HL Math Help

Version 1

Student-to-Student Tutoring

December 6 2020

Overview

Polynomial Theorems

- Factor Theorem
- Remainder Theorem
- Rational Root Theorem
- Complex Conjugate Root Theorem
- Sum and Product of Roots

Overview

Overview

- Derivatives
  - Power Rule

Overview

Chaos Theory
- Fixed Points
- Stability
- Cobweb Method
- n-cycles

Overview

Overview

Sequences and Series

basic Formulae

Overview

Mathematical Induction

I. Polynomial Theorems

I.i. Factor Theorem

If c is a root of $p(x)$, $(x - c)$ is a factor of $p(x)$.

I.ii. Remainder Theorem

The remainder when $p(x)$ is divided by c is $p(c)$.

I.iii. Rational Root Theorem

For the polynomial $p(x) = a_n x^2 + \cdots + a_0$, the rational roots are represented by $\frac{\text{factors}(a_0)}{\text{factors}(a_n)}$

Example

Given the polynomial $3x^3 + 4x^2 + 2x + 7$, the possible rational roots can be expressed as:

$$\text{p.r.z.} = \frac{\pm 1, \ \ \pm 7}{\pm 1, \ \ \pm 3}$$

This should be represented as:

$$\text{p.r.z.} = \left\{ \pm 1, \pm \frac{1}{3}, \pm 7, \pm \frac{7}{3} \right\}$$

I.iv. Complex Conjugate Root Theorem

Complex roots always come in conjugate pairs.

Example

If $x - mi$ is a factor of $f(x)$, then $x + mi$ must also be a factor of $f(x)$.

N.B.

Always remember to state the theorem you are using!

I.v. Sum and Product of Roots

Consider the following nth degree polynomial:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

The sum and product of roots can be found using the following formulae:

$$\Sigma\text{Roots} = \frac{-a_{n-1}}{a_n} \qquad \Pi\text{Roots} = \frac{(-1)^n a_0}{a_n}$$

II. Properties of Functions

II.i. Even and Odd-Powered Polynomials

Even-Powered Polynomials

Consider the Polynomial:

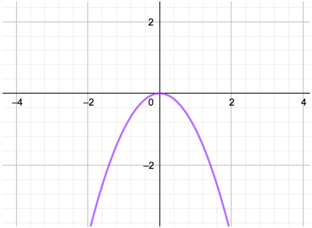
$$p(x) = a_n x^n + \cdots + a_1 x^1 + a_0$$

If n is an even number (i.e. if $2|n$), are two scenarios to consider:

1. Even-powered polynomials when $a_n < 0$ go from the bottom left, do something wonky, and come down on the bottom right.
2. Even-powered polynomials when $a_n > 0$ go from the top left, do something wonky, and come down on the top right.

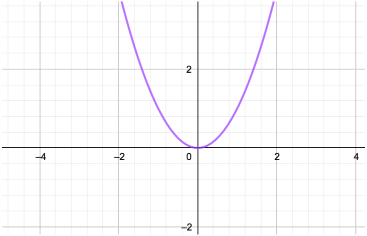
Example

Figure: x^2 when $a_n < 0$



Example

Figure: x^2 when $a_n > 0$



Odd-Powered Functions

Consider the same polynomial:

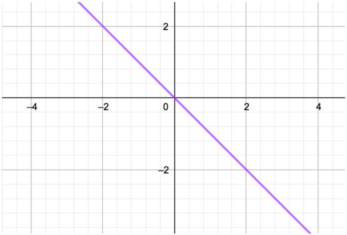
$$p(x) = a_n x^n + \cdots + a_1 x^1 + a_0$$

If n is an odd number, there are also two scenarios to consider:

1. Odd-powered polynomials when $a_n > 0$ go from the bottom left, do something wonky, and come up on the top right.
2. Odd-powered polynomials when $a_n < 0$ go from the top left, do something wonky, and come down on the bottom right.

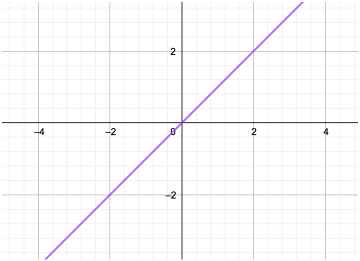
Example

Figure: x when $a_0 < 0$



Example

Figure: x when $a_0 > 0$



II.ii. Even and Odd Functions

The Odd Function

An odd function is a function with the following property:

$$f(x) = -f(-x)$$

This essentially means it is symmetrical over the $y = x$ line.

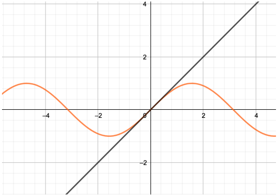
N.B.

An *odd function* is not the same as an *odd-powered polynomial*!

Example

The sin function is an example, as it is symmetrical over the $y = x$ line, as shown in the figure below:

Figure: The *sin* function is symmetrical over the $y = x$ line



The Even Function

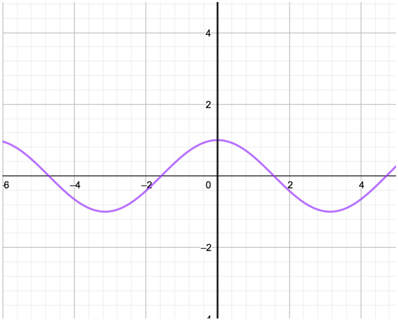
An even function is a function with the following property:

$$f(x) = f(-x)$$

This essentially means it is symmetrical over the y axis.

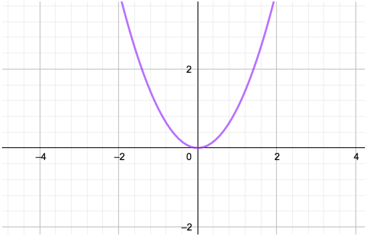
Example

Figure: The cos function



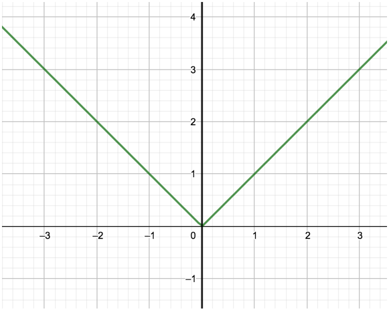
Example

Figure: The x^2 function



Example

Figure: The $|x|$ function



N.B.

Because an even function is symmetrical over the y axis, a vertical translation keeps it an even function. That said, a horizontal translation makes it no longer an even function. This means that x^2 and $x^2 + 1$ are both even functions, though $(x+1)^2$ is not.

II.iii. Function Transformations

Formulae

Given a function of the form:

$$g(x) = af(bx - h) + k$$

Any of its transformed coordinates can be determined by the following formula:

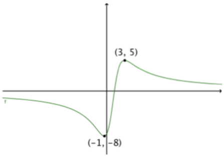
$$(x, y) \mapsto \left(\frac{x}{b} + h, ay + k\right)$$

Of course, an alternate method is to use logic.

Example Question

Given the graph of the function $f(x)$ below, determine the coordinates of the local extrema in the transformed function $g(x) = 2f(x-3)+1$.

Figure: Mystery Function $f(x)$



We start by noting down all relevant information about the function:

a	2
b	1
h	3
k	1

Then we apply the formula for point $(3, 5)$:

$$\begin{aligned} (x, y) &\mapsto \left(\frac{x}{b} + h, ay + k\right) \\ (3, 5) &\mapsto (3 + 3, 10 + 1) \\ &\mapsto (6, 11) \end{aligned}$$

And for point $(-1, -8)$:

$$\begin{aligned} (x, y) &\mapsto \left(\frac{x}{b} + h, ay + k\right) \\ (-1, -8) &\mapsto (2, -15) \end{aligned}$$

III. Derivatives

III.i. Power Rule

To find the derivative of $f(x)$, for each term, you multiply the coefficient by the power, make that the new coefficient, subtract one from the exponent, and make that the new exponent. As a result, constant terms are ignored, but negative exponents are not!

Example

Given $f(x) = 5x^3 + 3x^2 + 2x + 1$, the derivative is $f'(x) = 15x^2 + 6x + 2$.

Definition

The derivative of $f(x)$ is often written as $f'(x)$.

IV. Chaos Theory

IV.i. Fixed Points

A fixed point is when the function is equal to itself (i.e. the point is "trapped"). Fixed points are found by solving for $f(x) = x$ (i.e. the intersection between the function and the $y = x$ line).

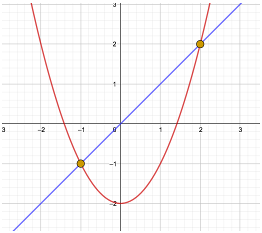
Example

The fixed points of $x^2 - 2$ would be:

$$
\begin{aligned}
x^2 - 2 - x &= 0 \\
(x-2)(x+1) &= \\
x &\in \{-1, 2\}
\end{aligned}
$$

So the fixed points are -1 and 2. This is also shown by the graph:

Figure: Fixed points of $x^2 - 2$



IV.ii. Stability

The stability of a fixed point can be determined using the derivative of the original function. Let $x_\star$ be a fixed point of $f(x)$. Let $g = |f'(x_\star)|$. If $g > 1$, it is unstable and repelling. If $g = 1$, it is neutral. If $g < 1$, it is stable and attracting.

Example

The fixed points of $f(x) = x^2 - 2$ are $x_\star \in \{-1, 2\}$. $f'(x) = 2x$, and let $g(x) = |f'(x)|$. For -1, $g(-1) = 2$, which is greater than 1. $g(2) = 4$, which is greater than 1. Both fixed points are unstable and repelling.

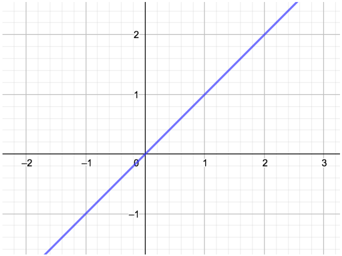
IV.iii. Cobweb Method

The cobweb method is a technique to find the fixed points of any given function. Essentially, you follow this procedure:

I Draw the the $y = x$ line on a cartesian plane.

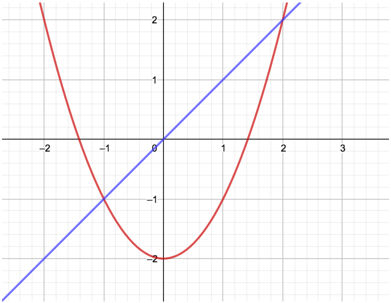
II Draw the function (we will use the example from earlier, $f(x) = x^2 - 2$)

III Choose a point on the x-axis near the fixed point you are testing (we are testing fixed point $x_\star = 2$).



The cobweb method is a technique to find the fixed points of any given function. Essentially, you follow this procedure:

I. Draw the the $y = x$ line on a cartesian plane.
II. Draw the function (we will use the example from earlier, $f(x) = x^2 - 2$)
III. Choose a point on the x-axis near the fixed point you are testing (we are testing fixed point $x_\star = 2$).

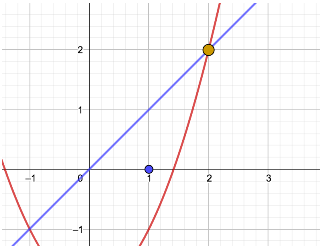


The cobweb method is a technique to find the fixed points of any given function. Essentially, you follow this procedure:

I Draw the the $y = x$ line on a cartesian plane.

II Draw the function (we will use the example from earlier, $f(x) = x^2 - 2$)

III Choose a point on the x-axis near the fixed point you are testing (we are testing fixed point $x_\star = 2$).

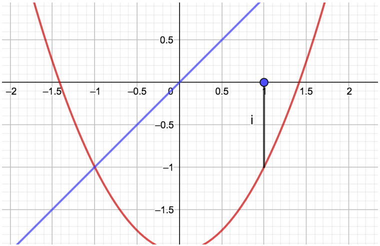


The cobweb method is a technique to find the fixed points of any given function. Essentially, you follow this procedure:

IV Draw a line from that point to the function.

V Draw a line from the function to the $y = x$ line.

VI Draw a line from the $y = x$ line to the function (in this case it converges already).

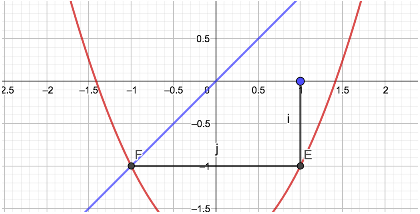


The cobweb method is a technique to find the fixed points of any given function. Essentially, you follow this procedure:

IV Draw a line from that point to the function.

V Draw a line from the function to the $y = x$ line.

VI Draw a line from the $y = x$ line to the function (in this case it converges already).

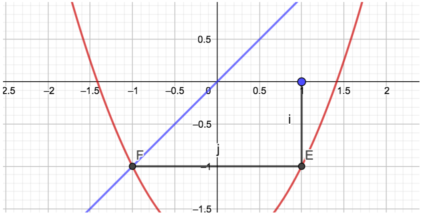


The cobweb method is a technique to find the fixed points of any given function. Essentially, you follow this procedure:

IV Draw a line from that point to the function.

V Draw a line from the function to the $y = x$ line.

VI Draw a line from the $y = x$ line to the function (in this case it converges already).



Repeat this process until you can see whether

1. It diverges (unstable and repelling)
2. It converges to the point (stable and attracting)
3. It behaves in a chaotic way (neutral)

IV.iv. n-cycles

To find an n-cycle, all one must do is plug the function into itself n times. An n^{th} cycle is often written as ${}^{n}f(x)$, for a given function $f(x)$.

It is important to note that all one-cycles are **trivial** n-cycles. This means that they will appear as n-cycles, even if really, they are one-cycles.

Also note that an n-cycle means that it gets stuck into a loop of sorts, where the function alternates between the values in the cycle.

Example

Find all 2-cycles of $f(x) = x^2 - 2$. To to so, we must compute $f(f(x)) - x = 0$.

$$\begin{aligned} {}^2f(x) - x &= 0 \\ f(f(x)) - x &= 0 \\ (x^2 - 2)^2 - 2 - x &= 0 \\ x^4 - 4x^2 - x + 2 &= 0 \end{aligned}$$

Example

Fortunately, we know that all one-cycles are trivial n-cycles, so $x^2 - x - 2$ is a root of $x^4 - 4x^2 - x + 2$. To find the other roots, we simply divide the two and factor the resulting quadratic:

$$
\begin{array}{r@{}l}
 & \quad x^2 + x - 1 \\
x^2 - x - 2 & \overline{)\, x^4 \qquad - 4x^2 - x + 2} \\
 & \underline{-x^4 + x^3 + 2x^2} \\
 & \qquad x^3 - 2x^2 - x \\
 & \qquad \underline{-x^3 + x^2 + 2x} \\
 & \qquad\qquad -x^2 + x + 2 \\
 & \qquad\qquad \underline{x^2 - x - 2} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Example

Now, we apply the quadratic formula:

$$\begin{aligned} x^2 - x - 1 &= 0 \\ x &= \frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2} \\ x &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Therefore, the **trivial** 2-cycles are -1 and 2, and the **nontrivial** 2-cycles are $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$

V. Advanced Rational Functions

V.i. Inequalities

What is the solution to the following?

$$\frac{x^2 + 5x}{x^2 - 4} > 0$$

Start by factoring to get:

$$\frac{x(x+5)}{(x-2)(x+2)} > 0$$

Note down some of the properties

$$\begin{aligned} \text{Zeroes:} \quad & x = 0 \\ & x = -5 \\ \text{Restrictions:} \quad & x \neq -2 \\ & x \neq 2 \end{aligned}$$

Now, note down these values on a number line, and identify the sections that must be tested:

A -5 B -2 C 0 D 2 E

Now, note down these values on a number line, and identify the sections that must be tested:

A	-5	B	-2	C	0	D	2	E

Testing for **A**

$$
\begin{aligned}
x &= -6 \\
&= \frac{-6(-6+5)}{(-6-2)(-6+2)} \\
&= \frac{- \cdot -}{- \cdot -} \\
&= + \\
+ &> 0
\end{aligned}
$$

This section does not work.

Now, note down these values on a number line, and identify the sections that must be tested:

A	-5	**B**	-2	C	0	D	2	E

Testing for **B**

$$
\begin{aligned}
x &= -3 \\
&= \frac{-3(-3+5)}{(-3-2)(-3+2)} \\
&= \frac{- \cdot +}{- \cdot -} \\
&= - \\
- &< 0
\end{aligned}
$$

This section works.

Now, note down these values on a number line, and identify the sections that must be tested:

A	-5	B	-2	**C**	0	D	2	E

Testing for **C**

$$\begin{aligned} x &= -1 \\ &= \frac{-1(-1+5)}{(-1-2)(-1+2)} \\ &= \frac{-\cdot+}{-\cdot+} \\ &= + \\ + &> 0 \end{aligned}$$

This section does not work.

Now, note down these values on a number line, and identify the sections that must be tested:

A	-5	B	-2	C	0	**D**	2	E

Testing for **D**

$$
\begin{aligned}
x &= 1 \\
&= \frac{1(1+5)}{(1-2)(1+2)} \\
&= \frac{+\cdot+}{-\cdot+} \\
&= - \\
- &< 0
\end{aligned}
$$

This section works.

Now, note down these values on a number line, and identify the sections that must be tested:

A	-5	B	-2	C	0	D	2	**E**

Testing for **E**

$$
\begin{aligned}
x &= 3 \\
&= \frac{3(3+5)}{(3-2)(3+2)} \\
&= \frac{+\cdot+}{+\cdot+} \\
&= + \\
+ &> 0
\end{aligned}
$$

This section does not work.

Now, note down these values on a number line, and identify the sections that must be tested:

A	-5	B	-2	C	0	D	2	E

In conclusion, $x < 0$ so long as $x \in \ ]-5, -2[\ \cup\ ]0, 2[$

VI. Sequences and Series

VI.i. basic Formulae

Geometric Sequence

Let $a = a_1, a_2, a_3, a_4, \cdots, a_n$ where there are n terms. If each element is r times the previous, the **sum** is:

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

The **recursive formula** is:

$$a_n = a_{n-1} \cdot r$$

The **closed formula** is:

$$a_n = a_1 \cdot r^{n-1}$$

Algebraic Sequence

Let $a = a_1, a_2, a_3, a_4, \cdots, a_n$ where there are n terms. If each element is d more than the previous, the **sum** is:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

The **recursive formula** is:

$$a_n = a_{n-1} + d$$

The **closed formula** is:

$$a_n = a_1 + d(n-1)$$

VII. Mathematical Induction

Example Question

Use mathematical induction to prove that
$5^n + 3^n \geq 2^{2n+1};\ \forall n \in \mathbb{Z}^\star$

Step I

$$\text{let } p(n):\ 5^n + 3^n \geq 2^{2n+1};\ \forall n \in \mathbb{Z}^\star$$

Step II

$$\begin{aligned} p(1):\ 5+3 &\geq 2^3 \\ 8 &\geq 8 \end{aligned}$$

$p(1)$ is true

Step III

$$p(k):\ 5^k + 3^k \geq 2^{2k+1}$$

Step IV

$$p(k+1):\ 5^{k+1} + 3^{K+1} \overset{?}{\geq} 2^{2k+3};\ \forall k \in \mathbb{Z}^\star$$

$LS = 5^{k+1} + 3^{k+1}$	Start on the left side
$5^k \cdot 5 + 3^k \cdot$	Expand.
$\geq (2^{2k+1} - 3^k) \cdot 5 + 3^k \cdot 3$	By the Inductive hypothesis, we have that if $5^k + 3^k \geq 2^{2k+1}$, $5^k \geq 2^{2k+1} - 3^k$
$= 5 \cdot 2^{2k+1} - 5 \cdot 3^k + 3 \cdot 3^k$	Simplify.
$= 5 \cdot 2^{2k+1} - 2 \cdot 3^k$	
$\geq 5 \cdot 2^{2k+1} - 2 \cdot 4^k$	Because $-3^k \geq -4^k$ (notice the brackets and watch out for negative signs).
$= 5 \cdot 2^{2k+1} - 2 \cdot 2^2 k$	
$= 5 \cdot 2^{2k+1} - \cdot 2^{2k+1}$	Because $4 = 2^2$

$= 4 \cdot 2^{2k+1}$	Simplify
$= 2^{2k+3}$	Because $4 = 2^2$
$= RS$	This is equal to the right side

Step V

By the principle of mathematical induction, $p(x)$ is true.

$\square$