Teoria dei sistemi.

State Space Analysis

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State Space Form

- So far, we have analysed in detail the IO representation of the system
- Intuitively, it captures the relation between Input and Output, but not the inner workings of the system
- It is time to re-open a different topic that we have just mentioned a few weeks ago

State Space Form of LTI sytems

State Space Form

The state space form of of a linear and time invariant system is given by:

$$\begin{aligned} \mathfrak{D}x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

The operator $\mathfrak{D}$ is defined as the differential (CT systems) or the difference (DT systems) operator.

- $x \in \mathbb{R}^n$ is a vector representing the n *states*
- $y \in \mathbb{R}^p$ is a vector representing the p *outputs*
- $u \in \mathbb{R}^m$ is a vector representing the m *inputs*

Advantages

- The State Space form is quite agnostic to the number of inputs and outputs (the I/O is best used for SISO systems)
- Not all that works within a system is visible as IO relation
- The state space form is easier to implement
- There is an entire area (state space control and estimation) that relies on the state space form.

Control Canonical Form

Let us start from the following problem:

Finding the State Space Form

Consider a SISO system expressed by

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i y(t) + \sum_{j=0}^{p} \beta_j \mathfrak{D}^j u(t), \qquad (2)$$

Find a state Space Form for the system.

RECALL: the initial condition and the input determine the evolution of the output

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RECALL: $p \leq n$ implies that the system is causal

RECALL: $p < n$ implies that the system is strictly causal

Case $p = 0$

- Let us start from the case $p = 0$:

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i y(t) + \beta_0 u(t).$$

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$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i y(t) + \beta_0 u(t).$$

- We can choose the following state variables $x_1 = y$, $x_2 = \mathfrak{D} y$, with the result

$$A = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \hline \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{array}\right], \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = \left[\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \end{array}\right], \quad \text{and} \quad D = 0.$$

with initial conditions

$$x_i(0) = \mathfrak{D}^{i-1} y(0).$$

Companion Form

Lower Companion Form

The matrix

$$A = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \hline \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{array}\right]$$

is called lower horizontal companion form.

The matrix is the *companion* of its own *characteristic polynomial* $\mathcal{P}(A)$, which appears with its coefficients (with opposite sign) in the last row

Case $p > 0$

- For the general case, we can use superimposition priniciple piling up the contributions of the different derivatives (differences) of $u(t)$ as different signals
- To do so, observe that if the system responds to u with y, it will respond to $\mathfrak{D}^k u$ with $\mathfrak{D}^k y$.

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- Indeed, consider the system

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$$y(t) = x(t)$$

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$$

Case $p > 0$

- If we apply the $\mathfrak{D}$ operator to both sides of the $x(t)$ equation, we find

$$\mathfrak{D}\mathfrak{D}^n x(t) = \mathfrak{D}\left(\sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i x(t) + u(t)\right) \text{ and}$$

$$\mathfrak{D}^n\left(\mathfrak{D}x(t)\right) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i\left(\mathfrak{D}x(t)\right) + \mathfrak{D}u(t).$$

Therefore $\mathfrak{D}x$ is solution to the equation with input $\mathfrak{D}u$ with output given by $y(t) = \mathfrak{D}x(t)$.

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  Therefore $\mathfrak{D}x$ is solution to the equation with input $\mathfrak{D}u$ with output given by $y(t) = \mathfrak{D}x(t)$.
- As a consequence, by applying $\omega(t) = \sum_{j=0}^{p} \beta_j \mathfrak{D}^j u(t)$, we produce as a result:

$$y(t) = \sum_{j=0}^{p} \beta_j \mathfrak{D}^j x(t).$$

Case $p > 0$

- By setting again $\mathfrak{D}^i x(t) = x_i(t)$, the matrices A and B do not change w.r.t. the simple case $p = 0$ and if the system is *strictly causal* systems $(p < n)$, we will have:

$$A = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \hline \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \end{array}\right], \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

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- We are just left with the case of non strictly causal system.

Non strictly causal system

- If $p = n$ (*non-strictly causal* systems), we can still say that $\mathfrak{D}^n x$ is the response to $\mathfrak{D}^n u$.
- However, $\mathfrak{D}^n x$ can be expressed from the differential equation as:

$$\mathfrak{D}^n x(t) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i x(t) + u(t).$$

- Therefore, the output $y(t)$ to $\omega(t) = \sum_{j=0}^{p} \beta_j \mathfrak{D}^j u(t)$, will be

$$\begin{aligned} y(t) &= \sum_{j=0}^{n} \beta_j \mathfrak{D}^j x(t) \\ &= \beta_n \mathfrak{D}^n x(t) + \sum_{j=0}^{n-1} \beta_j \mathfrak{D}^j x(t) \\ &= \sum_{i=0}^{n-1} \beta_n \alpha_i \mathfrak{D}^i x(t) + u(t) + \sum_{j=0}^{n-1} \beta_j \mathfrak{D}^j x(t). \end{aligned}$$

Non strictly causal system

State space form:

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$$C = \begin{bmatrix} \beta_0 + \beta_n\alpha_0 & \beta_1 + \beta_n\alpha_1 & \beta_2 + \beta_n\alpha_2 & \cdots & \beta_{n-1} + \beta_n\alpha_{n-1} \end{bmatrix}$$

$$\text{and} \quad D = \beta_n.$$

Example

Second order system

Consider the equation

$$a\ddot{y}(t) - by(t) = c\ddot{u}(t) - d\dot{u}(t) - eu(t),$$

where a, b, c, d, e are constant parameters.

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$$\ddot{y}(t) - b/ay(t) = c/a\ddot{u}(t) - d/a\dot{u}(t) - e/au(t),$$

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$$\ddot{y}(t) - b/ay(t) = c/a\ddot{u}(t) - d/a\dot{u}(t) - e/au(t),$$

- Repeating the computation for non strictly causal systems:

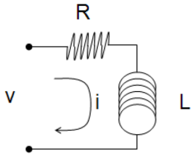
$$A = \left[\begin{array}{c|c} 0 & 1 \\ \hline \frac{b}{a} & 0 \end{array}\right], \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{cb-ea}{a^2} & -\frac{d}{a} \end{bmatrix}, \text{ and } D = \frac{c}{a}.$$

Example

RL circuit

Find the state space form of the circuit



- Dynamic equations:

$$L\frac{di(t)}{dt} + Ri(t) = v(t),$$

- Output $y(t) = i(t)$ and the input $u(t) = v(t)$.

Example - RL circuit

- IO Dynamic equations:

$$\dot{y}(t) = -\frac{R}{L}y(t) + \frac{u(t)}{L},$$

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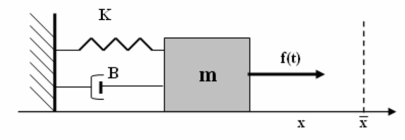
- State Space Form

$$\begin{aligned}\dot{x}_1(t) &= -\frac{R}{L}x_1(t) + \frac{u(t)}{L},\\ y(t) &= x_1(t)\end{aligned}$$

Example

Mass Spring Damper

Consider the mass spring damper system:



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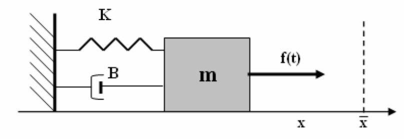

- Dynamic equations (Newton, Hooke and Rayleigh laws)

$$\frac{d^2p(t)}{dt^2}m + K(p(t) - \hat{p}) + B\frac{dp(t)}{dt} = f(t),$$

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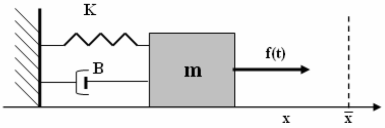

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$$\frac{d^2p(t)}{dt^2}m + K(p(t) - \hat{p}) + B\frac{dp(t)}{dt} = f(t),$$

- choosing $y(t) = p(t)$ and $u(t) = f(t)$, yields to

$$\ddot{y}(t) = -\frac{K}{m}(y(t) - \hat{y}) - \frac{B}{m}\dot{y}(t) + \frac{u(t)}{m}.$$

- Assuming without loss of generality $\hat{y} = 0$, we find

$$\dot{x}(t) = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\frac{K}{m} & -\frac{B}{m} \end{array}\right] x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} \frac{1}{m} & 0 \end{bmatrix} x(t).$$

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Discrete–time example

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- Canonical control form:

$$A = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\frac{a}{c} & \frac{b}{c} \end{array}\right], \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{da}{c^2} & \frac{ac-db}{c^2} \end{bmatrix}, \text{ and } D = -\frac{d}{c}.$$

Coordinate Transformation

- The state space description is dynamic relation between the states x, the inputs u and the outputs y
- We have seen how to define the inputs and the outputs looking at the differential or difference equations describing the system dynamics.
- One possible way to define the states is by using the outputs and their derivatives (or differences).
- However, this is not always possible and, more importantly, somethimes is not necessarily the best thing to do.

Coordinate Transformation

- the inputs u and the outputs y are, in some sense, constrained by the homogeneous representation
- the states x are to some extent "free choice".
- In general, it is always possible to define a *coordinates transformation* that binds a state variable x to another state variable, say z, using a *coordinates mapping*:

$$z = \Phi(x).$$

- A correct mapping $\Phi(\cdot)$ able to define alternative description of the states x has to be chosen with some care: it has it has to be bijective, so that any x is associated with only one z.
- Under these conditions, it is possible to define the inverse mapping $x = \Phi^{-1}(z)$.

Linear Coordinate Transformation

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- RECALL any linear trasformation from a vector space of dimension a to a vector space of dimension b is represented by a matrix in $b \times a$,
- it follows that a (time invariant) linear mapping between two state space representations is given by

$$z = Tx, \, T \in \mathbb{R}^{n \times n},$$

satisfying $\det(T) \neq 0$ in order to be bijective.

Linear Coordinate Transformation

- By substituting $x = T^{-1}z$ in

$$\begin{aligned} \mathfrak{D}x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

and recalling that T is time invariant, we find

$$\begin{aligned} \mathfrak{D}z(t) &= A_z z(t) + B_z u(t), \\ y(t) &= C_z z(t) + D_z u(t), \end{aligned}$$

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- $A_z = TAT^{-1}$, $B_z = TB$, $C_z = CT^{-1}$ and $D_z = D$.

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- All the possible state coordinates are said *similar* to each other and the system dynamic obtained are *equivalent*. Moreover, we also say that A_z is *similar* to A.
- There is not a representation that is "better" than another one, (true also for nonlinear systems) and the choice depends on the particular problem to solve.

Solution of Continuous Time State Space LTI Systems

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- the solutions of a linear differential equation can be found by *summing* up a particular solution of the differential equation to all possibile solution of the homogeneous equations (the one obtained setting $u(t) = 0$).

Solution of the homogeneous equation

- The *homogeneous* solution of the differential scalar equation, i.e., assuming $u(t) = 0$, is simply given by

$$x_u(t) = e^{at}x(0),$$

where $x(0)$ is the *initial condition* of the system.

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- we define in a similar way the solution of the multidimensional system:

$$x_u(t) = \left(I + At + \frac{A^2 t^2}{2} + \dots \right) x(0) = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!} x(0),$$

where I is the *identity* matrix of dimension n.

Solution of the homogeneous equation

- Is this a solution to $\dot{x}(t) = Ax(t)$? This is easily seen by deriving the solution:

$$\dot{x}_u(t) = \left(0 + A + At + \frac{A^2 t^2}{2} + \dots\right) x(0) =$$

$$= A\left(I + At + \frac{A^2 t^2}{2} + \dots\right) x(0) = Ax_u(t).$$

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- With this notation, we have

$$x_u(t) = e^{At}x(0) = \Phi(t)x(0). \tag{4}$$

- $\Phi(t)$ is the *state transition matrix*, which maps the initial state $x(0)$ in the state $x(t)$ with a linear transformation.

Example

Homogeneous equation

Compute the homogeneous solution of

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= -2x_1 - x_2 + u\end{aligned}$$

with initial conditions are $x_1(0) = 1$ and $x_2(0) = 2$.

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- By applying the definition Exponential Matrix, we get

$$e^{At} = I + At + I\frac{t^2}{2!} + A\frac{t^3}{3!} + I\frac{t^4}{4!} + A\frac{t^5}{5!} + \dots$$

Example

Homogeneous equation

Compute the homogeneous solution of

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$$\dot{x}_2 = -2x_1 - x_2 + u$$

with initial conditions are $x_1(0) = 1$ and $x_2(0) = 2$.

- By applying the definition Exponential Matrix, we get

$$e^{At} = I + At + I\frac{t^2}{2!} + A\frac{t^3}{3!} + I\frac{t^4}{4!} + A\frac{t^5}{5!} + \dots$$

- Therefore

$$e^{At} = \begin{bmatrix} 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \dots & 0 \\ -2\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right) & 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \dots \end{bmatrix}.$$

Example

- We observe that that

$$
\begin{aligned}
1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots &= e^t, \\
1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots &= e^{-t}, \\
t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots &= \frac{e^t - e^{-t}}{2},
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- This leads to

$$e^{At}=\begin{bmatrix} e^t & 0 \\ e^{-t}-e^t & e^{-t} \end{bmatrix}.$$

Example

- Wrapping up

$$x(t) = \begin{bmatrix} e^t & 0 \\ e^{-t} - e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t \\ 3e^{-t} - e^t \end{bmatrix}.$$

Computation of a particular solution of the differential equation

- It is now necessary to compute the *particular* solution of the differential equation in order to derive the evolution of the state variables.
- One possible way is the following (for scalar systems)

$$\dot{x}(t) = ax(t) + bu(t) \Rightarrow \dot{x}(t) - ax(t) = bu(t).$$

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- Since

$$\frac{de^{-at}x(t)}{dt} = e^{-at}\left(\dot{x}(t) - ax(t)\right),$$

  it follows that:

$$\frac{de^{-at}x(t)}{dt} = e^{-at}bu(t).$$

Computation of a particular solution of the differential equation

- Integrating both sides

$$\int_0^t \frac{de^{-a\tau}x(\tau)}{d\tau}d\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau,$$

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- As a consequence, we have that the particular solution is given by

$$x_f(t) = \int_0^t e^{a(t-\tau)}bu(\tau)d\tau.$$

The multivariable case

- For the multivariable case, it is necessary to discuss some properties of the matrix exponential
- $e^{A_1}e^{A_2} = e^{A_2}e^{A_1} = e^{A_1+A_2}$ iff $A_1A_2 = A_2A_1$. The proof follows from the definition;

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- Finally, $\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$, which again follows directly from the definition.

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- We can double check this

$$\begin{aligned}\frac{dx(t)}{dt} &= Ae^{At}x(0) + \frac{de^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau}{dt} \\ &= Ae^{At}x(0) + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-At}Bu(t) \\ &= Ax(t) + Bu(t).\end{aligned}$$

Example

Back to previous example

Compute the *response* of the system

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = -2x_1 - x_2 + u$$

initial conditions: $x_1(0) = 1$ and $x_2(0) = 2$ and input $u(t) = 10 \cdot \mathbf{1}(t)$.

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$$e^{At} = \begin{bmatrix} e^t & 0 \\ e^{-t} - e^t & e^{-t} \end{bmatrix},$$

- The unforced response is

$$x_u(t) = \begin{bmatrix} e^t & 0 \\ e^{-t} - e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t \\ 3e^{-t} - e^t \end{bmatrix}.$$

Example

- For the forced response, we have to compute

$$\int_0^t e^{A(t-\tau)} B u(\tau) d\tau = \int_0^t e^{A(t-\tau)} d\tau B \mathbf{1}(t),$$

since the input is a step function.

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- For the forced response, we have to compute

$$\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = \int_0^t e^{A(t-\tau)} d\tau B\mathbf{1}(t),$$

since the input is a step function.

- Therefore

$$\int_0^t e^{A(t-\tau)} d\tau B\mathbf{1}(t) = \begin{bmatrix} \int_0^t e^{t-\tau} d\tau & 0 \\ \int_0^t e^{\tau-t} - e^{t-\tau} d\tau & \int_0^t e^{\tau-t} d\tau \end{bmatrix} B\mathbf{1}(t) =$$

$$= \begin{bmatrix} e^t - 1 & 0 \\ 2 - e^t - e^{-t} & 1 - e^{-t} \end{bmatrix} B\mathbf{1}(t),$$

Example

- Observing that $B = [0 \;\; 1]^T$, we have

$$\begin{aligned} x(t) &= x_u(t) + x_f(t) = \\ &= \begin{bmatrix} e^t \\ 3e^{-t} - e^t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - e^{-t} \end{bmatrix} \mathbf{1}(t) = \\ &= \begin{bmatrix} e^t \\ 3e^{-t} - e^t + 10(1 - e^{-t}) \end{bmatrix}. \end{aligned}$$

Remarks

- The state space evolution comprises two terms.
  - a term depending only on the *initial condition* $x(0)$ is dubbed *unforced (free) response*.
  - a second term instead depends on the inputs $u(t)$ but not on the initial condition and therefore it is termed *forced response*.

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- For fixed t, Since e^{At} is a constant. Therefore the states are combined through a linear function.
- There are other alternative definitions of the matrix exponential. For example:

$$e^{At} = \lim_{k \to +\infty} \left(I + \frac{At}{k} \right)^k .$$

Connection with the Laplace Transform

- Consider the dynamic of a *autonomous system*:

$$\dot{x}(t) = Ax(t).$$

By applying the diffentiation rule, the Laplace Transform is:

$$\mathcal{L}\left(\dot{x}(t)\right) = sX(s) - x(0), \text{ and } \mathcal{L}\left(Ax(t)\right) = AX(s),$$

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$$sX(s) - x(0) = AX(s) \Rightarrow (sI - A)X(s) = x(0) \Rightarrow X(s) = (sI - A)^{-1}x(0). \tag{6}$$

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- The matrix $(sI - A)^{-1}$ is called the *resolvent* of A.
- The resolvent is defined for any $s \in \mathbb{C}$ except for the *eigenvalues* of A, which are the points in which $\det(sI - A) = 0$.

Connection with the Laplace Transform

- By applying the inverse Laplace Transform to (6), we have

$$x(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right) x(0),$$

The inverse of the resolvent

The inverse Laplace Transform of the resolvent of A *is* the matrix exponential e^{At} or, equivalently, the transition matrix $\Phi(t)$.

Remark

- An alternative way to show that $\mathcal{L}^{-1}\left((sI - A)^{-1}\right) = e^{At}$ is by considering the power series expansion of the inverse of a generic non-singular matrix $I - M$, i.e.,

$$(I - M)^{-1} = I + M + M^2 + M^3 + \dots,$$

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- In the case of the resolvent:

$$(sI - A)^{-1} = \left[s\left(I - \frac{A}{s}\right)\right]^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \frac{A^3}{s^4} + \dots$$

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- By applying the inverse Lapace Transform and the superimposition principle:

$$\mathcal{L}^{-1}\left((sI - A)^{-1}\right) = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \dots \triangleq e^{At}.$$

Example

Harmonic Oscillator

Consider the following

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x = Ax.$$

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- Since

$$sI - A = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix},$$

- We have that the eigenvalues are $\pm j$ and hence

$$(sI - A)^{-1} = \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}.$$

- The inverse transform produces a rotation matrix:

$$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0),$$

Example

- The same result would resutl from the application of the definition of exponential matrix

$$e^{At} = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!} = \begin{bmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{4!} + \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \dots & 1 - \frac{t^2}{2} + \frac{t^4}{4!} + \dots \end{bmatrix},$$

whose elements are the Taylor expansions of $\cos t$ and $\sin t$ around $t_0 = 0$.

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Double Integrator

Consider the following *double integrator*

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- Since

$$sI - A = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix},$$

- So, by the inverse Laplace Transform we have

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0).$$

Example

- Direct computation of the matrix exponentital

$$e^{At} = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!} = I + At,$$

  - the matrix A is nilpotent, i.e., there exists a number q such that $A^q \neq 0$ and $A^{\bar{q}} = 0 \ \forall \bar{q} > q$.

Transfer Function and State Space

- the output description given in (1), i.e.,

$$y(t) = C\left(e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) + Du(t). \quad (7)$$

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- For a SISO system and assuming that $u(\bar{t}) = 0 \ \forall \bar{t} < 0$, we can write

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$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{At} * Bu(t),$$

- As a consequence,

$$\mathcal{L}\left(e^{At} * Bu(t)\right) = (sI - A)^{-1}BU(s).$$

Transfer Function and State Space

- By applying the superimposition principle, we get

$$\mathcal{L}(y(t)) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}BU(s) + DU(s).$$

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- Of course, if the system starts at rest, as it is usually assumed for the frequency domain approach, we have $x(0) = 0$ and hence

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D.$$

Transfer Function and State Space

- The same would be obtained starting from

$$\begin{aligned} \mathfrak{D}x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{8}$$

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- Applying the Laplace transform of both sides:

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- Applying the Laplace transform of both sides:

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- Solving the equation

$$Y(s) = C(sI - A)^{-1}x(0) + \left[C(sI - A)^{-1}B + D\right] U(s),$$

The role of the eigenvalues in the transfer function

- For a SISO system, we know that for the transfer function

$$\frac{Y(s)}{U(s)} = G(s),$$

  the roots of the denominator (poles) play a prominent role.
  - e.g., the poles determine if a system is BIBO stable
- If we assume $x(0) = 0$, we have

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s),$$

- the direct coupling between input and output (given by D) does not have much to do with the system characteristics.
- we can focus on $G(s) = C(sI - A)^{-1}B$.

The role of the eigenvalues in the transfer function

- (i,j) element of the resolvent $(sI - A)^{-1}$ can be computed using the Cramer's rule
$$(-1)^{i+j}\frac{\det \mathrm{Adj}_{i,j}}{\mathcal{P}(A)},$$
where $\mathrm{Adj}_{i,j}$ is the *adjoint* matrix of $sI - A$, i.e., the matrix $sI - A$ with the j-th row and i-th column deleted.
- $\mathcal{P}(A)$ is instead the *characteristic polynomial* of A, defined as
$$\mathcal{P}(A) = \det(sI - A).$$
- $\mathcal{P}(A)$ satisfies the following properties:
  - $\mathcal{P}(A)$ is a polynomial of degree n, with leading (i.e., s^n) coefficient one;
  - The roots of $\mathcal{P}(A)$ are the eigenvalues of A;
  - $\mathcal{P}(A)$ has real coefficients, so eigenvalues are either real or occur in conjugate pairs;
  - There are n eigenvalues (if we count multiplicity as roots of $\mathcal{P}(A)$).

The role of the eigenvalues in the transfer function

- It follows that $\det \mathrm{Adj}_{i,j}$ has degree less than n.
- For SISO systems all the roots of the denominator of $G(s)$ are also the roots $\mathcal{P}(A)$, and, hence, are the eigenvalues of A.
- The converse is not true, due to possible *cancellation* between the roots of $\mathcal{P}(A)$ and the roots of $\det \mathrm{Adj}_{i,j}$.
- This is a problem that will have a direct impact on the analysis of the structural properties of a system.

The Role of the Eigenvalues for Continuous Time Systems

- If $v \in \mathbb{R}^n$ is an *eigenvector* of a matrix A associated with eigenvalue λ, $(Av = \lambda v)$ then v is also an eigenvector of the matrix A^k with eigenvalue λ^k, i.e., $A^k v = \lambda^k v$.

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- By using $e^{At} = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!}$, we have that v is also the eigenvector of e^{At} associated to the eigenvalue $e^{\lambda t}$, i.e., $e^{At} v = e^{\lambda t} v$;

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- By using $e^{At} = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!}$, we have that v is also the eigenvector of e^{At} associated to the eigenvalue $e^{\lambda t}$, i.e., $e^{At} v = e^{\lambda t} v$;
- For any given A and for any transformation matrix T, we have that $(T^{-1}AT)^k = T^{-1}A^k T$.From $e^{At} = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!}$ we have

$$e^{T^{-1}AT} = T^{-1}e^A T.$$

The Role of the Eigenvalues for Continuous Time Systems

- If $v \in \mathbb{R}^n$ is an *eigenvector* of a matrix A associated with eigenvalue λ, ($Av = \lambda v$) then v is also an eigenvector of the matrix A^k with eigenvalue λ^k, i.e., $A^k v = \lambda^k v$.
- By using $e^{At} = \sum_{i=0}^{+\infty} \frac{A^i t^i}{i!}$, we have that v is also the eigenvector of e^{At} associated to the eigenvalue $e^{\lambda t}$, i.e., $e^{At} v = e^{\lambda t} v$;
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- Bearing these properties in mind, we can now relate the exponential matrix to its eigenvalues.

Exponential Matrix for diagonal Matrices

- Let us first recall the definition of *diagonalisable matrix*.

Exponential Matrix for diagonal Matrices

- Let us first recall the definition of *diagonalisable matrix*.

Diagonalisable Matrices

Definition

A square matrix A is called *diagonalisable* if it is *similar* to a diagonal matrix, i.e., if there exists an invertible matrix T such that $T^{-1}AT$ is a *diagonal matrix*.

Diagonalisable Matrices

Theorem

Theorem

A matrix $A \in \mathbb{R}^{n\times n}$ is diagonalisable if and only if *there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A, i.e., the sum of the dimensions of its* eigenspaces *is equal to n. If such a basis exists, the matrix T such as $T^{-1}AT = \Lambda$ is diagonal has these basis vectors as columns. The Λ diagonal entries of this matrix are the eigenvalues of A.*

Exponential Matrix

- If a matrix is diagonalisable, the exponential matrix can be easily obtained. pause
- Let

$$T^{-1}AT = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

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- By definition

$$e^{\Lambda t} = \sum_{i=0}^{+\infty} \frac{\Lambda^i t^i}{i!} = \begin{bmatrix} \sum_{i=0}^{+\infty} \frac{\lambda_1^i t^i}{i!} & 0 & \dots & 0 \\ 0 & \sum_{i=0}^{+\infty} \frac{\lambda_2^i t^i}{i!} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{i=0}^{+\infty} \frac{\lambda_n^i t^i}{i!} \end{bmatrix},$$

Exponential Matrix

- Equivalently

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix},$$

Exponential Matrix

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- Finally

$$e^{At} = Te^{\Lambda t} T^{-1}.$$

Example

Free evolution

Compute the free evolution of the following system:

$$\dot{x} = Ax$$

$$A = \begin{bmatrix} -0.2000 & 0.3000 \\ 0.3000 & -0.2000 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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- First we compute the characteristic polynomial and the eigenvectors

$$det(\lambda I - A) = \lambda^2 + \frac{2}{5}\lambda - \frac{1}{20}$$

$$\lambda_1 = 0.1$$

$$\lambda_2 = -0.5$$

Example

- The next step is to compute the eigenvectors $Av_i = \lambda_i v_i$

$$\lambda_1 = 0.1 \to v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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- The matrix T and its inverse are given by

$$T = [v_1 v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

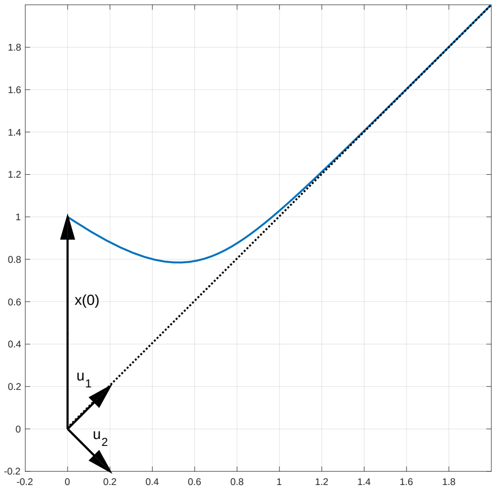
$$T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Example

- Finally

$$
\begin{aligned}
e^{At} &= T\begin{bmatrix} e^{0.1t} & 0 \\ 0 & e^{-0.5t} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{0.1t} & 0 \\ 0 & e^{-0.5t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{0.1t}+e^{-0.5t}}{2} & \frac{e^{0.1t}-e^{-0.5t}}{2} \\ \frac{e^{0.1t}-e^{-0.5t}}{2} & \frac{e^{0.1t}+e^{-0.5t}}{2} \end{bmatrix} \\
e^{At}x(0) &= \begin{bmatrix} \frac{e^{0.1t}-e^{-0.5t}}{2} \\ \frac{e^{0.1t}+e^{-0.5t}}{2} \end{bmatrix}
\end{aligned}
$$

Example - Plot



Observations

- The previous example suggests a few observations:
  - for real eigenvalues the state space trajectory can be studied starting from a coordinate system given by the eigenvectors
  - the component of the state along the eigenvector associated with negative eigenvalues vanish
  - the component associated with positive eigenvalue grows
  - the eigenvector associated with the largest eigenvalue is the asymptote

Exponential Matrix for Diagonal Matrices with Complex Eigenvalues

- If the matrix A has complex eigenvalues
  - the associated eigenvectors will be complex
  - the rows of matrix T will be complex

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- eventually the matrix exponential will be real.
- Therefore it is possible to find a coordinate trasnformation that transforms a diagonalisable matrix on the complex set into a *block diagonal matrix*, whose block dimension is at most 2.
- The diagonal blocks are associated to each complex and conjugated pairs.

Exponential Matrix for Diagonal Matrices with Complex Eigenvalues

- Consider a matrix A having a pairs of eigenvalues $\lambda_{1,2} = \sigma \pm j\omega$, i.e.,

$$A[v_r + jv_c] = (\sigma + j\omega)[v_r + jv_c],$$
$$A[v_r - jv_c] = (\sigma - j\omega)[v_r - jv_c],$$

  with v_r and v_c being the real and complex part of the eigenvectors

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- It then follows

$$e^{At}[v_r + jv_c \,|\, v_r - jv_c] = [v_r + jv_c \,|\, v_r - jv_c]e^{\sigma t}\begin{bmatrix} e^{j\omega t} & 0 \\ 0 & e^{-j\omega t} \end{bmatrix}.$$

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$$e^{At}[v_r + jv_c \,|\, v_r - jv_c] = [v_r + jv_c \,|\, v_r - jv_c] e^{\sigma t} \begin{bmatrix} e^{j\omega t} & 0 \\ 0 & e^{-j\omega t} \end{bmatrix}.$$

- By the Euler formula, we have that

$$e^{\sigma t + j\omega t} = e^{\sigma t}[\cos(\omega t) + j\sin(\omega t)],$$
$$e^{\sigma t - j\omega t} = e^{\sigma t}[\cos(\omega t) - j\sin(\omega t)].$$

Exponential Matrix for Diagonal Matrices with Complex Eigenvalues

- Introduce the invertible matrix

$$H = \frac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}, \quad H^{-1} = -j\begin{bmatrix} j & j \\ -1 & 1 \end{bmatrix},$$

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$$[v_r + jv_c \,|v_r - jv_c]H = [v_r|v_c],$$

- Observation 2:

$$H^{-1}\begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix} H = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix},$$

Exponential Matrix for Diagonal Matrices with Complex Eigenvalues

- Observation 3:

$$H^{-1}\begin{bmatrix} e^{j\omega t} & 0 \\ 0 & e^{-j\omega t} \end{bmatrix} H = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

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- From $AV = V\Lambda$ we have:

$$AVH = V\Lambda H =$$

$$= VHH^{-1}\Lambda H$$

$$A[v_r + jv_c \, | v_r - jv_c]H = A[v_r | v_c] = [v_r | v_c] \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Exponential Matrix for Diagonal Matrices with Complex Eigenvalues

- Therefore

$$e^{At}[v_r+jv_c\,|v_r-jv_c]H = e^{At}[v_r|v_c] = [v_r|v_c]e^{\sigma t}\begin{bmatrix}\cos(\omega t) & \sin(\omega t)\\ -\sin(\omega t) & \cos(\omega t)\end{bmatrix}.$$

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- We can combine a complex conjugate block with a real eigenvalues
- For instance, let $A \in \mathbb{R}^{4\times 4}$ with λ_1 and λ_4 being real and $\lambda_{2,3} = \sigma \pm j\omega$, we have

$$e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) & 0 \\ 0 & -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix}T^{-1}.$$

Example

Pure imaginary eigenvalues

Consider the matrix

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

- This is the real representation of a complex matrix.

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- To see this, let us compute the characteristic polynomial

$$\mathcal{P}(A) = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -\omega \\ \omega & \lambda \end{bmatrix}\right) =$$
$$(\lambda^2 + \omega^2) = (\lambda - j\omega)(\lambda + j\omega) = 0.$$

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- The eigenvector associated to $\lambda_1 = j\omega$ is

$$(\lambda_1 I - A)v_1 = 0 \Rightarrow \begin{bmatrix} j\omega & -\omega \\ \omega & j\omega \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} a \\ ja \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}.$$

Example

- The eigenvector associated to $\lambda_2 = -j\omega$ will be the complex conjugate of v_1, indeed

$$(\lambda_2 I - A)v_2 = 0 \Rightarrow \begin{bmatrix} -j\omega & -\omega \\ \omega & -j\omega \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \begin{bmatrix} a \\ -ja \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}.$$

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- We can now define the transformation matrix

$$T = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \quad T^{-1} = -\frac{1}{2j} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix},$$

and

$$T^{-1}AT = \Lambda = \begin{bmatrix} j\omega & 0 \\ 0 & -j\omega \end{bmatrix},$$

- We can re-write the above as

$$AT = \Lambda T = T\Lambda,$$

with $\Lambda = diag([j\omega, -j\omega])$.

Example

- Let us use the matrix

$$H = \frac{1}{2}\begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}, \; H^{-1} = -j\begin{bmatrix} j & j \\ -1 & 1 \end{bmatrix},$$

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- Therefore

$$AT H = T\Lambda H \Rightarrow A = THH^{-1}\Lambda H \Rightarrow A = H^{-1}\Lambda H = T\Lambda T^{-1},$$

which confirms that A is already in real form.

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$$ATH = T\Lambda H \Rightarrow A = THH^{-1}\Lambda H \Rightarrow A = H^{-1}\Lambda H = T\Lambda T^{-1},$$

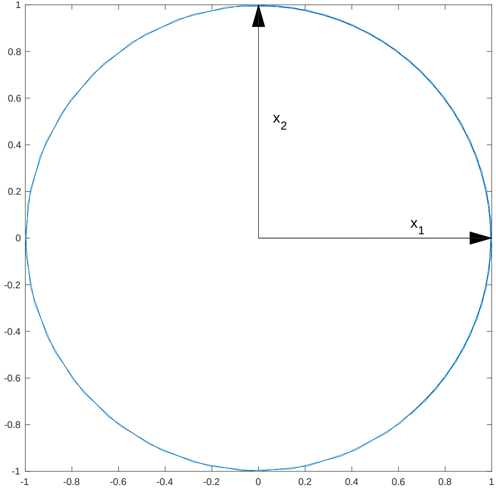
which confirms that A is already in real form.

- As a consequence

$$e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{j\omega t} & 0 \\ 0 & e^{-j\omega t} \end{bmatrix} T^{-1} = \begin{bmatrix} \frac{e^{j\omega t}+e^{-j\omega t}}{2} & \frac{e^{j\omega t}-e^{-j\omega t}}{2j} \\ -\frac{e^{j\omega t}-e^{-j\omega t}}{2j} & \frac{e^{j\omega t}+e^{-j\omega t}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix},$$

Example - Plot



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One real eigenvalue

Let us compute e^{At} for

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- The first step is to compute the eigenvalues:

$$\det(\lambda I - A) = 0 \Rightarrow \lambda(\lambda^2 - 4\lambda + 5) = \lambda[(\lambda - 2)^2 + 1] = 0,$$

which are $\lambda_{1,2} = 2 \pm j$ and $\lambda_3 = 0$

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- Then we compute the eigenvectors

$$(\lambda_1 I - A)v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 2 \\ 2-j \\ 1+j \end{bmatrix}$$

$$v_2 = v_1^* = v_2 = \begin{bmatrix} 2 \\ 2+j \\ 1-j \end{bmatrix}$$

$$(\lambda_3 I - A)v_3 = 0 \Rightarrow v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Example

- We can now define the transformation matrix

$$T = [v_{1,r} | v_{1,c} | v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad T^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \\ 3 & -1 & -1 \end{bmatrix},$$

which produces the following similar matrix for A

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- Hence

$$e^{At} = Te^{\Lambda t}T^{-1} = T\begin{bmatrix} e^{2t}\cos(t) & e^{2t}\sin(t) & 0 \\ -e^{2t}\sin(t) & e^{2t}\cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} T^{-1}.$$

Example

- Which produces:

$$e^{At} = \begin{bmatrix} \frac{e^{2t}(2\cos(t)+\sin(t))+3}{5} & \frac{e^{2t}(\cos(t)-2\sin(t))-1}{5} & \frac{e^{2t}(\cos(t)+3\sin(t))-1}{5} \\ \frac{e^{2t}(3\cos(t)+4\sin(t))-3}{5} & \frac{e^{2t}(4\cos(t)-3\sin(t))+1}{5} & \frac{e^{2t}(-\cos(t)+7\sin(t))+1}{5} \\ \frac{e^{2t}(3\cos(t)-\sin(t))-3}{5} & \frac{e^{2t}(-\cos(t)-3\sin(t))+1}{5} & \frac{e^{2t}(4\cos(t)+2\sin(t))+1}{5} \end{bmatrix}$$

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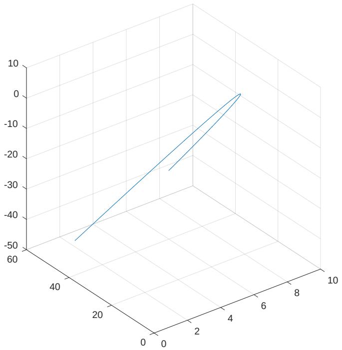

The Jordan Canonical Form for Defective Matrices

- Let us first state the definition of *defective matrix*.

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Defective Matrices

Definition
A square matrix A is called *defective* if it is *not diagonalisable*.

The Jordan Canonical Form for Defective Matrices

A few points

- Recall that a $n \times n$ matrix is diagonalisable iff its associated eigenvectors form a basis for $\mathbb{R}^n$

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- Recall that a $n \times n$ matrix is diagonalisable iff its associated eigenvectors form a basis for $\mathbb{R}^n$
- This is certainly true if the matrix has n *distinct* eigenvalues, i.e.,

$$\mathcal{P}(A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n),$$

since the eigenvectors associated with distinct eignevalues are independent.

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A few points

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  since the eigenvectors associated with distinct eignevalues are independent.
- However, this is only a sufficient condition. It can happen that a matrix has only $h \leq n$ distinct eigenvalues λ_i, $i = 1, \dots, h$:

$$\mathcal{P}(A) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2}\dots(\lambda - \lambda_h)^{r_h},$$

  where r_i is the *algebraic multiplicity* of the eigenvalue λ_i

Geometric multiplictiy

- In such cases it is important to understand the notion of geometric multiplicity

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Geometric Multiplicity

Definition

The *geometric multiplicity* of the eigenvalue λ_i is the number of linearly independent eigenvectors $v_{i,k}$ associated to λ_i.

A lemma

Lemma

Lemma

If the geometric multiplicity, *i.e., the number of linearly independent solutions of*

$$Av_{i,k} = \lambda_i v_{i,k} \Rightarrow (\lambda_i I - A)v_{i,k} = 0,$$

is equal to the algebraic multiplicity r_i, *i.e., there exists*

$$(\lambda_i I - A)v_{i,k} = 0, \;\; \textit{with } k = 1, \dots, r_i,$$

linearly independent solutions, the matrix is still diagonalisable.

General form for diagonlisable matrices

In the conditions described by the above lemma, we have:

$$T = [v_{1,1}|v_{1,2}|\dots|v_{1,r_1}|v_{2,1}|v_{2,2}|\dots|v_{2,r_2}|v_{3,1}|\dots|v_{h,1}|v_{h,2}|\dots|v_{1,r_h}],$$

and

$$\Lambda = T^{-1}AT = \begin{bmatrix} \lambda_1 I_{r_1} & 0 & \dots & 0 \\ 0 & \lambda_2 I_{r_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_h I_{r_h} \end{bmatrix}.$$

Defective Matrices

What if we cannot find a basis of eigenvectors

- It can happen that for some eigenvectors the geometric multiplicity remains below the algebraic multiplicity

Defective Matrices

What if we cannot find a basis of eigenvectors

- It can happen that for some eigenvectors the geometric multiplicity remains below the algebraic multiplicity
- There exists at least one eigenvalue λ_i such that

$$(\lambda_i I - A) v_{i,k} = 0, \text{ with } k = 1, \dots, q_i < r_i, \tag{9}$$

or, equivalently, for which the number of linearly independent eigenvectors is less then r_i.

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  or, equivalently, for which the number of linearly independent eigenvectors is less then r_i.
- In this case the idea of generalised eigenvectors comes to rescue and allows us to form a basis.

Generalised Eigenvectors

- The idea is that for each eigenvector we construct a *chain*
- Let us start from the second element of the chain

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- The obvious generalisation of the concept of eigenvector is the following:

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- Such an eigenvector is simply given by

$$(A - \lambda_i I)v_{i,k}^{(2)} = v_{i,k}^{(1)} = v_{i,k}.$$

- Indeed,

$$(A - \lambda_i I)^2 v_{i,k}^{(2)} = (A - \lambda_i I)v_{i,k}^{(1)} = 0.$$

Independence of the first element of the chain

Lemma

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Proof

Assume by contradiction that $v_{i,k}^{(2)} = \alpha v_{i,k}^{(1)}$. Then

$$(A - \lambda_i I)v_{i,k}^{(2)} = (A - \lambda_i I)\alpha v_{i,k}^{(1)} = 0,$$

which contradicts the hypotheses that

$$(A - \lambda_i I)v_{i,k}^{(2)} = v_{i,k}^{(1)} = v_{i,k}.$$

Longer Chains

- We can generalise the construction to longer chains, each one emanating from an eigenvector.

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- We can generalise the construction to longer chains, each one emanating from an eigenvector.
- The construction is then iterated until $m_{i,k}$, i.e.,

$$
\begin{aligned}
(A - \lambda_i I)v_{i,k}^{(2)} &= v_{i,k}^{(1)} = v_{i,k}, \\
(A - \lambda_i I)v_{i,k}^{(3)} &= v_{i,k}^{(2)}, \\
(A - \lambda_i I)v_{i,k}^{(4)} &= v_{i,k}^{(3)}, \\
&\vdots \\
(A - \lambda_i I)v_{i,k}^{(m_{i,k}+1)} &= v_{i,k}^{(m_{i,k})}.
\end{aligned}
$$

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Lemma

Each element $v_{i,k}^{(i)}$ is independent from the others.

Independence of the generalised eigenvectors

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Proof

This is true for i=2. Assume, by contradiction, that this is true up until h but not for $h+1$. Then $v_{i,k}^{h+1} = \alpha_1 v_{i,k}^{(1)} + \ldots + \alpha_h v_{i,k}^{(h)}$. By definition:

$$\begin{aligned}(A - \lambda_i I)v_{i,k}^{(h+1)} &= v_{i,k}^{(h)} \\ Av_{i,k}^{(h+1)} &= \lambda_i v_{i,k}^{(h+1)} + v_{i,k}^{(h)} \\ &= \alpha_1 \lambda_i v_{i,k}^{(1)} + \alpha_2 \lambda_i v_{i,k}^{(2)} + \ldots + (\lambda_i \alpha_h + 1) v_{i,k}^{(h)}\end{aligned}$$

....

Proof (continued)

By our hypotheses, we can write:

$$Av_{i,k}^{(h+1)} = A\alpha_1 v_{i,k}^{(1)} + \ldots + A\alpha_j v_{i,k}^{(h)}$$
$$= (\alpha_1\lambda_i - \alpha_2)v_{i,k}^{(1)} + (\alpha_2\lambda_i - \alpha_3)v_{i,k}^{(2)} \ldots + (\alpha_{h-1}\lambda_i - \alpha_h)\lambda_i v_{i,k}^{(h-1)} + \lambda_i\alpha_h v_{i,k}^{(h)}$$

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By equating the two terms:

$$\alpha_1\lambda_i v_{i,k}^{(1)} + \alpha_2\lambda_i v_{i,k}^{(2)} + \ldots + (\lambda_i\alpha_h + 1)v_{i,k}^{(h)} =$$
$$(\alpha_1\lambda_i - \alpha_2)v_{i,k}^{(1)} + (\alpha_2\lambda_i - \alpha_3)v_{i,k}^{(2)} \ldots + (\alpha_{h-1}\lambda_i - \alpha_h)\lambda_i v_{i,k}^{(h-1)} + \lambda_i\alpha_h v_{i,k}^{(h)}$$

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Equivalently

$$-\alpha_2 v_{i,k}^{(1)} - \alpha_3\lambda_i v_{i,k}^{(2)} + \ldots + v_{i,k}^{(h)} = 0,$$

for which $v_{i,k}^{(h)}$ would be linearly dependent from $v_{i,k}^{(1)}$, $v_{i,k}^{(2)}$, ..., $v_{i,k}^{(h-1)}$, which contradicts the hypotheses.

An additional result

Lemma

For each eigenvector λ_i and for each eigenvector $v_{i,k}$ let $m_{i,k}$ denote the length of the chain generated from $m_{i,k}$ and r_i be the algebraic multiplicity of λ_i. Then we can show:

$$r_i = \sum_{k=1}^{q_i} m_{i,k},$$

i.e., there exists a set of r_i generalised eigenvectors that are linearly independent.

Yet an additional result

Lemma

The generalised eignevector $v_{i,k}^{(f)}$ of the chain originated from each eigenvector $v_{i,k}$ is independent from the generalised eigenvalues of the chains generated by other eigenvectors (both related to the same and to different eigenvalues).

Jordan Decomposition

- Consider a square matrix $A \in \mathbb{R}^{n \times n}$. The following facts are true:
- there exist a complete basis of $\mathbb{R}^n$ made of generalised eigenvectors

Jordan Decomposition

- Consider a square matrix $A \in \mathbb{R}^{n \times n}$. The following facts are true:
- there exist a complete basis of $\mathbb{R}^n$ made of generalised eigenvectors
- By using the transformation matrix

$$T = [v_{1,1}^{(1)} | v_{1,1}^{(2)} | \dots | v_{1,1}^{(m_{1,1})} | v_{2,1}^{(1)} | v_{2,1}^{(2)} | \dots | v_{2,1}^{(m_{2,1})} | \dots | v_{h,q_h}^{(m_{h,q_h})}],$$

we obtain the following block-diagonal matrix

$$J = T^{-1}AT = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_h \end{bmatrix}$$

Jordan Decomposition

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is dubbed *Jordan Canonical Form*, and each block J_i is the *Jordan block* associated to λ_i and has dimension r_i.

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is dubbed *Jordan Canonical Form*, and each block J_i is the *Jordan block* associated to λ_i and has dimension r_i.

- Every Jordan block is a block-diagonal matrix
$$J_i = \begin{bmatrix} J_{i,1} & 0 & \dots & 0 \\ 0 & J_{i,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{i,q_i} \end{bmatrix},$$
where $J_{i,k}$, $k = 1, \dots, q_i$, is a *Jordan miniblock*.

Jordan Decomposition

- In
$$J_i = \begin{bmatrix} J_{i,1} & 0 & \dots & 0 \\ 0 & J_{i,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{i,q_i} \end{bmatrix},$$
Each miniblock is of the form
$$J_{i,k} = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix},$$

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- Its dimension is given by the number $m_{i,k}$ of linearly independent generalised eigenvectors of the k-th chain.

General Conditions for a matrix to be diagonalisable

Remark

Necessary and sufficient conditions for diagonalisability are:

- There exists n linearly independent eigenvectors;
- The algebraic multiplicity r_i of λ_i equal to the geometric multiplicity q_i;
- The dimension of each Jordan miniblock $J_{i,k}$ is unitary.

Exponential Matrix of Defective Matrices

Exponential of a block diagonal Matrix

- The exponential is given by the block-diagonal of the exponential of each block:

$$\begin{bmatrix} A_1 t & 0 & \dots & 0 \\ 0 & A_2 t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n t \end{bmatrix}^k = \begin{bmatrix} (A_1 t)^k & 0 & \dots & 0 \\ 0 & (A_2 t)^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (A_n t)^k \end{bmatrix},$$

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- ....from which ....

$$e^{\left(\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix} t\right)} = \begin{bmatrix} e^{A_1 t} & 0 & \dots & 0 \\ 0 & e^{A_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{A_n t} \end{bmatrix}.$$

Exponential Matrix of Defective Matrices

- We have seen that any defective matrix can be transformed into Jordan form.

Exponential Matrix of Defective Matrices

- We have seen that any defective matrix can be transformed into Jordan form.
- Therefore we need to compute the matrix exponential of each Jordan block, which in turn requires to compute the matrix exponential of each miniblock.

$$e^{J_{i,k}t} = e^{\left(\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} t\right)} = e^{\Lambda_i t + \bar{J}_{i,k} t} = e^{\Lambda_i t} e^{\bar{J}_{i,k} t},$$

where the last step is because the exponential matrices commute.

Exponential Matrix of A Jordan Miniblock

- Recall

$$e^{J_{i,k}t} = e^{\left(\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix} t\right)} = e^{\Lambda_i t + \bar{J}_{i,k} t} = e^{\Lambda_i t} e^{\bar{J}_{i,k} t},$$

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- $\bar{J}_{i,k}$ is a *nilpotent matrix* of order $p = m_{i,k}$, i.e., $\bar{J}_{i,k}^p = 0$ but $\bar{J}_{i,k}^{\bar{p}} \neq 0 \; \forall p > \bar{p} \geq 0$.

Exponential Matrix of A Jordan Miniblock

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- $\bar{J}_{i,k}$ is a *nilpotent matrix* of order $p = m_{i,k}$, i.e., $\bar{J}_{i,k}^p = 0$ but $\bar{J}_{i,k}^{\bar{p}} \neq 0 \ \forall p > \bar{p} \geq 0$.
- Hence,

$$e^{\bar{J}_{i,k} t} = I + \bar{J}_{i,k} t + \bar{J}_{i,k}^2 \frac{t^2}{2!} + \cdots + \bar{J}_{i,k}^2 \frac{t^{p-1}}{(p-1)!}.$$

Exponential Matrix of A Jordan Miniblock

- Finally

$$e^{J_{i,k}t} = e^{\Lambda_i t} e^{\bar{J}_{i,k}t} = e^{\Lambda_i t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{p-2}}{(p-2)!} & \frac{t^{p-1}}{(p-1)!} \\ 0 & 1 & t & \dots & \frac{t^{p-3}}{(p-3)!} & \frac{t^{p-2}}{(p-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Example

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Compute the matrix exponential of

$$A = \begin{bmatrix} 1.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

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- Characteristic polynomial:

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- Characteristic polynomial:

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  The algebraic multiplicity is 2
- By setting $(I - A)v = 0$ we find only one independent eigenvector $v_{1,1} = [1, -1]^T$

Example

- The second element of the chain is given by:

$$(A - I)v_{1,1}^{(2)} = v_{1,1}^{(1)} = v_{1,1}$$
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- Therefore we have:

$$A = T \begin{bmatrix} 1 & 1 \\ 0 & \end{bmatrix} T^{-1}$$

Example

- Finally:

$$e^{At} = T \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} T^{-1}$$

Example

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Compute the matrix exponential of

$$A = \begin{bmatrix} 2 & -0.5 & 0.5 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

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$$A = \begin{bmatrix} 2 & -0.5 & 0.5 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- Characteristic polynomial:

$$\lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = (\lambda - 1)^3(\lambda - 2)$$

- The algebraic multiplicity is of 1 is 3 and the algebraic multiplicity of 2 is 1

Example

- by setting $(2 * I - A)[v_1 v_2 v_3 v_4]^T = 0$, We find the following equations

$$
\begin{aligned}
0 \cdot v_1 + 0.5 \cdot v_2 - 0.5 \cdot v_3 + 1 \cdot v_4 &= 0 \\
0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 + 1 \cdot v_4 &= 0 \\
0 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3 - 1 \cdot v_4 &= 0 \\
0 \cdot v_1 + 0 \cdot v_2 + 0 v_3 + 1 \cdot v_4 &= 0
\end{aligned}
$$

Which has as a solution $v_2 = v_3 = v_4 = 0$, while v_1 can be chose freely. So we choose as eigenvalue $v_{1,1} = [1\,0\,0\,0]^T$

Example

- by setting $(I - A)[v_1 v_2 v_3 v_4]^T = 0$, We find the following equations

$$
\begin{aligned}
1 \cdot v_1 - 0.5 \cdot v_2 + 0.5 \cdot v_3 - 1 \cdot v_4 &= 0 \\
0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 - 1 \cdot v_4 &= 0 \\
0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 1 \cdot v_4 &= 0 \\
0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4 &= 0
\end{aligned}
$$

- From this we have $v_4 = 0$ and the equation

$$v_1 - 0.5v_2 + 0.5v_3 = 0$$

  , which has two free parameters.

- We can choose $v_1 = 0$ and $v_1 = 1$ obtaining the following two solutions: $v_{2,1} = [0\,1\,1\,0]^T$ and $v_{2,2} = [1\,1\,-1\,0]^T$

Example

- Observe that $(A - I)[v_1\, v_2\, v_3\, v_4]^T = [0\, 1\, 1\, 0]^T$, is impossible because it produces $v_4 = -1$ and $v_4 = 1$.

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- Therefore the chain can be computed starting from $v_{2,2}^{(1)} = v_{2,2}$. It is immediate to see that by choosing $v_{2,2}^{(2)} = [1\,2\,0\;-1]$ we have $(A-I)v_{2,2}^{(2)} = v_{2,2}^{(1)}$

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- Therefore

$$T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Implies

$$A = T \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} T^{-1}$$

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$$A = T \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} T^{-1}$$

Implies

$$e^{At} = T \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 \\ 0 & 0 & e^{t} & te^{t} \\ 0 & 0 & 0 & e^{t} \end{bmatrix} T^{-1}$$

The case of complex eigenvalues

- It can be possible for a Matrix A to have complex eigenvalues associated with generalised eigenvectors

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- In this case it is possible to combine the idea of generalised eigenvectors with the matrix transformation that we have seen for the case of complex eigenvalues
- Suppose $p = 2$ and define:

$$V = [v_r^{(1)} + jv_c^{(1)} \,|\, v_r^{(2)} + jv_c^{(2)} \,|\, v_r^{(1)} - jv_c^{(1)} \,|\, v_r^{(2)} - jv_c^{(2)}],$$

which is the matrix built with generalised eigenvectors

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  which is the matrix built with generalised eigenvectors
- Introduce the invertible matrix

$$H = \frac{1}{2}\begin{bmatrix} 1 & -j & 0 & 0 \\ 0 & 0 & 1 & -j \\ 1 & j & 0 & 0 \\ 0 & 0 & 1 & j \end{bmatrix}, \quad H^{-1} = -j\begin{bmatrix} j & 0 & j & 0 \\ -1 & 0 & 1 & 0 \\ 0 & j & 0 & j \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

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- We have:

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- Moreover, $AV = VJ$ where

$$J = \begin{bmatrix} \sigma + j\omega & 1 & 0 & 0 \\ 0 & \sigma + j\omega & 0 & 0 \\ 0 & 0 & \sigma - j\omega & 1 \\ 0 & 0 & 0 & \sigma - j\omega \end{bmatrix}.$$

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$$VH = [v_r^{(1)}|v_c^{(1)}|v_r^{(2)}|v_c^{(2)}].$$

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$$J = \begin{bmatrix} \sigma + j\omega & 1 & 0 & 0 \\ 0 & \sigma + j\omega & 0 & 0 \\ 0 & 0 & \sigma - j\omega & 1 \\ 0 & 0 & 0 & \sigma - j\omega \end{bmatrix}.$$

- Hence,

$$J_r = H^{-1}JH = \begin{bmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{bmatrix} = \begin{bmatrix} W & I \\ 0 & W \end{bmatrix}.$$

The case of complex eigenvalues

- For the exponential

$$J_r^k = \begin{bmatrix} W^k & kW^{k-1} \\ 0 & W^k \end{bmatrix},$$

hence

$$e^{J_r t} = \begin{bmatrix} I + Wt + W^2 \frac{t^2}{2!} + \dots & 0 + It + 2W \frac{t^2}{2!} + 3W^2 \frac{t^3}{3!} + \dots \\ 0 & I + Wt + W^2 \frac{t^2}{2!} + \dots \end{bmatrix} =$$

$$= \begin{bmatrix} e^{Wt} & te^{Wt} \\ 0 & e^{Wt} \end{bmatrix}.$$

The case of complex eigenvalues

- By recalling that

$$e^{Wt} = e^{\sigma t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix},$$

- we finally have

$$e^{J_r t} = \begin{bmatrix} e^{Wt} & te^{Wt} \\ 0 & e^{Wt} \end{bmatrix} =$$

$$e^{\sigma t}\begin{bmatrix} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} & t\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \\ 0 & \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \end{bmatrix}.$$

The case of complex eigenvalues

- If $p > 2$, we have that following the same steps we can express

$$J_r = H^{-1}JH = \begin{bmatrix} W & I & 0 & \dots & 0 & 0 \\ 0 & W & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & I \\ 0 & 0 & 0 & \dots & 0 & W \end{bmatrix},$$

The case of complex eigenvalues

- If $p > 2$, we have that following the same steps we can express

$$J_r = H^{-1}JH = \begin{bmatrix} W & I & 0 & \dots & 0 & 0 \\ 0 & W & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & W & I \\ 0 & 0 & 0 & \dots & 0 & W \end{bmatrix},$$

- ...which leads us to

$$e^{J_r t} = \begin{bmatrix} e^{Wt} & te^{Wt} & \frac{t^2}{2!}e^{Wt} & \dots & \frac{t^{p-2}}{(p-2)!}e^{Wt} & \frac{t^{p-1}}{(p-1)!}e^{Wt} \\ 0 & e^{Wt} & te^{Wt} & \dots & \frac{t^{p-3}}{(p-3)!}e^{Wt} & \frac{t^{p-2}}{(p-2)!}e^{Wt} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e^{Wt} & te^{Wt} \\ 0 & 0 & 0 & \dots & 0 & e^{Wt} \end{bmatrix}.$$