Automatic Control
Laboratory practice 1

M. Canale - S. M. Fosson

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Objectives: solution of LTI continuous-time dynamical systems by using state space equations and/or transfer function

Problem 1

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 5 & 8 \\ 1 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 4 \\ -1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 3 & -4 \end{pmatrix} x(t) + 7u(t)$$

(a) compute the analytical expressions of state and output responses, with initial conditions $x(0) = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ and null input $u(t) = 0$;

(b) repeat the exercise with a step signal of amplitude 4 as input.

Solution

(a) $x(t) = \begin{pmatrix} -2e^{7t} + 5e^{t} \\ -0.5e^{7t} - 2.5e^{t} \end{pmatrix} \varepsilon(t), \quad y(t) = \left(-4e^{7t} + 25e^{t}\right) \varepsilon(t)$

(b) $x(t) = \begin{pmatrix} -1.24e^{7t} + 15.67e^{t} - 11.43 \\ -0.31e^{7t} - 7.83e^{t} + 5.1429 \end{pmatrix} \varepsilon(t), \quad y(t) = \left(-2.48e^{7t} + 78.33e^{t} - 26.86\right) \varepsilon(t)$

Problem 2

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 27 & 0 \\ -23 & 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

compute the analytical expressions of state and output responses, with initial conditions $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and input $u(t) = \begin{pmatrix} 0 \\ \delta(t) \end{pmatrix}$.

Solution

$$x(t) = \begin{pmatrix} 1.15e^{-\frac{t}{2}} \cos(0.87t - 1.57) \\ 1.15e^{-\frac{t}{2}} \cos(0.87t + 0.52) \end{pmatrix} \varepsilon(t), \quad y(t) = \left(1.15e^{-\frac{t}{2}} \cos(0.87t - 1.57)\right) \varepsilon(t)$$

Problem 3

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix} x(t)$$

(a) compute the analytical expressions of the output response $y(t)$, with initial conditions $x(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^\top$ and input $u(t) = \left(1 + t\right) \varepsilon(t)$;

(b) compute the state response $x(t)$ with initial conditions $x(0) = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}^\top$ and null input $u(t) = 0$.

Solution

(a) $y(t) = \left(-0.16e^{-5t} + 1.15e^{-\frac{t}{2}} \cos(0.87t + 2.62) + 1.16 + 1.2t\right) \varepsilon(t)$

(b) $x(t) = \begin{pmatrix} 0 & 0 & e^{-10t} & e^{-5t} \end{pmatrix}^\top \varepsilon(t)$

Problem 4 (**)

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x(t)$$

(a) compute the analytical expressions of the output response, with initial conditions $x(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^\top$ and input $u(t) = \sin(t\sqrt{2})$.
(b) repeat the computation for $u(t) = \sin(t)$.
Hints:
$\mathcal{L}(\sin(\omega_0 t)) = \frac{\omega_0}{s^2+\omega_0^2}$

$\sqrt{2} = 1.4142\ldots, \frac{\sqrt{2}}{2} = 0.7071\ldots$

$\mathcal{L}^{-1}\left(\frac{R}{(s-\sigma_0-j\omega_0)^2} + \frac{R^\star}{(s-\sigma_0+j\omega_0)^2}\right) = 2|R|e^{\sigma_0 t} t\cos(\omega_0 t + \arg(R))\varepsilon(t).$

Solution

(a) $y(t) = \left(-\sqrt{2}\cos(t\sqrt{2}) + \sqrt{2}\cos(t)\right)\varepsilon(t)$
(b) $y(t) = \left(\frac{1}{2}t\sin(t)\right)\ \varepsilon(t)$

Extra questions: For all the exercises, check whether the state/output responses are bounded/unbounded.
Is there a relationship between the eigevanlues of A and the expressions of the state/output responses?

(**) Problem 4 will be retrieved and discussed in class.