Theory of Linear Systems
Exercises

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CHAPTER 1

Exercises on the Laplace Transform

Exercise 1

Consider the following system expressed with the differential equation

$$\ddot{y}(t) = -2\dot{y}(t) - 3y(t) - u(t) + 4\dot{u}(t),$$

determine if it is BIBO stable.

Solution

In order to determine the BIBO stability, we first compute the Laplace transform of the impulsive response imposing null initial conditions, i.e.,

$$s^2Y(s) = -2sY(s) - 3Y(s) - U(s) + 4sU(s),$$

that yields to

$$G(s) = \frac{4s-1}{s^2+2s+3}.$$

Since the roots of the denominator are $p_{12} = -1 \pm j\sqrt{2}$, it follows that the system is BIBO stable. This fact can be proved by assuming $u(t) = \mathbf{1}(t)$ and computing the forced response, i.e.,

$$Y(s) = \frac{4s-1}{s(s+1+j\sqrt{2})(s+1-j\sqrt{2})} = \frac{A_1}{s} + \frac{A_2}{s+1+j\sqrt{2}} + \frac{A_3}{s+1-j\sqrt{2}},$$

where

$$A_1 = -\frac{1}{3},\ \ A_2 = \frac{2+j13\sqrt{2}}{12},\ \ A_3 = \frac{2-j13\sqrt{2}}{12},$$

where $A_3 = A_2^*$ (the complex and conjugated). It then follows

$$\mathcal{L}^{-1}\left(Y(s)\right) = \left(-\frac{1}{3} + \frac{2+j13\sqrt{2}}{12}e^{(-1-j\sqrt{2})t} + \frac{2-j13\sqrt{2}}{12}e^{(-1+j\sqrt{2})t}\right)\mathbf{1}(t).$$

By rewriting

$$\frac{2+j13\sqrt{2}}{12} = Me^{j\phi},\ \text{ with }\ M = \frac{\sqrt{4+2\cdot 13^2}}{12},\ \ \phi = \arctan\left(\frac{13\sqrt{2}}{2}\right),$$

one has

$$\mathcal{L}^{-1}\left(Y(s)\right) = \left(-\frac{1}{3} + Me^{-t}e^{-j(\sqrt{2}t-\phi)} + Me^{-t}e^{j(\sqrt{2}t-\phi)}\right)\mathbf{1}(t) = \left(-\frac{1}{3} + 2Me^{-t}\cos(\sqrt{2}t-\phi)\right)\mathbf{1}(t),$$

which is of course bounded.

Exercise 2

Consider the following system expressed with the differential equation

$$\ddot{y}(t) = -2\dot{y}(t) - 3y(t) - u(t) + 4\dot{u}(t),$$

determine its unforced response given $\dot{y}(0) = 1$ and $y(0) = -1$.

Solution

To compute the unforced we need first to compute the Laplace transform of the system, that is

$$s^2Y(s) - sy(0) - \dot{y}(0) = -2sY(s) + 2y(0) - 3Y(s) - U(s) + 4sU(s) - 4u(0),$$

that yields to

$$Y(s) = \frac{4s-1}{s^2+2s+3}U(s) - \frac{4}{s^2+2s+3}u(0) + \frac{(s+2)y(0)+\dot{y}(0)}{s^2+2s+3}.$$

Since we are interested in the unforced response, we have

$$Y(s) = \frac{(s+2)y(0)+\dot{y}(0)}{s^2+2s+3} = \frac{-(s+2)+1}{s^2+2s+3} = \frac{A_1}{s+1+j\sqrt{2}} + \frac{A_2}{s+1-j\sqrt{2}},$$

where

$$A_1 = -\frac{1}{2}, \;\; A_2 = -\frac{1}{2},$$

where again $A_1 = A_2^*$. It then follows that

$$\mathcal{L}^{-1}\left(Y(s)\right) = -\frac{1}{2}\left(e^{(-1-j\sqrt{2})t} + e^{(-1+j\sqrt{2})t}\right)\mathbf{1}(t).$$

Hence

$$\mathcal{L}^{-1}\left(Y(s)\right) = -\frac{1}{2}e^{-t}\left(e^{-j\sqrt{2}t} + e^{j\sqrt{2}t}\right)\mathbf{1}(t) = -e^{-t}\cos(\sqrt{2}t)\mathbf{1}(t).$$

Exercise 3

Consider the following system expressed with the differential equation

$$\ddot{y}(t) = -\frac{5}{3}\dot{y}(t) - \frac{1}{4}y(t) - \frac{1}{3}u(t) + \frac{4}{3}\dot{u}(t),$$

determine its response to the input $u(t) = k\mathbf{1}(t)$ given $\dot{y}(0) = 0$, $y(0) = 1$ and $u(0) = 1$.

Solution

To compute the response we need first to compute the Laplace transform of the system, that is

$$3s^2Y(s) - 3sy(0) - 3\dot{y}(0) = -5sY(s) + 5y(0) - \frac{3}{4}Y(s) - U(s) + 4sU(s) - 4u(0),$$

that yields to

$$Y(s) = \frac{4s-1}{3s^2+5s+\frac{3}{4}}U(s) - \frac{4}{3s^2+5s+\frac{3}{4}}u(0) + \frac{(3s+5)y(0)+3\dot{y}(0)}{3s^2+5s+\frac{3}{4}}$$

that turns to

$$Y(s) = \frac{k(4s-1)}{s(3s^2+5s+\frac{3}{4})} - \frac{4}{3s^2+5s+\frac{3}{4}} + \frac{3s+5}{3s^2+5s+\frac{3}{4}},$$

i.e., the superposition of three different components. Therefore:

- For the first, we have

$$Y_1(s) = \frac{k(4s-1)}{s(3s^2+5s+\frac{3}{4})} = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{6}} + \frac{A_3}{s+\frac{3}{2}},$$

  where

$$A_1 = -k\frac{4}{3}, \;\; A_2 = k\frac{5}{2}, \;\; A_3 = -k\frac{7}{6}.$$

  It then follows

$$\mathcal{L}^{-1}(Y_1(s)) = k\left(-\frac{4}{3} + \frac{5}{2}e^{-\frac{1}{6}t} - \frac{7}{6}e^{-\frac{3}{2}t}\right)\mathbf{1}(t).$$

- For the second term, related to the initial condition of the input, we have

$$Y_2(s) = \frac{-4}{3s^2+5s+\frac{3}{4}} = \frac{A_4}{s+\frac{1}{6}} + \frac{A_5}{s+\frac{3}{2}},$$

  where

$$A_4 = -1, \;\; A_5 = 1.$$

  It then follows

$$\mathcal{L}^{-1}(Y_2(s)) = \left(-e^{-\frac{1}{6}t} + e^{-\frac{3}{2}t}\right)\mathbf{1}(t).$$

- For the third term, related to the initial condition of the output, we have
$$Y_3(s) = \frac{3s+5}{3s^2+5s+\frac{3}{4}} = \frac{A_6}{s+\frac{1}{6}} + \frac{A_7}{s+\frac{3}{2}},$$
where
$$A_6 = \frac{9}{8}, \;\; A_7 = -\frac{1}{8}.$$
It then follows
$$\mathcal{L}^{-1}\left(Y_3(s)\right) = \left(\frac{9}{8}e^{-\frac{1}{6}t} - \frac{1}{8}e^{-\frac{3}{2}t}\right)\mathbf{1}(t).$$

Therefore, we finally have by the superposition principle:
$$y(t) = \left[-\frac{4k}{3} + \frac{20k+1}{8}e^{-\frac{1}{6}t} - \frac{28k-21}{24}e^{-\frac{3}{2}t}\right]\mathbf{1}(t).$$

Exercise 4

Consider the following system

$$G(s) = \frac{3s+7}{s^2-2s-3}$$

and determine if it is BIBO unstable.

Solution

Since the coefficients of the polynomial at the denominator changes in sign, the system is BIBO unstable since it has one root with positive real part.

Exercise 5

Consider the following Laplace transform of two transfer functions

$$G_1(s) = \frac{3s-7}{s^2+2s+3}$$

$$G_2(s) = \frac{s+1}{s(s^2+2s+3)}$$

determine if they are BIBO stable.

Solution

Both of them are not unstable. However, the first one has two roots with negative real part (persistency of the sings of the polynomial), while the second has the same two roots *plus* a pole in the origin. Therefore, only the first one is BIBO stable. To verify this fact, it is sufficient to apply a unitary step as input and then compute the inverse Laplace transform of the output.

Exercise 6

Given a system whose transfer function is

$$G(s) = \frac{s-5}{(s+1)^2(s+2)}$$

determine the time evolution of the output when the input is $u(t) = e^{-3t} + \mathbf{1}(t-2)$.

Solution

One way to takle this problem is to first compute the Laplace transform of the input, which is given by

$$U(s) = \frac{1}{s+3} + \frac{e^{-2s}}{s},$$

and then apply the inverse Laplace transform to the output

$$Y(s) = \frac{s-5}{(s+1)^2(s+2)(s+3)} + \frac{(s-5)e^{-2s}}{s(s+1)^2(s+2)}.$$

Another, quite similar approach, would be to compute the inverse Laplace transform of the two output functions as they are applied at time $t = 0$, i.e.,

$$y_1(t) = \mathcal{L}^{-1}\left(\frac{s-5}{(s+1)^2(s+2)(s+3)}\right),$$
$$y_2(t) = \mathcal{L}^{-1}\left(\frac{s-5}{s(s+1)^2(s+2)}\right),$$

and then define

$$y(t) = y_1(t) + y_2(t-2).$$

CHAPTER 2

Exercises on the $\mathcal{Z}$-Transform

Exercise 1

Given the following $\mathcal{Z}$-Transform of an input sequence

$$U(z) = 2 + 3z^{-2} - z^{-5} + z^{-7},$$

find the corresponding signal representation

Solution

Straightforwardly, we apply the inverse $\mathcal{Z}$-Transform of the Kronecker delta function to find:

$$u(t) = 2\delta(t) + 3\delta(t-2) - \delta(t-5) + \delta(t-7).$$

Exercise 2

Given two sequences

$$u(t) = \mathbf{1}(t) - \mathbf{1}(t-4) \text{ and } h(t) = \mathbf{1}(t) - \mathbf{1}(t-5),$$

find $y(t) = h(t) * u(t)$.

Solution

We first recall that

$$y(t) = h(t) * u(t) = \mathcal{Z}^{-1}\left(H(z)U(z)\right).$$

Then, we noticed that

$$u(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3)$$

that yields

$$U(z) = 1 + z^{-1} + z^{-2} + z^{-3}.$$

Similarly

$$h(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \delta(t-4)$$
$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}.$$

As a consequence:

$$Y(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 4z^{-4} + 3z^{-5} + 2z^{-6} + z^{-7},$$

and then

$$y(t) = \delta(t) + 2\delta(t-1) + 3\delta(t-2) + 4\delta(t-3) + 4\delta(t-4) + 3\delta(t-5) + 2\delta(t-6) + \delta(t-7).$$

Exercise 3

Find the inverse $\mathcal{Z}$-Transform with ROC $|z| > 1$ for the following transfer function

$$G(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}.$$

Furthermore, determine if the system is BIBO stable.

Solution

We first rewrite $G(z)$ in the more suitable form

$$G(z) = \frac{z^2 + 2z + 1}{z^2 - \frac{3}{2}z + \frac{1}{2}}.$$

Being the roots of the denominator equal to $z_1 = \frac{1}{2}$ and $z_2 = 1$, we have

$$\frac{G(z)}{z} = \frac{A_1}{z} + \frac{A_2}{z - \frac{1}{2}} + \frac{A_3}{z - 1},$$

where

$$A_1 = 2, \;\; A_2 = -9, \;\; A_3 = 8.$$

It then follows that

$$\mathcal{Z}^{-1}\left(G(z)\right) = \mathcal{Z}^{-1}\left(2\right) - 9\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) + 8\mathcal{Z}^{-1}\left(\frac{z}{z - 1}\right)$$

and finally

$$\mathcal{Z}^{-1}\left(G(z)\right) = 2\delta(t) + \left[8 - 9\left(\frac{1}{2}\right)^t\right]\mathbf{1}(t).$$

Obviously, the system is BIBO unstable since $|z_2| = 1$. To further prove it, we compute the output of the system when the input $u(t)$ is a unitary step, i.e.,

$$Y(z) = G(z)U(z) = \frac{z^2 + 2z + 1}{z^2 - \frac{3}{2}z + \frac{1}{2}}\frac{z}{z - 1},$$

which yields to

$$\frac{Y(z)}{z} = \frac{A_2}{z - \frac{1}{2}} + \frac{A_3}{z - 1} + \frac{A_4}{(z - 1)^2},$$

with

$$\mathcal{Z}^{-1}\left(Y(z)\right) = 9\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 8\mathcal{Z}^{-1}\left(\frac{z}{z - 1}\right) + 8\mathcal{Z}^{-1}\left(\frac{z}{(z - 1)^2}\right)$$

and then

$$y(t) = \left[8t - 8 + 9\left(\frac{1}{2}\right)^t\right]\mathbf{1}(t).$$

Exercise 4

Consider the following difference equation representing the output $y(t)$ of a system subjected to the input $u(t)$

$$y(t+2) = 3u(t+2) - 2u(t+1) + u(t) + \frac{5}{6}y(t+1) - \frac{1}{6}y(t)$$

compute the impulse response of the system for null initial conditions.

Solution

Since the initial conditions are null, we have that $y(0) = y(1) = u(0) = u(1) = 0$. Therefore the $\mathcal{Z}$-Transform is given by

$$z^2Y(z) = 3z^2U(z) - 2zU(z) + U(z) + \frac{5}{6}zY(z) - \frac{1}{6}Y(z),$$

which yields to

$$Y(z) = \frac{3z^2 - 2z + 1}{z^2 - \frac{5}{6}z + \frac{1}{6}}U(z) = \frac{3z^2 - 2z + 1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}U(z).$$

The impulse response is then given by

$$H(z) = \frac{3z^2 - 2z + 1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}.$$

Hence,

$$\mathcal{Z}^{-1}\left(\frac{H(z)}{z}\right) = \frac{A_1}{z} + \frac{A_2}{z - \frac{1}{2}} + \frac{A_3}{z - \frac{1}{3}},$$

where

$$A_1 = 6, \quad A_2 = 9, \quad A_3 = -12.$$

It then follows that

$$\mathcal{Z}^{-1}\left(H(z)\right) = \mathcal{Z}^{-1}\left(6\right) + 9\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 12\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{3}}\right)$$

and finally

$$\mathcal{Z}^{-1}\left(H(z)\right) = 6\delta(t) + \left[9\left(\frac{1}{2}\right)^t - 12\left(\frac{1}{3}\right)^t\right]\mathbf{1}(t).$$

Comparison

In order to prove the correctness of the analysis, we can compute the time evolution of the output $y(t)$ when the input is a Kronecker delta $u(t) = \delta(t)$ using the difference equation and then compare it with the time evolution $\mathcal{Z}^{-1}\left(H(z)\right)$ computed for every t, as reported in Table 1. As shown in the Table, the correctness of the analysis carried out is further verified.

t	$y(t)$	$\mathcal{Z}^{-1}\left(H(z)\right)\big\|_t$
0	$3\delta(t) = 3$	$6\delta(t) + [9 - 12] = 3$
1	$-2\delta(t) + \frac{5}{6}y(0) = \frac{1}{2}$	$\left[\frac{9}{2} - \frac{12}{3}\right] = \frac{1}{2}$
2	$\delta(t) + \frac{5}{6}y(1) - \frac{1}{6}y(0) = \frac{11}{12}$	$\frac{9}{4} - \frac{12}{9} = \frac{11}{12}$
3	$\frac{5}{6}y(2) - \frac{1}{6}y(1) = \frac{49}{72}$	$\frac{9}{8} - \frac{12}{27} = \frac{49}{72}$
$\vdots$	$\vdots$	$\vdots$

TABLE 1. Comparison of the two impulse responses.

Exercise 5

Consider the following difference equation representing the output $y(t)$ of a system subjected to the input $u(t)$

$$y(t) = 3u(t) - 2u(t-1) + u(t-2) + \frac{5}{6}y(t-1) - \frac{1}{6}y(t-2)$$

compute the forced response of the system for

$$u(t) = \left(\frac{1}{4}\right)^t \mathbf{1}(t)$$

Solution

The forced response is computed for null initial conditions, i.e., $y(-1) = y(-2) = 0$. Therefore the $\mathcal{Z}$-Transform is given by

$$Y(z) = 3U(z) - 2z^{-1}U(z) + z^{-2}U(z) + \frac{5}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}Y(z),$$

which yields to

$$Y(z) = \frac{3z^2 - 2z + 1}{z^2 - \frac{5}{6}z + \frac{1}{6}}U(z) = \frac{z(3z^2 - 2z + 1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}.$$

Hence,

$$\mathcal{Z}^{-1}\left(\frac{Y(z)}{z}\right) = \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z - \frac{1}{3}} + \frac{A_3}{z - \frac{1}{4}},$$

where

$$A_1 = 18, \quad A_2 = -48, \quad A_3 = 33.$$

It then follows that

$$\mathcal{Z}^{-1}\left(Y(z)\right) = 18\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 48\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{3}}\right) + 33\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{4}}\right)$$

and finally

$$\mathcal{Z}^{-1}\left(Y(z)\right) = \left[18\left(\frac{1}{2}\right)^t - 48\left(\frac{1}{3}\right)^t + 33\left(\frac{1}{4}\right)^t\right]\mathbf{1}(t).$$

Comparison

In order to prove the correctness of the analysis, we can compute the time evolution of the output $y(t)$ when the input is $u(t)$ using the difference equation and then compare it with the time evolution $\mathcal{Z}^{-1}\left(Y(z)\right)$ computed for every t, as reported in Table 2. As shown in the Table, the correctness of the analysis carried out is further verified.

t	$y(t)$	$\mathcal{Z}^{-1}\left(H(z)\right)\big\vert_t$
0	3	$[18 - 48 + 33] = 3$
1	$\frac{3}{4} - 2 + \frac{5}{6}y(0) = \frac{5}{4}$	$\left[\frac{18}{2} - \frac{48}{3} + \frac{33}{4}\right] = \frac{5}{4}$
2	$\frac{3}{16} - \frac{1}{2} + 1 + \frac{5}{6}y(1) - \frac{1}{6}y(0) = \frac{59}{48}$	$\frac{9}{4} - \frac{48}{9} + \frac{33}{16} = \frac{59}{48}$
$\vdots$	$\vdots$	$\vdots$

TABLE 2. Comparison of the two vanishing step responses.

Exercise 6

Consider the following difference equation representing the output $y(t)$ of a system subjected to the input $u(t)$

$$y(t+2) = 3u(t+2) - 2u(t+1) + u(t) + \frac{5}{6}y(t+1) - \frac{1}{6}y(t)$$

compute the unforced response of the system for $y(1) = 0$ and $y(0) = -1$.

Solution

The $\mathcal{Z}$-Transform is given by

$$z^2Y(z) - z^2y(0) - zy(1) = 3z^2U(z) - 2z^1U(z) + U(z) + \frac{5}{6}zY(z) - \frac{5}{6}zy(0) - \frac{1}{6}z^{-2}Y(z),$$

which yields to

$$Y(z) = \frac{(3z^2 - 2z + 1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}U(z) + \frac{z(z - \frac{5}{6})y(0) + zy(1)}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}.$$

Hence $(U(z) = 0)$,

$$Y(z) = \frac{-z(z - \frac{5}{6})}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

and, therefore

$$\mathcal{Z}^{-1}\left(\frac{Y(z)}{z}\right) = \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z - \frac{1}{3}},$$

where

$$A_1 = 2, \;\; A_2 = -3.$$

It then follows that

$$\mathcal{Z}^{-1}\left(Y(z)\right) = 2\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{2}}\right) - 3\mathcal{Z}^{-1}\left(\frac{z}{z - \frac{1}{3}}\right)$$

and finally

$$\mathcal{Z}^{-1}\left(Y(z)\right) = \left[2\left(\frac{1}{2}\right)^t - 3\left(\frac{1}{3}\right)^t\right]\mathbf{1}(t).$$

Comparison

In order to prove the correctness of the analysis, we can compute the time evolution of the output $y(t)$ when the input is $u(t)$ using the difference equation and then compare it with the time evolution $\mathcal{Z}^{-1}\left(Y(z)\right)$ computed for every t, as reported in Table 3. As shown in the Table, the correctness of the analysis carried out is further verified.

t	$y(t)$	$\mathcal{Z}^{-1}\left(H(z)\right)\big\vert_t$
0	-1	$[2-3]=-1$
1	0	$[1-1]=0$
2	$\frac{5}{6}y(1)-\frac{1}{6}y(0)=\frac{1}{6}$	$\frac{2}{4}-\frac{3}{9}=\frac{1}{6}$
3	$\frac{5}{6}y(2)-\frac{1}{6}y(1)=\frac{5}{36}$	$\frac{2}{8}-\frac{3}{27}=\frac{5}{36}$
$\vdots$	$\vdots$	$\vdots$

TABLE 3. Comparison of the responses for the selected initial conditions.

CHAPTER 3

Exercises on the State Space Representation

Exercise 1
Given a continuous time linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) = Ax(t) + Bu(t),$$

compute the unforced response for a generic $x(0)$ and verify if the system is BIBO stable.

Solution
The unforced response is given by

$$x_u(t) = e^{At}x(0) = Te^{Jt}T^{-1}x(0),$$

where T is the transformation matrix that transforms the dynamic matrix A in the Jordan form J. Therefore, T is a base of generalised eigenvectors of the matrix A.

However, it is evident that the matrix A has two eigenvalues (i.e., it is upper triangular), that are $\lambda_1 = -1$ and $\lambda_2 = 0$. It is then clear that

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The transformation matrix $T = [v_1, v_2]$, where $v_i = [v_{i,1}, v_{i,2}]^T$, is then obtained by solving the eigenvector problem, i.e.,

$$Av_i = \lambda_i v_i \Rightarrow \begin{array}{l} \begin{cases} -v_{11} + v_{12} = -v_{11} \\ \qquad\qquad 0 = -v_{12} \end{cases} \\ \begin{cases} -v_{21} + v_{22} = 0 \\ \qquad\qquad 0 = 0 \end{cases} \end{array}$$

For the first eigenvector v_1, we have

$$v_1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix},$$

while for the second v_2, we have

$$v_2 = \begin{bmatrix} \beta \\ \beta \end{bmatrix},$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, and $\alpha \neq 0$ and $\beta \neq 0$. Therefore, a possible choice for T is

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

It then follows that for a generic $x(0) = [x_1(0), x_2(0)]^T$, we have

$$x_u(t) = Te^{Jt}T^{-1}x(0) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_2(0) + e^{-t}(x_1(0) - x_2(0)) \\ x_2(0) \end{bmatrix}.$$

For what concerns the BIBO stability, it is immediate to notice that the system has an eigenvalue in zero, so it is marginally stable. Therefore, it has a real pole in the origin, that corresponds to an integrator. It finally yields that the system is not BIBO stable.

Exercise 2

Given a continuous time linear system in Jordan form whose unforced response for the initial condition $x(0) = [-1, 1, 0, 3]^T$ is

$$x_u(t) = e^{Jt}x(0) = \begin{bmatrix} -\cos(t) + \sin(t) \\ \cos(t) + \sin(t) \\ 3te^{-2t} \\ 3e^{-2t} \end{bmatrix},$$

determine the explicit expression of the matrix J.

Solution

We first recall that a matrix in Jordan form is block diagonal. We then recall that in order to have sinusoidal functions in the expression of $x_u(t)$, at least two complex and conjugated eigenvalues are needed. Since the state has dimension 4 and the sinusoidal functions comes without the presence of the time t, we conclude that we have only two complex and conjugated eigenvalues that has geometric and algebraic multiplicity equals to one (i.e., eigenvalues $\lambda_1 = \sigma + j\omega$ and $\lambda_2 = \sigma - j\omega$). Hence, the submatrix associated to such eigenvalues is diagonalisable. On the contrary, the other two eigenvalues are coincident, with geometric multiplicity equals to one (since we have the explicit presence of the time t that multiplies the exponential function).

To summarise, we have two Jordan blocks, one associated to a pair of complex and conjugated eigenvalues, the other with only one mini-block (hence associated to a chain of generalised eigenvalues whose length is two). It then follows that

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 \\ 0 & e^{J_2 t} \end{bmatrix} = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) & 0 & 0 \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & te^{\lambda_3 t} \\ 0 & 0 & 0 & e^{\lambda_3 t} \end{bmatrix}.$$

Therefore, by recalling the initial condition $x(0)$ and the expression of $x_u(t)$, it follows that $\sigma = 0$, $\omega = 1$ and $\lambda_3 = -2$ and, then

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Exercise 3

Given the following continuous time linear invariant system

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) = Ax(t) + Bu(t),$$

compute the response of the system assuming an initial condition $x(0) = [0, 1]^T$ and an input $u(t) = \alpha \mathbf{1}(t - t_0)$, where $t_0 > 0$.

Solution

To solve this problem we have to recall the explicit form of the response of a linear continuous time system, i.e.,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Due to the particular form of the input, we have

$$x(t) = e^{At}x(0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{At}x(0) + \alpha \int_{t_0}^t e^{A(t-\tau)}d\tau B,$$

where the second equation is obtained by noticing that both B and $u(t)$ are constant for $t \geq t_0$. We then recall that the exponential of the matrix can be readily obtained using the Jordan form, i.e.,

$$\begin{aligned} x(t) &= Te^{Jt}T^{-1}x(0) + \alpha \int_{t_0}^t Te^{J(t-\tau)}T^{-1}d\tau B = \\ &= Te^{Jt}T^{-1}x(0) + \alpha T \int_{t_0}^t e^{J(t-\tau)}d\tau T^{-1}B \end{aligned}.$$

It is then evident that it is necessary to compute the Jordan form J of the matrix A to solve the described problem. To this end, let us first compute the eigenvalues, which are

$$\det(\lambda I - A) = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1,$$

so an eigenvalue with algebraic multiplicity equals to 2. Let us check the geometric multiplicity by tackling the eigenvector problem

$$Av_i = \lambda v_i \Rightarrow \begin{cases} v_{i1} - v_{i2} = -v_{i1} \\ 4v_{i1} - 3v_{i2} = -v_{i2} \end{cases}$$

which yields to only one constraint $v_{i2} = 2v_{i1}$, hence the equation is solved by eigenvectors linearly dependent with $v = [1, 2]^T$. Therefore, the geometric multiplicity is one, so the matrix is defective. It follows that the second generalised eigenvector will solve

$$(A - \lambda I)v^{(2)} = v^{(1)} = v \Rightarrow Av^{(2)} = -\lambda v^{(2)} + v^{(1)} \Rightarrow \begin{cases} v_1^{(2)} - v_2^{(2)} = -v_1^{(2)} + 1 \\ 4v_1^{(2)} - 3v_2^{(2)} = -v_2^{(2)} + 2 \end{cases}$$

which yields to $v_2^{(2)} = 2v_1^{(2)} - 1$, therefore a solution like $v^{(2)} = [1, 1]^T$ is linear independent to $v^{(1)} = [1, 2]^T$. We finally have that

$$T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix},$$

with

$$J = T^{-1}AT = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix},$$

that is one Jordan block with an associated mini block.

Finally, the unforced response is given by

$$x_u(t) = Te^{Jt}T^{-1}x(0) = \begin{bmatrix} -te^{-t} \\ e^{-t} - 2te^{-t} \end{bmatrix},$$

while the forced response is

$$x_f(t) = \alpha T \int_{t_0}^{t} e^{J(t-\tau)} d\tau T^{-1}B = \alpha T \begin{bmatrix} \int_{t_0}^{t} e^{\tau - t} d\tau & \int_{t_0}^{t} (t-\tau) e^{\tau - t} d\tau \\ 0 & \int_{t_0}^{t} e^{\tau - t} d\tau \end{bmatrix} T^{-1}B =$$

$$= \alpha T \begin{bmatrix} 1 - e^{t_0 - t} & (t_0 - 1)e^{t_0 - t} - t(e^{t_0 - t} - 1) - t + 1 \\ 0 & 1 - e^{t_0 - t} \end{bmatrix} T^{-1}B =$$

$$= \begin{bmatrix} -\alpha(t + e^{t_0 - t} + t(e^{t_0 - t} - 1) - e^{t_0 - t}(t_0 - 1) - 2) \\ -\alpha(2t + e^{t_0 - t} + 2t(e^{t_0 - t} - 1) - 2e^{t_0 - t}(t_0 - 1) - 3) \end{bmatrix}$$

Finally, we have

$$x(t) = x_u(t) + x_f(t).$$

Exercise 4

Given a discrete time linear system

$$x(k+1) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -3 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) = Ax(k) + Bu(k),$$

compute the unforced response for a generic $x(0)$ and the steady state value (if any).

Solution

The unforced response is given by

$$x_u(k) = A^k x(0) = TJ^kT^{-1}x(0),$$

where T is the transformation matrix that transforms the dynamic matrix A in the Jordan form J. Therefore, T is a base of generalised eigenvectors of the matrix A.

However, it is evident that the matrix A has two eigenvalues (i.e., it is lower triangular), that are $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = 1$. It is then clear that

$$J = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

The transformation matrix $T = [v_1, v_2]$, where $v_i = [v_{i,1}, v_{i,2}]^T$, is then obtained by solving the eigenvector problem, i.e.,

$$Av_i = \lambda_i v_i \Rightarrow \begin{array}{l} \begin{cases} -\frac{1}{2}v_{11} = -\frac{1}{2}v_{11} \\ -3v_{11} + v_{12} = -\frac{1}{2}v_{12} \end{cases} \\ \begin{cases} -\frac{1}{2}v_{21} = v_{21} \\ -3v_{21} + v_{22} = v_{22} \end{cases} \end{array}$$

For the first eigenvector v_1, we have

$$v_1 = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix},$$

while for the second v_2, we have

$$v_2 = \begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, and $\alpha \neq 0$ and $\beta \neq 0$. Therefore, a possible choice for T is

$$T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

It then follows that for a generic $x(0) = [x_1(0), x_2(0)]^T$, we have

$$x_u(k) = TJ^kT^{-1}x(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}^k x_1(0) \\ (-\frac{1}{2}^{k-1} - 2)x_1(0) + x_2(0) \end{bmatrix}.$$

For the steady state value, we have just to compute

$$\lim_{k \to +\infty} x_u(k) = \begin{bmatrix} 0 \\ -2x_1(0) + x_2(0) \end{bmatrix}.$$

Exercise 5

Given a discrete time linear system in Jordan form whose unforced response for the initial condition $x(0) = [-1, 1, 0, 3]^T$ is

$$x_u(k) = J^k x(0) = \begin{bmatrix} -\frac{1}{2}^k \\ 2^k \cos(k\frac{\pi}{2}) \\ -2^k \sin(k\frac{\pi}{2}) \\ -3\frac{1}{4}^k \end{bmatrix},$$

determine the explicit expression of the matrix J.

Solution

We first recall that a matrix in Jordan form is block diagonal. We then recall that in order to have sinusoidal functions in the expression of $x_u(k)$, at least two complex and conjugated eigenvalues are needed. Since the state has dimension 4 and the sinusoidal functions comes without a multiplication by the time k, we conclude that we have only two complex and conjugated eigenvalues that has geometric and algebraic multiplicity equals to one (i.e., eigenvalues $\lambda_2 = \sigma + j\omega$ and $\lambda_3 = \sigma - j\omega$). Hence, the sub-matrix associated to such eigenvalues is diagonalisable. By the particular form of the other two entries, we conclude that there are two additional and distinct real eigenvalues (λ_1 and λ_4).

To summarise, we have three Jordan blocks, one associated to a pair of complex and conjugated eigenvalues, and the other two associated with two distinct mini-blocks (in other words, the matrix is diagonalisable). It then follows that

$$J^k = \begin{bmatrix} J_1^k & 0 & 0 \\ 0 & J_2^k & 0 \\ 0 & 0 & J_3^k \end{bmatrix} = \begin{bmatrix} \lambda_1^k & 0 & 0 & 0 \\ 0 & \rho^k \cos(k\theta) & \rho^k \sin(k\theta) & 0 \\ 0 & -\rho^k \sin(k\theta) & \rho^k \cos(k\theta) & 0 \\ 0 & 0 & 0 & \lambda_4^k \end{bmatrix},$$

where $\rho = |\lambda_2| = \sqrt{\sigma^2 + \omega^2}$ and $\theta = \arctan(\frac{\mathbf{Imag}(\lambda_2)}{\mathbf{Real}(\lambda_2)}) = \arctan(\frac{\omega}{\sigma})$.

Therefore, by recalling the initial condition $x(0)$ and the expression of $x_u(k)$, it follows that $\sigma = 0$, $\omega = 2$, $\lambda_1 = \frac{1}{2}$ and $\lambda_4 = -\frac{1}{4}$, then

$$J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 2\cos(\frac{\pi}{2}) & 2\sin(\frac{\pi}{2}) & 0 \\ 0 & -2\sin(\frac{\pi}{2}) & 2\cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix}.$$

Exercise 6

Given the following discrete time linear invariant system

$$x(k+1) = \begin{bmatrix} 3 & -1 & -2 \\ 4 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) = Ax(k) + Bu(k),$$

compute the response of the system assuming an initial condition $x(0) = [1, 2, -1]^T$ and an input $u(k) = \alpha \mathbf{1}(k - k_0)$, with $k_0 \in \mathbb{Z}$ and $k_0 > 0$.

Solution

To solve this problem we have to recall the explicit form of the response of a linear discrete time system, i.e.,

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i).$$

Due to the particular form of the input, we have

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i) = A^k x(0) + \alpha \sum_{i=k_0}^{k-1} A^{k-i-1} B,$$

for $k > k_0$ and

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i) = A^k x(0),$$

for $k \leq k_0$. We then recall that the power of the matrix can be readily obtained using the Jordan form, i.e.,

$$\begin{aligned} x(k) &= TJ^kT^{-1}x(0) + \alpha \sum_{i=k_0}^{k-1} TJ^{k-i-1}T^{-1}B = \\ &= Te^{Jt}T^{-1}x(0) + \alpha T \left(\sum_{i=k_0}^{k-1} J^{k-i-1} \right) T^{-1}B \end{aligned}.$$

It is then evident that it is necessary to compute the Jordan form J of the matrix A to solve the described problem. To this end, let us first compute the eigenvalues, which are

$$\det(\lambda I - A) = 0 \Rightarrow \lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda = 1,$$

so an eigenvalue with algebraic multiplicity equal to 3. In other words, the Jordan form will have only one Jordan block. Let us check the geometric multiplicity by tackling the eigenvector problem

$$Av_i = \lambda v_i \Rightarrow \begin{cases} 3v_{i1} - v_{i2} - 2v_{i3} = v_{i1} \\ 4v_{i1} - v_{i2} - 4v_{i3} = v_{i2} \\ v_{i3} = v_{i3} \end{cases}$$

which yields to only one constraint $v_{i2} = 2v_{i1} - 2v_{i3}$, hence the equation is solved by 2 linearly independent eigenvectors, for example

$$v_1 = \begin{bmatrix} \frac{\eta}{2} \\ \eta \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} \beta \\ 0 \\ \beta \end{bmatrix}, \tag{3.1}$$

or the more straightforward

$$v_1 = \begin{bmatrix} \eta \\ 2\eta \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ -2\beta \\ \beta \end{bmatrix}. \tag{3.2}$$

In both cases, $\eta, \beta \in \mathbb{R}$ and $\eta \neq 0$ and $\beta \neq 0$. It is then obvious that the geometric multiplicity is 2, which is less than 3, hence we have two Jordan mini-blocks of dimensions 2 and 1, respectively, associated to the eigenvalue λ. We can then readily define the Jordan form J of the matrix A as

$$J = T^{-1}AT = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In order to fully characterise the Jordan form we need also to define the transformation matrix T, which is formed by a basis of generalised eigenvectors. To this end, let us first define $v_1^{(1)} = v_1$ and $v_2^{(1)} = v_2$, making the choice reported in (3.2). Then,

$$(A - \lambda I)v_1^{(2)} = v_1^{(1)} \Rightarrow \begin{cases} 2v_{11}^{(2)} - v_{12}^{(2)} - 2v_{13}^{(2)} = \eta \\ 4v_{11}^{(2)} - 2v_{12}^{(2)} - 4v_{13}^{(2)} = 2\eta \\ 0 = 0 \end{cases}$$

which yields to $v_{12}^{(2)} = 2v_{11}^{(2)} - 2v_{13}^{(2)} - \eta$. Following the same steps of the eigenvector, in this particular case we have two straightforward solutions

$$v^\star = \begin{bmatrix} \gamma \\ 2\gamma - \eta \\ 0 \end{bmatrix} \text{ and } v^\dagger = \begin{bmatrix} 0 \\ -2\delta \\ \delta \end{bmatrix},$$

or

$$v^\star = \begin{bmatrix} \gamma \\ 2\gamma \\ 0 \end{bmatrix} \text{ and } v^\dagger = \begin{bmatrix} 0 \\ -2\delta - \eta \\ \delta \end{bmatrix},$$

where $\gamma, \delta \in \mathbb{R}$ and $\gamma \neq 0$ and $\delta \neq 0$.

For the first choice, $v^\dagger$ is linearly dependent to $v_2^{(1)}$, hence $v_1^{(2)} = v^\star$. For the second choice, $v^\star$ is linearly dependent to $v_1^{(1)}$, hence $v_1^{(2)} = v^\dagger$. Notice that if one tries to continue the chain by computing $v_1^{(3)}$, it will obtain from the first choice $\eta = 0$, which violates the hypothesis, and from the second choice $\delta = 0$, which again violates the hypothesis. So the chain comprises only $v_1^{(1)}$ and $v_1^{(2)}$ (se the mini-block of dimension 2).

Finally, by computing $v_2^{(2)}$ starting from $v_2^{(1)}$, one immediately has that $\beta = 0$, which violates the hypothesis. It then follows that the matrix $T = [v_1^{(1)}, v_1^{(2)}, v_2^{(1)}]$ has two possible values

$$T^\star = \begin{bmatrix} \eta & \gamma & 0 \\ 2\eta & 2\gamma - \eta & -2\beta \\ 0 & 0 & \beta \end{bmatrix} \text{ and } T^\dagger = \begin{bmatrix} \eta & 0 & 0 \\ 2\eta & -2\delta - \eta & -2\beta \\ 0 & \delta & \beta \end{bmatrix},$$

that, by simply choosing $\eta = \beta = \gamma = \delta = 1$, yields to

$$T^\star = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T^{\star -1} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$T^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } T^{\dagger -1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -2 \\ -2 & 1 & 3 \end{bmatrix}.$$

Whatever is the choice made (i.e., $T = T^\star$ or $T = T^\dagger$), we have

$$J = T^{-1}AT = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow J^k = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

that is one Jordan block with two mini blocks. Notice that the same results would be obtained if the choice of (3.1) would be selected, instead (prove it on your own).

Finally, the unforced response is given by

$$x_u(k) = TJ^kT^{-1}x(0) = \begin{bmatrix} 1 + 2k \\ 2 + 4k \\ -1 \end{bmatrix},$$

while the forced response is

$$\begin{aligned} x_f(k) &= \alpha T \left(\sum_{i=k_0}^{k-1} J^{k-i-1} \right) T^{-1}B = \\ &= \alpha T \begin{bmatrix} \sum_{k_0}^{k-1} \lambda^{k-i-1} & \sum_{k_0}^{k-1} (k-i-1)\lambda^{k-i-2} & 0 \\ 0 & \sum_{k_0}^{k-1} \lambda^{k-i-1} & 0 \\ 0 & 0 & \sum_{k_0}^{k-1} \lambda^{k-i-1} \end{bmatrix} T^{-1}B = \\ &= \begin{bmatrix} -2\alpha k \\ -2\alpha k \\ \alpha \end{bmatrix}. \end{aligned}$$

Finally, we have

$$x(t) = x_u(t) + x_f(t).$$