Teoria dei sistemi.

Laplace transform

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November 9, 2016

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A few basic facts

- LTI system respond to exponential signals (e^{st} for CT systems and z^t for DT systems) in a special way.
- their corresponding output is the same function multiplied by an eigenvalue, which is given by:

$$H(s) = \int_0^{\infty} h(\tau)e^{-s\tau}d\tau \qquad \text{For CT systems}$$

$$H(z) = \sum_0^{\infty} h(\tau)z^{-\tau} \qquad \text{For DT systems.}$$

- This is the basis for a solution strategy based on the so-called Laplace and Z transform

Complex Exponential

Exponential function properties

An exponential function has a fundamental property:

$$e^a e^b = e^{a+b}$$
$$e^a / e^b = e^{a-b}$$
$$(e^a)^n = e^{na}.$$

Euler Exponential

THe fucntion

$$e^{j\theta} = \cos\theta + j\sin\theta,$$

has the very same properties.

Properties of Euler Exponential

Proof

$$
\begin{aligned}
e^{j\theta} e^{j\psi} &= (\cos\theta + j\sin\theta)(\cos\psi + j\sin\psi) \\
&= (\cos\theta\cos\psi - \sin\theta\sin\psi) + j(\sin\theta\cos\psi + \sin\psi\cos\theta) \\
&= (\cos(\theta+\psi) + j(\sin\theta+\psi) \\
&= e^{j(\theta+\psi)}.
\end{aligned}
$$

Real and Imaginary part

In obvius ways we define exponentials with real and imaginary part:

$$
\begin{aligned}
e^{\sigma + j\theta} &= e^{\sigma} e^{j\theta} \\
&= e^{\sigma}(\cos\theta + j\sin\theta).
\end{aligned}
$$

Properties of Euler Exponential

Computation of Powers

Computation of a power of a complex number z: if we express $z = \rho e^{j\theta}$, we have $z^n = \rho^n e^{jn\theta}$.

Complex Exponential Signals

- CT: let $s = \sigma + j\omega$, with $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}$, we define CT complex exponential the function:

$$e^{st} : \mathbb{R} \to \mathbb{C} = e^{(\sigma + j\omega)t} =$$
$$= e^{\sigma t} \left(\cos \omega t + j \sin \omega t \right).$$

  Oscillations with amplitude modulated by an exponential function (increasing if $\sigma > 0$ and decreasing if $\sigma < 0$).
- DT: Let $z = \rho e^{j\theta}$, we define the DT exponential as

$$z^t = \rho^t e^{jt\theta}.$$

Laplace Transform

Laplace Transform

The Laplace transform is an integral operator defined as

$$\mathcal{L}\left(f(t)\right) = F(s) = \int_0^{\infty} f(\tau)e^{-s\tau}d\tau.$$

The idea is to associate each signal $f(t)$ with a new function $F(s)$

Example 1: the step function

Laplace trasform of $\mathbf{1}(t)$

$$
\begin{aligned}
\mathcal{L}\left(\mathbf{1}(t)\right) &= \int_0^{\infty} \mathbf{1}(\tau) e^{-s\tau} d\tau = \\
&= \int_0^{\infty} e^{-s\tau} d\tau = \\
&= \frac{1}{s}\left(1 - \lim_{t\to\infty} e^{-st}\right).
\end{aligned}
$$

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\end{aligned}
$$

We can see

$$
lim_{t\to\infty}e^{-st} = \begin{cases} 0 & \textbf{Real}\left(s\right) > 0 \\ \text{Does not converge} & \textbf{Real}\left(s\right) \le 0. \end{cases}
$$

Therefore....

Example 1: the step function

Laplace trasform of $\mathbf{1}(t)$

$$\mathcal{L}\left(\mathbf{1}(t)\right) = \begin{cases} \frac{1}{s} & \text{if } \mathbf{Real}\left(s\right) > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Example 2: the truncated Exponential

Laplace trasform of $\mathbf{1}(t)e^{at}$, $a \in \mathbb{R}$

$$
\begin{aligned}
\mathcal{L}\left(\mathbf{1}(t)e^{at}\right) &= \int_0^{\infty} \mathbf{1}(t)e^{a\tau}e^{-s\tau}d\tau = \\
&= \int_0^{\infty} e^{-(s-a)\tau}d\tau = \\
&= \frac{1}{s-a}\left(1 - \lim_{t\to\infty} e^{-(s-a)t}\right).
\end{aligned}
$$

where

$$
lim_{t\to\infty}e^{-(s-a)t} = \begin{cases} 0 & \textbf{Real}\,(s) = \sigma > a \\ \text{Does not converge} & \textbf{Real}\,(s) = \sigma \leq a. \end{cases}
$$

Hence, ...

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Complex Exponential Functions

The same applies to complex exponential functions: $e^{(\sigma+j\omega)t}$ with $\sigma \in \mathbb{R}$ and $\omega \in \mathbb{R}$. In this case

$$\mathcal{L}\left(\mathbf{1}(t)e^{(\sigma+j\omega)t}\right) = \begin{cases} 1/(s-\sigma-j\omega) & \textit{if}\,\textbf{Real}\,(s) \geq \sigma. \\ \text{undefined} & \text{otherwise} \end{cases}$$

Example 3: the Dirac δ

Dirac δ

$$
\begin{aligned}
\mathcal{L}\left(\delta(t)\right) &= \int_0^\infty \delta(t) e^{-s\tau} d\tau = \\
&= \int_0^\infty \delta(t) e^{-s0} d\tau = \\
&= \int_0^\infty \delta(t) d\tau = \\
1
\end{aligned}
$$

This transform is defined for all possible s.

Lessons learned

Lessons Learned from the examples

- The Laplace transform of a CT signal is a function of a complex variable s,
- The Transform is defined on region of convergence (ROC) where the Transform makes sense.
- For the step function $\mathbf{1}(t)$ the ROC is $\textbf{Real}\,(s) \geq 0$.
- For an exponential signal $\mathbf{1}(t)e^{at}$ the ROC is $\textbf{Real}\,(s) > a$.

Issues to address

1. Is if the relation between a function and its Laplace transform is a bijection.
2. In the affirmative case, how to invert the Laplace transform?
3. What is the meaning and the practical use of this function.

Existence and Uniqueness of the Laplace Transform

In order to discuss existince and uniqueness, we need some definitions and results.

Definition of exponential order

A function $f(t)$ is said of exponential order γ if there exist a constant A such that

$$f(t) \leq Ae^{\gamma t}.$$

Theorem on Existence

Theorem

Theorem

Consider a function $f(t)$ and assume that: 1) $f(t)$ is continuous, 2) $f(t)$ is of exponential order γ. Then the Laplace transform:

$$\mathcal{L}\left(f(t)\right) = \int_0^\infty f(\tau)e^{-s\tau}d\tau,$$

exists and the ROC contains the half-space $\textbf{Real}\left(s\right) > \gamma$.

Almost equal functions

In order to deal with the invertibility problem, we need the following:

Definition of Almost Equality

Definition

Two functions $f(t)$ and $g(t)$ are said almost equal, $f(t) \approx g(t)$, if $f(t)$ and $g(t)$ are equal for all t except for a set of points of null measure.

Almost equal functions

Example

Consider the two functions $f(t) = e^{3t}$ and

$$g(t) = \begin{cases} 0 & \text{if } t = 2, 4, 6, 8 \ldots \\ e^{3t} & \text{otherwise.} \end{cases}$$

Almost equal functions

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$$g(t) = \begin{cases} 0 & \text{if } t = 2, 4, 6, 8 \dots \\ e^{3t} & \text{otherwise.} \end{cases}$$

The two functions are almost equal because the set $t = 2k$, with $k \in \mathbb{N}$ has null measure.

Invertibility of Laplace Transform

Theorem

Theorem (Lerch's Theorem)

Suppose f(t) and g(t) are continuous except for a countable number of isolated points, and that they are of exponential order γ. Then if $\mathcal{L}(f(t)) = \mathcal{L}(g(t))$ for all $s > \gamma$ the two functions are almost equal: $f(t) \approx g(t)$.

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Abuse of notation

Form an engineering perspective two almost equal functions are equal. We will therefore adopt the following abuse of notation: $f(t) = g(t)$.

Inverse Laplace Transform

In view of Lerch's theorem, we can invert Laplace transform under mild conditions.

Inverse Laplace Transform

The inverse is given by:

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds,$$

where $\mathbf{Real}(s) = \sigma$ belongs to the ROC.

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This integral is difficult to compute. We will find better ways to invert the Laplace Transform.

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- This integral is difficult to compute. We will find better ways to invert the Laplace Transform.
- However, it reveals quite clearly the idea of Laplace Transform as a way to express a signal using a basis of exponential functions.

A first application of the Laplace Transform

We will now see a couple of examples that are direct applications of the notion of eigenfucntions.

Example 1: Response to $e^{\alpha t}$

- Suppose a system has impulse response $\mathbf{1}(t)e^{-t}$
- Find The forces reqsponse to $u(t) = e^{\alpha t}$

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- As long as $\alpha > -1$, we have that $e^{\alpha t}$ is an eigenfucntion related to the eigenvalue $H(s)$ for $s = \alpha$, where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = 1/(s+1)$

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- Therefore we will have

$$y(t) = \frac{1}{\alpha + 1} e^{\alpha t}.$$

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$$y(t) = |H(j4)| \cos(4t + \angle H(j4).$$

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$$\begin{aligned} |H(j4)| &= \left|\frac{1}{j4+3}\right| = \\ &= \frac{|1|}{|3+j4|} = \\ &= \frac{1}{\sqrt{9+16}} = \\ &= \frac{1}{5}, \end{aligned}$$

$$\begin{aligned} \angle H(j4) &= \angle \frac{1}{j4+3} = \\ &= \angle 1 - \angle 3 + 4j \\ &= -\arctan 4/3. \end{aligned}$$

Properties of the Laplace Transform

- We have seen formulas for direct and inverse computation of the Laplace Transform

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    1. Use a few "elementary" transforms as building blocks
    2. Apply some properties to deal with more complex cases

Properties of the Laplace Transform

- We have seen formulas for direct and inverse computation of the Laplace Transform
- We will now look for a better way
  1. Use a few "elementary" transforms as building blocks
  2. Apply some properties to deal with more complex cases
- We will now go through some of these properties

Linearity

From the linearity of the integral operator descends the following:

Linearity of the Laplace Transform

Theorem

Let $F_1(s) = \mathcal{L}(f_1(t))$ *and* $F_2(s) = \mathcal{L}(f_2(t))$. *Then,*

$$\mathcal{L}(h_1 f_1(t) + h_2 f_2(t)) = h_1 F_1(s) + h_2 F_2(s).$$

Example

Example application of linearity

Let us compute the Laplace transform of $\cos 3t$.

$$
\begin{aligned}
\mathcal{L}\left(\mathbf{1}(t)\cos(3t)\right) &= \mathcal{L}\left(\mathbf{1}(t)\frac{e^{j3t}+e^{-j3t}}{2}\right) \\
&= \frac{1}{2}\left(\mathcal{L}\left(\mathbf{1}(t)e^{j3t}+\mathbf{1}(t)e^{-j3t}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{s-j3}+\frac{1}{s+j3}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2-(j3)^2}\right) \\
&= \frac{s}{s^2+9}.
\end{aligned}
$$

Time Shifting

Time Shifting for the Laplace Transform

Theorem

Let $F(s) = \mathcal{L}\left(\mathbf{1}(t)f(t)\right)$*. Then* $F(s)e^{-st_0} = \mathcal{L}\left(\mathbf{1}(t-t_0)f(t-t_0)\right)$.

Proof of the Time Shifting property

The Proof is through a simple change of variables

$$\begin{aligned}
\mathcal{L}\left(\mathbf{1}(t-t_0)f(t-t_0)\right) &= \int_0^\infty \mathbf{1}(t-t_0)f(t-t_0)e^{-st}dt \\
&= \int_0^\infty \mathbf{1}(t-t_0)f(t-t_0)e^{-st}e^{-st_0}e^{st_0}dt \\
&= e^{-st_0}\int_0^\infty \mathbf{1}(t-t_0)f(t-t_0)e^{-s(t-t_0)}dt \\
&= e^{-st_0}\int_{-t_0}^\infty \mathbf{1}(t')f(t')e^{-st'}dt' \\
&= e^{-st_0}\int_0^\infty \mathbf{1}(t')f(t')e^{-st'}dt' \\
&= e^{-st_0}F(s).
\end{aligned}$$

Example

Example of time shifting

Find $\mathcal{L}\left(f(t)\right)$ with

$$f(t) = \begin{cases} 1 & \text{If } t \in [0,\, 1] \\ 0 & \text{otherwise.} \end{cases}$$

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Computation

$f(t) = \mathbf{1}(t) - \mathbf{1}(t-1) \rightarrow$

$$\begin{aligned} \mathcal{L}\left(f(t)\right) &= \mathcal{L}\left(\mathbf{1}(t)\right) - \mathcal{L}\left(\mathbf{1}(t-1)\right) \\ &= \frac{1}{s} - \frac{e^{-s}}{s} \\ &= \frac{1-e^{-s}}{s}. \end{aligned}$$

Shifting in the Laplace Domain

We can also prove the following dual property.

Shifting in the Laplace domain

Theorem

Left $F(s) = \mathcal{L}(f(t))$. *Then* $F(s - s_0) = \mathcal{L}(f(t)e^{s_0 t})$.

Proof of Shifting in the Laplace Domain

Proof

$$\begin{aligned}
\mathcal{L}\left(f(t)e^{s_0 t}\right) &= \int_0^\infty f(\tau)e^{s_0\tau}e^{-s\tau}\,d\tau \\
&= \int_0^\infty f(\tau)e^{-(s-s_0)\tau}\,d\tau \\
&= F(s-s_0),
\end{aligned}$$

Example

Example 1

Let $\mathcal{L}\left(\mathbf{1}(t)\right) = 1/s$, then

- $\mathcal{L}\left(\mathbf{1}(t)e^{at}\right) = 1/(s-a)$.

Example 2

$\mathcal{L}\left(\mathbf{1}(t)\cos\omega t\right) = s/(s^2+\omega^2)$, then

- $\mathcal{L}\left(\mathbf{1}(t)e^{at}\cos\omega t\right) = (s-a)/((s-a)^2+\omega^2)$.

Time Scaling

Time Scaling for the Laplace Transform

Theorem

Let $F(s) = \mathcal{L}(f(t))$ *and let* $a \in \mathbb{R}^+$. *Then* $\mathcal{L}(f(at)) = \frac{1}{a}F(s/a)$.

Proof of the Time Scaling property

The Proof is through a direct application of the definition

$$\begin{aligned}
\mathcal{L}\left(f(at)\right) &= \int_0^\infty f(at)e^{-st}dt \\
&= \int_0^\infty f(t')e^{-st'/a}\frac{dt'}{a} \\
&= \frac{1}{a}F(s/a).
\end{aligned}$$

Example

Example 1

Let $\mathcal{L}\left(\mathbf{1}(t)\cos t\right) = s/(s^2+1)$ then

$$\begin{aligned}\mathcal{L}\left(\mathbf{1}(t)\cos\omega t\right) &= \frac{1}{\omega}\frac{s/\omega}{s^2/\omega^2+1}\\ &= \frac{s}{s^2+\omega^2}.\end{aligned}$$

Convolution

Convolution of two signals

Theorem

Let $f(t)$ and $h(t)$be two causal functions and let $\mathcal{L}(f(t)) = F(s)$ and $\mathcal{L}(h(t)) = H(s)$. Then,

$$\mathcal{L}(f(t) * h(t)) = F(s)H(s).$$

Proof

Step 1: Application of the definition

Let $g(t) = f(t) * h(t)$ then

$$\begin{aligned}\mathcal{L}\left(g(t)\right) &= \int_0^\infty e^{-st} f(t) * h(t) dt \\ &= \int_0^\infty e^{-st} \left(\int_0^t h(\tau) f(t-\tau)\tau \right) dt \\ &= \int_0^\infty \int_0^t e^{-st} h(\tau) f(t-\tau) d\tau dt\end{aligned}$$

Step 2: change of integration order

Integration in triangle $0 \leq \tau \leq t$. We can change the order:

$$\mathcal{L}\left(g(t)\right) = \int_{\tau=0}^\infty \int_{t=\tau}^\infty e^{-st} h(\tau) f(t-\tau) dt d\tau.$$

Proof

Step 3: change of variable

Let $\bar{t} = t - \tau$.

$$\begin{aligned}
\mathcal{L}\left(g(t)\right) &= \int_{\tau=0}^{\infty}\int_{\bar{t}=0}^{\infty} e^{-s(\bar{t}+\tau)}h(\tau)f(\bar{t})d\bar{t}d\tau \\
&= \left(\int_{\tau=0}^{\infty} e^{-s\tau}h(\tau)d\tau\right)\left(\int_{\bar{t}=0}^{\infty} e^{-s\bar{t}}f(\bar{t})d\bar{t}\right) \\
&= F(s)H(s).
\end{aligned}$$

Consequences

The forced response to any signal $u(t)$ can be found as follows:

1. compute the *Transfer Function* $H(s) = \mathcal{L}\left(h(t)\right)$,

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Consequences

The forced response to any signal $u(t)$ can be found as follows:

1. compute the *Transfer Function* $H(s) = \mathcal{L}(h(t))$,
   - how?
2. compute $U(s)$,
3. compute the inverse transform of $H(s)U(s)$.
   - how?

Differentiation

A key property is the following:

Differentiation Rule

Theorem

Let $F(s) = \mathcal{L}(f(t))$. *Then,*

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f(0).$$

Proof

Application of integration by parts

$$
\begin{aligned}
\mathcal{L}\left(\frac{df(t)}{dt}\right) &= \int_0^\infty \frac{df(t)}{dt} e^{-st} dt \\
&= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st}) dt \\
&= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\int_0^\infty f(t) e^{-st} dt \\
&= sF(s) - f(0) \quad \text{(in the ROC, } \lim_{t\to\infty} e^{-st} f(t) = 0\text{)}
\end{aligned}
$$

Consideration

- The differentiation rule offers a clear avenue to the solution of linear differential equations

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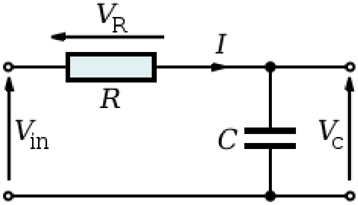
- To see this let us start from an example

Example



Let $u(t) = V_{in}(t)\mathbf{1}(t)$, $\tau = RC$, and $y(t) = V_c(t)$. Evolution:

$$\dot{y} = -\frac{y}{RC} + \frac{u(t)}{RC}$$

$$\mathcal{L}(\dot{y}) = \mathcal{L}\left(-\frac{y}{RC} + \frac{u(t)}{RC}\right)$$

$$sY(s) - y(0) = -\mathcal{L}\left(-\frac{y}{RC}\right) + \mathcal{L}\left(\frac{u(t)}{RC}\right)$$

$$sY(s) - y(0) = -\frac{Y(s)}{\tau} + \frac{U(s)}{\tau}$$

$$Y(s) = \frac{U(s)}{\tau(s + \frac{1}{\tau})} + \frac{y(0)}{s + \frac{1}{\tau}} =$$

$$Y(s) = \frac{1}{\tau s(s + \frac{1}{\tau})} + \frac{y(0)}{s + \frac{1}{\tau}}.$$

Example

Observations

- Automatics decomposition between forced evolution ($\frac{1}{\tau s(s+\frac{1}{\tau})}$) and free evolution ($\frac{y(0)}{s+\frac{1}{\tau}}$)

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- Since: 1. $\frac{U(s)}{\tau(s+\frac{1}{\tau})}$ and, 2. from the convolut $Y(s) = H(s)U(s)$, where $H(s)$ is the transfer function, **THEN** $H(s) = \frac{1}{\tau(s+\frac{1}{\tau})}$

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- Since: 1. $\frac{U(s)}{\tau(s+\frac{1}{\tau})}$ and, 2. from the convolut $Y(s) = H(s)U(s)$, where $H(s)$ is the transfer function, **THEN** $H(s) = \frac{1}{\tau(s+\frac{1}{\tau})}$
- We have

$$\begin{aligned} \frac{1}{\tau s(s+\frac{1}{\tau})} &= \frac{1}{\tau}\left(\frac{\tau}{s} - \frac{\tau}{s+\frac{1}{\tau}}\right) \\ &= \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}} \end{aligned}$$

In view of the invertibility of the Laplace transform:

$$y(t) = \mathbf{1}(t)(1 - e^{t/\tau}) + y(0)e^{-t/\tau}.$$

General form of the differentiation rule

By recursive application of the differentiation rule:

- For the case of the second derivative:

$$\mathcal{L}\left(\mathfrak{D}^2 f(t)\right) = s\mathcal{L}\left(\mathfrak{D} f(t)\right) - \mathfrak{D} f(0) = s^2 F(s) - sf(0) - \mathfrak{D} f(0).$$

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- In the general case:

$$\mathcal{L}\left(\mathfrak{D}^n f(t)\right) = s^n F(s) - s^{n-1} f(0) - s^{n-2}\mathfrak{D} f(0) - \ldots \mathfrak{D}^{n-1} f(0).$$

Integration Rule

Integration Rule

Theorem

Le $F(s) = \mathcal{L}(f(t))$, *then*

$$\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}.$$

Proof

Application of the convolution theorem

Observe that $\int_0^t f(\tau)d\tau = f(t) * \mathbf{1}(t)$ and apply the convolution rule ($\mathcal{L}(\mathbf{1}(t)) = \frac{1}{s}$)

Observation

Integral and differential operations in the time domain become simple algebraic operations in the Laplace domain.

Initial and Final Value

A lot can be read from the expression of the Laplace transform without computing the inverse.

Initial and Final Value

Theorem

Let $F(s) = \mathcal{L}(f(t))$. *Then:*

1. *If* $\lim_{t\to 0} f(t)$ *exists then* $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$,
2. *If* $\lim_{t\to\infty} f(t)$ *exists and is finite, then* $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Proof

First claim

From the differentiation rule:

$$\mathcal{L}(\mathfrak{D}f(t)) = \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt = sF(s) - f(0). \tag{1}$$

If we compute $\int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$ for $s \to \infty$, we find:

$$\int_0^\infty \frac{d}{dt} f(t) e^{-s\infty} dt = 0,$$

Therefore, we find $f(0) = \lim_{s\to\infty} sF(s)$.

Proof

Second claim

$$\lim_{s->0}\int_0^\infty \frac{d}{dt}f(t)e^{-st}dt = \int_0^\infty \lim_{s->0}\frac{d}{dt}f(t)e^{-st}dt \qquad =$$
$$= \frac{d}{dt}f(t)\Big|_0^\infty = f(\infty) - f(0).$$

Consider that

$$\mathcal{L}\left(\mathfrak{D}f(t)\right) = \int_0^\infty \frac{d}{dt}f(t)e^{-st}dt = sF(s) - f(0). \tag{2}$$

we find for $s \to 0$:

$$\lim_{s\to 0} sF(s) - f(0) = f(\infty) - f(0),$$

which leads us straight to the claim.

Example

Application of the final and initial value

Suppose that $F(s) = \frac{s}{s(s+2)}$. Then we have:

- $f(0) = \lim_{s\to\infty} sF(s) = 1$,
- $f(\infty) = \lim_{s\to 0} sF(s) = 0$.

Differentiation in the Laplace domain

Differentiation in the Laplace Domain

Theorem

Let $\mathcal{L}(f(t)) = F(s)$*. Then* $\mathcal{L}(-tf(t)) = dF(s)/ds$*.*

Proof

Differentiation of the formula

$$
\begin{aligned}
\frac{dF(s)}{ds} &= \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\
&= \int_0^\infty \frac{df(t)e^{-st}}{ds}dt \\
&= \int_0^\infty f(t)\frac{de^{-st}}{ds}dt \\
&= \int_0^\infty (-t)\cdot f(t)e^{-st}dt \\
&= \mathcal{L}(-tf(t))
\end{aligned}
$$

Corollary

Differentiation of n-th order in the Laplace Domain

Corollary

Let $\mathcal{L}(f(t)) = F(s)$*. Then* $\mathcal{L}((-1)^n t^n f(t)) = d^n F(s)/ds^n$*.*

The proof descends from the iterative application of the theorem on the differentiation in the Laplace domain.

Inversion of the Laplace Transform

- Consider a CT LTI

$$\mathfrak{D}^n y(t) = \sum_{i=0}^{n-1} \alpha_i \mathfrak{D}^i y(t) + \sum_{j=0}^{p} \beta_j \mathfrak{D}^j u(t), \tag{3}$$

with $p \leq n$ and initial conditions:

$$y(0), \mathfrak{D}y(0), \ldots, \mathfrak{D}^{n-1}y(0), \ldots, \mathfrak{D}^p u(0), \ldots, \mathfrak{D}u(0).$$

- By the differentiation rule

$$Y(s) = Y_{\text{forced}}(s) + Y_{\text{free}}(s) \qquad \text{with}$$

$$Y_{\text{forced}}(s) = \frac{\sum_{j=0}^{p} \beta_j s^j}{s^n - \sum_{i=0}^{n-1} \alpha_i s^i} U(s)$$

$$Y_{\text{free}}(s) = \frac{N_0(s)}{s^n - \sum_{i=0}^{n-1} \alpha_i s^i}$$

where $N_0(s)$ is a polynomial of degree $n-1$ whose coefficients are functions for the initial conditions.

Additional Observation

Looking $Y_{\text{forced}}(s)$, we have for $H(s)$ (i.e., the transform of the impulse response:

$$H(s) = \mathcal{L}(h(t)) = \frac{\sum_{j=0}^{p} \beta_j s^j}{s^n - \sum_{i=0}^{n-1} \alpha_i s^i}.$$

Furthermore

- for standard functions $u(t)$, $U(s)$ is given by a fraction of polynomials,
- we operate in conditions where the Laplace Transform is invertible

Solution of a CT LTI system

We conclude that *the solution of a differential equation of a CT LTI system requires the inversion of two Laplace functions* given by fractions of polynomial in the s variable.

Inversion of fractions of polynomials

Fraction of polynomials

General expression

$$A\frac{s^p + a_{p-1}s^{p-1} + a_{p-2}s^{p-2} + \ldots a_0}{s^n + b_{n-1}s^{n-1} + \ldots + b_0},$$

where $p \leq n$ for the causality of the system.

Inversion of fractions of polynomials

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Fundamental theorem of algebra

A polynomial of degree n with complex coefficients has exactly n complex roots.

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where $p \leq n$ for the causality of the system.

Fundamental theorem of algebra

A polynomial of degree n with complex coefficients has exactly n complex roots.

Fraction of polynomials in factorised form

$$A\frac{(s-z_1)(s-z_2)\dots(s-z_p)}{(s-p_1)(s-p_2)\dots(s-p_n)}.$$

z_i are called *zeros* while p_i are called *poles*.

The case of real and distinct poles

In this case

$$F(s) = A\frac{(s-z_1)(s-z_2)\dots(s-z_p)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$
$$= \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \dots + \frac{A_n}{s-p_n}$$

Each term $\frac{A_i}{s-p_i}$ is said a *partial fraction* and this is called *partial fraction expansion*.

Partial fraction expansion

Partial Fraction Expansion in case of real and distinct poles

Proposition

Consider a function

$$F(s) = A\frac{(s-z_1)(s-z_2)\dots(s-z_p)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

with distinct and real roots. Then the coefficients of its partial fraction expansion

$$F(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \dots + \frac{A_n}{s-p_n},$$

are given by:

$$A_i = F(s)(s-p_i)\big|_{s=p_i}.$$

Proof

We restrict for simplicity to A_1

$$F(s) = A\frac{(s-z_1)(s-z_2)\dots(s-z_p)}{(s-p_1)(s-p_2)\dots(s-p_n)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \dots + \frac{A_n}{s-p_n}$$

$$F(s)(s-p_1) = A_1 + \frac{A_2(s-p_1)}{s-p_2} + \dots + \frac{A_n(s-p_1)}{s-p_n}$$

If we evaluate the result at $s = p_1$, each of the terms $\left.\frac{A_j(s-p_1)}{s-p_j}\right|_{s=p_1} = 0$ (being the roots distinct). Therefore,

$$A_1 = F(s)(s-p_1)|_{s=p_1}$$

Example

Example of single real roots

Consider the system

$$\ddot{y} = -5\dot{y} - 4y - 4\dot{u} + u.$$

Study the system's evolution for

- $u(t) = \mathbf{1}(t)$,
- $y(0) = 1$, $\dot{y}(0) = 0$, $u(0) = 2$

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Study the system's evolution for

- $u(t) = \mathbf{1}(t)$,
- $y(0) = 1$, $\dot{y}(0) = 0$, $u(0) = 2$

Laplace Transform

$$\begin{aligned}
s^2 Y(s) - sy(0) - \dot{y}(0) &= -5sY(s) + 5y(0) - 4Y(s) - 4sU(s) + 4u(0) + U(s) \\
\left(s^2 + 5s + 4\right) Y(s) &= (1 - 4s)\, U(s) + sy(0) + 5y(0) + \dot{y}(0) + 4u(0) \\
Y(s) &= \frac{1 - 4s}{s^2 + 5s + 4} U(s) + \frac{sy(0) + 5y(0) + \dot{y}(0) + 4u(0)}{s^2 + 5s + 4} \\
Y(s) &= \frac{1 - 4s}{s^2 + 5s + 4} \frac{1}{s} + \frac{s + 5 + 8}{s^2 + 5s + 4}
\end{aligned}$$

Example

Poles

Observe that $s^2 + 5s + 4 = (s+4)(s+1)$ and
$Y(s) = \frac{1-4s}{s(s+4)(s+1)} + \frac{s+13}{(s+4)(s+1)}$

Example

Poles

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$Y(s) = \frac{1-4s}{s(s+4)(s+1)} + \frac{s+13}{(s+4)(s+1)}$

Partial Fraction Expansion

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+4} + \frac{B_1}{s+1} + \frac{B_2}{s+4}$$

$$A_1 = \frac{1-4s}{(s+4)(s+1)s} s \Big|_{s=0} = \frac{1}{4} \qquad A_2 = \frac{1-4s}{(s+4)(s+1)s}(s+1)\Big|_{s=-1} = \frac{-5}{3}$$

$$A_3 = \frac{1-4s}{(s+4)(s+1)s}(s+4)\Big|_{s=-4} = \frac{17}{12} \qquad B_1 = \frac{s+13}{(s+4)(s+1)}(s+1)\Big|_{s=-1} = 4$$

$$B_2 = \frac{s+13}{(s+4)(s+1)}(s+4)\Big|_{s=-4} = -3 \ .$$

Example

Back to the time domain

$$y(t) = \mathbf{1}(t)\left(\frac{1}{4} - \frac{5}{3}e^{-t} + \frac{17}{12}e^{-3t}\right) + \\ + \mathbf{1}(t)\left(4e^{-t} - 3e^{-3t}\right).$$

The case of complex conjugate poles

We start by a useful Lemma.

Conjugation of a polynomial with real coefficients

Lemma

Consider a polynomial $P(s)$ in a complex variable s with real coefficient. Then $P(\bar{s}) = \overline{P(s)}$.

Proof

Proof

$$
\begin{aligned}
P(\overline{p}) &= \overline{p}^n + a_{n-1}\overline{p}^{n-1} + a_{n-2}\overline{p}^{n-2} + \ldots + a_0 \\
&= \overline{p}^n + \overline{a_{n-1}p}^{n-1} + \overline{a_{n-2}p}^{n-2} + \ldots + \overline{a_0} \\
&= \overline{p^n + a_{n-1}p^{n-1} + a_{n-2}p^{n-2} + \ldots + a_0} \qquad = \overline{P(p)}.
\end{aligned}
$$

Attention

- First step is applicable because: the coefficients a_i are real and not affected by conjugation
- Second step applicable because the conjugate of a sum is the sum of the conjugates.

Consequence

Complex Conjugate roots

Theorem

Consider a polynomial in a complex variable s *with real coefficient. If* $s = p$ *is a root of the polynomial, then also its conjugate* $s = \overline{p}$ *is.*

Proof

- Consider the polynomial

$$P(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_0,$$

with $a_i \in \mathbb{R}$.

- Il p is a root, then

$$P(p) = p^n + a_{n-1}p^{n-1} + a_{n-2}p^{n-2} + \ldots + a_0 = 0.$$

- In view of Lemma 14 we have: $\overline{P(p)} = P(\overline{p})$.
- If we now observe that $P(p) = 0 \implies \overline{P(p)} = 0$

Implication

A direct implication of the theorem

Proposition

Consider a function $F(s)$ be a reation of polynomials with real coefficients. Let p_1 and $\overline{p_1}$ be a pair of complex conjugate poles.

$$F(s) = \frac{n(s)}{d(s)} = \frac{n(s)}{(s-p_1)(s-\overline{p_1})\dots}.$$

Let

$$F(s) = \frac{A_1}{s-p_1} + \frac{{A_1}'}{s-\overline{p_1}} + \dots$$

be its partial fraction expansion.
Then $A_1' = \overline{A_1}$.

Proof

Complex poles

Proposition 1 holds for any type of poles. Therefore, we can write:

$$A_1' = \lim_{s \to \overline{p_1}} \frac{n(s)}{d(s)} (s - \overline{p_1}).$$

Proof

Complex poles

Proposition 1 holds for any type of poles. Therefore, we can write:

$$A_1' = \lim_{s \to \overline{p_1}} \frac{n(s)}{d(s)}(s - \overline{p_1}).$$

Application of Lemma 14

$$\begin{aligned}
\overline{A_1'} &= \overline{\left.\frac{n(s)}{d(s)}(s - \overline{p_1})\right|_{s=\overline{p_1}}} \\
&= \frac{n(\overline{p_1})}{d_1(\overline{p_1})(\overline{p_1} - p_1)} \\
&= \frac{\overline{n(p_1)}}{\overline{d_1(p_1)}(-2j\mathbf{Imag}\,(p_1))} \\
&= \overline{\frac{n(p_1)}{d_1(p_1)(2j\mathbf{Imag}\,(p_1))}} \\
&= \overline{A_1}.
\end{aligned}$$

Inverse Transform of a Complex Conjugate Pair

If

$$F(s) = \frac{A_1}{s - p_1} + \frac{{A_1}'}{s - \overline{p_1}} + F_1(s),$$

with $p_1 = \sigma_1 + j\omega_1$, then

$$\begin{aligned}
\mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}\left(\frac{A_1}{s - p_1} + \frac{{A_1}'}{s - \overline{p_1}} + F_1(s)\right), \\
&= \mathcal{L}^{-1}\left(\frac{A_1}{s - p_1}\right) + \mathcal{L}^{-1}\left(\frac{{A_1}'}{s - \overline{p_1}}\right) + \mathcal{L}^{-1}(F_1(s)) \\
&= \mathcal{L}^{-1}\left(\frac{A_1}{s - p_1}\right) + \mathcal{L}^{-1}\left(\frac{\overline{A_1}}{s - \overline{p_1}}\right) + \mathcal{L}^{-1}(F_1(s)) \\
&= \mathbf{1}(t)A_1 e^{p_1 t} + \mathbf{1}(t)\overline{A_1} e^{\overline{p_1} t} + f_1(t) \\
&= \mathbf{1}(t)A_1 e^{p_1 t} + \mathbf{1}(t)\overline{A_1 e^{p_1 t}} + f_1(t) \\
&= 2\mathbf{1}(t)\mathbf{Real}\left(A_1 e^{p_1 t}\right) + f_1(t) \\
&= 2\mathbf{1}(t)\left|A_1\right| e^{\sigma_1 t}\cos(\omega_1 t + \angle A_1)) + f_1(t).
\end{aligned}$$

Example

Compute the response to $\mathbf{1}(t)$ of the following

$$\ddot{y} = \dot{y} - y + u(t)$$

Example

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$$\ddot{y} = \dot{y} - y + u(t)$$

Laplace Transform

$$\begin{aligned}
Y(s)(s^2 - s + 1) &= U(s) \\
Y(s) &= \frac{1}{(s^2 - s + 1)s} \\
&= \frac{1}{(s - \frac{1+\sqrt{-3}}{2})(s - \frac{1-\sqrt{-3}}{2})s} \\
&= \frac{1}{(s - \frac{1+j\sqrt{3}}{2})(s - \frac{1-j\sqrt{3}}{2})s} \\
&= \frac{A_1}{s} + \frac{A_2}{s - \frac{1+j\sqrt{3}}{2}} + \frac{\overline{A_2}}{s - \frac{1-j\sqrt{3}}{2}}
\end{aligned}$$

Example

Computation of the coefficients

$$
\begin{aligned}
A_1 &= \left.\frac{1}{(s^2 - s + 1)}\right|_{s=0} = 1 \\
A_2 &= \left.\frac{1}{(s - \frac{1-j\sqrt{3}}{2})s}\right|_{s=\frac{1+j\sqrt{3}}{2}} \\
&= \frac{1}{j\sqrt{3}\frac{1+j\sqrt{3}}{2}} \\
&= \frac{1}{-\frac{3}{2} + j\frac{\sqrt{3}}{2}} \\
&= \frac{-\frac{3}{2} - j\frac{\sqrt{3}}{2}}{\frac{9}{4} + \frac{3}{4}} \\
&= -\frac{1}{2} - j\frac{\sqrt{3}}{6}
\end{aligned}
$$

Example

Computation of the inverse transform

$$
\begin{aligned}
y(t) &= \mathbf{1}(t)\left(1 + 2\left|A_2\right| e^{\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t + \angle A_2)\right) \\
\left|A_2\right| &= \sqrt{\frac{1}{4} + \frac{3}{36}} \\
&= \frac{\sqrt{3}}{3} \\
\angle A_2 &= \operatorname{atan2}(-\frac{\sqrt{3}}{6}, -\frac{1}{2}) \\
&= -0.8571
\end{aligned}
$$

The case of multiple roots

- There are cases when the denominator has multiple roots.
- For instance, we could have $d(s) = (s+3)^2(s^2 - s + 1)$.

The case of multiple roots

- There are cases when the denominator has multiple roots.
- For instance, we could have $d(s) = (s+3)^2(s^2 - s + 1)$.
- We focus for simplicity on the case of a real pole p_1 with multiplicity h
- $F(s)$ could be

$$F(s) = A\frac{n(s)}{(s-p_1)^h d_1(s)}$$

where $d_1(s)$ does not divide $(s - p_1)$.

Result

A generalisation of the case of single roots

Let $F(s) = A\frac{n(s)}{(s-p_1)^h d_1(s)}$, with $p_1 \in \mathbb{R}$ and $h \in \mathbb{N}$. The $F(s)$ has the following partial fraction expansion:

$$\begin{aligned} F(s) &= A\frac{n(s)}{(s-p_1)^h d_1(s)} \\ &= \frac{A_{1,1}}{s-p_1} + \frac{A_{1,2}}{(s-p_1)^2} + \ldots + \frac{A_1, h}{(s-p_1)^h} + F_1(s) \end{aligned}$$

where $F_1(s)$ is found using the same rules that apply to single poles as we discussed above and $A_{1,i}$ is given by:

$$A_{1,h-r} = \frac{1}{(h-r)!}\,\frac{d^r}{ds^r}\left[F(s)(s-p_i)^h\right]\Big|_{s=p_i}.$$

Proof

- Multiply by $(s-p_1)^h$ both sides of

$$\frac{n(s)}{(s-p_1)^h d_1(s)} = \frac{A_{1,1}}{s-p_1} + \frac{A_{1,2}}{(s-p_1)^2} + \ldots + \frac{A_{1},h}{(s-p_1)^h} + F_1(s)$$

  and obtain

$$\frac{n(s)}{d_1(s)} = A_{1,1}(s-p_1)^{h-1} + \ldots + A_{1,h-1}(s-p_1) + A_{1,h} + F_1(s)(s-p_1)^h.$$

- Since neither $d_1(s)$ nor any denominator in $F_1(s)$ divides $s-p_1$, we can evaluate both sides in $s-p_1$ and get:

$$\frac{n(p_1)}{d_1(p_1)} = A_{1,h}.$$

Proof

- If we differentiate r times, we get:

$$\frac{d^r}{ds^r}\left[\frac{n(s)}{d_1(s)}\right] = A_{1,1}(h-1)(h-2)\dots(h-r)(s-p_1)^{h-1-r} + A_{1,2}(h-2)\dots$$
$$\dots + (h-r)(h-r-1)\dots 1\cdot A_{1,h-r} +$$
$$+ h(h-1)\dots(h-r)(s-p_1)^{h-r}F_1(s) + \frac{d^r}{ds^r}\left[F_1(s)\right](s-p_1$$

If we evaluate in $s = p_1$ we obtain our claim.

Another important fact

Inverse transform of a multiple simple fraction

Proposition

Let $F(s) = \frac{1}{(s-p)^h}$. *Then* $\mathcal{L}^{-1}(F(s)) = \frac{t^{h-1}}{h-1!}e^{pt}$.

The proof comes as a direct implication of the differentiation is the Laplace domain property.

Final Result

Putting together the two propositions.....

Theorem

Let $F(s) = A\frac{n(s)}{(s-p_1)^h d_1(s)}$ *with the partial fraction expansion:*

$$F(s) = \frac{A_{1,1}}{s-p_1} + \frac{A_{1,2}}{(s-p_1)^2} + \ldots + \frac{A_1,h}{(s-p_1)^h} + F_1(s)$$

then

$$\mathcal{L}^{-1}(F(s)) = \mathbf{1}(t)\left(A_{1,1}e^{p_1 t} + A_{1,2}te^{p_1 t} + A_{1,3}\frac{t^2}{2}e^{p_1 t} + \ldots + \right.$$
$$\left. = +A_{1,h}\frac{t^{h-1}}{(h-1)!}e^{p_1 t}\right) + \mathcal{L}^{-1}(F_1(s))$$

Example

- Compute the free evolution for $y(0) = -4$, $\dot{y}(0) = 2$ of $\ddot{y} = 2\dot{y} - y + u(t)$.

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- Setting $U(s) = 0$

$$
\begin{aligned}
&(s^2 - sy(0) - \dot{y}(0))Y(s) - (2s - 2y(0))Y(s) + Y(s) = 0 \\
&Y(s) = \frac{sy(0) + \dot{y}(0) - 2y(0)}{s^2 - 2s + 1} \\
&= \frac{-4s + 10}{(s-1)^2} \\
&= \frac{A_{1,1}}{s-1} + \frac{A_{1,2}}{(s-1)^2} \\
&A_{1,2} = (-4s + 10)\big|_{s=1} = 6 \\
&A_{1,1} = \frac{d}{ds}\left[(-4s + 10)\right]\Big|_{s=1} = -4
\end{aligned}
$$

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$$
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&= \frac{-4s + 10}{(s-1)^2} \\
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&A_{1,2} = (-4s + 10)\big|_{s=1} = 6 \\
&A_{1,1} = \frac{d}{ds}\left[(-4s + 10)\right]\Big|_{s=1} = -4
\end{aligned}
$$

- Result: $y(t) = \mathbf{1}(t)e^t\left(-4 + 6t\right)$.

Complex Conjugate Poles

We can treat each pole in the complex conjugate pair as in the real case by:

- finding the coefficients of the partial fraction expansion,
- observing that the coefficient related to the complex conjugate are complex conjugate,
- recombining the pairs related to the partial fraction with the same power and reducing them to real functions.

We will illustrate this idea through an example

Example

- Compute the inverse transform of $F(s) = \frac{1}{s(s^2 - s + 1)^2}$

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- Poles: 0, $p_1 = \frac{1 + j\sqrt{3}}{2}$, and $\overline{p}_1 = \frac{1 + j\sqrt{3}}{2}$

Example

- Compute the inverse transform of $F(s) = \frac{1}{s(s^2-s+1)^2}$
- Poles: 0, $p_1 = \frac{1+j\sqrt{3}}{2}$, and $\overline{p}_1 = \frac{1+j\sqrt{3}}{2}$
- Partial Fraction Expansion

$$\begin{aligned}
F(s) &= \frac{1}{s(s-p_1)^2(s-\overline{p_1})^2} \\
&= \frac{A_{1,1}}{s-p_1} + \frac{\overline{A_{1,1}}}{s-\overline{p_1}} + \\
&+ \frac{A_{1,2}}{(s-p_1)^2} + \frac{\overline{A_{1,2}}}{(s-\overline{p_1})^2} + \\
&+ \frac{A_2}{s}
\end{aligned}$$

Example

- Coefficients

$$
\begin{aligned}
A_2 &= F(s)s\big|_{s=0} = 1 \\
A_{1,2} &= F(s)(s-p_1)^2\Big|_{s=p_1} = \frac{1}{p_1(p_1-\overline{p_1})^2} \\
A_{1,1} &= \frac{d}{ds}\left[F(s)(s-p_1)^2\right]\Bigg|_{s=p_1} \\
&= \frac{-((s-\overline{p_1})^2+2s(s-\overline{p}_1)}{s^2(s-\overline{p_1})^4}\Bigg|_{s=p_1} \\
&= \frac{-(3s^2-\overline{p_1})}{s^2(s-\overline{p_1})^3}\Bigg|_{s=p_1} \\
&= \frac{-3p_1+\overline{p_1}}{p_1^2(p_1-\overline{p_1})^3}
\end{aligned}
$$

Example

- Observing that $p_1 = e^{j\pi/3}$ and $p_1 - \overline{p_1} = 2j\sqrt{3}2 = j\sqrt{3}$:

$$A_{1,2} = -\frac{1}{\frac{1+j\sqrt{3}}{2}3} = -\frac{1}{3e^{j\pi/3}} = \frac{1}{3}e^{-j\pi/3}$$

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- Computations

$$\begin{aligned}
A_{1,1} &= \frac{e^{-j\pi/3} - 3e^{j\pi/3}}{e^{j2\pi/3}(-j3\sqrt{3})} = \frac{e^{-j\pi/3} - 3e^{j\pi/3}}{e^{j2\pi/3}e^{-j\pi/2}3\sqrt{3}} \\
&= \frac{e^{-j\pi/3} - 3e^{j\pi/3}}{e^{j\pi/6}3\sqrt{3}} = \frac{e^{-j\pi/2} - 3e^{j\pi/6}}{3\sqrt{3}} \\
&= \frac{-j - 3\frac{\sqrt{3}}{2} - \frac{3}{2}j}{3\sqrt{3}} = \frac{-3\frac{\sqrt{3}}{2} - \frac{5}{2}j}{3\sqrt{3}} \\
&= -\frac{1}{2} - \frac{5}{6\sqrt{3}}j = \frac{1}{3}\sqrt{\frac{13}{3}}e^{j\operatorname{atan2}(-5/(6\sqrt{3}),-1/2)} = \frac{1}{3}\sqrt{\frac{13}{3}}e^{-j2.3754}
\end{aligned}$$

Example

- Combining the results...

$$
\begin{aligned}
f(t) &= A_2\mathbf{1}(t) + \\
&+ \mathbf{1}(t)\left(A_{1,1}e^{p_1 t} + \overline{A_{1,1}}e^{\overline{p_1}t}\right) + \\
&+ \mathbf{1}(t)\left(A_{1,2}te^{p_1 t} + \overline{A_{1,2}}te^{\overline{p_1}t}\right) \\
&= \mathbf{1}(t)\left(1 + 2\mathbf{Real}\left(A_{1,1}e^{p_1 t}\right) + 2\mathbf{Real}\left(A_{1,2}te^{p_1 t}\right)\right) = \\
&= \mathbf{1}(t)\Bigg(1 + 2\frac{1}{3}\sqrt{\frac{13}{3}}e^{1/2t}\cos\left(\frac{\sqrt{3}}{2}t - 2.3754\right) + \\
&+ \frac{2}{3}te^{1/2t}\cos\left(\frac{\sqrt{3}}{2}t - \pi/3\right)\Bigg)
\end{aligned}
$$