Optimal Trade Allocation with Power-Law Market Impact

1. Problem Formulation

We consider an **optimal execution** problem where a total of S shares must be purchased over a sequence of discrete times $t_1, t_2, \ldots, t_N$. At each time t_i, a trade of size x_i incurs a *temporary market impact cost*

$$g_{t_i}(x_i) = \alpha_{t_i} \, x_i^{\beta_{t_i}},$$

with $\alpha_{t_i} > 0$ and $\beta_{t_i} > 0$ given (pre-estimated from historical order book data). The objective is to **minimize the total impact cost** while fulfilling the volume constraint:

$$\text{minimize}_{\{x_i\}_{i=1}^N} \quad C_{\text{total}} = \sum_{i=1}^{N} \alpha_{t_i} \, x_i^{\beta_{t_i}}, \qquad \text{subject to} \quad \sum_{i=1}^{N} x_i = S,$$

with $x_i \geq 0$ for all i. We treat this as an offline optimization (all g_{t_i} are known in advance, and we ignore interim price dynamics or stochasticity). The goal is to find the optimal allocation x_i^* at each time t_i such that $\sum_i x_i^* = S$ and C_{total} is minimized.

2. Assumptions

- **Deterministic Impact:** The impact functions $g_{t_i}(x)$ are known deterministic functions (no randomness in cost). Permanent price impact and other costs are ignored; we consider only temporary impact costs.
- **Discrete Time Bins:** Time is discretized into N intervals with predetermined $\alpha_{t_i}, \beta_{t_i}$.
- **No Additional Constraints:** We assume no per-interval volume caps or other constraints beyond $x_i \geq 0$ and the total sum S.
- **Competitive Market:** The cost parameters $\alpha_{t_i}, \beta_{t_i}$ are fixed during execution (no feedback from our execution to future impact).

3. Convex vs. Concave Impact Functions

Convex Impact ($\beta_{t_i} > 1$): Spreading the trade reduces cost. The problem is convex and has a unique global minimum.
Linear Impact ($\beta_{t_i} = 1$): Cost is linear. Allocate all volume to the interval with lowest α_{t_i}.
Concave Impact ($\beta_{t_i} < 1$): Cost is sublinear. It is optimal to concentrate volume in one or few intervals with the lowest total cost $g_{t_i}(S)$.

4. Optimal Allocation via Lagrange Multipliers (Convex Case)

For convex costs ($\beta_{t_i} > 1$), the Lagrangian is:

$$\mathcal{L}(x_1, \ldots, x_N, \lambda) = \sum_{i=1}^{N} \alpha_{t_i} \, x_i^{\beta_{t_i}} + \lambda \left(S - \sum_{i=1}^{N} x_i \right).$$

The first-order condition yields:

$$\alpha_{t_i} \, \beta_{t_i} \, x_i^{\beta_{t_i} - 1} = \lambda \quad \text{for each } i. \tag{1}$$

Solving gives:

$$x_i = \left(\frac{\lambda}{\alpha_{t_i} \, \beta_{t_i}} \right)^{1/(\beta_{t_i} - 1)}. \tag{2}$$

Plug into the constraint $\sum_i x_i = S$ to solve for λ numerically. One can solve Equation (2) numerically by binary search on λ.

5. Non-Convex Scenario and Heuristics (Concave Case)

If $\beta_{t_i} < 1$, the problem is non-convex. The Lagrange conditions identify stationary points, but not necessarily minima.
Heuristic Strategy: Compare the cost of executing all S shares in each interval:

$$C_i = \alpha_{t_i} \, S^{\beta_{t_i}}.$$

Pick the interval with lowest C_i. Alternatively, use a greedy algorithm that iteratively allocates to the interval with lowest marginal cost.

6. Summary of Solution Techniques

- **Convex ($\beta > 1$):** Solve using Lagrangian approach. Numerically solve for λ.
- **Linear ($\beta = 1$):** Allocate all volume to the lowest-α interval.
- **Concave ($\beta < 1$):** Heuristically allocate all (or most) volume to the best interval (lowest $\alpha_{t_i} S^{\beta_{t_i}}$).

The solution depends heavily on the shape of $g_{t_i}(x)$: convex cost leads to smooth spreading, while concave cost leads to aggressive concentration.