Teoria dei sistemi.

Stability

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Theorem on stability of LTI systems

Theorem

Theorem

Consider a LTI system Σ with impulse response $h(t)$.

- *DT: it is BIBO stable if and only if there exists a constant S such that $\sum_{-\infty}^{\infty} |h(t)| = S < \infty$.*
- *CT: it is BIBO stable if and only if there exist a constant S such that $\int_{-\infty}^{\infty} |h(\tau)| d\tau = S < \infty$.*

From the discussion in the past lecture, we know that this has much to do with the poles.

BIBO stability form the transfer function

Theorem

Theorem

Consider a CT LTI system with transfer function:

$$H(s) = \frac{n(s)}{d(s)}.$$

Assume that no cancellation between poles and zero takes place: $\nexists p : n(p) = d(p) = 0$. *The system is BIBO stable if and only if all of its poles have negative real part.*

Proof

- We know that the system is BIBO stable iff $\int_0^\infty |h(t)|dt = L < +\infty$

Necessity

Let p_i be a pole with $\mathbf{Real}\,(p_i) \geq 0$ with multiplicity 1. Then $H(s) = \frac{A_i}{s-p_i} + \ldots$ with $A_i \neq 0$ because

$$A_i = \frac{n(p_i)}{(p_i - p_1)\ldots(p_i - p_{i-1})(p_i - p_{i+1})\ldots(p_i - p_n)},$$

and by our assumption $n(p_i) \neq 0$.
This fraction will give rise to an exponential function in $h(t)$ whose integral grows in an unbounded way.

Proof

Sufficiency

If $H(s) = \mathcal{L}(h(t))$, then $h(t) = \sum_{h=1}^{H} F_h h_h(t)$, with $h_h(t)$ given by:

$$h_h(t) = \begin{cases} e^{p_h t} & \text{Real root (single or multiple)} \\ e^{\mathbf{Real}(p_h)t} \cos\left(\mathbf{Imag}\left(p_h\right) t + \phi_h\right) & \text{Complex single or multiple root} \\ t^{n_h} e^{p_h t} & \text{Real multiple root} \\ t^{n_h} e^{\mathbf{Real}(p_h)t} \cos\left(\mathbf{Imag}\left(p_h\right) t + \phi_h\right) & \text{Complex multiple root,} \end{cases}$$

which can be interpreted as

$$|h_h(t)| \leq H_h(t) = \begin{cases} e^{p_h t} & \text{Real root (single or multiple)} \\ e^{\mathbf{Real}(p_h)t} & \text{Complex single or multiple root} \\ t^{n_h} e^{p_h t} & \text{Real multiple root} \\ t^{n_h} e^{\mathbf{Real}(p_h)t} & \text{Complex multiple root,} \end{cases}$$

Proof

Sufficiency

$$\int_0^{\infty} H_h(t)dt = \begin{cases} \lim_{K\to\infty} \frac{e^{p_h K}-1}{p_h}, & \text{For real roots} \\ \lim_{K\to\infty} \frac{e^{\textbf{Real}(p_h)K}-1}{\textbf{Real}(p_h)}, & \text{For complex roots} \\ \lim_{K\to\infty} \sum_{k=0}^{n_h} (-1)^{n_h-k} \frac{n_h!}{k! p_h^{n_h-k}} K^k e^{p_h K} - (-1)^{n_h} \frac{1}{p_h^{n_h}}, & \text{Real multiple root} \\ \lim_{K\to\infty} \sum_{k=0}^{n_h} (-1)^{n_h-k} \frac{n_h!}{k! \textbf{Real}(p_h)^{n_h-k}} K^k e^{\textbf{Real}(p_h)K} - (-1)^{n_h} \frac{1}{\textbf{Real}(p_h)^{n_h}}, & \text{Complex} \end{cases}$$

Proof

Sufficiency

If **Real** (p_h) is smaller than 0 for all poles, this leads to

$$\int_0^\infty H_h(t)dt = \begin{cases} \frac{-1}{p_h} & \text{For real roots} \\ \frac{-1}{\textbf{Real}(p_h)} & \text{For complex roots} \\ -\frac{(-1)^{n_h}}{p_h^{n_h}} & \text{Real multiple root} \\ -\frac{(-1)^{n_h}}{\textbf{Real}(p_h)^{n_h}} & \text{Real multiple root.} \end{cases}$$

In other words we can find a constant F that upper bounds all $H_h(t)$ for all h.

$$\int_0^\infty |h(t)|dt \leq |F_h| \int_0^\infty \sum |H_h(t)|dt \leq FM.$$

Example

Second order example

Transfer Function:

$$H(s) = \frac{s-1}{s^2 + 3s + 2}.$$

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$$p_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2} = \begin{cases} -2 \\ -1 \end{cases} .$$

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Conclusions: Both roots are negative. Hence, the system is BIBO stable.

General Case

Characteristic Polynomial and Equation

Definition

The polynomial $d(s)$ at the denominator of the transfer function is said *characteristic polynomial* and the equation $d(s) = 0$ is called *characteristic equation*.

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- All we need to compute is the sign of the real part
- we can do this by through the Routh-Hurwitz criterion

The Routh-Hurwitz Criterion

- Let the characteristic equation $d(s)$ be:

$$a_n s^n + a_{n-1} s^{n-1} + \dots a_0.$$

The Routh-Hurwitz Criterion

- Let the characteristic equation $d(s)$ be:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots a_0.$$

- Form the Routh Table

$$\begin{array}{l|cccccc}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots & a_2 & a_0 \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots & a_3 & a_1 \\
s^{n-2} & b_1 & b_2 & b_3 & \ldots & b_{n_2} & 0 \\
s^{n-2} & c_1 & c_2 & c_3 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & & & & \\
s^0 & q & & & & &
\end{array}$$

Routh Table

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$
$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$
$$b_3 = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$
$$\cdots$$
$$c_1 = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1}$$
$$c_2 = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1}$$
$$c_2 = \frac{b_1 a_{n-7} - b_4 a_{n-1}}{b_1}$$
$$\cdots .$$

The Routh-Hurwitz Theorem

Theorem

Theorem

Suppose that the Routh table can be built as:

$$\begin{array}{c|cccccc}
s^n & a_n & a_{n-2} & a_{n-4} & \dots & a_2 & a_0 \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \dots & a_3 & a_1 \\
s^{n-2} & b_1 & b_2 & b_3 & \dots & b_{n_2} & 0 \\
s^{n-2} & c_1 & c_2 & c_3 & \dots & 0 & 0 \\
\dots & \dots & \dots & & & & \\
s^0 & q & & & & &
\end{array}$$

Assume that the first column $(a_n, a_{n1}, b_1, c_1, \dots, q)$ *does not contain* 0 *elements. Then the number of sign changes in the first column corresponds to the number of poles in the right half of the complex plan.*

The Routh-Hurwitz Theorem

Theorem

Theorem

Assume that the first column of the Routh table does not contain 0*, a necessary and sufficient condition for all roots to be in the left a half plan is that all coefficients* a_i *be positive and that all the elements of the first column be positive.*

Example

Second Order Systems

Consider the classic second order equation:

$$a_2 s^2 + a_1 s + a_0.$$

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Consider the classic second order equation:

$$a_2 s^2 + a_1 s + a_0.$$

The Routh table is given by

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s & a_1 & 0 \\ s^0 & a_0. & \end{array}$$

Hence, in order for the system to be stable, all coefficients have to be positive.

Example

Second Order Systems – continued

This can be seen by considering that if the roots are negative, we can write the polynomial as

$$(s + r_1)(s + r_2) = s^2 + (r_1 + r_2)s + r_1 r_2$$

So if r_1 and r_2 are positive (meaning that the roots are negative) if and only if $(r_1 + r_2)$ and $r_1 r_2$ are positive in their turn.

Example

Third Order Systems

Consider the thrid degree characteristic equation

$$a_3 s^3 + a_2 s^2 + a_1 s + a0 = 0.$$

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Third Order Systems

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The Routh table is given by:

$$\begin{array}{c|ccc} s^3 & a_3 & a_1 & 0 \\ s^2 & a_2 & a_0 & 0 \\ s & \frac{a_2 a_1 - a_3 a_0}{a_2} & 0 & 0 \\ s^0 & a_0 & 0 & 0 \end{array}$$

Example

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Hence, along with $a_i > 0$, BIBO stability will also require $a_2 a_1 - a_3 a_0 > 0$.

Simplified Table

Simplified Routh Table

Proposition

If we multiply an entire row of the Routh table by a positive constant Theorem 5 still holds.

Corner Cases

1. The first element of a row is 0, thus preventing to derive the following row (which requires the division by the first element),

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Imaginary Roots

- Both cases reveal the presence of roots on the imaginary axis or in the positive half plan and therefore the loss of BIBO stability.

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Imaginary Roots

- Both cases reveal the presence of roots on the imaginary axis or in the positive half plan and therefore the loss of BIBO stability.
- Still we can use Routh criterion to identify the number of stable and unstable roots.

The case of 0 in the first element of the column

The first problem is solved by:

- perturbing the 0 element (setting it to ϵ) and
- evaluating the number of changes and permanence both for positive and negative ϵ.

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We see this by an example

Example

Third order example with missing term

Consider the polynomial

$$s^3 + s - 1.$$

Example

Third order example with missing term

Consider the polynomial

$$s^3 + s - 1.$$

Its associated table is

$$\begin{array}{c|ccc} s^3 & 1 & 1 & 0 \\ s^2 & 0 & -1 & 0 \\ s & ??? & & \\ s^0 & ??? & & \end{array}$$

Example

Third order example with missing term

Perturbed table:

$$
\begin{array}{c|ccc}
s^3 & 1 & 1 & 0 \\
s^2 & \epsilon & -1 & 0 \\
s & \frac{\epsilon+1}{\epsilon} & 0 & 0 \\
s^0 & -1 & 0 &
\end{array}
$$

Example

Third order example with missing term

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$\epsilon \to 0^+$: one sign change.

Example

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Perturbed table:

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$\epsilon \to 0^+$: one sign change.

$\epsilon \to 0^-$: one sign change.

Example

Third order example with missing term

Perturbed table:

$$\begin{array}{c|ccc} s^3 & 1 & 1 & 0 \\ s^2 & \epsilon & -1 & 0 \\ s & \frac{\epsilon+1}{\epsilon} & 0 & 0 \\ s^0 & -1 & 0 & \end{array}$$

$\epsilon \to 0^+$: one sign change.
$\epsilon \to 0^-$: one sign change.
Therefore we have got one root in the right half plan and two roots in the negative half plan.

Case of entire row 0

- When an entire row is null, two rows are proportional
- the characteristic polynomial $d(s)$ can be divided by the polynomial associated with the row immediately above the null row.
- The divisor polynomial is called “auxiliary polynomial” and call it $p(s)$.
- the auxiliary polynomial has roots that are symmetrical with respect to the imaginary axis.
- The number of changes above the line with zeros accounts for the stability of the roots of the “remainder” polynomial.

Case of entire row 0

- Strategy to complete the construction of the Routh table: to replace the null line with $\frac{dp(s)}{ds}$ and then continue.
- Looking the first column of the modified Routh table, the number of sign changes is still equal to the number of roots with positive real part.
- From the null row down, each sign change indicates a root with positive real part.
- Since roots are symmetric, there is be a corresponding root in the negative half plan.
- All roots not accounted in this way (i.e., no sign changes) are on the imaginary axis.

Example

Example with known roots

$$\begin{aligned} d(s) &= (s^2+1)(s+1)(s+3) \\ &= s^4 + 4s^3 + 4s^2 + 4s + 3 \end{aligned}$$

Two stable roots and a pair of imaginary roots.

- The first two rows of the Routh table are:

$$\begin{array}{c|ccc} s^4 & 1 & 4 & 3 \\ s^3 & 4 & 4 & 0 \\ \dots & & & \end{array}$$

Example

- We divide the third row by 3 and find:

$$\begin{array}{c|ccc} s^4 & 1 & 4 & 3 \\ s^3 & 1 & 1 & 0 \\ s^2 & 1 & 1 & 0 \\ s & 0 & 0 & 0 \end{array}$$

- The three positive elements of the first rows reveal two roots with negative real part.
- The auxiliary polynomial is given by $s^2 + 1$ and its derivative is $2s$.

Example

- Therefore we can complete the construction as:

$$
\begin{array}{c|ccc}
s^4 & 1 & 4 & 3 \\
s^3 & 1 & 1 & 0 \\
s^2 & 1 & 1 & 0 \\
s & 2 & 0 & 0 \\
1 & 1 & 0 & 0
\end{array}
$$

- The two new elements are positive. This means no changes and two roots on the imaginary axis.

Back to state space

- The notion of stability that we have introduced so far has to do with the Input/Output behaviour.
- A different notion of stability can be introduced “opening” the box and evaluating the internal workings of the system. This is what we have defined state space representation.
- We need a small digression in the realm of the state space representation.

Equilibrium point

- Consider a generic state space representation:

$$\dot{x} = f(x, u, t),$$

which can be simplified to

$$\dot{x} = f(x, u), \tag{1}$$

for time invariant systems.

Equilibrium Point

Definition

Consider the system in Equation (1). An equilibrium is a constant solution of the differential equation, i.e., a solution $\bar{x}, \bar{u}$ such that

$$f(\bar{x}, \bar{u}) = 0$$

Equilibrium Point

Mass Spring

Consider the usual mass spring system.

$$m\ddot{x} + c\dot{x} + kx = u \tag{2}$$

We can easily obtain a state space representation, $x_1 = x$, $x_2 = \dot{x}$:

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} = \ddot{x} &= (u - c\dot{x} - kx)/m \\ \dot{x_2} &= (u - cx_2 - kx_1)/m \end{aligned}$$

By setting $\dot{x}1 = 0$, $\dot{x_2} = 0$, we find:

$$\begin{aligned} \bar{x}_2 &= 0 \\ \bar{u} &= k\bar{x}_1 \end{aligned}$$

Stability

Stability

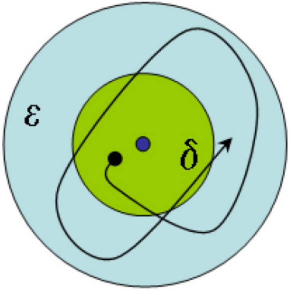
Definition

Consider the autonomous system $\dot{x} = f(x)$. Consider an equilibrium condition $\bar{x}$ such that

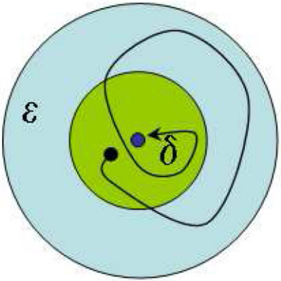
$$f(\bar{x}) = 0.$$

The Equilibrium is said stable if starting from an initial state $\bar{x} + \delta x$, for δx small enough, the trajectory remains close to the equilibrium: $\forall \epsilon > 0, \exists \delta x$ such that $x(0) = \bar{x} + \delta x \rightarrow |x(t) - \bar{x}| \leq \epsilon$ for all t, where $x(t) = \phi(\bar{x} + \delta x)$.
The Equilibrium is asymptotically stable of it is stable and if $\lim_{t \rightarrow \infty} \phi(\bar{x} + \delta x) = \bar{x}$, i.e., the trajectory eventually converges to the equilibrium.

Stability of an Equilibrium



Stable Equilibrium



Asymptoically Stable Equilibrium

Example

Mass Spring

Consider the usual mass spring system and suppose $u(t) = 0$. The equation is:

$$\dot{x_1} = x_2$$
$$\dot{x_2} = (-cx_2 - kx_1)/m$$

Let us consider the equilibrium $(x_1 = 0, x_2 = 0)$. Let us make a perturbation $(x_1 = 0.1, x_2 = 0.1)$. We can write it in matrix form:

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example

Mass Spring

Consider the usual mass spring system and suppose $u(t) = 0$.

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We can apply the differentiation rule on the entire state vector:

$$\begin{aligned} s \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} - \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= A \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} \\ \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} &= (sI - A)^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{aligned}$$

Example

Mass Spring

Suppose $m = 1,\ k = 4\ c = 5,\ (x_1 = 0.1, x_2 = 0.2)$.

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ k/m & s + c/m \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2 + (c/m)s + k/m} \begin{bmatrix} s + c/m & 1 \\ -k/m & s \end{bmatrix}$$

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{(s+c/m)x_1(0)}{s^2+(k/m)s+c/m} + \frac{x_2(0)}{s^2+(c/m)s+k/m} \\ -\frac{(k/m)x_1(0)}{s^2+(c/m)s+k/m} + \frac{sx_2(0)}{s^2+(c/m)s+k/m} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0.1(s+5)}{s^2+5s+4} + \frac{0.2}{s^2+5s+4} \\ -\frac{0.4}{s^2+5s+4} + \frac{0.2s}{s^2+5s+4} \end{bmatrix}$$

Example

Mass Spring

Inversion

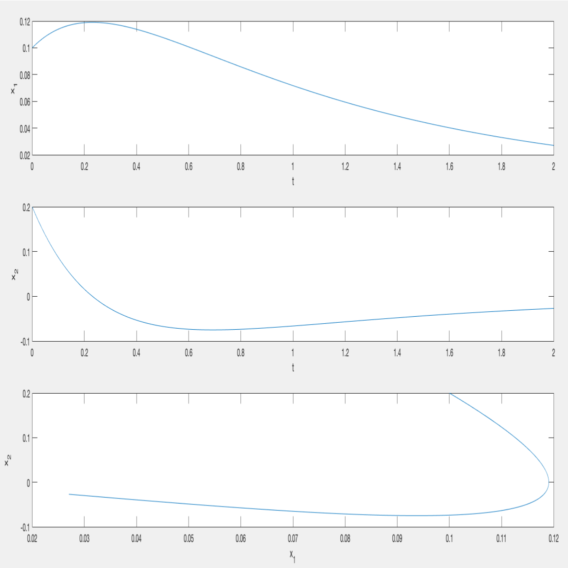
$$X(s) = \begin{bmatrix} \frac{0.1(s+5)}{(s+1)(s+4)} + \frac{0.2}{(s+1)(s+4)} \\ -\frac{0.4}{(s+1)(s+4)} + \frac{0.2s}{(s+1)(s+4)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{15} \cdot \frac{1}{s+1} - \frac{1}{30} \cdot \frac{1}{s+4} + \frac{1}{15} \cdot \frac{1}{s+1} - \frac{1}{15} \cdot \frac{1}{s+4} \\ \frac{2}{15} \cdot \frac{1}{s+4} - \frac{2}{15} \cdot \frac{1}{s+1} + \frac{4}{15} \cdot \frac{1}{s+4} - \frac{1}{15} \cdot \frac{1}{s+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} \cdot \frac{1}{s+1} - \frac{1}{10} \cdot \frac{1}{s+4} \\ \frac{2}{5} \cdot \frac{1}{s+4} - \frac{1}{5} \cdot \frac{1}{s+1} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \mathbf{1}(t) \left(\frac{1}{5} e^{-t} - \frac{1}{10} e^{-4t} \right) \\ \mathbf{1}(t) \left(\frac{2}{5} e^{-4t} - \frac{1}{5} e^{-t} \right) \end{bmatrix}$$

Example



0.12
0.1
0.08
0.06
0.04
0.02
x_1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t
0.2
0.1
0
-0.1
x_2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
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t
0.2
0.1
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x_2
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0.11
0.12
x_1

Stability as a local property

Generally speaking, a trajectory's stability is a local property: if we make small perturbation around the trajectory, we will remain close enough and we will eventually converge to the trajectory.
In general it is possible to define a region around the equilibrium called *region of asymptotic stability*, shuch that for any intial state within this region the trajectory will converge to the equilibrium.

Trajectories

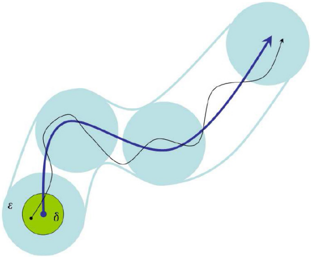
In the same way in which we evaluate the stability of an equilibrium, we can evaluate the stability of a trajectory.
Let $(\bar{x}(t), \bar{u}(t))$ be an equilibrium trajectory: $f(\bar{x}(t), \bar{u}(t)) = 0$. If we perturb the intitial state, we will have a new evolution for $x(t) = \bar{x}(t) + \delta x(t)$:

$$\begin{aligned}
\dot{x}(t) &= f(x, \bar{u}(t)) \\
\dot{\bar{x}}(t) + \dot{\delta x}(t) &= f(\bar{x} + \delta x, \bar{u}) \\
\dot{\delta x}(t) &= f(\bar{x} + \delta x, \bar{u}) - f(\bar{x} + \delta x, \bar{u}) \\
&= f(\bar{x} + \delta x, \bar{u}) \\
&= h(\delta x)
\end{aligned}$$

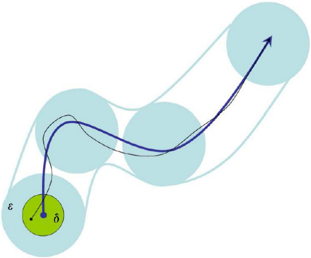
Fact

Consider a state space system $\dot{x} = f(x, u)$ and let $\bar{x}(t), \bar{u}(t)$ an equilibrium trajectory. The the stability around this trajectory can be studied as the sability of the origin of a new autonomous system $\dot{\delta x} = h(\delta x)$ where $h(\delta x) = f(\bar{x} + \delta x, \bar{u})$.

Stability of Trajectories



Stable Trajectory



Asymptotically Stable Trajectory

Stability of Linear and Time Invariant System

For a general system, stability is a property of a specific equilibrium point (or trajectories). Not so for Linear and Time Invariant systems.

Theorem

For a Linear and Time Invariant system,

- if an equilibrium point (or trajectory) is stable or asymptotically stable, then all the possible equilibrium points or trajectories are. In other words, stability (or asymptotic stability) is a property of the system.
- If a sytem is stable, then for any equilibrium the region of asymptotic stability consists of the whole state space (in other word stability is a global property).

Example

Mass Spring

Consider the usual mass spring system and suppose $u(t) = 10N/kg$, $c/m = 1$, $k/m = 10$ The equation is:

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_2 - 10x_1 + u(t)$$

Let us consider the equilibrium $(x_1 = 0, x_2 = 0)$. Let us make a perturbation $(x_1 = 100, x_2 = -1000)$. We can write it in matrix form:

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 10 \cdot \mathbf{1}(t)$$

The Laplace Transform is given by:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \left(sI - \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{10}{s} \end{bmatrix} \right)$$

Example

Mass Spring

Which is equal to:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{(s+1)100}{s^2+s+10} + \frac{\frac{10}{s}+1000}{s^2+s+10} \\ \frac{s\left(\frac{10}{s}+1000\right)}{s^2+s+10} - \frac{1000}{s^2+s+10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{49.5-168.05j}{s+0.5-3.1225j} + \frac{49.5+168.05j}{s+0.5+3.1225j} + \frac{1}{s} \\ \frac{500+238.59j}{s+0.5-3.1225j} + \frac{500+239.59j}{s+0.5+3.1225j} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{175.1929e^{-1.2843j}}{s+0.5-3.1225j} + \frac{175.1929e^{+1.2843j}}{s+0.5+3.1225j} + \frac{1}{s} \\ \frac{554.0087e^{0.4452j}}{s+0.5-3.1225j} + \frac{554.0087e^{-0.4452j}}{s+0.5+3.1225j} \end{bmatrix}$$

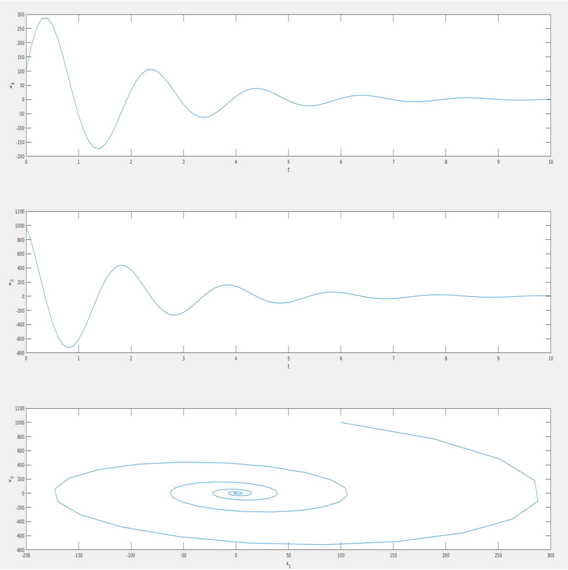
Example

Mass Spring

... in time domain, for $t \geq 0$ ..

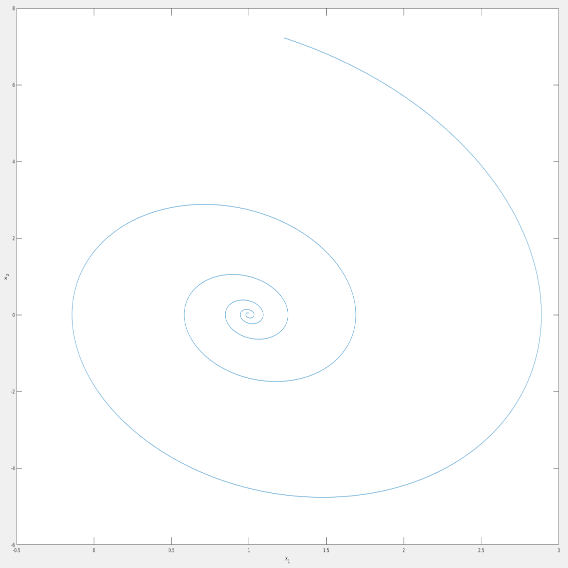
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{1}(t)\left(2 \cdot 175.1929 e^{-0.5t} \cos\left(3.1225t - 1.2843\right) + 1\right) \\ \mathbf{1}(t)\left(2 \cdot 554.0087 e^{-0.5t} \cos\left(3.1225t + 0.4452\right)\right) \end{bmatrix}$$

Example



Example

Behaviour for large values of t....



Example

Mass Spring

Consider the usual mass spring system and suppose $u(t) = 10N/kg$, $c/m = 0$, $k/m = 10$ (no damping). The equation is:

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -10x_1 + u(t)$$

Let us consider the equilibrium $(x_1 = 0, x_2 = 0)$. Let us make a perturbation $(x_1 = 100, x_2 = -1000)$. We can write it in matrix form:

$$\dot{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 10 \cdot \mathbf{1}(t)$$

The Laplace Transform is given by:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \left(sI - \begin{bmatrix} 0 & 1 \\ -10 & -0 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{10}{s} \end{bmatrix} \right)$$

Example

Mass Spring

... which corresponds to ...

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{100s}{s^2+10} + \frac{\frac{10}{s}+1000}{s^2+10} \\ \frac{s\left(\frac{10}{s}+1000\right)}{s^2+10} - \frac{1000}{s^2+10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{49.5-158.11j}{s-3.1623j} + \frac{49.5+158.11j}{s+3.1623j} + \frac{1}{s} \\ \frac{500+156.53j}{s-3.1623j} + \frac{500-156.53j}{s+3.1623j} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{165.6812e^{-1.2674j}}{s-3.1623j} + \frac{165.6812e^{+1.2674j}}{s+3.1623j} + \frac{1}{s} \\ \frac{523.93e^{0.3034j}}{s-3.1623j} + \frac{523.93e^{-0.3034j}}{s+3.1623j} \end{bmatrix}$$

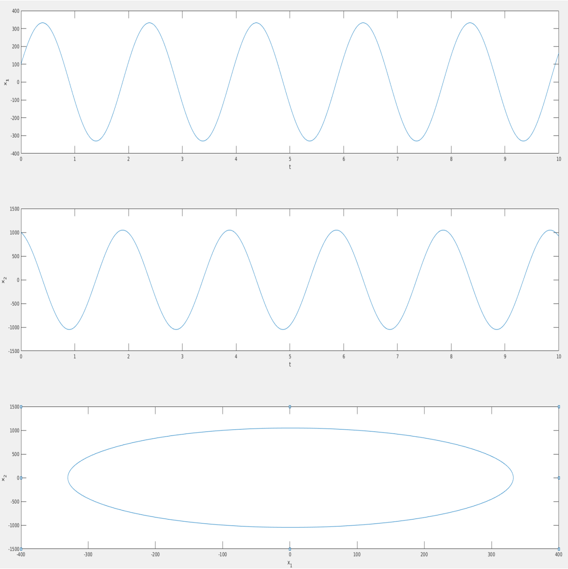
Example

Mass Spring

... in time domain, for $t \geq 0$ ..

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{1}(t)\left(2 \cdot 165.6812 \cos\left(3.1623t - 1.2674\right) + 1\right) \\ \mathbf{1}(t)\left(2 \cdot 523.93 \cos\left(3.1623t + 0.3034\right)\right) \end{bmatrix}$$

Example



400
300
200
100
0
-100
-200
-300
-400
x_1
0 1 2 3 4 5 6 7 8 9 10
t
1500
1000
500
0
-500
-1000
-1500
x_2
0 1 2 3 4 5 6 7 8 9 10
t
1500
1000
500
0
-500
-1000
-1500
x_2
-400 -300 -200 -100 0 100 200 300 400
x_1

Lessons Learned

- The Laplace transform can be used to study the trajectories of LTI in the state space
- A key role is played by the matrix $(sI - A)^{-1}$ and in particular by the denominator of its components $det(sI - A)$
- The roots of this denominator, which is the characteristic polynomial, are the matrix's eigenvalue
- Such eigenvalues give rise to exponential dyanmics and we call them pole

Stability of linear and time invariant systems

It is possible to show the following:

Theorem

Theorem

Consider a state space CT LTI system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Let p_i be the roots of $det(sI - A)$. Then the following are true:

- *If for all p_i, we have* $\mathbf{Real}\,(p_1) < 0$ *then the system is asymptotically stable*
- *If for all p_i, we have* $\mathbf{Real}\,(p_1) \leq 0$ *and if in the partial fraction expansion of all poles p_j such that* $\mathbf{Real}\,(p_j) = 0$ *the terms $\frac{1}{(s-p_i)^h}$ appear only with $h = 1$, then the system is stable.*
- *In all other cases the system is unstable.*

Stability of linear and time invariant systems – Discrete time counter part

It is possible to show the following:

Theorem

Theorem

Consider a state space CT LTI system:

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Let p_i be the roots of $det(sI - A)$. Then the following are true:

- *If for all p_i, we have $|p_1| < 1$ then the system is asymptotically stable*
- *If for all p_i, we have $|p_1| \leq 1$ and if in the partial fraction expansion of all poles p_j such that $|p_j| = 1$ the terms $\frac{1}{(s-p_i)^h}$ appear only with $h = 1$, then the system is stable.*
- *In all other cases the system is unstable.*

Observation

- We have seen two notions of stability: BIBO stability and Structural Stability (als said Lyapunov stability) for LTI systems.
- BIBO stability hinges on the position of the roots of the denominator of the transfer function $G(s)$, called poles.
- Lyapunov stability hinges on the positon of the roots of the roots of $det(sI - A)$, which we call....well....poles
- Any connection???

Transfer function from state space

- Consider a SISO LTI system with state space representation:

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

- Suppose we want to compute the forced evolution. We can resour to the Laplace Transform (assuming zero initial condition)

$$sX(s) = AX(s) + bU(s)$$
$$Y(s) = cX(s) + dU(s)$$

which leads to

$$Y(s) = c(sI - A)^{-1}bU(s)$$
$$Y(s) = G(s)U(s)$$

- Hence, $G(s) = c(sI - A)^{-1}b$.

Conclusions

- From $G(s) = c(sI - A)^{-1}b$, it follows that the poles of $G(s)$ are actually a subset of the eigenvalues of A (i.e., the poles of the state space representation)
- It could happen that the multiplication by c and b cancels some pole with a zero
- We call this event "loss of structural properties", which has to do with observability and controllability....a long story, which we do not have time to tell
- Essetially, when no such cancellation occurs, asymptotic stability amounts to BIBO stability.