname{Int} B$ *is needed*).

(2) $\exists c_1 > 0$, depending only on A_0 and c_D such that for all $f \in L^1_{\text{loc}}(E,\mu)$ and $\forall \lambda > 0$,

$$\mu\left\{M'_\mu f > \lambda\right\} \leqslant \frac{c_1}{\lambda} \int_{\{M'_\mu f\} > \lambda} \left|f\right| \mathrm{d}\mu.$$

(3) $\forall p \in (1, +\infty)$, $\exists c_p = c(p, c_1)$ such that for all $f \in L^p(E,\mu)$,

$$\|M'_\mu f\|_p \leqslant c_p \left\|f\right\|_p.$$

(4) The Lebesgue differentation: if $C_b(E) := \left\{f : E \to \mathbb{C} \text{ continuous with bounded support}\right\}$ is dense in $L^1(E,\mu)$, then $\forall f \in L^1_{\text{loc}}(E,\mu)$,

$$f(x) = \lim_{r(B)\to 0, x\in \operatorname{Int} B} \fint_{\operatorname{Int} B} \left|f\right| \mathrm{d}\mu, \qquad \mu\text{-a.e. } x \in E.$$

(5) If density above holds, $\forall f \in L^1_{\text{loc}}(E,\mu)$, $\left|f\right| \leqslant M'_\mu f$ for μ-a.e. $x \in E$.

4 Interpolation

Setup. There are two kinds of interpolations: The complex interpolation, *i.e.*, the Riesz–Thorin interpolation, has been discussed in the acceleration course *Elements of Functional Analysis*. So we will skip and discuss real interpolation: the Marcinkiewicz interpolation.

4.1 Weak L^p spaces and sublinear operators

Recall that the *strong L^p space* is defined as

$$L^p(M,\mu) := \left\{f : M \to \mathbb{C} : f \text{ measurable}, \left|f\right|^p \text{ integrable}\right\}, \quad L^\infty(M,\mu) := \left\{f : M \to \mathbb{C} : \exists \lambda = 0\, (\mu\left\{\left|f\right| > \lambda\right\} = 0)\right\}.$$

For $1 \leqslant p \leqslant \infty$, $(L^p(M,\mu), \|\cdot\|_p)$ is a Banach space. For $0 < p < 1$, $(L^p(M,\mu), \|\cdot\|_p)$ is a quasi-normed space and is complete for the distance $d_p(f,g) := \left\|f - g\right\|_p^p$.

Definition 4.1: Weak L^p space

Let (M,μ) be a measure space and $f : M \to \mathbb{C}$ is measurable. For $0 < p < \infty$, define the *weak L^p space* as

$$f \in L^{p,\infty} :\Longleftrightarrow \sup_{\lambda>0} \lambda^p \mu\left\{\left|f\right| > \lambda\right\} < +\infty.$$

And the *weak L^p norm* is defined as $\left\|f\right\|_{p,\infty} := \left(\sup_{\lambda>0} \lambda^p \mu\left\{\left|f\right| > \lambda\right\}\right)^{1/p}$. For $p = \infty$, we define $L^{\infty,\infty}(M,\mu) := L^\infty(M,\mu)$ and $\|\cdot\|_{\infty,\infty} = \left\|f\right\|_\infty$.

Proposition 4.2

The basic properties of weak L^p spaces are as follows:

(1) For $0 < p < \infty$, $(L^{p,\infty}, \|\cdot\|_{p,\infty})$ is a quasi-norm space.

(2) $d_p(f,g) := \left\{f - g\right\}_{p,\infty}^p$ is a complete quasi-metric on $L^{p,\infty}(M,\mu)$.

(3) $L^p(M,\mu) \subset L^{p,\infty}(M,\mu)$.

(4) The inclusion in (3) is strict in general.

Proof. (1) For all $\lambda > 0$, $\mu\{|f+g| > \lambda\} \leqslant \mu\{|f| > \lambda/2\} + \mu\{|g| > \lambda/2\}$. Thus

$$\|f+g\|_{p,\infty} \leqslant 2(\|f\|_{p,\infty}^p + \|g\|_{p,\infty}^p)^{1/p} \leqslant \begin{cases} 2(\|f\|_{p,\infty} + \|g\|_{p,\infty}), & p \geqslant 1, \\ 2^{1/p}(\|f\|_{p,\infty} + \|g\|_{p,\infty}), & 0 < p < 1. \end{cases}$$

(3) This is because for $f \in L^p(M,\mu)$,

$$\mu\{|f| > \lambda\} \leqslant \int_{\{|f|>\lambda\}} \left(\frac{|f|}{\lambda}\right)^p \mathrm{d}\mu \leqslant \frac{1}{\lambda^p} \int_M |f|^p \,\mathrm{d}\mu,$$

thus $\|f\|_{p,\infty} \leqslant \|f\|_p$.

(4) Let $M = \mathbb{R}^n$ and $\mu = \mathrm{d}x$ be the Lebesgue measure. For $\lambda > 0$, then $f(x) = |x|^{-\lambda}$ is weak $L^{n/\lambda}$ but not strong $L^{n/\lambda}$. This is because

$$f^{n/\lambda} = |x|^{-\lambda \cdot n/\lambda} = |x|^{-n} \implies \int_{\mathbb{R}^n} |x|^{-n} \,\mathrm{d}x = \infty.$$

But for $\alpha > 0$,

$$\left|\{|f| > \alpha\}\right| = \left|\{x \in \mathbb{R}^n : |x|^{-\lambda} > \alpha\}\right| = \left|\{x \in \mathbb{R}^n : |x| < \alpha^{-1/\lambda}\}\right| = \alpha^{-n/\lambda} |B(0,1)|.$$

Hence $\alpha^{n/\lambda} \left|\{|f| > \alpha\}\right| = |B(0,1)| < +\infty$. □

Let (M,μ) and (N,ν) be two measure spaces, $\mathscr{F}_M := \{f : M \to \mathbb{C} : f \text{ is } \mu\text{-measurable}\}$. Let $\mathscr{D}_M$ be a subspace of $\mathscr{F}_M$ and $T : \mathscr{D}_M \to \mathscr{F}_N$.

Definition 4.3: Sublinear operator

Say $T : \mathscr{D}_M \to \mathscr{F}_N$ is *sublinear* if $\forall f_1, f_2 \in \mathscr{D}_M$,

$$|T(f_1 + f_2)| \leqslant |Tf_1| + |Tf_2|.$$

We have all linear operators are sublinear directly from the definition. For example, the Hardy–Littlewood maximal operators $f \mapsto M_c f$ or Mf, $M_d f$ is sublinear, here $\mathscr{D}_M = L^1_{\mathrm{loc}}(\mathbb{R}^n, \mu)$.

Definition 4.4: Strong type and weak type

Let $T : \mathscr{D}_M \to \mathscr{F}_N$ be sublinear, $1 \leqslant p, q < \infty$.

(1) Say T is of *strong type* (p,q) if

$$T : \mathscr{D}_M \cap L^p(M,\mu) \to L^q(N,\nu), \qquad f \mapsto Tf$$

is bounded. In other words, $\exists c > 0$ such that $\forall f \in \mathscr{D}_M \cap L^p(M,\mu)$, $\|Tf\|_q \leqslant c\|f\|_p$.

(2) For $q < \infty$, say T is of *weak type* (p,q) if

$$T : \mathscr{D}_M \cap L^p(M,\mu) \to L^{q,\infty}(N,\nu), \qquad f \mapsto Tf$$

is bounded. In other words, $\exists c > 0$ such that $\forall f \in \mathscr{D}_M \cap L^p(M,\mu)$, $\|Tf\|_{q,\infty} \leqslant c\|f\|_p$.

(3) Say T is of *weak type* (p,∞) if it is of strong type (p,∞).

If T is of strong type (p,q), then it is of weak type (p,q). But the reverse is false.

Example 4.5

(1) The Hardy–Littlewood operator is of weak type $(1,1)$, but not strong type $(1,1)$. While it is of strong type (p,p) for $1<p<\infty$.

(2) By Hardy–Littlewood–Sobolev theorem, $v_\lambda(x)=|x|^{-\lambda}$, the operator $T: f\mapsto v_\lambda * f$ is of weak type $(1,n/\lambda)$ for $0<\lambda<n$, and of strong type (p,q) for $0<\lambda<n$ and $1/p-1/q=1-\lambda/n$.

4.2 The real interpolation – Marcinkiewicz's theorem

Theorem 4.6: Marcinkiewicz's interpolation

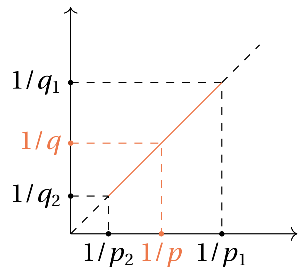
Let $T:\mathscr{D}_M\to\mathscr{F}_N$ be a sublinear operator, $\mathscr{D}_M$ be stable under multiplication by $\mathbb{1}_A$ for $A\subset M$ measurable. Let $1\leqslant p_1<p_2\leqslant\infty$, $1\leqslant q_1<q_2\leqslant\infty$ with $p_1\leqslant q_1$ and $p_2\leqslant q_2$. Assume that T is of weak type (p_1,q_1) and (p_2,q_2).

Then for all $p_1<p<p_2$, T is of strong type (p,q) with

$$\frac{1}{q}=\frac{1-\theta}{q_1}+\frac{\theta}{q_2},$$

when $1/p=(1-\theta)/p_1+\theta/p_2$ for $0<\theta<1$.

Proof. We only prove the diagonal case: when $p_1=q_1$ and $p_2=q_2$. For the general case, see Stein–Weiss theorem.



Case 1: $p_2=q_2<\infty$. We have

$$\|Tf\|_{p_i}\leqslant c_i\|f\|_{p_i},\qquad i=1,2,\ f\in\mathscr{D}_M\cap L^{p_i}(M,\mu).$$

The best c_i is called the *weak type* (p_i,q_i) *constant*. Fix $f\in\mathscr{D}_M\cap L^p(M,\mu)$,

$$\|Tf\|_p^p=\int_N|Tf|^p\,\mathrm{d}\nu=p\int_0^\infty\lambda^{p-1}\nu\{|Tf|>\lambda\}\,\mathrm{d}\lambda.$$

We use cut-offs of f at height λ. Denote $g_\lambda=f\mathbb{1}_{\{|f|>\lambda\}}\in\mathscr{D}_M$, $h_\lambda=f\mathbb{1}_{\{|f|\leqslant\lambda\}}\in\mathscr{D}_M$, and $f=g_\lambda+h_\lambda$. We estimate

$$|h_\lambda|\leqslant\min\{|f|,\lambda\}\implies h_\lambda\in L^p(M,\mu)\cap L^\infty(M,\mu)\subset L^{p_2}(M,\mu),$$
$$|g_\lambda|\leqslant|f|\implies g_\lambda\in L^p(M,\mu)\cap L^1(M,\mu)\subset L^{p_1}(M,\mu),$$

because

$$\int_M|g_\lambda|\,\mathrm{d}\mu=\int_M f\mathbb{1}_{\{|f|>\lambda\}}\,\mathrm{d}\mu\leqslant\int_M|f|\left|\frac{f}{\lambda}\right|^{p-1}\mathbb{1}_{\{|f|/\lambda\}>1}\,\mathrm{d}\mu\leqslant\frac{1}{\lambda^p}\|f\|_p^p<+\infty.$$

Since T is sublinear, $|Tf|\leqslant|Tg_\lambda|+|Th_\lambda|$ for ν-a.e. $y\in N$. We estimate

$$\begin{aligned}\|Tf\|_p^p&\leqslant p\int_0^\infty\lambda^{p-1}\nu\left\{|Tg_\lambda|>\frac{\lambda}{2}\right\}\mathrm{d}\lambda+p\int_0^\infty\lambda^{p-1}\nu\left\{|Th_\lambda|>\frac{\lambda}{2}\right\}\mathrm{d}\lambda\\&\leqslant\underbrace{p\int_0^\infty\lambda^{p-p_1-1}(2c_1)^{p_1}\|g_\lambda\|_{p_1}^{p_1}}_{\text{(I)}}+\underbrace{p\int_0^\infty\lambda^{p-p_2-1}(2c_2)^{p_2}\|h_\lambda\|_{p_2}^{p_2}}_{\text{(II)}}\end{aligned}$$

with

$$
\begin{aligned}
\text{(I)} &= p(2c_1)^{p_1}\int_0^\infty \lambda^{p-p_1-1}\int_M |f|^{p_1}\mathbb{1}_{\{|f|>\lambda\}}\,\mathrm{d}\mu\,\mathrm{d}\lambda \\
&= p(2c_1)^{p_1}\int_M |f|^{p_1}\int_0^{|f|}\lambda^{p-p_1-1}\,\mathrm{d}\lambda\,\mathrm{d}\mu = \frac{p}{p-p_1}(2c_1)^{p_1}\int_M |f|^p\,\mathrm{d}\mu, \\
\text{(II)} &= p(2c_2)^{p_2}\int_0^\infty \lambda^{p-p_2-1}\int_M |f|^{p_2}\mathbb{1}_{\{|f|\leqslant\lambda\}}\,\mathrm{d}\mu\,\mathrm{d}\lambda \\
&= p(2c_2)^{p_2}\int_M |f|^{p_2}\int_{|f|}^{\infty}\lambda^{p-p_2-1}\,\mathrm{d}\lambda\,\mathrm{d}\mu = \frac{p}{p_2-p}(2c_2)^{p_2}\int_M |f|^p\,\mathrm{d}\mu,
\end{aligned}
$$

since $p-p_1>0$ and $p-p_2<0$. Then

$$
\|Tf\|_p^p \leqslant \left(\frac{p(2c_1)^{p_1}}{p-p_1}+\frac{p(2c_2)^{p_2}}{p_2-p}\right)\|f\|_p^p .
$$

Case 2. $p_2=q_2=\infty$. In this case, $Th_\lambda\in L^\infty(M,\mu)$ by the same argument, thus

$$
\|Th_\lambda\|_\infty \leqslant c_2\|h_\lambda\|_\infty \leqslant c_2\frac{\lambda}{2c_2}
$$

if $h_\lambda = f\mathbb{1}_{\{|f|\leqslant\lambda/(2c_2)\}}$. Hence $\nu\{|Th_\lambda|>\lambda/2\}=0$. We only need to estimate $\nu\{|Tg_\lambda|>\lambda/2\}$ with $g_\lambda = f\mathbb{1}_{\{|f|>\lambda/(2c_2)\}}$, which follows by the same proof. □

Remark. This proof gives that the strong type (p,p) constant $c_{p,p}$ is bounded by

$$
c_{p,p} \leqslant \left(\frac{p(2c_1)^{p_1}}{p-p_1}+\frac{p(2c_2)^{p_2}}{p_2-p}\right)^{1/p}.
$$

The right hand side blows up when $p\searrow p_1$, and $p\nearrow p_2$ if $p_2<\infty$.

If we modify g_λ to $f\mathbb{1}_{\{|f|>a\lambda\}}$ for $a>0$, the estimation in the proof above then will become

$$
\int_M |g_\lambda|\,\mathrm{d}\mu \leqslant \frac{1}{a(p-p_1)}\|f\|_p^p .
$$

And then

$$
\|Tf\|_p^p \leqslant \left(\frac{a^{p_1-p}p(2c_1)^{p_1}}{p-p_1}+\frac{a^{p_2-p}p(2c_2)^{p_2}}{p_2-p}\right)\|f\|_p^p .
$$

By optimising the coefficient, one has $c(p,p_1,p_2)\leqslant c_1^{1-\theta}c_2^{\theta}$.

*Remark. The complex interpolation case gives: assuming strong type (p_1,p_1) and (p_2,p_2), then it is of strong type (p,p) with constant $c_{p,p}\leqslant c_1^{1-\theta}c_2^{\theta}$. For a proof, see [**Bergh–Löfstrom**].*

5 Calderón–Zygmund Operators

Setup. Consider an operator with kernel representation

$$
Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y, \qquad x\in\mathbb{R}^n
$$

defined on some function f. If $K:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ is mesurable, and

$$
\operatorname*{ess\,sup}_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,\mathrm{d}x = c_1<+\infty, \qquad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,\mathrm{d}y = c_\infty<+\infty,
$$

then the integral makes sense a.e. for $f\in L^p(\mathbb{R}^n)$ for $1\leqslant p\leqslant\infty$. For $f\in L^1(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)||f(y)|\,\mathrm{d}y\,\mathrm{d}x \leqslant c_1\int_{\mathbb{R}^n}|f(y)|\,\mathrm{d}y = c_1\|f_1\|.
$$

And for $f \in L^\infty$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)||f(y)|\,\mathrm{d}y\,\mathrm{d}x \leqslant c_\infty \|f\|_\infty .$$

Thus

$$T: L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)+L^\infty(\mathbb{R}^n), \qquad f \mapsto Tf : x \mapsto \int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y$$

defines a linear operator. By interpolation, T can be defined for $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$, and of strong type (p,p) because it of strong type $(1,1)$ and (∞,∞).

But what if the finite esssup condition is not satisfied?

5.1 Hilbert transform and Riesz transforms

We introduce two kinds of transforms on $\mathbb{R}^n$. The Hilbert transform is in $\mathbb{R}$ and Riesz transforms is in $\mathbb{R}^n$. Their kernels have origins from PDE (for $n \geqslant 2$) and complex analysis (for $n = 1$).

Denote $\mathscr{S}(\mathbb{R}^n)$ the Schwartz functions and $\mathscr{S}'(\mathbb{R}^n)$ the tempered distribution. Also, denote $(\cdot,\cdot)$ the bilinear duality form.

Definition 5.1: Hilbert transform

Define

$$H: \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}), \qquad f \mapsto Hf := \text{p.v.}\,\frac{1}{\pi x} * f,$$

i.e., $\forall g \in \mathscr{S}(\mathbb{R})$, $(Hf, g) = (\text{p.v.}\frac{1}{\pi x}, \check{f} * g)$, where $\check{f}(x) = f(-x)$, and the limit

$$\left(\text{p.v.}\,\frac{1}{\pi x}, f\right) := \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\{|x|>\varepsilon\}} \frac{f(x)}{x}\,\mathrm{d}x$$

exists when $f \in \mathscr{S}(\mathbb{R})$.

By the definition above,

$$\begin{aligned}(Hf,g) &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\{|y|>\varepsilon\}} \frac{(\check{f} * g)(y)}{y}\,\mathrm{d}y = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\{|y|>\varepsilon\}}\frac{1}{y}\int_{\mathbb{R}} f(x-y)g(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{1}{\pi}\int_{\{|y|>\varepsilon\}}\frac{f(x-y)}{y}\,\mathrm{d}y\,g(x)\,\mathrm{d}x =: \lim_{\varepsilon\to 0}\int_{\mathbb{R}} H_\varepsilon f(x) g(x)\,\mathrm{d}x.\end{aligned}$$

We want $\tilde{f}(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x)$ exists for each $x \in \mathbb{R}$, here $H_\varepsilon f(x) := \frac{1}{\pi}\int_{\{|y|>\varepsilon\}}\frac{f(x-y)}{y}\,\mathrm{d}y$.

Proposition 5.2

For $f \in \mathscr{S}(\mathbb{R})$, $Hf \in C^\infty(\mathbb{R})$ with

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\{|y|>\varepsilon\}}\frac{f(x-y)}{y}\,\mathrm{d}y.$$

Proof. One can show that $\tilde{f} \in C^\infty(\mathbb{R})$ and that $H_\varepsilon f$ have slow growth at infinity, *i.e.*,

$$\exists M > 0\, \exists c_f < +\infty\, \forall x \in \mathbb{R}\, \forall \varepsilon > 0 \left(\left|H_\varepsilon f(x) \leqslant c_f(1+|x|)^M\right|\right).$$

Since $g \in \mathscr{S}(\mathbb{R})$, apply DCT, we obtain the conclusion. □

Thus, the Hilbert transform is an integral operator with kernel

$$K(x,y) = \frac{1}{\pi(x-y)},$$

which cannot satisfty the finite esssup condition.

Definition 5.3: Riesz transforms

For $j \in [\![1,n]\!]$, define

$$R_j : \mathscr{S}(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n), \qquad f \mapsto R_j f,$$

with

$$(R_j f, g) = \left(\text{p.v.} c_n \frac{x_j}{|x|^{n+1}}, \check{f} * g\right), \qquad \forall f, g \in \mathscr{S}(\mathbb{R}^n).$$

The Riesz transforms are integral operators with kernels

$$K_j(x,y) = c_n \frac{x_j - y_j}{\left|x-y\right|^{n+1}}, \qquad \forall x, y \in \mathbb{R}^n \setminus \Delta,\ j \in [\![1,n]\!].$$

Hence Hilbert transform can be seen as a special case of Riesz transform in the case $n = 1$ and $K(x,y) = \pi \frac{x-y}{|x-y|^2}$. Similarly, we have

Proposition 5.4

For $f \in \mathscr{S}(\mathbb{R}^n)$, $R_j f \in C^\infty(\mathbb{R}^n)$ with slow growth, and

$$R_j f(x) = \lim_{\varepsilon \to 0} c_n \int_{\{|y|>\varepsilon\}} f(x-y) \frac{y_j}{\left|y\right|^{n+1}} \, \mathrm{d}y,$$

where c_n is not relevant to j and comes from calculation.

To prove the proposition, we need a lemma.

Lemma 5.5

(i) For $\xi \in \mathbb{R}$, $\widehat{Hf}(\xi) = -\mathrm{i}\frac{\xi}{|\xi|}\hat{f}(\xi)$.

(ii) For $\xi \in \mathbb{R}^n$, $\widehat{R_j f}(\xi) = -\mathrm{i}\frac{\xi_j}{|\xi|}\hat{f}(\xi)$.

Remark. The right hand side $\hat{f} \in \mathscr{S}$, multiplied by an L^∞ function not defined at $\xi = 0$. And the left hand side $\widehat{Hf} \in \mathscr{S}'(\mathbb{R})$, $\widehat{R_j f} \in \mathscr{S}'(\mathbb{R}^n)$. The equality here means they identify to an L^1_{loc} function.

Proof. (1) follows from (2) in the 1 dimensional case. So it suffices to prove (2). Set $K_j = \text{p.v.}\ c_n \frac{x_j}{|x|^{n+1}} \in \mathscr{S}'(\mathbb{R}^n)$, then

$$R_j f = K_j * f \quad (\text{in } \mathscr{S}'(\mathbb{R}^n)) \implies \widehat{R_j f} = \widehat{K_j * f} = \widehat{K_j}\hat{f} \quad (\text{in } \mathscr{S}'(\mathbb{R}^n)),$$

where $(\widehat{K_j}\hat{f}, g) = (\widehat{K_j}, \hat{f} g)$ for all $g \in \mathscr{S}(\mathbb{R}^n)$. It is enough to show that $\widehat{K_j} \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $\widehat{K_j}(\xi) = -\mathrm{i}\frac{\xi_j}{|\xi|}$ for $\xi \neq 0$.

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$(\widehat{K_j}, \varphi) = (K_j, \hat{\varphi}) = \lim_{\varepsilon \to 0} c_n \int_{\{|x|>\varepsilon\}} \frac{x_j}{|x|^{n+1}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} \hat{\varphi}(\xi) \, \mathrm{d}\xi \, \mathrm{d}x$$

and by DCT, $\int_{\{|x|>1/\varepsilon\}} 1 \, \mathrm{d}x \to 0$ as $\varepsilon \to 0$. Then denote

$$I_\varepsilon(x) = c_n \int_{\{\varepsilon<|x|<1/\varepsilon\}} \mathrm{e}^{-\mathrm{i}x\cdot\xi} \frac{x_j}{|x|^{n+1}} \, \mathrm{d}x,$$

we can rewrite $(K_j, \hat{\varphi}) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} I_\varepsilon(\xi)\varphi(\xi) \, \mathrm{d}\xi$. Then we show

(a) $|I_\varepsilon(\xi)| \leqslant c$ uniformly in $\varepsilon \in (0,1]$ and $\xi \in \mathbb{R}^n$, and note $I_\varepsilon(0) = 0$ for all $\varepsilon \in (0,1)$.

(b) $\forall \xi \in \mathbb{R}^n \setminus \{0\}$, $\lim_{\varepsilon\to 0} I_\varepsilon(\xi) = -\mathrm{i}\frac{\xi_j}{|\xi|}$.

(a) By symmetry and oddness, denote $\omega = \frac{\xi}{|\xi|} \in \mathbb{S}^{n-1}$, $x = \frac{r}{|\xi|}\theta$ with $r > 0$ and $\theta \in \mathbb{S}^{n-1}$, then

$$I_\varepsilon(\xi) = -\mathrm{i}c_n \int_{\{\varepsilon<|x|<1/\varepsilon\}} \sin(x\cdot\xi)\frac{x_j}{|x|^{n+1}}\,\mathrm{d}x = -\mathrm{i}c_n \int_{\mathbb{S}^{n-1}} \int_{\varepsilon/|\xi|}^{1/\varepsilon|\xi|} \sin(r\theta\cdot\omega)\frac{\mathrm{d}r}{r}\,\mathrm{d}\sigma(\theta).$$

For all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sup_{0<a<b<\infty}\left|\int_a^b \frac{\sin t}{t}\,\mathrm{d}t\right| \leqslant c < +\infty, \qquad \int_a^b \frac{\sin t}{t}\,\mathrm{d}t \to \frac{\pi}{2} \quad \text{when } a\to 0^+, b\to+\infty.$$

Thus

$$|I_\varepsilon(\xi)| < c_n\cdot c\int_{\mathbb{S}^{n-1}} \left|\theta_j\right|\mathrm{d}\sigma(\theta)$$

with the right hand side finite and independent of ε.

(b) By DCT,

$$\lim_{\varepsilon\to 0} I_\varepsilon(\xi) = -\mathrm{i}\frac{\pi}{2}c_n\int_{\mathbb{S}^{n-1}} \operatorname{sgn}(\theta\cdot\omega)\theta_j\,\mathrm{d}\sigma(\theta) =: a_j.$$

Then by rotation invariance of $\mathbb{S}^{n-1}$, we rotate e_1 to ω and get

$$a\cdot\omega = \sum_{j=1}^n a_j\omega_j = -\mathrm{i}\frac{\pi}{2}c_n\int_{\mathbb{S}^{n-1}} \operatorname{sgn}(\theta\cdot\omega)(\theta\cdot\omega)\,\mathrm{d}\sigma(\theta) = -\mathrm{i}\frac{\pi}{2}\int_{\mathbb{S}^{n-1}} |\theta_1|\,\mathrm{d}\sigma(\theta).$$

Choose c_n such that $\frac{\pi}{2}c_n\int_{\mathbb{S}^{n-1}}|\theta_1|\,\mathrm{d}\sigma(\theta) = 1$, then $a\cdot\omega = -\mathrm{i}$. We decompose

$$\theta = (\theta - (\theta\cdot\omega)\omega) + (\theta\cdot\omega)\omega \in \{\omega\}^\perp + \{\omega\}.$$

Then the map $h:\theta\mapsto(\theta-(\theta\cdot\omega)\omega)\operatorname{sgn}(\theta\cdot\omega)$ is odd w.r.t. $\{\omega\}^\perp$, thus $\int_{\mathbb{S}^{n-1}} h(\theta)\,\mathrm{d}\sigma(\theta) = 0$. Now

$$\begin{cases} a = -\mathrm{i}\frac{\pi}{2}c_n\int_{\mathbb{S}^{n-1}}\operatorname{sgn}(\theta\cdot\omega)(\theta\cdot\omega)\,\mathrm{d}\sigma(\theta)\cdot\omega, \\ a\cdot\omega = -\mathrm{i}, \end{cases}$$

implies $a = -\mathrm{i}\omega$, and $a_j = -\mathrm{i}\omega_j = -\mathrm{i}\frac{\xi_j}{|\xi|}$.

Then by (a) and (b),

$$(K_j,\hat{\varphi}) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n} I_\varepsilon(\xi)\varphi(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} -\mathrm{i}\frac{\xi_j}{|\xi|}\varphi(\xi)\,\mathrm{d}\xi,$$

then we conclude the proof. □

Proof of the proposition.

Recall that the Fourier transform $\mathscr{F}$ extends as $\mathscr{S}'(\mathbb{R}^n) \xrightarrow{\sim} \mathscr{S}'(\mathbb{R}^n)$ by the duality from $(\hat{f}, g) = (f, \hat{g})$ for $g \in \mathscr{S}(\mathbb{R}^n)$.

(1) The lemma says that Hf identifies to an L^2 function and $\left\|\widehat{Hf}\right\|_2 = \|\hat{f}\|_2$. By Planchernel identity, $\|f\|_2 = (\sqrt{2\pi})^{-n}\|\hat{f}\|_2$ in $L^2(\mathbb{R}^n)$, thus $\|Hf\|_2 = \|f\|_2$.

(2) Same and $\left\|\widehat{R_jf}\right\|_2 \leqslant \|\hat{f}\|_2$. □

Corollary 5.6

H and R_j, $j\in[\![1,n]\!]$ extend to bounded operators on $L^2(\mathbb{R}^n)$, denoted as $\tilde{H}$, $\tilde{R}_j$.

Proof. This is because $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and the linearity implies Lipschitz continuity. Use extension theorem of Lipschitz maps. □

We shall see that $\forall f \in L^2(\mathbb{R})$,

$$\tilde{H}f(x) = \int_{\mathbb{R}} \frac{1}{\pi} \cdot \frac{f(y)}{x-y}\,\mathrm{d}y, \qquad \text{a.e. } x \notin \operatorname{supp} f.$$

For example, take $f = \mathbb{1}_{[0,1]} \in L^2(\mathbb{R})$,

$$\tilde{H}f(x) = \frac{1}{\pi}\log\left|\frac{x-1}{x}\right|, \qquad \text{a.e. } x \notin [0,1].$$

Here we note that $\log\left|\frac{x-1}{x}\right| \sim \frac{1}{|x|} \notin L^1(\mathbb{R})$, hence $\tilde{H}$ cannot be bounded on $L^1(\mathbb{R})$. Also $\log\left|\frac{x-1}{x}\right| \notin L^\infty(\mathbb{R})$ near 0 and 1, hence $\tilde{H}$ cannot be bounded on $L^\infty(\mathbb{R})$.

5.2 Calderón–Zygmund operators and kernels

Let $\Delta = \{(x,x) : x \in \mathbb{R}^n\}$ be the diagonal of $\mathbb{R}^n$.

Definition 5.7: Calderón–Zygmund kernel

For $0 < \alpha \leqslant 1$, a *Calderón–Zygmund kernel* of order α is a continuous function $K : \Delta^c \to \mathbb{C}$ such that $\exists c \in (0,+\infty)$,

(i) $\forall x, y \in \Delta^c$, $\left|K(x,y)\right| \leqslant \frac{c}{|x-y|^n}$;

(ii) $\forall x, y, y'$ with $x \neq y$ and $\left|y-y'\right| \leqslant \frac{1}{2}\left|x-y\right|$, $\left|K(x,y)-K(x,y')\right| \leqslant c\left(\frac{|y-y'|}{|x-y|}\right)^\alpha \frac{1}{|x-y|^n}$;

(iii) $\forall x, x', y$ with $x \neq y$ and $|x-x'| \leqslant \frac{1}{2}\left|x-y\right|$, $\left|K(x,y)-K(x,y')\right| \leqslant c\left(\frac{|x-x'|}{|x-y|}\right)^\alpha \frac{1}{|x-y|^n}$.

And denote $\|K\|_\alpha := \inf\{c > 0 : \text{ (i) – (iii) hold}\}$ and $\mathrm{CZK}_\alpha = \{K : \Delta^c \to \mathbb{C} : \|K\|_\alpha < +\infty\}$.

Thus, CZK_α is a vector space for $0 < \alpha \leqslant 1$, and it is stable by adjoint rule: let $K^*(x,y) := \overline{K(y,x)}$, then $K \in \mathrm{CZK}_\alpha$ implies $K^* \in \mathrm{CZK}_\alpha$. For example, the kernels we discussed in section 5.1,

- $K(x,y) = \frac{1}{\pi} \cdot \frac{1}{x-y} \in \mathrm{CZK}_1$.
- $K_j(x,y) = c_n \frac{x_j - y_j}{|x-y|^{n+1}} \in \mathrm{CZK}_1$.

Definition 5.8: Calderón–Zygmund operator

A *Calderón–Zygmund operator* of order $\alpha \in [0,1]$ is a linear operator T with the following properties:

(i) T is continuous: $T \in \mathscr{B}(L^2(\mathbb{R}^n, \mathrm{d}x))$.

(ii) There exists $K \in \mathrm{CZK}_\alpha$ such that $\forall f \in L^2(\mathbb{R}^n, \mathrm{d}x)$ with compact support,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y, \qquad \text{a.e. } x \notin \operatorname{supp} f.$$

Denote the set of Calderón–Zygmund operators of order α as CZO_α.

The condition (ii) in the above definition makes sense as follows:

- $Tf \in L^2(\mathbb{R}^n, \mathrm{d}x)$ implies $Tf(x)$ exists a.e..
- The integral exists $\mathrm{d}x$-a.e. $x \notin \operatorname{supp} f$.

In fact, if $g \in L^2(\mathbb{R}^n, \mathrm{d}x)$ with compact support and $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$, then

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|K(x,y)\right|\left|f(y)\right|\left|g(x)\right|\mathrm{d}x\,\mathrm{d}y &\leqslant \Big(\sup_{x \in \operatorname{supp} f, y \in \operatorname{supp} g} \left|K(x,y)\right|\Big) \int_{\operatorname{supp} f} \left|f\right|\mathrm{d}x \int_{\operatorname{supp} g} \left|g\right|\mathrm{d}y \\ &\leqslant \frac{c}{d(\operatorname{supp} f, \operatorname{supp} g)^n} \left|\operatorname{supp} f\right|^{1/2} \left\|f\right\|_2 \left|\operatorname{supp} g\right|^{1/2} \left\|g\right\|_2 < +\infty.\end{aligned}$$

In particular, for dx-a.e. (*and if K is continuous, for all*) $x \notin \operatorname{supp} f$, $\int_{\mathbb{R}^n} |K(x,y)||f(y)| \, \mathrm{d}y < +\infty$.

Example 5.9

The bounded extensions of Hilbert transform and Riesz transforms are CZO_1.

For all $f \in \mathscr{S}(\mathbb{R})$, $\forall x \notin \operatorname{supp} f$, we have proved

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, \mathrm{d}y.$$

If $f \in L^2(\mathbb{R}, \mathrm{d}x)$ with compact support, $\exists (f_k)_{k\geqslant 1} \subset C_0^\infty(\mathbb{R})$ and $\operatorname{supp} f_k \subset \operatorname{supp} f + \bar{B}(0, 1/k)$ with $f_k \to f$ in $L^2(\mathbb{R}, \mathrm{d}x)$. Thus

$$\left\| Hf_k - \tilde{H}_f \right\| \to 0 \implies Hf_{\varphi(k)} \to \tilde{H}f \text{ a.e., for a subsequence } \varphi(k).$$

Wlog, assume $\varphi(k) = k$ by choosing subsequence properly. When $1/k < \delta/2$,

$$\lim_{k\to\infty} \int_{\operatorname{supp} f + \bar{B}(0,\delta/2)} \frac{f_k(y)}{x-y} \, \mathrm{d}y = \int_{\operatorname{supp} f + \bar{B}(0,\delta/2)} \frac{f(y)}{x-y} \, \mathrm{d}y, \qquad \forall x \notin \operatorname{supp} f.$$

Let $g(y) = \mathbb{1}_{\operatorname{supp} f + \bar{B}(0,\delta/2)}(y) \frac{1}{\pi(x-y)} \in L^2(\mathbb{R}, \mathrm{d}x)$, then $\langle f_k, g\rangle \to \langle f, g\rangle$ by L^2-continuity.

Remark. *The argument above holds with $\frac{1}{\pi(x-y)}$ replaced by any $K \in \mathrm{CZK}_\alpha$ on $\mathbb{R}^n$.*

If $T \in \mathscr{B}(L^2(\mathbb{R}^n, \mathrm{d}x))$ and $\forall f \in C_0^\infty(\mathbb{R}^n)$, $\forall x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \, \mathrm{d}y.$$

Then $T \in \mathrm{CZO}_\alpha$, it is enough to check the representation for $f \in C_0^\infty(\mathbb{R}^n)$ by density and continuity instead of $f \in L^2(\mathbb{R}^n, \mathrm{d}x)$ with compact support.

5.3 Weak type $(1,1)$ estimate for CZO_α

The main theorem of this subsection is as follows:

Theorem 5.10: Calderón–Zygmund

Let $T \in \mathrm{CZO}_\alpha$ with associated kernel $K \in \mathrm{CZK}_\alpha$, there exists a constant $c = c(n, \alpha, \|T\|, \|K\|_\alpha) \in (0, +\infty)$ such that $\forall f \in L^1(\mathbb{R}^n, \mathrm{d}x) \cap L^2(\mathbb{R}^n, \mathrm{d}x)$, $\forall \lambda > 0$,

$$\left| \{ |Tf| > \lambda \} \right| \leqslant \frac{c}{\lambda} \|f\|_1 .$$

Remark. *For the (extended) Riesz transforms, they are not bounded on $L^1(\mathbb{R}^n, \mathrm{d}x)$, hence they are not strong-type $(1,1)$.*

We prove the *Calderón–Zygmund decomposition* of L^1-functions.

Lemma 5.11: Calderón–Zygmund decomposition

There exists $c(n) = c \in (0, \infty)$ such that $\forall f \in L^1(\mathbb{R}^n, \mathrm{d}x)$, $\forall \lambda > 0$,

$$f = g + b \qquad \text{a.e. on } \mathbb{R}^n$$

with

(1) the *good part* $g \in L^\infty(\mathbb{R}^n, \mathrm{d}x)$ with $\|g\|_\infty \leqslant c\lambda$.

(2) the *bad part* $b = \sum_{i \geqslant 1} b_i$, and there are open balls $\{B_i\}_{i\geqslant 1}$ with

  (i) $b_i = 0$ on B_i^c, *i.e.*, $\operatorname{supp} b_i \subset B_i$.

  (ii) $\int_{B_i} |b_i(y)| \,\mathrm{d}y \leqslant c\lambda |B_i|$.

  (iii) $\int_{\mathbb{R}^n} b_i(y)\,\mathrm{d}y = 0$.

  (iv) The balls B_i have bounded overlaps: $\sum_{i\geqslant 1} \mathbb{1}_{B_i} \leqslant c$.

  (v) $\sum_{i \geqslant 1} |B_i| \leqslant \frac{c}{\lambda} \|f\|_1$.

The decomposition $f = g + b$ is called the *Calderón–Zygmund decomposition* of f.

Proof. The constant c may be different in the proof. We take the maximum of the constants appeared in the proof to be the desired c.

Let Mf be the centred maximal function w.r.t. Lebesgue measure. Apply maximal theorem to $\Omega_\lambda = \{Mf > \lambda\}$, which is an open set with $|\Omega_\lambda| \leqslant \frac{c_M}{\lambda} \|f\|_1$ for $\Omega_\lambda \neq \mathbb{R}^n$.

If $\Omega_\lambda = \varnothing$, then $|f| \leqslant Mf$ a.e. on $\mathbb{R}^n$. We take $g = f$ and $b = 0$, done.

Thus we assume $\Omega_\lambda \neq \varnothing$ in the following discussion. We have to construct g and b on Ω_λ. Apply Whitney covering theorem, there exist balls $(\tilde{B}_i)_{i\geqslant 1}$ and $c_1 > 0$ such that

- $\tilde{B}_i$ are mutually disjoint, and $\Omega_\lambda = \bigcup_{i\geqslant 1} c_1 \tilde{B}_i$.
- $c_1 \tilde{B}_i \subset \Omega_\lambda$ with bounded overlap, denote the constant $c(n)$.
- $4c_1 \tilde{B}_i \cap \Omega_\lambda^c \neq \varnothing$.

Set $B_i = c_1 \tilde{B}_i$, then (iv) is true by construction, and (v) is derived by

$$\sum_{i\geqslant 1} |B_i| = c_1^n \sum_{i \geqslant 1} |\tilde{B}_i| = c_1^n \left| \bigcup_{i\geqslant 1} \tilde{B}_i \right| \leqslant c_1^n |\Omega_\lambda| \leqslant \frac{c_M c_1^n}{\lambda} \|f\|_1 .$$

Now we start to define b_i for $i \geqslant 1$. First, let

$$\varphi_i(x) = \begin{cases} \mathbb{1}_{B_i}(x), & x \in \Omega_\lambda, \\ \sum_{j\geqslant 1} \mathbb{1}_{B_j}(x), & x \in \Omega_\lambda^c. \end{cases}$$

Then $(\varphi_i)_{i\geqslant 1}$ is a partition of unity on Ω_λ, *i.e.*, $\sum_{i\geqslant 1} \varphi_i = \mathbb{1}_{\Omega_\lambda}$. Then set

$$b_i = \begin{cases} f\varphi_i - \fint_{B_i} f\varphi_i, & x \in B_i, \\ 0, & x \in B_i^c. \end{cases}$$

Then (i) holds by construction. As for (ii), note that

$$\int_{B_i} |f\varphi_i| \,\mathrm{d}x \leqslant \int_{B_i} |f| \,\mathrm{d}x \leqslant \int_{4B_i} |f| \,\mathrm{d}x.$$

Since $4B_i \cap \Omega_\lambda^c \neq \varnothing$, $\exists z \in 4B_i$ with $Mf(z) \leqslant \lambda$. While

$$\int_{4B_i} |f| \,\mathrm{d}x \leqslant Mf(z) |4B_i| \implies \int_{B_i} |f| \,\mathrm{d}x \leqslant \lambda \cdot 4^n |B_i|.$$

Thus $\int_{B_i} |b_i| \,\mathrm{d}x \leqslant 2\lambda \cdot 4^n |B_i|$. Hence (ii) holds. And (iii) is because $\int_{\mathbb{R}^n} b_i(y)\,\mathrm{d}y = \int_{B_i} b_i(y)\,\mathrm{d}y$.

We define $b = \sum_{i\geqslant 1} b_i$, then

$$\int_{\mathbb{R}^n} |b(y)| \,\mathrm{d}y = \sum_i \geqslant 1 \int_{\mathbb{R}^n} |b_i(y)| \,\mathrm{d}y \leqslant \sum_{i\geqslant 1} 2\lambda \cdot 4^n |B_i| \leqslant 2\lambda \cdot 4^n \frac{c}{\lambda} \|f\|_1 = 2c \cdot 4^n \|f\|_1 .$$

Hence $b \in L^1(\mathbb{R}^n, \mathrm{d}x)$ with $\|b\|_1 \leqslant \tilde{c} \|f\|_1$.

Set $g = f - b \in L^1(\mathbb{R}^n, \mathrm{d}x)$. By construction,

$$g = \begin{cases} f, & x \in \Omega_\lambda^c, \\ \sum_{i \geqslant 1} \left(\fint_{B_i} f\varphi_i \right) \mathbb{1}_{B_i}, & x \in \Omega_\lambda. \end{cases}$$

On Ω_λ^c, $|f| \leqslant Mf < \lambda$ a.e.. And on Ω_λ,

$$|g| \leqslant \sum_{i \geqslant 1} \left(\fint_{B_i} |f\varphi_i| \right) \mathbb{1}_B \leqslant 4^n \lambda \sum_{i \geqslant 1} \mathbb{1}_{B_i} = 4^n \lambda c.$$

Hence $g \in L^\infty(\mathbb{R}^n, \mathrm{d}x)$. □

Remark. *The constant $c(n)$ in the statement is independent of f and λ.*

We have seen that $b \in L^1(\mathbb{R}^n, \mathrm{d}x)$ with $\|b\|_1 \leqslant 2 \cdot 4^n c(n) \|f\|_1$. Hence $g \in L^1(\mathbb{R}^n, \mathrm{d}x)$ with $\|g\|_1 \leqslant (1 + 2 \cdot 4^n c(n)) \|f\|_1$. So $g \in L^1 \cap L^\infty \subset L^2$ with

$$\|g\|_2 \leqslant \sqrt{\|g\|_1 \|g\|_\infty} \leqslant c''(n) \lambda^{1/2} \|f\|_1^{1/2}.$$

Lemma 5.12

If $K \in \mathrm{CZK}_\alpha$, then

$$\sup_{(y,y') \in \Delta^c} \int_{\{|x-y| \geqslant 2|y-y'|\}} |K(x,y) - K(x,y')| \, \mathrm{d}x \leqslant c(n, \alpha, \|K\|_\alpha).$$

Proof. For convenience, we denote $I = \int_{\{|x-y| \geqslant 2|y-y'|\}} |K(x,y) - K(x,y')| \, \mathrm{d}x$. Then estimate

$$\begin{aligned} I &\leqslant \int_{\{|x-y| \geqslant 2|y-y'|\}} \|K\|_\alpha \left(\frac{y-y'}{x-y} \right)^\alpha \frac{1}{|x-y|^n} \, \mathrm{d}x \\ &\leqslant \sum_{j \geqslant 0} \int_{\{2^j |y-y'| \leqslant x < 2^{j+2} |y-y'|\}} \|K\|_\alpha \left(\frac{y-y'}{x-y} \right)^\alpha \frac{1}{|x-y|^n} \, \mathrm{d}x \\ &\leqslant \|K\|_\alpha \sum_{j \geqslant 0} \frac{|y-y'|^\alpha}{(2^{j+1}|y-y'|)^{n+\alpha}} \left|2^{j+2}|y-y'|\right|^n |B(0,1)| \\ &= |B(0,1)| \, \|K\|_\alpha \sum_{j \geqslant 0} \frac{(2^{j+2})^n}{(2^{j+1})^{n+\alpha}}, \end{aligned}$$

which is finite. □

Now, let us prove the main theorem.

Proof of theorem 5.10.

Since $f \in L^1(\mathbb{R}^n, \mathrm{d}x) \cap L^2(\mathbb{R}^n, \mathrm{d}x)$ and $\lambda > 0$, by Calderón–Zygmund decomposition, $f = g + b$. We have shown that $f \in L^2$, $g \in L^2$, thus $b \in L^2$ with $Tf = Tg + Tb$. So

$$|\{|Tf| > \lambda\}| \leqslant \left| \left\{ |Tg| > \frac{\lambda}{2} \right\} \right| + \left| \left\{ |Th| > \frac{\lambda}{2} \right\} \right|.$$

For the good part, we have

$$\left| \left\{ |Tg| > \frac{\lambda}{2} \right\} \right| \leqslant \frac{1}{(\lambda/2)^2} \int_{\mathbb{R}^n} |Tg|^2 \, \mathrm{d}x \leqslant \frac{\|T\|_2^2}{(\lambda/2)^2} \|g\|_2^2 \leqslant \frac{4\|T\|^2}{\lambda^2} c''(n)^2 \|f\|_1 \lambda = \frac{4\|T\|_2^2 c''(n)^2}{\lambda} \|f\|_1.$$

As for the bad part $b = \sum_{i \geqslant 1}$, it is convergent in L^2, so

$$\left| b - \sum_{i=1}^k b_i^2 \right| = \left| \sum_{i \geqslant k+1} b_i^2 \right| \leqslant \left| \sum_{i \geqslant k+1} b_i \mathbb{1}_{B_i} \right|^2 \leqslant \left(\sum_{i \geqslant k+1} |b_i|^2 \right) \left(\sum_{i \geqslant k+1} \mathbb{1}_{B_i} \right)^2 \leqslant c(n) \left(\sum_{i \geqslant k+1} |b_i|^2 \right) \to 0.$$

By Calderón–Zygmund decomposition, $\left|\fint_{B_i} f\varphi_i\right|^2 \leqslant c\lambda^2$, and $|f\varphi_i|^2 \leqslant |f|^2 \mathbb{1}_{B_i}$ by definition of φ_i. Thus

$$|b_i|^2 = \left|\left(f\varphi_i - \fint_{B_i} f\varphi_i\right)\mathbb{1}_{B_i}\right|^2 \leqslant 2|f\varphi_i|^2 + 2\left|\fint_{B_i} f\varphi_i\right|\mathbb{1}_{B_i} \leqslant (2|f|^2 + 2c\lambda^2)\mathbb{1}_{B_i}.$$

Hence

$$\sum_{i\geqslant 1} |b_i|^2 \leqslant (2|f|^2 + c\lambda^2)\sum_{i\geqslant 1} \mathbb{1}_{B_i} \leqslant (2|f|^2 + c\lambda^2)c(n)\mathbb{1}_{\Omega_\lambda} \in L^1(\mathbb{R}^n, \mathrm{d}x).$$

Since $f \in L^2(\mathbb{R}^n, \mathrm{d}x)$ and $|\Omega_\lambda| < +\infty$, by DCT,

$$\int_{\mathbb{R}^n} \left|b - \sum_{i=1}^k b_i\right|^2 \mathrm{d}x \to 0.$$

Thus $Tb = \sum_{i\geqslant 1} Tb_i \in L^2(\mathbb{R}^n, \mathrm{d}x)$ since T is continuous. Thus $|Tb| \leqslant \sum_{i\geqslant 1} |Tb_i|$ a.e.. Let $c > 1$ to be chosen,

$$\left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right| \leqslant \left|\bigcup_{j\geqslant 1} cB_j\right| + \left|(\mathbb{R}^n \setminus \bigcup_{j\geqslant 1} cB_j) \cap \left\{|Tb| > \frac{\lambda}{2}\right\}\right|.$$

By (v),

$$\left|\bigcup_{j\geqslant 1} cB_j\right| \leqslant c^n \sum_{j\geqslant 1} |B_j| \leqslant c^n c(n)\frac{\|f\|_1}{\lambda}.$$

And the latter term $A := \left|(\mathbb{R}^n \setminus \bigcup_{j\geqslant 1} cB_j) \cap \{|Tb| > \frac{\lambda}{2}\}\right|$ satisfies

$$A \leqslant \left|\left\{x \notin \bigcup_{j\geqslant 1} cB_j\right\} : \sum_{i\geqslant 1}\left|Tb_i(x) > \frac{\lambda}{2}\right|\right| \leqslant \frac{1}{\lambda}\int_{\mathbb{R}^n \setminus \bigcup_{j\geqslant 1} cB_j} \sum_{i\geqslant 1} |Tb_i(x)|\,\mathrm{d}x \leqslant \frac{1}{\lambda}\sum_{i\geqslant 1}\int_{\mathbb{R}^n\setminus cB_i} |Tb_i(x)|\,\mathrm{d}x.$$

For $i \geqslant 1$ and a.e. $x \in \mathbb{R}^n \setminus cB_i$,

$$Tb_i(x) = \int_{\mathbb{R}^n} K(x,y)b_i(y)\,\mathrm{d}y = \int_{B_i} K(x,y)b_i(y)\,\mathrm{d}y = \int_{B_i} (K(x,y) - K(x,y_i))b_i(y)\,\mathrm{d}y,$$

the last equality is because $\int_{B_i} b_i(y)\,\mathrm{d}y = 0$ with y_i the centre of B_i. Therefore

$$\int_{\mathbb{R}^n\setminus cB_i} |Tb_i(x)|\,\mathrm{d}x \leqslant \int_{B_i} |b_i(y)| \int_{\{x\in\mathbb{R}^n\setminus cB_i : |x-y_i| \geqslant cr(B_i) > c|y-y_i|\}} |K(x,y) - K(x,y_i)|\,\mathrm{d}x\,\mathrm{d}y.$$

Pick $c = 2$, the RHS above $\leqslant \int_{B_i} |b_i(y)|\, c(n,\alpha,\|K\|_\alpha)\,\mathrm{d}y$. Thus

$$A \leqslant \frac{c(n,\alpha,\|K_\alpha\|)}{\lambda}\sum_{i\geqslant 1}\int_{B_i} |b_i(y)|\,\mathrm{d}y \leqslant \frac{c(n,\alpha,\|K\|_\alpha)}{\lambda}\sum_{i\geqslant 1} c\lambda|B_i| \leqslant \frac{c'}{\lambda}\|f\|_1.$$

Combining all the estimates above, done. □

Remark. *For K, we use size and regularity w.r.t. y instead of x.*

Lemma 5.13

If $T \in \mathrm{CZO}_\alpha$, then $T^* \in \mathrm{CZO}_\alpha$ with its kernel K^* satisfies $K^*(x,y) = \overline{K(x,y)}$.

Proof. Let $f, g \in L^2$ with compact and disjoint support,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)||f(y)||g(x)|\,\mathrm{d}y\,\mathrm{d}x < +\infty.$$

Use Fubini's theorem, note that $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$,

$$\begin{aligned}\int_{\mathbb{R}^n} Tf(x)\overline{g(x)}\,\mathrm{d}x &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y\,\overline{g(x)}\,\mathrm{d}x\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(y)\overline{\int_{\mathbb{R}^n} K^*(y,x)g(x)\,\mathrm{d}x}\,\mathrm{d}y = \int_{\mathbb{R}^n} f(y)\overline{T^*g(y)}\,\mathrm{d}y.\end{aligned}$$

Since f is arbitary with compact support, which is disjoint from $\operatorname{supp} g$, thus $T^*g(y) = \int_{\mathbb{R}^n} K^*(y,x)g(x)\,\mathrm{d}x$ a.e. □

Corollary 5.14: Extrapolation

If $T \in \mathrm{CZO}_\alpha$, then $\forall p \in (1,\infty)$, T is of strong type (p,p).

Proof. *If $1<p<2$:* Note that $L^1 \cap L^2$ is stable under multiplicating characteristic functions,

$$\begin{aligned} T \text{ has weak type } (1,1) &\implies \left|\left\{\left|Tf\right| > \lambda\right\}\right| \leqslant \frac{c}{\lambda}\left\|f\right\|_1, \quad f \in L^1 \cap L^2, \\ T \text{ has strong type } (2,2) &\implies \left\|Tf\right\|_2 \leqslant \|T\|_{2,2}\left\|f\right\|_2, \qquad f \in L^2. \end{aligned}$$

Use Marcinkiewicz interpolation, for $1<p<2$, T has strong type (p,p), *i.e.*,

$$\left\|Tf\right\|_p \leqslant c(p,n,\|T\|)\left\|f\right\|_p, \qquad f \in L^1 \cap L^2.$$

And note that $L^1 \cap L^2$ is dense in L^p, T is linear, hence T has a bounded extension T_p on L^p.

If $2<p<\infty$: We use duality. Let $f \in L^p \cap L^2$, T^* the adjoint of $T \in \mathscr{B}(L^2)$, *i.e.*,

$$\int_{\mathbb{R}^n} Tf(x)\overline{g(x)}\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\overline{T^*g(x)}\,\mathrm{d}x.$$

Then $T^* \in \mathrm{CZO}_\alpha$ because $T \in \mathrm{CZO}_\alpha$. Since $2<p<\infty$, we have $1<p'<2$ and apply the first case to T^*,

$$\left|\int_{\mathbb{R}^n} Tf(x)\overline{g(x)}\,\mathrm{d}x\right| \leqslant \left\|f\right\|_p \left\|T^*g\right\|_{p'} \leqslant \left\|f\right\|_p c' \left\|g\right\|_{p'}.$$

Since $L^1 \cap L^2$ is dense in $L^{p'}$, $Tf \in L^p$ with $\left\|Tf\right\|_p \leqslant c'\left\|f\right\|_p$ due to the Riesz duality between L^p and $L^{p'}$. SO this is true for $f \in L^p \cap L^2$. We can extend by density to a bounded operator T_p on L^p. □

Example 5.15

Extensions of Hilbert and Riesz transforms on L^2 are of strong type (p,p) for $1<p<\infty$. In particular, $\exists c(n,p) < +\infty$ such that $\forall f \in \mathscr{S}(\mathbb{R}^n)$,

$$\left\|R_j f\right\|_p \leqslant c(n,p,j)\left\|f\right\|_p, \qquad j \in [\![1,n]\!].$$

The Fourier transform gives

$$\widehat{\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} f\right)} = (-\mathrm{i}\xi_j)(-\mathrm{i}\xi_k)\hat{f} = -\xi_j\xi_k\hat{f}.$$

The Laplacian $\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$, thus $\widehat{\Delta f} = -|\xi|^2\hat{f}$. When $\xi \neq 0$,

$$(-\mathrm{i}\xi_j)(-\mathrm{i}\xi_k)\hat{f} = -\left(\frac{-\mathrm{i}\xi_j}{|\xi|}\right)\left(\frac{-\mathrm{i}\xi_k}{|\xi|}\right)(-|\xi|^2\hat{f}) = -(\tilde{R}_j\tilde{R}_k\Delta f)\widehat{}\,.$$

Here $\tilde{R}_j$ denotes the bounded extension of R_j on L^2. Thus $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} f = -R_jR_k\Delta f$, hence

$$\left\|\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} f\right\|_p \leqslant c(n,p,j,k)\left\|\Delta f\right\|_p, \qquad \forall f \in \mathscr{S}(\mathbb{R}^n).$$

5.4 Mikhlin multiplier theorem

Definition 5.16: Mikhlin multiplier

Let $m \in L^\infty(\mathbb{R}^n, \mathrm{d}x)$, define an operator

$$T_m : L^2(\mathbb{R}^n, \mathrm{d}x) \to L^2(\mathbb{R}^n, \mathrm{d}x), \qquad f \mapsto T_m f,$$

where $\widehat{T_m f} = m \cdot \hat{f}$. Then T_m is called a *Mikhlin multiplier* and m is called its *symbol*.

By Planchernel identity,

$$\|T_m f\|_2 = \frac{1}{(2\pi)^{n/2}} \left\|\widehat{T_m f}\right\|_2 = \frac{1}{(2\pi)^{n/2}} \|m\hat{f}\|_2,$$

thus $\|T_m f\|_2 \leqslant \|m\|_\infty \|f\|_2$, which shows that T_m is a bounded operator on $L^2(\mathbb{R}^n, \mathrm{d}x)$. The Mikhlin multiplier can be seen as the generalisation of Riesz transforms, just set $m_j(\xi) = -\mathrm{i}\frac{\xi_j}{|\xi|}$.

Definition 5.17: Mikhlin symbol

Say m is a *Mikhlin symbol* if

- $m \in L^\infty(\mathbb{R}^n)$.
- $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and for all $\alpha \in \mathbb{N}^*$, $\exists c(\alpha) > 0$ such that when $\xi \neq 0$, $|\partial^\alpha m(\xi)| \leqslant c(\alpha)/|\xi|^{|\alpha|}$.

Theorem 5.18

If m is a Mikhlin symbol, then $T_m \in \mathrm{CZO}_1$. Hence T_m is of strong type (p,p) for $1 < p < \infty$.

Proof. We have to calculate the inverse Fourier transform of m, denote it by K. Our goal is that:

- Show $K \in C^1(\mathbb{R}^n \setminus \{0\})$ and $|K(x)| \leqslant c|x|^{-n}$, $|\nabla K(x)| \leqslant c|x|^{-(n+1)}$.
- Show that $T_m f(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\,\mathrm{d}y$ for all $f \in C_0^\infty(\mathbb{R}^n)$ and a.e. $x \notin \operatorname{supp} f$.

Use the reasoning same as R_j, the second goal follows from the first. Then $T_m \in \mathrm{CZO}_1$ follows from $\|T_m\| \leqslant \|m\|_\infty < +\infty$.

We first compute in $\mathscr{S}(\mathbb{R}^n)$. Let $\psi \in \mathscr{S}(\mathbb{R}^n)$. We want to calculate and estimate $\check{m} = k$ in $\mathscr{S}'(\mathbb{R}^n)$. Thus, we accomplish the smooth partition of unity associated to annuli. (The *Littlewood–Paley decomposition*).

Let $w : \mathbb{R}_+ \to [0,1]$ be a smooth and positive function, $\operatorname{supp} w \in [1/2, 4]$, $w|_{[1,2]} = 1$. Define

$$W(t) = \frac{w(t)}{\sum_{j\in\mathbb{Z}} w(2^{-j}t)}, \qquad \forall t \in \mathbb{R}_+.$$

Then W is smooth because $t \mapsto \sum_{j\in\mathbb{Z}} w(2^{-j}t)$ is smooth on $\mathbb{R}_+$, which is actually a finite sum with no more than 3 non-zero terms in the series for each t. From $\bigcup_{k\in\mathbb{Z}} [2^k, 2^{k+1}] = \mathbb{R}_+$, we also have

$$1 \leqslant \sum_{j\in\mathbb{Z}} w\left(\frac{t}{2^j}\right) \leqslant 3.$$

Thus W satisfies:

- $W : \mathbb{R}_+ \to [0,1]$ is a smooth function with $\operatorname{supp} W \in [1/2, 4]$.
- $\forall k \in \mathbb{N}$,

$$\sup_{t\in\mathbb{R}_+} \left|W^{(k)}(t)\right| = \sup_{t\in[1/2,4]} \left|W^{(k)}(t)\right| = c_k < +\infty,$$

because $W^{(k)}$ is a continuous function supported on a compact set.

Define $\varphi : \mathbb{R}^n \to [0,1]$, $\xi \mapsto W(|\xi|)$, then

- φ is smooth with $\operatorname{supp}\varphi = C_0 := \{\xi \in \mathbb{R}^n : 1/2 \leqslant |\xi| \leqslant 4\}$.
- φ is radial.
- $\forall \alpha \in \mathbb{N}^n$, $\|\partial^\alpha \varphi\|_\infty = c_\alpha < +\infty$ because $\partial^\alpha\varphi$ is a continuous function supported on a compact set.
- $\forall \xi \in \mathbb{R}^n$,
$$\sum_{j\in\mathbb{Z}} \varphi\left(\frac{\xi}{2^j}\right) = \sum_{j\in\mathbb{Z}} W\left(\frac{|\xi|}{2^j}\right) = \begin{cases} 1, & \text{if } \xi \neq 0,\\ 0, & \text{if } \xi = 0.\end{cases}$$
So $\varphi(2^{-j}\xi)$ is a partition of unity up to $\mathrm{d}x$-null sets.

In particular, if we calculate
$$\frac{\partial^\alpha}{\partial\xi^\alpha}\varphi\left(\frac{\xi}{2^j}\right) = \frac{1}{2^{j|\alpha|}}(\partial^\alpha\varphi)(\xi).$$
Then $\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\varphi(2^{-j}\xi)\right| \leqslant 2^{-j|\alpha|}c_\alpha \sim |\xi|^\alpha$.

Now calculate
$$\begin{aligned}\langle \check{m}, \psi\rangle = \langle m, \check{\psi}\rangle &= \int_{\mathbb{R}^n} m(\xi)\check{\psi}(\xi)\,\mathrm{d}\xi = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} m(\xi)\varphi\left(\frac{\xi}{2^j}\right)\check{\psi}(\xi)\,\mathrm{d}\xi\\ &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\left(\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} m(\xi)\varphi\left(\frac{\xi}{2^j}\right)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi\right)\psi(x)\,\mathrm{d}x =: \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n} k_j(x)\psi(x)\,\mathrm{d}x = \left\langle \sum_{j\in\mathbb{Z}} k_j, \psi\right\rangle.\end{aligned}$$
Then $\check{m} = k = \sum_{j\in\mathbb{Z}} k_j$ in $\mathscr{S}'(\mathbb{R}^n)$, and $k_j(\xi) = (m(\xi)\varphi(2^{-j}\xi))^{\vee} \in C^\infty(\mathbb{R}^n)$.

We claim that $\forall\alpha\in\mathbb{N}^n$, $\forall M\in\mathbb{N}$, $\exists c = c(\alpha, N)$ such that
$$\left|\partial^\alpha k_j(x)\right| \leqslant c\frac{2^{nj}2^{j|\alpha|}}{(1+|2^j x|)^M}.$$
We admit the claim then, if $M > n+|\alpha|$,
$$\sum_{j\in\mathbb{Z}}\frac{2^{nj}2^{j|\alpha|}}{(1+|2^jx|)^M} \leqslant \frac{c(n,|\alpha|,M)}{|x|^{n+|\alpha|}}.$$
Fix $x\neq 0$, let j_0 be the largest integer with $2^{j_0}|x| = |2^{j_0}x| \leqslant 1$. Then
$$\sum_{j\leqslant j_0}\frac{2^{nj}2^{j|\alpha|}}{(1+|2^jx|)^M} \leqslant \sum_{j\geqslant j_0} 2^{jn}2^{j|\alpha|} = c(n+|\alpha|)2^{j_0 n}2^{j_0|\alpha|} \leqslant \frac{c(n+|\alpha|)}{|x|^{n+|\alpha|}}.$$
And
$$\begin{aligned}\sum_{j>j_0}\frac{2^{nj}2^{j|\alpha|}}{(1+|2^jx|)^M} &\leqslant \frac{1}{|x|^M}\sum_{j>j_0}\frac{2^{jn}2^{j|\alpha|}}{2^{jM}} = \frac{1}{|x|^M}c(M-n-|\alpha|)2^{(j_0+1)(n+|\alpha-M|)}\\ &= c(M-n-|\alpha|)\left|2^{j_0+1}x\right|^{n+|\alpha|-M}\frac{1}{|x|^{n+|\alpha|}} \leqslant \frac{c(M-n-|\alpha|)}{|x|^{n+\alpha}}.\end{aligned}$$
For the last inequality, note that $|2^{j_0+1}x| \geqslant 1$ and $n+|\alpha|-M\leqslant 0$. Hence $k\in C^\infty(\mathbb{R}^n\setminus\{0\})$ with
$$|\partial^\alpha k(x)| \leqslant \frac{c(n,|\alpha|,M)}{|x|^{n+\alpha}}, \qquad \forall x\neq 0,\ \forall\alpha\in\mathbb{N}^n.$$
Therefore, we conclude $K(x,y) = k(x-y)$, $K\in\mathrm{CZK}_1$.

Now we prove the claim. Recall that
$$k_j(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} m(\xi)\varphi\left(\frac{\xi}{2^j}\right)\mathrm{e}^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi.$$
When $\alpha = 0$,
$$\|k_j\|_\infty \leqslant \frac{1}{(2\pi)^n}\|m\cdot\varphi(2^{-j}\cdot)\|_1 \leqslant \frac{\|m\|_\infty}{(2\pi)^n}\|\varphi(2^{-j}\cdot)\|_1 \leqslant \frac{\|m\|_\infty}{(2\pi)^n}2^{jn}\|\varphi\|_1.$$

For $\ell \in \mathbb{N}$, we have

$$|x|^{2\ell} k_j(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (-\Delta_\xi)^\ell (m(\xi)\varphi(2^{-j}\xi)) e^{ix\cdot\xi} \, d\xi,$$

and

$$\partial_\xi^\beta (m(\xi)\varphi(2^{-j}\xi)) = \sum_{\gamma+\delta=\beta} \binom{\gamma}{\delta} (\partial_\xi^\gamma m)(\partial_\xi^\delta \varphi(2^{-j}\xi)) \leqslant \sum_{\gamma+\delta=\beta} \binom{\gamma}{\delta} c_{m,j} |\xi|^{-|\gamma|} c_j 2^{-j|\delta|} \lesssim 2^{-j(|\gamma|+|\delta|)}.$$

Thus

$$\sup_{\xi\in\mathbb{R}^n} \left| \partial_\xi^\beta (m(\xi)\varphi(2^{-j}\xi)) \right| \leqslant c \cdot 2^{-j|\beta|}$$

and it is supported on $\frac{1}{2} \leqslant \frac{\xi}{2^j} \leqslant 4$. Hence

$$\left\| \partial_\xi^\beta (m \cdot \varphi(2^{-j}\cdot)) \right\|_1 \leqslant \left\| \partial_\xi^\beta (m \cdot \varphi(2^{-j}\cdot)) \right\|_\infty \left| \operatorname{supp}(m \cdot \varphi(2^{-j}\cdot)) \right| \leqslant c 2^{-j|\beta|} \cdot c' 2^{jn} \leqslant c \cdot 2^{-j|\beta|} 2^{jn}.$$

Let $|\beta| = 2\ell$, then

$$\left| |x|^{2\ell} k_j(x) \right| \leqslant c 2^{-j\cdot 2\ell} 2^{jn} \implies \left| \left| 2^j x \right|^{2\ell} k_j(x) \right| \leqslant c 2^{jn}.$$

So for $\ell \in \mathbb{N}$, $\sup_{x\in\mathbb{R}^n} (1 + |2^j x|)^{2\ell} |k_j(x)| \leqslant c 2^{jn}$.

The α-derivatives of $k_j(x)$ can be considered from the α-derivative of $m \cdot \varphi(2^{-j}\cdot)$.

$$m(\xi)\varphi(2^{-j}\xi)(i\xi)^\alpha = m(\xi)\varphi_\alpha(2^{-j}\xi) 2^{j|\alpha|},$$

where $\varphi_\alpha(\xi) = \varphi(\xi)(i\xi)^\alpha$. Note that φ_α has the same support as φ and $\varphi_\alpha \in C^\infty(\mathbb{R}^n)$. So the same estimation goes. $\square$

6 BMO and H^1

Setup. We have proved CZOs do not act as bounded operators from L^1 to L^1, nor L^∞ to L^∞, but L^p to L^p when $1 < p < \infty$. So the question is, is there any « substitution » of L^1 and L^∞?

We shall develop two spaces this section, the Hardy space H^1 to replace L^1, and the space BMO to replace L^∞.

6.1 The space BMO

We always work on measure space $(\mathbb{R}^n, dx)$.

Definition 6.1: BMO functions

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we say that f is of *bounded mean oscallation*, or *BMO* for short, if

$$\|f\|_* := \sup_{\text{cubes } Q} \fint_Q |f - m_Q f| < +\infty,$$

where $m_Q f = \fint_Q f = \frac{1}{|Q|}\int_Q f =: f_Q$. The cubes (*open or closed*) are with sides parallel to axes. The space of all BMO functions on $\mathbb{R}^n$ is denoted as $\mathrm{BMO}(\mathbb{R}^n)$, or just BMO when the space is clear.

Proposition 6.2

It is not hard to check the basic properties of BMO functions.

(1) $L^\infty \subset \mathrm{BMO}$ with $\|f\|_* \leqslant 2\|f\|_\infty$. (*But false reversely, for example,* $\log|x| \in \mathrm{BMO}(\mathbb{R})$ but not L^∞).

(2) BMO is a semi-normed vector space:

$$\|f+g\|_* \leqslant \|f\|_* + \|g\|_*, \qquad \|\lambda f\|_* = |\lambda| \, \|f\|_*,$$

and $\|f\|_* = 0 \Longleftrightarrow f = m_Q f$ a.e. $\Longleftrightarrow f = \text{const}$ a.e..

(3) BMO/$\mathbb{K}$ is a Banach space with $\|[f]\|_* = \|f\|_*$.

(4) If $f \in$ BMO, $x_0 \in \mathbb{R}^n$, $t > 0$. Let $g = f((x - x_0)/t)$, then $\|g\|_* = \|f\|_*$. Moreover, if $f \in L^\infty$, then $\|g\|_\infty = \|f\|_\infty$.

Lemma 6.3

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that for any cube Q, $\exists c_Q \in \mathbb{K}$ such that

$$A = \sup_Q \fint_Q |f - c_Q| < +\infty,$$

then $f \in$ BMO with $\|f\|_* \leqslant 2A$.

Proof. This is because

$$f - m_Q f = f - c_Q + c_Q - m_Q f = (f - c_Q) + m_Q(c_Q - f).$$

Thus

$$\fint_Q |f - m_Q f| \leqslant \fint_Q |f - c_Q| + |m_Q(c_Q - f)| \leqslant 2A,$$

which gives $\|f\|_* \leqslant 2A < +\infty$. Hence $f \in$ BMO. $\square$

Proposition 6.4

We denote $\text{BMO}_{\mathbb{K}}$ the BMO functions that are $\mathbb{K}$-valued, here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

(1) If $f \in$ BMO, then $|f| \in$ BMO with $\||f|\|_* \leqslant 2\|f\|_*$.

(2) $f \in \text{BMO}_{\mathbb{C}}$ if and only if $\operatorname{Re} f$, $\operatorname{Im} f \in \text{BMO}_{\mathbb{R}}$. And $\|f\|_* \sim \|\operatorname{Re} f\|_* + \|\operatorname{Im} f\|_*$.

(3) When $\mathbb{K} = \mathbb{R}$, we do the cut-off as follows: for $N > 0$, define

$$f_N := \begin{cases} -N, & f < -N, \\ f, & -N \leqslant f \leqslant N, \\ N, & f > N \end{cases} \implies f_N \in L^\infty(\mathbb{R}^n).$$

If $f \in$ BMO, then $f_N \in$ BMO with $\|f_N\|_* \leqslant 2\|f\|_*$.

(4) If $f \in$ BMO, Q, R are two cubes with $Q \subset R$, then

$$|m_Q f - m_R f| \leqslant \frac{|R|}{|Q|} \|f\|_*,$$

note that $\{m_Q f : Q \text{ cubes}\}$ is not a bounded family.

Proof. (1) Let $c_Q = |m_Q f|$, then

$$\fint_Q \big||f| - |m_Q f|\big| \leqslant \fint_Q |f - m_Q f| \leqslant \|f\|_*.$$

Hence $\|f\|_* \geqslant A$, then apply the previous lemma.

(2) Obvious.

(3) For any $x, y \in \mathbb{R}^n$, we have $\left|f_N(x) - f_N(y)\right| \leqslant \left|f(x) - f(y)\right|$. Thus

$$\begin{aligned}\fint_Q \left|f_N(x) - m_Q f_N(x)\right| &= \fint_Q \left|f_N(x) - \fint_Q f_N(y)\,\mathrm{d}y\right| \mathrm{d}x \\ &\leqslant \fint_Q \fint_Q \left|f_N(x) - f_N(y)\right| \mathrm{d}y\,\mathrm{d}x \leqslant \fint_Q \fint_Q \left|f(x) - m_Q f\right| + \left|m_Q f - f(y)\right| \mathrm{d}y\,\mathrm{d}x \leqslant 2\left\|f\right\|_* .\end{aligned}$$

The same reasoning implies $f \in \mathrm{BMO} \implies f^+,\ f^- \in \mathrm{BMO}$. Also, if $f, g \in \mathrm{BMO}$, then $\max\{f, g\}$, $\min\{f, g\} \in \mathrm{BMO}$.

(4) This is because

$$\begin{aligned}\left|m_Q f - m_R f\right| &= \left|m_Q(f - m_R f)\right| \leqslant \fint_Q \left|f - m_R f\right| \\ &= \frac{|R|}{|Q|}\frac{1}{|R|}\int_Q \left|f - m_R f\right| \leqslant \frac{|R|}{|Q|}\fint_R \left|f - m_R f\right| \leqslant \frac{|R|}{|Q|}\left\|f\right\|_* .\end{aligned}$$

The second inequality comes from $Q \subset R$. □

Theorem 6.5: John–Niremburg inequality

If there exists $c = c(n) > 0$, $\alpha = \alpha(n) > 0$ such that $\forall f \in \mathrm{BMO}$ with $\left\|f\right\|_* \neq 0$, $\forall Q$ cube, $\forall \lambda > 0$,

$$\left|\left\{x \in Q : \left|f(x) - m_Q f\right| > \lambda\right\}\right| \leqslant c \exp\left(-\alpha \frac{\lambda}{\left\|f\right\|_*}\right)|Q|.$$

Remark. If we use Markov's inequality,

$$\left|\left\{x \in Q : \left|f(x) - m_Q f\right| > \lambda\right\}\right| \leqslant \frac{1}{\lambda}\int_Q \left|f - m_Q f\right| \leqslant \frac{\left\|f\right\|_*}{\lambda}|Q|.$$

Done because mean oscallations are controlled for all cubes.

Proof. Wlog, we may assume:

- $Q = Q_0 = [0,1)^n$ by applying to $f((x-a)/\ell)$ for $Q = \prod_{k=1}^n [a_k, a_k + \ell)$ and the scalar-invariant inequality.
- $\left\|f\right\|_* = 1$ by multiplying $f / \left\|f\right\|_*$.
- $m_{Q_0} f = 0$ by $\left\|f\right\|_* = \left\|f - m_{Q_0} f\right\|_*$.
- $f \in L^\infty(\mathbb{R}^n)$ by cut-off functions, (3) in the previous proposition.

If $\lambda > 1$ or $c = 1,\ \alpha = 1$, then we have done. Denote

$$\mathscr{D}(Q) = \left\{\text{dyadic subcubes of } Q\right\}, \qquad \mathscr{E}_\lambda = \left\{x \in Q : M_{d,Q} f(x) > \lambda\right\} = \bigcup_i Q_{i,\lambda}.$$

Here $\mathscr{E}_\lambda$ becomes smaller as λ grows. $Q_{i,\lambda}$ are dyadic subcubes of Q that are maximal for $\fint_{Q_{i,\lambda}} \left|f\right| > \lambda$.

If we can show $|\mathscr{E}_\lambda| \leqslant c\mathrm{e}^{-\alpha\lambda}$, we are close since

$$\left|\left\{x \in Q : \left|f(x)\right| > \lambda\right\}\right| \leqslant |\mathscr{E}_\lambda|.$$

Since $\left|f\right| \leqslant M_{d,Q} f$ a.e. on Q, and $|\mathscr{E}_\lambda| = \sum_i \left|Q_{i,\lambda}\right|$, we have

$$m_Q \left|f\right| = m_Q \left|f - m_Q f\right| \leqslant \left\|f\right\|_* = 1 < \lambda.$$

Denote $\widehat{Q_{i,\lambda}}$ the ancestor of $Q_{i,\lambda}$, then $m_{Q_{i,\lambda}} \left|f\right| > \lambda$, $m_{\widehat{Q_{i,\lambda}}} \left|f\right| \leqslant \lambda$ by construction, and

$$\left|m_{Q_{i,\lambda}} f - m_{\widehat{Q_{i,\lambda} f}}\right| \leqslant c_0 \left\|f\right\|_*$$

for some $c_0 > 0$. Now define $R_0 := Q_{i,\lambda}$ and construct a sequence of cubes $R_0, \dots, R_n$ with $R_k \subset R_{k+1}$ and $|R_{k+1}| / |R|_k = 2$ for $k \leqslant n$, $R_{n-1} \subset \widehat{Q_{i,\lambda}} \subset R_n$. Then

$$\left| m_{R_0} f - m_{\widehat{Q_{i,\lambda}} f} \right| \leqslant \sum_{k=0}^{n-2} \left| m_{R_k} f - m_{R_{k+1}} f \right| + \left| m_{R_{n-1} f} - m_{\widehat{Q_{i,\lambda}}} f \right| \leqslant \sum_{k=0}^{n-2} 2 + 2 = 2n.$$

And $\left| m_{\widehat{Q_{i,\lambda}}} f - m_{R_n} f \right| \leqslant 2$. Hence

$$\left| m_{Q_{i,\lambda}} f \right| \leqslant 2n + \left| m_{\widehat{Q_{i,\lambda}}} f \right| \leqslant 2n + \lambda.$$

Pick $\delta > 2n+1$, $\mathscr{E}_{\lambda+\delta} = \bigcup_j Q_{j,\lambda+\delta} \subset \mathscr{E}_\lambda$. For all j, there exists a unique index i such that $Q_{j,\lambda+\delta} \subset Q_{i,\lambda}$. Then

$$\begin{aligned}
\left| \mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda} \right| &= \sum_{j: Q_{j,\lambda+\delta} \subset Q_{i,\lambda}} \left| Q_{j,\lambda+\delta} \right| \\
&\leqslant \frac{1}{\lambda+\delta} \sum_{j: Q_{j,\lambda+\delta} \subset Q_{i,\lambda}} \int_{Q_{j,\lambda+\delta}} |f| && \text{(by maximality)} \\
&= \frac{1}{\lambda+\delta} \int_{\mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda}} |f| && \text{(by disjointness)} \\
&\leqslant \frac{1}{\lambda+\delta} \int_{\mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda}} \left| f - m_{Q_{i,\lambda}} f \right| + \frac{1}{\lambda+\delta} \left| m_{Q_{i,\lambda}} f \right| \left| \mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda} \right|.
\end{aligned}$$

Thus

$$\left| \mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda} \right| \leqslant \frac{1}{\lambda+\delta} \left| Q_{i,\lambda} \right| + \frac{2n+\lambda}{\lambda+\delta} \left| Q_{i,\lambda} \right| \implies \left| \mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda} \right| \leqslant \frac{\left| Q_{i,\lambda} \right|}{\delta - 2n}.$$

Hence

$$|\mathscr{E}_{\lambda+\delta}| = \sum_i \left| \mathscr{E}_{\lambda+\delta} \cap Q_{i,\lambda} \right| \leqslant \sum_i \frac{\left| Q_{i,\lambda} \right|}{\delta - 2n} = \frac{|\mathscr{E}_\lambda|}{\delta - 2n}.$$

Let $\lambda = 2$, then $|\mathscr{E}_{2+k\delta}| \leqslant (\delta - 2n)^{-k} |\mathscr{E}_2|$.

Let f_N be the cut-off function of f between $-N$ and N, we know that $\|f_N\|_* \leqslant 2\|f\|_* = 2$. We have shown that for $N \geqslant 1$,

$$\left| \left\{ x \in Q_0 : \left| f_N(x) - m_{Q_0} f_N \right| > \lambda \right\} \right| \leqslant c \exp\left(-\alpha \frac{\lambda}{\|f_N\|_*} \right) \leqslant c \mathrm{e}^{-\alpha\lambda/2}.$$

Then let $N \to \infty$, $f_N \to f$ with $|f_N| \leqslant |f|$, and $m_Q f_N \to m_Q f = 0$. Thus

$$\left| f_N - m_{Q_0} f_N \right| \to |f|$$

in $L^1_{\mathrm{loc}}([0,1)^n)$ with domination on $[0,1)^n$. Done. □

Corollary 6.6

On BMO, we have the following equivalent semi-norms: for $p \in [1, \infty)$,

$$\|f\|_{*,p} := \sup_{\text{cubes } Q} (m_Q(|f - m_Q f|^p))^{1/p}.$$

Proof. By Jensen inequality,

$$m_Q |f - m_Q f| \leqslant (m_Q(|f - m_Q f|^p))^{1/p},$$

hence $\|f\|_* \leqslant \|f\|_{*,p}$.

Conversely, assume $\|f\|_* \neq 0$ or it would be trivial. Since

$$\begin{aligned}
\int_Q |f - m_Q f|^p \,\mathrm{d}x &= p \int_0^\infty \left| \{ x \in Q : |f(x) - m_Q f| > \lambda \} \right| \lambda^{p-1} \,\mathrm{d}\lambda \\
&= pc \int_0^\infty \exp\left(-\alpha \frac{\lambda}{\|f\|_*} \right) \left(\frac{\lambda}{\|f\|_*} \right)^p \frac{\mathrm{d}\lambda}{\lambda} \|f\|_*^p |Q| = pc \int_0^\infty \mathrm{e}^{-\alpha\mu} \mu^p \frac{\mathrm{d}\mu}{\mu} \|f\|_*^p |Q|,
\end{aligned}$$

here $\mu = \lambda / \|f\|_*$. Hence $\|f\|_{*,p} \leqslant \left(pc\int_0^\infty \mathrm{e}^{-\alpha\mu}\mu^{p-1}\,\mathrm{d}\mu\right)^{1/p}\|f\|_*$. $\square$

Remark. *The exponential inequality also holds when* $\sup_Q m_Q(\mathrm{e}^{\beta|f-m_Q f|}) < +\infty$ *if* $0 < \beta < \alpha$. *Any* $f \in \mathrm{BMO}$ *is locally* L^p*-integrable.*

6.2 Hardy space H^1

Definition 6.7: p-atoms

Let $p \in (1,\infty]$, a p-*atom* on cube $Q \subset \mathbb{R}^n$ is a function $a : Q \to \mathbb{R}$ satisfying

(1) $a = 0$ a.e. on Q^c.

(2) $\|a\|_{L^p(\mathbb{R}^n)} \leqslant \|a\|_{L^p(Q)} \leqslant |Q|^{-(1-1/p)}$.

(3) $\int_Q a\,\mathrm{d}x = 0$.

Note that (1) and (2) implies $\|a\|_{L^1(\mathbb{R}^n)} \leqslant \|a\|_{L^1(Q)} \leqslant 1$ by Hölder inequality. We denote

$$\mathscr{A}_Q^p := \{p\text{-atoms on } Q\}, \qquad \mathscr{A}^p = \bigcup_{Q\subset\mathbb{R}^n} \mathscr{A}_Q^p.$$

Definition 6.8: $H^{1,p}$ norm

For $p \in (1,\infty]$, $f \in L^1(\mathbb{R}^n)$. Say $f \in H^{1,p}(\mathbb{R}^n)$ if $\exists(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ and $(a_i)_{i\in\mathbb{N}} \in (\mathscr{A}^p)^{\mathbb{N}}$ such that $f = \sum_{i\geqslant 0}\lambda_i a_i$ a.e. (*convergent in* $L^1(\mathbb{R}^n)$). Define

$$\|f\|_{H^{1,p}} := \inf\left\{\sum_{i\geqslant 0}|\lambda_i| : f = \sum_{i\geqslant 0}\lambda_i a_i\right\}.$$

The infimum is taken over all representations of f of this form.

If $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$, then

$$\sum_{i\geqslant 0}\|\lambda_i a_i\|_1 \leqslant \sum_{i\geqslant 0} < +\infty,$$

and normal convergence implies convergence in the sense of $L^1(\mathbb{R}^n)$. We say $f = \sum_{i\geqslant 0}\lambda_i a_i$ a p-*atomic decomposition* of f.

Remark. $H^{1,p}(\mathbb{R}^n) \not\subset L^1(\mathbb{R}^n)$. *Also, although*

$$\int_{\mathbb{R}^n}\sum_{i\geqslant 0}\lambda_i a_i\,\mathrm{d}x = \sum_{i\geqslant 0}\lambda_i\int_{\mathbb{R}^n} a_i\,\mathrm{d}x = 0,$$

$H^{1,p}(\mathbb{R}^n) \not\subset L_0^1(\mathbb{R}^n) := \left\{f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f\,\mathrm{d}x = 0\right\}$.

Theorem 6.9

Assume $p \in (1,\infty]$.

(1) $(H^{1,p}, \|\cdot\|_{H^{1,p}})$ is a Banach space.

(2) If $1 < p < q \leqslant \infty$, $H^{1,\infty} \subset H^{1,q} \subset H^{1,p} \subset L^1$ with $\|f\|_1 \leqslant \|f\|_{H^{1,p}} \leqslant \|f\|_{H^{1,q}} \leqslant \|f\|_{H^{1,\infty}}$.

(3) If $p \neq \infty$, $H^{1,p} = H^{1,\infty}$ with equivalent norms.

Proof. (1) It is not difficult to check $(H^{1,p}, \|\cdot\|_{H^{1,p}})$ is a normed vector space. For the completeness, consider the series $\sum_{k\geqslant 0}\|f_k\|_{H^{1,p}} < +\infty$, which implies $\sum_{k\geqslant 0} f_k$ is convergent to $f \in H^{1,p}$.

(2) Exercise.

(3) We have known that $H^{1,\infty}\subset H^{1,p}$. It suffices to prove $H^{1,p}\subset H^{1,\infty}$.

Step 1. Do intermediate decompositions for p-atoms. We prove an intermediate inclusion: for $a\in\mathscr{A}^p$, there exist b,g with

- $b\in H^{1,p}$ with $\|b\|_{H^{1,p}}\leqslant\frac{1}{2}$,
- $g\in H^{1,\infty}$ with $\|g\|_{H^{1,\infty}}\leqslant c(n,p)$.

Note that $a\in\mathscr{A}^p$ implies that $\|a\|_{H^{1,p}}\leqslant 1$.

Fix $Q\subset\mathbb{R}^n$ a cube such that $a\in\mathscr{A}^p_Q$, we do C–Z decomposition of a with duadic subcubes of $Q(=\mathscr{D}(Q))$.

$$m_Q|a|^p=\frac{1}{|Q|}\int_Q|a|^p\,\mathrm{d}x\leqslant\frac{1}{|Q|}\left(\frac{1}{|Q|^{1-1/p}}\right)^p=\frac{1}{|Q|^p}.$$

Let $\alpha>0$ with $\alpha^p>m_Q|a|^p$ (and choose α later). Define

$$\mathscr{E}_\alpha=\left\{x\in Q:(M_{d,Q}|a|^p)^{1/p}(x)>\alpha\right\}=\left\{x\in Q:(M_{d,Q}|a|^p)(x)>\alpha^p\right\}=\sup_{Q'\in\mathscr{D}(Q)}m_{Q'}|a|^p.$$

Then $\mathscr{E}_\alpha$ is a disjoint union of maximal dyadic subcubes. $Q_i\subset\mathscr{D}(Q)$ for $m_{Q_i}|a|^p>\alpha^p$. For $i=1,2,,\dots,$ set $b_i=(a-m_{Q_i}a)\mathbb{1}_{Q_i}$, and define

$$b=\sum_{i\geqslant1}b_i,\qquad g=a-b.$$

Then we need to check the properties. Firstly, $b_i=0$ on Q^c and $\int_{Q_i}b_i\,\mathrm{d}x=0$. Secondly,

$$|b_i|_{L^p(Q_i)}\leqslant|a|_{L^p(Q_i)}+\left|m_{Q_i}a\right||Q_i|^{1/p}\leqslant2(m_{Q_i}|a|^p)^{1/p}=\frac{\lambda_i}{|Q|^{1-1/p}}.$$

Set $\lambda_i=2(m_{Q_i}|a|^p)^{1/p}|Q_i|$, then $\lambda_i>0$ such that $b_i/\lambda_i\in\mathscr{A}^p_{Q_i}$. Thus

$$\begin{aligned}
\|b\|_{H^{1,p}}&=\left\|\sum_i\lambda_i\cdot\frac{b_i}{\lambda_i}\right\|_{H^{1,p}}\leqslant\sum_i|\lambda_i|=2\sum_i(m_{Q_i}|a|^p)^{1/p}|Q_i|\\
&=2\sum_i\left(\int_{Q_i}|a|^p\,\mathrm{d}x\right)^{1/p}|Q_i|^{1-1/p}\overset{\text{Hölder}}{\leqslant}2\left(\sum_i\int_{Q_i}|a|^p\,\mathrm{d}x\right)^{1/p}\left(\sum_i|Q_i|\right)^{1/p'}\\
&\leqslant2\left(\sum_i\int_{Q_i}|a|^p\,\mathrm{d}x\right)^{1/p}|E_\alpha|^{1/p'}\leqslant2\left(\int_Q|a|^p\,\mathrm{d}x\right)^{1/p}\left(\frac{1}{\alpha^p}\int_Q|a|^p\,\mathrm{d}x\right)^{1/p'}\\
&=\frac{2}{\alpha^{p-1}}\int_Q|a|^p\,\mathrm{d}x=\frac{2}{\alpha^{p-1}}\left(\frac{1}{|Q|^{1-1/p}}\right)^p=\frac{2}{(\alpha|Q|)^{p-1}}.
\end{aligned}$$

Now pick α such that $2(\alpha|Q|)^{-(p-1)}=\frac{1}{2}$.

Then it suffices to check $\alpha^p>m_Q|a|^p$, this is because

$$m_Q|a|^p=\frac{1}{|Q|}\int_Q|a|^p\leqslant\frac{1}{|Q|}\left(\frac{1}{|Q|^{1-1/p}}\right)^p=\frac{1}{|Q|^p}=\frac{1}{(\alpha|Q|)^p}\alpha^p.$$

Note that $\alpha|Q|<1$, then $\|b\|_{H^{1,p}}\leqslant\frac{1}{2}$.

By definition, the good part

$$g=\begin{cases}a, & x\in Q\setminus\mathscr{E}_\alpha,\\ \sum_{m_{Q_i}a\cdot\mathbb{1}_{Q_i}}, & x\in\mathscr{E}_\alpha\\ 0, & x\in Q^c\end{cases}$$

Then on $\mathscr{E}_\alpha$, $\left|g\right|^p=|a|^p\leqslant M_{d,Q}|a|^p\leqslant\alpha^p$ a.e.. And on Q_i, denote the parent of Q_i as $\widehat{Q_i}$,

$$\left|g\right|^p=\left|m_{Q_i}a\right|^p\leqslant m_{Q_i}|a|^p\leqslant2^nm_{\widehat{Q_i}}|a|^p\leqslant2^n\alpha^p.$$

Thus $\left\|g\right\|_\infty\leqslant2^{n/p}\alpha$, so $\left\|\frac{g}{2^{n/p}\alpha}\cdot\frac{1}{|Q|}\right\|_\infty\leqslant\frac{1}{|Q|}$. Hence

$$\int_Q\frac{g}{2^{n/p}\alpha}\frac{1}{|Q|}\,\mathrm{d}x=0,$$

because $\int g\,\mathrm{d}x = \int a\,\mathrm{d}x - \int b\,\mathrm{d}x = 0-0$, here $\int a\,\mathrm{d}x = 0$ by $a \in \mathscr{A}^p$ and $\int b\,\mathrm{d}x = 0$ by $b \in H^{1,p}$. Thus

$$\frac{g}{2^{n/p}\alpha|Q|} \in \mathscr{A}_Q^\infty \implies \|g\|_{H^{1,\infty}} \leqslant 2^{n/p}\alpha|Q| = 2^{n/p}4^{1/(p-1)}.$$

<u>*Step 2.*</u> We want an intermediate decomposition for $f \in H^{1,p}$. Let $f_0 \in H^{1,p}$, we claim that there exists f_1, $g^{(0)}$ with

$$\|f_1\|_{H^{1,p}} \leqslant \frac{2}{3}\|f_0\|_{H^{1,p}}, \qquad \|g^{(0)}\|_{H^{1,\infty}} \leqslant \frac{4}{3}c(n,p)\|f_0\|_{H^{1,p}}.$$

Let $\varepsilon > 0$, there exists $f_0 = \sum_j \lambda_j a_j$ be a p-atom decomposition with $\sum|\lambda_j| \leqslant \|f_0\|_{H^{1,p}} + \varepsilon$. Apply Step 1 to a_j, we obatin a decomposition $a_j = b_j + g_j$. Set $f_1 = \sum_j \lambda_j b_j$, $g^{(0)} = \sum_j \lambda_j g_j$. Then

$$\|f_1\| \leqslant \sum_j \|\lambda_j b_j\|_{H^{1,p}} \leqslant \frac{1}{2}(\|f_0\|_{H^{1,p}} + \varepsilon).$$

Take $\varepsilon = \frac{1}{3}\|f_0\|_{H^{1,p}}$, then $\|f_1\|_{H^{1,p}} \leqslant \frac{2}{3}\|f_0\|_{H^{1,p}}$.

$$\|g^{(0)}\|_{H^{1,\infty}} \leqslant \sum_j |\lambda_j| c(n,p) = \frac{4}{3}c(n,p)\|f_0\|_{H^{1,p}}.$$

<u>*Step 3.*</u> Do iteration:

$$f_0 = f_1 + g^{(0)}, \quad f_1 = f_2 + g^{(1)}, \quad \cdots,$$

For $k \geqslant 1$, $f_0 = f_k + g^{(0)} + \cdots + g^{(k-1)}$ with $\|f_k\|_{H^{1,p}} \leqslant \left(\frac{2}{3}\right)^k \|f_0\|_{H^{1,p}}$, which tends to 0 as $k \to \infty$. And

$$\sum_{\ell=0}^{k-1} \|g^{(\ell)}\|_{H^{1,\infty}} \leqslant \sum_{\ell=0}^{k-1} \frac{4}{3}c(n,p)\|f_\ell\|_{H^{1,p}} \leqslant \sum_{\ell=0}^{k-1} \frac{4}{3}c(n,p)\left(\frac{2}{3}\right)^\ell \|f_0\|_{H^{1,p}} \leqslant 4c(n,p)\|f_0\|_{H^{1,p}}.$$

Thus $\sum_{\ell=0}^{k-1} g^{(\ell)}$ is convergent in $H^{1,\infty}$ to some g by completeness of $H^{1,\infty}$. Since $H^{1,\infty} \subset H^{1,p}$, the convergence holds in $H^{1,p}$. Thus $f_0 = g$ in $H^{1,p}$. Since $g \in H^{1,\infty}$, $f_0 \in H^{1,\infty}$, and $\|f_0\|_{H^{1,\infty}} \leqslant 4c(n,p)\|f_0\|_{H^{1,p}}$. □

We shall denote the real Hardy space $(H^{1,\infty}(\mathbb{R}^n), \|\cdot\|_{H^{1,\infty}})$ as $H^1(\mathbb{R}^n)$.

Corollary 6.10

All norms $\|\cdot\|_{H^{1,p}}$, $p > 1$ are equivalent on $H^1(\mathbb{R}^n)$.

6.3 The H^1-BMO Duality

Theorem 6.11: Fefferman

The topological dual of H^1 is isomorphic to $\mathrm{BMO}/\mathbb{K}$, *i.e.*, there exists a bijective continuous linear operator $T : \mathrm{BMO}/\mathbb{K} \to (H^1)'$ with continuous inverse.

Proof. Equip H^1 with $\|\cdot\|_{H^{1,2}}$. Actually, any p finite would do the same rule, but not $\|\cdot\|_{H^{1,\infty}}$. Equip BMO with $\|\cdot\|_{*,2}$ semi-norm.

<u>*Step 1. Define a linear map* $\mathrm{BMO}/\mathbb{K} \to (H^1)'$.</u> Define

$$L_b f := \int_{\mathbb{R}^n} b(x) f(x)\,\mathrm{d}x$$

whenever $|bf|$ is integrable.

<u>*Case 1:* $b \in L^\infty$, $f \in H^1$.</u> Thus $f \in L^1$ with $\|f\|_1 \leqslant \|f\|_{H^{1,2}}$, $|L_b f| \leqslant \|b\|_\infty \|f\|_1$. If $f = \sum_i \lambda_i a_i$ is a 2-atomic decomposition, $\|f\|_{H^{1,2}} \leqslant \sum_i |\lambda_i|$. Thus

$$|L_b f| \leqslant \|b\|\infty \|f\|_1 \leqslant \|b\|_\infty \sum_i |\lambda_i|.$$

<u>*Case 2: $b\in$ BMO with a_i be a 2-atom*</u>. If $a\in\mathscr{A}^2_Q$, $ba\in L^1(\mathbb{R}^n)$ since $b\in L^p_{\mathrm{loc}}$ and $a\in L^p$. In this case,

$$|L_b a| = \left|\int_Q (b(x)-m_Q b)a(x)\,\mathrm{d}x\right| \leqslant \|b\|_{*,2}|Q|^{1/2}\|a\|_2 \leqslant \|b\|_{*,2}.$$

Here $|Q|^{1/2}\|a\|_2\leqslant 1$ results from $a\in\mathscr{A}^2_Q$.

<u>*Case 3: Combine case 1 and 2*</u>. If $b\in L^\infty$, $f\in H^1$, $L_b f=\sum_i\lambda_i L_b a_i$. Then

$$\left|L_b f\right| \leqslant \sum_i |\lambda_i||L_b a_i| \leqslant \|b\|_{*,2}\left\|f\right\|_{H^{1,2}}.$$

<u>*Case 4: $b\in$ BMO and $f\in\mathrm{Vect}\mathscr{A}^2$*</u>. Assume $\mathbb{K}=\mathbb{R}$ by decomposing $f=\mathrm{Re}\,f+\mathrm{iIm}\,f$, and let $(b_k)_{k\geqslant 1}$ be a sequence of cut-offs of b. Write $f=\sum_{i=1}^m a_i$ with a_i being 2-atoms, $m\in\mathbb{N}^*$ and $\lambda_i\in\mathbb{R}$. First, $bf\in L^1(\mathbb{R}^n)$ and $b_k f\to bf$ in L^1. $\left|b_k f\right|\leqslant\left|bf\right|$, $L_b f=\lim_{k\to\infty}L_{b_k}f$. Then by case 3,

$$\left|L_{b_k}f\right| \leqslant \|b_k\|_{*,2}\left\|f\right\|_{H^{1,2}}.$$

Since $\|b\|_{*,2}\leqslant c(n)\|b_k\|_{*,2}\leqslant c(n)2\|b\|_{*,2}\leqslant\tilde{c}(n)\|b\|_{*,2}$, we obtain

$$\left|L_b f\right| \leqslant \tilde{c}(n)\|b\|_{*,2}\left\|f\right\|_{H^{1,2}}.$$

Now since $\mathrm{Vect}\mathscr{A}^2$ is dense in $(H^1,\|\cdot\|_{H^{1,2}})$, there exists a linear extension $\tilde{L}_b$ of L_b with $\tilde{L}_b\in(H^{1,2})'$. Set $\tilde{T}:\mathrm{BMO}\to(H^{1,2})'$, $b\mapsto\tilde{L}_b$. Then $\tilde{T}$ is a continuous linear map. If b is a constant, $L_b=0$ implies $\tilde{L}_b=0$. We may pass to the quotient

$$\begin{tikzcd}
\mathrm{BMO} \arrow[r, "\tilde{T}"] \arrow[d] & (H^{1,2})' \\
\mathrm{BMO}/\mathbb{K} \arrow[ur, dashed, "T"'] &
\end{tikzcd}$$

<u>*Step 2. Injetivity of T*</u>. Let $b\in$ BMO such that $\tilde{L}_b=0$. Fix $Q\subset\mathbb{R}^n$ cubes, let $L^2_0(Q)=\left\{f\in L^2(Q):\int_Q f\,\mathrm{d}x=0\right\}$ and extend $f\in L^2(Q)$ by 0 on Q^c. If $f\in L^2_0(Q)$, there exists $\lambda>0$ such that $f/\lambda\in\mathscr{A}^2_Q$. Take $\lambda=\left\|f\right\|_2|Q|^{1/2}$, thus $f\in\mathrm{Vect}\mathscr{A}^2$,

$$0=\tilde{L}_b f=L_b f=\int_Q b(x)f(x)\,\mathrm{d}x.$$

Hence $b|_Q\in L^2_0(Q)^\perp\cong\mathbb{K}$, *i.e.*, b is a constant on Q. So b is a constant on $\mathbb{R}^n$.

<u>*Step 3. Surjecitivity of T*</u>. Take $L\in(H^{1,2})'$, we want a $b\in$ BMO such that $L=\tilde{L}_b$. Fix a cube $Q\subset\mathbb{R}^n$ and then $L^2_0(Q)\subset\mathrm{Vect}\mathscr{A}^2\subset H^{1,2}$. Now $L|_{L^2_0(Q)}$ is linear, $f\in L^2_0(Q)$. If $\lambda=\left\|f\right\|_2|Q|^{1/2}$,

$$\begin{cases} \left|L|_{L^2_0(Q)}f\right| \leqslant \left|Lf\right| \leqslant \|L\|\left\|f\right\|_{H^{1,2}}, \\ \left\|f\right\|_{H^{1,2}}=\lambda\left\|\frac{f}{\lambda}\right\|_{H^{1,2}}\leqslant\lambda=\left\|f\right\|_2|Q|^{1/2} \end{cases} \implies L|_{L^2_0(Q)}\in(L^2_0(Q))'.$$

By Riesz representation theorem, there exists $b_Q\in L^2_0(Q)$ such that for all $f\in L^2_0(Q)$, $L|_{L^2_0(Q)}f=\int_Q b_Q f\,\mathrm{d}x$ with

$$\left\|b_Q\right\|_{L^2_0(Q)}=\left\|b_Q\right\|_{L^2(Q)}\leqslant|Q|^{1/2}\|L\|.$$

Then we define b, if $Q'\subset Q$, we claim $b_Q-b_{Q'}$ is a constant on Q. Indeed $f\in L^2_0(Q)\subset L^2_0(Q')$,

$$\int_Q b_Q f\,\mathrm{d}x=Lf=\int_{Q'}b_{Q'}f\,\mathrm{d}x=\int_Q b_{Q'f}\,\mathrm{d}x.$$

This implies $(b_Q-b_{Q'})|_Q$ is a constant. Set

$$b(x)=\begin{cases} b_{[-1,1]^n}m & x\in[-1,1]^n, \\ b_{[-2^j,2^j]^n}+c_j, & x\in[2^{-j},2^j]^n\setminus[-2^{j-1},2^{j-1}]^n \end{cases}$$

with $c_j \in \mathbb{C}$ such that $b_{[-2^j,2^j]^n} + c_j = b_{[-1,1]^n}$ on $[-1,1]^n$. If $j \geqslant k \geqslant 0$, $b_{[-2^j,2^j]^n} + c_j = b_{[-2^k,2^k]^n} + c_k$ on $[-2^k,2^k]^n$ since both of them are equal to $b_{[-1,1]^n}$ on $[-1,1]^n$. Thus, $b = b_{[-2^j,2^j]^n} + c_j$ on $[-2^j,2^j]^n$.

We want $b \in \mathrm{BMO}$. Fix a cube Q, take $j \in \mathbb{N}$ such that $Q \subset [-2^j,2^j]^n$. Then $b = b_{[-2^j,2^j]^n} + c_j = b_Q + c_Q$ on Q. Because $c_Q = m_Q c_Q$ and $m_Q b_Q = 0$,

$$b - m_Q b = b_Q - c_Q - m_Q(b_Q + c_Q) = b_Q.$$

Thus

$$\left(\int_Q \left|b - m_Q b\right|^2\right)^{1/2} = \left\|b_Q\right\|_{L^2(Q)} \leqslant |Q|^{1/2} \|L\|.$$

If $f \in \mathrm{Vect}\mathscr{A}^2$, we know that

$$La = \int_Q b_Q a \,\mathrm{d}x = \int_Q (b - m_Q b) a \,\mathrm{d}x = \int_{\mathbb{R}^n} ba \,\mathrm{d}x = L_b a = \tilde{L}_b a.$$

So $L = \tilde{L}_b$ on $\mathrm{Vect}\mathscr{A}^2$ can be extended to $H^{1,2}$. □

7 Littlewood–Paley Theory

7.1 Vector-valued Calderón–Zygmund operator

Let H be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, with the first component linear and the second anti-linear. Choose an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H.

Definition 7.1: Strongly measurable

Let $f : \mathbb{R}^n \to H$. Say f is *strongly measurable* if $\forall i \in \mathbb{N}$, the map from $\mathbb{R}^n$ to $\mathbb{C}$, $x \mapsto \langle f(x), e_i\rangle$ is measurable.
For $p \in [1,\infty)$, define

$$L^p(\mathbb{R}^n, H) := \left\{ f : \mathbb{R}^n \to H : f \text{ strongly measurable and } \int_R \left|f(x)\right|_H^p \mathrm{d}x < +\infty \right\}.$$

Here $\left|f(x)\right|_H^2 = \sum_{i\geqslant 0} \left|\langle f(x), e_i\rangle\right|^2$ is the norm induced by the inner product on H. Set $\|f\|_p := \left\| |f|_H \right\|_p$.
For $p = \infty$, define

$$L^\infty(\mathbb{R}^n, H) := \left\{ f : \mathbb{R}^n \to H : f \text{ strongly measurable with } |f|_H \in L^\infty(\mathbb{R}^n) \right\}.$$

Set $\|f\|_\infty := \left\| |f|_H \right\|_\infty$.

It is obvious that strong measurability is independent of the choice of the basis.

Proposition 7.2

$L^p(\mathbb{R}^n, H)$ are Banach spaces for $p \in [1,\infty]$. And $L^2(\mathbb{R}^n, H)$ is a Hilbert space with inner product $\langle f, g\rangle := \int_{\mathbb{R}^n} \langle f, g\rangle_H \,\mathrm{d}x$.

Proof. Obvious. □

Definition 7.3: CZO between Hilbert spaces

For $0<\alpha\leqslant 1$, say $K:\mathbb{R}^n\times\mathbb{R}^n\to\mathscr{B}(H_1,H_2)$ is a *Calderón–Zygmund kernel* if $\forall x,y\in\Delta^c$, one has the estimations by replacing the modulus $|\cdot|$ on $\mathbb{C}$ by the norm $\|\cdot\|_{\mathscr{B}(H_1,H_2)}$ subordinate to the norms on H_1 and H_2.

$$\left\|K(x,y)\right\| = \sup_{h_1\in H_1,\ h_1\neq 0}\frac{\left\|K(x,y)h_1\right\|_{H_2}}{\|h_1\|_{H_1}}.$$

Say $T\in \mathrm{CZO}_\alpha(H_1,H_2)$ if

(1) $T\in\mathscr{B}(L^2(\mathbb{R}^n,H_1),L^2(\mathbb{R}^n,H_2))$.

(2) There exists $K\in\mathrm{CZK}_\alpha(H_1,H_2)$ such that $\forall f\in L^2(\mathbb{R}^n,H_1)$ with compact support,

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)\,\mathrm{d}y \qquad \text{a.e. } x\notin\operatorname{supp}f.$$

Here $\operatorname{supp}f:=\operatorname{supp}\|f\|_H$

Integration theory for H-valued strongly measurable functions $f:\mathbb{R}^n\to H$ is induced from $\left|f\right|_H$. If $\int_{\mathbb{R}^n}\left|f\right|_H\mathrm{d}x$ exists, then $\int_{\mathbb{R}^n}f\,\mathrm{d}x\in H$ exists and is defined by

$$\int_{\mathbb{R}^n}f\,\mathrm{d}x=\sum_{i\geqslant 0}\int_{\mathbb{R}^n}\left\langle f(x),e_i\right\rangle_H\mathrm{d}x\cdot e_i.$$

Here x fixed with $x\notin\operatorname{supp}f$, $f\in L^2(\mathbb{R}^n,H_1)$, $g(y)=K(x,y)f(y)\in H_2$ continuous w.r.t. y and $g:\mathbb{R}^n\to H_2$ is strongly measurable with $\int_H\left|g(y)\right|_{H_2}\mathrm{d}y<+\infty$.

Theorem 7.4

Any $\mathrm{CZO}_\alpha(H_1,H_2)$ operator has strong type (p,p) for $p\in(1,\infty)$, and weak type $(1,1)$. Hence it has unique extensions $L^p(\mathbb{R}^n,H_1)\to L^p(\mathbb{R}^n,H_2)$ for $p\in(1,\infty)$ and $L^1(\mathbb{R}^n,H_1)\to L^{1,\infty}(\mathbb{R}^n,H_2)$.

Proof. Exactly the same with the usual case. □

7.2 Littlewood–Paley estimates

Definition 7.5: Homogeneous L.P. family

A *homogeneous Littlewood–Paley family* is a family of functions $\mathscr{G}=\left\{g_j\right\}_{j\in\mathbb{Z}}$ with $0<a<b<\infty$ such that

(1) For all $j\in\mathbb{Z}$, $g_j\in\mathscr{S}(\mathbb{R}^n)$ with $\hat{g}_j$ supported in $C_{a,b}:=\{x\in\mathbb{R}^n: a\leqslant|\xi|\}\leqslant b$.

(2) $\forall\alpha\in\mathbb{N}^n$, $\sup_{j\in\mathbb{Z}}\left\|\partial_\varepsilon^\alpha\hat{g}_j\right\|_\infty=c_\alpha<+\infty$.

Set Q_j is the Fourier multiplier with $\hat{g}_j(2^{-j}\cdot)$, *i.e.*, $\forall f\in\mathscr{S}'(\mathbb{R}^n)$, $Q_jf\in\mathscr{S}'(\mathbb{R}^n)$, $\widehat{Q_jf}=\hat{g}_j(2^{-j}\cdot)\hat{f}$ or $Q_jf=2^{nj}g_j(2^j\cdot)*f$, which is, $Q_jf(x)=\int_{\mathbb{R}^n}2^{nj}g_j(2^j(x-y))f(y)\,\mathrm{d}y$.

For example, let $g_j=\varphi$, $j\in\mathbb{Z}$ as in the proof of Mikhlin's theorem, then $Q_j=\Delta_j$, which implies $\mathscr{G}=\{\varphi\}$.

We first give two theorems to obatin a quadratic estimate.

Theorem 7.6

For all $p \in (1,\infty)$, there exists $c = c(n,p,\mathscr{G})$ such that $\forall f \in L^p(\mathbb{R}^n)$,

$$\left\| \Big(\sum_{j\in\mathbb{Z}} |Q_j f| \Big)^{1/2} \right\|_p \leqslant c \|f\|_p,$$

which is called the *almost orthogonal condition.*

Theorem 7.7

Suppose there exists a second homogeneous L.P. family $\tilde{\mathscr{G}} = \{\tilde{g}_j\}_{j\in\mathbb{Z}}$ with

$$\sum_{j\in\mathbb{Z}} \hat{\tilde{g}}_j(2^{-j}\xi)\hat{g}_j(2^{-j}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^n, \tag{7.1}$$

i.e., the non-degeneracy condition. Then $\forall p \in (1,\infty)$, $\exists \tilde{c} = c(n,p,\mathscr{G})$ such that $\forall f \in L^p(\mathbb{R}^n)$,

$$\|f\|_p \leqslant \tilde{c} \left\| \Big(\sum_{j\in\mathbb{Z}} |Q_j f|^2 \Big)^{1/2} \right\|_p.$$

Combine the above theorems, when $|j - j'|$ is not small, such that $\operatorname{supp}\widehat{Q_j f} \cap \operatorname{supp}\widehat{Q_{j'} f} = \varnothing$,

$$\int_{\mathbb{R}^n} \widehat{Q_j f}\,\overline{\widehat{Q_{j'} f}}\,\mathrm{d}\xi = 0 \tag{$\perp$}$$

for nice functions f. When $p = 2$, by Plancherel's theorem,

$$\left\| \Big(\sum_{j\in\mathbb{Z}} |Q_j f|^2 \Big)^{1/2} \right\|_2^2 = \int_{\mathbb{R}^n} \Big(\sum_{j\in\mathbb{Z}} |\widehat{Q_j f}|^2 \Big)\,\mathrm{d}\xi$$

If ($\perp$) holds for all $j \neq j'$, then

$$\int_{\mathbb{R}^n} \Big(\sum_{j\in\mathbb{Z}} \widehat{Q_j f} \Big)\Big(\sum_{j'\in\mathbb{Z}} \overline{\widehat{Q_{j'} f}} \Big)\,\mathrm{d}\xi = \int_{\mathbb{R}^n} \Big(\sum_{j\in\mathbb{Z}} |\widehat{Q_j f}|^2 \Big)\,\mathrm{d}\xi.$$

With the non-degeneracy condition, we expect that the above formula is equivalent to $\int_{\mathbb{R}^n} |\hat{f}|^2\,\mathrm{d}\xi$. This works if $\tilde{g}_j = g_j$ for all $j \in \mathbb{Z}$ even without ($\perp$) for all $j \neq j'$.

Proposition 7.8

There exists a homogeneous L.P. family $\tilde{\mathscr{G}}'$ with (7.1) if

$$\inf_{\xi\in\mathbb{R}^n\setminus\{0\}} \sum_{j\in\mathbb{Z}} \left|\hat{g}_j(2^{-j}\xi)\right|^2 > 0.$$

Proof. Let $\omega(\varepsilon) = \sum_{j\in\mathbb{Z}} \left|\hat{g}_j(2^{-j\varepsilon})\right|^2$. Then $\omega \in C^\infty(\mathbb{R}^n \setminus \{0\})$, the cardinal of non-zero terms depends on a, b. So $\omega \in L^\infty(\mathbb{R}^n)$ by (1) and (2) in the defintion for $\alpha = 0$, and $\omega \geqslant \delta > 0$ on $\mathbb{R}^n \setminus \{0\}$.

Define $\tilde{g}_j$ as the inverse Fourier transform of $\xi \mapsto \frac{\hat{g}_j(\xi)}{\sqrt{\omega(2^j\xi)}}$, then we are done. □

Proof of Theorem 7.6.

Set

$$T : L^2(\mathbb{R}^n,\mathbb{C}) \to L^2(\mathbb{R}^n,\ell^2(\mathbb{Z})), \qquad f \mapsto (Q_j f)_{j\in\mathbb{Z}}.$$

For all $j \in \mathbb{Z}$, $Q_j \in \mathscr{B}(L^2(\mathbb{R}^n,\mathbb{C}))$. We want T be well-defined, linear and bounded. Note that $|Tf|_{\ell^2(\mathbb{Z})} = \left(\sum_{j\in\mathbb{Z}} |Q_j f|^2\right)^{1/2}$.

We want to show first that

$$\int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left|Q_j f\right|^2 \mathrm{d}x \leqslant c \int_{\mathbb{R}^n} \left|f\right|^2 \mathrm{d}x. \tag{7.2}$$

If (7.2) holds, then T is well-defined, linear and bounded. Now

$$\begin{aligned}(7.2) \Longleftrightarrow & \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left|\widehat{Q_j f}\right|^2 \mathrm{d}\xi \leqslant c \int_{\mathbb{R}^n} \left|\hat{f}\right|^2 \mathrm{d}x \\ \Longleftrightarrow & \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left|\hat{g}_j(2^{-j}\xi)\right|^2 \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi \leqslant c \int_{\mathbb{R}^n} \left|\hat{f}\right|^2 \mathrm{d}x.\end{aligned}$$

Note that $\sum_{j\in\mathbb{Z}} \left|\hat{g}_j(2^{-j}\xi)\right|^2 = \omega(\xi) \leqslant \operatorname{ess\,sup} \omega \cdot \int_{\mathbb{R}^n} \left|f\right|^2 \mathrm{d}x$. Take $c = \operatorname{ess\,sup} \omega$ because of Proposition 7.8, then (7.2) holds.

Then we want to show $T \in \mathrm{CZO}_1(\mathbb{C}, \ell^2(\mathbb{Z}))$. Denote $Tf = (Q_j f)_{j\in\mathbb{Z}} = (k_j * f)_{j\in\mathbb{Z}}$, where $k_j(x) = 2^{nj} g_j(2^j x)$. We set for $x, y \in \mathbb{R}^n$ that $K(x,y) := (k_j(x-y))_{j\in\mathbb{Z}}$, then $K \in \mathrm{CZK}_1(\mathbb{C}, \ell^2(\mathbb{Z}))$ because

$$\left\|K(x,y)\right\|_{\mathscr{B}(\mathbb{C},\ell^2(\mathbb{Z}))} = \left\|(k_j(x-y))_{j\in\mathbb{Z}}\right\|_{\ell^2(\mathbb{Z})} = \left(\sum_{j\in\mathbb{Z}} \left|k_j(x-y)\right|^2\right)^{1/2}.$$

For (2) of $\mathscr{G}$, for all $M > 0$, there exists $c_M < +\infty$ such that

$$\left|g_j(x)\right| \leqslant \frac{c_M}{(1+|x|)^M}, \qquad \forall j \in \mathbb{Z},\ \forall x \in \mathbb{R}^n. \tag{7.3}$$

When $M > n$, we have

$$\left\|K(x,y)\right\|_{\mathscr{B}(\mathbb{C},\ell^2(\mathbb{Z}))} \leqslant \left(\sum_{j\in\mathbb{Z}} \frac{c_M 2^{nj}}{(1+\left|2^j(x-y)\right|)}^{2M}\right)^{1/2} \leqslant c_M c(n,M) \left|x-y\right|^{-n}.$$

Estimate the gradiants:

$$\nabla_x K(x,y) = (\nabla_x k_j(x-y))_{j\in\mathbb{Z}} = ((\nabla k_j)(x-y))_{j\in\mathbb{Z}},$$

while

$$\nabla k_j(x-y) = 2^{j+1} \nabla g_j(2^j(x-y)).$$

Note that ∇g_j has the same properties as g_j in (1) and (2), so (7.3) holds for ∇g_j. Hence for $M > n+1$, we have

$$\left\|\nabla_x K(x,y)\right\|_{\mathscr{B}(\mathbb{C},\ell^2(\mathbb{Z}))} \leqslant C_{M,n} \left|x-y\right|^{-n-1}.$$

And $\nabla_y K(x,y) = -\nabla_x K(x,y)$, the same argument works. □

Proof of Theorem 7.7.

Use the duality principle. For $f \in L^p \cap L^2$, $h \in L^{p'} \cap L^2$,

$$\begin{aligned}\int_{\mathbb{R}^n} f\bar{h}\,\mathrm{d}x &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}\bar{\hat{h}}\,\mathrm{d}\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \hat{\tilde{g}}_j\left(\frac{\xi}{2^j}\right) \hat{g}_j\left(\frac{\xi}{2^j}\right) \hat{f}(\xi)\overline{\hat{h}(\xi)}\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \widehat{Q_j f}(\xi) \overline{\widehat{\tilde{Q}_j h}(\xi)}\,\mathrm{d}\xi = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} Q_j f(x) \overline{\tilde{Q}_j h(x)}\,\mathrm{d}x = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} Q_j f(x) \overline{\tilde{Q}_j h(x)}\,\mathrm{d}x.\end{aligned}$$

Since $Q_j f$, $\tilde{Q}_j h \in L^2(\mathbb{R}^n, \ell^2(\mathbb{Z}))$, using Hölder inequality and theorem 7.6,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f\bar{h}\,\mathrm{d}x\right| &\leqslant \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} \left|Q_j f(x)\right|^2\right)^{1/2} \left(\sum_{j\in\mathbb{Z}} \left|\tilde{Q}_j h(x)\right|^2\right)^{1/2} \\ &\leqslant \left\|Tf\right\|_p \left\|\tilde{T}h\right\|_p \leqslant \left\|Tf\right\|_p \cdot c(n,p',\mathscr{G}) \left\|h\right\|_{p'}.\end{aligned}$$

Thus

$$\|f\|_p = \sup_{h\in L^{p'}\cap L^2,\ h\neq 0} \frac{\left|\int_{\mathbb{R}^n} f\bar{h}\,\mathrm{d}x\right|}{\|h\|_{p'}} \leqslant c(n,p',\mathscr{G})\,\|Tf\|_p.$$

For the general case, $f\in L^p$, take $(f_k)_{k\geqslant 1}\subset L^p\cap L^2$ such that $f_k\to f$ in L^p. Apply the estimate for f_k. Since $T: L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n,\ell^2(\mathbb{Z}))$ is continuous, we still have the same estimate for f. □

Remark. (1) $f\mapsto \|Tf\|_p$ *is an equivalent norm on* $L^p(\mathbb{R}^n)$ *if both theorem 7.6 and 7.7 hold.*

(2) In theorem 7.7, we assumed $f\in L^p(\mathbb{R}^n)$*. But* $(\sum_{j\in\mathbb{Z}}|Q_jf|^2)^{1/2}$ *exists as a measurable function for all* $f\in\mathscr{S}'(\mathbb{R}^n)$*. What is the conclusion if we assume it belongs to* $L^p(\mathbb{R}^n)$*? The answer is NO. In fact, there exists a polynomial* P *of* n *variables such that* $f+P\in L^p(\mathbb{R}^n)$*, but*

$$\|f+P\| \leqslant c(n,p',\mathscr{G})\,\|Tf\|_p.$$

Note that $\widehat{Q_jP} = \hat{g}_j(2^{-j}\cdot)\hat{P}$*, but* $\operatorname{supp}\hat{P}\subset\{0\}$ *iff* P *is a polynomial. And* $\operatorname{supp}\hat{g}_j(2^{-j}\cdot)$ *is supported away from 0. Thus* $\widehat{Q_jP}=0$*, hence* $Q_jP=0$*.*

Definition 7.9: Inhomogeneous L.P. family

An *inhomogeneous L.P. family* is a family of functions $\mathscr{G}_0 = \{g_j\}_{j\in\mathbb{N}}\cup\{\Phi\}$ with

(1) g_j satisfies (1) and (2) in the definition of homogeneous L.P. family.

(2) $\Phi\in\mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\Phi}\subset\bar{B}(0,c)$ for some $c>0$.

Denote Q_j the same and $P_0f=\Phi * f$.

Similarly, we have

Theorem 7.10

For all $p\in(1,\infty)$, $f\in L^p$,

$$\|P_0f\| + \left\|\left(\sum_{j\in\mathbb{N}}|Q_jf|^2\right)^{1/2}\right\|_p \lesssim \|f\|_p.$$

Proof. Almost the same as Theorem 7.6. □

Theorem 7.11

If there exists another family $\tilde{\mathscr{G}}_0$ such that $\forall\xi\in\mathbb{R}^n$,

$$1 = \hat{\tilde{\Phi}}(\xi)\hat{\Phi}(\xi) + \sum_{j\in\mathbb{N}} \hat{\tilde{g}}_j\left(\frac{\xi}{2^j}\right)\hat{g}_j\left(\frac{\xi}{2^j}\right),$$

then for all $f\in\mathscr{S}'(\mathbb{R}^n)$,

$$\|f\|_p \leqslant c(n,p',\tilde{\mathscr{G}}_0)\left(\|P_0f\|_p + \left\|\left(\sum_{j\geqslant 0}|Q_jf|^2\right)^{1/2}\right\|_p\right).$$

Proof. Almost the same as Theorem 7.7 because for $h\in\mathscr{S}'(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} f\bar{h}\,\mathrm{d}x \to {}_{\mathscr{S}'}\langle f,h\rangle_{\mathscr{S}}, \qquad \|f\|_p = \sup\frac{|{}_{\mathscr{S}'}\langle f,h\rangle_{\mathscr{S}}|}{\|h\|_{\mathscr{S}'}}.$$

Done. □

Example 7.12

Consider the L.P. family $\{\varphi\}$ constructed before, $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\operatorname{supp}\hat{\varphi} = C_0 = \left\{\xi : \frac{1}{2} \leqslant |\xi| \leqslant 4\right\}, \qquad \sum_{j\in\mathbb{Z}} \left|\hat{\varphi}\left(\frac{\xi}{2^j}\right)\right|^2 = 1, \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Consider F defined as

$$F(\xi) = \begin{cases} \sum_{j=-\infty}^{0} \left|\hat{\varphi}(2^{-j}\xi)\right|^2, & \xi \neq 0, \\ 1, & \xi = 0. \end{cases}$$

Since $\hat{\varphi}$ is radial, $\hat{\varphi} \geqslant 0$, $\operatorname{supp}\hat{\varphi} \subset C_0$, and when $\xi = 0$, $\sum_{k\in\mathbb{Z}} \left|\varphi(2^{\tilde{-}k}\xi)\right|^2 = 1$. If we add the assumption $\inf_{1\leqslant|\xi|\leqslant 2} \hat{\varphi}(\xi) = \delta > 0$, then $\operatorname{supp} F \subset \bar{B}(0,4)$. Now

- $F(\xi) = 1$ if $|\xi| \leqslant 1$.
- $F(\xi) \geqslant \left|\hat{\varphi}(\xi)\right|^2 \geqslant \delta^2 > 0$ if $1 \leqslant |\xi| \leqslant 2$.
- $F(\xi) = \left|\hat{\varphi}(\xi)\right|^2 = \hat{\varphi}(\xi)$ if $|\xi| \geqslant 2$.
- $\sqrt{F}$ is smooth on $B(0,4)$.
- $\sqrt{F} = \hat{\varphi}$ on $\bar{B}(0,2)^c$.

So $\sqrt{F}$ is also smooth on $\bar{B}(0,2)^c$, this implies $\sqrt{F}$ is smooth on $\mathbb{R}^n$. Then we can apply Theorem 7.7 with such a decomposition.

7.3 Sobolev spaces

For $s \in \mathbb{R}$, set

$$m_s : \mathbb{R}^n \to [0,+\infty), \qquad \xi \mapsto (1+|\xi|^2)^{-s/2}.$$

Then $m_s \in C^\infty(\mathbb{R}^n)$ and $\partial_\xi^\alpha m_s$ has at most polynomial growth at infinity for all $\alpha \in \mathbb{N}^n$. We set

$$J_s : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \qquad f \mapsto \mathscr{F}^{-1}(m_s \cdot \hat{f})$$

a Fourier multiplier.

Lemma 7.13

J_s is a bounded linear operator on $\mathscr{S}(\mathbb{R}^n)$, equipped with the semi-norm giving its topology.

Proof. Just note that $J_s = (1+\Delta)^{-s/2}$. □

Now $J_0 = \mathrm{id}$, $J_{s+t} = J_s J_t$ for $s,t \in \mathbb{R}$. So J_s^{-1} exists and $J_s^{-1} = J_{-s}$. We extend J_s to $\mathscr{S}'(\mathbb{R}^n)$ by duality: if $f \in \mathscr{S}'(\mathbb{R}^n)$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$(J_s f, \varphi) := (f, J_s\varphi),$$

where $(\cdot,\cdot)$ denotes the bilinear duality. Since m_s is $\mathbb{R}$-valued and even, $\overline{J_s\varphi} = J_s\bar{\varphi}$. We also have $(J_s f, \bar{\varphi}) = (f, \overline{J_s\varphi})$. Hence we can set a sesqualinear form $\langle f, \varphi\rangle := (f, \bar{\varphi})$.

Corollary 7.14

$J_s : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is a invertible continuous linear operator on $\mathscr{S}'(\mathbb{R}^n)$, with $J_s^{-1} = J_{-s}$.

Definition 7.15: Sobolev space

For $p \in [1,\infty)$, $s \in \mathbb{R}$, denote

$$L_s^p(\mathbb{R}^n) := J_s(L^p(\mathbb{R}^n)) \subset \mathscr{S}(\mathbb{R}^n).$$

And equip a norm $\|f\|_{L_s^p} = \|h\|_p$, where $f = J_s h$.

Since J_s is an isomorphism on $\mathscr{S}'(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ and preserved by J_s. One has $(L_s^p(\mathbb{R}^n), \|\cdot\|_{L_s^p})$ is a Banach space and $\mathscr{S}(\mathbb{R}^n)$ is dense in $L_s^p(\mathbb{R}^n)$. It can be shown that $\mathscr{D}(\mathbb{R}^n)$ is dense in $L_s^p(\mathbb{R}^n)$, but more difficult.

For $p = 2$, $f \in L_s^2(\mathbb{R}^n)$ iff $\hat{f} = m_s \cdot \bar{h}$, where $h = J_{-s} f \in L^2(\mathbb{R}^n)$. In particular, $\hat{f} \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. By Planchernel identity,

$$\|f\|_{L_s^2}^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \left|\hat{h}(\xi)\right| \mathrm{d}\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}\left|\hat{f}(\xi)\right| \mathrm{d}\xi.$$

For $p \neq 2$, there is no such formula to compute $\|f\|_{L_s^p}$. So we need to look for something *computable*.

Introduce the inverse Fourier transform of $m_s \in \mathscr{S}'(\mathbb{R}^n)$, denoted as $G_s \in \mathscr{S}'(\mathbb{R}^n)$. The function G_s is often called a *Bessel potential*.

Proposition 7.16

For $s > 0$, G_s is a $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ function, non-negative ($G_s > 0$ a.e.), radial with the following estimates.

(1) There exists $c = c(n,s) > 0$ such that $G_s(x) \leqslant c\exp(-|x|/2)$ a.e. on $x \in \bar{B}(0,2)^c$.

(2) There exists $c = c(n,s) > 0$ such that $\frac{1}{c} \leqslant \frac{G_s(x)}{H_s(x)} \leqslant c$ a.e. $x \in B(0,2)$ with

$$H_s(x) = \begin{cases} |x|^{-(n-1)}, & 0 < s < n, \\ \log(2/|x|) + 1, & s = n, \\ 1, & s > n. \end{cases}$$

In particular, for all $s > 0$, $G_s \in L^1(\mathbb{R}^n)$ and

$$\|G_s\|_1 = \int_{\mathbb{R}^n} G_s(x)\,\mathrm{d}x = \hat{G}_s(0) = 1.$$

Corollary 7.17

For $s > 0$, $L_s^p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with $\|f\|_p \leqslant \|f\|_{L_s^p(\mathbb{R}^n)}$.

Proof. Let $f = J_s h$, $h \in L^p(\mathbb{R}^n)$, $\|h\|_p = \|f\|_{L_s^p}$. And $\hat{f} = m_s \cdot \hat{h} = \hat{G}_s \cdot \hat{h}$ in $\mathscr{S}'(\mathbb{R}^n)$. But $G_s \in L^1$, $h \in L^p$, thus $G_s * h \in L^p$ and $\widehat{G_s * h} = \hat{G}_s \cdot \hat{h}$ in $\mathscr{S}'(\mathbb{R}^n)$. By invertibility of $\mathscr{F}$ in $\mathscr{S}'(\mathbb{R}^n)$, $f = G_s * h$ in $\mathscr{S}'(\mathbb{R}^n)$. Therefore $f \in L^p(\mathbb{R}^n)$ and

$$\|f\|_p \leqslant \|G_s\|_1 \|h\|_p = \|h\|_p = \|f\|_{L_s^p}.$$

Done. □

Theorem 7.18: Sobolev embedding

Assume $1 < p \leqslant q < \infty$, $0 \leqslant s < n$ such that

$$0 \leqslant \frac{1}{p} - \frac{1}{q} \leqslant \frac{s}{n} < 1 \implies \begin{cases} p \leqslant q \leqslant \frac{np}{n-sp}, & sp < n, \\ p \leqslant q < \infty, & sp \geqslant n. \end{cases}$$

Then $L^p_s(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ with $\|f\|_q \leqslant c(n,p,q,s)\,\|f\|_{L^p_s}$. This gives better integrability because $p \leqslant q$.

Proof. Let $f = J_s h$, $h \in L^p(\mathbb{R}^n)$, we have shown that $f = G_s * h$. For $0 < s < n$,

$$G_s(x) \leqslant c\,|x|^{-(n-s)} \qquad \text{a.e. } \mathbb{R}^n,$$

because exponential decay is faster than polynomial decay. We can use the Hardy–Littlewood–Sobolev inequality.

$$\|G_s * h\|_q \leqslant c(n,s)\,\left\||x|^{-(n-s)} * h\right\|_q \leqslant c(n,s,p,q)\,\|h\|_q, \qquad \text{when } \frac{1}{p} - \frac{1}{q} = \frac{s}{n}.$$

This concludes the proof when $sp < n$. Indeed $f \in L^p$ implies $f \in L^{p_*}$, where $p_* = (np)/(n-sp)$. Thus $f \in L^q$ for any q with $p \leqslant q \leqslant p_*$.

Case when $sp \geqslant n$,

$$|G_s * h| \leqslant (cH_s \mathbb{1}_{\bar{B}(0,2)}) * |h| + (c\mathrm{e}^{-|x|/2}\mathbb{1}_{\bar{B}(0,2)^c}) * |h| \in L^q.$$

here $H_s \mathbb{1}_{\bar{B}(0,2)} \in L^r$ for $r \in [1,(n-s)/n]$, $|h| \in L^p$, by Young's inequality, $L^r * L^p \subset L^q$ because $1/r + 1/p = 1 + 1/q$. Also, $\mathrm{e}^{-|x|/2}\mathbb{1}_{\bar{B}(0,2)^c} \in L^r$ for $r \in [1,\infty]$, then $L^1 * L^p \subset L^p$, $L^{p'} * L^p \subset L^\infty$. So $L^p \subset L^q$ since $p \leqslant q < \infty$.

Fix $q \in [p,\infty)$, let $s^* = \frac{n}{p} - \frac{n}{q} \in (0,n)$. First case applied to s^*, $L^*_s \hookrightarrow L^q$. Then $s \geqslant s^*$ thus $L^p_s \hookrightarrow L^p_{s^*}$ because $L^p_{s-s^*} \subset L^p$. So $L^p_s = J_{s^*}(L^p_{s-s^*}) \subset J_{s^*}(L^p) = L^p_{s^*}$. □

Remark. *If $p = 1$, $0 < s < n$ gives $L^1_s(\mathbb{R}^n) \subset L^{n/(n-s),\infty}(\mathbb{R}^n)$ and this is optimal. $L^1_s(\mathbb{R}^n) \notin L^{n/(n-s)}(\mathbb{R}^n)$. For $f = G_s * h$, $0 \leqslant G_s \leqslant c\,|x|^{-(1-s)}$, thus $\|f\|_{n/(n-s),\infty} \leqslant c\,\|h\|_1$ by Hardy–Littlewood–Sobolev inequality.*

Can $f \in L^{n/(n-s)}(\mathbb{R}^n)$? If $\|f\|_{n/(n-s)} \leqslant c\,\|h\|_1$. For all $h \in L^1$, choose $h(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ for $\varepsilon > 0$ with $\int \varphi \,\mathrm{d}x = 1$, $\varphi \geqslant 0$, $\varphi \in C^\infty_0(\mathbb{R}^n)$. Then

$$G_s * \frac{1}{\varepsilon^n}\varphi\left(\frac{\cdot}{\varepsilon}\right) \to G_s \text{ a.e. as } \varepsilon \to 0.$$

Since $\|\varepsilon^{-n}\varphi(\varepsilon^{-1}\cdot)\|_1 = \|\varphi\|_1 = 1$. By Fatou lemma, $G_s \in L^{n/(n-s)}$ with $\|G_s\|_{n/(n-s)} \leqslant c$. But

$$G_s(x) \geqslant c^{-1}\,|x|^{-(n-s)} \text{ a.e. } x \in \bar{B}(0,2).$$

Here $|x|^{-(n-s)} \in L^{n/(n-s)}$.

1 TD I – Covering theorems and maximal functions

Exercise 1. A disjoint version of Besicovitch's theorem

Let $C(n)$ be the constant (integer) appearing in the Besicovitch's theorem in $\mathbb{R}^n$. We set $Q(n) = 4^n C(n)+1$.

Let $A \subset \mathbb{R}^n$ be a bounded set, $\mathscr{B}$ a family of closed non-degenerate balls such that every point of A is the centre of some $B \in \mathscr{B}$. Our purpose is to show the existence of $Q(n)$ families $\mathscr{B}_1, \dots, \mathscr{B}_{Q(n)} \subset \mathscr{B}$ which is at most countable, with $\mathscr{B}_i$ mutually disjoint, such that

$$A \subset \bigcup_{j=1}^{Q(n)} \bigcup_{B \in \mathscr{B}_j} B =: \bigcup_{j=1}^{Q(n)} \bigcup \mathscr{B}_j.$$

Part I

We start from a family $\{B_i : i \in \mathbb{N}\} \subset \mathscr{B}$ that contains at most countable balls given by Besicovitch's theorem, *i.e.*, to verify

$$\mathbb{1}_A \leqslant \sum_{i \in \mathbb{N}} \mathbb{1}_{B_i} \leqslant C(n).$$

Then proceed as follows:

1-1. Show that for each $\varepsilon > 0$, the set $\{i \in \mathbb{N} : r(B_i) \geqslant \varepsilon\}$ is finite.

1-2. Deduce that we can assume $r(B_1) \geqslant r(B_2) \geqslant \cdots$.

Proof.

1-1. For $\varepsilon > 0$, denote $I_\varepsilon := \{i \in \mathbb{N} : r(B_i) \geqslant \varepsilon\}$.

If $\sup\{r(B_i) : i \in \mathbb{N}\} = +\infty$, then one ball is enough. Otherwise,

$$\# I_\varepsilon \, |B(0,\varepsilon)| \leqslant \sum_{i \in I\varepsilon} |B(x_i,\varepsilon)| \leqslant \left| \bigcup_{i \in I_\varepsilon} B(x_i,\varepsilon) \right| \leqslant |A + B(0,\varepsilon)| < +\infty.$$

Thus $\# I_\varepsilon < +\infty$.

1-2. Since $\{B_i : i \in \mathbb{N}\}$ is countable, by **1-1**, the only possible limit point of $\{r(B_i) : i \in \mathbb{N}\}$ is 0. Thus, by properly relabelling the balls, one can assume that

$$r(B_1) \geqslant r(B_2) \geqslant \cdots \geqslant r(B_n) \geqslant \cdots.$$

□

Part II

We then set $B_{1,1} := B_1$, and inductively assuming that $B_{1,1}, \dots, B_{1,j}$ have been constructed. Set $B_{1,j+1} := B_{k_j}$, where

$$k_j = \min\left\{ i \in \mathbb{N} : B_i \cap \bigcup_{k=1}^{j} B_{1,k} = \varnothing \right\}.$$

Then we define $\mathscr{B}_1 = \{B_{1,1}, B_{1,2}, \dots\}$ and we observe that the result is demonstrated if $A \subset \bigcup \mathscr{B}_1$.

If that is not the case, we set $B_{2,1} := B_k$ where $k = \min\{i \in \mathbb{N} : B_i \notin B_1\}$. Then we define step by step, assuming $B_{2,1}, \dots, B_{2,j}$ have been constructed, set $B_{2,j} := B_{k_j}$, where

$$k_j = \min\left\{ i \in \mathbb{N} : B_i \notin \mathscr{B}_1,\ B_i \cap \bigcup_{k=1}^{j} B_{2,k} = \varnothing \right\}.$$

And define $\mathscr{B}_2 = \{B_{2,1}, B_{2,2}, \dots\}$. We iterate this process.

Our purpose is to show that if $m \in \mathbb{N}$ such that

$$A \setminus \bigcup_{j=1}^{m} \bigcup \mathscr{B}_j \neq \varnothing,$$

then we have $m \leqslant 4^n C(n)$. To do this, we fix $x \in A \setminus \bigcup_{j=1}^m \bigcup \mathscr{B}_j$ and proceed as follows.

1-3. Show that there exists an index i for which we have $x \in B_i$, but $B_i \notin \mathscr{B}_j$ for each $1 \leqslant j \leqslant m$.

1-4. Deduce that for all $1 \leqslant j \leqslant m$, there exists $k_j \in \mathbb{Z}$ such that

$$B_i \cap B_{j,k_j} \neq \varnothing, \quad \text{and} \quad r(B_{j,k_j}) \geqslant r(B_i).$$

1-5. Deduce that for all $1 \leqslant j \leqslant m$, there exists a ball B_j' of radius $r(B_i)/2$ such that

$$B_j' \subset 2B_i \cap B_{j,k_j}.$$

1-6. Deduce that $\sum_{j=1}^m \mathbb{1}_{B_j'} \leqslant C(n) \mathbb{1}_{\bigcup_{j=1}^m B_j'}$.

1-7. Deduce from the previous point and from the inclusion $B_j' \subset 2B_i$, verify that for each $1 \leqslant j \leqslant m$, one has $m \leqslant 4^n C(n)$.

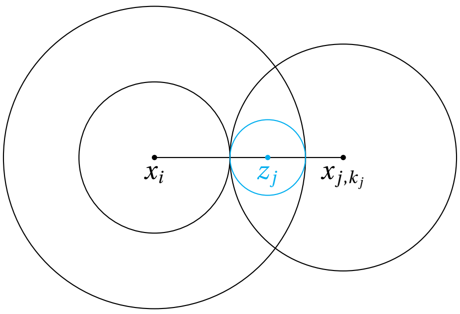
Proof.

1-3. Since $A \subset \bigcup_{i\in\mathbb{N}} B_i$, $\exists i \in \mathbb{N}$ such that $x \in B_i$. Trivially $B_i \notin \mathscr{B}_j$, $j \in \{1,\dots,m\}$ because $x \in A \setminus \bigcup_{j=1}^m \bigcup \mathscr{B}_j$.

1-4. Let $j \in \{1,\dots,m\}$. Consider the balls in $\mathscr{B}_j \subset \mathscr{B}$ with $r(B) \geqslant r(B_i)$. If $B_i \cap B = \varnothing$ for all $B \in \mathscr{B}_j$, then B should have been put in $\mathscr{B}_j$. Hence there exists k_j such that $B_i \cap B_{j,k_j} \neq \varnothing$. We assume that $(r(B_i))_{i\geqslant 1}$ is a decreasing sequence in **1-2** hence $r(B) \geqslant r(B_i)$. For convenience, take the largest ball $B \in B_j$ for which $r(B) \geqslant r(B_i)$.

1-5. We have

$$\left|z_j - x_{j,k_j}\right| = \left|x_i - x_{j,k_j}\right| - \left|z_j - x_i\right| \leqslant r(B_i) + r(B_{j,k_j}) - \frac{3}{2} r(B_i) \leqslant r(B_{j,k_j}) - \frac{1}{2} r(B_i) \leqslant \frac{1}{2} r(B_{j,k_j}).$$



Hence $B_j' = B(z_j, r(B_i)/2) \subset 2B_i \cap B_{j,k_j}$.

1-6. Because

$$\sum_{j=1}^m \mathbb{1}_{B_j'} \leqslant \mathbb{1}_{\bigcup_{j=1}^m B_j'} \sum_{j=1}^m \mathbb{1}_{B_j'} \leqslant \mathbb{1}_{\bigcup_{j=1}^m B_j'} \sum_{j=1}^m \mathbb{1}_{B_{j,k_j}} \leqslant \mathbb{1}_{\bigcup_{j=1}^m B_j'} \cdot C(n).$$

1-7. For $j \in \{1,\dots,m\}$, $B_j \subset 2B_i$ and disjoint. $r(B_j') = r(B_i)/2$. Do the volume counting argument,

$$m \cdot \left|B\left(0, \frac{r_i}{2}\right)\right| \leqslant C(n) \cdot |B(0, 2r_i)| \implies m \leqslant C(n) \frac{|B(0,2r_i)|}{|B(0, r_i/2)|} = 4^n C(n).$$

Thus we can take $Q(n) = m + 1 = 4^n C(n) + 1$ and conclude. $\square$

Exercise 2. A variant of Vitali's Lemma

Assume $A \subset \mathbb{R}^n$ and $\mathscr{F}$ a collection of non-degenerate (*i.e.*, $r > 0$) closed balls $\bar{B}(x,r)$, such that for all $x \in A$, $\inf\{r > 0 : \bar{B}(x,r) \in \mathscr{F}\} = 0$. We want to show that there is a subcollection $\mathscr{G} = \{B_i\}_{i\geqslant 1} \subset \mathscr{F}$ at most countable, such that $\mathscr{G}$ pairwise disjoint, and $A \setminus \bigcup_{i\geqslant 1} B_i$ is a Lebesgue null set.

2-1. Show that the result for unbounded A follows from that for bounded A.

2-2. Now suppose A to be bounded, conclude if $\sup\{r > 0 : \bar{B}(x,r) \in \mathscr{F}\} = \infty$.

2-3. Now assume $\sup\{r > 0 : \bar{B}(x,r) \in \mathscr{F}\} < \infty$. Consider $\mathscr{G}$ the subcollection of $\mathscr{F}$ constructed as in the proof of Vitali's lemma, and write $\mathscr{G} = \{B_i\}_{i\geqslant 1}$. Let $N \in \mathbb{N}_{\geqslant 1}$, and $x \in A \setminus \bigcup_{i=1}^N B_i$.

- **(a)** Show that there exists $r > 0$ such that $\bar{B}(x,r) \cap B_i = \varnothing$ for each $i \in \{1,\dots,N\}$.
- **(b)** Verify that the proof of Vitali's lemma implies that $\bar{B}(x,r)$ must intersect some $B_\beta \in \mathscr{G}$ of radius at least $r/2$.
- **(c)** Deduce from this that $A \setminus \bigcup_{i=1}^N B_i \subset \bigcup_{j>N} 5B_j$ and conclude.

Proof.

2-1. Denote $V_0 = B(0,1)$, $V_k = B(0,2^k) \setminus \bar{B}(0,2^{k-1})$ for $k \geqslant 1$. Denote $A_k = A \cap V_k$ for $k \in \mathbb{N}$, then define

$$\mathscr{F}_k := \left\{\bar{B}(x,r) : x \in A \cap V_k,\ \bar{B}(x,r) \subset V_k\right\}.$$

Then $\mathscr{F}_k \neq \varnothing$ for each k by assumption. Apply the result of bounded case for each A_k with collection $\mathscr{F}_k$, one has

- $|A_k \setminus \bigcup \mathscr{G}_k| = 0$ and $\mathscr{G}_k \subset \mathscr{F}_k$.
- For $B \in \mathscr{G}_k$, $B' \in \mathscr{G}'_k$ with $k \neq k'$, then $B \subset V_k$ and $B' \subset V_{k'}$ implies $B \cap B' = \varnothing$.

Note that $A = (\bigcup_{k\geqslant 0} A_k) \cup \left(\bigcup_{k\geqslant 0} A \cap \partial B(0,2^k)\right)$, where $\left|\bigcup_{k\geqslant 0} A \cap \partial B(0,2^k)\right| = 0$. Thus

$$\mathscr{G} = \bigcup_{k\geqslant 0} \mathscr{G}_k \implies \left|A \setminus \bigcup \mathscr{G}\right| = 0,$$

which concludes the result when A unbounded.

2-2. $M = \operatorname{diam} A < +\infty$ since A is bounded. Because $\sup\{r > 0 : \bar{B}(x,r) \in \mathscr{F}\} = +\infty$, choose $r > 2M$ and take $\mathscr{G} = \{\bar{B}(x,r)\}$ for some $x \in A$, then $A \setminus \bar{B}(x,r) = \varnothing$ is a Lebesgue null set.

2-3. Now assume $M = \sup\{r > 0 : \bar{B}(x,r) \in \mathscr{F}\} < +\infty$. Apply Vitali's lemma, $\exists \mathscr{G} \subset \mathscr{F}$ such that $A \subset \bigcup \mathscr{G}$. Then

$$\sum_{i\in\mathscr{G}} |5B_i| \leqslant 5^n \sum_{i\in\mathscr{G}} |B_i| \leqslant 5^n \left|\bigcup \mathscr{G}\right|.$$

Note that $\bigcup \mathscr{G} \subset \bigcup_{i\in\mathscr{G}} 5B_i \subset A + \bar{B}(0,5M)$, thus

$$\sum_{i\in\mathscr{G}} |5B_i| \leqslant 5^n \left|A + \bar{B}\right| < +\infty.$$

Hence $\mathscr{G}$ is at most countable.

(a) For $x \in A \setminus \bigcup_{i=1}^N B_i$, $\exists r_0 \geqslant 0$ such that $B(x,r_0) \cap \bigcup_{i=1}^N B_i = \varnothing$. Hence $\bar{B}(x,r) \in \mathscr{F}$, $\bar{B}(x,r) \subset B(x,r_0)$. Then $\bar{B}(x,r) \cap \bigcup_{i=1}^N B_i = \varnothing$.

(b) Otherwise, $\bar{B}(x,r)$ should have been selected and $j > N$ by **(3a)**.

(c) We know that if $j > N$, $\bar{B}(x,r) \cap B_j \neq \varnothing$ and $r(B_j) \geqslant \frac{r}{2}$ from **(3b)**. So $y \in B(x,r)$, $d(y,x_j) \leqslant d(y,x) + d(x,x_j) \leqslant r + r + r(B_j) \leqslant 5r(B_j)$. Hence $B(x,r) \subset \bigcup_{j>N} 5B_j$.

Now

$$\left|A \setminus \bigcup_{i=1}^N B_i\right| \leqslant \left|\bigcup_{j>N} 5B_j\right| \to 0$$

as $N \to \infty$, thus $|A \setminus \bigcup_{i\geqslant 1} B_i| = 0$. $\square$

Exercise 3. Counterexample to Besicovitch Theorem, in the Heisenburg group

Part I – Definitions and fundamental properties.

For (z,t), $(z',t') \in \mathbb{H} := \mathbb{C} \times \mathbb{R}$, we define

$$(z,t)\cdot(z',t') = (z+z', t+t'+2\mathrm{Im}\,(z\bar{z}')).$$

We also define for $(z,t) \in \mathbb{H}$,

$$\|(z,t)\| = (|z|^4 + t^2)^{1/4}.$$

For $\rho > 0$, we define a function

$$\delta_p : \mathbb{H} \to \mathbb{H}, \qquad (z,t) \mapsto (\rho z, \rho^2 t).$$

3-I1. Show that $(\mathbb{H},\cdot)$ is a group, called the *Heisenburg group*.

3-I2. Show that for $a \in \mathbb{H}$ and $\rho > 0$, we have

$$\|a\| = \left\|a^{-1}\right\|, \qquad \text{and} \qquad \|\delta_t(a)\| = t\,\|a\|.$$

3-I3. Show that the function

$$d : \mathbb{H} \to \mathbb{H} \to \mathbb{R}_+, \qquad (a,b) \mapsto \left\|a^{-1}\cdot b\right\|$$

is a distance on $\mathbb{H}$ and that the following properties are satisfied:

(i) $\delta_p(0) = 0$ for all $\rho > 0$;

(ii) $\delta_{\rho\rho'}(a) = \delta_\rho(\delta_{\rho'}(a))$ for all $\rho,\ \rho' > 0$ and $a \in \mathbb{H}$;

(iii) $d(\delta_\rho(a), \delta_\rho(b)) = \rho d(a,b)$ for all $\rho > 0$ and $a, b \in \mathbb{H}$.

Hint: To establish the triangle inequality, begin by showing that it is sufficient to prove $\left\|xy\right\| \leqslant \|x\| + \left\|y\right\|$ *for all* $x, y \in \mathbb{H}$*, and observe that for* $x = (z,t)$, $y = (w,s)$, *we has*

$$\left\|xy\right\|^4 = \left||z+w|^2 + \mathrm{i}(t+s+2\mathrm{Im}\,(z\bar{w}))\right|^2.$$

Part II

We define $\mathbb{S} := \{a \in \mathbb{H} : \|a\| = 1\}$.

3-II1. Show that if $a, b \in \mathbb{S}$ are given with $a = (z,t)$, $b = (z',t')$, we have

$$d(\delta_\rho(a), b)^4 = 1 - 4\rho(\left|z'\right|^2 \mathrm{Re}\,(z\bar{z}') + t'\mathrm{Im}\,(z\bar{z}')) + \rho^2 R(\rho),$$

where $R(\rho)$ is a polynomial of degree 2 whose coefficients are integers that is independent of a and b.

3-II2. Fix $b = (z',t') \in \mathbb{S} \setminus \{(0,\pm 1)\}$, write $t' + \mathrm{i}|z'|^2 = \mathrm{e}^{\mathrm{i}\psi}$ with $\psi \in [0, 2\pi)$, and show that if $\alpha > 0$ is given, there exists $\rho_\alpha > 0$ satisfying the following property: for all $a \in \mathbb{S} \setminus \{(0,\pm 1)\}$ such that $\mathrm{Im}\,(\mathrm{e}^{\mathrm{i}\psi} z\bar{z}') \leqslant -\alpha$, and all $0 < \rho < \rho_\alpha$, we have

$$d(\delta_\rho(a), b) > 1.$$

Part III

We define for $j \in \mathbb{N}^*$,

$$\psi_j := \pi - \frac{\pi}{2}\frac{1}{(j+1)^2}, \quad \theta_j := \frac{\pi}{2}\cdot\frac{j-1}{j}, \quad z_j := \mathrm{e}^{\mathrm{i}\theta_j}\sqrt{\sin\psi_j}, \quad t_j := \cos\psi_j.$$

Observe that in particular, $e^{i\psi_j} = t_j + i\left|z_j\right|^2$.

3-III1. Show that for integers $n > j$, we have

$$\operatorname{Im}(e^{i\psi_j} z_n \bar{z}_j) \leqslant \operatorname{Im}(e^{i\psi_j} z_{j+1} \bar{z}_j) < 0.$$

3-III2. Construct a sequence $(\rho_j)_{j\in\mathbb{N}^*}$ by induction, such that it decreases towards 0 and satisfies the following property: for all $\rho \leqslant \frac{\rho_{j+1}}{\rho_j}$ and $(z,t) \in \mathbb{S}$ satisfying

$$\operatorname{Im}(z\bar{z}_j e^{i\psi_j}) \leqslant \operatorname{Im}(z_{j+1}\bar{z}_j e^{i\psi_j}),$$

we have

$$d(\delta_\rho(z,t),(z_j,t_j)) > 1.$$

3-III3. Conclude that we have for any pair of integers $n > j$ and any $\rho \leqslant \frac{\rho_{j+1}}{\rho_j}$, we have

$$d(\delta_\rho(z_n,t_n),(z_j,t_j)) > 1.$$

Part IV

Let us define, for each $j \in \mathbb{N}^*$,

$$a_j := \delta_{\rho_j}(z_j,t_j).$$

Show that the family $\left\{B(a_j,\rho_j) : j \in \mathbb{N}^*\right\}$ cannot satisfy the statement of the Besicovitch theorem in the metric space $(\mathbb{H},d)$.

Exercise 4. Measurability of the centred maximal function

We consider a function $M_c(\mu) : \mathbb{R}^n \to [0,\infty]$ with respect to a positive and locally finite Borel measure μ, and a positive Borel measure ν.

4-1. Show that if $f : \mathbb{R}^n \to [0,\infty]$ and $g : \mathbb{R}^n \to [0,\infty)$ are Borel functions, then $f/g : \mathbb{R}^n \to [0,\infty]$ is a Borel function, with convention that $0/0 = 0$.

4-2. Show that if $r > 0$ is fixed, then $x \mapsto \mu(B(x,r))$ and $x \mapsto \nu(B(x,r))$ are lower semi-continuous functions on $\mathbb{R}^n$.

4-3. Show that for all $x \in \mathbb{R}^n$, $M_c(\nu)(x) = \sup_{r>0,\ r\in\mathbb{Q}} \frac{\nu(B(x,r))}{\mu(B(x,r))}$. (*Use monotone convergence theorem and discuss by cases.*)

4-4. Conclude that $M_c(\nu)$ is a Borel function.

Proof.

4-1. This is because

$$\frac{f}{g} = f\cdot\frac{1}{g}\mathbb{1}_{\{g>0\}} + 0\cdot\mathbb{1}_{\{g=0\}\cap\{f=0\}} + \infty\cdot\mathbb{1}_{\{g=0\}\cap\{f>0\}},$$

and each term is Borel.

4-2. For $\lambda > 0$, denote $V = \left\{x : \mu(B(x,r)) > \lambda\right\}$. For all $x_0 \in V$, since $\mu(B(x_0, r-1/n))$ is an increasing sequence that converges to $\mu(B(x_0,r))$, there exists $n \in \mathbb{N}$ such that $\mu(B(x_0,r-1/n)) > \lambda$. For $x \in B(x_0,1/n)$,

$$B(x,r) \supset B\left(x_0, r-\frac{1}{n}\right) \implies \mu(B(x,r)) \geqslant \mu\left(B\left(x_0, r-\frac{1}{n}\right)\right) > \lambda,$$

which shows that V is an open set. Hence $x \mapsto \mu(B(x,r))$ is lower semi-continuous. We do not use the locally finite property of μ, hence similar argument can be applied to ν.

4-3. It is trivial that $M_{c,\mathbb{Q}}(\nu)(x) = \sup_{r>\mathbb{Q}_+} \frac{\nu(B(x,r))}{\mu(B(x,r))} \leqslant M_c(\mu)(x)$. To show the other direction, let $r \in \mathbb{R}\setminus\mathbb{Q}$, $r>0$. Take $(r_k)_{k\geqslant 1} \subset \mathbb{Q}$ and $r_k \to r$ increasingly.

If $\mu(B(x,r)) = 0$, then $\mu(B(x,r_k)) = 0$ for all k and $\frac{\nu(B(x,r_k))}{\mu(B(x,r_k))} = \frac{\nu(B(x,r))}{\mu(B(x,r))}$. Otherwise, for k large enough,

$$\frac{\nu(B(x,r_k))}{\mu(B(x,r_k))} \to \frac{\nu(B(x,r))}{\mu(B(x,r))}, \qquad k\to\infty.$$

Thus $M_c(\nu) \leqslant M_{c,\mathbb{Q}}(\nu)$,

4-4. By **4-1** and **4-2**, for each fixed $r \in \mathbb{Q}_{>0}$, the function

$$g_r : \mathbb{R}^n \to [0,\infty], \qquad x \mapsto \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

is a Borel function. And note that for all $\alpha > 0$, one has

$$M_{c,\mathbb{Q}}(\nu)^{-1}(\alpha,+\infty] = (\sup g_r)^{-1}(\alpha,+\infty] = \bigcup_{r\in\mathbb{Q}_{>0}} g_r^{-1}(\alpha,+\infty].$$

Each g_r is Borel implies $M_{c,\mathbb{Q}}(\nu)$ is Borel. Then by **4-3**, $M_c(\nu)$ is Borel. □

Exercise 5. (Non-)integrability of maximal functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable. We define the centred maximal function $M_c f : \mathbb{R}^n \to [0,\infty]$ as in the Hardy–Littlewood maximal function. In this exercise, we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *trivial* if $f = 0$ almost everywhere on $\mathbb{R}^n$.

We aim to show that if the integrable function (in the sense of Lebesgue) $f : \mathbb{R}^n \to \mathbb{R}$ is non-trivial, then its maximal function $M_c f$ is never integrable over $\mathbb{R}^n$.

Part I

First, we fix $f : \mathbb{R}^n \to \mathbb{R}$ an integrable function (in the sense of Lebesgue) on $\mathbb{R}^n$.

5-1. Show that $|f| \leqslant M_c f$ almost everywhere on $\mathbb{R}^n$. (*Hint: you can use Lebesgue's differentiation theorem*)

5-2. Deduce that $M_c f$ is non-trivial if f is non-trivial.

Proof.

5-1. The Lebesgue's differentiation theorem told us that

$$f(x) = \lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(x)\,\mathrm{d}\mu(x)$$

for x μ-a.e.. Hence

$$\begin{aligned}|f(x)| &= \left|\lim_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(x)\,\mathrm{d}\mu(x)\right| \\ &\leqslant \limsup_{r\to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(x)|\,\mathrm{d}\mu(x) \leqslant \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(x)|\,\mathrm{d}\mu(x) = (M_c f)(x)\end{aligned}$$

for x μ-a.e..

5-2. If $f \neq 0$ a.e., then $\exists B$ such that $\frac{1}{\mu(B)}\int_B |f| > 0$. Hence $Mf(x) \geqslant \frac{1}{\mu(2B)}\int_{2B} |f| > 0$ for all $x \in B$. □

Part II

Now, suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is integrable on $\mathbb{R}^n$ and f is non-trivial. We aim to show that under these conditions, Mf is not integrable on $\mathbb{R}^n$. We proceed to this end in stages, and we denote by $\omega_n := |B(0,1)|$ the measure of the n-dimensional unit ball of $\mathbb{R}^n$.

5-3. Show that there exist real numbers $R > 0$ and $\varepsilon > 0$ such that

$$\int_{B(0,R)} |f| \geqslant \varepsilon.$$

5-4. Show that if $x \in \mathbb{R}^n$ satisfies $|x| > R$, then

$$M_c f(x) \geqslant \frac{1}{2^n \omega_n |x|^n} \int_{B(0,R)} |f|.$$

Hint: What can you say about the set $B(x, 2|x|)$?

5-5. Deduce that $M_c f$ is not integrable over $\mathbb{R}^n$.

Proof.

5-3. Since f is non-trivial, there exists a subset $E \subset \mathbb{R}^n$ with positive measure such that $|f| > 0$ on E. Choose $R > 0$ large enough such that $\mu(B(0,R) \cap E) > 0$, then

$$\int_{B(0,R)} |f| \geqslant \int_{B(0,R)\cap E} |f| > 0.$$

Take $\varepsilon = \int_{B(0,R)\cap E} |f|$ then we are done.

5-4. Since $|x| > R$, $B(x, 2|x|) \supset B(0,R)$, then

$$\begin{aligned}(M_c f)(x) &= \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \\ &\geqslant \frac{1}{\mu(B(x,2|x|))} \int_{B(x,2|x|)} |f| \geqslant \frac{1}{2^n |x|^n \omega_n} \int_{B(0,R)} |f|.\end{aligned}$$

5-5. By **5-4**, $M_c f \geqslant 2^{-n} |x|^{-n} \omega_n^{-1} \varepsilon > 0$ outside $\bar{B}(0,R)$. Hence $M_c f$ cannot be integrable. □

Exercise 6. Local integrability of $M_c f$ in $L \log L$

We define a function

$$\phi : \mathbb{R}_+ \to \mathbb{R}_+, \qquad t \mapsto \begin{cases} 0, & \text{if } 0 \leqslant t \leqslant 1, \\ t \log t, & \text{if } t > 1. \end{cases}$$

Note that $\phi(t)$ behaves as $t \log_+ t$ for $t > 0$, where $\log_+ t$ is the positive part of the logarithmic function, *i.e.*, $\log_+ t = \max\{\log t, 0\}$. If $f : \mathbb{R}^n \to \mathbb{R}$ is integrable, we define

$$\int_{\mathbb{R}^n} |f| \log_+ |f| := \int_{\mathbb{R}^n} \phi(|f|),$$

where $\int_{\mathbb{R}^n} \phi(|f|)$ is interpreted in a general sense (if $\phi(|f|)$ is not integrable, then we take the convention $\int_{\mathbb{R}^n} \phi(|f|) = +\infty$).

We propose to show step-by-step, that if $f : \mathbb{R}^n \to \mathbb{R}$ is integrable over $\mathbb{R}^n$, for any bounded subset B of $\mathbb{R}^n$ (which is also measurable), the following inequality holds:

$$\int_B M_c f \leqslant 2|B| + C \int_{\mathbb{R}^n} |f| \log_+ |f|, \tag{$\star$}$$

where $C > 0$ is a constant independent of f and B.

6-1. Show that for any measurable function $g : \mathbb{R}^n \to \mathbb{R}_+$ and any measurable subset $A \subset \mathbb{R}^n$,

$$\int_A g = \int_0^\infty \left|\{x \in A : g(x) > t\}\right| \mathrm{d}t.$$

Proof.

Note that $t \mapsto \left|\{x \in A : g(x) > t\}\right|$ is monotone, hence measurable. Use Fubini–Tonelli theorem,

$$\begin{aligned}\int_0^\infty \left|\{x \in A : g(x) > t\}\right| \mathrm{d}t &= \int_0^\infty \int_A \mathbb{1}_{\{g(x)>t\}} \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_A \int_0^\infty \mathbb{1}_{\{g(x)>t\}} \,\mathrm{d}t\,\mathrm{d}x = \int_A \int_0^{g(x)} \mathrm{d}t\,\mathrm{d}x = \int_A g(x)\,\mathrm{d}x.\end{aligned}$$

Then we conclude the proof. □

Now, fix a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(|f|)$ is Lebesgue integrable over $\mathbb{R}^n$, and a bounded and measurable subset $B \subset \mathbb{R}^n$. For $t > 0$, we define $f_t : \mathbb{R}^n \to \mathbb{R}$ as $f_t := f\mathbb{1}_{\{|f|>t\}}$, that is to say for $x \in \mathbb{R}^n$,

$$f_t(x) := \begin{cases} f(x), & \text{if } |f(x)| > t, \\ 0, & \text{else.} \end{cases}$$

We shall establish inequality ($\star$) as follows:

6-2. Show that

$$\int_B M_c f \leqslant 2|B| + 2\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t.$$

6-3. Show that for all $t \geqslant 1$, we have

$$\{x \in B : M_c f(x) > 2t\} \subset \{x \in B : M_c(f_t)(x) > t\}.$$

Hint: We can first show that we have $M(f - f_t) \leqslant t$ everywhere on $\mathbb{R}^n$.

6-4. Show that there exists a constant $C > 0$ independent of f and B, and whose origin will be specified, such that we have

$$\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \leqslant C \int_1^\infty \mathrm{d}t \frac{1}{t} \int_{\{|f|>t\}} |f(x)|\,\mathrm{d}x.$$

6-5. Deduce that we have

$$\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \leqslant C \int_{\mathbb{R}^n} \phi(|f|),$$

and conclude that ($\star$) is verified.

Proof.

6-2. This is because

$$\begin{aligned}\int_B M_c f &= \int_0^\infty \left|\{x \in B : M_c f(x) > t\}\right| \mathrm{d}t = 2\int_0^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \\ &= 2\int_0^1 \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t + 2\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \\ &= 2\int_0^1 \int_B \mathbb{1}_{\{M_c f(x) > 2t\}} \,\mathrm{d}x\,\mathrm{d}t + 2\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \\ &\leqslant 2\int_0^1 \int_B \mathrm{d}x\,\mathrm{d}t + 2\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \\ &= 2|B| + 2\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t.\end{aligned}$$

6-3. By the definition of f_t, $f - f_t = f\mathbb{1}_{\{|f(x)|\leqslant t\}}$. Therefore

$$\begin{aligned}M_c(f - f_t) &= \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_t|\,\mathrm{d}x \\ &= \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f\mathbb{1}_{|f(x)\leqslant t|}\,\mathrm{d}x \leqslant \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} t\,\mathrm{d}x = t.\end{aligned}$$

By the sub-additivity of the maximal function, if $x \in B$ such that $M_c f(x) > 2t$,

$$M_c(f) \leqslant M_c(f_t) + M_c(f - f_t) \implies M_c(f_t) \geqslant M_c(f) - M_c(f - f_t) > 2t - t = t,$$

thus $x \in \{x \in B : M_c f_t(x) > t\}$.

6-4. $f_t \in L^1(\mathrm{d}x)$ because $\phi(|f|)$ is integrable. Apply the maximal theorem, there exists a constant C that only depends on the dimension n, such that

$$\left|\{x \in B : M_c f_t(x) > t\}\right| \leqslant \frac{C}{t} \int_{\mathbb{R}^n} |f_t| \,\mathrm{d}x = \frac{C}{t} \int_{\{|f|>t\}} |f| \,\mathrm{d}x.$$

Thus by **6-3**,

$$\int_1^\infty \left|\{x \in B : M_c f(x) > 2t\}\right| \mathrm{d}t \leqslant \int_1^\infty \left|\{x \in B : M_c f_t(x) > t\}\right| \mathrm{d}t \leqslant \int_1^\infty \frac{C}{t} \int_{\{|f|>t\}} |f| \,\mathrm{d}x \,\mathrm{d}t.$$

6-5. Use Fubini–Tonelli theorem:

$$\begin{aligned}\int_1^\infty \frac{1}{t} \int_{\{|f(x)|>t\}} |f(x)| \,\mathrm{d}x \,\mathrm{d}t &= \int_1^\infty \frac{\mathrm{d}t}{t} \int_{\mathbb{R}^n} |f(x)| \mathbb{1}_{\{|f|>t\}} \,\mathrm{d}x = \int_{\mathbb{R}^n} |f(x)| \int_1^\infty \mathbb{1}_{\{|f|>t\}} \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} |f(x)| \int_1^{|f(x)|} \frac{\mathrm{d}t}{t} \,\mathrm{d}x = \int_{\mathbb{R}^n} |f(x)| \log_+ |f(x)| \,\mathrm{d}x = \int_{\mathbb{R}^n} \phi(|f(x)|) \,\mathrm{d}x.\end{aligned}$$

So ($\star$) holds. □

2 TD II – Maximal functions and spaces of homogeneous type

Exercise 1. Comparison of Maximal Functions

We aim to study the comparability between dyadic and Hardy–Littlewood maximal functions, both global and local, defined with respect to the Lebesgue measure. Let Mf, $M_c f$ denote the non-centred and the centred Hardy–Littlewood maximial functions on $\mathbb{R}^n$ of a function f, and $M_d f$ denote the dyadic maximal function of f on $\mathbb{R}^n$.

1-1. Verify that there exists two constants C, C' depending on n, such that

$$M_c f \leqslant Mf \leqslant C M_c f, \qquad M_d f \leqslant C' M f.$$

1-2. Show that there cannot exist a constant $c > 0$ such that for all $f \in L^1(\mathbb{R}^n)$, we have almost everywhere on $\mathbb{R}^n$,

$$M_d f > c M f.$$

Hint: Show that there exists a non-trivial function $f \in L^1(\mathbb{R}^n)$ such that $M_d f = 0$ on a non-trivial cube.

1-3. Let $M_{d,[0,1)}$ be the dyadic maximal function on $[0,1)$ and $M_{[0,1]}$ the Hardy–Littlewood maximal function centred on the homogeneous space $([0,1], |\cdot|, \mathrm{d}x)$. Can there exists a constant $c > 0$ such that for all $f \in L^1(0,1)$, we have almost everywhere on $[0,1]$:

$$M_{d,[0,1)} f > c M_{[0,1]} f?$$

Proof.

1-1. $M_c f \leqslant M f$ is trivial. On the other hand, for any ball containing x with radius r, $B \subset B(x, 2r)$ and $\mu(B)/\mu(B, 2r) = 2^{-n}$. Hence

$$\frac{1}{\mu(B)} \int_B |f| \,\mathrm{d}\mu \leqslant \frac{\mu(B(x,2r))}{\mu(B)} \cdot \frac{1}{\mu(B(x,2r))} \int_{B(x,2r)} |f| \,\mathrm{d}\mu,$$

thus $Mf \leqslant 2^n M_c f$.

For dyadic cube Q containing x, denote its diamater $\operatorname{diam} Q$, $B(x, \operatorname{diam} Q) \supset Q$. Hence

$$\frac{1}{\mu(Q)} \int_Q |f| \, \mathrm{d}\mu \leqslant \frac{\mu(B(x, \operatorname{diam} Q))}{\mu(Q)} \cdot \frac{1}{\mu(B, \operatorname{diam} Q)} \int_{B(x, \operatorname{diam} Q)} |f| \, \mathrm{d}\mu,$$

thus $M_d f \leqslant C' M_c f \leqslant C' M f$, where

$$C' = \frac{\mu(B(x, \operatorname{diam} Q))}{\mu(Q)} = \frac{(n\pi)^{n/2}}{\Gamma(n/2+1)}.$$

1-2. Take $f = \mathbb{1}_{[0,1]^n}$. Then for $x \in \mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) : x_i \geqslant 0\}$, denote $k = \min\{m \in \mathbb{N} : x \in [0, 2^k)^n\}$, one has

$$M_d f(x) = \sup_{x \in Q} \frac{1}{\mu(Q)} \int_Q |f| \, \mathrm{d}\mu = \frac{1}{\mu([0, 2^k)^n)} \int_{[0,2^k)^n} |\mathbb{1}_{[0,1]^n}(x)| \, \mathrm{d}x = \frac{|[0,1]^n|}{|[0, 2^k)^n|} = \frac{1}{2^{nk}} > 0.$$

And for $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, $M_d f(x) = 0$. While $Mf(x) > 0$ for $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$.

1-3. Choose a sequence $(f_n)_{n \geqslant 1}$, for example, $f_n = \mathbb{1}_{[1/2-1/n, 1/2]}$ for $n \geqslant 2$, then

$$M_{d,[0,1)} f_n\left(\frac{1}{2}\right) = \frac{1}{2}, \qquad M_c f_n\left(\frac{1}{2}\right) = \frac{n}{2},$$

because $\frac{1}{2} \in Q$, $Q \cap \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right] = \varnothing$ and Q dyadic implies that $Q = [0, 1)$. □

Exercise 2. Hölder Regularity of Maximal Functions

Part A. Hölder Regularity in the Euclidean Case

Given $u : \mathbb{R}^n \to \bar{\mathbb{R}}$, define a function $u_h : \mathbb{R}^n \to \bar{\mathbb{R}}$ for $h \in \mathbb{R}^n$ as

$$u_h(x) := u(x+h), \qquad \forall x \in \mathbb{R}^n.$$

2-A1. Show that if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, then for $h \in \mathbb{R}^n$,

$$(Mu)_h = M(u_h),$$

where Mu is the centred maximal function of u.

2-A2. Deduce from the previous point and the sublinearity of M that if $u : \mathbb{R}^n \to \mathbb{R}^n$ is Hölder continuous with exponent α (where $0 < \alpha \leqslant 1$) and constant C, then

$$|Mu(x+h) - Mu(x)| \leqslant C|h|^\alpha,$$

whenever $x, h \in \mathbb{R}^n$ are given.

Proof.

2-A1. This is because Lebesgue measure is translation invariant, thus

$$(Mu)_h(x) = Mu(x+h) = \sup_{x+h \in B} \frac{1}{\mu(B)} \int_B |u(t)| \, \mathrm{d}t = \sup_{x \in B} \frac{1}{\mu(B)} \int_{B+h} |u(t)| \, \mathrm{d}t = Mu_h(x).$$

2-A2. Since u is Hölder continuous, $|u(x+h) - u(x)| = |(u_h - u)(x)| \leqslant c|h|^\alpha$ for some $c > 0$. Thus by $|u_h| \leqslant |u| + |u_h - u|$, one has $Mu_h \leqslant Mu + M(u_h - u)$ and $Mu \leqslant Mu_h + M(u - u_h)$. Thus

$$|Mu_h - Mu| \leqslant M(u_h - u) \leqslant c|h|^\alpha.$$

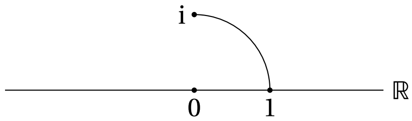
Take $C = c$ then we are done. □

Part B. Counterexample to Lipschitz Regularity in Metric Spaces

Consider the measured metric space (X, d, μ) defined as follows: Let

$$X := \mathbb{R} \cup \Gamma, \text{ where } \Gamma = \left\{ z \in \mathbb{C} : |z| = 1,\ 0 \leqslant \arg z \leqslant \frac{\pi}{2} \right\},$$

and let d be the Euclidean distance restricted to X.



Denote $\mu = \mathscr{H}^1|_X$ the 1-dimensional Hausdorff measure restricted to X. Thus, for any integrable or positive measurable function,

$$\int_X f \, \mathrm{d}\mu = \int_{\mathbb{R}} f(t) \, \mathrm{d}t + \int_0^{\pi/2} f(\exp(\mathrm{i}\theta)) \, \mathrm{d}\theta.$$

2-B1. Show that (X, d, μ) is a space of homogeneous type.

Proof.

2-B1. Note that $d : X \times X \to [0, \infty)$ is a distance because it is restricted from a Euclidean distance. It suffices to prove the measure is doubling.

Case 1: $x \in \mathbb{R}$. In this case, $\mu(B(x,r)) = |(x-r, x+r)| + \mu(B(x,r) \cap \Gamma)$. Hence we have the lower bound $\mu(B(x,r)) \geqslant 2r$. Then we discuss the upper bound.

Case 1(1): $x < 0$. For $x < 0$, $B(x,r) \cap \Gamma \neq \varnothing \implies r \geqslant 1$, hence

$$\mu(B(x,r)) \leqslant \begin{cases} 2r, & r < 1, \\ 2r + \frac{\pi}{2} \leqslant \left(2 + \frac{\pi}{2}\right) r, & r \geqslant 1. \end{cases}$$

Case 1(2): $0 < x < 2r$. Now $B(x,r) \subset B(x, 3r)$, hence

$$\mu(B(x,r)) \leqslant \mu(B(0,3r)) \leqslant 6r + \frac{\pi}{2} \leqslant cr.$$

Case 1(3): $x > 2r$.

- If $r > 1$, $B(x,r) \cap \Gamma = \varnothing$, hence $\mu(B(x,r)) = 2r$.
- If $r < \frac{1}{2}$, $B(x,\mu) \cap \Gamma$ has length $\leqslant cr$, thus $\mu(B(x,r)) \leqslant (c+2)r$.
- If $\frac{1}{2} \leqslant r \leqslant 1$, $\mu(B(x,r)) \leqslant 2 + \frac{\pi}{2} \leqslant (4+\pi)r$.

Thus, $\forall x \in \mathbb{R}$, $\forall r > 0$, $2r \leqslant \mu(B(x,r)) \leqslant cr$.

Case 2: $x \in X \setminus \mathbb{R} \subset \Gamma$. Denote $x = a + b\mathrm{i}$, then $|x| = 1$ and $d(x, \mathbb{R}) = b \leqslant 1$.

Case 2(1): $r \geqslant \max\{|x-1|, |x-\mathrm{i}|\} \geqslant \frac{\sqrt{2}}{2}$. In this case, $B(x,r) \supset \Gamma$, hence

$$\mu(B(x,r)) = \frac{\pi}{2} + 2\sqrt{r^2 - b^2}, \qquad \mu(B(x,2r)) = \frac{\pi}{2} + 2\sqrt{4r^2 - b^2}.$$

Then

- If $r \leqslant 2$, $\mu(B(x,2r)) \leqslant \frac{\pi}{2} + 2 \cdot 2r \leqslant 8 + \frac{\pi}{2}$; $\mu(B(x,r)) \leqslant \frac{\pi}{2}$.
- If $r \geqslant 2$, then $\sqrt{r^2 - b^2} \geqslant \sqrt{r^2 - 1} \geqslant \frac{\sqrt{3}}{2} r$. Then $\mu(B(x,2r)) \leqslant 4r + \frac{\pi}{2} + \left(4 + \frac{\pi}{4}\right) r$, $\mu(B(x,r)) \geqslant \sqrt{3} r$.

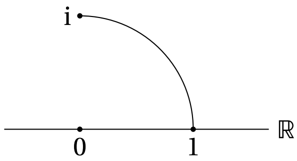
Case 2(2): $r < \min\{|x-1|, |x-\mathrm{i}|\}$. In this case, $|x-1| = \sqrt{2-2a}$, $|x-\mathrm{i}| = \sqrt{2-2b}$.

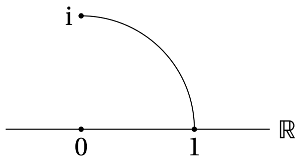
- If $2r \leqslant b$, $B(x,2r) = B(x,r) \cap \Gamma$, then $\mu(B(x,2r)) = \frac{\pi}{2} \cdot 2r$, $\mu(B(x,r)) = \frac{\pi}{2} r$.
- If $b \leqslant r \leqslant \sqrt{2} \min\left\{ \sqrt{1-a}, \sqrt{1-b} \right\} = m$:

- When $|x-\mathrm{i}| \leqslant |x-1|$, we have $b \geqslant \frac{\sqrt{2}}{2}$, and $\mu(B(x,2r)) \leqslant 2m+\frac{\pi}{2}$. Then $\mu(B(x,r)) \geqslant \mu(B(x,b)) \geqslant b$ because the length of red curve $\geqslant$ radius.
- When $|x-1| \leqslant |x-\mathrm{i}|$, $\mu(B(x,r)) \geqslant r$ because the blue curve $\geqslant$ radius. And

$$B(x,2r) \subset B(1,2r+|x-1|) \subset B(1,2r+\sqrt{2}b) \subset B(1,(2+\sqrt{2})r).$$

By case 1, $\mu(B(x,2r)) \leqslant c(2+\sqrt{2})r$.





Thus $\forall x \in X \setminus \mathbb{R} \subset \Gamma$, $\forall r > 0$, $c'r \leqslant \mu(B(x,r)) \leqslant cr$.

The argument above implies that μ is a doubling measure. $\square$

Now, fix a Lipschitz function $u : X \to [0,1]$ such that $u(z) = 0$ when $z \in \mathbb{R}$ or $\arg z \leqslant \frac{\pi}{5}$; and $u(z) = 1$ when $\arg z \geqslant \frac{\pi}{4}$. Let $M_c u$ denote the centred maximal function of u with respect to μ.

2-B2. Show that

$$M_c u(0) = \frac{1}{2+\pi/2} \int_{B(0,1)} u \,\mathrm{d}\mu.$$

2-B3. Deduce that

$$M_c u(0) \leqslant \frac{3\pi}{20+5\pi}.$$

2-B4. Show that if $x \in X$ is real and satisfies $x < 0$, then

$$B(x, d(x, \mathrm{e}^{\mathrm{i}\pi/4})) \cap \Gamma = \left\{ z \in \Gamma : \arg z \geqslant \frac{\pi}{4} \right\}.$$

2-B5. Deduce that

$$\lim_{x \to 0^-} M_c u(x) \geqslant \frac{\pi}{8+\pi} > M_c u(0).$$

Proof.

2B-2. Now $u : X \to [0,1]$ is Lipschitz. Consider $B(0,r) \cap X$. When $r < 1$,

$$u|_{B(0,r)} = u|_{(-r,r)} = 0 \implies \int_{B(0,r)} u \,\mathrm{d}\mu = 0;$$

when $r \geqslant 1$,

$$\int_{B(0,r)} u \,\mathrm{d}\mu = \int_{B(0,1)} u \,\mathrm{d}\mu,$$

but for any $r \geqslant 1$, $\mu(B(0,r)) \geqslant \mu(B(0,1))$. Thus

$$M_c u(0) = \frac{1}{\mu(B(0,1))} \int_{B(0,1)} u \,\mathrm{d}\mu = \frac{1}{2+\frac{\pi}{2}} \int_{B(0,1)} u \,\mathrm{d}\mu.$$

2-B3. By definition of u, $\operatorname{supp} u \subset \left\{ z \in \mathbb{S}^1 : \frac{\pi}{5} \leqslant \arg z \leqslant \frac{\pi}{2} \right\}$. Thus

$$\int_{B(0,1)} u \,\mathrm{d}\mu = \int_0^{\pi/2} u(\mathrm{e}^{\mathrm{i}\theta}) \,\mathrm{d}\theta = \int_{\pi/5}^{\pi/2} u(\mathrm{e}^{\mathrm{i}\theta}) \,\mathrm{d}\theta \leqslant \left(\frac{\pi}{2} - \frac{\pi}{5} \right) \|u\|_\infty = \frac{3\pi}{10}.$$

Therefore

$$M_c u(0) = \frac{1}{2+\frac{\pi}{2}} \int_{B(0,1)} u \,\mathrm{d}\mu \leqslant \frac{3\pi}{20+5\pi}.$$

2-B4. When $x \in \mathbb{R}$, $x < 0$, for $z = e^{i\theta}$ with $\theta < \frac{\pi}{4}$, one has $d(x,z) > d(x, e^{i\pi/4})$.

2-B5. For $x \in \mathbb{R}$, $x < 0$,

$$\begin{aligned} M_c u(x) &\geqslant \frac{1}{\mu(B(x, d(x, e^{i\pi/4})))} \int_{B(x,d(x,e^{i\pi/4}))} u \,\mathrm{d}\mu = \frac{1}{\mu(B(x, d(x, e^{i\pi/4})))} \int_A 1 \,\mathrm{d}\mu \\ &= \frac{1}{\mu(B(x, d(x, e^{i\pi/4})))} \cdot \frac{\pi}{4} \geqslant \frac{1}{2 + \frac{\pi}{4}} \cdot \frac{\pi}{4} = \frac{\pi}{8+\pi} > \frac{3\pi}{20 + 5\pi} \geqslant M_c u(0). \end{aligned}$$

□

Main Part. Regularity in "Regular" Metric Spaces

Now given $0 < \delta \leqslant 1$, and (X, d, μ) is a locally finite measured metric space that satisfies the following annular decay property: there exists $K_\delta > 0$ such that for all $x \in X$, $R > 0$ and $0 < h < R$, we have

$$\mu(B(x,R) \setminus B(x, R-h)) \leqslant K_\delta \left(\frac{h}{R}\right)^\delta \mu(B(x,R)) \tag{B.1}$$

(*Examples for $\delta = 1$: $\mathbb{R}^n$ and Heisenberg group, etc.*)

Verify that under these assumptions, (X, d, μ) is a space of homogeneous type. For $0 < \alpha \leqslant 1$, we equip the space $C_0^\alpha(X)$ with the norm

$$\|u\|_{0,\alpha} := \|u\|_\infty + H_\alpha(u), \text{ where } H_\alpha(u) := \sup_{x \neq y} \frac{|u(x) - u(y)|}{d(x,y)^\alpha}.$$

We aim to show that, set $\beta := \min\{\alpha, \delta\}$, the centred maximal operator M associated with μ maps $C_0^\alpha(X)$ into $C_0^\beta(X)$ as a bounded operator.

We fix $u \in C_0^\alpha(X)$ such that $\|u\|_{0,\alpha} = 1$. And denote $⨍_E f \,\mathrm{d}\mu = \frac{1}{\mu(E)} \int_E f \,\mathrm{d}\mu$ for convenience.

2-1. Show that it suffices to prove that there exists a constant $C > 0$, independent of u, such that for any given $x, y \in X$, we have

$$M_c u(x) - M_c u(y) \leqslant C d(x,y)^\beta.$$

2-2. Show that we may assume $d(x,y) \leqslant 1$.

<u>*Proof.*</u>

Setup. Since (X, d, μ) is a sht, let $K_\delta \left(\frac{h}{R}\right)^\delta = \alpha$, then $h = R\left(\frac{\alpha}{K_\delta}\right)^{1/\delta}$. Choose $\alpha \in (0,1)$ such that $\left(\frac{\alpha}{K_\delta}\right)^{1/\delta} = \frac{1}{2}$. Hence

$$\mu\left(B(x,R) \setminus B\left(x, \frac{R}{2}\right)\right) \leqslant \mu(B(x,R)) \implies \mu(B(x,R)) \leqslant \frac{1}{1-\alpha} \mu\left(B\left(x, \frac{R}{2}\right)\right).$$

2-1. If this holds, then

$$H_\beta(M_c u) = \sup_{x \neq y} \frac{|M_c u(x) - M_c u(y)|}{d(x,y)^\beta} \leqslant C.$$

And by definition, $\forall x \in X$, $M_c u(x) = \sup_{r>0} ⨍_{B(x,r)} |u| \,\mathrm{d}\mu \leqslant \|u\|_\infty$, hence $\|M_c u\|_\infty \leqslant \|u\|_\infty = 1$,

$$\|M_c u\|_{0,\beta} = \|M_c u\|_\infty + H_\beta(M_c u) \leqslant \|M_c u\|_\infty + C \leqslant (C+1)\|u\|_{0,\alpha}.$$

This implies $M_c : C_0^\alpha(X) \to C_0^\beta(X)$ is bounded since C is independent of u.

2-2. If one proves the inequality for $d(x,y) \leqslant 1$, then assume $d(x,y) \geqslant 1$,

$$|M_c u(x) - M_c u(y)| \leqslant M_c u(x) + M_c u(y) \leqslant 2\|M_c u\|_\infty \leqslant 2 \leqslant 2|x-y|^\beta.$$

Then for all $x, y \in X$ the inequality holds. □

Now fix $x, y \in X$ with $d(x,y) \leqslant 1$, and choose $r > 0$ such that

$$\fint_{B(x,r)} |u|\,\mathrm{d}\mu \geqslant M_c u(x) - d(x,y)^\beta.$$

Case 1. Suppose first that $d(x,y) \geqslant r$**.**

2-3. Show that $|u(z) - u(w)| \leqslant 3^\alpha d(x,y)^\alpha$ for all $z, w \in B(x,r) \cup B(y,r)$.

2-4. Deduce that

$$\left| \fint_{B(x,r)} |u|\,\mathrm{d}\mu - \fint_{B(y,r)} |u|\,\mathrm{d}\mu \right| \leqslant 3^\alpha d(x,y)^\beta.$$

2-5. Conclude that

$$M_c u(y) \geqslant M_c u(x) - (3^\alpha + 1) d(x,y)^\beta.$$

Proof.

2-3. If $z, w \in B(x,r)$ or $z, w \in B(y,r)$,

$$|u(z) - u(w)| \leqslant d(z,w)^\alpha \leqslant (2r)^\alpha \leqslant 2^\alpha d(x,y)^\alpha.$$

If $z \in B(x,r)$, $w \in B(y,r)$, then

$$\begin{aligned} |u(z) - u(w)| &\leqslant |u(z) - u(x)| + \big|u(x) - u(y)\big| + \big|u(y) - u(w)\big| \\ &\leqslant d(z,x)^\alpha + d(x,y)^\alpha + d(y,w)^\alpha \leqslant r^\alpha + d(x,y)^\alpha + r^\alpha \leqslant 3d(x,y)^\alpha \leqslant 3^\alpha d(x,y)^\alpha \end{aligned}$$

since $\alpha \leqslant 1$. Similar argument applies to $z \in B(y,r)$, $w \in B(x,r)$ and we obtain the same conclusion.

2-4. Since $d(x,y) \leqslant 1$ and $0 < \beta \leqslant \alpha$,

$$\begin{aligned} \left| \fint_{B(x,r)} |u|\,\mathrm{d}\mu - \fint_{B(y,r)} |u|\,\mathrm{d}\mu \right| &\leqslant \fint_{B(x,r)} \left| u(z) - \fint_{B(y,r) u(w)} \mathrm{d}\mu(w) \right| \mathrm{d}\mu(z) \\ &\leqslant \fint_{B(x,r)} \fint_{B(y,r)} |u(z) - u(w)|\,\mathrm{d}\mu(w)\,\mathrm{d}\mu(z) \\ &\leqslant \fint_{B(x,r)} \fint_{B(y,r)} 3^\alpha d(x,y)^\alpha\,\mathrm{d}\mu(w)\,\mathrm{d}\mu(z) \leqslant 3^\alpha d(x,y)^\beta. \end{aligned}$$

2-5. This is because

$$M_c u(y) \geqslant \fint_{B(y,r)} |u|\,\mathrm{d}\mu \geqslant \fint_{B(x,r)} |u|\,\mathrm{d}\mu - \left| \fint_{B(x,r)} |u|\,\mathrm{d}\mu - \fint_{B(y,r)} |u|\,\mathrm{d}\mu \right| \geqslant M_c u(x) - 3^\alpha d(x,y)^\beta - d(x,y)^\beta.$$

by our choice of r. □

Case 2. Suppose $d(x,y) < r$**.**

For simplicity, let $a := d(x,y)$. For any $c > 0$, let S_c be the set of functions $v \in L^1(B(x, r+2a))$ such that

$$\fint_{B(x,r)} |v|\,\mathrm{d}\mu - \fint_{B(y,r+a)} |v|\,\mathrm{d}\mu \leqslant c d(x,y)^\beta.$$

2-6. Show that it suffices to establish the existence of $c > 0$, independent of x and y, such that $u|_{B(x,r+2a)} \in S_c$.

Proof.

2-6. If $u|_{B(x,r+2a)} \in S_c$, note that $B(x,r) \subset B(y, r+a) \subset B(x, r+2a)$,

$$M_c u(x) \leqslant \fint_{B(x,r)} |u|\,\mathrm{d}\mu + d(x,y)^\beta \leqslant \fint_{B(y,r+a)} |u|\,\mathrm{d}\mu + d(x,y)^\beta + c d(x,y)^\beta.$$

Thus $M_c u(x) \leqslant M_c u(y) + (c+1) d(x,y)^\beta$. □

Let $A := \min\{1, (6r)^\alpha\}$ and define

$$F := \left\{ v \in L^1(B(x, r+2a)) : \|v\|_{L^\infty(B(x,r+2a))} \leqslant A,\ H_\alpha(v) \leqslant 1 \right\}.$$

We begin by showing that it suffices to prove that $F \subset S_c$. Suppose that $F \subset S_c$ for some $c > 0$, and now show that $u|_{B(x,r+2a)} \in S_c$. To do this, set

$$m_1 := \inf_{B(x,r+2a)} u, \qquad m_2 := \sup_{B(x,r+2a)} u, \qquad u_1 := u - m_1, \qquad u_2 := u - m_2.$$

2-7. Show that the result is clear if $A = 1$.

2-8. Now suppose $A < 1$, show that $u_1 \in F$ and $u_2 \in F$.

2-9. If $u(z_0) > A$ for some $z_0 \in B(x, r+2a)$, then u and u_1 are positive on $B(x, r+2a)$, deduce that $u \in S_c$.

2-10. If $u(z_0) \leqslant -A$ for some $z_0 \in B(x, r+2a)$, show that u and u_2 are negative on $B(x, r+2a)$; conclude similarly that $u \in S_c$.

2-11. Using the annular decay property (B.1) of μ, show that for $v \in F$,

$$\fint_{B(x,r)} |v| \,\mathrm{d}\mu - \fint_{B(y,r+a)} |v| \,\mathrm{d}\mu \leqslant \left(\frac{1}{\mu(B(x,r))} - \frac{1}{\mu(B(y,r+a))} \right) \int_{B(x,r)} |v| \,\mathrm{d}\mu \leqslant 2^\delta c_\mu A K_\delta \left(\frac{a}{r+2a} \right)^\delta.$$

2-12. Conclude that $v \in S_c$ for a well-chosen $c > 0$, distinguishing between the cases $\alpha \leqslant \delta$ and $\alpha > \delta$.

Proof.

2-7. Now we assume $A = 1$, then consider $u|_{B(x,r+2a)}$. We have

$$\left\| u|_{B(x,r+2a)} \right\|_{L^\infty(B(x,r+2a))} \leqslant \|u\|_\infty \leqslant 1, \qquad H_\alpha(u|_{B(x,r+2a)}) \leqslant H_\alpha(u) \leqslant 1$$

by our assumption. Hence $u|_{B(x,r+2a)} \in F \subset S_c$, then done by **2-6**.

2-8. $A < 1$ means $A = (6r)^\alpha$ and $6r < 1$. One has

$$\|u_1\|_{L^\infty(B(x,r+2a))} = m_2 - m_1 =: u(z) - u(w),$$

where $z, w \in \bar{B}(x, r+2a)$ because u is continuous. Then

$$|u(z) - u(w)| \leqslant d(z,w)^\alpha \leqslant (2(r+2a))^\alpha = 2^\alpha (r + d(x,y))^\alpha \leqslant 2^\alpha (3r)^\alpha = (6r)^\alpha.$$

And $H_\alpha(u_1) = H_\alpha(u - m_1) = H_\alpha(u) \leqslant 1$. So $u_1 \in F$. The estimation of $\|u_2\|_{L^\infty(B(x,r+2a))}$ is similar to that of u_1. And $u_2 \leqslant 0$ on $B(x, r+2a)$, hence $H_\alpha(u_2) \leqslant H_\alpha(u) \leqslant 1$, which implies $u_2 \in F$.

2-9. We have known $u_1 \geqslant 0$, and $m_2 \geqslant u(z_0) \geqslant A \geqslant m_2 - m_1$. Hence $m_1 \geqslant m_2 - A \geqslant 0$, so $u \geqslant 0$. Thus

$$\begin{aligned} \fint_{B(x,r)} |u| \,\mathrm{d}\mu - \fint_{B(y,r+a)} |u| \,\mathrm{d}\mu &= \fint_{B(x,r)} u \,\mathrm{d}\mu - \fint_{B(y,r+a)} u \,\mathrm{d}\mu \\ &= \fint_{B(x,r)} u_1 \,\mathrm{d}\mu - \fint_{B(x,r)} u_1 \,\mathrm{d}\mu \leqslant c d(x,y)^\beta \end{aligned}$$

because $u_1 \in F$ by **2-8**.

2-10. Almost the same as **2-9**.

2-11. If $v \in F$, note that $B(y, r+a) \subset B(x, r+2a)$,

$$\begin{aligned}
\fint_{B(x,r)} |v|\,\mathrm{d}\mu - \fint_{B(y,r+a)} |v|\,\mathrm{d}\mu &= \left(\frac{1}{\mu(B(x,r))} - \frac{1}{\mu(B(y,r+a))}\right)\int_{B(x,r)} |v|\,\mathrm{d}\mu - \frac{1}{\mu(B(y,r+a))}\int_{B(y,r+a)\setminus B(x,r)} |v|\,\mathrm{d}\mu \\
&\leqslant \frac{\mu(B(y,r+a)\setminus B(x,r))}{\mu(B(x,r))\mu(B(y,r+a))}\int_{B(x,r)} |v|\,\mathrm{d}\mu \\
&\leqslant \frac{\mu(B(x,r+2a)\setminus B(x,r))}{\mu(B(y,r+a))} A \\
&\leqslant K_\delta\left(\frac{2a}{r+2a}\right)^\delta A\cdot\frac{\mu(B(x,3r))}{\mu(B(x,r))} \leqslant K_\delta 2^\delta A c_\mu\left(\frac{a}{r+2a}\right)^\delta .
\end{aligned}$$

2-12. Since $6r < 1$ and $\beta \leqslant \alpha$, $A = (6r)^\alpha \leqslant (6r)^\beta$. By the result of **2-11**,

$$\fint_{B(x,r)} |v|\,\mathrm{d}\mu - \fint_{B(y,r+a)} |v|\,\mathrm{d}\mu \leqslant K_\delta\left(\frac{2d(x,y)}{r+2d(x,y)}\right)^\delta (6r)^\beta c_\mu. \tag{B.2}$$

- If $\beta = \alpha \leqslant \delta$, $\left(\frac{2d(x,y)}{r+2d(x,y)}\right)^\delta \leqslant \left(\frac{2d(x,y)}{r+2d(x,y)}\right)^\beta$, hence (B.2) is bounded by $K_\delta \cdot 12^\beta c_\mu d(x,y)^\beta$.
- If $\beta = \delta \leqslant \alpha$, $\frac{r^\delta}{(r+2d(x,y))^\beta} = \left(\frac{r}{r+2d(x,y)}\right)^\beta \leqslant 1$, hence (B.2) is bounded by $K_\delta \cdot 12^\beta c_\mu d(x,y)^\beta$.

Then let $c = 12^\beta K_\delta c_\mu$, we conclude the proof. □

Exercise 3. Hardy operator: weak type and strong type

Let $H(f)(t) = \frac{1}{t}\int_0^t |f(s)|\,\mathrm{d}s$ defined for measurable f on $(0,+\infty)$ and $t > 0$. H is called the *Hardy operator.*

3-1. Assume $f \in L^1((0,+\infty),\mathrm{d}t)$. Let $\Omega_\lambda = \{t > 0 : Hf(t) > \lambda\}$ for $\lambda > 0$. Show that

$$|\Omega_\lambda| \leqslant \frac{1}{\lambda}\int_{\Omega_\lambda} |f(s)|\,\mathrm{d}s.$$

Hint: Observe that Ω_λ is a union of interval using the function $F(t) - \lambda t$ where $F(t) = tHf(t)$. Use a graphic representation.

3-2. Deduce that for $f \in L^p(0,+\infty)$, $1 < p \leqslant +\infty$, $\|Hf\|_p \leqslant \frac{p}{p-1}\|f\|_p$.

3-3. Let $\gamma > 0$ and $K_\gamma(t,s) = \left(\frac{s}{t}\right)^{1+\gamma}\mathbb{1}_{\{s<t\}}$ for $s, t > 0$. Check that

$$\int_0^\infty K_\gamma(t,s)\frac{\mathrm{d}s}{s} = \frac{1}{1+\gamma} = \int_0^\infty K_\gamma(t,s)\frac{\mathrm{d}t}{t}.$$

Deduce using Hölder inequality that $(T_\gamma f)(t) = \int_0^\infty K_\gamma(t,s) f(s)\frac{\mathrm{d}s}{s}$ satisfies

$$\|T_\gamma f\|_{L^p((0,\infty),\mathrm{d}t/t)} \leqslant \frac{1}{1+\gamma}\|f\|_{L^p((0,\infty),\mathrm{d}t/t)}.$$

3-4. Relate the operators H and T_γ and recover the result of **3-2** from the inequality in **3-3**.

Proof.

3-1. Note that

$$\Omega_\lambda = \{t > 0 : Hf(t) > \lambda\} = \{t > 0 : tHf(t) > \lambda t\} = \{t > 0 : F(t) - \lambda t > 0\}.$$

Since $f \in L^1(\mathbb{R}_+)$, $F(t) = \int_0^t |f(t)|\,\mathrm{d}t$ is monotonically increasing, bounded and continuous. For any fixed $\lambda > 0$, since $\lim_{t\to\infty} F(t) - \lambda t = -\infty$, Ω_λ is a union of interval and cannot contain interval of the form $(a,+\infty)$. Denote $\Omega_\lambda = \bigcup_{i\in I}(a_i, b_i)$. Then

$$F(b_i) - F(a_i) = \lambda(b_i - a_i), \qquad \forall i \in I.$$

Thus

$$\frac{1}{\lambda}\int_{a_i}^{b_i}\left|f(s)\right|\mathrm{d}s=b_i-a_i=|(a_i,b_i)|.$$

Sum from $i\in I$, we obtain the result. Actually this implies $|\Omega_\lambda|=\frac{1}{\lambda}\int_{\Omega_\lambda}\left|f(s)\right|\mathrm{d}s$, the inequality comes from the case that H' is defined as $H(f)(t)=\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$. This implies that H is of weak type (1,1).

3-2. This is because

$$\begin{aligned}\int_0^\infty\left|Hf(t)\right|^p\mathrm{d}t&=p\int_0^\infty\lambda^{p-1}|\Omega_\lambda|\,\mathrm{d}\lambda=p\int_0^\infty\lambda^{p-1}\frac{1}{\lambda}\int_{\Omega_\lambda}\left|f(s)\right|\mathrm{d}s\,\mathrm{d}\lambda\\&=p\int_0^\infty\lambda^{p-2}\int_{\Omega_\lambda}\left|f(s)\right|\mathrm{d}s\,\mathrm{d}\lambda=p\int_0^\infty\left|f(s)\right|\int_0^{Hf(s)}\lambda^{p-2}\,\mathrm{d}\lambda\,\mathrm{d}s\\&=p\int_0^\infty\left|f(s)\right|\frac{(Hf(s))^{p-1}}{p-1}\mathrm{d}s\leqslant\frac{p}{p-1}\left\|f\right\|_p\left\|Hf\right\|_p^{p-1}.\end{aligned}$$

This implies H is of strong type (p,p).

3-3. Let $p\in(1,\infty)$ and q such that $1/p+1/q=1$. Then

$$\begin{aligned}\left\|T_\gamma f\right\|_{p,\mathrm{d}t/t}^p&=\int_0^\infty\left(\int_0^\infty K_\gamma(t,s)f(s)\frac{\mathrm{d}s}{s}\right)^p\frac{\mathrm{d}t}{t}=\int_0^\infty\left(\int_0^\infty K_\gamma(t,s)^{1/q}K_\gamma(t,s)^{1/p}f(s)\frac{\mathrm{d}s}{s}\right)^p\frac{\mathrm{d}t}{t}\\&\leqslant\left\|K_\gamma^{1/p}f\right\|_{p,\mathrm{d}t/t}^p\left\|K_\gamma^{1/q}\right\|_{p,\mathrm{d}t/t}^p\frac{\mathrm{d}t}{t}=\left(\frac{1}{1+\gamma}\right)^{p/q}\int_0^\infty\int_0^\infty K_\gamma(t,s)\left|f(s)\right|^p\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\\&=\left(\frac{1}{1+\gamma}\right)^{p/q}\int_0^\infty\left|f(s)\right|^p\frac{\mathrm{d}s}{s}\int_0^\infty K_\gamma(t,s)\frac{\mathrm{d}t}{t}=\left(\frac{1}{1+\gamma}\right)^{p/q}\int_0^\infty\frac{1}{1+\gamma}\left|f(s)\right|^p\frac{\mathrm{d}s}{s}\\&=\left(\frac{1}{1+\gamma}\right)^{1+p/q}\left\|f\right\|_{p,\mathrm{d}t/t}^p.\end{aligned}$$

Therefore

$$\left\|T_\gamma f\right\|_{p,\mathrm{d}t/t}\leqslant\left(\frac{1}{1+\gamma}\right)1/p(1+p/q)\left\|f\right\|_{p,\mathrm{d}t/t}=\frac{1}{1+\gamma}\left\|f\right\|_{p,\mathrm{d}t/t}.$$

It actually holds for $\gamma>-1$.

3-4. Take $f\in L^p(\mathbb{R}^n,\frac{\mathrm{d}t}{t})$, with loss of generality, assume $f\geqslant 0$. By **3-3**,

$$\left\|t^{-1/p}T_\gamma(s^{1/p}f)\right\|_p=\left\|T_\gamma(s^{-1/p}f)\right\|_{p,\mathrm{d}t/t}\leqslant\frac{1}{1+\gamma}\left\|t^{1/p}f\right\|_{p,\mathrm{d}t/t}=\frac{1}{1+\gamma}\left\|f\right\|_p.$$

And let $\gamma=-1/p>-1$,

$$t^{-1/p}T_\gamma(s^{1/p}f)(t)=t^{-1/p}\int_0^t\left(\frac{s}{t}\right)^{1+\gamma}s^{1/p}f(s)\frac{\mathrm{d}s}{s}=\frac{1}{t}\int_0^t\left(\frac{s}{t}\right)^{\gamma+1/p}f(s)\,\mathrm{d}s=\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s=Hf(t).$$

Thus

$$\left\|Hf\right\|_p\leqslant\frac{1}{1-1/p}\left\|f\right\|_p=\frac{p}{p-1}\left\|f\right\|_p.$$

This concludes the proof. It cannot hold when $p=1$ because in this case, $\gamma=-1$ and **3-3** cannot apply. □

Exercise 4. Reverse doubling property

Let (E,d,μ) be a space of homogeneous type. Assume that E is connected and unbounded and that $y\mapsto d(y,x)$ is continuous for all $x\in E$.

4-1. Show that any ball $B(x,cr)$ contains a ball of radius comparable to r disjoint from $B(x,r)$, where c is a constant.

4-2. Deduce that E has the *reverse doubling property*: there exists $\varepsilon>0$ such that $\mu(B(x,cr))\geqslant(1+\varepsilon)\mu(B(x,r))$ for all $x\in E$ and $r>0$.

Proof.

4-1. If we want $B(x, cr)$ contains $B(z, c'r)$ which is disjoint from $B(x, r)$, let $y \in B(z, c'r)$, $y \notin B(x, r)$. We assume $d(x, z) = \alpha r$ for some $\alpha > A_0$.

$$d(x,z) = \alpha r \leqslant A_0 d(x,y) + A_0 d(x,z) \leqslant A_0 d(x,y) + A_0 c' r \implies d(x,y) \geqslant A_0^{-1}(\alpha - A_0 c')r,$$
$$d(x,y) \leqslant A_0(d(x,z) + d(z,y)) \leqslant A_0(\alpha + c')r \implies d(x,y) \leqslant A_0(\alpha + c')r.$$

Solve

$$A_0^{-1}(\alpha - A_0 c') \geqslant 1, \qquad A_0(\alpha + c') \leqslant c,$$

we can choose $c' \leqslant \min\left\{\frac{\alpha - A_0}{A_0}, \frac{c}{A_0} - \alpha\right\}$.

4-2. Since E is connected and unbounded, $z \mapsto d(x,z)$ continuous, there exists $z_0 \in E$ such that $d(x,z) = \alpha r$. Let c' be in **4-1**, then

$$B(z, c'r) \cap B(x, r) = \varnothing, \qquad B(z, c'r) \subset B(x, A_0(\alpha + c')r).$$

Thus

$$\mu(B(x, cr)) \geqslant \mu(B(x,r)) + \mu(B(z, c'r)) \geqslant (1+\varepsilon)\mu(B(x,r)),$$

where $\varepsilon = \varepsilon(\alpha, c', c_D) > 0$ due to Property 2 in the lecture notes. □

Exercise 5. Geometric doubling

Consider two different notions of geometric doubling in a quasi-metric space (E, d) with quasi-constant $A_0 \geqslant 1$. Using the following properties for integers $a > 1$ and $N \geqslant 1$:

- $P(a, N)$: any ball of radius R can be covered by at most N balls of radius R/a;
- $Q(a, N)$: any ball of radius R contains at most N centres of disjoint balls of radius R/a.

Let $n = \log_2 N$, show that the following conditions weaken:

(1) $P(2, N)$;

(2) For all $a > 1$, $P(a, Na^n)$;

(3) For all $a > 1$, $Q(a, Na^n)$;

(4) $P(2, N(4A_0)^n)$.

Proof.

(1) $\implies$ (2): By induction we have $P(2^k, N^k)$ for all $k \in \mathbb{N}$. For $a > 1$, let $k = \lceil \log_2 a \rceil$, then

$$N^k = N^{\lceil \log_2 a \rceil} \leqslant N^{\log_2 a + 1} = N \cdot N^{\log_2 a} = N a^{\log_2 N} = N a^n.$$

Since $R/a > R/2^k$, then $P(2^k, N^k) \implies P(a, Na^n)$ (*because we use bigger and more balls to cover*).

(2) $\implies$ (3): Because $P(a, Na^n)$ holds, let $\{B(x_i, R/a)\}_{i=1}^{Na^n}$ be a cover of $B(x, R)$. We prove by contradiction. If there are $m > Na^n$ disjoint balls $\{B(x_j', R/a)\}_{j=1}^m$ such that their centres lie in $B(x, R)$, then by construction

$$d(x_i', x_j') \geqslant \frac{2R}{a}, \qquad \forall i, j \in \{1, \ldots, m\}.$$

Since $\{B(x_i, R/a)\}_{i=1}^{Na^n}$ covers $B(x, R)$, there exists a $\varphi(i) \in \{1, \ldots, Na^n\}$ such that $x_i' \in B(x_{\varphi(i)}, R/a)$. Then the contradiction comes from the pigeonhole principle.

(3) $\implies$ (4): We want a cover of $B(x, R)$ using balls of radius $R/2$. By $Q(a, Na^n)$, one can find Na^n balls of radius R/a, say $\{B(x_i, R/a)\}_{i=1}^{Na^n}$, mutually disjoint, and each $x_i \in B(x, R)$. Then $\{B(x_i, 2R/a)\}_{i=1}^{Na^n}$ is a cover of $B(x, R)$, otherwise $\exists z \in B(x, R)$, $\forall i \in \{1, \ldots, Na^n\}$, $d(z, x_i) > 2R/a$ contradicts the maximality of Na^n.

Denote $2^k < a < 2^{k+1}$, then $2^{-k}R < R/a < 2^{-k+1}R$.

Exercise 6. Existence of a metric equivalent to a power of the quasi-distance

Let (E,d) be a quasi-metric space with quasi-constant $A_0 > 1$. Let $0 < p \leqslant 1$ and define for $x, y \in E$,

$$\rho(x,y) = \inf\left\{\sum_{i=0}^{n-1} d(x_i, x_{i+1})^p : x = x_0, x_1, \ldots, x_n = y,\ n \geqslant 1\right\}.$$

6-1. Show that ρ is symmetric, satisfies the triangle inquality and the inequality $\rho(x,y) \leqslant d(x,y)^p$.

6-2. Take $p \in (0,1]$ defined by $(2A_0)^p = 2$, we want to show by induction on $n \geqslant 2$ that for all chains $x_0, \ldots, x_n$ of points of E, we have

$$d(x_0,x_n)^p \leqslant 2\left(d(x_0,x_1)^p + 2\sum_{i=1}^{n-2} d(x_i,x_{i+1})^p + d(x_{n-1},x_n)^p\right). \tag{B.3}$$

Note that if $n = 2$, the inner sum disappears. We proceed as follows:

(1) Show that $d(x,z)^p \leqslant 2\max\left\{d(x,z)^p, d(z,y)^p\right\}$ for all $x, y, z \in E$.

(2) Check the equality (B.3) for $n = 2$.

(3) Assume the induction holds for all k with $2 \leqslant k \leqslant n$. Let $x = x_0, x_1, \ldots, x_n = y$, and m be the largest number in $\{0, \ldots, n\}$ such that $d(x,y)^p \leqslant 2d(x_m,y)^p$. Show that $d(x,y)^p \leqslant d(x,x_{m+1})^p + d(x_m,y)^p$. Conclude the induction.

6-3. Deduce that $d(x,y)^p \leqslant 4\rho(x,y)$, hence ρ is a distance equivalent to d^p.

6-4. Deduce that on E there exists an equivalent quasi-distance that is Hölder continuous with exponent p.

Proof.

6-1. d is symmetric implies ρ is symmetric. $\rho(x,y) \leqslant d(x,y)^p$ is by definition. The triangle inequality holds because

$$\begin{aligned}
\rho(x,y) &= \inf\left\{\sum d(x_i,x_{i+1})^p : x_0 = x,\ x_n = y\right\} \\
&\leqslant \inf\left\{\sum d(x_i,x_{i+1})^p : x_0 = x,\ x_n = y,\ \exists j\,(x_j = z)\right\} \\
&\leqslant \inf\left\{\sum d(x_i,x_{i+1})^p : x_0 = x,\ x_j = z\right\} + \inf\left\{\sum d(x_i,x_{i+1})^p : x_j = z,\ x_n = y\right\} \\
&= \rho(x,z) + \rho(z,y).
\end{aligned}$$

6-2. We have

$$d(x,y)^p \leqslant (A_0(d(x,z)+d(z,y)))^p \leqslant (A_0 \cdot 2\max\left\{d(x,z), d(z,y)\right\})^p = 2\max\left\{d(x,z), d(z,y)\right\}^p.$$

This proves (1). When $n = 2$,

$$d(x,y)^p \leqslant 2\max\left\{d(x,z)^p, d(z,y)^p\right\} \leqslant 2(d(x,z)^p + d(z,y)^p),$$

hence (2) is true. When $1 \leqslant m \leqslant n-1$,

$$d(x,y)^p \leqslant 2d(x_m,y)^p \implies d(x_{m+1},y) < \frac{1}{2}d(x,y)^p.$$

Thus

$$d(x,y)^p \leqslant \max\left\{2d(x,x_{m+1})^p, 2d(x_{m+1},y)^p\right\} \leqslant 2d(x,x_{m+1})^p.$$

Therefore,

$$\begin{aligned}
&d(x,y)^p \leqslant d(x,x_{m+1})^p + d(x_m,y)^p \\
&\quad\leqslant 2\left(d(x_0,x_1)^p + 2\sum_{i=1}^{m-1} d(x_i,x_{i+1})^p + d(x_m,x_{m+1})^p\right) + 2\left(d(x_m,x_{m+1})^p + 2\sum_{i=m+1}^{n-1} d(x_i,x_{i+1})^p + d(x_n,x_{n+1})^p\right) \\
&\quad= 2\left(d(x_0,x_1)^p + 2\sum_{i=1}^{n-1} d(x_i,x_{i+1})^p + d(x_n,x_{n+1})^p\right).
\end{aligned}$$

Exercise 7. A non-continuous quasi-metric

Let $E = \mathbb{N}$, $\varepsilon > 0$ and $d : E \times E \to [0, \infty)$ with

$$d(n,n) = 0, \quad d(n,m) = d(m,n), \quad d(0,1) = 1,$$

and for $m, n \geqslant 2$,

$$d(0,m) = 1 + \varepsilon, \qquad d(1,m) = \frac{1}{m}, \qquad d(n,m) = \frac{1}{n} + \frac{1}{m}.$$

7-1. Check that d is a quasi-distance on $E \times E$ with quasi-constant $A_0 = 1 + \varepsilon$.

7-2. Show that $B(0, 1 + \varepsilon/2) = \{0, 1\}$ and that for all $\eta > 0$, $B(1, \eta)$ contains infinitely many points. Deduce that $B(0, 1 + \varepsilon/2)$ is not an open set.

7-3. Show that $m \mapsto d(0, m)$ is not continuous at $m = 1$.

Proof.

7-1. By symmetry we only need to consider the case $m \geqslant 2$. We have the following chart:

	(m,n)	$d(m,n)$	$d(m,\ell) + d(\ell,n)$
$m = n$		0	0
$m > n$	(1,0)	1	1+0, $\ell = 0$
			0+1, $\ell = 1$
			$\frac{1}{\ell} + 1 + \varepsilon$, $\ell \geqslant 2$
	$(\geqslant 2, 0)$	$1 + \varepsilon$	$1 + \varepsilon + 0$, $\ell = 0$
			$\frac{1}{m} + 1$, $\ell = 1$
			$\frac{1}{m} + \frac{1}{\ell} + 1 + \varepsilon$, $\ell \geqslant 2$
	$(\geqslant 2, 1)$	$\frac{1}{m}$	$\frac{1}{m} + 1$, $\ell = 0$
			$\frac{1}{m} + 0$, $\ell = 1$
			$\frac{1}{m} + \frac{1}{\ell} + 1 + \varepsilon$, $\ell \geqslant 2$
	$(\geqslant 2, \geqslant 2)$	$\frac{1}{m} + \frac{1}{n}$	$1 + \varepsilon + 1 + \varepsilon$, $\ell = 0$
			$\frac{1}{m} + \frac{1}{n}$, $\ell = 1$
			$\frac{1}{m} + \frac{1}{\ell} + \frac{1}{n} + \frac{1}{\ell}$, $\ell \geqslant 2$

So one has $d(m,n) \leqslant d(m,\ell) + d(\ell,n)$ except for $(m,\ell,n) = (\geqslant 2, 1, 0)$. In this case, $d(m,n) \leqslant (1 + \varepsilon)(d(m,\ell) + d(\ell,n))$ is the best estimation possible.

7-2. When $m \geqslant 2$, $d(0,m) = 1+\varepsilon > 1+\frac{\varepsilon}{2}$, hence $B\big(0, 1 + \frac{\varepsilon}{2}\big) = \{0,1\}$. For $\eta > 0$, $B(1,\eta) = \big\{m \in \mathbb{N} : m > 1/\eta\big\}$ has infinitely many elements. For the last claim, note that $1 \in B(0, 1+\varepsilon/2)$, but there is no non-empty open ball B centred at 1 such that $B \subset B(0, 1 + \varepsilon/2)$.

7-3. Denote $f(m) = d(0,m)$. For $m \geqslant 2$, $d(1,m) = \frac{1}{m}$, so $d(1,m) \to 0$ as $m \to +\infty$. But $\big|f(1) - f(m)\big| \geqslant \varepsilon$, hence f is not continuous at $m = 1$. $\square$

3 TD III – Interpolation and Calderón–Zygmund operators

Exercise 1. Paley's Inequality

Let $1 < p < 2$, use Marcinkiewicz's interpolation, the Fourier transform defined on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ has an extension on $u \in L^p(\mathbb{R}^n)$. Prove

$$\int_{\mathbb{R}^n} \left|\mathscr{F}_p u(\xi)\right|^p |\xi|^{n(p-2)} \,\mathrm{d}\xi \leqslant c_p \int_{\mathbb{R}^n} |u(x)|^p \,\mathrm{d}x.$$

We consider an operator $T : u \to Tu$, $Tu(\xi) := |\xi|^n \,\hat{u}(\xi)$ defined on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and introduce a new measure $\mathrm{d}\nu(\xi) = |\xi|^{-2n} \,\mathrm{d}\xi$ to study the type of T.

Proof.

Denote $X = (\mathbb{R}^n, \mathrm{d}x)$, $Y = (\mathbb{R}^n, |x|^{-2n} \,\mathrm{d}x)$. For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, by Planchernel's theorem,

$$\int_Y \left|Tf(\xi)\right|^2 \mathrm{d}\nu(\xi) = \int_{\mathbb{R}^n} |\xi|^{2n} \left|\hat{f}(\xi)\right|^2 \frac{\mathrm{d}\xi}{|\xi|^{2n}} = \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right| \mathrm{d}\xi = \int_X \left|f(x)\right|^2 \mathrm{d}x.$$

Hence T is of strong type $(2,2)$.

For each $\xi \in \mathbb{R}^n$, $\left|\hat{f}(\xi)\right| \leqslant \|f\|_1$, $\forall \lambda > 0$,

$$\left\{\left|Tf(\xi)\right| > \lambda\right\} = \left\{\left|\hat{f}(\xi)\right| |\xi|^n > \lambda\right\} \subset \left\{|\xi| > \left(\frac{\lambda}{\|f\|_1}\right)^{1/n}\right\} =: A_\lambda.$$

Denote $a = (\lambda / \|f\|_1)^{1/n}$. Then

$$\begin{aligned} \nu(\{\left|Tf(\xi)\right| > \lambda\}) &= \int_{A_\lambda} \mathrm{d}\nu(\xi) = \int_{A_\lambda} \frac{\mathrm{d}\xi}{|\xi|^{2n}} \\ &= \int_{\{|\xi|>a\}} \frac{\mathrm{d}\xi}{|\xi|^{2n}} = c \cdot \int_a^\infty r^{n-1} \frac{\mathrm{d}r}{r^{2n}} = c \cdot \int_a^\infty r^{-n-1} \,\mathrm{d}r = \frac{c}{n} a^{-n} = \frac{c}{n\lambda} \|f\|_1. \end{aligned}$$

Here we use $\xi = |\xi| \cdot n$, $n \in \mathbb{S}^{n-1}$ to calculate the integral. Hence T is of weak type $(1,1)$.

Note that T is subadditive, by Marcinkiewicz interpolation, T is of strong type (p,p) for $1 < p \leqslant 2$, hence there exists $c_p > 0$ such that

$$\|Tf\|_{L^p(Y)} \leqslant c_p^{1/p} \|f\|_{L^p(X)} \iff \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^p |\xi|^{n(p-2)} \,\mathrm{d}\xi \leqslant c_p \int_{\mathbb{R}^n} \left|f(x)\right|^p \mathrm{d}x,$$

which concludes the proof. □

Exercise 2. Extension of Calderón–Zygmund operators

Let $T \in \mathrm{CZO}_\alpha$ and $K \in \mathrm{CZK}_\alpha$ the associated kernel. $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is of weak type (1,1) and strong type (p,p) for $1 < p < \infty$. Recall that $L^{1,\infty}$ is complete.

2-1. Weak type has been shown in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Show that there exists a linear extension $T_1 : L^1(\mathbb{R}^n) \to L^{1,\infty}$ that satisfies

$$T_1 f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \,\mathrm{d}y, \qquad \text{a.e. } x \in \operatorname{supp} f \tag{C.1}$$

for $f \in L^1$ with compact support.

2-2. Show that there exist linear extensions $T_p : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ that satisfy

$$T_p f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) \,\mathrm{d}y, \qquad \text{a.e. } x \in \operatorname{supp} f \tag{C.2}$$

for $f \in L^p$ with compact support.

Proof.

2-1. Since the subspace of compactly supported functions in $L^1 \cap L^2$ is dense in L^1, and $L^{1,\infty}$ is complete, one can take any $f \in L^1$ and approximate it by a sequence $(f_n)_{n\geqslant 1} \subset L^1 \cap L^2$. By the weak-type estimate, $(Tf_n)_{n\geqslant 1}$ is a Cauchy sequence in $L^{1,\infty}$. Hence it converges to some $g \in L^{1,\infty}$. Define $T_1 f := g$, then $T_1 f = \lim_{n\to\infty} Tf_n$ in the $L^{1,\infty}$ sense. The boundedness gives that the definition does not depend on the particular approximating sequence, and the map $f \mapsto T_1 f$ is linear. By construction, for $h \in L^\infty(\mathbb{R}^n)$ with compact support, $\operatorname{supp} f \cap \operatorname{supp} h = \varnothing$,

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)||f(y)||h(x)|\,\mathrm{d}y\,\mathrm{d}x &\leqslant \left(\sup|K(x,y)|\right)\int_{\operatorname{supp} f}|f|\,\mathrm{d}x\int_{\operatorname{supp} h}|h|\,\mathrm{d}x \\ &\leqslant \frac{c}{d(\operatorname{supp} f,\operatorname{supp} h)^n}\|f\|_1 |\operatorname{supp} g| \|g\|_\infty < +\infty.\end{aligned}$$

Hence $T_1 f(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y$ makes sense for a.e. $x \notin \operatorname{supp} f$. So the extension is exactly what was required.

2-2. For compactly supported $f \in L^p$, Tf is given by the integral $Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y$ a.e. $x \notin \operatorname{supp} f$. The strong-type (p,p) gives $\|Tf\|_p \leqslant c_p\|f\|_p$. Approximate a general $f \in L^p$ by compactly supported $f_n \in L^p$, the uniform bound shows that $(Tf_n)_{n\geqslant 1}$ is Cauchy in L^p. Hence it converges to some g in L^p. Define $T_p f := g$, then $T_p f = \lim_{n\to\infty} Tf_n$. Similarly, the boundedness gives that the definition does not depend on the particular approximating sequence, and the map $f \mapsto T_p f$ is linear. □

Exercise 3. Cotlar inequality

Part A. A result of Kolmogorov

Let S be an operator of weak-type $(1,1)$, $0 < \nu < 1$ and $A \subset \mathbb{R}^n$ Lebesuge measurable set with finite measure. Use Cavalieri's principle to show

$$\int_A |Sf(x)|^\nu\,\mathrm{d}x \leqslant c|A|^{1-\nu}\|f\|_1^\nu.$$

Proof.

Take $g(x) = |Sf(x)|$, then Cavalieri's principle gives

$$\int_A |Sf(x)|^\nu\,\mathrm{d}\mu = \nu\int_0^\infty |\{x\in A\}:|Sf(x)|>\lambda|\lambda^{\nu-1}\,\mathrm{d}\lambda.$$

Then $Sf \in L^{1,\infty}$, so $\sup_{\lambda>0}\lambda|\{|Sf(x)|>\lambda\}| < +\infty$, the integral is well-defined.

$$\|Sf\|_{1,\infty} = \sup_{\lambda>0}\lambda|\{|Sf(x)|>\lambda\}| \leqslant c\|f\|_1 \implies |\{|Sf(x)|>\lambda\}| \leqslant \frac{c}{\lambda}\|f\|_1.$$

Therefore

$$\int_0^\infty |\{x\in A:|Sf(x)|>\lambda\}|\lambda^{\nu-1}\,\mathrm{d}\lambda \begin{cases} \leqslant |A|\lambda^{\nu-1}\,\mathrm{d}\lambda, & \text{for small } \lambda, \nu-1>-1, \\ \leqslant c\lambda^{\nu-2}\|f\|_1\,\mathrm{d}\lambda, & \text{for large } \lambda, \nu-1<0.\end{cases}$$

So for a nice $\lambda_0 > 0$,

$$\begin{aligned}\int_0^\infty |\{x\in A:|Sf(x)|>\lambda\}|\lambda^{\nu-1}\,\mathrm{d}\lambda &= \left(\int_0^{\lambda_0}+\int_{\lambda_0}^\infty\right)|\{x\in A:|Sf(x)|>\lambda\}|\lambda^{\nu-1}\,\mathrm{d}\lambda \\ &\leqslant \int_0^{\lambda_0}|A|\lambda^{\nu-1}\,\mathrm{d}\lambda + \int_{\lambda_0}^\infty c\lambda^{\nu-2}\|f\|_1\,\mathrm{d}\lambda \\ &= \frac{\lambda_0^\nu}{\nu}|A| + \frac{1}{\nu-1}c\|f\|_1\lambda_0^{\nu-1}.\end{aligned}$$

So

$$\int_A |Sf(x)|^\nu\,\mathrm{d}x = \lambda_0^\nu|A| + \frac{\nu}{\nu-1}c\|f\|_1\lambda_0^{\nu-1}.$$

Now $\lambda_0 = c\|f\|_1/|A|$ is a wise choice, then

$$\int_A |Sf(x)|^\nu\,\mathrm{d}x \leqslant \frac{c\|f\|_1^\nu}{|A|^\nu}|A| + \frac{\nu}{\nu-1}c\|f\|_1\frac{\|f\|_1^{\nu-1}}{|A|^{\nu-1}} = c\|f\|_1^\nu|A|^{\nu-1}.$$

Then we are done. $\square$

Part B. An estimation on CZO_α

Let $T \in \mathrm{CZO}_\alpha$, the associated kernel $K \in \mathrm{CZK}_\alpha$. Set for $\varepsilon > 0$, $f \in L^1(\mathbb{R}^n)$ with compact support, and $x \in \mathbb{R}^n$,

$$(T_\varepsilon f)(x) = \int_{\{y:|x-y|\geqslant\varepsilon\}} K(x,y)f(y)\,\mathrm{d}y, \qquad (T^* f)(x) = \sup_{\varepsilon>0} \left|(T_\varepsilon f)(x)\right|.$$

Let $f \in L^1(\mathbb{R}^n)$ with compact support, we want to show for $0 < \nu \leqslant 1$, one has

$$\left|(T_\varepsilon f)(0)\right| \leqslant c(M(\left|Tf\right|^\nu(0))^{1/\nu} + Mf(0)),$$

where M is the maximal operator. $c = c(\varepsilon) > 0$ is a constant. Fix $\varepsilon > 0$, denote $B = B(0,\varepsilon/2)$, $f_1 = f \cdot \mathbb{1}_{2B}$, $f_2 = f - f_1$.

3-B1. Show that $Tf_2(0) = T_\varepsilon f(0)$.

3-B2. Fix $z \in B$, and show

$$\left|Tf_2(z) - Tf_2(0)\right| \leqslant cMf(0),$$

where $c > 0$ is a constant depending on n, α and $\|K\|_\alpha$.

(*Hint: Decompose* $\{y \in \mathbb{R}^n : |y| > \varepsilon\} = \bigcup_{k\in\mathbb{N}} \{y \in \mathbb{R}^n : 2^k\varepsilon < |y| \leqslant 2^{k+1}\varepsilon\}$.)

3-B3. To conclude that we have $\left|T_\varepsilon f(0)\right| \leqslant c' Mf(0) + \left|Tf(z)\right| + \left|Tf_1(z)\right|$.

Proof.

3-B1. This is because

$$Tf_2(0) = Tf(0) - Tf_1(0) = \int_{\mathbb{R}^n} K(0,y)f(u)\,\mathrm{d}y - \int_{B(0,\varepsilon)} K(0,y)f(y)\,\mathrm{d}y = \int_{\{y:|y|>\varepsilon\}} K(0,y)f(y)\,\mathrm{d}y = T_\varepsilon f(0).$$

3-B2. Note that $\operatorname{supp} f_2 \subset B(0,\varepsilon)^c$, $z \in B(0,\varepsilon/2)$, hence $z \notin \operatorname{supp} f_2$. Therefore

$$Tf_2(z) = \int_{\mathbb{R}^n} K(z,y)f_2(y)\,\mathrm{d}y \qquad \text{a.e.},$$

and $z \mapsto Tf_2(z)$ is continuous on B. Thus Tf_2 agrees with a continuous f a.e. on $B(0,\varepsilon/2)$. We identify

$$\begin{aligned}
\left|Tf_2(z) - Tf_2(0)\right| &\left|\int_{\mathbb{R}^n} K(z,y)f_2(y)\,\mathrm{d}y - \int_{\mathbb{R}^n} K(0,y)f_2(y)\,\mathrm{d}y\right| \leqslant \int_{\{y:|y|>\varepsilon\}} \left|K(z,y) - K_0(y)\right|\left|f_2(y)\right|\mathrm{d}y \\
&= \sum_{k\geqslant0} \int_{\{2^k\varepsilon<|y|\leqslant2^{k+1}\varepsilon\}} \left|K(z,y) - K_0(y)\right|\left|f_2(y)\right|\mathrm{d}y \leqslant \sum_{k\geqslant0} \|K\|_\alpha \left(\frac{|z|}{|y|}\right)^\alpha \frac{1}{|y|^n} \int_{\{2^k\varepsilon<|y|\leqslant2^{k+1}\varepsilon\}} \left|f_2(y)\right|\mathrm{d}y \\
&\leqslant \|K\|_\alpha \left(\frac{1}{2}\right)^\alpha \sum_{k\geqslant0} \frac{1}{(2^k\varepsilon)^n} \int_{\{2^k\varepsilon<|y|\leqslant2^{k+1}\varepsilon\}} \left|f_2(y)\right|\mathrm{d}y
\end{aligned}$$

And note that

$$Mf(0) \geqslant \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \left|f_2(y)\right|\mathrm{d}y, \qquad x = 0,\ 2^k\varepsilon < r \leqslant 2^{k+1}\varepsilon.$$

Done.

3-B3. Thus,

$$\left|Tf_\varepsilon(0)\right| = \left|Tf_2(0)\right| \leqslant \left|Tf_2(z)\right| + \left|Tf_2(z) - Tf_2(0)\right| \leqslant cMf(0) + \left|Tf(z)\right| + \left|Tf_1(z)\right|,$$

which gives what we want. $\square$

4 TD IV – Calderón–Zygmund operators, continued

We work in $\mathbb{R}^n$ with Lebesgue measure. Let $T \in \mathrm{CZO}_\alpha$ and K be the associated kernel. We use the Euclidean norm to define balls and the estimates on the kernel. Denote H^1 the Hardy space and BMO the space of bounded mean oscillation functions.

Exercise 1. Action on H^1

Part I. Boundedness for H^1 to L^1.

1-1. Let ψ be a bounded measurable function and a ball B with $\psi = 0$ on B^c and $\int_B \psi(x)\,\mathrm{d}x = 0$. Show that $T\psi \in L^1((2B)^c)$ with $\int_{(2B)^c} |T\psi|\,\mathrm{d}x \leqslant c(n,\alpha,\|K\|_\alpha)$.

1-2. Let $a \in \mathscr{A}^\infty$ be an ∞-atom. Show that $Ta \in L^1$, with $\|Ta\|_1 \leqslant c(n,\alpha,\|T\|_\alpha) < +\infty$.

1-3. Let $f \in H^1 \cap L^2$ and let $\sum_{i\in\mathbb{N}} \lambda_i a_i$ be an ∞-atomic decomposition of f. Show that $\sum_{i\geqslant 0} \lambda_i Ta_i$ converges in L^1 and its sum equals Tf a.e.. (*Hint: Use weak-type* $(1,1)$ *to prove the equality.*)

1-4. Deduce that Tf is in L^1 with $\|Tf\|_1 \leqslant c(n,\alpha,\|T\|_\alpha)\,\|f\|_{H^1}$.

1-5. Consider $\tilde{T}$, the extension to L^1 of $T : L^1 \cap L^2 \to L^{1,\infty}$ defined in class. Show that $\tilde{T}$ is bounded from H^1 to L^1 and that it is the unique linear extension to H^1 of $T : H^1 \cap L^2 \to L^1$. (*Hint: Explain using the lectures why $H^1 \cap L^2$ is dense in H^1.*)

Proof.

1-1. Since $\psi = 0$ on B^c, $\operatorname{supp}\psi \subset B$ with $\int_B \psi = 0$, we have $\psi \in L^2(\mathbb{R}^n, \mathrm{d}x)$ with compact support. Denote y_B the centre of B, then

$$T\psi(x) = \int_{\mathbb{R}^n} K(x,y)\psi(y)\,\mathrm{d}y = \int_B (K(x,y) - K(x,y_B))\psi(y)\,\mathrm{d}y, \qquad \text{a.e. } x \in (2B)^c.$$

And

$$\begin{aligned}\int_{(2B)^c} |T\psi(x)|\,\mathrm{d}x &\leqslant \int_{(2B)^c}\int_B |K(x,y) - K(x,y_B)|\,|\psi(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\int_{(2B)^c}\int_B \|K\|_\alpha \left(\frac{|y-y_B|}{x-y}\right)^\alpha \frac{1}{|x-y|^n}\,|\psi(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \|K\|_\alpha \frac{cr^\alpha}{(2r)^{n+\alpha}} \int_B |\psi(y)|\,\mathrm{d}y = c(n,\alpha,\|K\|_\alpha)\,\|\psi\|_1.\end{aligned}$$

1-2. Now $a \in \mathscr{A}^\infty \subset L^2(\mathbb{R}^n,\mathrm{d}x)$, there exists a cube $Q \subset \mathbb{R}^n$ such that a is an ∞-atom on Q. Choose a ball $B \supset Q$, and $\int_B a\,\mathrm{d}x = \int_Q a\,\mathrm{d}x = 0$. Take $\psi = a$ in **1-1**, we have $Ta \in L^1((2B)^c)$ with

$$\int_{(2B)^c} |Ta|\,\mathrm{d}x \leqslant c(n,\alpha,\|K\|_\alpha)\,\|a\|_1 \leqslant c(n,\alpha,\|K\|_\alpha)$$

and by Hölder inequality,

$$\int_{2B} |Ta|\,\mathrm{d}x = \int_{2B} |Ta|\cdot 1\,\mathrm{d}x \leqslant \|Ta\|_2\,\|1\|_2 \leqslant \|T\|\,\|a\|_2\,|2B|^{1/2}.$$

Combine the estimates above, then we obtain what we want.

1-3. By **1-2**, one has $\|Ta_i\|_1 \leqslant c(n,\alpha,\|T\|)$ for all i. Since $T \in \mathrm{CZO}_\alpha$ is of weak-type (1,1), $\{|Tf| > \lambda\} \leqslant \frac{c}{\lambda}\|f\|_1$. So

$$\Big\|\sum_{i\in\mathbb{N}} T(\lambda_i a_i)\Big\|_1 \leqslant \sum_{i\in\mathbb{N}} |\lambda_i|\,\|Ta_i\|_1 \leqslant c(n,\alpha,\|T\|)\sum_{i\in\mathbb{N}} |\lambda_i| < +\infty.$$

Hence $\sum_{i\in\mathbb{N}} \lambda_i Ta_i$ converges in L^1 to some $g \in L^1$. And $g_N = g - \sum_{i=0}^N \lambda_i Ta_i \to 0$ in L^1. Similarly, denote $f_N = f - \sum_{i=0}^N \lambda_i Ta_i$, then $f_N \to 0$ in L^1 and $g_N - Tf_N = g - Tf$. So using Markov inequality and weak-type

(1,1),
$$\begin{aligned}\left|\{|g-Tf|>\lambda\}\right| &= \left|\{|g_N-Tf_N|>\lambda\}\right|\\ &\leqslant \left|\left\{|g_N|>\frac{\lambda}{2}\right\}\right|+\left|\left\{|Tf_N|>\frac{\lambda}{2}\right\}\right| \leqslant \frac{2}{\lambda}\|g_N\|_1+\frac{2c}{\lambda}\|f_N\|_1\to 0\end{aligned}$$
as $N\to+\infty$, and holds for all $\lambda>0$. So $g=Tf$ a.e..

1-4. $Tf\in L^1$ is obtained from **1-3** because $g\in L^1$. And for atomic decomposition $f=\sum_{i\in\mathbb{N}}\lambda_i a_i$,
$$\|Tf\|_1=\left\|\sum_{i\in\mathbb{N}}T(\lambda_i a_i)\right\|_1\leqslant c(n,\alpha,\|T\|)\sum_{i\in\mathbb{N}}|\lambda_i|<+\infty,$$
take infimum over all atomic decompositions, we have $\|Tf\|_1\leqslant c(n,\alpha,\|T\|)\|f\|_{H^1}$.

1-5. We know that each atom $a_i\in L^2$, so for $f=\sum_{i\in\mathbb{N}}\lambda_i a_i\in H^1$, one can use the partial sum $(f_n)_{n\geqslant 1}\subset H^1\cap L^2$ to approximate $f\in H^1$ in L^2-norm. But by the definition of H^1-norm,
$$\|f-f_n\|_{H^1}\leqslant\sum_{i>n}|\lambda_i|\to 0,$$
hence $f\in\overline{H^1\cap L^2}^{\|\cdot\|_{H^1}}$. We have proved in **1-4** that $T:H^1\cap L^2\to L^1$ is bounded, so we can extend by density that $\tilde{T}:H^1\to L^1$. $\square$

Part II. H^1 **molecules.**

Let $\varepsilon>0$, we say that $m:\mathbb{R}^n\to\mathbb{C}$ is an ε*-molecule* associated to a ball $B=B(x,r)$ if

(a) $\int_{2B}|m|^2\,\mathrm{d}y\leqslant|2B|^{-1}$;

(b) For every $j\in\mathbb{N}_{\geqslant 1}$, we have $\sup_{C_j}|m|\leqslant 2^{-j\varepsilon}\left|2^jB\right|^{-1}$, where $C_j=2^{j+1}B\setminus 2^jB$.

(c) $\int_{\mathbb{R}^n}m(y)\,\mathrm{d}y=0$.

1-6. Verify that condition (c) makes sense given conditions (a) and (b). And show that $\left|\int_{2^jB}m\,\mathrm{d}y\right|\leqslant c(n,\varepsilon)2^{-j\varepsilon}$ for all $j\geqslant 1$.

1-7. Set $C_0=2B$. Let m_j be the mean value of m on C_j for $j\geqslant 0$. Show that the series $g=\sum_{j\geqslant 0}(m-m_j)\mathbb{1}_{C_j}$ converges in H^1 and that its norm is bounded by a constant $c'(n,\varepsilon)$. (*Hint: Establish that each term is proportional to a* 2 *or* ∞*-atom.*)

1-8. Show that $h=\sum_{j\geqslant 0}m_j\mathbb{1}_{C_j}$ can be written as $h=\sum_{j\geqslant 1}\left(\int_{2^jB}m\,\mathrm{d}y\right)f_j$ with $f_j=\frac{\mathbb{1}_{C_{j-1}}}{|C_{j-1}|}-\frac{\mathbb{1}_{C_j}}{|C_j|}$.

1-9. Show that there is a constant $\lambda=\lambda(n)$ such that f_j/λ is an ∞-atom. Deduce that $h\in H^1$ with a bound $c(n,\varepsilon)\lambda$.

1-10. Conclude that $m\in H^1$, with a bound controlled by a constant depending only on n and ε.

Proof.

1-6. Note that $|C_j|=\left|2^{j+1}B\right|-\left|2^jB\right|=(2^n-1)\left|2^jB\right|$. So
$$\begin{aligned}\left|\int_{\mathbb{R}^n}m\right| &\leqslant\int_{2B}|m|+\sum_{j\geqslant 1}\int_{C_j}|m|\leqslant\int_{2B}|m|\cdot 1+\sum_{j\geqslant 1}\frac{|C_j|(2^{-j\varepsilon})}{|2^jB|}\\ &\leqslant\left(\int_{2B}|m|^2\right)^{1/2}|2B|^{1/2}+\sum_{j\geqslant}\frac{2^n-1}{2^{j\varepsilon}}=1+(2^n-1)\sum_{j\geqslant 1}\frac{1}{2^{j\varepsilon}}<+\infty.\end{aligned}$$
This implies $m\in L^1$, so (c) makes sense. Then since $2^jB=2B\sqcup C_1\sqcup\cdots\sqcup C_{j-1}$,
$$\left|\int_{2^jB}m\right|=\left|\int_{(2^jB)^c}m\right|=\left|\sum_{k\geqslant j}\int_{C_k}m\right|\leqslant\sum_{k\geqslant j}\int_{C_k}|m|\leqslant\sum_{k\geqslant j}\frac{2^{-k\varepsilon}|C_k|}{|2^kB|}\leqslant(2^n-1)\sum_{k\geqslant j}2^{-k\varepsilon}=c(n,\varepsilon)2^{-j\varepsilon}.$$
1-7. When $j=0$,
$$\int_{2B}|m-m_0|^2\leqslant 2\left(\int_{2B}|m|^2+m_0^2|2B|\right)\leqslant\frac{4}{|2B|},$$

which is, $\left\|(m-m_0)\mathbb{1}_{C_0}\right\|_2 \leqslant 2/|C_0|^{1/2}$. Hence $(m-m_0)\mathbb{1}_{C_0}$ is a 2-atom.

When $j \geqslant 1$,

$$\left\|(m-m_j)\mathbb{1}_{C_j}\right\|_\infty \leqslant \frac{2^{-j\varepsilon}}{|2^j B|} + |m_j| + \left|\fint_{C_j} m\right| \leqslant 2\cdot\frac{2^{-j\varepsilon}}{|2^j B|} \leqslant \frac{2^{-j\varepsilon}c(n)}{|C_j|}.$$

So $(m-m_j)\mathbb{1}_{C_j}$ is an ∞-atom. While the norms $\|\cdot\|_{H^{1,2}}$ and $\|\cdot\|_{H^{1,\infty}}$ are equivalent,

$$\sum_{j\geqslant 0}\left\|(m-m_j)\mathbb{1}_{C_j}\right\|_{H^{1,\infty}} \leqslant c'(n,\varepsilon).$$

The Hardy space H^1 is a Banach space, so absolute convergent implies convergent.

1-8. Just calculate

$$h = \sum_{j\geqslant 0} m_j \mathbb{1}_{C_j} = \sum_{j\geqslant 0}\frac{\mathbb{1}_{C_j}}{|C_j|}\int_{C_j} m = \sum_{j\geqslant 0}\left(\frac{\mathbb{1}_{C_j}}{|C_j|} - \frac{\mathbb{1}_{C_{j-1}}}{|C_{j-1}|}\right)\int_{2^{j+1}B} m = \sum_{j\geqslant 1} f_j \int_{2^j B} m.$$

1-9. For $j \geqslant 1$,

$$\|f_j\|_\infty = \left\|\frac{\mathbb{1}_{C_j}}{|C_j|} - \frac{\mathbb{1}_{C_{j-1}}}{|C_{j-1}|}\right\|_\infty = \frac{1}{|C_j|}$$

because $C_{j-1}\cap C_j = \varnothing$. And for $Q \supset C_j$, $\int_Q f_j = 1-1 = 0$. Therefore f_j is an ∞-atom. Rescaling f_j, then f_j/λ satisfies the definition of an ∞-atom. Then

$$h = \sum_{j\geqslant 1}\left(\int_{2^j B} m\right) f_j \implies \|h\|_{H^{1,\infty}} \leqslant \sum_{j\geqslant 1}\left(\int_{2^j B} m\right)\lambda \leqslant \lambda c(n,\varepsilon)\sum_{j\geqslant 1} 2^{-j\varepsilon} = c(n,\varepsilon)\lambda.$$

This gives $h \in H^1$ and the norm estimate.

1-10. We have $m = g+h$. By **1-7**, $\|g\|_{H^{1,\infty}} \leqslant c'(n,\varepsilon)$. By **1-9**, $\|h\|_{H^{1,\infty}} \leqslant c(n,\varepsilon)\lambda(n)$. Hence

$$\|m\|_{H^{1,\infty}} \leqslant \|g\|_{H^{1,\infty}} + \|h\|_{H^{1,\infty}} \leqslant c'(n,\varepsilon) + c(n,\varepsilon)\lambda(n),$$

where right hand side is a constant depending on n and ε. □

Part III. $H^1 \to H^1$ **boundedness.**

1-11. Show that if a is an ∞-atom, then there is a positive constant $c = c(n,\alpha,\|T\|_\alpha)$ such that $\frac{Ta}{c}$ verifies the conditions (a) and (b) of an α-molecule associated to some ball.

1-12. Show that if $\int_{\mathbb{R}^n} Ta\,\mathrm{d}y = 0$ for all ∞-atoms a, then $\tilde{T}$, defined in Part I, is bounded from H^1 to H^1.

1-13. Show that the converse holds.

Proof.