Numerical Methods

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Contents

Unit 4: Solution of Linear Algebraic Equations

Unit 4 **Solution of Linear Algebraic Equations** **10 Hrs.**

Review of the existence of solutions and properties of matrices, Consistency of a Linear System of Equations, Gaussian Elimination Method, Gauss-Jordan Method, Inverse of matrix using Gauss Elimination Method, Method of factorization, Iterative Methods(Jacobi & Gauss-Seidel Iteration),Power Method.

System of linear equations

- A system of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
\vdots \qquad\qquad \vdots \qquad\qquad &\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}
$$

where a_{jk} and b_j are given (real or complex) numbers.

◆ **Augumented matrix**

$$
[\mathbf{A} : \mathbf{B}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : & b_n \end{bmatrix}.
$$

System of linear equations

Solution of System of linear equations

1. Analytic approach
   - Gauss-Elimination method (Echelon form)
   - Gauss-Jordan method

2. Iteration approach
   - Jacobi's iteration method
   - Gauss-Seidal method

Gauss Elimination method

Ex.1 Applying Gauss elimination method to solve the system of equations $2x+3y-z=5$, $3x-y+7z=22$, $5x+y-2z=1$

Gauss-Jordan method

- **Example** Applying Gauss-Jordan method to solve the system of equations $3x - 2y + z = 0$, $x - 3y - 5z = -19$, $2x - y + 3z = 9$

System of linear equations

Ex.2 Applying Gauss elimination method to solve the system of equations $3x+5y-z=7$, $6x-7y+13z=12$, $9x-5y+10z=14$

Applying Gauss-Jordan method to solve the system of equations $11x-13y+5z=6$, $x+2y-z=2$, $3x+y-7z=10$,

System of linear equations

Example Applying Gauss-Jordan method to solve the system of equations $3x - 2y + z = 0$, $x - 3y - 5z = -19$, $2x - y + 3z = 9$

Class work

Applying Gauss-Jordan method to solve the system of equations $11x - 13y + 5z = 6$, $x + 2y - z = 2$, $3x + y - 7z = 10$,

System of linear equations

Applying Jacobi's iterative method to solve the system of equations $16x+y-2z=30$, $x-3y+10z=24$, $x+15y-4z=39$ correct to two decimal places

System of linear equations

The other approximations are shown in the following table:

Iterations	x	y	z
1	1.875	2.6	2.4
2	2.0125	3.115	2.9925
3	2.054375	3.263833	3.13325
4	2.062667	3.298575	3.173712
5	2.065553	3.308812	3.183306
6	2.066112	3.311178	3.186088

The values of x, y and z in fifth and sixth iterations are same till two decimal places. Hence, the roots are $x = 2.06$, $y = 3.31$, $z = 3.18$.

Gauss – Seidal iteration

Applying Gauss-Seidal iterative method to solve system of linear equations $10x + y - z = 11, \quad x + 10y + z = 28, \quad -x + y + 10z = 35$ correct to two decimal places

Gauss – Seidal iteration

The other approximations are the following table:

Iterations	x	y	z
1	1.1	2.69	3.341
2	1.1651	2.3494	3.3816
3	1.2032	2.3415	3.3862
4	1.2045	2.3409	3.3864

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Eigenvalue and Eigenvector - Power method

- **Example** **Find the Largest Eigenvalue and Eigenvector of the matrix** $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

Eigenvalue and Eigenvector – Power method

Find the largest eigenvalue and the correspon ding eigenvector of the matrix $\begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix}$

Eigenvalue and Eigenvector – Power method

First Iteration:

$$AX = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.50 \\ 0.25 \end{bmatrix} = \lambda^{(1)} X^{(1)}.$$

Second iteration:

$$AX^{(1)} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 3.25 \\ 2.0 \end{bmatrix} = 4.5 \begin{bmatrix} 1 \\ 0.7222 \\ 0.4444 \end{bmatrix} = \lambda^{(2)} X^{(2)}.$$

Third iteration:

$$AX^{(2)} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.7222 \\ 0.4444 \end{bmatrix} = \begin{bmatrix} 4.7222 \\ 4.2220 \\ 2.4444 \end{bmatrix} = 4.7222 \begin{bmatrix} 1 \\ 0.89410.5176 \end{bmatrix} = \lambda^{(3)} X^{(3)}.$$

Eigenvalue and Eigenvector – Power method

Fourth iteration:

$$AX^{(3)} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8941 \\ 0.5176 \end{bmatrix} = \begin{bmatrix} 4.8941 \\ 4.5880 \\ 2.7882 \end{bmatrix} = 4.8941 \begin{bmatrix} 1 \\ 0.9375 \\ 0.5697 \end{bmatrix} = \lambda^{4)} X^{(4)}.$$

Sixth iteration:

$$AX^{(5)} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9820 \\ 0.5823 \end{bmatrix} = \begin{bmatrix} 4.9820 \\ 4.9115 \\ 2.9640 \end{bmatrix} = 4.9820 \begin{bmatrix} 1 \\ 0.9858 \\ 0.5949 \end{bmatrix} = \lambda^{(6)} X^{(6)}.$$

Eigenvalue and Eigenvector – Power method

Seventh iteration:

$$AX^{(6)} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9858 \\ 0.5949 \end{bmatrix} = \begin{bmatrix} 4.9858 \\ 4.9745 \\ 2.9716 \end{bmatrix} = 4.9858 \begin{bmatrix} 1 \\ 0.9977 \\ 0.5960 \end{bmatrix} = \lambda^{(7)} X^{(7)}.$$

Eighth iteration:

$$AX^{(7)} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9977 \\ 0.5960 \end{bmatrix} = \begin{bmatrix} 4.9977 \\ 4.9800 \\ 2.9954 \end{bmatrix} = 4.9977 \begin{bmatrix} 1 \\ 0.9965 \\ 0.5994 \end{bmatrix} = \lambda^{(8)} X^{(8)}.$$

Exercise

Jacobi's iteration method

1.
$$10x + y - z = 14$$
$$-x + 20y + 7z = 23$$
$$3x - 2y + 15z = 33$$

2.
$$6x - z = 9$$
$$-x + 5y + z = 6$$
$$-y + 4z = 9$$

Gauss-Seidal iteration method

3.
$$9x + 2y + 3z = 17$$
$$-2x + 7y + 3z = 11$$
$$x - 2y + 13z = 9$$

4.
$$10x + 3y + z = 19$$
$$x + 20y + 2z = 21$$
$$2x + y + 10z = 20$$

Applying power method to find the largest Eigenvalue and corresponding Eigenvector of the following matrices

9. $\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 5 & -1 & 0 \\ 2 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$