# Performance Evaluation and Applications

POLITECNICO DI MILANO

## Advanced Distributions

# Gamma

The *Gamma* distribution is an extension of the *Erlang distribution* where the number of stages can be any positive real number.

It is called in this way because it uses the *Gamma function* in its PDF.

$$f_{Gamma<\theta,k>}(t) = \frac{t^{k-1}e^{-t/\theta}}{\theta^k \Gamma(k)}$$

The Gamma distribution has the following mean and variance.

$$\mu_{Gamma<\theta,k>} = k\theta$$

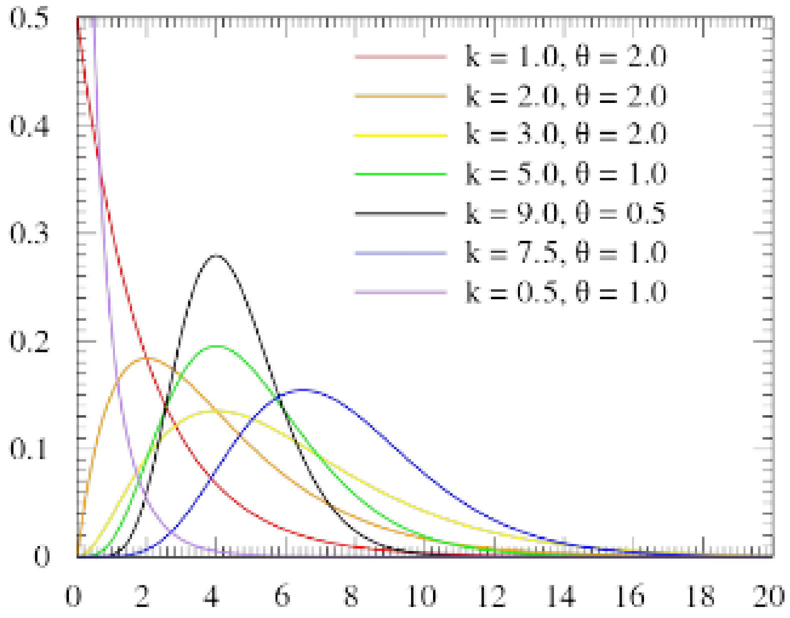
$$\sigma^2{}_{Gamma<\theta,k>} = k\theta^2$$

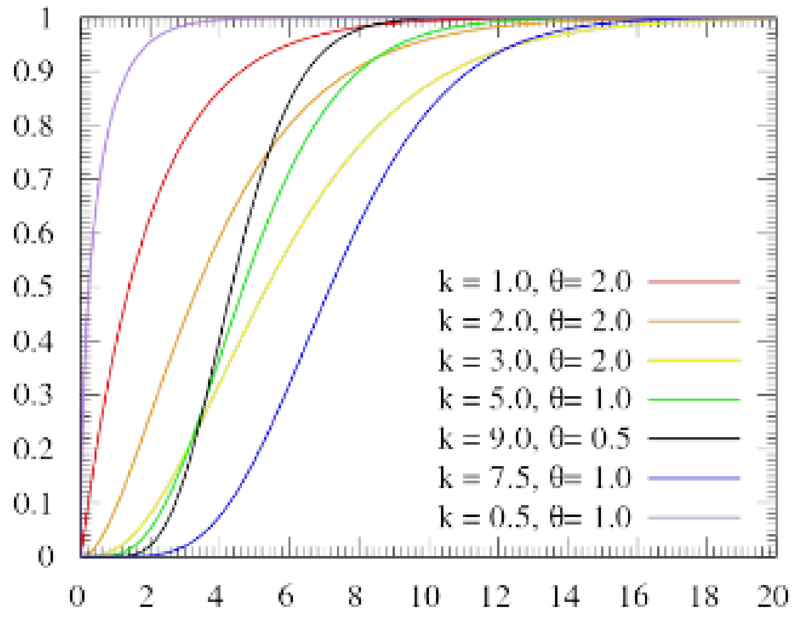
$$f_{Erlang<\lambda,k>}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}$$

# Gamma

Although very complex, it still has analytical results available: for this reason it is sometimes used to obtain random variables with a given mean and c.v.

$$k = \frac{1}{\sqrt{c_v}} \qquad \theta = \frac{E[X]}{k}$$



# Weibull

The *Weibull distribution* is characterized by two parameters: the *shape* $k$ and the *scale* $\lambda$. It similar to an exponential distribution, but its exponent is raised to the power of $k$.

$$f_{Weibull<\lambda,k>}(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^{k}}$$

$$F_{Weibull<\lambda,k>}(t) = 1 - e^{-\left(\frac{t}{\lambda}\right)^{k}}$$

It is has the following mean, variance and moments:

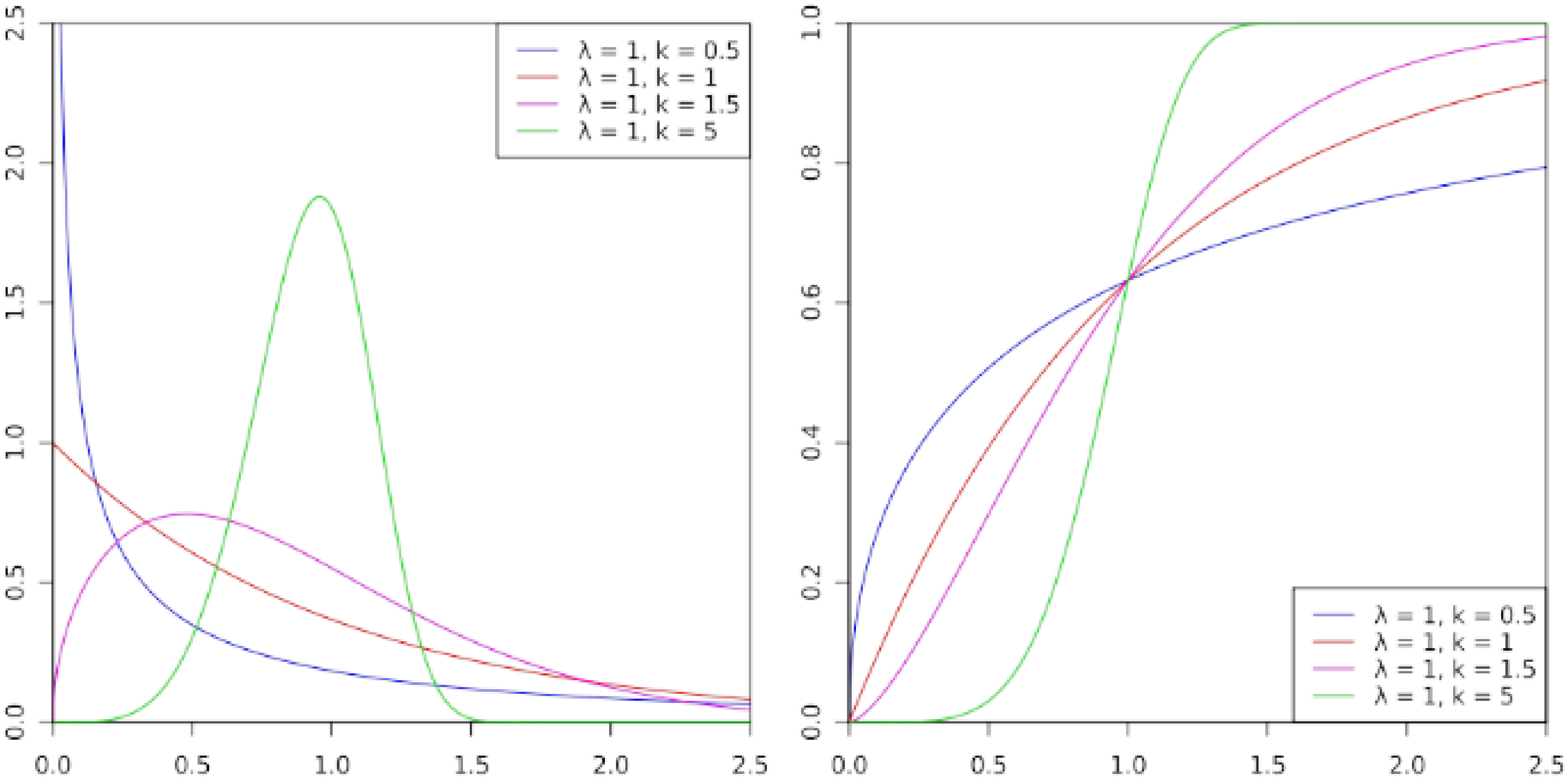
$$\mu_{Weibull<\lambda,k>} = \lambda\Gamma\left(1+\frac{1}{k}\right)$$

$$\sigma^2{}_{Weibull<\lambda,k>} = \lambda^2\left[\Gamma\left(1+\frac{2}{k}\right) - \left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]$$

$$E[X^n] = \lambda^n\Gamma\left(1+\frac{n}{k}\right)$$

# Weibull

The PDF and the CDF have the following shape:



# Weibull

It is used for modeling the lifetime of components subject to aging.

# Pareto

*The Pareto distribution* has a simple monomial expression that is characterized by a shape parameter *a* and a scale parameter *m*:

$$f_{Pareto<\alpha,m>}(t) = \begin{cases} \dfrac{\alpha m^{\alpha}}{t^{\alpha+1}} & t \geq m \\ 0 & t < m \end{cases}$$

$$F_{Pareto<\alpha,m>}(t) = \begin{cases} 1 - \left(\dfrac{m}{t}\right)^{\alpha} & t \geq m \\ 0 & t < m \end{cases}$$

Since the integral of a PDF must equal to 1, the Pareto is a proper distribution only for *a > 0*, and it generates values in the range *[m,∞]*.

$$\int_{m}^{\infty} \frac{\alpha m^{\alpha}}{x^{\alpha+1}} dx = \alpha m^{\alpha} \int_{m}^{\infty} x^{-\alpha-1} dx = -\frac{\alpha m^{\alpha}}{\alpha} \left(x^{-\alpha}\right|_{m}^{\infty}$$

$$= \begin{cases} \infty & \alpha \leq 0 \\ \dfrac{\alpha m^{\alpha}}{\alpha} m^{-\alpha} = 1 & \alpha > 0 \end{cases}$$

# Pareto

The distribution has a finite mean only for α > 1.

$$E\left[X_{Pareto<\alpha,m>}\right] = \int_m^\infty x \frac{\alpha m^\alpha}{x^{\alpha+1}} dx = \alpha m^\alpha \int_m^\infty x^{-\alpha} dx = \frac{\alpha m^\alpha}{1-\alpha} \left(x^{1-\alpha}\right|_m^\infty$$

$$\mu_{Pareto<\alpha,m>} = \begin{cases} \infty & \alpha \le 1 \\ \dfrac{\alpha m}{\alpha - 1} & \alpha > 1 \end{cases}$$

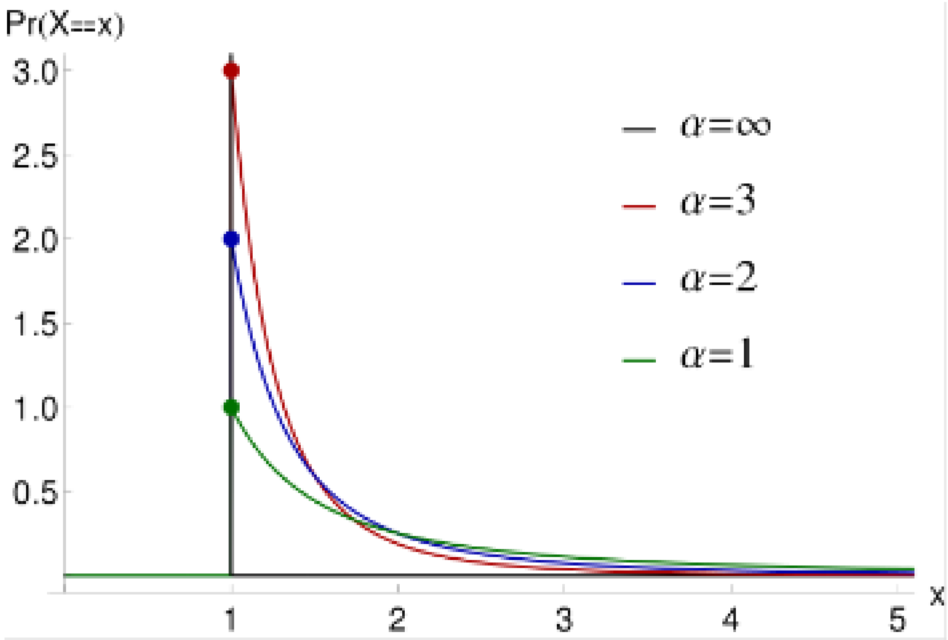
It has finite mean, but infinite variance for 1 < α <= 2. The distribution has finite variance only for α > 2.

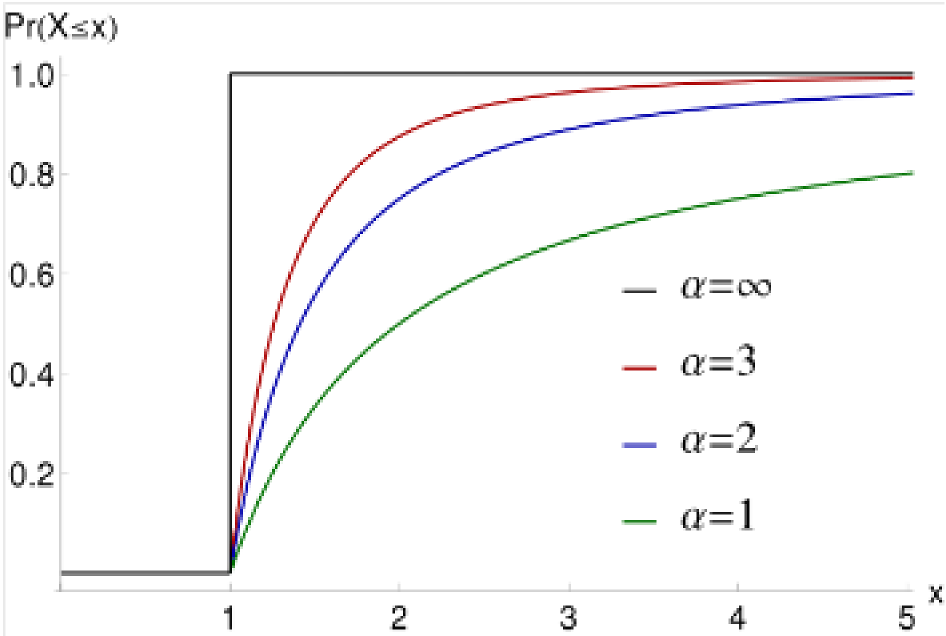
$$E\left[{X^2}_{Pareto<\alpha,m>}\right] = \int_m^\infty x^2 \frac{\alpha m^\alpha}{x^{\alpha+1}} dx = \alpha m^\alpha \int_m^\infty x^{1-\alpha} dx = \frac{\alpha m^\alpha}{2-\alpha} \left(x^{2-\alpha}\right|_m^\infty$$

$$E\left[{X^2}_{Pareto<\alpha,m>}\right] = \begin{cases} \infty & \alpha \le 2 \\ \frac{\alpha m^2}{\alpha - 2} & \alpha > 2 \end{cases} \qquad {\sigma^2}_{Pareto<\alpha,m>} = \begin{cases} \infty & \alpha \le 2 \\ \frac{\alpha m^2}{(\alpha-1)^2(\alpha-2)} & \alpha > 2 \end{cases}$$

# Pareto

The distribution has the following PDF and CDF.

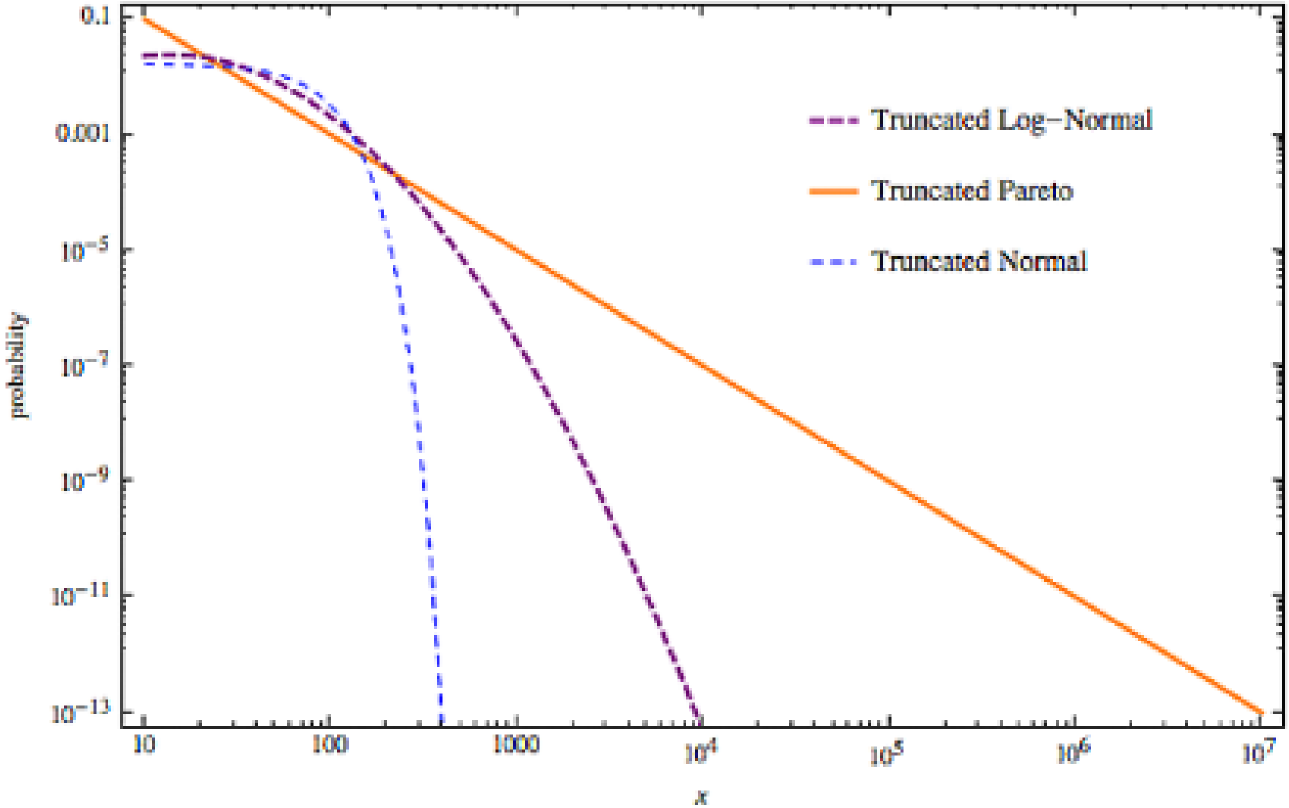


# Pareto

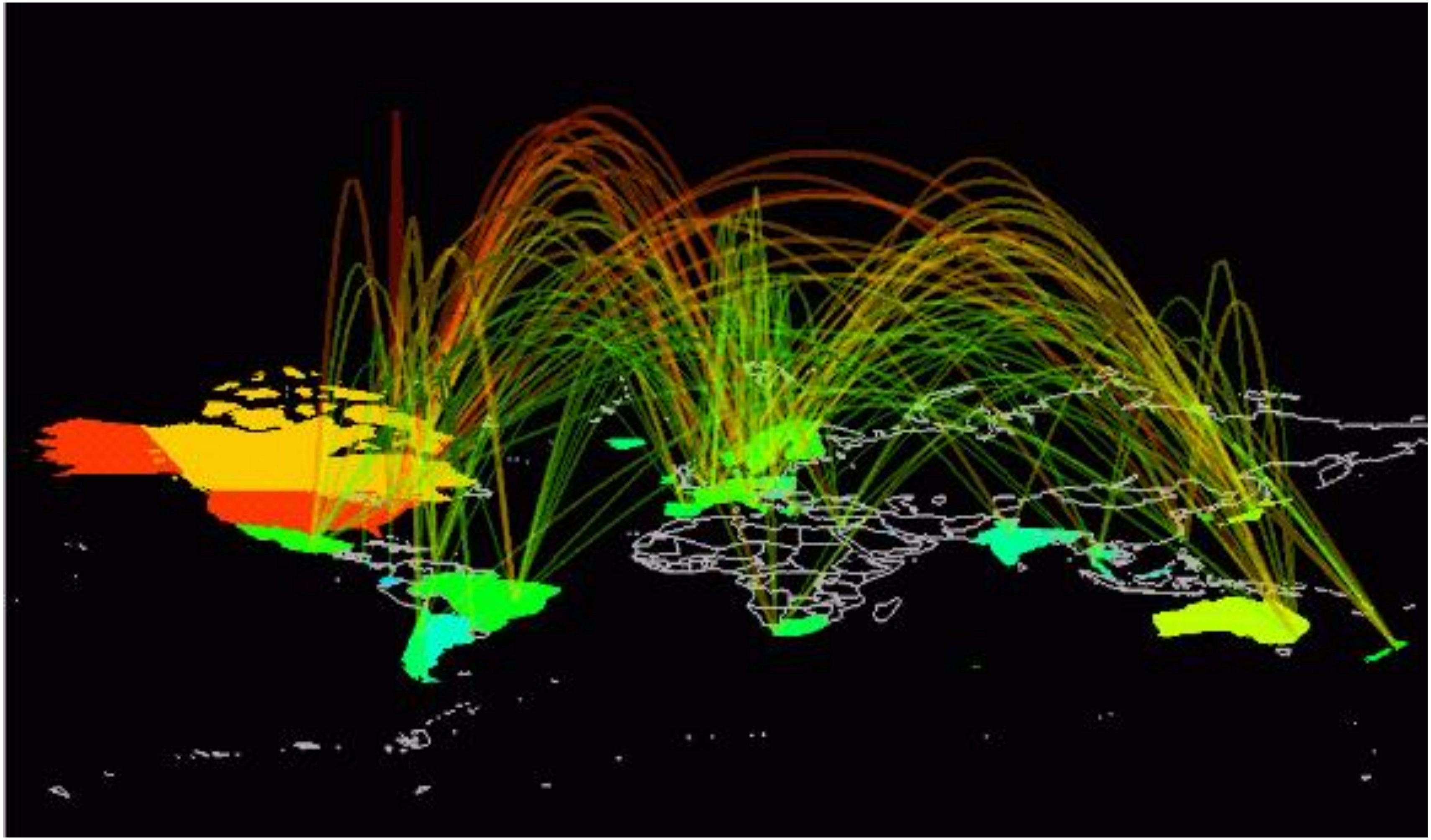
It is used to model systems where very large values can have a non-negligible probability to occur.

This phenomenon is called *Heavy tail*, and it can shown plotting *1-F(x)* in log-log-scale.



# Pareto

Heavy tail behavior has been observed in Internet traffic, where sometimes very large files are transferred.



# Pareto

When the either the average or the variance go to the infinity, many of the common techniques that we will see in this course might no longer be used.

Specific care should be taken, and special techniques must be employed to correctly estimate measures like the number of jobs in the queue and the response time.

In this course we will not consider these topics, but we must be aware of the risks that might occur when using Heavy Tail distributions.

# Normal

The *Normal (or Gaussian) distribution* is defined by its mean and its variance.

$$f_{N<\mu,\sigma^2>}(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

$$F_{N<\mu,\sigma^2>}(t) = \frac{1}{2}\left[1 + erf\left(\frac{t-\mu}{\sigma\sqrt{2}}\right)\right]$$

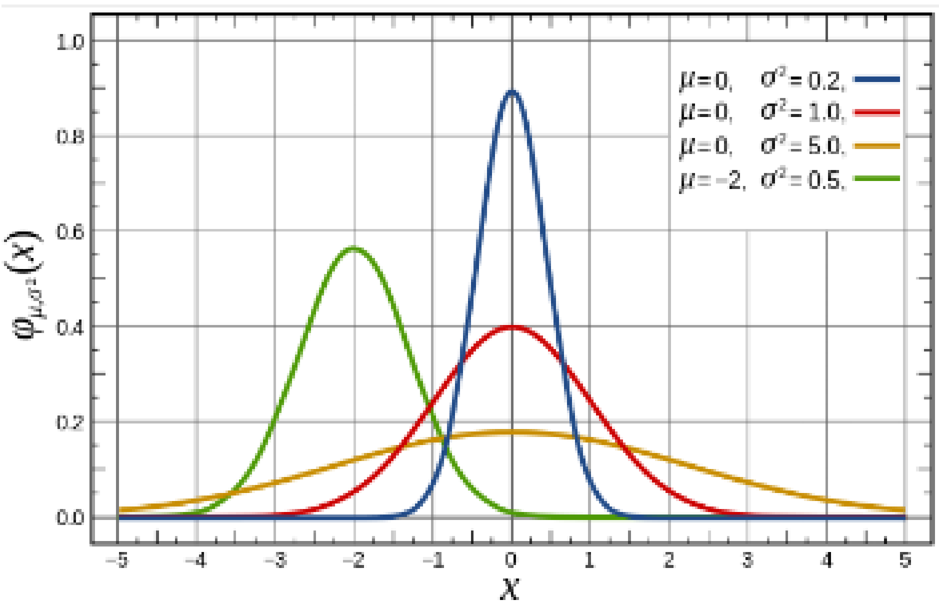
$$\mu_{N<\mu,\sigma^2>} = \mu \qquad {\sigma^2}_{N<\mu,\sigma^2>} = \sigma^2$$

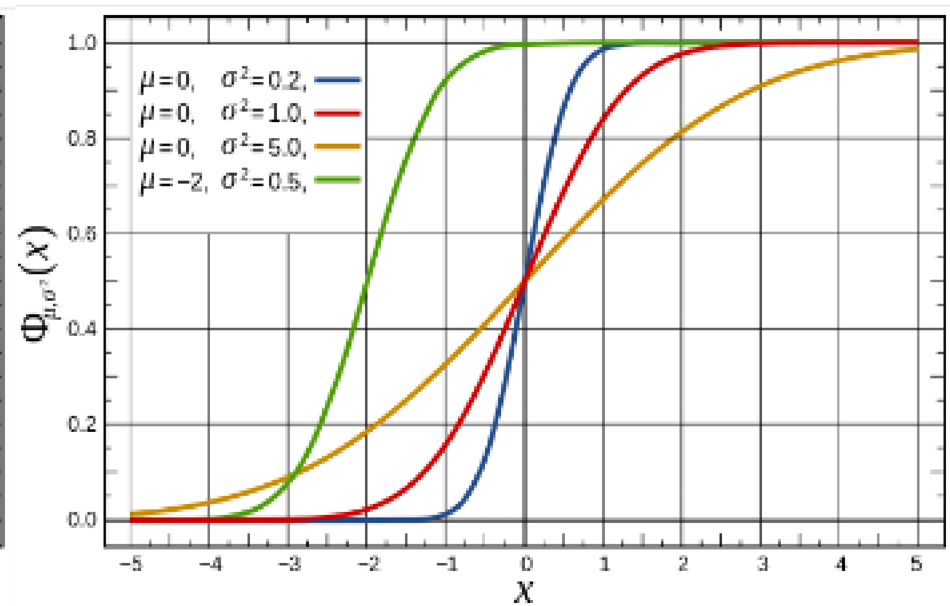
It is very important since, for the *central limit theorem*, the sum of a large number of independent random variables tends to a Normal distribution.

$$\sum_i X_i \cong X_{N<\sum_i \mu_i, \sum_i {\sigma^2}_i>}$$

# Normal

It has the following PDF and CDF:



# Normal

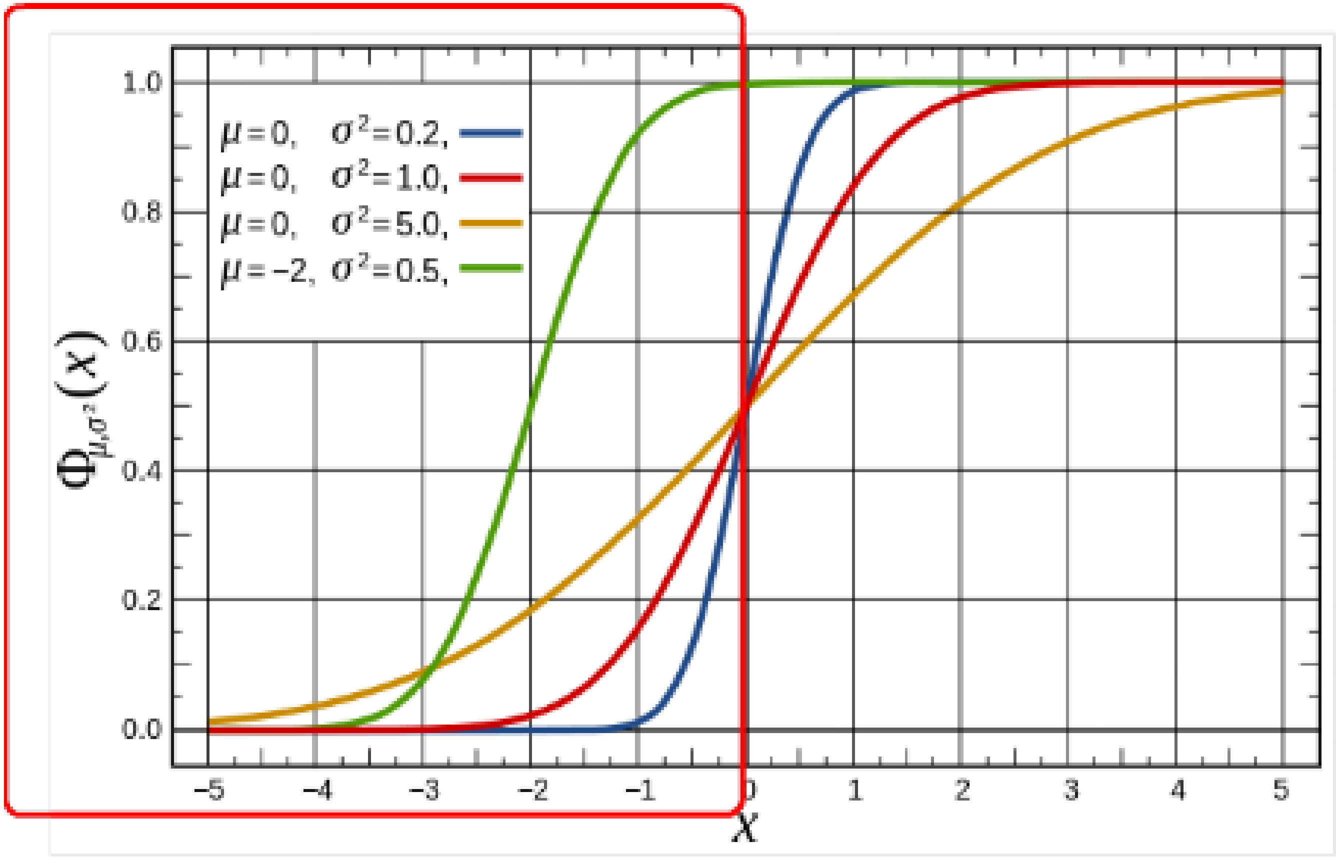
When $\mu$=*0* and $\sigma^2$=*1*, the distribution is called *Standard Normal*.

If $N_{<0,1>}$ is a Standard Normal distribution, then we can obtain normal distributions with different average and variance with a simple linear transform:

$$X_{N<\mu,\sigma^2>} = \mu + \sigma \cdot X_{N<0,1>}$$

# Truncated normal

The main problem with the Normal distribution is that it has an infinite support which can produce negative numbers that are not meaningful as service or inter-arrival times.



# Truncated normal

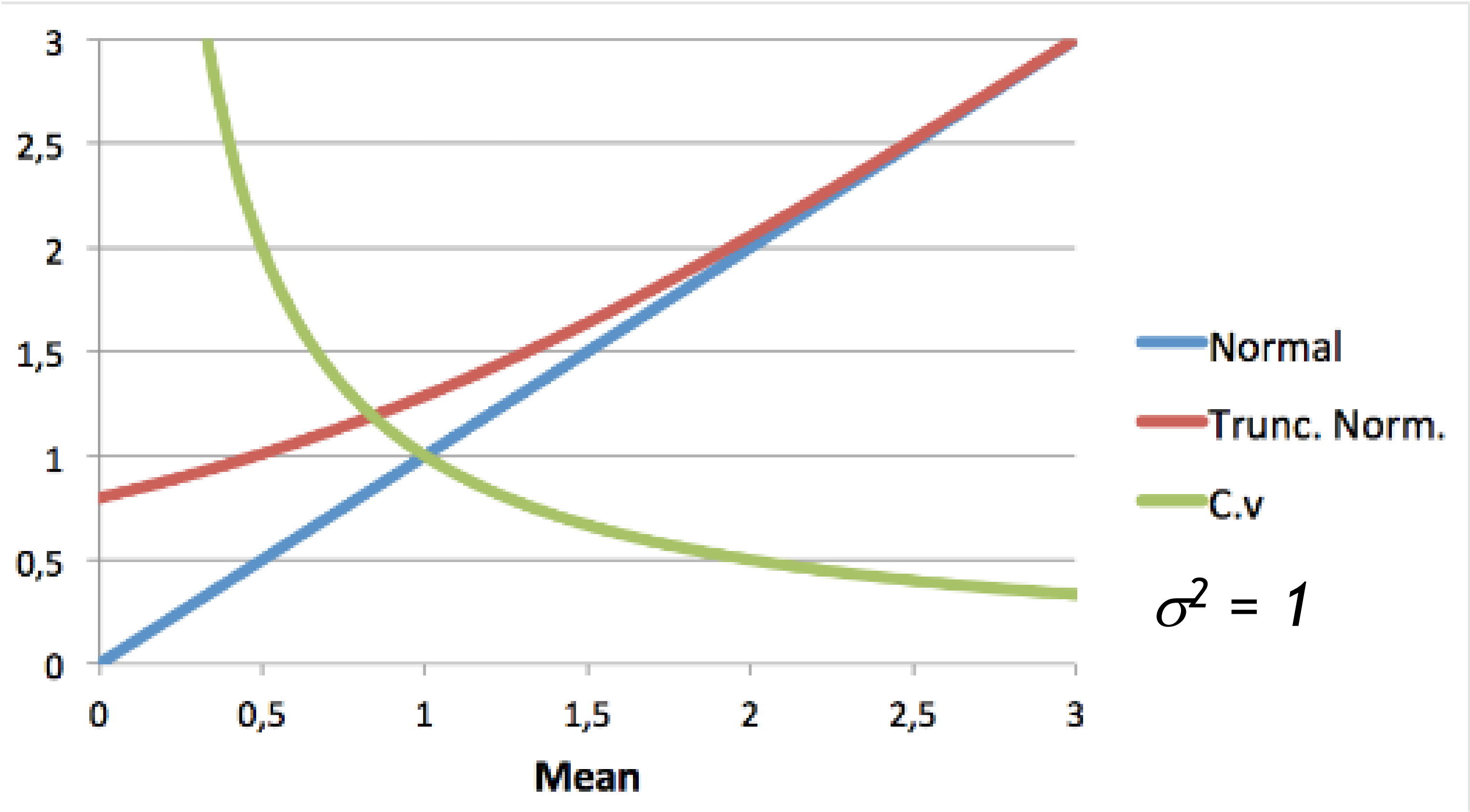
The *Truncated Normal distribution* discards all the values that are negative when sampling a conventional Normal distribution.

$$f_{TruncN<\mu,\sigma^2>}(t) = \frac{f_{N<\mu,\sigma^2>}(t)}{1 - F_{N<\mu,\sigma^2>}(0)}$$

# Truncated normal

The truncated distribution, however, has different mean and variance with respect to the original Normal distribution.

It can still be a good approximation if the c.v. is not too large, since the difference between the two values can be negligible.



# Truncated normal

To be more precise, solving a non-linear system of equations, it would be possible to obtain parameters of a truncated Normal distribution to match a given average and standard deviation (if possible).

$$\mathbb{E}_{\mu,\sigma}[X] = \mu + \frac{\varphi(\mu/\sigma)}{1-\Phi(-\mu/\sigma)}\sigma$$

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\xi^2\right)$$

$$\Phi(x) = \frac{1}{2}\left(1+\operatorname{erf}(x/\sqrt{2})\right).$$

$$\operatorname{var}_{\mu,\sigma}(X) = \sigma^2\left[1 - \frac{\mu\varphi(\mu/\sigma)/\sigma}{1-\Phi(-\mu/\sigma)} - \left(\frac{\varphi(\mu/\sigma)}{1-\Phi(-\mu/\sigma)}\right)^2\right]$$

These derivations are however quite involved, and we will consider them outside the scope of this course.

# Log Normal

The Log Normal distribution, is computed by using a Normal distribution sample (with a given average and standard deviation) as the exponent for the exponential function.

$$X_{LogNormal<\mu,\sigma^2>} = e^{X_{N<\mu,\sigma^2>}}$$

Thanks to the properties of the exponential, it produces only positive numbers.

Its PDF, CDF, average and variance are:

$$f_{LogNormal<\mu,\sigma^2>}(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln(t)-\mu)^2}{2\sigma^2}}$$

$$F_{LogNormal<\mu,\sigma^2>}(t) = \frac{1}{2}\left[1 + erf\left(\frac{\ln(t) - \mu}{\sigma\sqrt{2}}\right)\right]$$

$$\mu_{LogN<\mu,\sigma^2>} = e^{\mu+\frac{\sigma^2}{2}} \quad \sigma^2{}_{LogN<\mu,\sigma^2>} = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

# Log Normal

Given a target average *E[X]* and coefficient of variation $c_v$, its parameters can be computed in the following way:

$$\mu = ln\left(\frac{E[X]}{\sqrt{1 + c_v^2}}\right) \qquad \sigma^2 = \sqrt{ln(1 + c_v^2)}$$

# Log Normal

It has the following shape: