Algoritmi avanzati
A.A. 2012-2013

AVVERTENZA: lucidi da usare come ausilio mnemonico e lista degli argomenti svolti a lezione.

Non sostituiscono in alcun modo il libro di testo che va usato per lo studio approfondito.

An example in politics

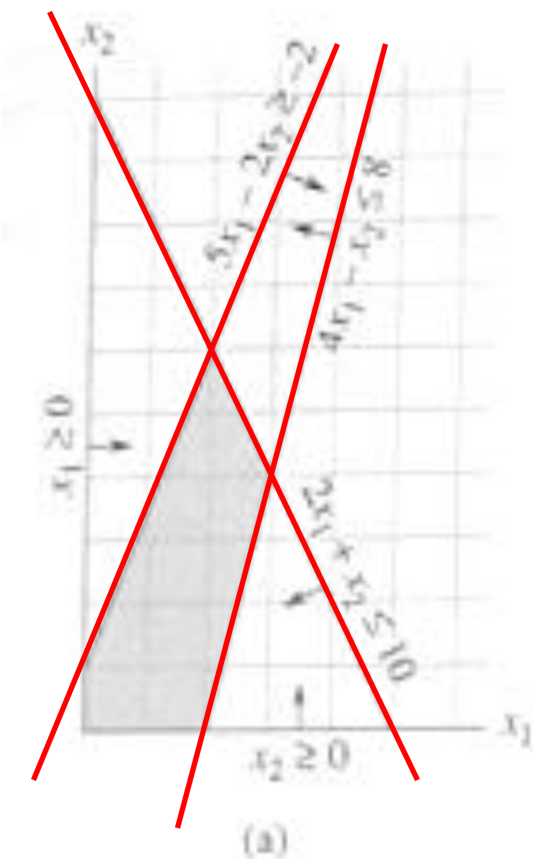
- How to minimize advertising money and win the elections

Programmazione lineare

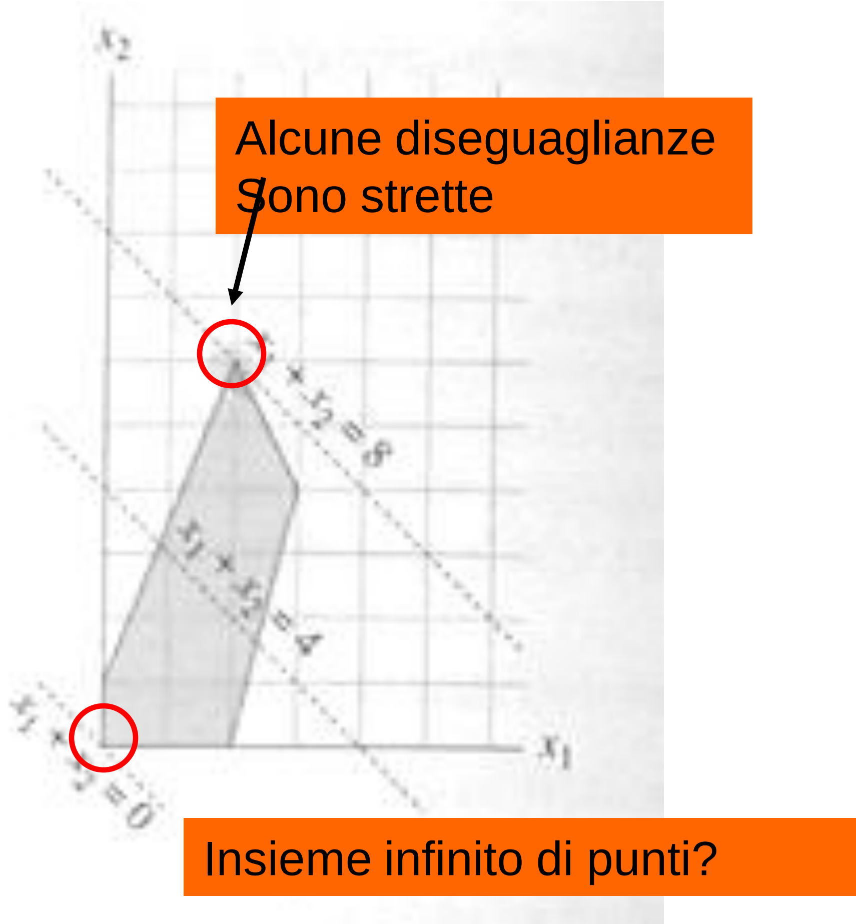
- Minimizzare una funzione lineare
- Soggetta a vincoli lineari (eguaglianza o diseguaglianza)
- Applicazioni rilevanti (e.g., flight crews, oil drilling, politics, diet, multicommodity flow, …)…Taylor

- Definizioni e Trasformazioni (forma standard, forma slack,…)
- Algoritmo del simplesso
- Dualità

$$
\begin{array}{lrcrcr}
\text{maximize} & x_1 & + & x_2 & & \\
\text{subject to} & & & & & \\
& 4x_1 & - & x_2 & \leq & 8 \\
& 2x_1 & + & x_2 & \leq & 10 \\
& 5x_1 & - & 2x_2 & \geq & -2 \\
& & x_1, x_2 & & \geq & 0\,.
\end{array}
$$



(a)



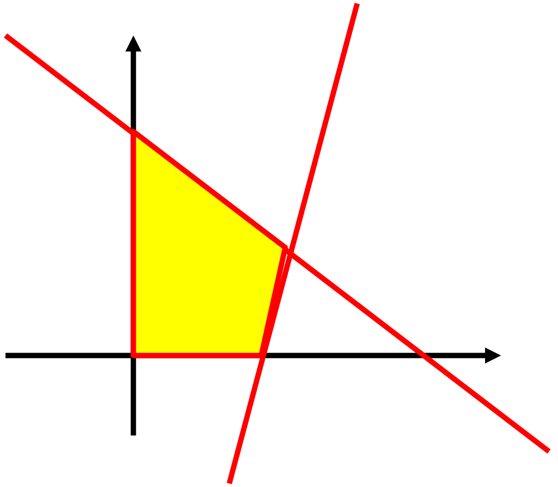
Programmazione lineare

- Funzione obiettivo, soluzione ottimale
- Soluzione fattibile, regione fattibile
- Illimitato se soluzione fattibile ma nessun valore ottimale finito
- Convessità (intersezione di insiemi convessi ,simplesso)
- Minimo locale è minimo globale
- La soluzione ottimale è un vertice del simplesso
- Da problema continuo a problema discreto
- Algoritmo del simplesso
  - Inizia da un punto fattibile su un vertice
  - Ripeti fino a quando minimo locale:
    - **Muovi ad un vertice vicino con valore di f inferiore**

Algoritmi per LP

- Simplex (can require exponential time, fast in most practical cases)
- Ellipsoid algorithm (first poly. time algorithm)
- Interior point methods, poly. time, move in the interior (Karmakar), competitive with simplex for large dimensions
- NOTE: integer linear program is NP-hard

LP forme



standard

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \\ & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \ldots, m \\ & x_j \geq 0 \quad \text{for } j = 1, 2, \ldots, n . \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & \\ & Ax \leq b \\ & x \geq 0 . \end{aligned}$$

Conversione a forma standard

- L equivalent to L' (feasible x, f(x) =z → feasible x', f(x')=z and viceversa, not necessarily one-one)

- Min to max … negative coefficients
- Non-negativity contraints… $x_j = x_j' - x_j''$, $x_j' \geq 0$, $x_j'' \geq 0$
- Equality into inequality f(x)=b… f(x)≥b, f(x)≤ b
- Less-than-or-equal contraints … multiply by -1

Variabili slack

- Turn each inequality constraint (apart from non-negativity) into an equality constraint by adding a slack variable *(variabile di scarto)*

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \Rightarrow \quad \begin{aligned} s &= b_i - \sum_{j=1}^{n} a_{ij} x_j , \\ s &\ge 0 . \end{aligned}$$

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j ,$$

$$x_{n+i} \ge 0.$$

i=1,..,m one slack var. for each contraint

Forma Slack (esempio)

Objective function

$$
\begin{aligned}
z &= & & 2x_1 &-& 3x_2 &+& 3x_3 \\
x_4 &= & 7 &- x_1 &-& x_2 &+& x_3 \\
x_5 &= & -7 &+ x_1 &+& x_2 &-& x_3 \\
x_6 &= & 4 &- x_1 &+& 2x_2 &-& 2x_3
\end{aligned}
$$

Basic variables (B)
Related to constraints

Non-basic variables (N)

Non-negativity contraints omitted

Basic solution → all non-basic variables zero, on a vertex of the simplex

LP forma slack

slack

$|N| = n$, $|B| = m$, and $N \cup B = \{1, 2, \ldots, n+m\}$.

tuple $(\underline{N, B}, \underline{A, b}, \underline{c, v})$

$$z = v + \sum_{j \in N} c_j x_j$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B ,$$

Forma slack: esempio

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} .
\end{aligned}
$$

we have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$
A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}
$$

$$
b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix} ,
$$

$c = (\, c_3 \;\; c_5 \;\; c_6 \,)^{\mathrm{T}} = (\, -1/6 \;\; -1/6 \;\; -2/3 \,)^{\mathrm{T}}$, and $v = 28$.

Formulazioni LP di problemi

- Cammini minimi singolo paio $s \rightarrow t$
  - Max d[t]
  - Subject to:

    d[v] <= d[u]+ w(u,v) for each edge (u,v) in graph

    d[s] =0

Bellman-Ford

- Massimo flusso (source s, sink t)
  - Max $\Sigma_{v \text{ in } V}$ f(s,v)
  - Subject to:
    - f(u,v) <= c(u,v) capacity constraints
    - f(u,v) = -f(v,u) skew symmetry
    - $\Sigma_{v \text{ in } V}$ f(u,v) = 0 for each u in V – {s,t} flow conservation

L'algoritmo del simplesso

- Geometric interpretation vs. algebraic view
- A kind of "Gaussian elimination for inequalities"… rewrite the system until solution is simple
- Use slack form (everything becomes an equality)
- Basic solution: put *non*-basic variables to zero (corr. to one vertex of the simplex)
- Transform slack form so that value of basic feasible solution does not decrease (usually increases)
- When rewriting the role of basic vs non-basic changes

Simplesso: esempio

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}
$$

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 .
\end{array}
$$

Basic solution is feasible (in this case)

Positive!

Simplesso: esempio

ENTER!

LEAVE!

$$
\begin{aligned}
z &= & 3x_1 &+ x_2 &+ 2x_3 \\
x_4 &= 30 &- x_1 &- x_2 &- 3x_3 \\
x_5 &= 24 &- 2x_1 &- 2x_2 &- 5x_3 \\
x_6 &= 36 &- 4x_1 &- x_2 &- 2x_3 .
\end{aligned}
$$

Reformulate so that basic solution has greater objective value
By how much can I increase x1 ? Third constraint is the most binding!
Solve for x1 from third constraint and substitute

Move along the edge of the simplex

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} .
\end{aligned}
$$

Basic solution goes from 0 to 27!

Simplesso: esempio

ENTER!

LEAVE!

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} .
\end{aligned}
$$

Move along the edge of the simplex

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} .
\end{aligned}
$$

Which variable do I pick next?

Simplesso: esempio

ENTER!

LEAVE!

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} .
\end{aligned}
$$

x2 can grow arbitrarily

Move along the edge of the simplex

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} .
\end{aligned}
$$

Which variable do I pick next? Value of basic solution cannot be increased!
Basic solution is optimal. Values of slack variables say degree of constraint satisfaction

Pivoting

l= leaving, e=entering

PIVOT(N, B, A, b, c, v, l, e)

▷ Compute the coefficients of the equation for new basic variable x_e.
$\widehat{b}_e \leftarrow b_l / a_{le}$
for each $j \in N - \{e\}$
    do $\widehat{a}_{ej} \leftarrow a_{lj} / a_{le}$
$\widehat{a}_{el} \leftarrow 1 / a_{le}$
▷ Compute the coefficients of the remaining constraints.
for each $i \in B - \{l\}$
    do $\widehat{b}_i \leftarrow b_i - a_{ie} \widehat{b}_e$
        for each $j \in N - \{e\}$
            do $\widehat{a}_{ij} \leftarrow a_{ij} - a_{ie} \widehat{a}_{ej}$
        $\widehat{a}_{il} \leftarrow -a_{ie} \widehat{a}_{el}$
▷ Compute the objective function.
$\widehat{v} \leftarrow v + c_e \widehat{b}_e$
for each $j \in N - \{e\}$
    do $\widehat{c}_j \leftarrow c_j - c_e \widehat{a}_{ej}$
$\widehat{c}_l \leftarrow -c_e \widehat{a}_{el}$
▷ Compute new sets of basic and nonbasic variables.
$\widehat{N} = N - \{e\} \cup \{l\}$
$\widehat{B} = B - \{l\} \cup \{e\}$
return $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$

xe on l.h.s.

substitute xe in r.h.s. of other equations

Effect of pivoting on basic solution

Consider a call to PIVOT(N, B, A, b, c, v, l, e) in which $a_{le} \neq 0$. Let the values returned from the call be $(\widehat{N}, \widehat{B}, \widehat{A}, \widehat{b}, \widehat{c}, \widehat{v})$, and let $\bar{x}$ denote the basic solution after the call. Then

1. $\bar{x}_j = 0$ for each $j \in \widehat{N}$.
2. $\bar{x}_e = b_l / a_{le}$.
3. $\bar{x}_i = b_i - a_{ie}\widehat{b}_e$ for each $i \in \widehat{B} - \{e\}$.

Alg. del simplesso: formalizzazione

- Is the LP feasible?
- If feasible, how do we get a feasible initial basic solution?

Initialize-Simplex

- How do we determine if LP is unbounded?
- How do we choose entering and leaving variables?

Simplesso

```
SIMPLEX(A, b, c)
(N, B, A, b, c, v) ← INITIALIZE-SIMPLEX(A, b, c)
while some index j ∈ N has c_j > 0
    do choose an index e ∈ N for which c_e > 0
       for each index i ∈ B
           do if a_ie > 0
                then Δ_i ← b_i / a_ie
                else Δ_i ← ∞
       choose an index l ∈ B that minimizes Δ_i
       if Δ_l = ∞
          then return "unbounded"
          else (N, B, A, b, c, v) ← PIVOT(N, B, A, b, c, v, l, e)
for i ← 1 to n
    do if i ∈ B
          then x̄_i ← b_i
          else x̄_i ← 0
return (x̄_1, x̄_2, ..., x̄_n)
```

Most binding constraint if any

Simplesso

- If properly started, if it terminates, feasible solution or unbounded
- It terminates!
- Returned solution is optimal (see duality)

Use loop invariant:

At the start of each iteration of the **while** loop of lines 2–11,

1. the slack form is equivalent to the slack form returned by the call of INITIALIZE-SIMPLEX,
2. for each $i \in B$, we have $b_i \geq 0$, and
3. the basic solution associated with the slack form is feasible.

Simplesso

Loop invariant bi ≥ 0

First, we observe that $\widehat{b}_e \geq 0$ because $b_l \geq 0$ by the loop invariant, $a_{le} > 0$ by line 5 of SIMPLEX, and $\widehat{b}_e = b_l/a_{le}$ by line 2 of PIVOT.

For the remaining indices $i \in B - l$, we have that

$$\begin{aligned} \widehat{b}_i &= b_i - a_{ie}\widehat{b}_e && \text{(by line 8 of PIVOT)} \\ &= b_i - a_{ie}(b_l/a_{le}) && \text{(by line 2 of PIVOT)} . \end{aligned} \tag{29.79}$$

We have two cases to consider, depending on whether $a_{ie} > 0$ or $a_{ie} \leq 0$. If $a_{ie} > 0$, then since we chose l such that

$$b_l/a_{le} \leq b_i/a_{ie} \quad \text{for all } i \in B , \tag{29.80}$$

Most binding constraint picked!

we have

$$\begin{aligned} \widehat{b}_i &= b_i - a_{ie}(b_l/a_{le}) && \text{(by equation (29.79))} \\ &\geq b_i - a_{ie}(b_i/a_{ie}) && \text{(by inequality (29.80))} \\ &= b_i - b_i \\ &= 0 , \end{aligned}$$

and thus $\widehat{b}_i \geq 0$. If $a_{ie} \leq 0$, then because a_{ie}, b_i, and b_l are all nonnegative, equation (29.79) implies that $\widehat{b}_i$ must be nonnegative, too.

Simplesso

- Slack form uniquely determined by the sets of basic variables

$$z = v + \sum_{j \in N} c_j x_j \tag{29.82}$$

$$x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B \, , \tag{29.83}$$

and the second as

$$z = v' + \sum_{j \in N} c'_j x_j \tag{29.84}$$

$$x_i = b'_i - \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B \, . \tag{29.85}$$

Consider the system of equations formed by subtracting each equation in line (29.85) from the corresponding equation in line (29.83). The resulting system is

$$0 = (b_i - b'_i) - \sum_{j \in N} (a_{ij} - a'_{ij}) x_j \quad \text{for } i \in B$$

or, equivalently,

$$\sum_{j \in N} a_{ij} x_j = (b_i - b'_i) + \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B \, .$$

Simplesso: terminazione

- If each iteration increases objective value of basic solution we are done
- Only case is degeneracy: objective value unchanged → possible (endless) cycles
- If SIMPLEX fails to terminate in at most $\binom{n+m}{m}$

iterations it cycles

- Cycling is extremely rare, can be avoided by breaking ties by choosing variable wih smallest index

Dualità

- Under suitable assumptions, SIMPLEX terminates
- Does it find the optimal solution?
- Demonstration uses duality (a related minimization problem s.t. they have the same optimal objective value)

LP Primale

$$\begin{aligned}
&\text{maximize} && \sum_{j=1}^{n} c_j x_j \\
&\text{subject to} && \\
& && \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for } i = 1, 2, \ldots, m \\
& && x_j \ge 0 \quad \text{for } j = 1, 2, \ldots, n.
\end{aligned}$$

LP Duale

$$\begin{aligned}
&\text{minimize} && \sum_{i=1}^{m} b_i y_i \\
&\text{subject to} && \\
& && \sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad \text{for } j = 1, 2, \ldots, n, \\
& && y_i \ge 0 \quad \text{for } i = 1, 2, \ldots, m.
\end{aligned}$$

Dualità debole

- Value feasible sol. primal ≤ value feasible sol. dual

$$\sum_{j=1}^{n} c_j \bar{x}_j \le \sum_{i=1}^{m} b_i \bar{y}_i$$

$$\begin{aligned}
\sum_{j=1}^{n} c_j \bar{x}_j &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \bar{y}_i \right) \bar{x}_j \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \bar{x}_j \right) \bar{y}_i \\
&\le \sum_{i=1}^{m} b_i \bar{y}_i
\end{aligned}$$

Dualità debole: corollario

- If value fasible sol. primal = Value fasible sol dual…

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i$$

They are both optimal solutions of their problems

Dalla soluzione del primale alla soluzione del duale

- Assume last slack form of the primal:

$$z = v' + \sum_{j \in N} c'_j x_j$$

$$x_i = b'_i - \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B .$$

- Optimal dual solution is:

$$\bar{y}_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N , \\ 0 & \text{otherwise} . \end{cases}$$

LP dualità

- Suppose SIMPLEX returns X, and Y defined as before, then they are both optimal solutions of their problems

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i .$$

- Consider final slack form:

$$z = v' + \sum_{j \in N} c'_j x_j$$

$c'_j \leq 0 \quad \text{for all } j \in N$

$c'_j = 0 \quad \text{for all } j \in B$ by definition

LP dualità

- One can rewrite

$$
\begin{aligned}
z &= v' + \sum_{j \in N} c'_j x_j \\
&= v' + \sum_{j \in N} c'_j x_j + \sum_{j \in B} c'_j x_j \\
&= v' + \sum_{j=1}^{n+m} c'_j x_j
\end{aligned}
$$

- Given equivalence of slack forms

$$
\begin{aligned}
\sum_{j=1}^{n} c_j \bar{x}_j &= v' + \sum_{j=1}^{n+m} c'_j \bar{x}_j \\
&= v' + \sum_{j \in N} c'_j \bar{x}_j + \sum_{j \in B} c'_j \bar{x}_j \\
&= v' + \sum_{j \in N} (c'_j \cdot 0) + \sum_{j \in B} (0 \cdot \bar{x}_j) \\
&= v' .
\end{aligned}
$$

LP dualità

$$
\begin{aligned}
\sum_{j=1}^{n} c_j x_j
&= v' + \sum_{j=1}^{n+m} c'_j x_j \\
&= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{j=n+1}^{n+m} c'_j x_j \\
&= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} c'_{n+i} x_{n+i} \\
&= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} (-\bar{y}_i)\, x_{n+i} \quad \text{slack} \\
&= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} (-\bar{y}_i) \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \\
&= v' + \sum_{j=1}^{n} c'_j x_j - \sum_{i=1}^{m} b_i \bar{y}_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} x_j)\, \bar{y}_i \\
&= v' + \sum_{j=1}^{n} c'_j x_j - \sum_{i=1}^{m} b_i \bar{y}_i + \sum_{j=1}^{n} \sum_{i=1}^{m} (a_{ij} \bar{y}_i)\, x_j \\
&= \left(v' - \sum_{i=1}^{m} b_i \bar{y}_i \right) + \sum_{j=1}^{n} \left(c'_j + \sum_{i=1}^{m} a_{ij} \bar{y}_i \right) x_j \,,
\end{aligned}
$$

LP dualità

$$v' - \sum_{i=1}^{m} b_i \bar{y}_i = 0\,,$$

$$c'_j + \sum_{i=1}^{m} a_{ij} \bar{y}_i = c_j \quad \text{for } j = 1, 2, \ldots, n\,.$$

negative

Soluzione iniziale fattibile

- The initial slack form may not have an initial feasible solution

$$
\begin{array}{lrcrcr}
\text{maximize} & 2x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & - & x_2 & \leq & 2 \\
& x_1 & - & 5x_2 & \leq & -4 \\
& & x_1, x_2 & & \geq & 0 .
\end{array}
$$

- Define an auxiliary Laux to know if feasible solution exists and to initialize

Soluzione iniziale fattibile

$$\begin{aligned}
\text{maximize} \quad & -x_0 \\
\text{subject to} \quad & \\
& \sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i \quad \text{for } i = 1, 2, \ldots, m , \\
& x_j \ge 0 \quad \text{for } j = 0, 1, \ldots, n .
\end{aligned}$$

- L is feasible iff **Laux** has optimal objective value = 0

Initialize-Simplex

```
INITIALIZE-SIMPLEX(A, b, c)
  let l be the index of the minimum b_i
  if b_l ≥ 0                                   ▷ Is the initial basic solution feasible?
     then return ({1, 2, ..., n}, {n + 1, n + 2, ..., n + m}, A, b, c, 0)
  form L_aux by adding −x_0 to the left-hand side of each equation
          and setting the objective function to −x_0
  let (N, B, A, b, c, v) be the resulting slack form for L_aux
  ▷ L_aux has n + 1 nonbasic variables and m basic variables.
  (N, B, A, b, c, v) ← PIVOT(N, B, A, b, c, v, l, 0)
  ▷ The basic solution is now feasible for L_aux.
  iterate the while loop of lines 2–11 of SIMPLEX until an optimal solution
          to L_aux is found
  if the basic solution sets x̄_0 = 0
     then return the final slack form with x_0 removed and
                 the original objective function restored
     else return "infeasible"
```

“Most dangerous” contraint

x0 enters

Teorema fondamentale di LP

- Anly LP given in standard form, either
  1. Has optimal solution with finite obj. value
  2. Is infeasible
  3. Is unbounded

The SIMPLEX algorithm completely solves the problem