Teoria dei sistemi.

Introduction

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IO representation

Suppose our system is described by an a differential (or by a difference) equation

$$\mathfrak{D}^n y(t) = F(y(t), \mathfrak{D} y(t), \mathfrak{D}^2 y(t), \ldots, \mathfrak{D}^n y(t), u, \mathfrak{D} u(t), \ldots, \mathfrak{D}^p u(t), t),$$

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Restriction

Let us restrict for simplicity to $p = 1$.

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Derivation of state space representation

Set $x_i = \mathfrak{D}^{i-1} y$, with the obvious implication that $\mathfrak{D}x_i = \mathfrak{D}\mathfrak{D}^i y = \mathfrak{D}^{i+1} y = x_{i+1}$. We can write the equation as:

$$\mathfrak{D} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \ldots \\ F(x_1, x_2, \ldots, x_n, u, t) \end{bmatrix} \tag{1}$$

$$y = x_1. \tag{2}$$

Deriving state space from IO representation

IO representation

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with $p > 1$, things are not so simple.

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Example

Consider a system whose I/O representation is given by

$$\dot{y} = y + 5\dot{u} - u.$$

Suppose, that we set $x_1 = y - 5u$. We have,

$$\begin{aligned}
\dot{x_1} &= \dot{y} - 5\dot{u} \\
&= y - u = \\
&= y - 5u + 4u \\
&= x_1 + 4u \\
y &= x_1 + 5u
\end{aligned}$$

Example (Continued)

A first State Space Description

Consider a system whose I/O representation is given by

$$\dot{y} = y + 5\dot{u} - u.$$

Our state space model

$$\begin{aligned} \dot{x}_1 &= x_1 + 4u \\ y &= x_1 + 5u \end{aligned}$$

An equivalent description

If you now consider

$$\begin{aligned} \dot{x}_1 &= x_1 + 4u \\ \dot{x}_2 &= 3u \\ y &= x_1 + 5u \end{aligned}$$

From an IO perspective the behaviours generated by the two systems are absolutely the same.

Lessons Learned

Lessons Learned

- There are more than one state space representation for the same I/O description,
- Some systems (that we will learn to qualify as linear) allow for a much more compact state

Linear Systems

Recap

An abstract system $\Sigma(t_0)$ is defined as a set of pairs $(u_0(t), y_0(t))$ with $u_0(t) \in \mathcal{U}^{\mathcal{T}(t_0)}$ and $y_0(t) \in \mathcal{Y}^{\mathcal{T}(t_0)}$

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Additional Assumption

The sets $\mathcal{U}$ and $\mathcal{Y}$ are vector spaces defined over the real set $\mathbb{R}$. Clearly the range sets U and Y where the functions take value have the same algebraic structure.

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Additional Assumption

If $u_1(t) \in \mathcal{U}$ and $u_2(t) \in \mathcal{U}$, then $\alpha u_1(t) + \beta u_2(t) \in U$ for any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ (and the same for any $y_1(t)$ and $y_2(t) \in \mathcal{Y}$).

Linear Systems: definition

Linear Systems

The system $\Sigma(t_0)$ is linear if given any two pairs $(u_1(t), y_1(t))$, $(u_2(t), y_2(t))$, if they both belong to the system $\Sigma(t_0)$, so will their linear combination: $\alpha\,(u_1(t), y_1(t)) + \beta\,(u_2(t), y_2(t)) \in \Sigma(t_0)$, where $\alpha, \beta \in \mathbb{R}$.

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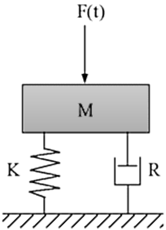
Linearity and Parametric description

Consider now a parametric description of a system $(X_{t_0}, \pi(\cdot,\, \cdot))$. For a given parameter x_0, the system responds to an input u_1 with the output $y_1 = \pi_{t_0}(x_0, u_1)$ and to the input u_2 with the output $y_2 = \pi_{t_0}(x_0, u_2)$. The intuitive meaning of linearity is:

- *Superimposition*: the response to the combination $u_1(t) + u_2(t)$ will be: $y_1(t) + y_2(t)$
- *Scaling*: the response to $\alpha u_1(t)$ is $\alpha y_1(t)$.

Example



The Newton equation leads us to following differential equation

$$m\ddot{x} = F(t) - mg - K(x - x_{rest}) - R\dot{x}$$

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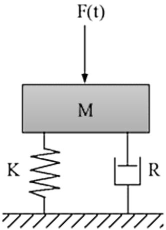

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$$m\ddot{x} = F(t) - mg - K(x - x_{rest}) - R\dot{x}$$

Some of two signals

Starting from a fixed initial condition, $x_1(t)$ is the evolution for $F_1(t)$ and $x_2(t)$ is the evolution for $F_2(t)$.

$$m\ddot{x}_1 = F_1(t) - Kx_1 - R\dot{x}_1$$
$$m\ddot{x}_2 = F_2(t) - Kx_2 - R\dot{x}_2$$

It turns out that $\alpha x_1(t) + \beta x_2(t)$ is a solution to $\alpha F_1(t) + \beta F_2(t)$.

Observation

Differentiation is a linear operation

The considerations in the example before apply because the differential equation describing the system is linear and the derivative is a linear operator: $\mathfrak{D}(\alpha f(t) + \beta g(t)) = \alpha\mathfrak{D}(f(t)) + \beta\mathfrak{D}(g(t))$.

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Inverted pendulum

If we repeat the same line of reasoning with the inverted pendulum, we will fail because the equation is not linear.

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Inverted pendulum

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Saturation

Even if we consider the mass spring example but we impose a maximum value for the force $F(t)$, the scaling principle will not apply and the linearity will be lost.

Linear Ordinary Differential Equations

The same argument of the mass-spring system can be repeated for *any* linear difference/differential equation

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i(t)\mathfrak{D}^i y(t) + \sum_{j=0}^{p} \beta_j(t)\mathfrak{D}^j u(t)$$

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Solution

For well-behaved $\alpha_i(t)$,$\beta_i(t)$ and $u(t)$ function and for causal system, the equation has a unique solution, which is a function of $u(\cdot)$ and of the initial conditions:

$$y(t) = \phi\left(t, t_0, y(0), \mathfrak{D}y(0), \ldots, \mathfrak{D}^{n-1}y(0), u(0), \ldots, \mathfrak{D}^p u(0), u|_{[t_0,t]}\right).$$

Two fundamental lemmas

Lemma

For the Equation

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i(t)\mathfrak{D}^i y(t) + \sum_{j=0}^{p} \beta_j(t)\mathfrak{D}^j u(t)$$

define its associated homogeneous equation as:

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_h(t)\mathfrak{D}^i y(t) \tag{3}$$

The solution to this equation form a vector space and the dimension of this space is n. This applies to both differential and difference equations

Two fundamental lemmas

Lemma

For the Equation

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i(t)\mathfrak{D}^i y(t) + \sum_{j=0}^{p} \beta_j(t)\mathfrak{D}^j u(t),$$

suppose that $\tilde{y}$ is a particular solution related to the input $\tilde{u}$. If we sum $\tilde{y}$ to any solution $\hat{y}$ of the homoegeneous system, we still obtain a solution of the differential equation.

General Expression for the solution

General solution

By choosing any basis $\{\hat{y}_1, \ldots, \hat{y}_n\}$ of the vector space of the solutions of the homegeneous equaiton, we can write a general solution of of the differential eqquation as

$$y(t) = \tilde{y}(t) + H_1\hat{y}_1(t) + \ldots + H_n\hat{y}_n(t)$$

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The H_i coeeficients can be found by imposing n conditions derived from the initial conditions.

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Observation

The *linearity* of the system allows breaking down the solution in two pieces:

- Forced Evolution $\tilde{y}$ (determined by the input $\tilde{u}$)
- Free evolution $H_1\hat{y}_1(t) + \ldots + H_n\hat{y}_n(t)$ (depending on the initial conditions).

State Space Representation for Linear Systems

State Space Representation

A state space *explicit representation* is given by a couple of functions ϕ and η, defined as:

$$\begin{aligned} \phi : \mathcal{T} \times \mathcal{T} \times X \times \mathcal{U} \to X & \qquad x(t) = \phi(t, t_0, x_0, u) \\ \eta : \mathcal{T} \times X \times U \to X & \qquad y(t) = \eta(t, x, u) \end{aligned}$$

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$$\eta : \mathcal{T} \times X \times U \to X \qquad y(t) = \eta(t, x, u)$$

Linearity

A representation of this type is linear if both functions are linear with respect to their arguments $X \times \mathcal{U}$.

Explicit State Space Representation for Linear Systems

Linear ϕ

$$\begin{cases} \phi(t, t_0, x_0, u_1|_{[t_0,t)}) = x_1 \\ \phi(t, t_0, x_0, u_2|_{[t_0,t)}) = x_2 \end{cases} \implies \phi(t, t_0, x_0, \alpha u_1|_{[t_0,t)} + \beta u_2|_{[t_0,t)}) = \alpha x_1 + \beta x_2$$

$$\begin{cases} \phi(t, t_0, x_{0,1}, u|_{[t_0,t)}) = x_1 \\ \phi(t, t_0, x_{0,2}, u|_{[t_0,t)}) = x_2 \end{cases} \implies \phi(t, t_0, \alpha x_{0,1} + \beta x_{0,2}, u) = \alpha x_1 + \beta x_2.$$

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Linear η

$$\begin{cases} \eta(t, x_0, u_1) = y_1 \\ \eta(t, x_0, u_2) = y_2 \end{cases} \implies \eta(t, x_0, \alpha u_1 + \beta u_2) = \alpha y_1 + \beta y_2$$

$$\begin{cases} \eta(t, x_{0,1}, u) = y_1 \\ \eta(t, x_{0,2}, u) = y_2 \end{cases} \implies \eta(t, \alpha x_{0,1} + \beta x_{0,2}, u) = \alpha y_1 + \beta y_2.$$

Implicit State Space Representation for Linear Systems

Implicit representantion

$$
\begin{aligned}
f &: \mathcal{T} \times X \times U \to X & \qquad \mathfrak{D}x(t) &= f(t, x, u) \\
\eta &: \mathcal{T} \times X \times U \to X & y(t) &= \eta(t, x, u)
\end{aligned}
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Implicit State Space Representation for Linear Systems

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\end{aligned}
$$

Linerarity

f has to be linear with respect to $X \times U$. For ϕ we require linearity with respect to $X \times \mathcal{U}$ (where $\mathcal{U}$ is a space of signals).
For f and for η we require linearity with respect to $X \times U$ (where U is the range of the input signals $\mathcal{U}$).

Implicit State Space Representation for Linear Systems

Matrix Representation

The linearity of f and η can be expressed in matrix notation as:

$$\begin{aligned} \mathfrak{D}x(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned} \tag{4}$$

Time Invariance

General definition of time invariance for an abstract system $\Sigma(t_0)$ defined as a set of pairs $(u_0(t), y_0(t))$.

Definition

A system $\Sigma(t_0)$ is time invariant iff

$$(u_0(t), y_0(t)) \in \Sigma(t_0) \implies (u_0(t-t_0), y_0(t-t_0)) \in \Sigma(t_0-t_0).$$

If a behaviour $(u_0(t), y_0(t))$ is generated by the system, so will the same behaviour with u_0 and y_0 shifted forward or backward in time of the same amount.

Time Invariance and IO Representations

Time Invariant Differential and DIfference Equations

An IO representation of a system is time invariant if the differential or the difference equation do not explicitly depend on t.

Time Invariance and IO Representations

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Banking Account Example

$$C((k+1)T) = \begin{cases} \left(1 + I_+ \cdot \frac{T}{12}\right)\left(C(kT) + S(kT)\right) & \text{Se } C(kT) + S(kT) \geq 0 \\ \left(1 + I_- \cdot \frac{T}{12}\right)\left(C(kT) + S(kT)\right) & \text{Se } C(kT) + S(kT) < 0 \end{cases}$$

- I_+, I_- constant in time $\rightarrow$ time invariant system.

Time Invariance and IO Representations

Mass Spring

$$m\ddot{x}_1 = F_1(t) - Kx_1 - R\dot{x}_1$$

- m, K, R constant in time $\rightarrow$ time invariant system.

Time Invariance and State Representations

Explicit Form

A state representation in explicit form is time invariant if and only if *phi* does not depends singularly on t and t_0, but it depends on their difference, and if η does not depend on t

$$\begin{aligned} \phi : \mathcal{T} \times \mathcal{T} \times X \times \mathcal{U} \to X \quad & x(t) = \phi(t, t_0, x_0, u) & = \phi(t - t_0, x_0, u) \\ \eta : \mathcal{T} \times X \times U \to X \quad & y(t) = \eta(t, x, u) & = \eta(x, u) \end{aligned}$$

Implicit Form

If we derive the implicit form, it is easy to see that this implies that f and η do not depend on t.

$$\begin{aligned} f : \mathcal{T} \times X \times U \to X \qquad & \mathfrak{D}x(t) = f(t, x, u) & = f(x, u) \\ \eta : \mathcal{T} \times X \times U \to X \qquad & y(t) = \eta(t, x, u) & = \eta(x, u) \end{aligned}$$

Time Invariance and State Representations

Linear Systems

In the case of linear system, time invariance means that matrices A, B, C, D are constants

$$\begin{aligned} \mathfrak{D}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{5}$$

Hybrid SYstems

Intuition

In addition to CT, DT and CE systems, we have hybrid combinations of the different model.

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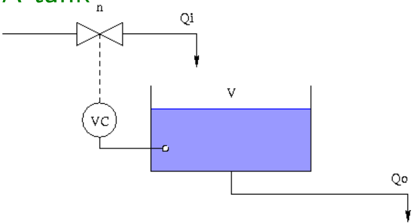
A tank



In the tank in the picture the valve makes the system switch between two dynamics and the switches are ruled by a state machine. These systems are out fo the scope of this course.