Teoria dei sistemi.

Introduction

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IO Representation for MIMO systems

IO representation for MIMO systems

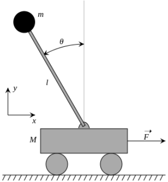
IO representations can be generalised to MIMO systems by introdusing a number of differential (difference) equations.

$$\mathfrak{D}^n y_j(t) = F(y_1(t), \ldots, \mathfrak{D}^n y_1(t), \ldots, y_m(t), \ldots, \mathfrak{D}^n y_m(t), u_1, \mathfrak{D} u_1(t), \ldots,$$
$$\ldots, \mathfrak{D}^p u_1(t), \ldots u_h, \mathfrak{D} u_m(t), \ldots, \mathfrak{D}^p u_h(t), t), j = 1, \ldots m \tag{1}$$

for a system with m output functions and h input functions.

IO Representation for MIMO systems

Example



- M be the mass of the cart, m the mass of the pendulum,
- l the length of the pendulum,
- y_1 the position x of the centre of mass of the cart, y_2 the angle θ of the pendulum with the vertical line,
- u be the force F applied to the cart.

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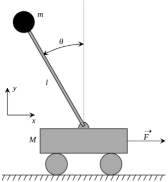

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IO representation

$$\ddot{y}_1 = \frac{ml}{M+m}\left(\ddot{y}_2 \cos y_2 - \dot{y}_2^2 \sin y_2\right) + \frac{1}{M+m}u$$

$$\ddot{y}_2 = \frac{1}{l}\left(g \sin y_2 + \ddot{y}_1 \cos y_2\right)$$

Lessons Learned

- for a system the same input function can be associated with multiple output functions,
- param eteric representation: for each t_0 given an input function, the associated output function can be "disambiguated" through parameter x_0,
- for a class of systems (causal systems), this parametric representation is causal
  - for a x_0, the output at any given time $t' > t_0$ only depends on the values of the input function in the interval $[t_0, t')$ (or $[t_0, t']$

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Qualitative Observations

Observation

t_0 can be chosen freely $\rightarrow$

1. a causal system cannot "foresee" the future,
2. the parameter x_0 summarises all the story of the system previous to t_0.

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From qualitative considerations to definitions

These considerations are the basis for the defintion of a state space representation....but we need some more theory.

Consideration n. 1

Parameter set

- For each t_0 we can theoretically find a different x_0 parameter living in a different space.

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IO representation

This is true for the parameters associated to IO representation, which are essentially the initial conditions.

Consideration n. 2

Consistency

- to identify the systems story up to t_0 with x_0, we we need some form of consistency between the parameter x_0 at time t_0 and the parameter x_1 at time any other time t_1.
- We will assume the existence of a function ϕ, such that

$$x_1 = \phi(t_1, t_0, x_0, u),$$

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Intuition

ϕ tells us how to transform the story up until t_0 into the story up until t_1, given the knowledge of the input in between t_0 and t_1.

Definition of ϕ

Definition

Let $(T \times T)^*$ be defined as:

$$(T \times T)* = \{(t, t_0) : t, t_0 \in \mathcal{T} \text{ and } t \geq t_0\}.$$

The function ϕ is defined as

$$\begin{aligned} \phi : (T \times T) * \times X \times \mathcal{U} \to X \\ x(t) = \phi(t, t_0, x_0, u), \end{aligned} \tag{2}$$

and it is required to satisfy three properties.

Properties of ϕ

Consistency

$$\forall t \in T, \forall u \in \mathcal{U}, \phi(t, t, x, u) = x,$$

Causality

$$\forall t, t_0 \in \mathcal{T}, \forall x_0 \in X \ u|_{[t_0,t)} = u'|_{[t_0,t)} \implies \phi(t, t_0, x_0, u) = \phi(t, t_0, x_0, u')$$

Separation

$$\forall (t, t_0), \forall x_0 \in X, \forall u \in \mathcal{U} \tag{3}$$

$$t > t_1 > t_0 \implies \phi(t, t_0, x_0, u) = \phi(t, t_1, \phi(t_1, t_0, x_0, u), u). \tag{4}$$

Link between state and output

Parametric representation

The notion of state derives from the parametric representation of systems

Output

The parametric representation

$$\forall t \in T, \forall u \in \mathcal{U}, \phi(t, t, x, u) = x,$$

leads us to

$$\forall t_0, y_0(t) = \left[\pi_{t_0}(x_0, u_0)\right](t) \forall t \geq t_0,$$

Output function

Re-writing the above for a generic time t brings us to the definition of the function η

$$\begin{aligned} \eta : \mathcal{T} \times X \times U \rightarrow Y \\ y(t) = \eta(t, x(t), u(t)). \end{aligned} \tag{5}$$

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Although systems with finite states exist (example sequential networks), we will consider systems with dense state space.

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Finite and dense state variables

Although systems with finite states exist (example sequential networks), we will consider systems with dense state space.

Example

Sometimes the state space can be very large.....

`https://www.youtube.com/watch?v=teQwViKMnxw`

Input Signals

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Input signals are all quantities that influence the system evolution

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Output Signals

Output Signals: performance specs

Output signals are measurable quantities and can be used to formulate design specifications.

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An output signal conveys useful elements of information that allow reconstructing the system state when the latter is not directly measurable.

- "observer" system used to estimate the whole state based on partial observations.

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Example

Estimating position of a drone from inertial data.

`https://www.youtube.com/watch?v=EmUWJQ-mLmY`

Existence of a State

Parametric description corresponding to state space

The functions ϕ and η generate a system:

$$\Sigma(t_0) = \Big\{(u_0,\ y_0) \in \mathcal{U}^{\mathcal{T}(t_0)} \times \mathcal{Y}^{\mathcal{T}(t_0)}, \tag{6}$$

$$u_0 = u|_{\mathcal{T}(t_0)}, y_0 : y_0(t) = \eta(t,\ \phi(t, t_0, x_0, u), u(t)),$$

$$\text{with } u_0 \in \mathcal{U} \text{ and } x_0 \in X\Big\}. \tag{7}$$

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Building a state space description is possible for an abstract system under mild conditions.

- input functions have the same range
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Note

More details on existence and uniqueness of a state space byond our scope. We will always assume a state space exists.

Implicit and Explicit Representations

Explicit Representation

$$
\begin{aligned}
x(t) &= \phi(t, t_0, x_0, u) \\
y(t) &= \eta(t, x(t), u(t)).
\end{aligned}
$$

The functions ϕ and η compound the so called "explicit" representation of the system.

- Given x_0 and $u(\cdot)$, I compute $x(t)$ and $y(t)$ by direct application of the formulas

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Implicit Representation

For an *implicit* representation the computation the output requires the solution of a system of difference or of differential equations.

Implicit representation of a DT system

Explicit Representatnion

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The time space $\mathcal{T}$ is $\mathcal{Z}$ (.)

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Implicit Representation

The implicit rperesentation can be found by simply evaluation ϕ across one step.

$$
x(t+1) = \phi(t+1, t, x(t), u(t)) = f(t, x(t), u(t)) \tag{8}
$$

The function $f(\cdot, \cdot, \cdot)$ is said *generator function*. The representation (X, f, η) is said *implicit representation*.

Example

Scalar DT sytems

$$x(t+1) = a(t)x(t) + b(t)u(t).$$

State space ha dimension 1.

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Iterative Solution

$$\begin{aligned}
x(t_0+1) &= a(t_0)x_0 + b(t_0)u(t_0) \\
&\dots \\
x(t_0+3) &= a(t_0+2)x(t_0+2) + b(t_0+2)u(t_0+2) = \\
&= a(t_0+2)a(t_0+1)a(t_0)x_0 + a(t_0+2)a(t_0+1)b(t_0)u(t_0)+ \\
&+ a(t_0+2)b(t_0+1)u(t_0+1) + b(t_0+2)u(t_0+2) \\
&\dots\dots
\end{aligned}$$

Example

General Step

By induction we can prove

$$x(t_0 + H) = \phi(t_0 + H, t_0, x_0, u) =$$

$$= \psi(t_0 + H, t_0)x_0 + \sum_{t=t_0}^{t_0+H-1} G(t_0 + H - 1, t)u(t)$$

where

$$\psi(t, \tau) = \begin{cases} \prod_{t'=\tau}^{t-1} a(t') & \text{if } t > \tau \\ 1 & \text{if } t \leq \tau \end{cases}$$

$$G(t, \tau) = \psi(t + 1, \tau + 1)b(\tau)$$

Implicit representation of a CT system

Explicit Representation

$$
\begin{aligned}
x(t) &= \phi(t, t_0, x_0, u) \\
y(t) &= \eta(t, x(t), u(t)).
\end{aligned}
$$

- The time space $\mathcal{T}$ is $\mathbb{R}$.
- It is reasonable to use differential equation.
- This requires differentiability of the function ϕ.

Implicit representation of a CT system

Explicit Representation

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- The time space $\mathcal{T}$ is $\mathbb{R}$.
- It is reasonable to use differential equation.
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Implicit Representation

If we admit that ϕ is the solution to the equation:

$$\frac{\partial}{\partial t}\phi = f(t, \phi, u(t)),$$

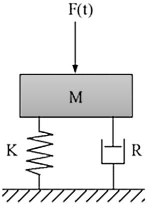
then our implicit representation is given by:

$$\dot{x} = f(t, x(t), u(t)).$$

Generally speaking the state is a vector and f is vector valued.

Example of state space representation for a continuos time system

Example



Let $\tilde{x}$ represent the vertical position of the mass. State

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{x} - x_{rest} \\ \frac{d(\tilde{x} - x_{rest})}{dt} \end{bmatrix}$$

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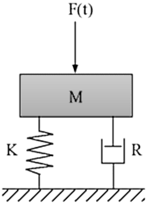

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IO representation

The use of standard laws of mechanics leads us to

$$\dot{x}(t) = \dot{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}} = \begin{bmatrix} x_2 \\ F(t) - Kx_1 - Rx_2 \end{bmatrix}.$$