Automatic Control
Laboratory practice 2

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Objectives: study of the natural modes, internal stability and BIBO stability of LTI continuous-time dynamical systems

Problem 1

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability;
(d) repeat (b) and (c) with $A = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
(e) if possible, compute the time constant for $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and for $A = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Which system has natural modes with faster convergence rate?

Solution
(a) The system is internally asymptotically stable.
(b) The natural modes are of kind $e^{-\frac{1}{2}t}\cos(\dots) \to$ (exponentially) convergent.
(c) The system is BIBO stable.
(d) The system is internally asymptotically stable ($\Rightarrow$ BIBO stable); the natural modes are of kind $e^{-\frac{1}{6}t}\cos(\dots) \to$ (exponentially) convergent.

(e) For a convergent mode associated with pole p, the time constant is

$$\tau = \frac{1}{|Re(p)|}.$$

In this problem, the time constants are $\tau = 2$ and $\tau = 6$; the natural modes of the first system have faster convergence rate.

Problem 2

Given the LTI system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 1 & 3 \\ 6 & 4 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)\end{aligned}$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability.

Solution
(a) The system is internally unstable.
(b) The natural modes are e^{-2t} (exponentially) convergent and e^{7t} (exponentially) divergent
(c) The system is BIBO unstable (the tf is $H(s) = \frac{2}{s-7}$).

Problem 3

Given the LTI system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) - 2u(t)\end{aligned}$$

(a) study the internal stability;
(b) perform the modal analysis, i.e., classify the natural modes;
(c) study the BIBO stability.

Solution

(a) The eigenvalues of A are $\lambda_1 = -2$ and $\lambda = 0$ with $\mu_2' = 2$. The system is internally unstable.

(b) The natural modes are $e^{-2t} \rightarrow$ (exponentially) convergent; $1 \rightarrow$ bounded (constant); $t \rightarrow$ (linearly) divergent.

(c) The system is BIBO stable (the tf is $H(s) = \frac{-2s-3}{s+2}$).

Problem 4

Given the LTI system

$$\dot{x}(t) = \begin{pmatrix} 5 & -1 & 2 \\ 3 & 1 & 0 \\ -5 & 4 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} u(t)$$
$$y(t) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 3 \end{pmatrix} x(t)$$

(a) study the internal stability;

(b) perform the modal analysis, i.e., classify the natural modes;

(c) study the BIBO stability.

Solution

(a) The system is internally unstable.

(b) The natural modes are $e^{3.6t} \rightarrow$ (exponentially) divergent; $e^{1.19t}\cos(\dots)1 \rightarrow$ $\rightarrow$ (exponentially) divergent.

(c) The system is BIBO unstable (all the eigevanlues of A have positive real part; then, all the poles of the tf, that are a subset of $eig(A)$, have positive real part).

Problem 5

Given $p \in \mathbb{R}$, study the internal stability of an LTI system with

$$A = \begin{pmatrix} p^2 - 1 & 0 & 0 \\ 0 & p-3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(Hint: A is diagonal)

Solution

For $p \in (-1, 1) \to$ asymptotically stable; for $p \in \{-1, +1\} \to$ internally stable; for $|p| > 1 \to$ internally unstable.

Problem 6

Given $p \in \mathbb{R}$, study the BIBO stability of an LTI system with tf

$$H(s) = \frac{4}{s^2 + (p+1)s + 4p - 2}$$

Hint: Use the Decartes' rule of signs

For polynomials of 2nd order, the Decartes' rule of signs is as follows: given $ax^2 + bx + c$, consider the coefficients a, b, c. Reading them from the highest degree to the lowest degree, count how many changes of signs there are, i.e.,

a	b	c	changes of signs
> 0	> 0	> 0	0
> 0	> 0	< 0	1
> 0	< 0	> 0	2
$\dots$	$\dots$	$\dots$	$\dots$

The number of changes of signs is equal to the number of positive real roots. (If the roots are complex conjugate, the real part is $\frac{-b}{2a}$)

Solution
The system is BIBO stable for $p > \frac{1}{2}$.

Extra problems

Consider the LTI systems given in LAB 1. Study their internal stability and BIBO stability.

Solution
P1 Internally unstable; BIBO unstable
P2 Internally asymptotically stable; BIBO stable
P3 Internally asymptotically stable; BIBO stable
P4 Internally unstable; BIBO unstable.