Teoria dei sistemi.

Impulse Response and Convolutions

Luigi Palopoli

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Scope

IO Representations

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Differential/Difference Equation

IO representations are associated with a differential/difference equation.

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i y(t) + \sum_{j=0}^{p} \beta_j \mathfrak{D}^j u(t), \tag{1}$$

Starting from today and for several weeks, we will focus on IO Representations of this kind.

A few basic facts

- time invariance requires that α_i and β_i be constant.
- given the vector of initial conditions

$$y(0), \mathfrak{D}y(0), \ldots, \mathfrak{D}^{n-1}y(0), \ldots, \mathfrak{D}^{p}u(0), \ldots, \mathfrak{D}u(0)$$

  and the input $u|_{[t_0,t]}$, we can generally find a unique solutions.
- The linearity of the system allows us to split the evolution in two separate terms:

$$y(t) = y_{\text{free}}(t) + y_{\text{forced}}(t)$$

with

$$y_{\text{free}}(t) = \mathcal{F}_{\text{free}}(y(0), \mathfrak{D}y(0), \ldots, \mathfrak{D}^{n-1}y(0), \ldots, \mathfrak{D}^{p}u(0), \ldots, \mathfrak{D}u(0))$$
$$y_{\text{forced}}(t) = \mathcal{F}_{\text{forced}}(u|_{[t_0,\, t]}).$$

- Let us focus, for some time, on the forced evolution

Discrete–time systems

- We start our discussion from discrete–time ssytems
- It is useful to intruduce a strange function

Kronecker δ

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0. \end{cases} \tag{2}$$

Properties of the δ function

Sampling

For any discrete–time signal f we have the following property:

$$f(t)\delta(t-t_0) = \begin{cases} f(t_0) & t = t_0 \\ 0 & t \neq t_0 \end{cases} \tag{3}$$

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Observation

Multiplying δ shifted to t_0 by a signal generates a signal that is zero at all times except for t_0 (where it is $f(t_0)$).

Properties of the δ function

Expression of a signal

Another property is:

$$\sum_{\tau=-\infty}^{\tau=\infty} f(\tau)\delta(t-\tau) = f(t) \tag{4}$$

Properties of the δ function

Expression of a signal

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$$\sum_{\tau=-\infty}^{\tau=\infty} f(\tau)\delta(t-\tau) = f(t) \tag{4}$$

Observation

In other words we can express any signal as a linear combination of infinite δ, each translated to a different instant.

Impulse Response

- Our final goal is to compute the response to any input signal $u(t)$
- Let us start from the response to a $\delta(t)$

Impulse Response

Let us define $h(t)$ the forced response to the signal $\delta(t)$.

$$h(t) = \mathcal{F}_{\text{forced}}(\delta).$$

An application of linearity and time invariance

- We can express the input as

$$u(t) = \sum_{\tau=-\infty}^{\tau=\infty} u(\tau)\delta(t-\tau)$$

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- Application of time–invariance.

$$\mathcal{F}_{\text{forced}}(u) = \sum_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)$$

Convolution sum

Convolution

The operation

$$\mathcal{F}_{\text{forced}}(u) = \sum_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)$$

is called *convolution sum* and is denoted by $*$:

$$u(t) * h(t) = \sum_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)$$

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$$u(t) * h(t) = \sum_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)$$

Meaning

The meaning of $y(t) = u(t) * h(t)$ is:

1. Compute the sequence $h(-\tau)$,
2. Shift the obtained sequence by t and compute the product element–wise;
3. Sum up the products, and this produces $y(t)$.

Example 1

First Example

Compute the system response for

$$h(t) = \begin{cases} 1 & t = 0 \\ 2 & t = 1 \\ -2 & t = 2 \\ 0 & \text{Otherwise} \end{cases} \qquad u(t) = \begin{cases} 5 & t = 0 \\ -3 & t = 1 \\ 0 & \text{Otherwise} \end{cases}$$

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Compute the system response for

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Solution

$$y(t) = \sum_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)$$

$$= 5h(t) - 3h(t-1) = \qquad = \begin{cases} 5 & t = 0 \\ 10 - 3 & t = 1 \\ -10 - 6 & t = 2 \\ 6 & t = 3. \end{cases}$$

Example 1

Observation

If the support of $h(t)$ is $[0, M]$ and the support of $u(t)$ is $[0, N]$, the support of $h(t) * u(t)$ will be $[0, N + M]$.

Example 2

Second Example

Suppose $h(t) = 1(t)a^t$, compute the forced response to $u(t) = 1(t)b^t$, where

$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Example 2

Second Example

Suppose $h(t) = 1(t)a^t$, compute the forced response to $u(t) = 1(t)b^t$, where

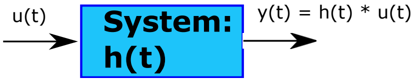
$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Solution

$$\begin{aligned} \mathcal{F}_{\text{forced}}(u) &= \sum_{\tau=-\infty}^{\tau=\infty} 1(\tau) b^\tau 1(t-\tau) a^{t-\tau} = \\ &= \sum_{\tau=1}^{\tau=t-1} b^\tau a^{t-\tau} = a^t \sum_{\tau=1}^{\tau=t-1} \left(\frac{b}{a}\right)^\tau = \\ &= a^t \frac{(b/a) - (b/a)^t}{1-(b/a)} = \frac{b}{a-b} a^t - \frac{a}{a-b} b^t. \end{aligned}$$

A Block Scheme

- The examples above can be summarised in the following block scheme:



- The box represents the system and the arrows the input and the output signals.

Properties

Theorem

The convolution sum enjoys the following properties:

1. Commutative Property: $h(t) * u(t) = h(t) * u(t)$.
2. Distributive Property:
   $(h_1(t) + h_2(t)) * u(t) = h_1(t) * u(t) + h_2(t) * u(t)$
3. Associative Property:
   $h_1(t) * (h_2(t) * u(t)) = (h_1(t) * h_2(t)) * u(t)$.

Proof (Commutative Property)

Commutative Property

Considering that $h(t) * u(t) = \sum_{\tau=-\infty}^{+\infty} h(\tau)u(t-\tau)$, by setting $t - \tau = \tau_1$ we have :

$$\begin{aligned} \sum_{\tau=-\infty}^{+\infty} h(\tau)u(t-\tau) &= \sum_{\tau_1=\infty}^{-\infty} h(t-\tau_1)u(\tau_1) \\ &= u(t) * h(t). \end{aligned}$$

Proof (Distributive Property)

Distributive Property

It comes as a direct consequence of the the linearity of the convolution operator:

$$
\begin{aligned}
(h_1(t) + h_2(t)) * u(t) &= \sum_{\tau=-\infty}^{+\infty} (h_1(\tau) + h_2(\tau))\, u(t-\tau) = \\
&= \sum_{\tau_1=\infty}^{-\infty} h_1(\tau)u(t-\tau) + \sum_{\tau_1=\infty}^{-\infty} h_2(\tau)u(t-\tau) = \\
&= h_1(t) * u(t) + h_2(t) * u(t)
\end{aligned}
$$

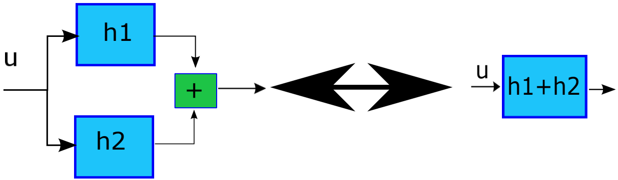
Proof (Associative Property)

Associative Property

$$
\begin{aligned}
(h_1(t) * h_2(t)) * u(t) = \sum_{\tau_2=-\infty}^{\infty} u(t-\tau_2) \sum_{\tau_1=-\infty}^{\infty} h_1(\tau_1) h_2(\tau_2-\tau_1) &= \\
= \sum_{\tau_2=-\infty}^{\infty} \sum_{\tau_1=-\infty}^{\infty} u(t-\tau_2) h_1(\tau_1) h_2(\tau_2-\tau_1) &= \\
= \sum_{\tau_1=-\infty}^{\infty} h_1(\tau_1) \sum_{\tau_2'=-\infty}^{\infty} u(\tau_2') h_2(t-\tau_2'-\tau_1) &= \\
= \sum_{\tau_1=-\infty}^{\infty} h_1(\tau_1)(u(t) * h_2(t))|_{t-\tau_1} & \\
= h_1(t) * (u(t) * h_2(t)) &
\end{aligned}
$$

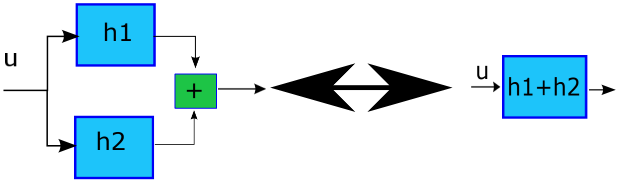
Meaning of the distributive property

- While the first property is merely operational, the distributive and the associative property have a clear "physical" meaning.
- The distributive property gives information on parallel composition.
  $(h_1(t) + h_2(t)) * u(t) = h_1(t) * u(t) + h_2(t) * u(t)$



Meaning of the associative property

- The distributive property gives information on the series composition. $h_1(t) * (h_2(t) * u(t)) = (h_1(t) * h_2(t)) * u(t)$



Exponential Functions

- There is a special class of functions that receives a special "treatment" from linear systems

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Exponential Functions

Consider the signal $u(t) = z^t$. The response to this input is given by:

$$
\begin{aligned}
\sum_{\tau=-\infty}^{+\infty} h(\tau)u(t-\tau) &= \sum_{\tau=-\infty}^{\infty} u(t-\tau)h(\tau) \\
&= \sum_{\tau=-\infty}^{\infty} z^{t-\tau} h(\tau) \\
&= z^t \sum_{\tau=-\infty}^{\infty} z^{-\tau} h(\tau) \\
&= z^t H(z)
\end{aligned}
$$

where $H(z) = \sum_{\tau=-\infty}^{\infty} z^{-\tau} h(\tau)$.

Eigenfunctions

- Whenever an exponential signal z^t is processed by a DT LTI system, the result is the same signal scaled by a constant $H(z) = \sum_{\tau=-\infty}^{\infty} z^{-\tau} h(\tau)$ as far as the series converge.
- *This applies both to real and complex z.*

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Eigenfunctions

z^t is called an *eigenfunction*. An eigenfunction is essentially an eigenvector, with $H(z)$ being its eigenvalue.

Properties of the eigenvectors

- Consider linear application defined from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that it is associated to a matrix A:

$$y = Ax.$$

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- Let $\hat{x}$ be the coordinates in this basis of a generic vector x expressed in the canonical basis. We have:

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- By combining the two conditions:

$$\begin{aligned}\hat{y} &= M^{-1}y = \\ &= M^{-1}Ax = \\ &= M^{-1}AM\tilde{x}.\end{aligned} \tag{5}$$

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- It can easily be seen that $M^{-1}AM$ is a diagonal matrix.

What about eigenfucntions

- if we express a vector using a basis of eigenvectors, the system operates on each component in a decoupled way.
- The same holds if we express any signal as a linear combination of Eigenfunctions.
- This will lead us to the notion of Z-transform

Forced Evolution of Continuous TIme Systems

- We now move to studying the forced evolution of Continuous Time Systems
- Our first problem is to correctly define impulse function

Dirac δ

- The impulse function is unusual for the continuous time domain
- In CT we are used to continuous and differentiable functions
- The simplest possible definition for an impulse can be the following.

Dirac δ

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) \tag{6}$$

$$\delta_\Delta(t) = \begin{cases} 0 & t \notin \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \\ \frac{1}{\Delta} & t \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right] \end{cases} \tag{7}$$

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- The Dirac δ has some important properties

Property 1

Property 1

If we compute the integral of Dirac δ on any interval enclosing the origin, we get 1.0:

$$\forall a > 0,\ b > 0 \int_{-a}^{b} \delta(\tau) d\tau = 1.$$

Property 2

Property 2

Multiplying a $\delta(t-\tau)$ by any function has the effect of "sampling" the value of the function in τ:

$$f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau).$$

Property 3

Property 3

Any function can be expressed as an integral of impulse functions.

$$\forall \epsilon > 0,\ f(t) = \int_{t-\epsilon}^{t+\epsilon} f(\tau)\delta(t-\tau)d\tau.$$

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Proof

$$\begin{aligned}
\int_{t-\epsilon}^{t+\epsilon} f(\tau)\delta(t-\tau)d\tau &= \int_{t-\epsilon}^{t+\epsilon} f(t)\delta(t-\tau)d\tau \\
&= f(t)\int_{t-\epsilon}^{t+\epsilon} \delta(t-\tau)d\tau \\
&= f(t).
\end{aligned}$$

Property 4

Property 4

The integral from any negative number to a generic instant t produces a step function:

$$\forall \epsilon > 0, \int_{-\epsilon}^{t} \delta(\tau) d\tau = 1(t),$$

where $1(t)$ is defined as

$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Property 5

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We define

$$\delta(t) = \frac{d}{dt} 1(t).$$

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Abuse of Notation

This is an obvious abuse of notation because the step function is not differentiable.

Impulse Response

- We can now repeat the same arguments of the DT case
- Define $h(t)$ as the impulse response of the system

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Convolution Integral

Using linearity and time–invariance:

$$\begin{aligned} y(t) &= \mathcal{F}_{\text{forced}}(u) \\ &= \mathcal{F}_{\text{forced}}\Big(\int_{\tau=-\infty}^{\tau=\infty} u(\tau)\delta(t-\tau)d\tau\Big) \\ &= \int_{\tau=-\infty}^{\tau=\infty} u(\tau)\mathcal{F}_{\text{forced}}(\delta(t-\tau))d\tau, \\ &= \int_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)d\tau. \end{aligned}$$

Convolution Integral

Convolution Integral

The integral:

$$\int_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)d\tau.$$

is called "convolution integral" and is denoted by $h(t) * u(t)$

Computation of the Convolution Integral

Computation

The computation of the convolution integral requires the following steps:

1. Compute the "reflection" of $h(\tau)$ through $\tau = 0$,
2. Translate the result to the right of t (to the left if t is negative).
3. Compute the product by $u(\tau)$ and then the integral of the function thus obtained.

Example

Computation

Let $h(t) = 1(t)e^{-3t}$ and $u(t) = 1(t)$. The response $y(t)$ can be found as follows:

$$
\begin{aligned}
y(t) &= \mathcal{F}_{\text{forced}}(u) \\
&= \int_{\tau=-\infty}^{\tau=\infty} u(\tau)h(t-\tau)d\tau \\
&= \int_{\tau=-\infty}^{\tau=\infty} 1(t)1(t-\tau)e^{-3t+3\tau}d\tau \\
&= \int_{0}^{\tau=t} e^{-3t+3\tau}d\tau \\
&= e^{-3t}\frac{1}{3}e^{3\tau}|_{\tau=0}^{t} \\
&= \frac{1-e^{-3t}}{3}.
\end{aligned}
$$

Properties of the convolution Integral

Properties

The convolution integral has the same three properties as the convolution sum for DT system. And, the proof of these properties is absolutely simialr to the DT case.

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3. Associative Property:
   $h_1(t) * (h_2(t) * u(t)) = (h_1(t) * h_2(t)) * u(t)$.

Eigenfunctions

The DT case

For the DT case we have seen that z^t is an eigenfunction.

Eigenfunctions

The DT case

For the DT case we have seen that z^t is an eigenfunction.

The CT case

For the CT case e^{st} are eigenfunctions. ... let us see why!!

Eigenfunctions

Response to e^{st}

$$
\begin{aligned}
y(t) &= \int_{\tau=-\infty}^{+\infty} h(\tau)u(t-\tau)d\tau \\
&= \int_{\tau=-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= e^{st}\int_{\tau=-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \\
&= e^{st}H(s).
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Eigenfunctions

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$$

Eigenvalues

Assuming that the integral $H(s) = \int_{\tau=-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$ converges. We can say that e^{st} is an Eigenfunctions related to the eigenvalue $H(s)$.

Harmonic functions

- The results above apply to real and complex exponentials alike.
- Now let us consider an harmonic function $u(t) = \cos \omega t$.

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

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$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

Computation

$$\begin{aligned}
y(t) &= \int_{\tau=-\infty}^{+\infty} h(\tau) u(t-\tau) d\tau \\
&= \int_{\tau=-\infty}^{+\infty} h(\tau) \frac{e^{j\omega(t-\tau)} + e^{-j\omega(t-\tau)}}{2} d\tau \\
&= \frac{1}{2} \int_{\tau=-\infty}^{+\infty} h(\tau) e^{j\omega(t-\tau)} d\tau + \frac{1}{2} \int_{\tau=-\infty}^{+\infty} e^{-j\omega(t-\tau)} d\tau \\
&= \frac{1}{2} e^{j\omega t} H(j\omega) + \frac{1}{2} e^{-j\omega t} H(-j\omega),
\end{aligned}$$

where $H(j\omega) = \int_{\tau=-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$.

Recap: Properties of the complex numbers

Let $\overline{z}$ represent the complex conjugate of a complex number z. If z_1 and z_2 are two complex numbers and α is a real. We can show

Properties of Complex Numbers

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
3. $\overline{\alpha z_1} = \alpha \overline{z_1}$.

Back to Harmonic functions

Computation of $H(-j\omega)$

Applying these properties, we can see:

$$\begin{aligned}\overline{H(j\omega)} &= \overline{\int_{\tau=-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau} \\ &= \int_{\tau=-\infty}^{+\infty} \overline{h(\tau)e^{-j\omega\tau}}d\tau \\ &= \int_{\tau=-\infty}^{+\infty} h(\tau)\overline{e^{-j\omega\tau}}d\tau \\ &= H(-j\omega)\end{aligned}$$

Back to Harmonic functions

..and

As a consequence of $\overline{H(j\omega)} = H(-j\omega)$, we have

$$\begin{aligned} y(t) &= \frac{1}{2}e^{j\omega t}H(j\omega) + \frac{1}{2}e^{-j\omega t}H(-j\omega) \\ &= \frac{1}{2}e^{j\omega t}H(j\omega) + \frac{1}{2}e^{-j\omega t}\overline{H(j\omega)}. \end{aligned}$$

Back to Harmonic functions

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As a consequence of $\overline{H(j\omega)} = H(-j\omega)$, we have

$$\begin{aligned} y(t) &= \frac{1}{2}e^{j\omega t}H(j\omega) + \frac{1}{2}e^{-j\omega t}H(-j\omega) \\ &= \frac{1}{2}e^{j\omega t}H(j\omega) + \frac{1}{2}e^{-j\omega t}\overline{H(j\omega)}. \end{aligned}$$

Final Result

The result can be relaborated using mudlus/phase representation:

$$\begin{aligned} H(j\omega) &= |H(j\omega)|\, e^{j\angle H(j\omega)} \\ y(t) &= \frac{1}{2}e^{j\omega t}H(j\omega) + \frac{1}{2}e^{-j\omega t}\overline{H(j\omega)} \\ &= \frac{|H(j\omega)|}{2}\left(e^{j(\angle H(j\omega)+\omega t)} + e^{-j(\angle H(j\omega)+\omega t)}\right) \\ &= |H(j\omega)| \cos\left(\omega t + \angle H(j\omega)\right). \end{aligned}$$

Theorem of Harmonic Functions

The discussion above can be summarised in the following:

Theorem

Theorem

Consider a TC LTI system. If $\int_{\tau=-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau$ *converges to a value* $H(j\omega)$*, then the system responds to an harmonic input function* $\cos\omega t$ *with an harmonic output function having the same frequency.*

Properties

Many important properties of LTI systems can be read from their impulse response.

Causality

Theorem

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Let $h(t)$ be the impulse response of a system Σ. The system is causal if and only if $h(t) = 0$ for $t < 0$.

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Proof

Necessity: if we choose $u(t) = \delta(t)$, causality requires that $h(t) = 0$ for $t < 0$.

Sufficiency:if $h(t) = 0$ for $t < 0$ then

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau = \int_{-\infty}^{t} h(t-\tau)u(\tau)d\tau.$$

Therefore, $y(t)$ is only affected by $u(\cdot)$ until t.

BIBO stability

There are several notions of stability. We start by BIBO stability meaning that if we apply "small" input, we will have "small output". More formally:

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Definition

Definition (BIBO stability)

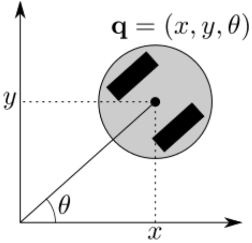
A system is BIBO stable iff for all $\epsilon > 0$ there exists a positive real $\delta > 0$ such that

$$|u(t)| \leq \epsilon \implies |y(t)| < \delta.$$

BIBO stability

Example of Unicycle Robot

Consider a cylindrical robot that moves with constrant forward speed $v(t)$ and the angular speed $\omega(t)$ (which is the input variable).



BIBO stability

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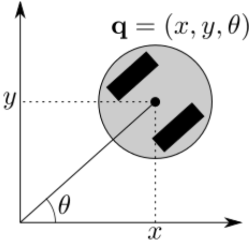

Differential Equations

Kinematics of the system:

$$\dot{y} = v \sin \theta$$

$$\dot{\theta} = \omega.$$

Suppose that our output is y

BIBO stability

Example of Unicycle Robot

Suppose we apply the following signal:

$$\omega = \begin{cases} \epsilon & t \in [0, 0.1s] \\ 0 & t \notin [0, 0.1s] \end{cases}$$

Evolution

- the system changes slightly its orientation
- then y starts to grow unbounded even if ϵ is very small.
- this is the perfect example of a BIBO unstable system.

BIBO stability for LTI systems

For LTI systems we have got a very simple criterion expressed by:

Theorem

Theorem

Consider a LTI system Σ with impulse response $h(t)$.

- *If the system is DT then it is BIBO stable if and only if there exists a constant S such that $\sum_{-\infty}^{\infty} |h(t)| = S < \infty$.*
- *If the system is CT then it is BIBO stable if and only if there exist a constant S such that $\int_{-\infty}^{\infty} |h(\tau)| d\tau = S < \infty$.*

Proof (Sufficiency)

Proof of sufficiency for DT systems

Assume that for some $\epsilon > 0$, $u(t) \leq \epsilon, \forall t$.

$$
\begin{aligned}
y(t) &= \sum_{\tau=-\infty}^{\infty} h(\tau)u(t-\tau) \\
&\leq \sum_{\tau=-\infty}^{\infty} |h(\tau)u(t-\tau)| \\
&\leq \sum_{\tau=-\infty}^{\infty} |h(\tau)||u(t-\tau)| \\
&\leq \sum_{\tau=-\infty}^{\infty} |h(\tau)|\epsilon \\
&\leq S\epsilon.
\end{aligned}
$$

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&\leq \sum_{\tau=-\infty}^{\infty} |h(\tau)|\epsilon \\
&\leq S\epsilon.
\end{aligned}
$$

Consequence:

$$|u(t)| \leq \epsilon \implies |y(t)| \leq \delta = S\epsilon,$$

Proof (Necessity)

Proof of Necessity for DT systems

Cnsider the input signal $u(t) = \epsilon \text{sign}(h(-t))$. Compute $y(0)$:

$$
\begin{aligned}
y(0) &= \sum_{\tau=-\infty}^{\infty} h(\tau)u(-\tau) \\
&\leq \sum_{\tau=-\infty}^{\infty} \epsilon h(\tau)\text{sign}(h(\tau)) \\
&\leq \epsilon \sum_{\tau=-\infty}^{\infty} |h(\tau)|
\end{aligned}
$$

Therefore, if $\sum_{\tau=-\infty}^{\infty} |h(\tau)|$ diverges, so will $y(0)$, even for a bounded signal $u(t)$.