Performance Evaluation and Applications

POLITECNICO DI MILANO

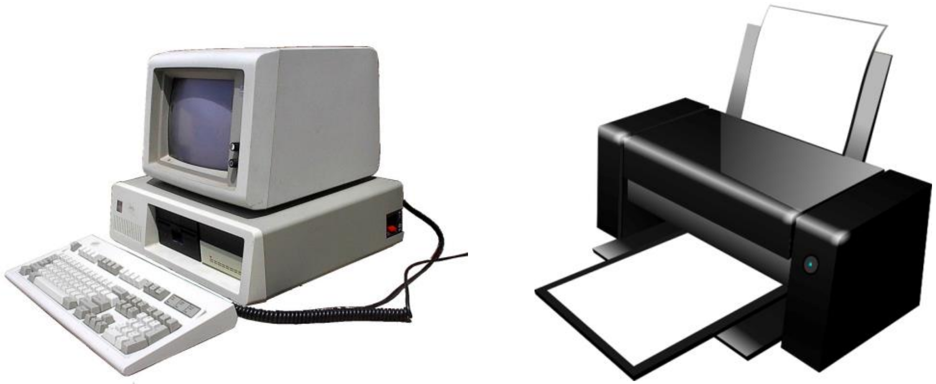




Steady State and Transient Analysis of CTMC

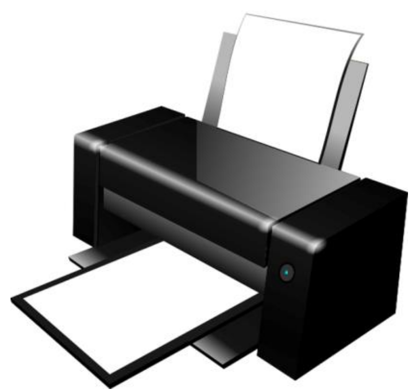
Motivating example

A system is composed by a PC and a Printer. They both alternate between a high power consumption state (*200 W* for the PC and *300 W* for the Printer), and a low power consumption state (*50 W* for both). The average time in high power is *10 min* for the PC and *5 min* for the Printer. The time in low power is instead *10 min* for the PC and *25 min* for the printer.



Motivating example

- Which is the average power consumption of the system? How many times per hour, the system reaches a state with the minimum power consumption?
- Supposing that the system starts always with the PC in high power state, and with the printer 50% of the times in low power, which are the expected average and maximum power consumptions?



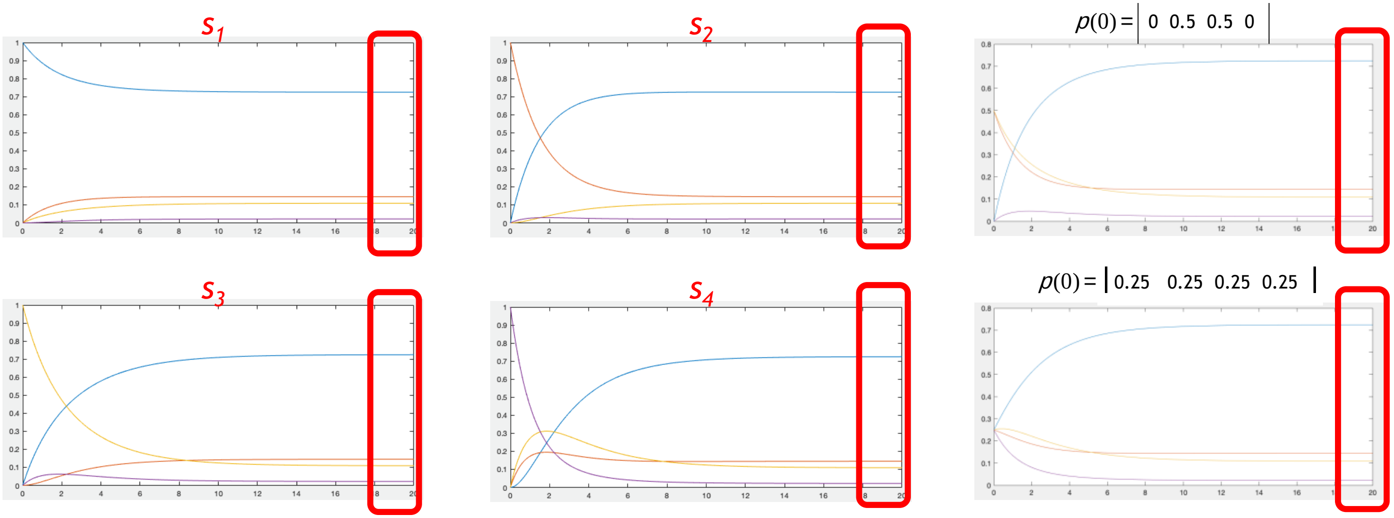
?

Steady-state distribution of a CTMC

Under very common hypothesis, the transient probability $\pi(t)$ becomes a fixed vector π as t tends to the infinity regardless of the distribution of the initial state $\pi(0)$:

$$\lim_{t\to\infty} \pi(t) = \pi, \quad \forall \pi(0)$$

This limit distribution π, is called the *Steady-state distribution* of the CTMC.



Steady-state distribution of a CTMC

When the system is in steady-state, the distribution of its states does not change, meaning that the derivative over time is zero:

$$\frac{d\pi(t)}{dt} = 0$$

This allow us to write down an equation to determine such limit (the state probability) distribution:

$$\frac{d\pi(t)}{dt} = \pi(t) \cdot Q$$

$$\pi \cdot Q = 0$$

Steady-state distribution of a CTMC

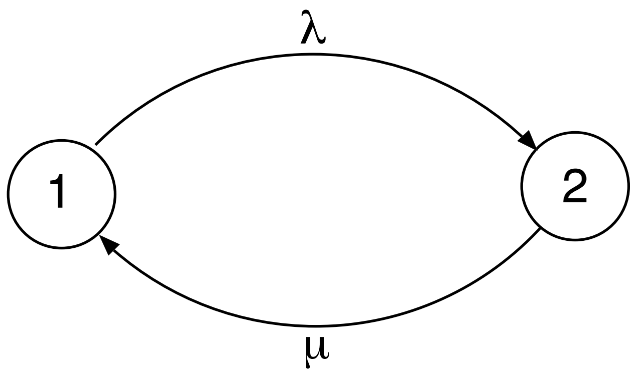
Since the rows of matrix *Q* sums up to 0, they are not linear independent and the system has an infinite number of solutions.

We then use the so called *Normalizing Condition*, that accounts for the fact that the sum of the probabilities of all the states is equal to 1.

$$\begin{cases} \pi \cdot Q = 0 \\ \boxed{\sum_{i=1}^{n} \pi_i = 1} \end{cases}$$

Example: availability of a component

Let us model as simple component that works for an exponentially distributed amount of time of rate λ, and then it undertakes maintenance at rate μ. We have:



$$Q = \begin{vmatrix} -\lambda & \lambda \\ \mu & -\mu \end{vmatrix} \qquad \begin{cases} \pi \cdot Q = 0 \\ \sum_{i=1}^{n} \pi_i = 1 \end{cases}$$

Let us call $\pi = |A \quad U|$, and let us ignore equation the second equation λA-μU=*0*, due to the linear dependency.

$$\begin{aligned} -\lambda A + \mu U &= 0 \\ A + U &= 1 \end{aligned} \qquad \begin{aligned} \mu U &= \lambda A \\ A + \frac{\lambda}{\mu} A &= 1 \end{aligned}$$

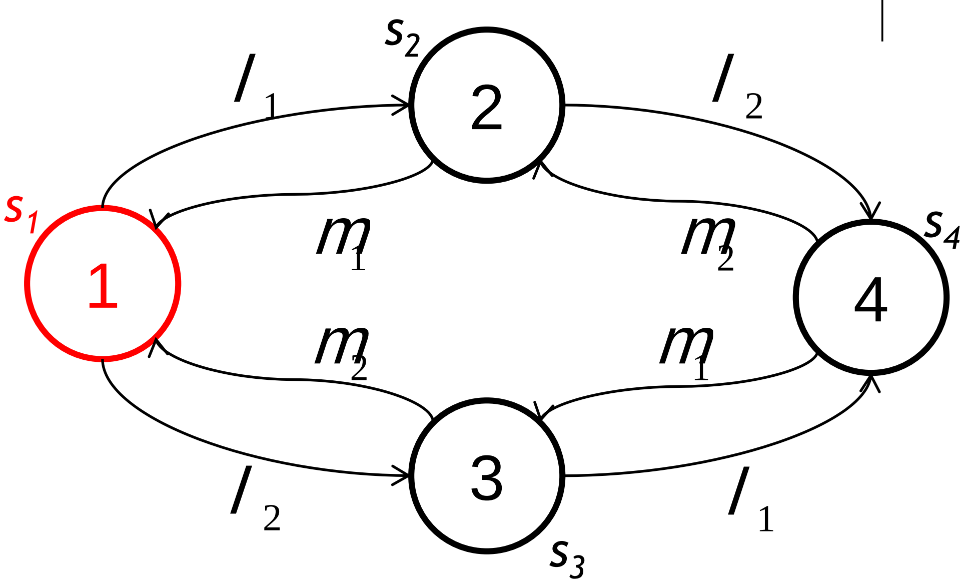
$$A = \frac{\mu}{\lambda + \mu} = \frac{MTTF}{MTTR + MTTF}$$

$$U = \frac{\lambda}{\lambda + \mu} = \frac{MTTR}{MTTR + MTTF}$$

Computation of the stationary solution

Let us consider the simple four state system introduced last time.

$$
\begin{array}{c|cccc|c}
 & s_1 & s_2 & s_3 & s_4 & \\
\hline
 & -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & & s_1 \\
Q = & \mu_1 & -\mu_1-\lambda_2 & & \lambda_2 & s_2 \\
 & \mu_2 & & -\mu_2-\lambda_1 & \lambda_1 & s_3 \\
 & & \mu_2 & \mu_1 & -\mu_2-\mu_1 & s_4
\end{array}
$$



$$
\begin{array}{c|cccc|}
 & s_1 & s_2 & s_3 & s_4 \\
p(0) = & 1 & 0 & 0 & 0
\end{array}
$$

Computation of the stationary solution

In this case we can replace one column of the matrix (e.g. the first) with a column of ones to express the normalization condition, invert it, and multiply with a vector that has one in the same place, and zero otherwise.

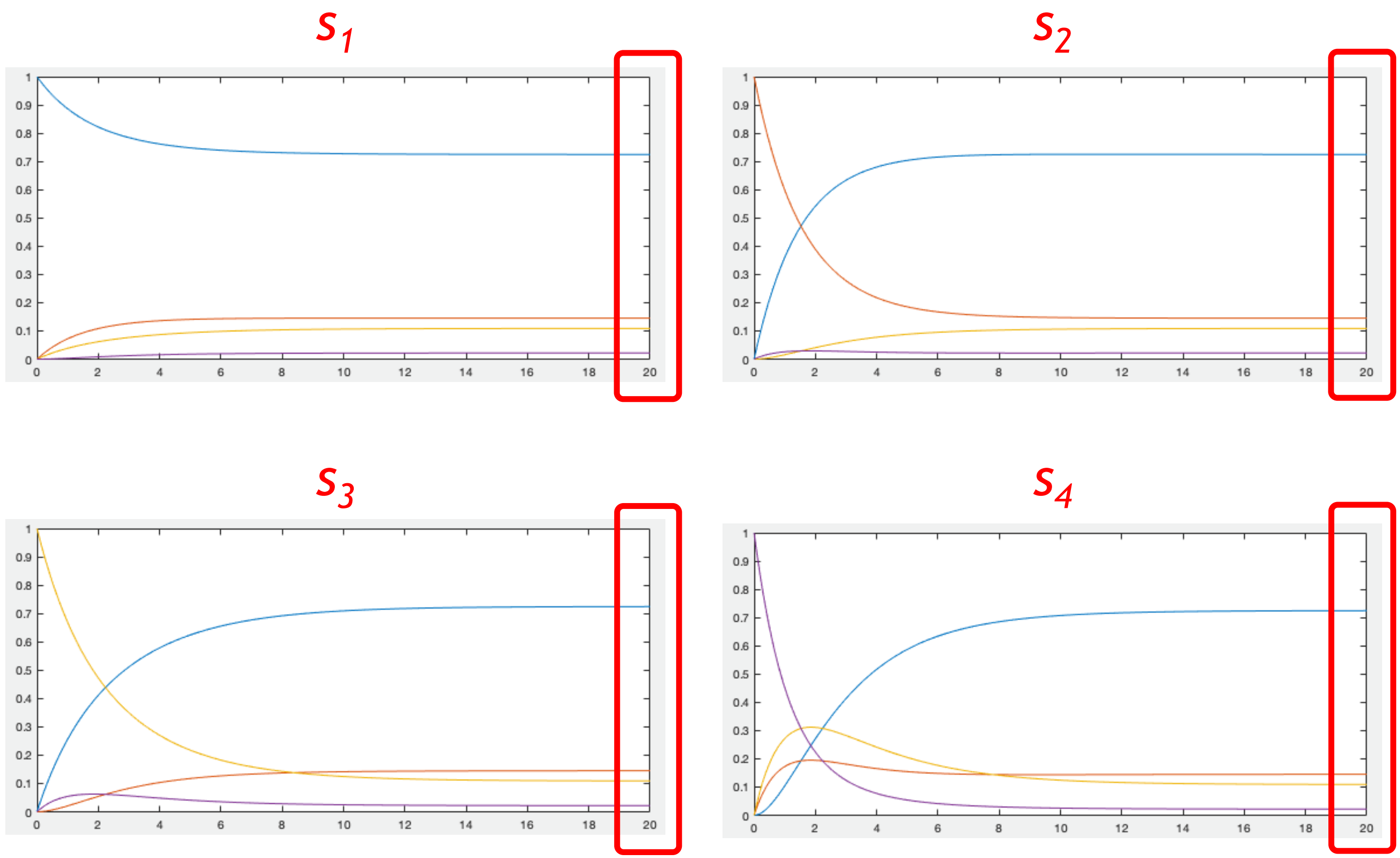
```
MTTF1 = 10;
MTTF2 = 20;
MTTR1 = 2;
MTTR2 = 3;

l1 = 1/MTTF1;
l2 = 1/MTTF2;
m1 = 1/MTTR1;
m2 = 1/MTTR2;

Q = [-l1-l2,  l1   ,  l2   ,  0;
        m1 ,-m1-l2,   0   , l2;
        m2 ,   0   ,-m2-l1, l1;
         0 ,  m2   ,  m1   ,-m2-m1];

u = [1, 0, 0, 0];

Q(:,1) = ones(4,1);
pi = u * inv(Q)
```

$$Q' = \begin{vmatrix} 1 & \lambda_1 & \lambda_2 & \\ 1 & -\mu_1-\lambda_2 & & \lambda_2 \\ 1 & & -\mu_2-\lambda_1 & \lambda_1 \\ 1 & \mu_2 & \mu_1 & -\mu_2-\mu_1 \end{vmatrix}$$

$$u = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix} \qquad \pi = u \cdot Q'^{-1}$$

Computation of the stationary solution

Please note that the solution of the system indeed corresponds to the limit to which CTMC tends.



$\pi_1 = 0.7246$

$\pi_2 = 0.1449$

$\pi_3 = 0.1087$

$\pi_4 = 0.0217$

Performance indices

On Markov chains, performance metrics can be computed associating *rewards* to the states and the transitions.

States rewards represent measures that are proportional to the time the system spends in each state.

Transitions rewards account instead for performance indices that are proportional to the number of events that occur per time unit.

Performance indices

Examples of measures computed using state rewards are:

- Utilization
- Average number of jobs in the system / in the queue
- Average energy consumption
- Availability
- Time dependent costs / revenues

Examples of transition rewards are instead:

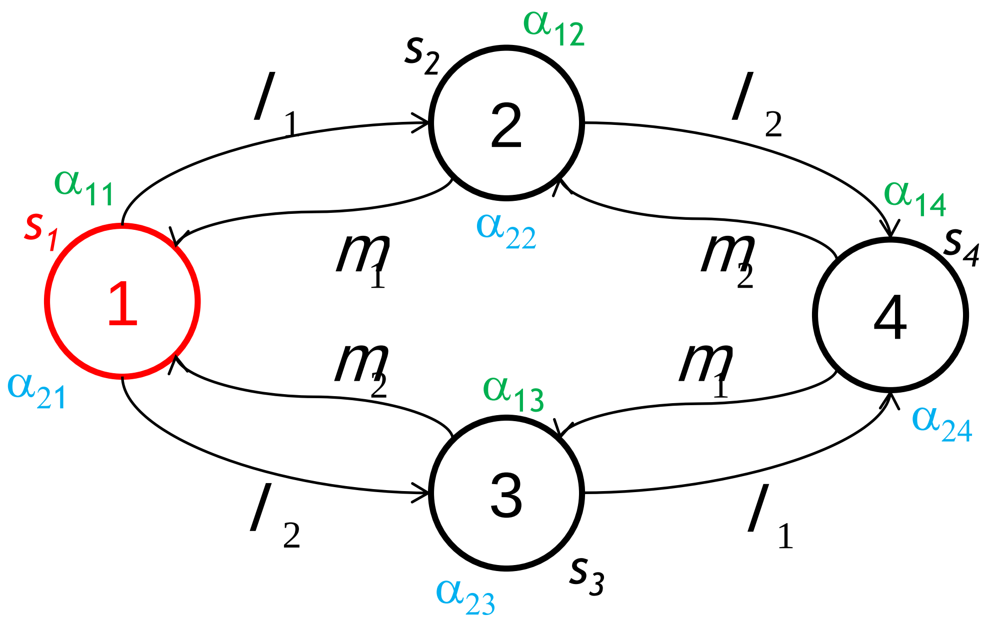
- Throughput
- Loss rate / blocking rate (presented in a future lesson)
- Failure and repair rates
- Event dependent costs / revenues

Performance indices

A model can have several *state rewards* α_k, each one defined by a vector that has a component per state of the CTMC $\boldsymbol{\alpha}_k = | \alpha_{k1} \ldots \alpha_{kN} |$.

α_k represents the reward that is obtained when the system is in state *k*.

It can be *positive*, *negative* or *zero*, depending that it represents a *gain*, a *loss*, or the fact that the *no reward* is obtained in the state.



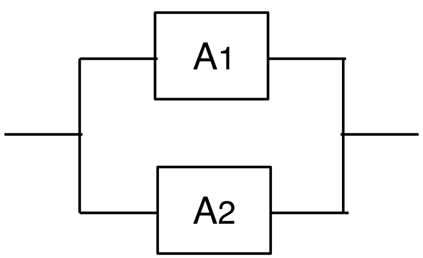
$\boldsymbol{\alpha}_1 = | \alpha_{11}\ \alpha_{12}\ \alpha_{13}\ \alpha_{14} |$

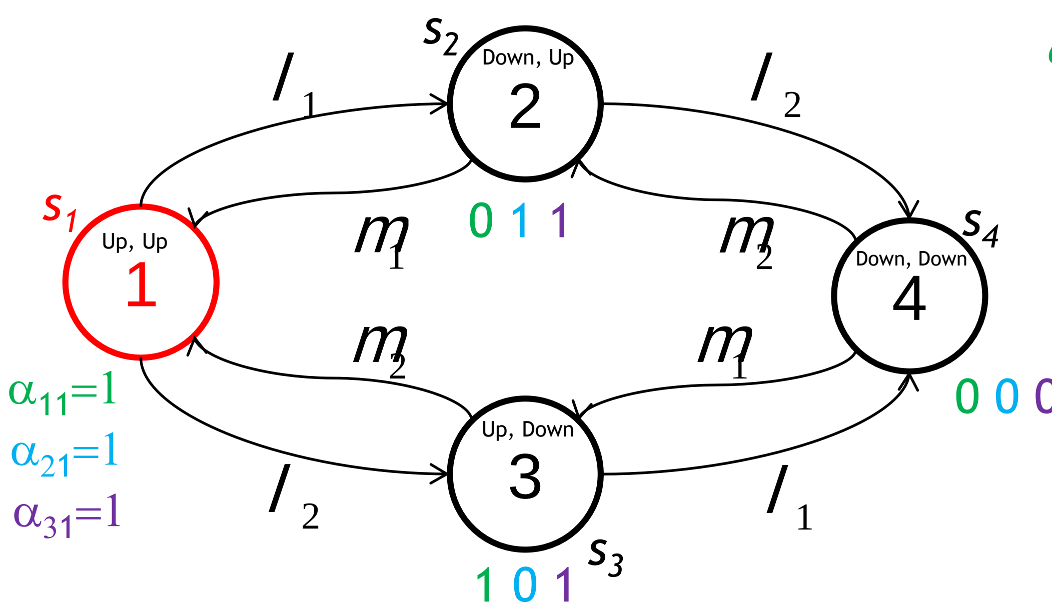
$\boldsymbol{\alpha}_2 = | \alpha_{21}\ \alpha_{22}\ \alpha_{23}\ \alpha_{24} |$

Performance indices

For example, if our model represents a system with two repairable components in parallel, breaking at rate λ_i, and repaired at rate μ_i, the reward rates α_{ki} could be *Boolean* values that represent whether a component or the entire system are working in state *i*.

- α_1 : *Component 1 is working*
- α_2 : *Component 2 is working*
- α_3 : *The parallel system is working*





$\alpha_1 = |1\ 0\ 1\ 0|$

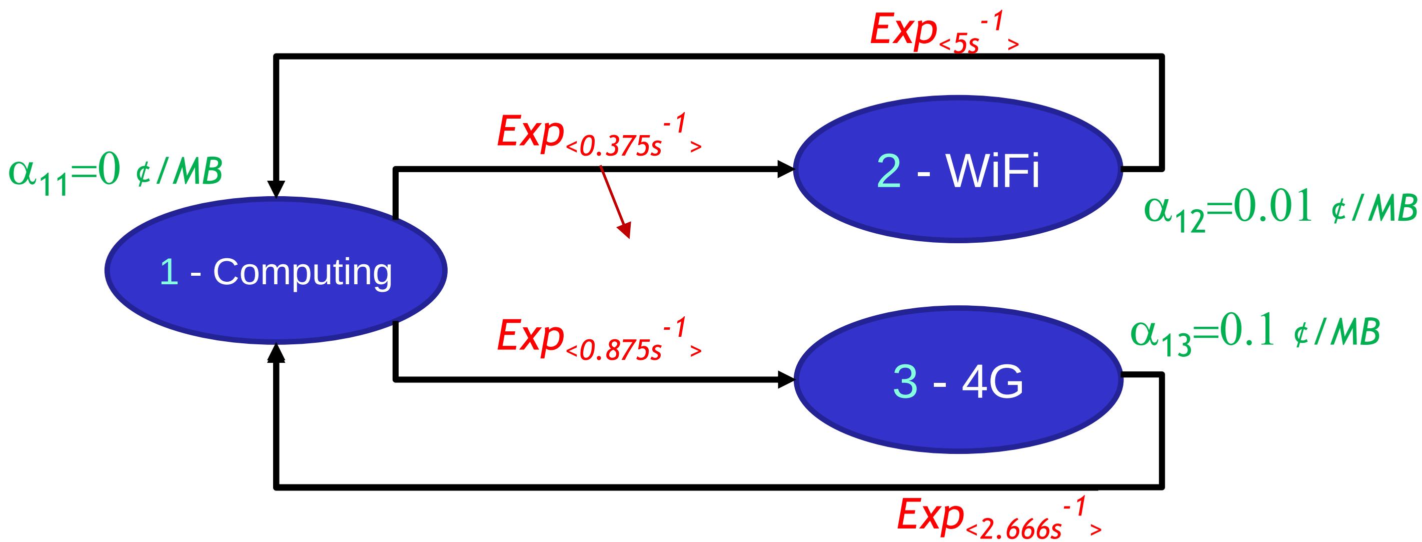
$\alpha_2 = |1\ 1\ 0\ 0|$

$\alpha_3 = |1\ 1\ 1\ 0|$

Performance indices

In the App model, α_{1i} could represent the communication cost, expressed in ¢/MB, in each state.

$\alpha_1 = |0 \;\; 0.01 \;\; 0.1|$

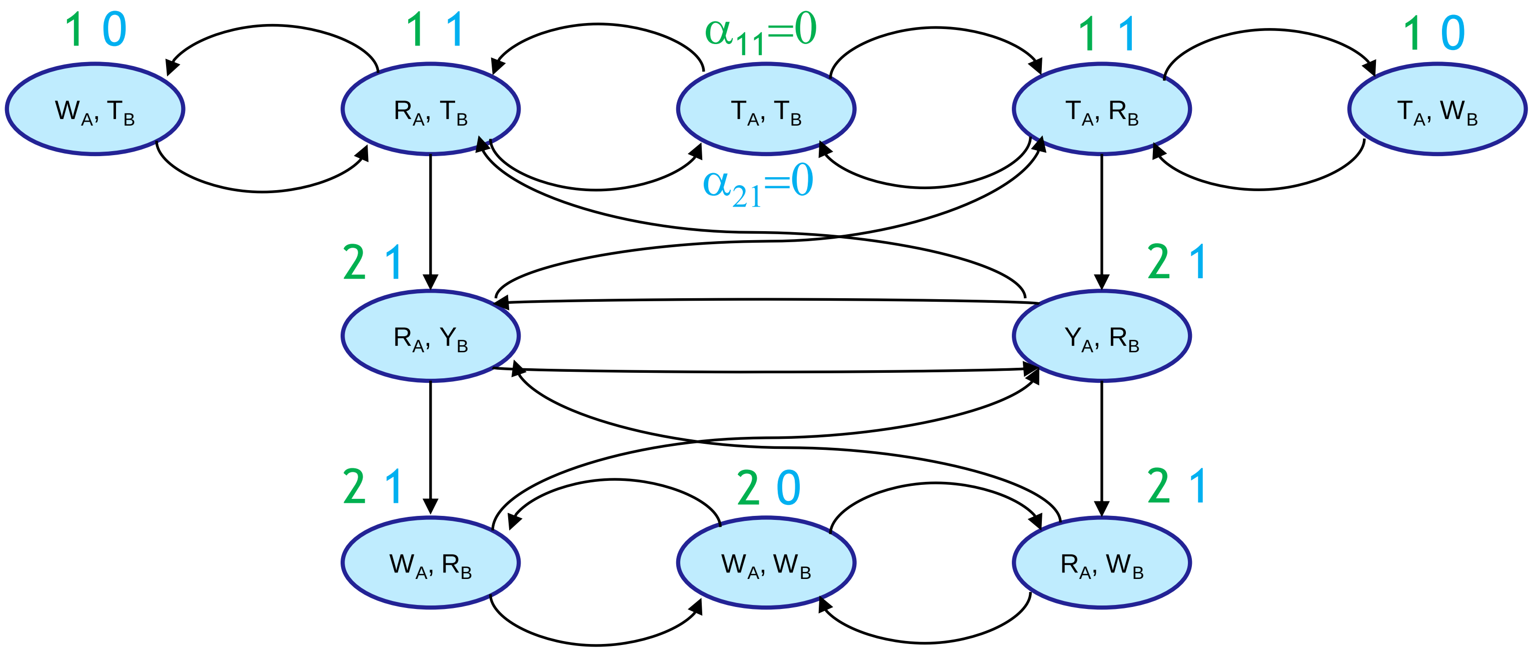


Performance indices

In the *Operative System Processes* model, α_{1i} could represent the number of jobs in memory in each state, and α_{2i} could be a *Boolean* variable accounting whether the system is working or idle – i.e. it can be used to compute the *Utilization*.

$\alpha_1 = |0\ 1\ \ldots\ 2|$

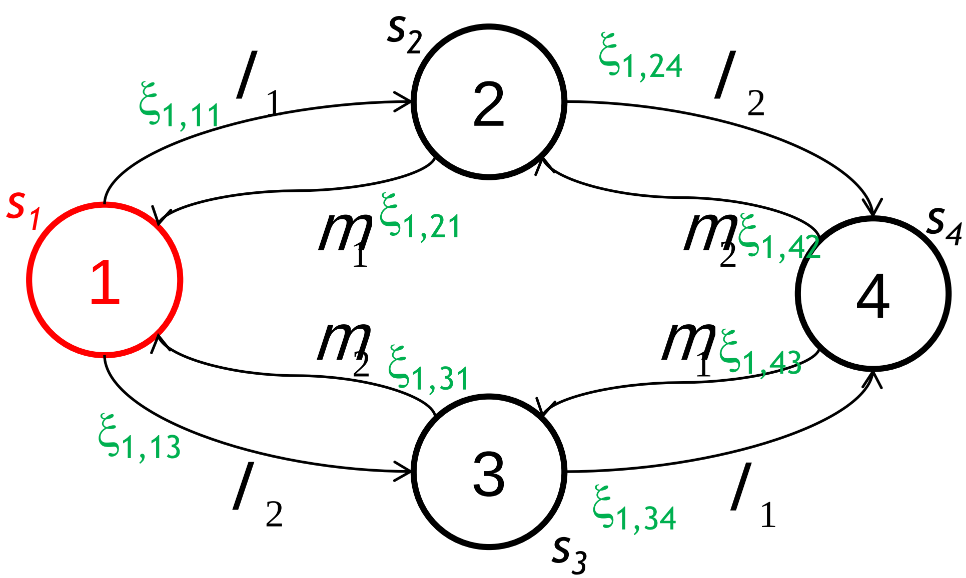
$\alpha_2 = |0\ 1\ \ldots\ 0|$



Performance indices

Transition rewards ξ_k , are associated to transitions, and are defined by matrices ξ_k = | $\xi_{k,ij}$ |.

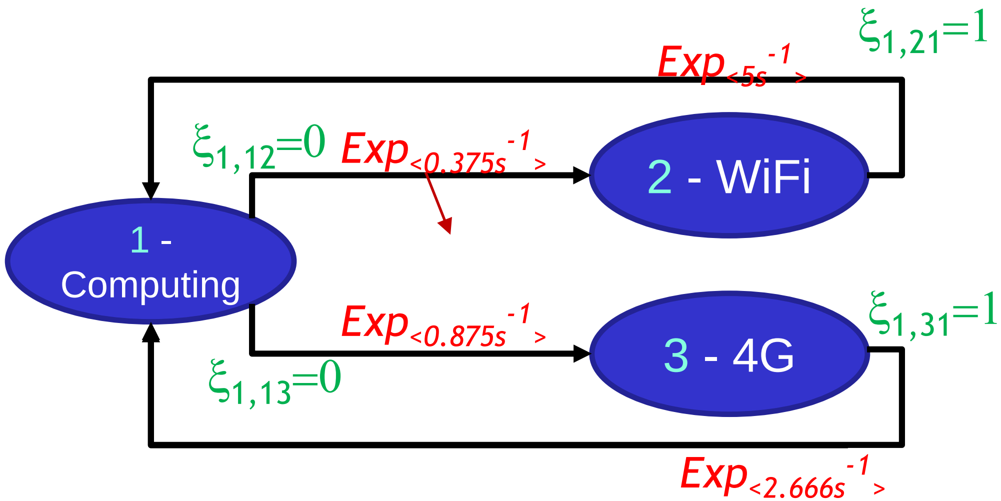
They represent the reward gained or lost when a transition *from state i to state j* occurs.

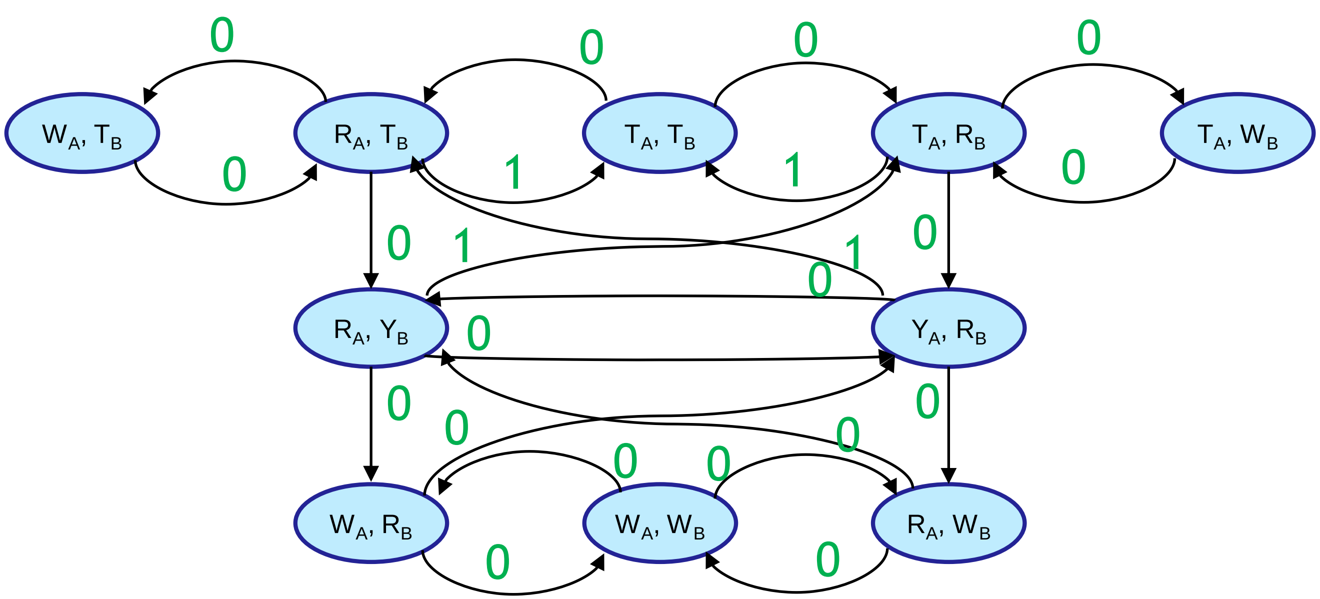


$$\xi_1 = \begin{vmatrix} & \xi_{1,12} & \xi_{1,13} & \\ \xi_{1,21} & & & \xi_{1,24} \\ \xi_{1,31} & & & \xi_{1,34} \\ & \xi_{1,42} & \xi_{1,43} & \end{vmatrix}$$

Performance indices

In both the *App* and the *Operative System Process* it can represent the end of a job, and it can be used to compute the system throughput.





Performance indices

Starting from the steady state solution π_i :

- for state rewards, we can compute their average value:

$$E[\boldsymbol{\alpha}_k] = \sum_{i=1} \pi_i \cdot \alpha_{ki}$$

- for transition reward, we can compute their average per time unit:

$$E[\boldsymbol{\xi}_k] = \sum_{i=1} \pi_i \sum_{j \neq i} q_{ij} \cdot \xi_{k,ij}$$

Performance indices

Both measures can be computed in steady state to asses long-time performance metrics that give a general view of the system:

$$E[\boldsymbol{\alpha}_k] = \sum_{i=1} \pi_i \cdot \alpha_{ki} \qquad E[\boldsymbol{\xi}_k] = \sum_{i=1} \pi_i \sum_{j \neq i} q_{ij} \cdot \xi_{k,ij}$$

They can also be computed in transient, to evaluate the corresponding property in a specific point in time *t*:

$$E[\boldsymbol{\alpha}_k(t)] = \sum_{i=1} \pi_i(t) \cdot \alpha_{ki} \qquad E[\boldsymbol{\xi}_k(t)] = \sum_{i=1} \pi_i(t) \sum_{j \neq i} q_{ij} \cdot \xi_{k,ij}$$

Measures like response time require more advanced techniques: we will not cover them in this course. However, *Little's Law* can be used in most of the case to derive their average from the measures previously presented.

State classification

In a CTMC, a state can either be:

» Ergodic

» Transient

» Absorbing

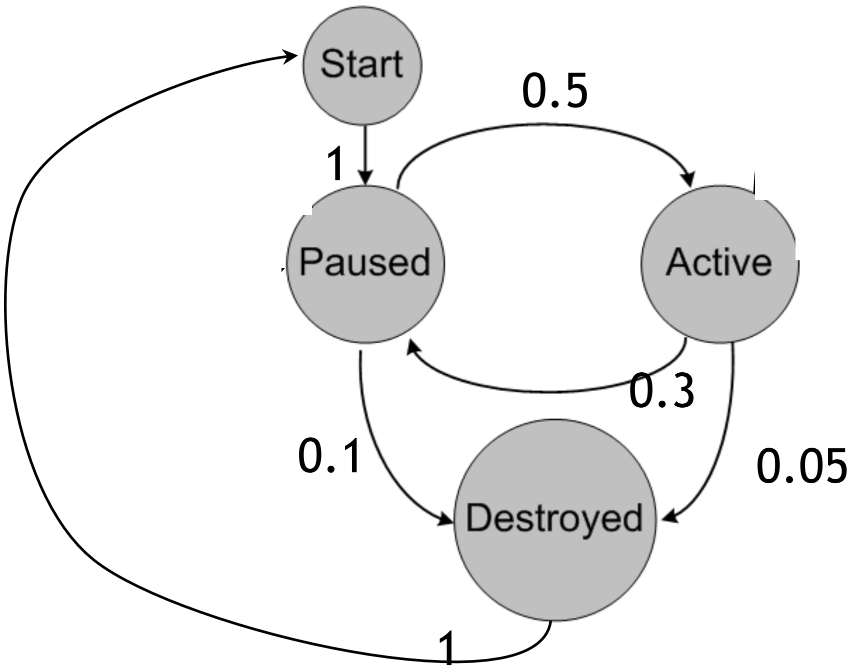
Knowing these type of states can help to identify possible modelling errors, and understand issues that can arise when studying the model.

Ergodic States

An **Erogodic** (or *recurrent*) state, is a state into which the system will always return in the future.

Ergodic are the "normal" states of a system.

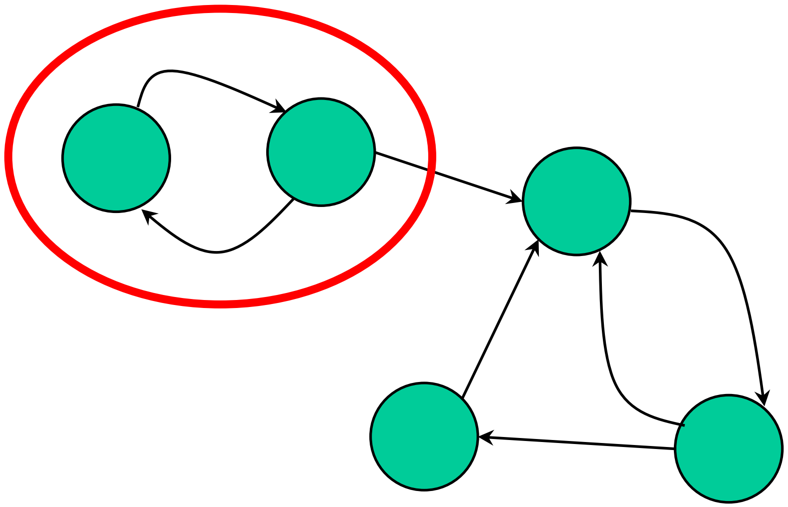
For ergodic states it is meaningful to compute the *steady-state* distribution.



Transient States

Transient states are the ones for which their probability tends to zero as the time goes to infinity.

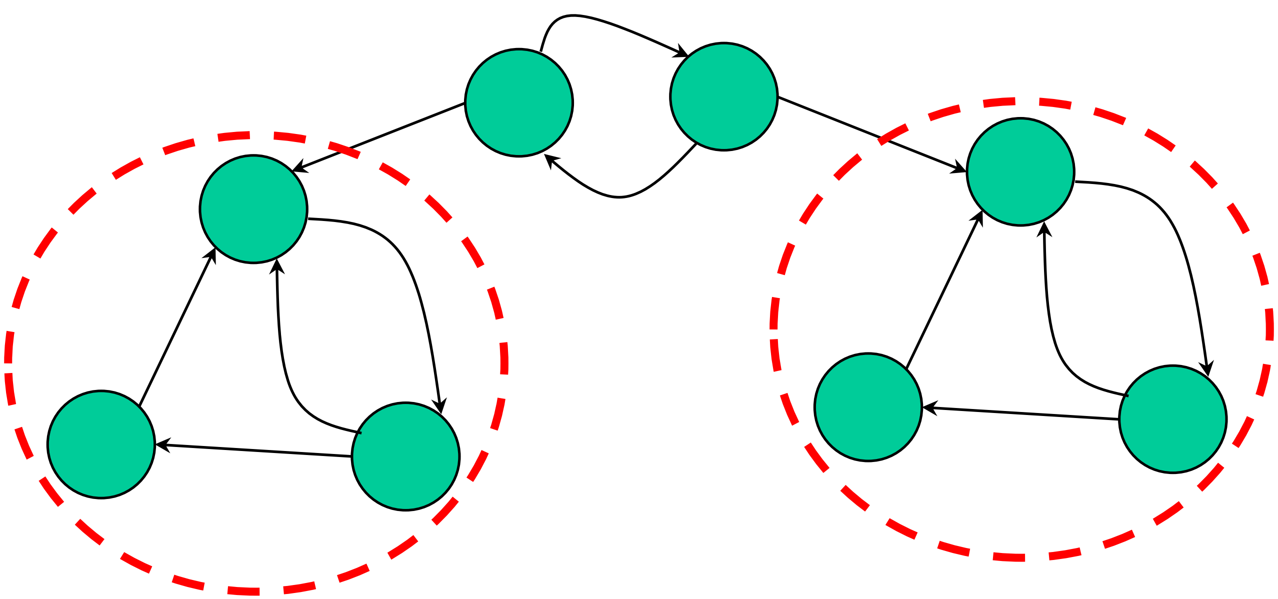
They are the states that sooner or later will be abandoned, and in which the system will never be able to return again.



Transient States

If there are transient states, the chain may have more than one *Strong component*: a subset of the states where the system might be "trapped" inside.

If there are more than one strong component, the steady-state solution is not unique, and depends on the initial state of the model. Moreover it has a different meaning since, once one of the components has been chosen, the others will have zero probability.

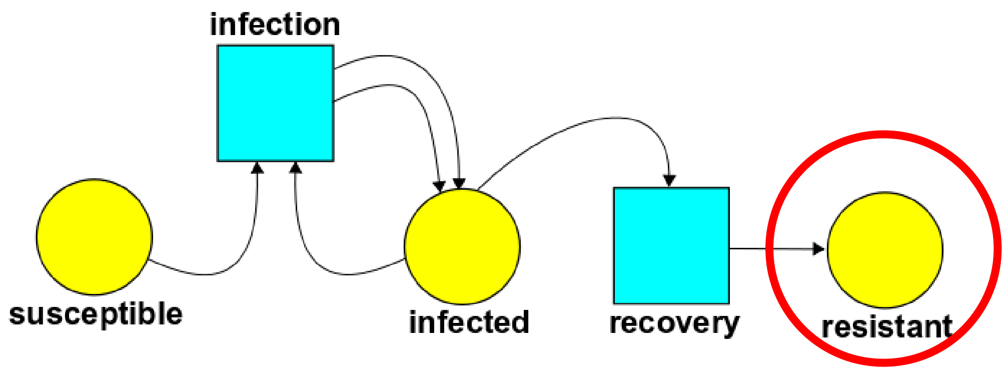


Absorbing States

A state in which there are no output transitions is called an **Absorbing state**.

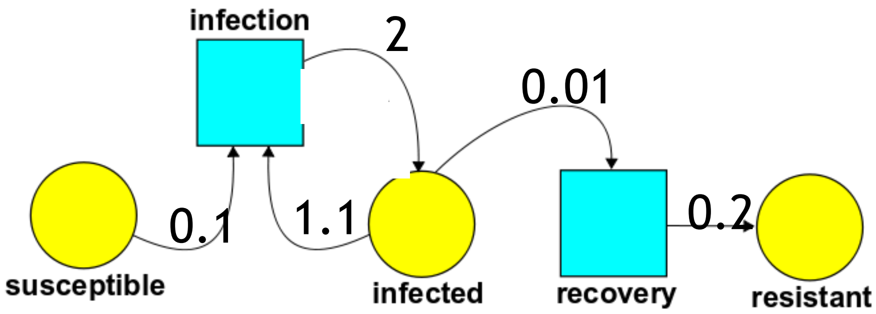
An absorbing state is a *strong component*, composed by a single state.

When the system reaches an absorbing state, it will be trapped inside it forever.



Absorbing States

In the infinitesimal generator of a CTMC, the presence of absorbing states corresponds to lines entirely composed of zeros.



$$Q = \begin{vmatrix} -0.1 & 0.1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1.1 & -1.11 & 0.01 & 0 \\ 0 & 0 & 0 & -0.2 & 0.2 \\ \boxed{0 \quad 0 \quad 0 \quad 0 \quad 0} \end{vmatrix}$$

Absorbing States

If there is a single absorbing state, its steady-state probability is (usually) 1, and the steady-state probability of all the other states is 0.

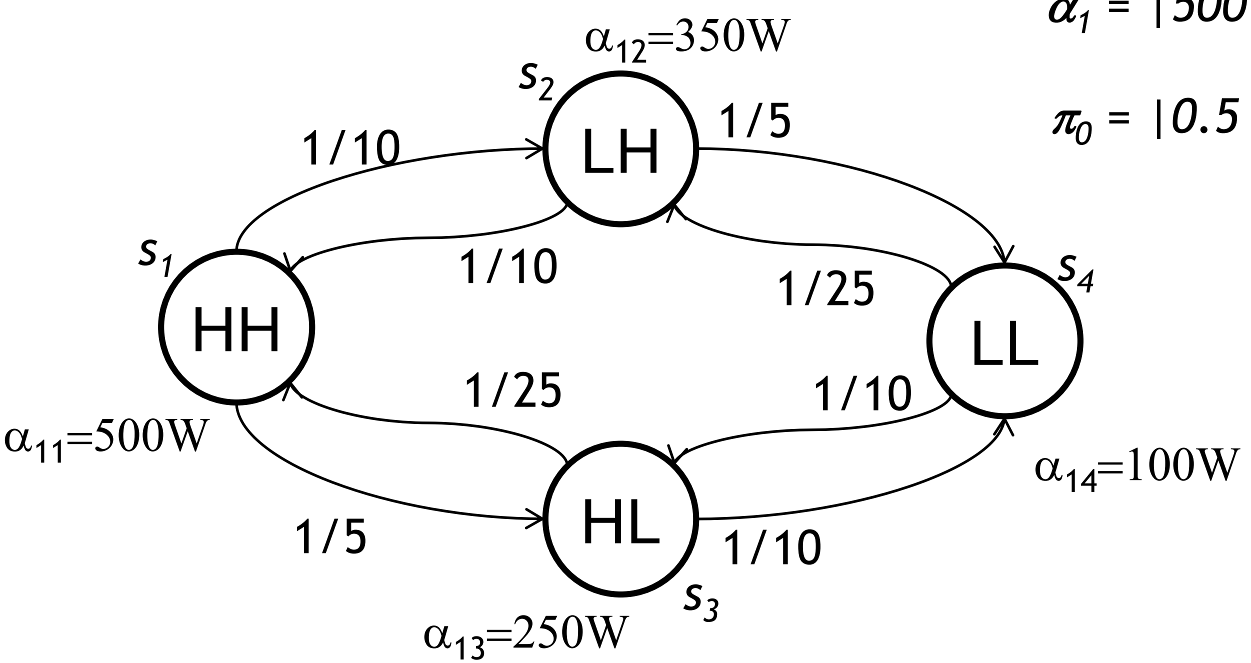
- If there are more than one absorbing state, since they are strong components, their steady state distribution depends on the initial state of the model.

Even if their steady-state distribution is trivial, the evolution of the states probabilities over time can be used to study interesting properties of a system: for example, it can be used to determine the *reliability* in a *dependability* study.

For time constraints, we will not investigate this topic further.

Analysis of Motivating Example

We can model the system with a 4 states CTMC, and evaluate the average power consumption in steady state (for the average) and during transient (for the maximum).

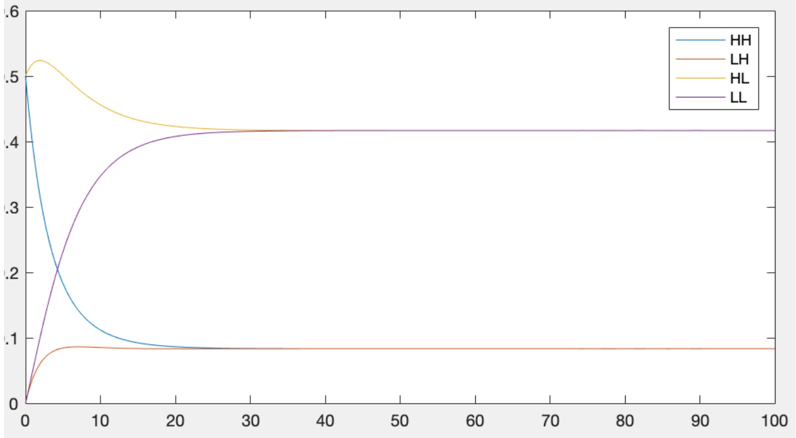


α_1 = |500 350 250 100|

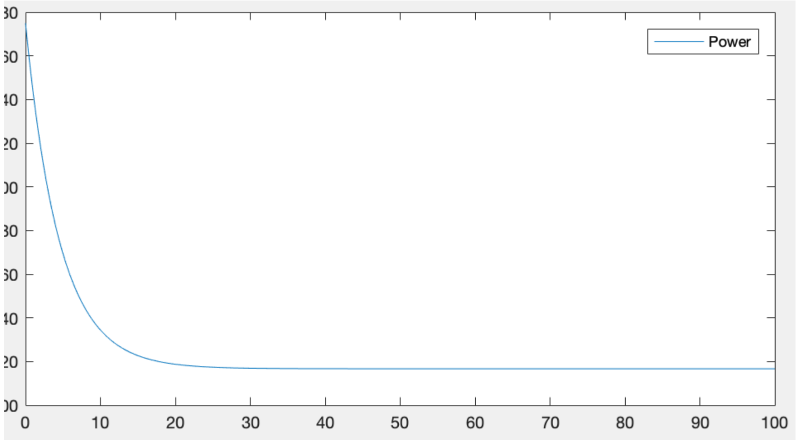
π_0 = |0.5   0   0.5   0|

Analysis of Motivating Example

Results are the following:

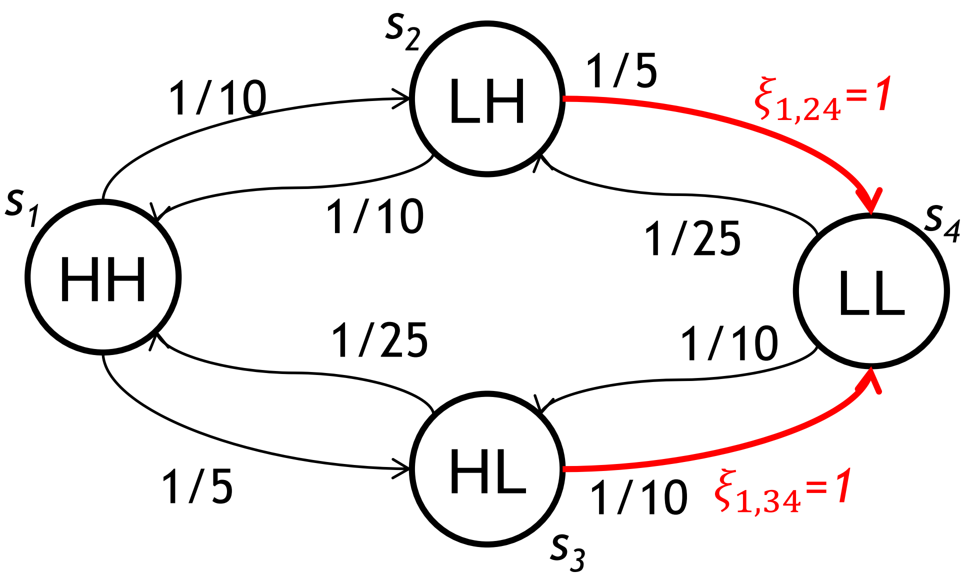


- Average power consumption:
    - 216.7 Watt
- Maximum average power consumption (determined studying the resulting function):
    - 375 Watt



Analysis of Motivating Example

The number of times the system reaches the minimum power state, can be computed with a transition reward.



$$\xi_1 = \left|\begin{array}{cccc} & 0 & 0 & \\ 0 & & & 1 \\ 0 & & & 1 \\ & 0 & 0 & \end{array}\right|$$