Teoria dei sistemi.

Z transform

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Table of contents

Z Transform

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- The Z-transform is the discrete time counter–part of the Laplace tranform.
- the Z-Transform provides analytical methods for the solution of a difference equation,
- it offers direct insight into the transient and steady state behaviour of a DT signal
- it can be used to evaluate the stability of a system.

Definition

Definition of the Z-Transform

Definition

The definition of the z-Transform is the following:

$$\mathcal{Z}\left(f(t)\right) = \sum_{0}^{\infty} f(t) z^{-t}.$$

- This time we associate a DT signal with a function of the complex variable z.
- We will find it convenient to use the polar representation: $z = \rho e^{j\phi}$.

Z-Transform of $\mathbf{1}(t)$

$$\begin{aligned}
\mathcal{Z}\left(\mathbf{1}(t)\right) &= \sum_{0}^{\infty} \mathbf{1}(t) z^{-t} \\
&= \sum_{0}^{\infty} z^{-t} \\
&= \lim_{H\to\infty} \sum_{0}^{H} z^{-t} \\
&= \lim_{H\to\infty} \frac{1 - z^{-H}}{1 - z^{-1}}.
\end{aligned}$$

Z-Transform of $\mathbf{1}(t)$

Setting $z = \rho e^{j\theta}$, we have $z^{-H} = \rho^{-H} e^{-jH\theta}$. We have two cases:

$$\lim_{H\to\infty} z^{-H} = \begin{cases} 0 & \text{if } \rho = |z| > 1 \\ \infty & \text{if } \rho = |z| < 1 \\ e^{-jH\theta} & \text{if } \rho = |z| = 1 \end{cases}$$

$$\mathcal{Z}\left(\mathbf{1}(t)\right) = \lim_{H\to\infty} \frac{1 - z^{-H}}{1 - z^{-1}} = \begin{cases} \frac{1}{1-z^{-1}} & \text{if } |z| > 1 \\ \text{is not defined} & \text{otherwise} \end{cases}$$

Z-Transform of $\mathbf{1}(t)a^t$

$$\begin{aligned}
\mathcal{Z}\left(\mathbf{1}(t)a^t\right) &= \sum_0^{\infty} \mathbf{1}(t)a^t z^{-t} \\
&= \sum_0^{\infty} a^t z^{-t} \\
&= \lim_{H\to\infty} \sum_0^{H} \left(\frac{a}{z}\right)^t \\
&= \lim_{H\to\infty} \frac{1-\left(\frac{a}{z}\right)^H}{1-\frac{a}{z}}.
\end{aligned}$$

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&= \lim_{H\to\infty} \frac{1-\left(\frac{a}{z}\right)^H}{1-\frac{a}{z}}.
\end{aligned}
$$

- $a > 0$: Setting $z = \rho e^{j\theta}$, we have $\left(\frac{a}{z}\right)^H = \left(\frac{a}{\rho}\right)^H e^{-jH\theta}$.
- $a < 0$: we have $\left(\frac{a}{z}\right)^H = \left(\frac{a}{\rho}\right)^H e^{j\pi - jH\theta}$.

Z-Transform of $\mathbf{1}(t)a^t$

In conclusion:

$$\mathcal{Z}\left(\mathbf{1}(t)a^t\right) = \begin{cases} \frac{z}{z-a} & \text{if } |z| > |a| \\ \text{is not defined} & \text{otherwise} \end{cases}$$

Observations

- The z-Transform of a DT signal is a function of a complex variable z

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- As for the Laplace transform, it is not typically possibile to define the z-Transform for all values of z, but only for a subset that we define Region of Convergence (ROC).

Observations

- The z-Transform of a DT signal is a function of a complex variable z
- As for the Laplace transform, it is not typically possibile to define the z-Transform for all values of z, but only for a subset that we define Region of Convergence (ROC).
- Whereas for the Laplace transform the ROC is typically an half space ($\mathbf{Real}\,(s) > \alpha$) for the z-Transform it is an anulus $|z| \geq \rho$.

Existence of the z-Transform

The existence of the z - Transform is guaranteed by the following:

Theorem

Theorem

Consider a function $f(t)$ and assume that one of the following limits exists:

$$R_f = \lim_{t\to\infty} |f(t)|^{1/t}$$

$$R_f = \lim_{t\to\infty} \frac{f(t+1)}{f(t)}.$$

Then:

1. *the z-Transform $\mathcal{Z}(f(t))$ exists and converges for $|z| \geq R_f$.*
2. *the z-Trasform is analytic, i.e., continuous and infinitely differentiable w.r.t. z, for $|z| \geq R_f$.*

Observations

- The result is very difficult to prove in the context of an elementary course

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- Suffice it to say that the existence of a limit $R_f = \lim_{t\to\infty} |f(t)|^{1/t}$ is equivalent to that of an exponential upper bound for the function

$$f(t) \leq AR_f^t,$$

for some A.

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$$f(t) \leq AR_f^t,$$

  for some A.
- There is an inverse theorem, as shown next.

Inverse theorem

Invertibility of the z- Transform

Theorem

If $F(z)$ and $G(z)$ are z-Transofrms of two functions $f(t)$ and $g(t)$ and if $F(z) = G(z)$ for all $|z| > R$, for some $R > 0$ then $f(t) = g(t)$ for $t = 0, 1, 2, \ldots$.

Proof

- If $G(z) = F(z)$ for all $|z| > R$ then

$$\sum_{t=0}^{\infty} f(t) z^{-t} = \sum_{t=0}^{\infty} g(t) z^{-t} \leftrightarrow$$

$$\sum_{t=0}^{\infty} (f(t) - g(t))\, z^{-t} = 0 \leftrightarrow \sum_{t=0}^{\infty} a_t w^t = 0$$

where we have set $w = 1/z$ and $a_t = f(t) - g(t)$.

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  where we have set $w = 1/z$ and $a_t = f(t) - g(t)$.
- It is known that if we have a power series $\sum_{t=0}^{\infty} a_t w^t$ then if $\sum_{t=0}^{\infty} a_t w^t = 0$ for all $|w| \leq W$ then $a_t = 0$.

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- It is known that if we have a power series $\sum_{t=0}^{\infty} a_t w^t$ then if $\sum_{t=0}^{\infty} a_t w^t = 0$ for all $|w| \leq W$ then $a_t = 0$.
- Using this result, we conclude $f(t) = g(t)$.

Inverse z - Transform

Inverse z- Transform

Theorem

the inverse transform of the z-Transform is defined by the following line integral.

$$f(t) = \mathcal{Z}^{-1}(F(z)) = \oint_{|z|=R} F(z) z^{t-1} dz$$

where the line integral is computed along a circle included in the ROC.

Observations

- This formula is impractical and the inverse z-Transform is best computed using a different procedure
- However, this formula reveals that the z-Transform is in some sense a decomposition of a function using a basis of functions of type z^n.

Properties of the z-Transform

- The properties of z - Transform are very similar to the ones of the Laplace transform
- We state them in the following theorem

Properties of the z - Transform

Properties of the z- Transform (1)

Theorem

Let $X(z)$ have ROC R, $X_1(z)$ have ROC R_1 and $X_2(z)$ have ROC R_2.

Linearity $\mathcal{Z}\left(\alpha_1 x_1(t) + \alpha_2 x_2(t)\right) = \alpha_1 X_1(z) + \alpha_2 X_2(z)$ (ROC $R' = R_1 \cap R_2$)

Time-shifting if $x(t)$ is a causal signal and $k > 0$ then $\mathcal{Z}\left(x(t-k)\right) = z^{-k} X(z)$ (ROC $R' \supset R \cap \{0 < |z| < \infty\}$, and $\mathcal{Z}\left(x(t+k)\right) = z^k X(z) - z^k x(0) - z^{k-1} x(1) - \ldots - z x(K-1)$ (ROC $R' = R$)

Multiplication by exponential $\mathcal{Z}\left(z_0^t x(t)\right) = X(\frac{z}{z_0})$ (ROC $R' = |z_0|\, R$

Multiplication by t $\mathcal{Z}\left(t x(t)\right) = -z\frac{dX(z)}{dz}$ (ROC $R' = R$)

Properties of the z - Transform

Properties of the z- Transform (2)

Theorem

Let $X(z)$ have ROC R, $X_1(z)$ have ROC R_1 and $X_2(z)$ have ROC R_2.

- *Time Scaling* $\mathcal{Z}\left(x(t/t_0)\right) = X(z^{t_0})$ *for t_0 positive integer.*
- *Convolution* $\mathcal{Z}\left(x_1(t) * x_2(t)\right) = X_1(z)X_2(z)$. *(ROC $R' \supset R_1 \cap R_2$).*
- *Accumulation* $\mathcal{Z}\left(\sum_{\tau=0}^{t} x(\tau)\right) = \frac{z}{z-1}X(z)$ *(ROC $R' = R \cap \{|z| > 1\}$)*
- *Initial value* $x(0) = \lim_{z\to\infty} X(z)$
- *Final value* $x(\infty) = \lim_{z\to 1}(z-1)X(z)$, *if $x(\infty)$ exists.*

Proof

The proof very similar to Laplace transform

We give a couple of examples.

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We give a couple of examples.

Time shifting (with positive shift)

$$\begin{aligned}
\mathcal{Z}\left(x(t+k)\right) &= \sum_{t=0}^{\infty} x(t+k)z^{-t} = \sum_{t=0}^{\infty} x(t+k)z^{-(t+k)}z^{k} = \\
&= z^{k}\sum_{t'=k}^{\infty} x(t')z^{-t'} = \\
&= z^{k}\left(\sum_{t'=0}^{\infty} x(t')z^{-t'}\right) - x(0)z^{k} - x(1)z^{k-1} - \ldots zx(k-1) \\
&= z^{k}X(z) - x(0)z^{k} - x(1)z^{k-1} - \ldots zx(k-1)
\end{aligned}$$

Proof

Convolution (case of causal signals)

$$
\begin{aligned}
\mathcal{Z}\left(x_1(t) * x_2(t)\right) &= \sum_{t=0}^{\infty} x_1(t) * x_2(t) z^{-t} = \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} x_1(\tau) x_2(t-\tau) z^{-t} = \\
&= \sum_{\tau=0}^{\infty} \sum_{t=0}^{\infty} x_1(\tau) x_2(t-\tau) z^{-t} = \sum_{\tau=0}^{\infty} x_1(\tau) \sum_{t=0}^{\infty} x_2(t-\tau) z^{-t} = \\
&= \sum_{\tau=0}^{\infty} x_1(\tau) \sum_{t=0}^{\infty} x_2(t-\tau) z^{-(t-\tau)} z^{-\tau} \\
&= \sum_{\tau=0}^{\infty} x_1(\tau) z^{-\tau} \sum_{t=-\tau}^{\infty} x_2(t') z^{-t'} = \\
&= X_1(z) X_2(z)
\end{aligned}
$$

Example

We can use the properties to construct complex z - Transform from simpler ones.

$s(t) = \mathbf{1}(t)\cos\Omega t$

We can use the properties as follows:

$$
\begin{aligned}
\mathcal{Z}\left(\mathbf{1}(t)\cos\Omega t\right) &= \frac{1}{2}\mathcal{Z}\left(\mathbf{1}(t)e^{j\Omega t}\right) + \frac{1}{2}\mathcal{Z}\left(\mathbf{1}(t)e^{-j\Omega t}\right) \\
&= \frac{1}{2}\frac{z}{z-e^{j\Omega}} + \frac{1}{2}\frac{z}{z-e^{j\Omega}} = \\
&= \frac{1}{2}\frac{z(z-e^{j\Omega}+z-e^{-j\Omega})}{z^2-z(e^{j\Omega}+e^{-j\Omega})+1} \\
&= \frac{z(z-\cos\Omega)}{z^2-2z\cos\Omega+1}
\end{aligned}
$$

Transform table

We can proceed in a similar way to compute a table of transforms.

z-Transform of known Signals

Signal	z-Tranform	ROC
$\delta(t)$	1	$z \in \mathbb{C}$
$\delta(t-t_0)$	z^{-t_0}	$z \in \mathbb{C}\backslash 0$
$\mathbf{1}(t)$	$\frac{z}{z-1}$	$\|z\| > 1$
$\mathbf{1}(t)t$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$\mathbf{1}(t)t^2$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
$\mathbf{1}(t)a^t$	$\frac{z}{(z-a)}$	$\|z\| > \|a\|$
$\mathbf{1}(t)a^t$	$\frac{z}{(z-a)}$	$\|z\| > \|a\|$
$\mathbf{1}(t)ta^t$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$\mathbf{1}(t)t^2a^t$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\| > \|a\|$

Transform table

We can proceed in a similar way to compute a table of transforms.

z-Transform of known Signals		
Signal	z-Tranform	ROC
$\mathbf{1}(t)\cos\Omega t$	$\frac{z(z-\cos\Omega)}{z^2-2z\cos\Omega+1}$	$\lvert z\rvert > 1$
$\mathbf{1}(t)\sin\Omega t$	$\frac{z\sin\Omega}{z^2-2z\cos\Omega+1}$	$\lvert z\rvert > 1$
$a^t\cos\Omega t$	$\frac{z(z-a\cos\Omega)}{z^2-2az\cos\Omega+a^2}$	$\lvert z\rvert > a$
$\mathbf{1}(t)a^t\sin\Omega t$	$\frac{za\sin\Omega}{z^2-2az\cos\Omega+a^2}$	$\lvert z\rvert > a$

Application to difference equations

An interesting application is shown through the different example.

Difference Equations

Consider the following difference equation.

$$y(t+2) = 3y(t+1) - 2y(t) + u(t+1) - 3u(t).$$

Let us find $Y(z)$ for $u(t) = \mathbf{1}(t)$, $y(1) = 1$, $y(0) = -1$, $u(0) = 1$.

Application to difference equations

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- The z - Transform is the following

$$\mathcal{Z}\left(y(t+2)\right) = \mathcal{Z}\left(3y(t+1) - 2y(t) + u(t+1) - 3u(t)\right).$$

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- The z - Transform is the following

$$\mathcal{Z}\left(y(t+2)\right) = \mathcal{Z}\left(3y(t+1) - 2y(t) + u(t+1) - 3u(t)\right).$$

- Application of time shifting rule

$$z^2 Y(z) - z^2 y(0) - zy(1) =$$
$$3zY(z) - 3zy(0) - 2Y(z) + zU(z) - zu(0) - 3U(z)$$

Applications to difference equations

- The latter equation becomes

$$
\begin{aligned}
Y(z) &= \frac{U(z)(z-3)}{z^2-3z+2} + \frac{z^2y(0)+z(y(1)-3y(0)-u(0))}{z^2-3z+2} \\
&= \frac{z(z-2)}{(z-1)(z^2-3z+2)} + \frac{z^2y(0)+z(y(1)-3y(0)-u(0))}{z^2-3z+2}.
\end{aligned}
$$

Applications to difference equations

- The latter equation becomes

$$\begin{aligned} Y(z) &= \frac{U(z)(z-3)}{z^2-3z+2} + \frac{z^2 y(0) + z(y(1) - 3y(0) - u(0))}{z^2-3z+2} \\ &= \frac{z(z-2)}{(z-1)(z^2-3z+2)} + \frac{z^2 y(0) + z(y(1) - 3y(0) - u(0))}{z^2-3z+2}. \end{aligned}$$

- We will soon see how to invert this.

Inversion of the z - Transform

- Also for LTI DT systems the evolution of the system is compounded by a free evolution and by a forced evolution.
- We have to deal with a ratio of polynomial with the numerator that is typically proportional to z.
- One of the possible ways to deal with this case is by using the same technique (partial fraction expansion) that we have used for the Laplace transform (with some care)
- We will see this trhough some examples

Back to the example

Free evolution

Let us go to the example above and compute the free evolution for $y(1) = 1$, $y(0) = -1$, $u(0) = 1$.

Back to the example

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Let us go to the example above and compute the free evolution for $y(1) = 1$, $y(0) = -1$, $u(0) = 1$.

- the z - Transform

$$Y(z) = \frac{z^2 y(0) + z(y(1) - 3y(0) - u(0))}{z^2 - 3z + 2}$$
$$= \frac{-z^2 + 3z}{(z-2)(z-1)}$$

Back to the example

Free evolution

Let us go to the example above and compute the free evolution for $y(1) = 1,\ y(0) = -1,\ u(0) = 1$.

- the z - Transform

$$\begin{aligned} Y(z) &= \frac{z^2 y(0) + z(y(1) - 3y(0) - u(0))}{z^2 - 3z + 2} \\ &= \frac{-z^2 + 3z}{(z-2)(z-1)} \end{aligned}$$

- It is convenient to divide by z and then proceed with partial fraction expansion:

$$\frac{Y(z)}{z} = \frac{-z+3}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{2}{z-1}$$

Back to the example

- And finally....

$$
\begin{aligned}
Y(z) &= \frac{z}{z-2} - \frac{2z}{z-1} \\
y(t) &= \mathbf{1}(t)\left(2^t - 2\right).
\end{aligned}
$$

Back to the example

- And finally....

$$Y(z) = \frac{z}{z-2} - \frac{2z}{z-1}$$
$$y(t) = \mathbf{1}(t)\left(2^t - 2\right).$$

- Now let us compute the response for $u(t) = \mathbf{1}(t)$ Let us compute the forced evolution for $u(t) = \mathbf{1}(t)$.
- z -Transform

$$Y(z) = \frac{z(z-3)}{(z-1)^2(z-2)}.$$

Back to the example

- It is convenient to divide by z and then proceed with partial fraction expansion:

$$
\begin{aligned}
\frac{Y(z)}{z} &= \frac{(z-3)}{(z-1)^2(z-2)} = \\
&= \frac{A_{1,1}}{(z-1)^2} + \frac{A_{1,2}}{z-1} + \frac{A_2}{z-2} = \\
&= \frac{A_{1,1}}{(z-1)^2} + \frac{A_{1,2}}{z-1} + \frac{A_2}{z-2} = \\
&= \frac{2}{(z-1)^2} + \frac{1}{z-1} - \frac{1}{z-2} =
\end{aligned}
$$

Back to the example

- It is convenient to divide by z and then proceed with partial fraction expansion:

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&= \frac{A_{1,1}}{(z-1)^2} + \frac{A_{1,2}}{z-1} + \frac{A_2}{z-2} = \\
&= \frac{2}{(z-1)^2} + \frac{1}{z-1} - \frac{1}{z-2} =
\end{aligned}
$$

- Which leads to

$$
\begin{aligned}
Y(z) &= \frac{2z}{(z-1)^2} + \frac{z}{z-1} - \frac{z}{z-2} \\
y(t) &= \mathbf{1}(t)\left(t + 1 - 2^t\right)
\end{aligned}
$$

Another example

Another Example

Let us compute the forced step response of the following:

$$y(k+3)+0.1y(k+2)-0.12y(k+1)+0.04y(k)=u(k).$$

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Let us compute the forced step response of the following:

$$y(k+3) + 0.1y(k+2) - 0.12y(k+1) + 0.04y(k) = u(k).$$

- The z-Transform produces:

$$z^3Y(z) + 0.2z^2Y(z) - 0.12zY(z) + 0.04Y(z) = U(z)$$
$$Y(z) = \frac{1}{z^3 + 0.2z^2 - 0.12z + 0.04}U(z) =$$
$$= \frac{1}{z^3 + 0.1z^2 - 0.12z + 0.04}\frac{z}{z-1}$$

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- The z-Transform produces:

$$z^3Y(z)+0.2z^2Y(z)-0.12zY(z)+0.04Y(z)=U(z)$$
$$Y(z)=\frac{1}{z^3+0.2z^2-0.12z+0.04}U(z)=$$
$$=\frac{1}{z^3+0.1z^2-0.12z+0.04}\frac{z}{z-1}$$

Another Example

- The partial fraction expansion is as follows:

$$\frac{Y(z)}{z} = \frac{1}{(z+0.5)(z-1)(z-0.2-0.2j)(z-0.2+0.2j)}$$

$$= \frac{1}{(z+0.5)(z-1)(z-0.2-0.2j)(z-0.2+0.2j)}$$

$$= \frac{A_1}{z-1} + \frac{A_2}{z+0.5} + \frac{A_3}{z-0.2-0.2j} + \frac{\overline{A_3}}{z-0.2+0.2j}$$

$$A_1 = \frac{1}{(1+0.5)(1-0.2-0.2j)(1-0.2+0.2j)} = 0.9804$$

$$A_2 = \frac{1}{(-0.5-1)(-0.5-0.2-0.2j)(-0.5-0.2+0.2j)} = -1.2579$$

$$A_3 = \frac{1}{(0.5+0.2+0.2j)(0.2+0.2j-1)((0.2+0.2j-0.2+0.2j)} =$$

$$= \frac{1}{0.008-0.24j} = \frac{0.08+0.24j}{0.0577} = 0.1386+4.1594j.$$

Another Example

- Therefore,

$$y(t) = \mathbf{1}(t)\left(A_1 + A_2(-0.5)^t + A_3(0.2+0.2j)^t + \overline{A_3}(0.2-0.2j)^t\right).$$

- Since

$$0.2 + 0.2j = 0.2828e^{j\frac{\pi}{4}}$$
$$0.2 - 0.2j = 0.2828e^{-j\frac{\pi}{4}}$$

- we have

$$\begin{aligned} y(t) &= \mathbf{1}(t)\left(A_1 + A_2(-0.5)^t + A_3 0.2828^t e^{j\frac{\pi}{4}t} + \underline{A_3} 0.2828^t e^{-j\frac{\pi}{4}t}\right) \\ &= \mathbf{1}(t)\left(A_1 + A_2(-0.5)^t + |A_3|\, 0.2828^t \left(e^{j\frac{\pi}{4}t + j\angle A_3} + e^{-j\frac{\pi}{4}t - j\angle A_3}\right)\right) \\ &= \mathbf{1}(t)\left(A_1 + A_2(-0.5)^t + 2\,|A_3|\, 0.2828^t \cos\left(\frac{\pi}{4}t + \angle A_3\right)\right) \end{aligned}$$

Natural Modes

Each pole determines an evolution of the system (natural modes) described by an exponential function as for CT systems, as shown below:

Natural modes associated with the different ples

Pole	CT modes	DT modes
Single real pole p	e^{pt}	p^t
Multiple real pole p (multipl. m)	$e^{pt}, te^{pt}, \ldots, t^{m-1}e^{pt}$	$p^t, tp^t, \ldots, t^{m-1}p^t$
Single complex pair $p, \overline{p}$	$e^{\mathbf{Real}(p)t} \cos\left(\mathbf{Imag}\,(p)\,t + \phi\right)$	$\|p\|^t \cos\left(\angle pt + \phi\right)$

Real poles

- For DT systems a real pole p, when negative, gives rise to an exponential mode p^t that oscillates.
- For CT systems, on the contrary, oscillating behaviours are only possible for complex conjugate pairs.

BIBO stability of DT systems

The discusssion on the modes is synthesised in the following.

Theorem

Theorem

Consider a DT LTI system with transfer function:

$$H(z) = \frac{n(z)}{d(z)}.$$

and assume that no zero pole cancellation takes place.
Then the system is BIBO stable if and only if all poles have modules smaller than 1: $\forall p s.t.\ d(p) = 0,\ \text{we have}\ |p| < 1.$

Remarks

- The proof of this result is the same as the proof of the similar results that we have given for CT systems.
- The stability could be checked with a criterion very similar to the Rout-Hurwitz criterion (called Jury criterion), but this is out of the scope

Example

Consider the following system

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$$y(k+2) - 3y(k+1) + 2y(k) = 3u(k+1) - u(k).$$

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- The transfer function of the system is

$$\begin{aligned} Y(z)(z^2 - 3z + 2) &= 3zU(z) - U(z) \\ Y(z) &= \frac{3z - 1}{z^2 - 3z + 2}. \end{aligned}$$

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- Both poles are outside the unit circle. Hence, the system is BIBO unstable.

z-Transform of sampled data signals

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z-Transform of sampled data signals

- The z-Transform and Laplace transform are in fact close relatives.
- Consider a CT signal $f(t)$, its Laplace transform is given by: $F(s) = \int_0^\infty f(\tau)e^{-s\tau}d\tau$.
- Suppose we transform the signal into a DT sequence by taking a sample every T time units by multiplying the signal by a sequence of Dirac δ: $f_D(t) = f(t)\sum_{k=0}^{\infty}\delta(t-kT)$.

z-Transform of sampled data signals

- If we compute the Laplace transform of this signal, we find:

$$
\begin{aligned}
\mathcal{L}\left(f_D(t)\right) &= \int_0^\infty f_D(\tau)e^{-s\tau}d\tau = \int_0^\infty f(t)\sum_{k=0}^{\infty}\delta(t-kT)e^{-s\tau}d\tau \\
&= \int_0^\infty \sum_{k=0}^{\infty} f(kT)\delta(t-kT)e^{-s\tau}d\tau \\
&= \int_0^\infty \sum_{k=0}^{\infty} f(kT)\delta(t-kT)e^{-skT}d\tau \\
&= \sum_{k=0}^{\infty}\int_{(k-1)T}^{kT} f(kT)\delta(t-kT)e^{-skT}d\tau \\
&= \sum_{k=0}^{\infty} f(kT)e^{-skT}\int_{(k-1)T}^{kT}\delta(t-kT)d\tau \\
&= \sum_{k=0}^{\infty} f(kT)e^{-skT} = \sum_{k=0}^{\infty} f(kT)(e^{-sT})^k
\end{aligned}
$$

z-Transform of sampled data signals

- If we set $e^{sT} = z$ we find

$$F(z) = \sum_{k=0}^{\infty} f(kT) z^k,$$

we find the definition of the z-Transform.