Teoria dei sistemi.

Stability

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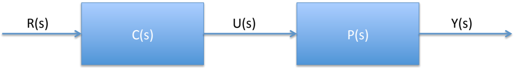
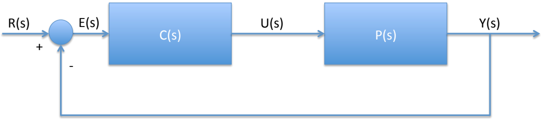
Use of Laplace Transform for Control Design

- The Laplace transform is a very useful tool for designing systems that control other systems.
- This is done taking advantage of some properties of the connections of systems.
- The two simplest ways of connecting two systems are shown below



(A)



(B)

Series Connection

- For series connection $y(t) = p(t) * c(t) * r(t)$.
- In the Laplace domain this property can be translated as:

$$Y(s) = P(s)C(s)R(s),$$

because convolutions correspond to products in the Laplace domain.

- This connection can be used in different ways
- We see this through an example

Example

First order system

Consider a system

$$P(s) = \frac{1}{s+5}.$$

Suppose we want it to track with 0 error constant reference signals. Formally:

$$r(t) = A\mathbf{1}(t) \to \lim_{t\to\infty} y(t) = A.$$

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If we make a series connection with a controller $C(s)$:

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By the final value theorem

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sP(s)C(s)\frac{A}{s} \\ &= AP(0)C(0).\end{aligned}$$

Example

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- Hence, $C(0) = \frac{1}{P(0)} = 5$.
- We can choose $C(s) = 5$.
- Clearly, if the pole is known with some error Δ the steady state error will be affected by an error of the same amount.
- What if we also require a response as fast as e^{-10t}.
- One possibility is to choose the controller

$$C(s) = 10\frac{s+5}{s+10}.$$

Notice that gain has been chosen equal to 10 in order to have $P(0)C(0) = 1$.

Another Example

Pole/zero cancellation

Pole zero cancellation can change the dynamics but is not always safe.

Unstable systems

Consider the system

$$P(s) = \frac{1}{s-5}.$$

Suppose we want it to track with 0 error constant the reference signal is $r(t) = A\mathbf{1}(t)$.

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- Suppose we make a small error in estimating the pole

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  with $\Delta << 1$.
- The overall response to $U(s) = \frac{1}{s}$ is given by:

$$P(s)C(s)R(s) = p\frac{s-5}{s(s+p)(s-5+\Delta)} = \frac{5}{5-\Delta}\frac{1}{s} + \\ - \frac{p+5}{(p+5-\Delta)}\frac{1}{s+p} - p\frac{-\Delta}{(5-\Delta)(5-\Delta+p)}\frac{1}{s-5+\Delta}$$

Another Example

- Although very small,

$$H\frac{1}{s-5+\Delta},$$

with $H = -p\frac{-\Delta}{(5-\Delta)(5-\Delta+p)}$ is not null.

- Therefore even for a very small perturbation we will have an unstable exponential that sooner or later will make the system diverge.

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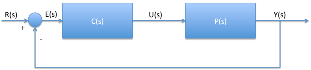
- This example teaches us that stabilising a system trhough a serier connection and cancellation of the unstable pole is not a good idea.
- BTW, why is it possible to turn the system into a wire by simply setting $C(s) = 1/P(s)$ if $P(s)$ is stable?
  - No, because the controller should be non causal (degree of denominator greater than degree of numerator)

Feedback connection

(A)



(B)

- A more robust solution is given by the feedback connection for which we have

$$\begin{aligned} Y(s) &= P(s)U(s) \\ U(s) &= C(s)E(s) \\ E(s) &= R(s) - Y(s). \end{aligned}$$

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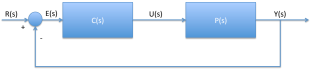

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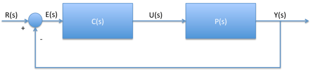

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Feedback connection

Closed Loop

Definition

The equation

$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} R(s).$$

is called closed–loop equation and

$$\frac{P(s)C(s)}{1 + P(s)C(s)}$$

is the closed–loop transfer function.

Steady state error

- If we want to enforce a 0 steady state error to $U(s) = \frac{A}{s}$ through the final value theorem:

$$\begin{aligned}
\lim_{t \to \infty} y(t) &= \lim s \to 0 sY(s) \\
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- If $\lim_{s \to 0} P(s)C(s) = \infty$ then

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no matter how imprecise the knowledge of $P(0)$ is.

- *Either* $P(0) = \frac{1}{s}P_1(s)$ *or* $C(0) = \frac{1}{s}C_1(s) \to$ error equal to zero.

Internal principle model

Internal model

If we want to use a feedback connection to follow a step function with steady state error 0, an integrator $\frac{1}{s}$ has to be found in $P(s)$ or in $C(s)$

This is a particular instance of a more general principle called "internal model principle".

Stabilisation

- The feedback connection allows us to shift the poles (rather than cancelling them).
- This is much more reliable to achieve stability in the presence of noise or uncertainties.
- As usual we see this throguh an example

Example

First order system with uncertainties in the pole and in the gain

Consider the system

$$P(s) = \frac{10 + \Delta_0}{s - 5 - \Delta_1}$$

where Δ_0 and Δ_1 are bounded but unknown. Suppose we choose the feedback controller:

$$C(s) = \frac{10(s+5)}{s}.$$

Let us see whether or not it achieves stability and zero steady state error

Example

- Because $C(s)$ has a pole in $s = 0$ the requirement on zero steady state error is met as long as the system is stable.
- The closed loop response is given by

$$
\begin{aligned}
\frac{P(s)C(s)}{1+P(s)C(s)} &= \frac{\frac{10+\Delta_0}{s-5-\Delta_1}\frac{10(s+5)}{s}}{1+\frac{10+\Delta_0}{s-5-\Delta_1}\frac{10(s+5)}{s}} \\
&= \frac{(10+\Delta_0)(10(s+5))}{(s-5-\Delta_1)s+10(s+5)(10+\Delta_0)} \\
&= \frac{(10+\Delta_0)(10(s+5))}{s^2+(10\Delta_0-\Delta_1+95)s+500+50\Delta_0}
\end{aligned}
$$

Example

- By Routh-Hurwitz criterion, we know that all coefficients have to be positive:

$$\Delta_0 > -10$$
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$$\Delta_0 > -10$$
$$\Delta_1 < 10\Delta_0 + 95.$$

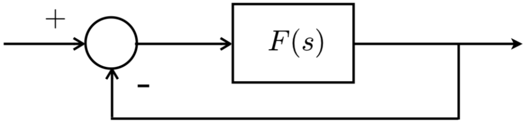
- As long as these conditions are met the system is both BIBO stable and responds with null error to a step input.
- We emphasise that if we use a series connection scheme, even a small Δ_1 destroys stability and $\Delta_0 \neq 0$ determines errors in he steady state behaviour.

Introduction to the Nyquist Criterion

- The use of the Nyquist plot lends itself to an important application.
- the situation we consider is the one in the Figure
- $L(s)$ is the open loop transfer function and it includes plant and controller:

$$F(s) = P(s)C(s)$$

- Our objective is to infer the closed loop stability property starting fromm the open loop properties of $W(s)$ and from its Nyquist plot.



A first fact

Closed Loop

Proposition

The closed loop system has poles with $\mathbf{Real}\,(\cdot) = 0$ *if and only if the Nyquist plot of* $F(j\omega)$ *passes through the critical point* $(0, -1)$

A second (more important) fact

Closed Loop

Proposition

Let us assume that the system has no open loop pole on the imaginary axis (i.e., with $\mathbf{Real}(\cdot) = 0$*). Define:*

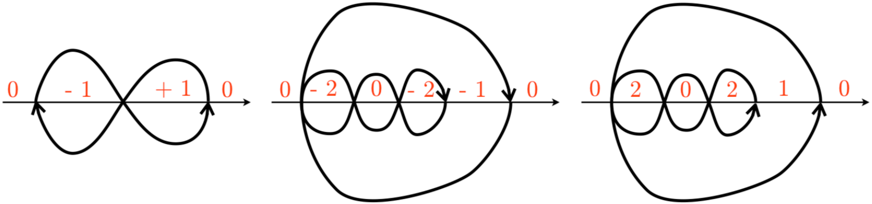
- n_F^+ *the number of open loop poles with positive real part*
- n_W^+ *the number of closed loop polse with positive real part*
- N_c *the number of times the Nyquist plot of* $F(j\omega)$ *encircles the critical point* $(-1, 0)$ *counter clock wise.*

We have:

$$N_c = n_F^+ - n_W^+.$$

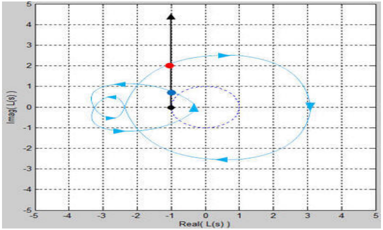
Encirclements

Number of encirclement depending on the position of the critical point.



Encirclements

There is an easy way to count the encirclements. An easier way to determine the number of encirclements of the $-1 + j0$ point is to simply draw a line out from the point, in any directions.

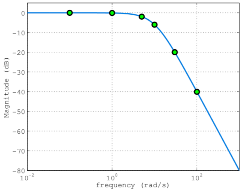


If you count the number of times that the Nyquist path crosses the line in the clockwise direction (i.e., left to right in the image, and denoted by a red circle) and subtract the number of times it crosses in the counterclockwise direction (the blue dot), you get the number of clockwise encirclements of the $-1 + j0$ point. A negative number indicates counterclockwise encirclements.

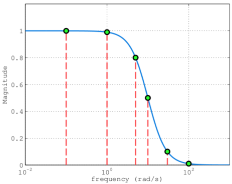
Example - 1

Let us consider the function

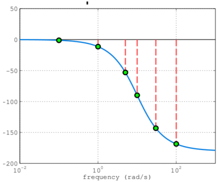
$$F(s) = \frac{100}{(s+10)^2} = \frac{1}{(1+s/10)^2}$$



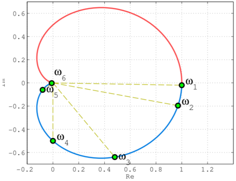
Magnitude (dB)



Magnitude (Linear)



Phase



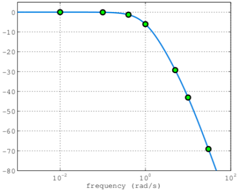
Nyquist

$$N_c = 0 = n_F^+ - n_W^+ \rightarrow n_W^+ = 0$$

Example - 2

Let us consider the function

$$F(s) = \frac{10}{(s+10)(s+1)^2} = \frac{1}{(s/10+1)(1+s)^2}$$



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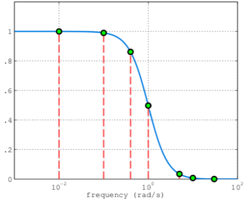

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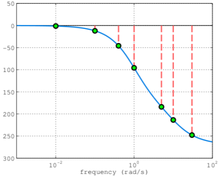

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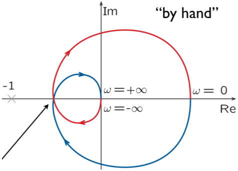


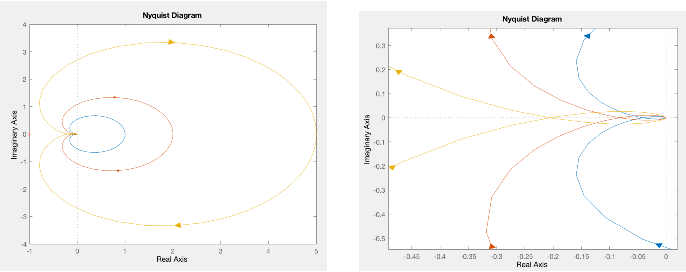
Nyquist

$$N_c = 0 = n_F^+ - n_W^+ \rightarrow n_W^+ = 0$$

Example - 2 (cont)

Suppose that we have a gain k in the loop:

$$F(s) = \frac{10k}{(s+10)(s+1)^2} = \frac{k}{(s/10+1)(1+s)^2}$$



Changing the gain the plot expands and finally encircles the critical point two times (corresponding to two unstable poles).

Example - 2 (cont)

If we compute the closed loop response of

$$W(s) = \frac{F(s)}{1 + F(s)}$$

the denominator becomes $s^3 + 12s^2 + 21s + 10k + 10$
The Routh table is

$$\begin{array}{c|ccc} s^3 & 1 & 21 & 0 \\ s^2 & 12 & 10(k+1) & 0 \\ s & \frac{12 \cdot 21 - 10(k+1)}{12} & 0 & 0 \\ s^0 & 10(k+1) & 0 & \end{array}$$

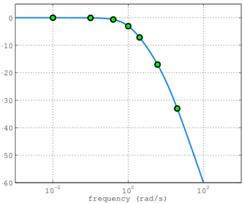
from which we find that the system is stable for

$$24.2 > k > -1$$

Example - 3

Let us consider the function

$$F(s) = \frac{-10(s-1)}{(s+10)(s+1)^2} = \frac{1-s}{(s/10+1)(1+s)^2}$$



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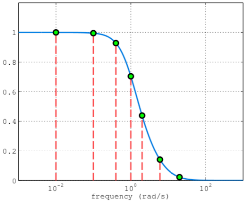

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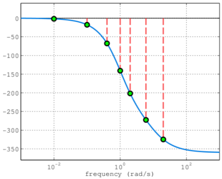

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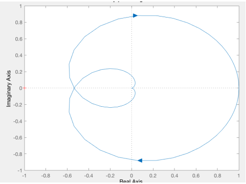


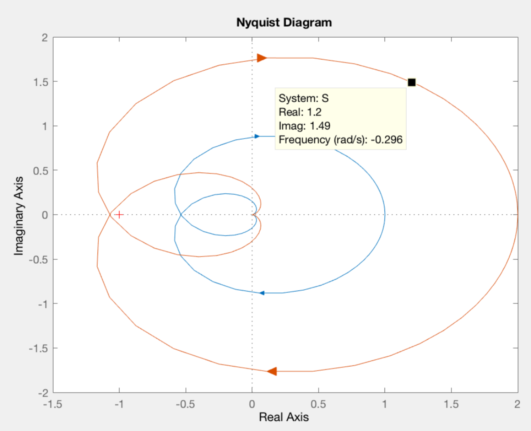
Nyquist

$$N_c = 0 = n_F^+ - n_W^+ \rightarrow n_W^+ = 0$$

Example - 3 (cont)

Suppose that we have a gain k in the loop:

$$F(s) = \frac{k(1-s)}{(s/10+1)(s+1)^2}$$

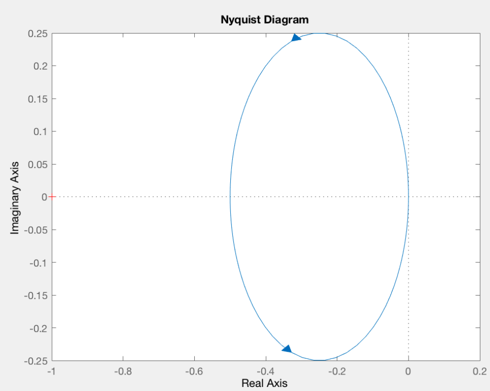


Changing the gain the plot expands and finally encircles the critical point two times (corresponding to two unstable poles).

Example - 4

Let us consider the function

$$F(s) = \frac{0.5}{s-1} = \frac{-0.5}{1-s} =$$

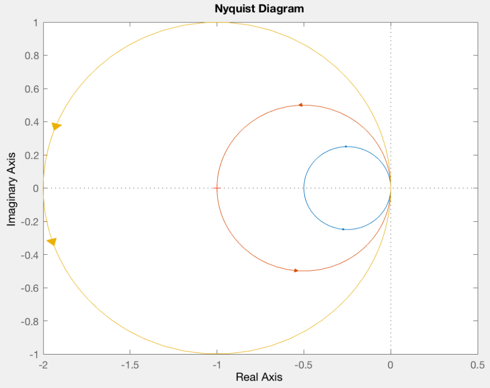


The sytem has an unstable open loop pole. No encirclement of the critical point. Hence, also the closed loop system is unstable.

Example - 4

Suppose we have a gain

$$F(s) = \frac{0.5k}{s-1} = \frac{-0.5k}{1-s} =$$



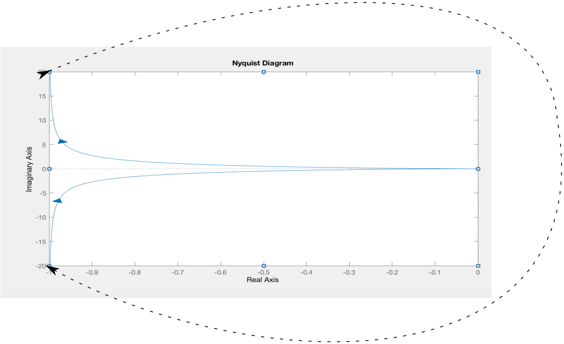
By increasing the gain we can find a point (using the Routh criterion) in which the plot encircles the critical point. When this happens, we have one open loop unstable pole and one encirclement, which ensures stability $N_c = 1 = n_F^+ - n_W^+ = 1 - n_W^+ \rightarrow n_W^+ = 0$

Closure at infinity

- So far, we have assumed no poles on the imaginary axis (i.e., with **Real** $(\cdot)$)
- Poles on the imaginary axis can be integrator or give rise to resonance ($\pm j\omega_n$).
- Such poles introduce a discontinuity in the phase
  - From $\pi/2$ to $-\pi/2$ when ω swithches from 0^- to 0^+
  - From 0 to π when ω switches from ω_n^- to ω_n^+
- The magnitude goes to ∞
- In order to obtain a closed Nyquist plot we introduce the notion of "closure" at ∞, which consists of rotating of π clockwise, once for every pole with $\Re\cdot = 0$.

Example 5

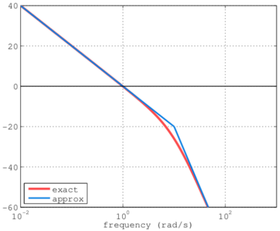
$$F(s) = \frac{1}{s(s+1)}$$

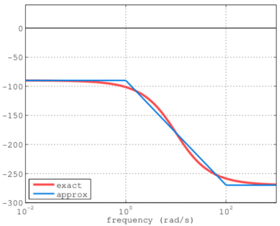


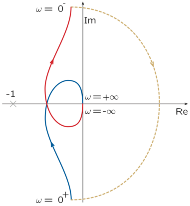
This diagram, even if expanded, wull never ecnricle the critical point. Hence the system is stable and will remain stable even with arbitrary positive gain.

Example 6

$$F(s) = \frac{100}{s(s+10)^2}$$



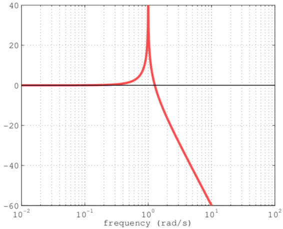


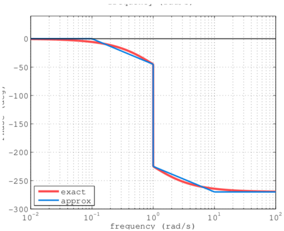


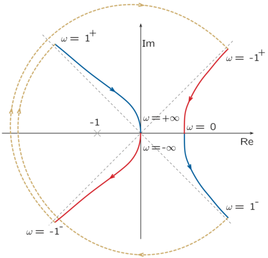
$N_c = n_F^+ = 0$ The system is stable. If we increase the gain at some point we will encircle the critical point.

Example 7

$$F(s) = \frac{1}{(s+1)(s^2+1)}$$







Two complete rotations and no open loop pole implies two unstable closed loop poles

Typical specifications for control design

The typical specification for control design are:

- Stability
- Steady state response
- Transient performance
- Immunity (low sensitivity) to noise and disturbance

We want to design a controller $C(s)$, such that when connected in feedback with the plant $G(s)$ it meets all of these specs.

Procedure

The approach is to shape the open loop response $L(s) = C(s)G(s)$ so that the clodes loop response $\frac{L(s)}{1+L(s)}$ has the desired behaviour.

On stability

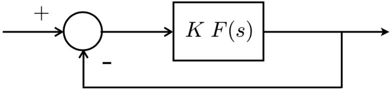
Apparently, we said all we had to with Nyquist criterion.....
TOO easy!!! The problem is that the system is not exactly as we model it:

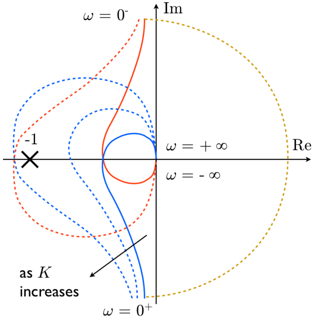
- Non linear effects
- Time varying effects
- Partial knowledge of phyisical parameters.

We have to make sure that our stability claims resit when they are challenged by these effects: this is waht we call *robust stability*

On stability

Considera a (rather typical) situation in which the open loop system is stable. Therefore, the closed–loop system is stable if and only if there is no encirclement of the critical point $(-1, 0)$ Consider the following setup (in which the open loop function $L(s) = kF(s)$)





As K increases, at some point the system becomes unstable.

Gain Margin

The proximity to the critical point $(-1, 0)$ is an indicator of the proximity to instability of the closed loop system.

Gain Margin

Definition

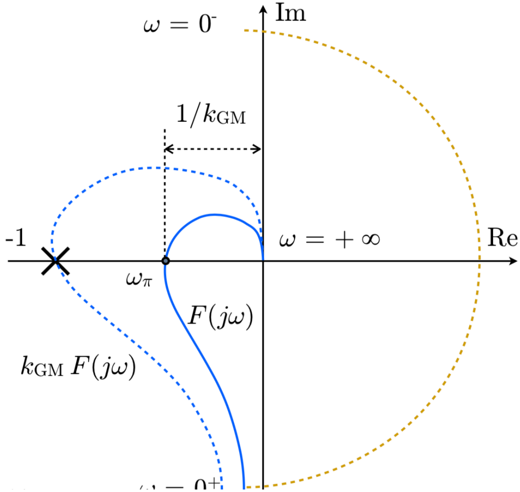
Let the open loop function be $L(s) = kF(s)$. Define ω_π as the frequency such that $\angle F(\jmath\omega_\pi) = -\pi$. The gain margin k_{GM} is

$$k_{GM} = \frac{1}{|F(j\omega_\pi)|}$$

and

$$k_{GM}|_{\mathrm{dB}} = - \left|F(j\omega_\pi)\right||_{\mathrm{dB}}$$

Gain Margin – Example



The gain margin acconts for the smallest multiplicative uncertainty that the closed loop system can tolerate before losing stability (it is a measure of robustness).

Phase Margin

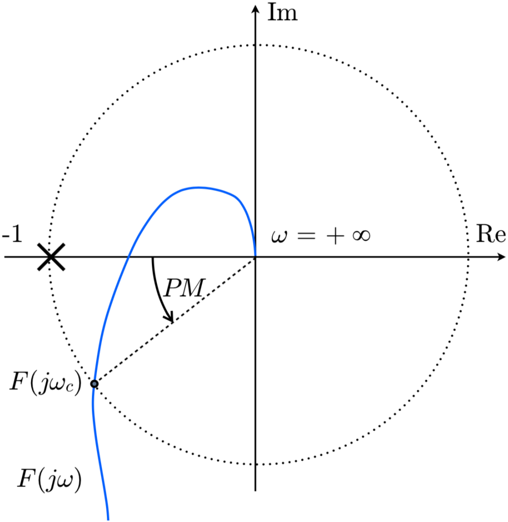
The proximity to the critical point $(-1, 0)$ is an indicator of the proximity to instability of the closed loop system.

Gain Margin

Definition

Let the open loop function be $L(s) = kF(s)$. Define the crossover frequency ω_c as the frequency such that $|F(\jmath\omega_c)| = 1$ (or 0dB). The phase margin PM is defined as $PM = \pi + \angle F(j\omega_c)$.

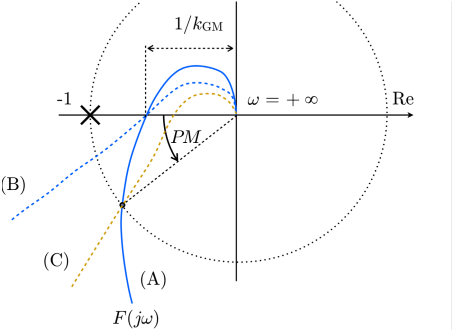
Phase Margin – Example



The phase margin accounts for the smallest phase error (e.g., delay) that the closed loop can tolerate before losing stability (it is another measure of robustness).

Other examples

Phase margins and gain margins need to be considered together to assess the robustness of the closed loop stability.



B and A have the same gain margins but different phase margins. C and A have the same phase margins but different gain margis