# Teoria dei sistemi.

## Z transform

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November 21, 2016

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- The Z-transform is the discrete time counter–part of the Laplace tranform.
- the Z-Transform provides analytical methods for the solution of a difference equation,
- it offers direct insight into the transient and steady state behaviour of a DT signal
- it can be used to evaluate the stability of a system.

# Definition

## Definition of the Z-Transform

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The definition of the z-Transform is the following:

$$\mathcal{Z}\left(f(t)\right) = \sum_{0}^{\infty} f(t) z^{-t}.$$

- This time we associate a DT signal with a function of the complex variable $z$.
- We will find it convenient to use the polar representation: $z = \rho e^{j\phi}$.

# Z-Transform of $\mathbf{1}(t)$

$$
\begin{aligned}
\mathcal{Z}\left(\mathbf{1}(t)\right) &= \sum_{0}^{\infty} \mathbf{1}(t) z^{-t} \\
&= \sum_{0}^{\infty} z^{-t} \\
&= \lim_{H \to \infty} \sum_{0}^{H} z^{-t} \\
&= \lim_{H \to \infty} \frac{1 - z^{-H}}{1 - z^{-1}}.
\end{aligned}
$$

# Z-Transform of $\mathbf{1}(t)$

Setting $z = \rho e^{j\theta}$, we have $z^{-H} = \rho^{-H} e^{-jH\theta}$. We have two cases:

$$\lim_{H\to\infty} z^{-H} = \begin{cases} 0 & \text{if } \rho = |z| > 1 \\ \infty & \text{if } \rho = |z| < 1 \\ e^{-jH\theta} & \text{if } \rho = |z| = 1 \end{cases}$$

$$\mathcal{Z}\left(\mathbf{1}(t)\right) = \lim_{H\to\infty} \frac{1 - z^{-H}}{1 - z^{-1}} = \begin{cases} \frac{1}{1-z^{-1}} & \text{if } |z| > 1 \\ \text{is not defined} & \text{otherwise} \end{cases}$$

# Z-Transform of $\mathbf{1}(t)a^t$

$$
\begin{aligned}
\mathcal{Z}\left(\mathbf{1}(t)a^t\right) &= \sum_{0}^{\infty} \mathbf{1}(t)a^t z^{-t} \\
&= \sum_{0}^{\infty} z^{-t} \\
&= \lim_{H\to\infty} \sum_{0}^{H} \left(\frac{a}{z}\right)^t \\
&= \lim_{H\to\infty} \frac{1-\left(\frac{a}{z}\right)^H}{1-\frac{a}{z}}.
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$$
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&= \sum_0^\infty z^{-t} \\
&= \lim_{H\to\infty} \sum_0^H \left(\frac{a}{z}\right)^t \\
&= \lim_{H\to\infty} \frac{1-\left(\frac{a}{z}\right)^H}{1-\frac{a}{z}}.
\end{aligned}
$$

- $a > 0$: Setting $z = \rho e^{j\theta}$, we have $\left(\frac{a}{z}\right)^H = \left(\frac{a}{\rho}\right)^H e^{-jH\theta}$.
- $a < 0$: we have $\left(\frac{a}{z}\right)^H = \left(\frac{a}{\rho}\right)^H e^{j\pi - jH\theta}$.

# Z-Transform of $\mathbf{1}(t)a^t$

In conclusion:

$$\mathcal{Z}\left(\mathbf{1}(t)a^t\right) = \begin{cases} \frac{z}{z-a} & \text{if } |z| > |a| \\ \text{is not defined} & \text{otherwise} \end{cases}$$

# Observations

- The z-Transform of a DT signal is a function of a complex variable $z$

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- As for the Laplace transform, it is not typically possibile to define the z-Transform for all values of $z$, but only for a subset that we define Region of Convergence (ROC).

# Observations

- The z-Transform of a DT signal is a function of a complex variable $z$
- As for the Laplace transform, it is not typically possibile to define the z-Transform for all values of $z$, but only for a subset that we define Region of Convergence (ROC).
- Whereas for the Laplace transform the ROC is typically an half space ($\textbf{Real}\,(s) > \alpha$) for the z-Transform it is an anulus $|z| \geq \rho$.

# Existence of the z-Transform

The existence of the z - Transform is guaranteed by the following:

**Theorem**

**Theorem**

*Consider a function $f(t)$ and assume that one of the following limits exists:*

$$R_f = \lim_{t \to \infty} |f(t)|^{1/t}$$
$$R_f = \lim_{t \to \infty} \frac{f(t+1)}{f(t)}.$$

*Then:*

1. *the z-Transform $\mathcal{Z}(f(t))$ exists and converges for $|z|\ geqR_f$.*
2. *the z-Trasform is analytic, i.e., continuous and infinitely differentiable w.r.t. $z$, for $|z| \geq R_f$.*

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- Suffice it to say that the existence of a limit $R_f = \lim_{t\to\infty} |f(t)|^{1/t}$ is equivalent to that of an exponential upper bound for the function

$$f(t) \leq AR_f^t,$$

for some $A$.

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- Suffice it to say that the existence of a limit $R_f = \lim_{t\to\infty} |f(t)|^{1/t}$ is equivalent to that of an exponential upper bound for the function

$$f(t) \leq AR_f^t,$$

  for some $A$.
- There is an inverse theorem, as shown next.

# Inverse theorem

## Invertibility of the z- Transform

**Theorem**

*If $F(z)$ and $G(z)$ are z-Transofrms of two functions $f(t)$ and $g(t)$ and if $F(z) = G(z)$ for all $|z| > R$, for some $R > 0$ then $f(t) = g(t)$ for $t = 0, 1, 2, \ldots$.*

# Proof

- If $G(z) = F(z)$ for all $|z| > R$ then

$$\sum_{t=0}^{\infty} f(t) z^{-t} = \sum_{t=0}^{\infty} g(t) z^{-t} \leftrightarrow$$

$$\sum_{t=0}^{\infty} \left(f(t) - g(t)\right) z^{-t} = 0 \leftrightarrow \sum_{t=0}^{\infty} a_t w^t = 0$$

where we have set $w = 1/z$ and $a_t = f(t) - g(t)$.

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  where we have set $w = 1/z$ and $a_t = f(t) - g(t)$.
- It is known that if we have a power series $\sum_{t=0}^{\infty} a_t w^t$ then if $\sum_{t=0}^{\infty} a_t w^t = 0$ for all $|w| \leq W$ then $a_t = 0$.

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where we have set $w = 1/z$ and $a_t = f(t) - g(t)$.
- It is known that if we have a power series $\sum_{t=0}^{\infty} a_t w^t$ then if $\sum_{t=0}^{\infty} a_t w^t = 0$ for all $|w| \leq W$ then $a_t = 0$.
- Using this result, we conclude $f(t) = g(t)$.

# Inverse z - Transform

## Inverse z- Transform

### Theorem

*the inverse transform of the z-Transform is defined by the following line integral.*

$$f(t) = \mathcal{Z}^{-1}(F(z)) = \oint_{|z|=R} F(z) z^{t-1} dz$$

*where the line integral is computed along a circle included in the ROC.*

# Observations

- This formula is impractical and the inverse z-Transform is best computed using a different procedure
- However, this formula reveals that the z-Transform is in some sense a decomposition of a function using a basis of functions of type $z^n$.

# Properties of the z-Transform

- The properties of z - Transform are very similar to the ones of the Laplace transform
- We state them in the following theorem

# Properties of the z - Transform

## Properties of the z- Transform (1)

### Theorem

*Let $X(z)$ have ROC $R$, $X_1(z)$ have ROC $R_1$ and $X_2(z)$ have ROC $R_2$.*

**Linearity** $\mathcal{Z}\left(\alpha_1 x_1(t) + \alpha_2 x_2(t)\right) = \alpha_1 X_1(z) + \alpha_2 X_2(z)$ *(ROC $R' = R_1 \cap R_2$)*

**Time-shifting** *if $x(t)$ is a causal signal and $k > 0$ then* $\mathcal{Z}\left(x(t-k)\right) = z^{-k}X(z)$ *(ROC $R' \supset R \cap \{0 < |z| < \infty\}$, and* $\mathcal{Z}\left(x(t+k)\right) = z^k X(z) - z^k x(0) - z^{k-1}x(1) - \ldots - zx(K-1)$ *(ROC $R' = R$)*

**Multiplication by exponential** $\mathcal{Z}\left(z_0^t x(t)\right) = X\left(\frac{z}{z_0}\right)$ *(ROC $R' = |z_0|\, R$*

**Multiplication by $t$** $\mathcal{Z}\left(tx(t)\right) = -z\frac{dX(z)}{dz}$ *(ROC $R' = R$)*

# Properties of the z - Transform

## Properties of the z- Transform (2)

### Theorem

*Let $X(z)$ have ROC $R$, $X_1(z)$ have ROC $R_1$ and $X_2(z)$ have ROC $R_2$.*

| | |
|---|---|
| Time Scaling | $\mathcal{Z}\left(x(t/t_0)\right) = X(z^{t_0})$ *for* $t_0$ *positive integer.* |
| Convolution | $\mathcal{Z}\left(x_1(t) * x_2(t)\right) = X_1(z)X_2(z)$. *(ROC* $R' \supset R_1 \cap R_2$*).* |
| Accumulation | $\mathcal{Z}\left(\sum_{\tau=0}^{t} x(\tau)\right) = \frac{z}{z-1}X(z)$ *(ROC* $R' = R \cap \{\lvert z\rvert > 1\}$*)* |
| Initial value | $x(0) = \lim_{z\to\infty} X(z)$ |
| Final value | $x(\infty) = \lim_{z\to 1}(z-1)X(z)$, *if* $x(\infty)$ *exists.* |

# Proof

The proof very similar to Laplace transform

We give a couple of examples.

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## Time shifting (with positive shift)

$$\mathcal{Z}\left(x(t+k)\right) = \sum_{t=0}^{\infty} x(t+k)z^{-t} = \sum_{t=0}^{\infty} x(t+k)z^{-(t+k)}z^{k} =$$

$$= z^k \sum_{t'=k}^{\infty} x(t')z^{-t'} =$$

$$= z^k \left(\sum_{t'=0}^{\infty} x(t')z^{-t'}\right) - x(0)z^k - x(1)z^{k-1} - \ldots zx(k-1)$$

$$= z^k X(z) - x(0)z^k - x(1)z^{k-1} - \ldots zx(k-1)$$

# Proof

## Convolution (case of causal signals)

$$\mathcal{Z}\left(x_1(t) * x_2(t)\right) = \sum_{t=0}^{\infty} x_1(t) * x_2(t) z^{-t} = \sum_{t=0}^{\infty} \sum_{\tau=0}^{\infty} x_1(\tau) x_2(t-\tau) z^{-t} =$$

$$= \sum_{\tau=0}^{\infty} \sum_{t=0}^{\infty} x_1(\tau) x_2(t-\tau) z^{-t} = \sum_{\tau=0}^{\infty} x_1(\tau) \sum_{t=0}^{\infty} x_2(t-\tau) z^{-t} =$$

$$= \sum_{\tau=0}^{\infty} x_1(\tau) \sum_{t=0}^{\infty} x_2(t-\tau) z^{-(t-\tau)} z^{-\tau}$$

$$= \sum_{\tau=0}^{\infty} x_1(\tau) z^{-\tau} \sum_{t=-\tau}^{\infty} x_2(t') z^{-t'} =$$

$$= X_1(z) X_2(z)$$

# Example

We can use the properties to construct complex z - Transform from simpler ones.

$s(t) = \mathbf{1}(t)\cos\Omega t$

We can use the properties as follows:

$$
\begin{aligned}
\mathcal{Z}\left(\mathbf{1}(t)\cos\Omega t\right) &= \frac{1}{2}\mathcal{Z}\left(\mathbf{1}(t)e^{j\Omega t}\right) + \frac{1}{2}\mathcal{Z}\left(\mathbf{1}(t)e^{-j\Omega t}\right) \\
&= \frac{1}{2}\frac{z}{z-e^{j\Omega}} + \frac{1}{2}\frac{z}{z-e^{j\Omega}} = \\
&= \frac{1}{2}\frac{z(z-e^{j\Omega}+z-e^{-j\Omega})}{z^2-z(e^{j\Omega}+e^{-j\Omega})+1} \\
&= \frac{z(z-\cos\Omega)}{z^2-2z\cos\Omega+1}
\end{aligned}
$$

# Transform table

We can proceed in a similar way to compute a table of transforms.

**z-Transform of known Signals**

| Signal | z-Tranform | ROC |
|---|---|---|
| $\delta(t)$ | $1$ | $z \in \mathbb{C}$ |
| $\delta(t-t_0)$ | $z^{-t_0}$ | $z \in \mathbb{C} \backslash 0$ |
| $\mathbf{1}(t)$ | $\frac{z}{z-1}$ | $\lvert z \rvert > 1$ |
| $t$ | $\frac{z}{(z-1)^2}$ | $\lvert z \rvert > 1$ |
| $t^2$ | $\frac{z(z+1)}{(z-1)^3}$ | $\lvert z \rvert > 1$ |
| $a^t$ | $\frac{z}{(z-a)}$ | $\lvert z \rvert > \lvert a \rvert$ |
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| $ta^t$ | $\frac{az}{(z-a)^2}$ | $\lvert z \rvert > \lvert a \rvert$ |
| $t^2a^t$ | $\frac{az(z+a)}{(z-a)^3}$ | $\lvert z \rvert > \lvert a \rvert$ |

# Transform table

We can proceed in a similar way to compute a table of transforms.

| **z-Transform of known Signals** | | |
|---|---|---|
| Signal | z-Tranform | ROC |
| $\cos \Omega t$ | $\frac{z(z-\cos\Omega)}{z^2-2z\cos\Omega+1}$ | $\lvert z\rvert > 1$ |
| $\sin \Omega t$ | $\frac{z\sin\Omega}{z^2-2z\cos\Omega+1}$ | $\lvert z\rvert > 1$ |
| $a^t \cos \Omega t$ | $\frac{z(z-a\cos\Omega)}{z^2-2az\cos\Omega+a^2}$ | $\lvert z\rvert > a$ |
| $a^t \sin \Omega t$ | $\frac{za\sin\Omega}{z^2-2az\cos\Omega+a^2}$ | $\lvert z\rvert > a$ |

# Application to difference equations

An interesting application is shown through the different example.

### Difference Equations

Consider the following difference equation.

$$y(t+2) = 3y(t+1) - 2y(t) + u(t+1) - 3u(t).$$

Let us find $Y(z)$ for $u(t) = \mathbf{1}(t)$, $y(1) = 1$, $y(0) = -1$, $u(0) = 0$.

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- The z - Transform is the following

$$\mathcal{Z}\left(y(t+2)\right) = \mathcal{Z}\left(3y(t+1) - 2y(t) + u(t+1) - u(t)\right).$$

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$$\mathcal{Z}\left(y(t+2)\right) = \mathcal{Z}\left(3y(t+1) - 2y(t) + u(t+1) - u(t)\right).$$

- Application of time shifting rule

$$z^2 Y(z) - z^2 y(0) - zy(1) =$$
$$3zY(z) - 3zy(0) - 2Y(z) + zU(z) - zu(0) - 3U(z)$$

# Applications to difference equations

- The latter equation becomes

$$
\begin{aligned}
Y(z) &= \frac{U(z)(z-3)}{z^2-3z+2} + \frac{z^2y(0)+z(y(1)-3y(0)-u(0))}{z^2-3z+2} \\
&= \frac{z(z-2)}{(z-1)(z^2-3z+2)} + \frac{z^2y(0)+z(y(1)-3y(0)-u(0))}{z^2-3z+2}.
\end{aligned}
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$$\begin{aligned} Y(z) &= \frac{U(z)(z-3)}{z^2-3z+2} + \frac{z^2y(0)+z(y(1)-3y(0)-u(0))}{z^2-3z+2} \\ &= \frac{z(z-2)}{(z-1)(z^2-3z+2)} + \frac{z^2y(0)+z(y(1)-3y(0)-u(0))}{z^2-3z+2}. \end{aligned}$$

- We will soon see how to invert this.

# Inversion of the z - Transform

- Also for LTI DT systems the evolution of the system is compounded by a free evolution and by a forced evolution.
- We have to deal with a ratio of polynomial with the numerator that is typically proportional to $z$.
- One of the possible ways to deal with this case is by using the same technique (partial fraction expansion) that we have used for the Laplace transform (with some care)
- We will see this trhough some examples

# Back to example

## Free evolution

Let us go to the example above and compute the free evolution for $y(1) = 1$, $y(0) = -1$, $u(0) = 0$.

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- the z - Transform

$$\begin{aligned} Y(z) &= \frac{z^2 y(0) + z(y(1) - 3y(0) - u(0))}{z^2 - 3z + 2} \\ &= \frac{-z^2 + 4z}{(z-2)(z-1)} \end{aligned}$$

# Back to example

## Free evolution

Let us go to the example above and compute the free evolution for $y(1) = 1,\ y(0) = -1,\ u(0) = 0.$

- the z - Transform

$$Y(z) = \frac{z^2 y(0) + z(y(1) - 3y(0) - u(0))}{z^2 - 3z + 2} = \frac{-z^2 + 4z}{(z-2)(z-1)}$$

- It is convenient to divide by $z$ and then proceed with partial fraction expansion:

$$\frac{Y(z)}{z} = \frac{-z+4}{(z-2)(z-1)} = \frac{2}{z-2} - \frac{3}{z-1}$$

# Back to the example

- And finally....

$$
\begin{aligned}
Y(z) &= \frac{2z}{z-2} - \frac{3z}{z-1} \\
y(t) &= \mathbf{1}(t)\left(2 \cdot 2^t - 3\right).
\end{aligned}
$$

# Back to the example

- And finally....

$$Y(z) = \frac{2z}{z-2} - \frac{3z}{z-1}$$
$$y(t) = \mathbf{1}(t)\left(2 \cdot 2^t - 3\right).$$

- Now let us compute the response for $u(t) = \mathbf{1}(t)$ Let us compute the forced evolution for $u(t) = \mathbf{1}(t)$.
- z -Transform

$$Y(z) = \frac{z(z-3)}{(z-1)^2(z-2)}.$$

# Back to the example

- It is convenient to divide by $z$ and then proceed with partial fraction expansion:

$$
\begin{aligned}
\frac{Y(z)}{z} &= \frac{(z-3)}{(z-1)^2(z-2)} = \\
&= \frac{A_{1,1}}{(z-1)^2} + \frac{A_{1,2}}{z-1} + \frac{A_2}{z-2} = \\
&= \frac{A_{1,1}}{(z-1)^2} + \frac{A_{1,2}}{z-1} + \frac{A_2}{z-2} = \\
&= \frac{2}{(z-1)^2} + \frac{1}{z-1} - \frac{1}{z-2} =
\end{aligned}
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&= \frac{A_{1,1}}{(z-1)^2} + \frac{A_{1,2}}{z-1} + \frac{A_2}{z-2} = \\
&= \frac{2}{(z-1)^2} + \frac{1}{z-1} - \frac{1}{z-2} =
\end{aligned}
$$

- Which leads to

$$
\begin{aligned}
Y(z) &= \frac{2z}{(z-1)^2} + \frac{z}{z-1} - \frac{z}{z-2} \\
y(t) &= \mathbf{1}(t)\left(t + 1 - 2^t\right)
\end{aligned}
$$