Teoria dei sistemi.

Introduction

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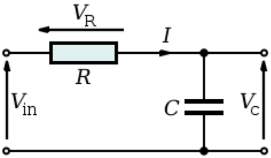
Scope

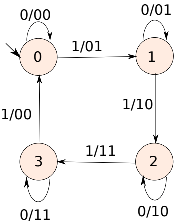
- This course is about *timed systems*
- A system is a physical or artificial entity that *evolves in time*
- The evolution of the system is studied through the relation between the evolution of some quantities
- In some cases we can have "oriented systems" with input and output quantities clearly identified. In others we do not.
- Our framework is *mathematical*, meaning that the evolution of the quantities is described function from "time" (suitably defined) to a "value"
  - Such functions are called *signals*

Example 1



- In the RC circuit shown in the picture relevant quantities are:
  - V_{in}: input voltage
  - V_c: voltage across the capacitor (output voltage)
  - I: current through the resistor
- The notion of time used in this case is physical time

Example 2



- In the modulo 4 counter shown here we have
  - 1 bit input (could be '0', or '1')
  - 2 bit output (the number of '1' counted)
- The notion of time does not necessarily coincide with physical time.
  - We are only interested in which input comes first and which one comes after in the sequence
  - This is an abstract specification (mapping to a synchronous notion of time is only a possible a implementation)

Example 3

- Consider a banking account, in which operations and capitalisation takes place upon the multiple integer of one fiscal year.
- Let T be the number of months corresponding to the time between two operations
- The evolution is given by

$$C((k+1)T) = \begin{cases} \left(1 + I_{+} \cdot \frac{T}{12}\right)\left(C(kT) + S(kT)\right) & \text{Se } C(kT) + S(kT) \geq 0 \\ \left(1 + I_{-} \cdot \frac{T}{12}\right)\left(C(kT) + S(kT)\right) & \text{Se } C(kT) + S(kT) < 0 \end{cases}$$

- In this case we observe the evolution at discrete points in time, but the these instants are linked to physical time.

Signals

- A signal is s a function from a time space $\mathcal{T}$ to a set $\mathbb{U}$.
- At the moment, we make no restriction on the set $\mathbb{U}$ (we will quite soon)
- Calligraphic letter to denote classes of signals. For example,

$$\mathcal{U} = \{u(\cdot) : \mathcal{T} \to \mathbb{U}\}.$$

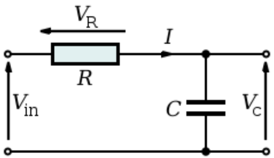
Time

- What it the time set $\mathcal{T}$?
- There us a huge literature on this.
- The three most important notions for systems theory are
  - Continuous time
  - Discrete Event
  - Discrete time

Continuous Time

- Continuous time is used to describe the evolution of physical quantities
- Given two events, it is relevant to know which comes first and which after...
- but it is also relevant to know how much physical time elapses between two events.

Back to the Example



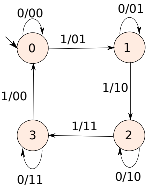
- In the RC circuit shown in the picture relevant quantities are:
  - $V_{in}(t)$: is the signal corresponding to the input
  - $V_c(t)$: is the signal corresponsing to the output
  - I: current through the resistor
- If I switch in a 5V battery (Event 1) and then out (Event 2), it makes a lot of difference if $t_2 - t_1 = 1$ms or $t_2 - t_1 1$s.
- Moreover, I can be interested in knowing $V_c(t)$ at all time (e.g., to understand when a threshold is reached to start a relais).

Continuous Time

- A more formal description of continuous time is as follows:
- The time space $\mathcal{T}$ has to be:
  1. totally ordered,
     - *I can know which event comes first*
  2. metric,
     - *I can measure distances between events and durations*
  3. a continuum (in the mathematical sense).
     - *I can use differential calculus*
- a Typical choice is to choose $\mathcal{T}$ as a subset of $\mathbb{R}$

Discrete Events

- For Discrete Events Signals the connection between the time space $\mathcal{T}$ and the physical time is very shallow
- All we need to know on the timing of the event is condensed in their order.
- **EXAMPLE**: the input sequence 011 produces a sequence of states and of output than the sequence 101
  - the input sequence (e.g., $0 \cdot 1 \cdot 1$) will produce the same output sequence whatever the time interval between the events



Discrete Events

- For discrete time systems the time space $\mathcal{T}$ has to be:
  1. totally ordered,
     - *I can know which event comes first*
  2. such that between any two events we can find a finite number of events
     - *Quite technical to prevent Zeno effects and simplify simulation*
- it is quite natural to use $\mathbb{N}$ as time space $\mathcal{T}$

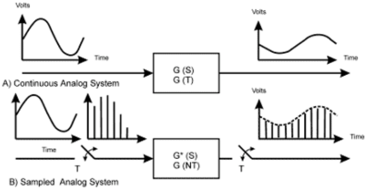
Discrete Time

- Discrete Time signals are a particular class of discrete event signals with synchronous time instants
- Instants are not necessarily synchronised with a periodic time base (although this is the typical choice
- The set has to be
  1. totally ordered,
     - *I can know which event comes first*
  2. An abelian group
     - *We need to compute sums and differences of events*
- Quite natural to choose $\mathcal{Z}$ (a)s time space

Sampled Data

- a different type of systems is compounded of a collection of heterogeneous subsystems, each one associated
- sample data systems, which are DT systems obtained from CT systems restricting the points in time where certain quantities can be measured (sampled) or certain input variables be changed.

Sampled Data



- A physical quantity is sampled with some period

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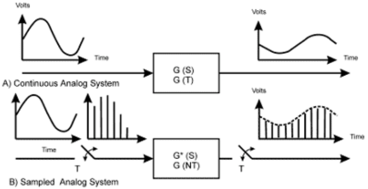

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- The outcome is a sequence of numbers that is a DT signal

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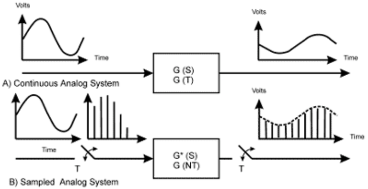

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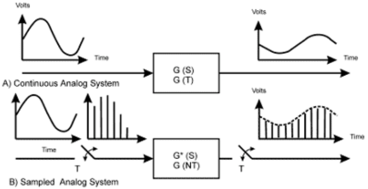

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- The latter signal is converted back into a CT signal through a ZoH DAG converter

Abstract Definition

- In this course we will deal only with DT and CT systems
- However the definitions of systems below are very general
- Let $\mathcal{U}$ represent a class of input signals taking value in the set U
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Definition

A system is a binary relation between $\mathcal{U}$ and $\mathcal{Y}$: $\mathcal{S} \subseteq \mathcal{U} \times \mathcal{Y}$.

Binary Relations

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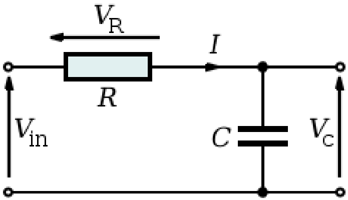
Uniqueness

In defining a system as a relation we allow for multiple output functions associated with the same input.

Example: Electrical Circuit



- Assume that we apply the input

$$V_{in}(t) = V_f \cdot \mathbf{1}(t) = \begin{cases} 0 & \text{if } t < 0 \\ V_f & \text{if t } \geq 0. \end{cases}$$

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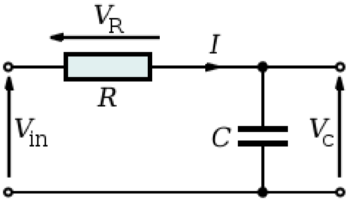

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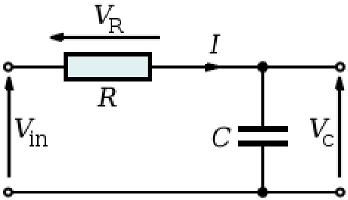

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$$V_C(t) = V_c(0)e^{-t/RC} + (1 - e^{-t/RC})V_f$$

- Depending on the initial charge on the capacitor $V_c(0)$, we will have a different evolution of the output for the same input

Banking Account

- Consider the banking account described by the equation

$$C((k+1)T) = \begin{cases} \left(1 + I_+ \cdot \frac{T}{12}\right)\left(C(kT) + S(kT)\right) & \text{Se } C(kT) + S(kT) \geq 0 \\ \left(1 + I_- \cdot \frac{T}{12}\right)\left(C(kT) + S(kT)\right) & \text{Se } C(kT) + S(kT) < 0 \end{cases}$$

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- Suppose that we deposit a constant amount of money S at every period, starting from a capital C_0.
- The evolution of the capital is given by

$$c(kT) = (1 + I_+ \cdot \frac{T}{12})^k C_0 + S\frac{(1 + I_+ \cdot \frac{T}{12})^{k+1} - 1}{I_+ \cdot \frac{T}{12}}.$$

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- The evolution will be much different depending on the initial capital C_0

Accessory Definitions

- For an instant t_0, denote by $\mathcal{T}(t_0)$ the subset of the time space
$$\mathcal{T}(t_0) = \{t \in \mathcal{T} : t \geq t_0\}\,.$$
- Denote by $\mathcal{W}^{T(t_0)}$ the set of functions from $\mathcal{T}(t_0)$ to W:
$$\mathcal{W}^{\mathcal{T}(t_0)} = \{w_0(\cdot) : \forall t \geq t_0, t \to w_0(t) \in W\}\,.$$
- By $w0|_{T(t_1)}$ we will denote the truncation of the function from $t_1 > t_0$ onward.

Formal Definition of Systems

Definition

An abstract dynamic system is a 3-tuple $\{\mathcal{T}, \mathcal{U} \times \mathcal{Y}, \Sigma\}$ where

- $\mathcal{T}$ is the time space
- $\mathcal{U}$ is the set of input functions
- $\mathcal{Y}$ is the set of output functions

and

$$\Sigma = \left\{ \Sigma(t_0) \subset \mathcal{U}^{T(t_0)} \times \mathcal{Y}^{T(t_0)} : t_0 \in \mathcal{T} \text{and CRT is satisfied} \right\},$$

where CRT stands for closure with respect to truncation: i.e., $\forall t_1 \geq t_0$

$$(u_0, y_0) \in \Sigma(t_0) \implies \left(u_0|_{\mathcal{T}(t_1)}, y_0|_{\mathcal{T}(t_1)} \right) \in \Sigma(t_1).$$

Parametric Representation of Binary Relations

In order to study Abstract Systems, we can apply the following general result

Lemma

Given a binary relation R, it is possible to define a set P and a function $\pi : P \times D(R) \to R(R)$ such that

$$(a, b) \in R \implies \exists p : b = \pi(p, a) \tag{1}$$

$$p \in P, a \in D(R) \implies (a, \pi(p, a)) \in R \tag{2}$$

π is said *parametric representation* and (P, π) is said *parametrisation* of the relation.

Parametric Representation of Abstract Systems

The Lemma cited above leads us to the following:

Theorem

Consider a system defined as above. It is possible to identify a parametrisation (X_{t_0}, π) such that

$$\pi = \{\pi_{t_0} : X_{t_0} \times D(\Sigma(t_0)) \to R(\Sigma(t_0))/t_0 \in \mathcal{T}\} \tag{3}$$

satisfying the following properties:

$$(u_0, y_0) \in \Sigma(t_0) \implies \exists x_0 : y_0 = \pi_{t_0}(x_0, y_0) \tag{4}$$

$$x_0 \in X_{t_0}, u_0 \in D(\Sigma(t_0)) \implies (u_0, \pi_{t_0}(x_0, u_0)) \in \Sigma(t_0). \tag{5}$$

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$$x_0 \in X_{t_0}, u_0 \in D(\Sigma(t_0)) \implies (u_0, \pi_{t_0}(x_0, u_0)) \in \Sigma(t_0). \tag{5}$$

Meaning of the parameter

By using this result we can remove the ambiguity. Given input + parameter we identify a single output.

Example

It is quite intuitive that initial conditions are a perfect means to define a parametric definition.

Electrical circuit

The initial charge in the capacitor (or equivalently the initial voltage) is a possible parameter and the function π_{t_0} for the step input function is

$$V_C(t) = V_c(t_0)e^{-(t-t_0)/RC} + (1 - e^{-(t-t_0)/RC})V_f$$

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Banking Account

The initial capital can be used as parameter. the π_{t_0} for the step input function is

$$c(kT) = (1 + I_+ \cdot \frac{T}{12})^{(k-k_0)}C_0 + S\frac{(1 + I_+ \cdot \frac{T}{12})^{k+1-k_0} - 1}{I_+ \cdot \frac{T}{12}}.$$

Definition of Causality

Definition

Let $u|_{[t_0,\bar{t}]}$ be the restriction of the function u to the closed interval $[t_0,\ \bar{t}]$. A system is causal if it has a representation (X_{t_0}, π) such that

$$\forall t_0 \in \mathcal{T}, \forall x_0 \in X_{t_0}, \forall \bar{t} \in \mathcal{T} \tag{6}$$

$$u_{[t_0,\ \bar{t}]} = u'_{[t_0,\ \bar{t}]} \implies [\pi_{t_0}(x_0, u)](\bar{t}) = [\pi_{t_0}(x_0, u')](\bar{t}). \tag{7}$$

A parametric representation of this type is said causal. If instead of the closed interval $[t_0,\ \bar{t}]$ we use the semi-open interval $[t_0,\ \bar{t})$, the parametric representation and the system is said strictly causal.

Definition of Causality

Remark

- π is a functional
- So $\pi_{t_0}(x_0, u)$ is the output function associated to the parameter x_0 and to the function u
  - $[\pi_{t_0}(x_0, u)](\bar{t})$ is the value it takes at time $\bar{t}$.
- Therefore for a causale system the values of u beyond $\bar{t}$ do not affect the value of the output at time $\bar{t}$.
- A simple way to put it is that a causal system does not foresee the future.
- If the system is strictly the output at time $\bar{t}$ is only affected by the input at time *strictly* smaller than $\bar{t}$.

Classification based on the number input/output signals number

- So far no specific assumptions on the range U of the input functions $\mathcal{U}$ and on the range Y of the output functions $\mathcal{Y}$.
- In some cases such quantitities can be scalar, in other they can be vecotrs.
- This gives rise to the following taxonomy
  1. Single Input Single Output (SISO): bot input and output are scalars
  2. Multiple Input Single Output (MISO): $\mathcal{U}$ is a vector, $\mathcal{Y}$ is a scalar
  3. Single Input Multiple Output (SIMO): $\mathcal{U}$ is a scalar, $\mathcal{Y}$ is a vector
  4. Multiple Input Multiple Output (MIMO): both $\mathcal{U}$ and $\mathcal{Y}$ are vectors