Teoria dei sistemi.

State Space Analysis

Luigi Palopoli

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Modes Analysis

Continuous Time Systems

The discrete time case

- Consider the evolution

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\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t).
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x(1) &= Ax(0) + Bu(0),\\
x(2) &= Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1),\\
&\vdots\\
x(k) &= Ax(k-1) + Bu(k-1) = A^kx(0) + A^{k-1}Bu(0) + \cdots + ABu(k-2
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- ...which can be written as

$$\begin{aligned} x(k) &= A^kx(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i), \\ y(k) &= CA^kx(0) + C\sum_{i=0}^{k-1} A^{k-1-i}Bu(i) + Du(k). \end{aligned} \tag{1}$$

Observations

Free and forces evolution

As for the continuous–time systems, we can identify a free evolution (depending on the initial condition $x(0)$) and a forced evolution (resulting from the application of the input function).

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Role of A^k

- Since A^k is a constant once k is fixed, it turns out that the states are combined through a time invariant linear transformation.
- The combination of forced and unforced response is an application of the *superposition principle* between the states and the inputs.

Connection with $Z-$transform

- Consider the autonomous system:

$$x(k+1) = Ax(k),$$

whose Z-Transform are given by

$$\mathcal{Z}\left(x(k+1)\right) = zX(z) - zx(0), \text{ and } \mathcal{Z}\left(Ax(k)\right) = AX(z),$$

that has been obtained by applying the *time-shifting rule*.

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- Hence,

$$\begin{aligned} & zX(z) - zx(0) = AX(z) \Rightarrow (zI - A)X(z) = zx(0) \\ & \Rightarrow X(z) = (zI - A)^{-1} zx(0). \end{aligned}$$

Inverse Z-transform

- The sequence $x(k)$ can be recoverd via the inverse Z-trasnform.

$$x(k) = \mathcal{Z}^{-1}\left(z(zI - A)^{-1}\right) x(0),$$

Proposition

The matrix A^k can be computed through the inverse of Z-Transform:

$$A^k = \mathcal{Z}^{-1}\left(z(zI - A)^{-1}\right).$$

Example

Harmonic Oscillator

Consider the following *harmonic oscillator*

$$x(k+1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(k) = Ax.$$

Compute the evolution.

- Since
$$zI - A = \begin{bmatrix} z & 1 \\ -1 & z \end{bmatrix},$$
- we have that the eigenvalues are $\pm j$ and hence
$$\begin{aligned}
(zI - A)^{-1} &= \frac{1}{z^2+1} \begin{bmatrix} z & -1 \\ 1 & z \end{bmatrix} \\
A^k &= \mathcal{Z}^{-1}\left(z(zI - A)^{-1}\right) \\
&= \mathcal{Z}^{-1}\left(\begin{bmatrix} \frac{z^2}{z^2+1} & -\frac{z}{z^2+1} \\ \frac{z}{z^2+1} & \frac{z^2}{z^2+1} \end{bmatrix}\right)
\end{aligned}$$

Example

- …finally…

$$A^k = \begin{bmatrix} \cos(\frac{\pi}{2}k) & -\sin(\frac{\pi}{2}k) \\ \sin(\frac{\pi}{2}k) & \cos(\frac{\pi}{2}k) \end{bmatrix}$$

Discrete convolution Summation

- The computation of the forced response requires the extension to matrices of the *discrete convolution sum*

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- Consider a SISO system and assume that $u(\bar{k}) = 0 \ \forall \bar{k} < 0$, we can write

$$\sum_{i=0}^{k-1} A^{k-1-i} B u(i) = A^{k-1} * B u(k),$$

Discrete convolution Summation

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$$\sum_{i=0}^{k-1} A^{k-1-i} B u(i) = A^{k-1} * B u(k),$$

- As a consequence,

$$\mathcal{Z}\left(A^{k-1} * B u(k)\right) = (zI - A)^{-1} B U(z).$$

Discrete convolution Summation

- Applying the superposition principle, we get

$$\mathcal{Z}\left(y(k)\right) = Cz(zI - A)^{-1}x(0) + C(zI - A)^{-1}BU(z) + DU(z).$$

Discrete convolution Summation

- Applying the superposition principle, we get

$$\mathcal{Z}\left(y(k)\right) = Cz(zI - A)^{-1}x(0) + C(zI - A)^{-1}BU(z) + DU(z).$$

- If we consider $x(0) = 0$ we find the expression for the trasfer function.

$$\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D.$$

Example

Second order system

For the linear system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.5 & 1 \\ 0 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 1 & -1 \end{bmatrix} x(k), \end{aligned}$$

compute the transfer function.

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Second order system

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\end{aligned}
$$

compute the transfer function.

- The result is in the following function:

$$
G(z) = [1 \quad -1]\,(zI - A)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{-z + 1.5}{z^2 - 0.25}
$$

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- Hence, assuming $x(0) = 0$ as in the transfer function analysis case, we have

$$Y(z) = C(zI - A)^{-1}BU(z) + DU(z),$$

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The Role of the Eigenvalues in the discrete transfer function

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$$Y(z) = C(zI - A)^{-1}BU(z) + DU(z),$$

  where D can be neglected since it does not involve the system characteristics.
- We can restrict to $G(z) = C(zI - A)^{-1}BU(z)$.

The Role of the Eigenvalues in the discrete transfer function

- the (i,j) element of $(zI - A)^{-1}$ can be computed using the Cramer's rule
$$(-1)^{i+j}\frac{\det \mathrm{Adj}_{i,j}}{\mathcal{P}(A)},$$
where $\mathcal{P}(A)$ is the *characteristic polynomial* of A, defined as
$$\mathcal{P}(A) = \det(zI - A).$$
- $\det \mathrm{Adj}_{i,j}$ has degree less than n and there is still the possibility of having a *cancellation* between the roots of $\mathcal{P}(A)$ and the roots of $\det \mathrm{Adj}_{i,j}$.

The Role of the Eigenvalues for Discrete Time Systems

Let us start with some properties of the matrix power and its eigenvalues that will be useful:

- $A_1^k A_2^k = A_2^k A_1^k = (A_1 A_2)^k$, $\forall k$, iff $A_1 A_2 = A_2 A_1$;

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- If A is invertible, then $\left(A^k\right)^{-1} = \left(A^{-1}\right)^k = A^{-k}$;
- If $v \in \mathbb{R}^n$ is an *eigenvector* of a matrix A associated with the eigenvalue λ, i.e., $Av = \lambda v$, than it immediately follows that v is also an eigenvector of the matrix A^k associated with the eigenvalue λ^k, i.e., $A^k v = \lambda^k v$;

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- For any given A and for any transformation matrix T, we have that $(T^{-1}AT)^k = T^{-1}A^k T$;
- $\det(A^k) = [\det(A)]^k$. From this property, from $\det(AB) = \det(A)\det(B)$ it follows that $\det(AA^{-1}) = \det(A)\det(A^{-1}) = 1$, which is of course $\det(I) = 1$. Hence, the *eigenvalues does not change for similarity transformation*.

Matrix Powers for Diagonal Matrices

- If a matrix is diagonalisable, the matrix power can be easily obtained. Indeed, let

$$T^{-1}AT = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

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- We have that

$$A^k = T\Lambda^k T^{-1} = T \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix} T^{-1}.$$

Matrix Powers for Diagonal Matrices with Complex Eigenvalues

- If the matrix A has complex eigenvalues, it is possible to transforms a diagonalisable matrix on the complex set into a *block diagonal matrix*, whose block dimension is at most 2.

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Matrix Powers for Diagonal Matrices with Complex Eigenvalues

- If the matrix A has complex eigenvalues, it is possible to transforms a diagonalisable matrix on the complex set into a *block diagonal matrix*, whose block dimension is at most 2.
- The diagonal blocks are associated to each complex and conjugated pair.
- As for the continuous time case, since $AV = V\Lambda$ we have $AVH = V\Lambda H = VHH^{-1}\Lambda H$, we have

$$A[v_r + jv_c \,|v_r - jv_c]H = A[v_r|v_c] = [v_r|v_c]\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix},$$

- The matrix can be further transformed using the modulus $\rho = \sqrt{\sigma^2 + \omega^2}$ and phase $\theta = \arctan\left(\frac{\omega}{\sigma}\right)$, i.e.,

$$A[v_r|v_c] = [v_r|v_c]\rho\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Matrix Powers for Diagonal Matrices with Complex Eigenvalues

- ...which leads us to...

$$A^k[v_r|v_c] = [v_r|v_c]\rho^k \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix}.$$

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A 4 X 4 example

let A be a matrix defined in $\mathbb{R}^{4\times 4}$, with λ_1 and λ_4 real and $\lambda_{2,3} = \sigma \pm j\omega$, we have

$$A^k = T\Lambda^k T^{-1} = T \begin{bmatrix} \lambda_1^k & 0 & 0 & 0 \\ 0 & \rho^k \cos(k\theta) & \rho^k \sin(k\theta) & 0 \\ 0 & -\rho^k \sin(k\theta) & \rho^k \cos(k\theta) & 0 \\ 0 & 0 & 0 & \lambda_4^k \end{bmatrix} T^{-1}.$$

Defective matrices

- We can still use the Jordan form

Defective matrices

- We can still use the Jordan form
- Indeed for a block diagonal matrix we have:

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}^k = \begin{bmatrix} A_1^k & 0 & \dots & 0 \\ 0 & A_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n^k \end{bmatrix}.$$

- Hence, the name of the game becomes to understand how to compute the power of *Jordan miniblocks*.

Defective matrices

- We can see that

$$J_{i,l}^k = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}^l = (\Lambda_i + \bar{J}_{i,l})^k = \sum_{q=0}^{k} C_q^k \lambda_i^{k-q} \bar{J}_{i,l}^q,$$

where

$$\begin{cases} C_q^k = \left(\begin{array}{c} k \\ q \end{array} \right) = \dfrac{k!}{q!(k-q)!} & \text{if } k \geq q, \\ C_q^k = 0 & \text{if } k < q, \end{cases},$$

that is a polynomial function of k of degree q, i.e., the *binomial coefficient*.

Defective matrices

- Therefore, for a miniblock of dimension $p = m_{i,k}$, we have

$$J_{i,l}^k = \begin{bmatrix} \lambda_i^k & C_1^k \lambda_i^{k-1} & C_2^k \lambda_i^{k-2} & \dots & C_{q-2}^k \lambda_i^{k-q+2} & C_{q-1}^k \lambda_i^{k-q+1} \\ 0 & \lambda_i^k & C_1^k \lambda_i^{k-1} & \dots & C_{q-3}^k \lambda_i^{k-q+3} & C_{q-2}^k \lambda_i^{k-q+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i^k & C_1^k \lambda_i^{k-1} \\ 0 & 0 & 0 & \dots & 0 & \lambda_i^k \end{bmatrix}.$$

An example

A 3 X 3 Matrix

Let us consider the matrix

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

An example

A 3 X 3 Matrix

Let us consider the matrix

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

- By applying the formulas, it follows that

$$A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}.$$

Matrix Powers of Defective Matrices with Complex Eigenvalues

- If we has generalised eigenvectors associated with complex eigenvalues, we can repeat the same reasoning of the continuous time case.

Matrix Powers of Defective Matrices with Complex Eigenvalues

- If we has generalised eigenvectors associated with complex eigenvalues, we can repeat the same reasoning of the continuous time case.
- For the case $p = 2$ we have

$$J_r = H^{-1}JH = \begin{bmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{bmatrix} = \begin{bmatrix} W & I \\ 0 & W \end{bmatrix}.$$

- For the matrix power, we recall that

$$J_r^k = \begin{bmatrix} W^k & kW^{k-1} \\ 0 & W^k \end{bmatrix} = \begin{bmatrix} W^k & C_1^k W^{k-1} \\ 0 & W^k \end{bmatrix},$$

Matrix Powers of Defective Matrices with Complex Eigenvalues

Which corresponds to

$$J_r^k = \begin{bmatrix} \rho^k \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix} & k\rho^{k-1} \begin{bmatrix} \cos((k-1)\theta) & \sin((k-1)\theta) \\ -\sin((k-1)\theta) & \cos((k-1)\theta) \end{bmatrix} \\ 0 & \rho^k \begin{bmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{bmatrix} \end{bmatrix}.$$

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We can generalise this to $p > 2$....

Mode Analysis

- The unforced response of a linear system is just a linear composition of *only* the functions that can be found in the real Jordan form matrix.
- Such functions are called the *modes* of the system.

Continuous Time Systems

For CT systems, such modes are given by:

- Simple exponential functions $e^{\lambda t}$ given by Jordan miniblocks of dimension one with real eigenvalue λ
  - converge to 0 if $\lambda < 0$, constant if $\lambda = 0$, diverges if $\lambda > 0$
- Composite exponential functions $t^k e^{\lambda t}$ given by Jordan miniblocks of dimension $m > 1$ with real eigenvalue λ.
  - converge to 0 if $\lambda < 0$, diverge polynomially if $\lambda = 0$, diverge exponentially if $\lambda > 0$
- Oscillating functions of the type $e^{\sigma t} \cos(\omega t)$ and $e^{\sigma t} \sin(\omega t)$ given by two miniblocks of dimension 1 associated to complex and conjugated eigenvalues $\lambda = \sigma \pm j\omega$.
  - converge to 0 if $\sigma < 0$, persistently oscillate if $\lambda = 0$, diverge exponentially if $\sigma > 0$

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- Oscillating functions of the type $t^k e^{\sigma t} \cos(\omega t)$ and $t^k e^{\sigma t} \sin(\omega t)$ given by Jordan miniblocks of dimension $m > 1$ (and hence $k = 0, \ldots, m-1$) with complex and conjugated eigenvalues $\lambda = \sigma \pm j\omega$.
  - converge $\sigma < 0$, diverge polynomially if $\sigma = 0$ and $k > 0$ or diverge exponentially if $\sigma > 0$

Discrete Time Systems

For DT systems, such modes are given by:

- Powers λ^k given by Jordan miniblocks of dimension one with real eigenvalue λ.
  - converge $|\lambda| < 1$, constant if $|\lambda| = 1$ or diverge exponentially if $|\lambda| > 1$. We have an oscillation if $\lambda < 0$

Discrete Time Systems

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- Powers λ^k given by Jordan miniblocks of dimension one with real eigenvalue λ.
  - converge $|\lambda| < 1$, constant if $|\lambda| = 1$ or diverge exponentially if $|\lambda| > 1$. We have an oscillation if $\lambda < 0$
- Series of functions $C_q^k \lambda^{k-q}$ given by Jordan miniblocks of dimension $m > 1$ (and hence $k = 0, \dots, m-1$) with real eigenvalue λ.
  - Converge if $|\lambda| < 1$, diverge polinomially if $\lambda = 1$, diverge exponentially of $|\lambda| > 1$. We still have an oscillation if $\lambda < 0$.

Discrete Time Systems

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- Series of functions $C_q^k \lambda^{k-q}$ given by Jordan miniblocks of dimension $m > 1$ (and hence $k = 0, \ldots, m-1$) with real eigenvalue λ.
  - Converge if $|\lambda| < 1$, diverge polinomially if $\lambda = 1$, diverge exponentially of $|\lambda| > 1$. We still have an oscillation if $\lambda < 0$.
- Oscillating sequence of the type $\rho^k \cos(k\theta)$ and $\rho^k \sin(k\theta)$ given by two miniblocks of dimension 1 associated to complex and conjugated eigenvalues $\lambda = \sigma \pm j\omega = \rho e^{\pm j\theta}$.
  - Converge if $\rho < 1$, oscillate persistently if $\rho = 1$, diverge exponentially id $\rho > 1$

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  - Converge if $\rho < 1$, diverge polynomially if $\rho = 1$ and $k > 0$ or diverge exponentially if $\rho > 1$.