Teoria dei sistemi.

Introduction

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Formal Definition of Systems

Definition

An abstract dynamic system is a 3-tuple $\{\mathcal{T}, \mathcal{U} \times \mathcal{Y}, \Sigma\}$ where

- $\mathcal{T}$ is the time space
- $\mathcal{U}$ is the set of input functions
- $\mathcal{Y}$ is the set of output functions

and

$$\Sigma = \left\{ \Sigma(t_0) \subset \mathcal{U}^{T(t_0)} \times \mathcal{Y}^{T(t_0)} : t_0 \in \mathcal{T} \text{and CRT is satisfied} \right\},$$

where CRT stands for closure with respect to truncation: i.e., $\forall t_1 \geq t_0$

$$(u_0, y_0) \in \Sigma(t_0) \implies \left(u_0|_{\mathcal{T}(t_1)}, y_0|_{\mathcal{T}(t_1)} \right) \in \Sigma(t_1).$$

Parametric Representation of Binary Relations

In order to study Abstract Systems, we can apply the following general result

Lemma

Given a binary relation R, it is possible to define a set P and a function $\pi : P \times D(R) \to R(R)$ such that

$$(a, b) \in R \implies \exists p : b = \pi(p, a) \tag{1}$$

$$p \in P, a \in D(R) \implies (a, \pi(p, a)) \in R \tag{2}$$

π is said *parametric representation* and (P, π) is said *parametrisation* of the relation.

Parametric Representation of Abstract Systems

The Lemma cited above leads us to the following:

Theorem

Consider a system defined as above. It is possible to identify a parametrisation (X_{t_0}, π) such that

$$\pi = \{\pi_{t_0} : X_{t_0} \times D(\Sigma(t_0)) \to R(\Sigma(t_0))/t_0 \in \mathcal{T}\} \tag{3}$$

satisfying the following properties:

$$(u_0, y_0) \in \Sigma(t_0) \implies \exists x_0 : y_0 = \pi_{t_0}(x_0, y_0) \tag{4}$$

$$x_0 \in X_{t_0}, u_0 \in D(\Sigma(t_0)) \implies (u_0, \pi_{t_0}(x_0, u_0)) \in \Sigma(t_0). \tag{5}$$

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Meaning of the parameter

By using this result we can remove the ambiguity. Given input + parameter we identify a single output.

Example

It is quite intuitive that initial conditions are a perfect means to define a parametric definition.

Electrical circuit

The initial charge in the capacitor (or equivalently the initial voltage) is a possible parameter and the function π_{t_0} for the step input function is

$$V_C(t) = V_c(t_0)e^{-(t-t_0)/RC} + (1 - e^{-(t-t_0)/RC})V_f$$

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Banking Account

The initial capital can be used as parameter. the π_{t_0} for the step input function is

$$c(kT) = (1 + I_+ \cdot \frac{T}{12})^{(k-k_0)} C_0 + S \frac{(1 + I_+ \cdot \frac{T}{12})^{k+1-k_0} - 1}{I_+ \cdot \frac{T}{12}}.$$

Definition of Causality

Definition

Let $u|_{[t_0,\bar{t}]}$ be the restriction of the function u to the closed interval $[t_0,\ \bar{t}]$. A system is causal if it has a representation (X_{t_0}, π) such that

$$\forall t_0 \in \mathcal{T}, \forall x_0 \in X_{t_0}, \forall \bar{t} \in \mathcal{T} \tag{6}$$

$$u_{[t_0,\ \bar{t}]} = u'_{[t_0,\ \bar{t}]} \implies [\pi_{t_0}(x_0, u)](\bar{t}) = [\pi_{t_0}(x_0, u')](\bar{t}). \tag{7}$$

A parametric representation of this type is said causal. If instead of the closed interval $[t_0,\ \bar{t}]$ we use the semi-open interval $[t_0,\ \bar{t})$, the parametric representation and the system is said strictly causal.

Definition of Causality

Remark

- π is a functional
- So $\pi_{t_0}(x_0, u)$ is the output function associated to the parameter x_0 and to the function u
  - $[\pi_{t_0}(x_0, u)](\bar{t})$ is the value it takes at time $\bar{t}$.
- Therefore for a causale system the values of u beyond $\bar{t}$ do not affect the value of the output at time $\bar{t}$.
- A simple way to put it is that a causal system does not foresee the future.
- If the system is strictly the output at time $\bar{t}$ is only affected by the input at time *strictly* smaller than $\bar{t}$.

Classification based on the number input/output signals number

- So far no specific assumptions on the range U of the input functions $\mathcal{U}$ and on the range Y of the output functions $\mathcal{Y}$.
- In some cases such quantitities can be scalar, in other they can be vecotrs.
- This gives rise to the following taxonomy
  1. Single Input Single Output (SISO): bot input and output are scalars
  2. Multiple Input Single Output (MISO): $\mathcal{U}$ is a vector, $\mathcal{Y}$ is a scalar
  3. Single Input Multiple Output (SIMO): $\mathcal{U}$ is a scalar, $\mathcal{Y}$ is a vector
  4. Multiple Input Multiple Output (MIMO): both $\mathcal{U}$ and $\mathcal{Y}$ are vectors

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- We now restrict the focus to DT and CT systems
- In particular, our goal is to understand *how* to find a parametric representation for a DT or CT system
- A possible way is *denotational*
  - Specifying a mathematical relation between input and output variables

Differential Equation and Difference Equations

Continuous Time systems

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General form

CT (DT) systems can be expressed thorugh an appropriate differential(difference) equation an appropriate order n:

$$F(y(t), \mathfrak{D}y(t), \mathfrak{D}^2 y(t), \ldots, \mathfrak{D}^n y(t), u(t), \mathfrak{D}u(t), \ldots, \mathfrak{D}^p u(t), t) = 0, \tag{8}$$

where the operator $\mathfrak{D}$, when applied to a generic function f, is defined as:

$$\mathfrak{D}^k f = \begin{cases} \frac{d^k f}{dt^k} & \text{for CT systems} \\ y(t+k) & \text{for DT systems.} \end{cases} \tag{9}$$