Digital Communications - HW3

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21/05/2018

We have to implement six different versions of the receiver structure in a QPSK modulation scheme. First we present the setup of the transmitter and the channel as given, then we analyze the different configurations one by one and give a brief discussions of the resulting probabilities of symbol error obtained from simulation over different values of the SNR at the channel output, Γ.

Transmitter and Channel

The system takes a sequence of input symbols a_k at sampling time $T = 1$ and applies an upsampling of factor 4, obtaining a'_k at $T/4$. This new sequence is then filtered by q_c as described by the following difference equation:

$$s_c(nT/4) = 0.67 s_c((n-1)T/4) + 0.7424 a_{n-5} \tag{1}$$

After the filtering white noise is added. The SNR at the channel output for all the configurations in this first phase is $\Gamma = 10$ dB, so from the following relations we can derive σ_w^2, the variance of the complex valued Gaussian noise:

$$\Gamma = \frac{M_{s_c}}{N_0 \frac{1}{T}} = \frac{\sigma_a^2 E_{q_c}}{\sigma_w^2} \longrightarrow \sigma_w^2 = \frac{\sigma_a^2 E_{q_c}}{\Gamma} = 2\sigma_I^2 \tag{2}$$

where σ_I^2 is the variance per component. In addition we can also compute the PSD as $N_0 = \sigma_w^2 T_c = \sigma_w^2/4$, because the sampling time T_c at which we add the noise is $T/4$. In figure 1 we plot the impulse response and the frequency response of the filter q_c. This implementation of the transmitter is the same for all the following discussion.

Point A

In point A at the receiver we have a matched filter g_M (see figure 2), obtained from q_c as $g_M = q_c^*(t_0 - t)$. For simplicity in the last formula we have denoted the filters as is they were defined on continuous time while in the actual simulation they are at $T/4$.
The output of the matched filter is then sampled at T starting from an initial offset called *timing phase* t_0. In our case the choice of t_0 is made easy by the presence of the matched filter, as we can just choose the value $\bar{t}_0$, multiple of $T/4$, that is the index of the peak of the correlation between q_c and g_M, then t_0 will be equal to $\bar{t}_0 T/4$. Following this reasoning we chose $\bar{t}_0 = 17$, equal also to the length of g_M (see figure 2).

The signal is then passed to a linear equalizer (LE) derived by a particular case of a Decision Feedback Equalizer (DFE) where we only have the feedforward filter c (see point B for the detailed analysis of the DFE). The signal at this point in the receiver system is called x_k and is the result of the convolution of the input sequence a_k with the overall impulse response $h_i = q_c * g_M$ that goes from $-N_1$ to N_2. We will call precursors the taps of h that go from $-N1$ to -1 and postcursors the taps from 1 to N_2. To obtain the coefficients of c we used the Wiener approach on the input random process and solved the Wiener-Hopf equation $\boldsymbol{c}_{opt} = \boldsymbol{R}^{-1}\boldsymbol{p}$ using the matrix $\boldsymbol{R}$ and vector $\boldsymbol{p}$ as in equations 5 and 6 with the parameter M_2 (the order of the feedback filter) set to 0 because we have no feedback filter in this case. The free parameters that we have to choose are M_1, the order of filter c and D the delay that it will introduce on the input sequence. We carried out this choice looking at the value of the cost function J_{min} (see equation 7) for each combination of the two parameters, preferring low values if possible to avoid increasing the complexity. The best choice in this case was $M_1 = 5$ and $D = 2$. In figure 3 we plot the impulse response c_i at sampling time T as obtained from the Wiener solution.
The aim of filtering with c is to obtain an overall impulse response of the system that satisfies the Nyquist conditions for the absence of ISI at time T. This implies that in the ideal case $\psi = h * c$ is a delayed impulse centered on D that is the delay. In our case the ψ obtained is shown in figure 4. We can see the result is pretty good as all the precursors and postcursors are almost canceled by the equalizer.
The detected symbols a_{k-D} are chosen using a threshold detector that analyzes the sign of the imaginary and real part of the input complex value. Not that the same detector is also used at point B, C and D.

Point B

For point B the system is the same as in point A up to the equalizer, so the choice of $\bar{t}_0 = 17$ is the same. The matched filter is the same as in point A, see figure 5. However in this case we equalize with a DFE, that is made of two filters called feedforward and feedback filter denoted by c and b. The feedforward filter has the role of equalizing only the precursors of the overall impulse response, while the ISI due to postcursors will be canceled by filter b positioned on a feedback loop between the output of the threshold detector and its input.
The computation of the optimal filters c and b is carried out using the Wiener filter approach. The relation between the input random process and the output is:

$$\begin{aligned} y_k &= x_{FF,k} + x_{FB,k} \\ &= \sum_{i=0}^{M_1-1} c_i x_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j} \end{aligned} \tag{3}$$

where M_1 is the order of the feedforward filter, M_2 is the order of the feedback filter and a_{k-D} are the already detected past symbols fed back through b. Defining postcursors and precursors as in point A, we have that we can apply the Wiener-Hopf equations on the process:

$$y_k = \sum_{i=0}^{M_1-1} c_i \left(x_{k-i} - \sum_{j=1}^{M_2} h_{j+D-i} a_{k-j-D} \right) \tag{4}$$

The result can be easily computed as $c_{opt} = \boldsymbol{R}^{-1}\boldsymbol{p}$ once we find the autocorrelation matrix $\boldsymbol{R}$ and the correlation vector $\boldsymbol{p}$, expressed as [1]:

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left(\sum_{j=-N_1}^{N_2} h_j h^*_{j-(p-q)} - \sum_{j=1}^{M_2} h_{j+D-q} h^*_{j+d-p} \right) + r_{\tilde{w}}(p-q) \tag{5}$$

$$[\mathbf{p}]_p = \sigma_a^2 h^*_{D-p} \qquad\qquad p = 0, 1, \ldots, M_1 - 1 \tag{6}$$

where for a QPSK scheme $\sigma_a^2 = 2$ because it is the sum of two orthogonal components each with power 1. The values of $r_{\tilde{w}}$ are the result of the autocorrelation of the noise after being filtered by g_M, so being the noise white we have $r_{\tilde{w}}(n) = N_0 r_{g_M}(nT)$. At this point we can define the overall impulse response up to the threshold detector $\psi = h * c_{opt}$ and derive the optimal coefficients for filter b as $b_i = -\psi_{i+D}$ for $i = 1, \ldots, M_2$.
The value of the cost function J_{min} obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left(1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{D-l} \right) \tag{7}$$

Again the parameters to choose are the order of filter c, M_1, and the delay introduced D. This is because the order of b can be chosen in such a way that all the postcursors are canceled by the feedback: $M_2 = N_2 + M_1 - D - 1$, and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional J_{min}, this time being $M_1 = 5$ and $D = 4$, and consequently $M_2 = 4$ because $N_2 = 4$. In figure 6, 7 and 8 we plot the resulting filters c, ψ and b at sampling time T.

Point C

In the receiver at point C we use a different type of approach. Before sampling the received signal we use an anti-aliasing filter instead of a matched filter. This is not an optimal solution but can be useful in some situation where the channel is not known or varies in time. Also the downsampling is at $T/2$ instead of T and this gives more degrees of freedom int the equalization and more robustness with respect to the choice of the timing phase. The anti-aliasing filter has to avoid overlapping in frequency when we downsample the signal at the output of the channel. Our signal has frequency content that is periodic of period $4/T$, and has the shape of the Fourier transform of filter q_c. Therefore its bandwidth is not limited and our anti-aliasing filter g_{AA} will remove useful information. The sampling at $T/2$ causes the frequency content of the signal to be repeated every $2/T$, causing an overlap. To avoid this, the cutting frequency of g_{AA} has to be around $1/T$. In figure 9 we show the magnitude of the frequency response $|G_{AA}|$, where we chose the passband at $0.45 \cdot 2/T$ and the stopband at $0.55 \cdot 2/T$.

The sampling then has to start after an offset $\bar{t}_0$ equal to the peak of the overall impulse response $q_c * g_{AA}$ at time $T/4$ that in this case was equal to 21. In this kind of configuration we also add a digital matched filter after the sampling that is different from the previous points, now the matched filter is given by flipping and taking the conjugate of the whole impulse response $q_c * g_{AA}$ at sampling time $T/2$:

$$g_M = \{q_c * g_{AA}\}^* \left(t_0 + i\frac{T}{2}\right) \tag{8}$$

And the inpulse response is depicted in figure 10. In this case we use again a DFE. To derive the coefficients of the equalizer. Now the c filter works at $T/2$ so the equations used for the Wiener solution aren't the same used for the points A and B. The equations 4, 5 and 6 become

$$y_k = \sum_{i=0}^{M_1-1} c_i \left(x_{2k-i} - \sum_{j=1}^{M_2} h_{2(j+D)-i} a_{k-j-D} \right) \tag{9}$$

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left(\sum_{n=-\infty}^{\infty} h_{2n-q} h_{2n-p}^* - \sum_{j=1}^{M_2} h_{2(j+D)-q} h_{2(j+D)-p}^* \right) + r_{\tilde{w}}(p-q) \tag{10}$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{2D-p}^* \qquad\qquad p, q = 0, 1, \ldots, M_1 - 1 \tag{11}$$

where for a QPSK scheme $\sigma_a^2 = 2$ because it is the sum of two orthogonal components each with power 1. The values of $r_{\tilde{w}}$ are the result of the autocorrelation of the noise after being filtered by g_{AA} and g_M, so being the noise white we have $r_{\tilde{w}}(n) = N_0 r_{g_M * g_{AA}}(nT/2)$. At this point we can define the overall impulse response up to the threshold detector $\psi = h * c_{opt}$ and derive the optimal coefficients for filter b as $b_i = -\psi_{2(i+D)}$ for $i = 1, \ldots, M_2$ because filter b works at T.
The value of the cost function J_{min} obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left(1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{2D-l}\right) \tag{12}$$

Again the parameters to choose are the order of filter c, M_1, and the delay introduced D. This is because the order of b can be chosen in such a way that all the postcursors are canceled by the feedback: $M_2 = N_2 + M_1 - D - 1$, and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional J_{min}, this time being $M_1 = 9$ and $D = 4$, and consequently $M2 = 16$ because $N_2 = 12$
In figure 11, 12 and 13 we plot the resulting filters c, ψ at sampling time $T/2$ and b at sampling time T. In the fractionally spaced version of the DFE the aim is again to obtain an overall pulse ψ that satisfies the Nyquist conditions at time T so the samples at even multiples of $T/2$ are not used in the equalizer and are therefore a degree of freedom of the system.

Point D

The following table sums up the final parameters for the 4 configurations A, B, C, D.

Point	Length of h		Parameters			
	N_1	N_2	M_1	M_2	D	$\bar{t}_0$
A	4	4	5	0	2	17
B	4	4	5	4	4	17
C	10	12	9	16	4	21
D	10	12	9	16	4	21

Table 1: Choices for the various parameters in the different receiver implementations with respect to the length of h.

Point E

In this point we use the Viterbi algorithm (VA) to implement the Maximum Likelihood criterion for data detection. For the execution of the algorithm we consider the signal y_k at the output of filter c in the schema at point B:

$$y_k = \psi_D a_{k-D} + \psi_{D+1} a_{k-D-1} + \cdots + \psi_{D+M_2} a_{k-D-M_2} + w_k \tag{13}$$

where w_k includes the residual ISI and the noise. The useful signal u_k can then be taken as $u_k = \psi_D a_{k-D} + \psi_{D+1} a_{k-D-1} + \cdots + \psi_{D+M_2} a_{k-D-M_2}$ and if we call $y_k = z_k$ we have the usual setup for VA: $z_k = u_k + w_k$. We will have then that the number of states is $N_s = M^{M_2}$ where M is 4, the cardinality of the QPSK constellation. In the specific case of point B $M_2 = 4$ so $N_s = 256$. We define the state basing ourselves of the choice of u_k so:

$$\begin{aligned} \boldsymbol{s}_k = \{a_{k-D}, a_{k-D-1}, a_{k-D-2}, \ldots, a_{k-(M_2-1)}\} = \\ = \{a_{k-D}, a_{k-D-1}, a_{k-D-2}, a_{k-D-3}\} \end{aligned} \tag{14}$$

Point F

Simulation results

References

[1] Nevio Benvenuto, Giovanni Cherubini, *Algorithms for Communication Systems and their Applications.* Wiley, 2002.

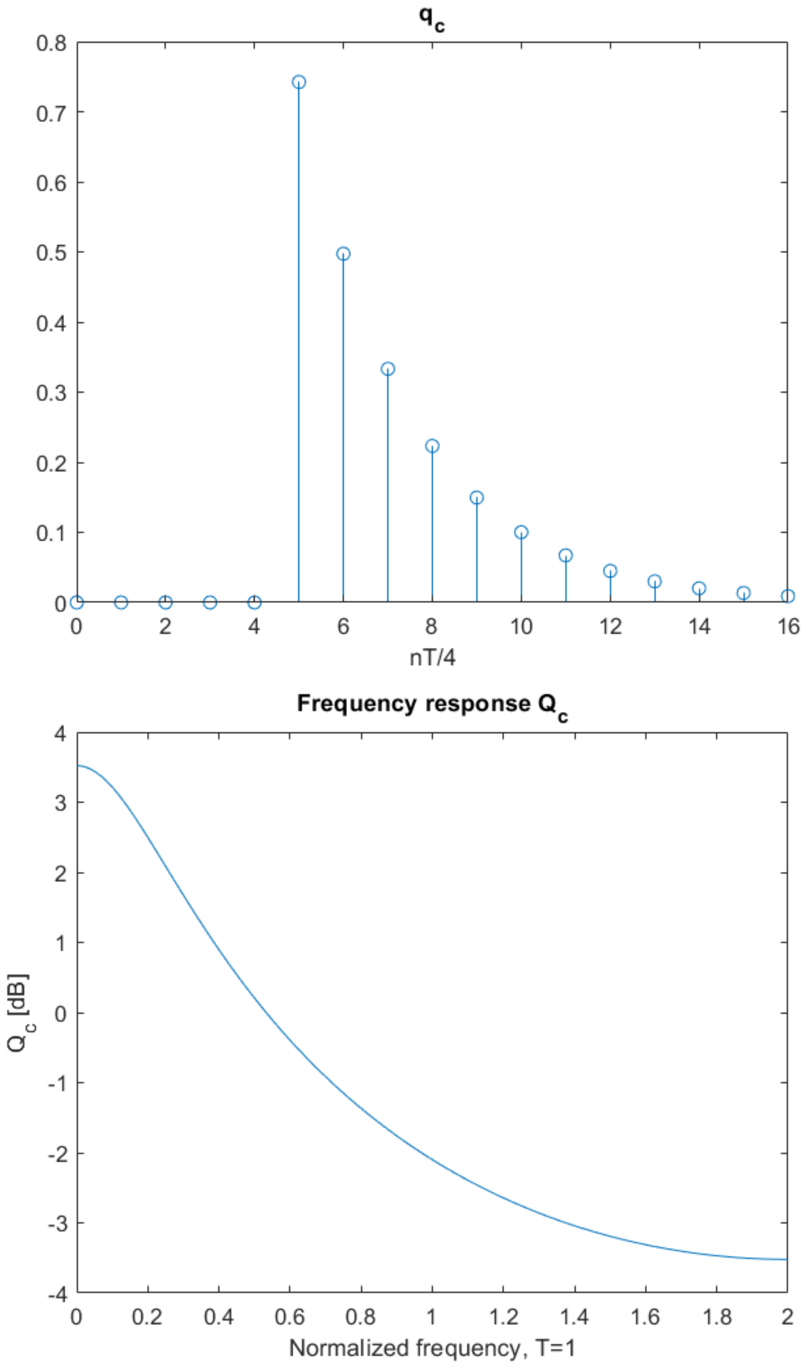

Figure 1: Impulse and frequency response of the filter q_c at $T/4$.

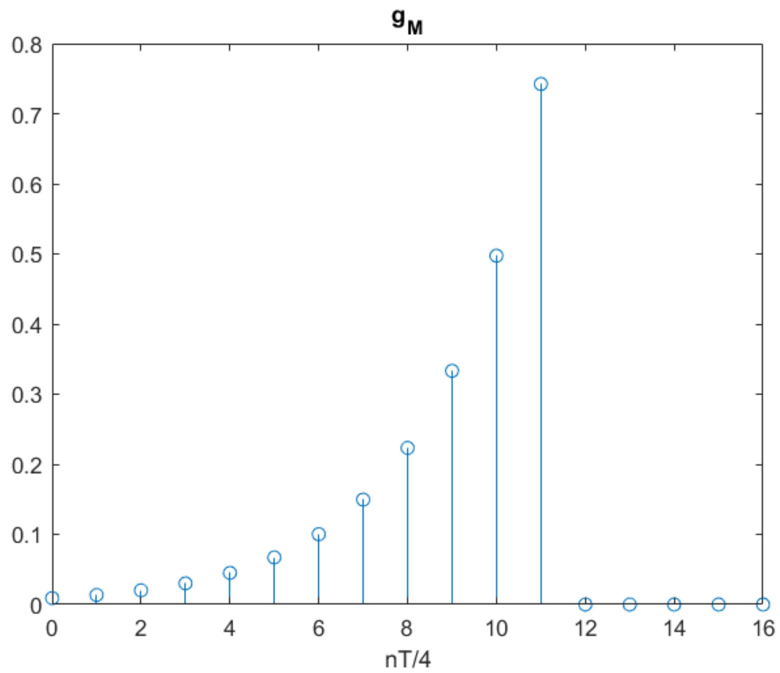

Figure 2: Impulse response of the matched filter g_M for the receiver in point A.

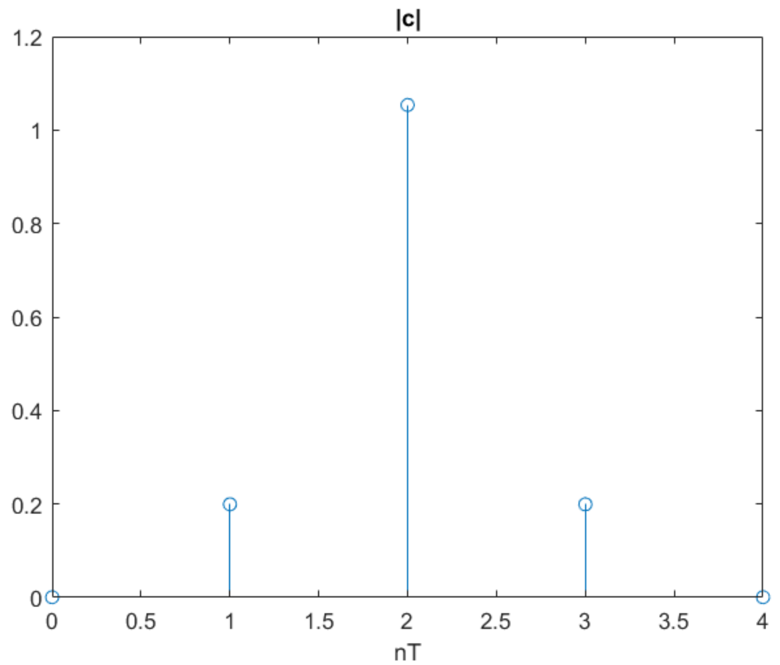

Figure 3: Magnitude of the impulse response of filter c in point A.

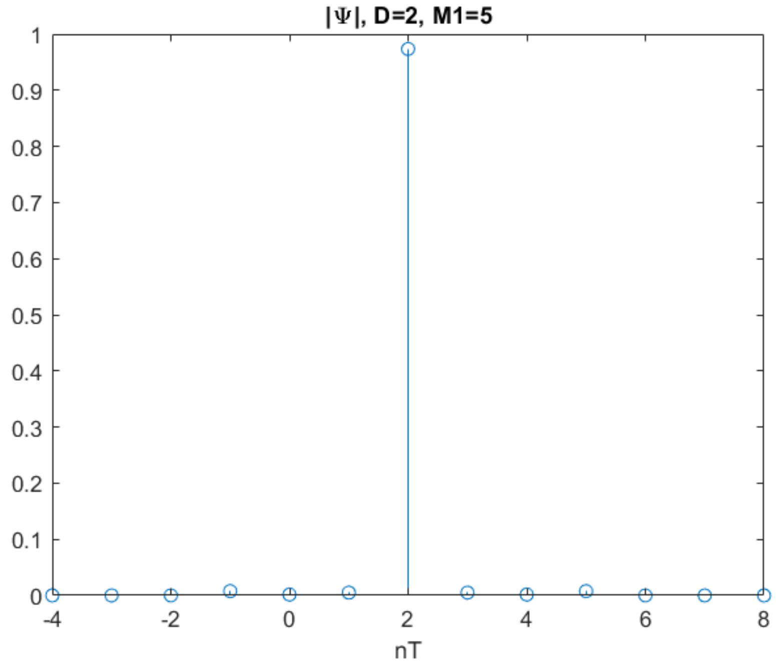

Figure 4: Magnitude of the impulse response of the system ψ in point A.

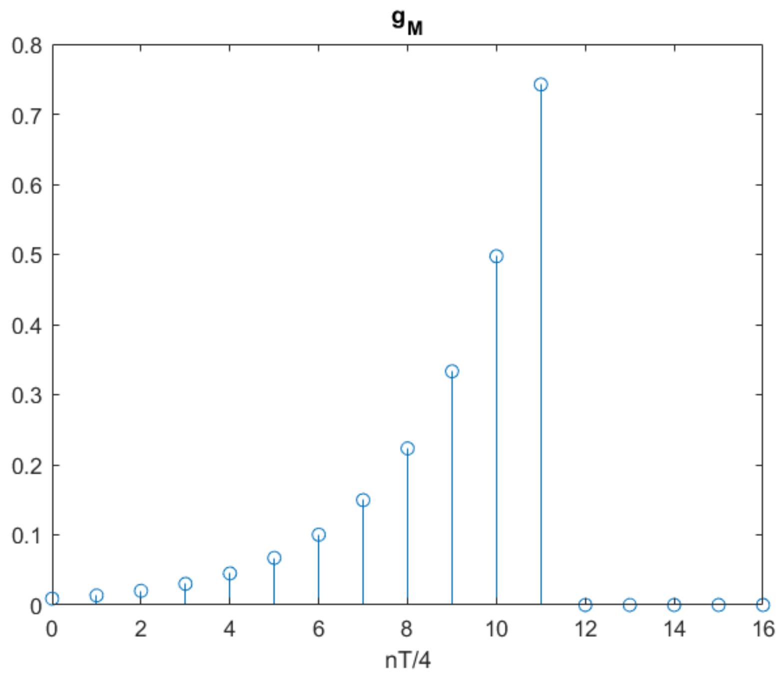

Figure 5: Impulse response of the matched filter g_M for the receiver in point B.

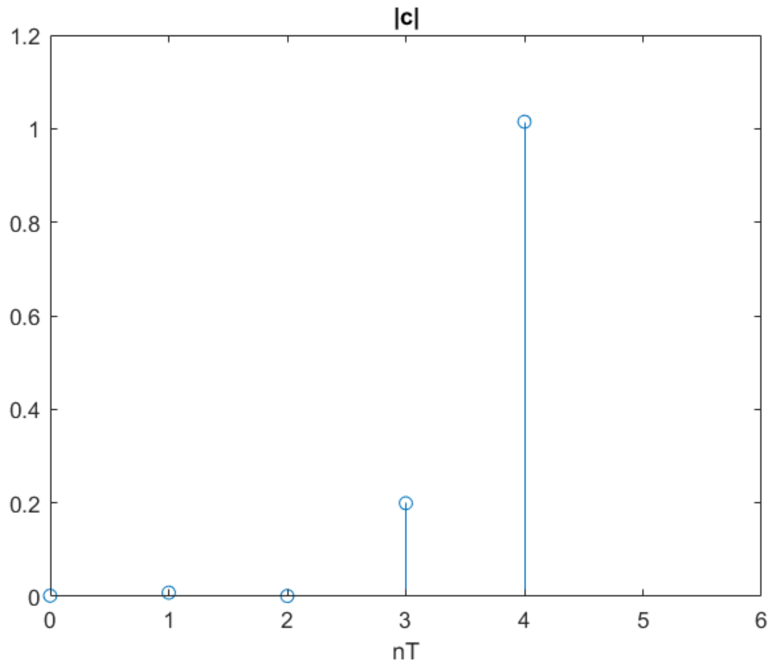

Figure 6: Magnitude of the impulse response of the filter c (feedforward filter) for the receiver in point B.

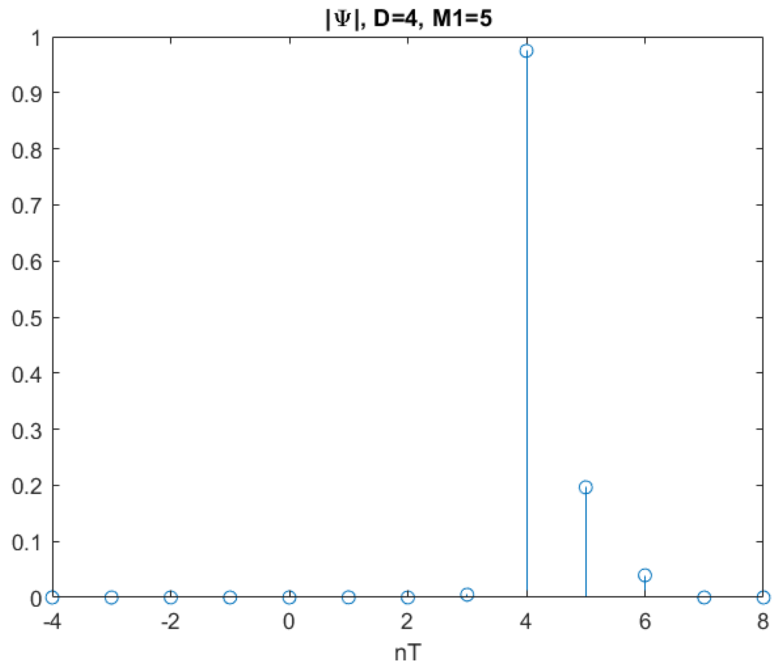

Figure 7: Magnitude of the impulse response of the system ψ in point B.

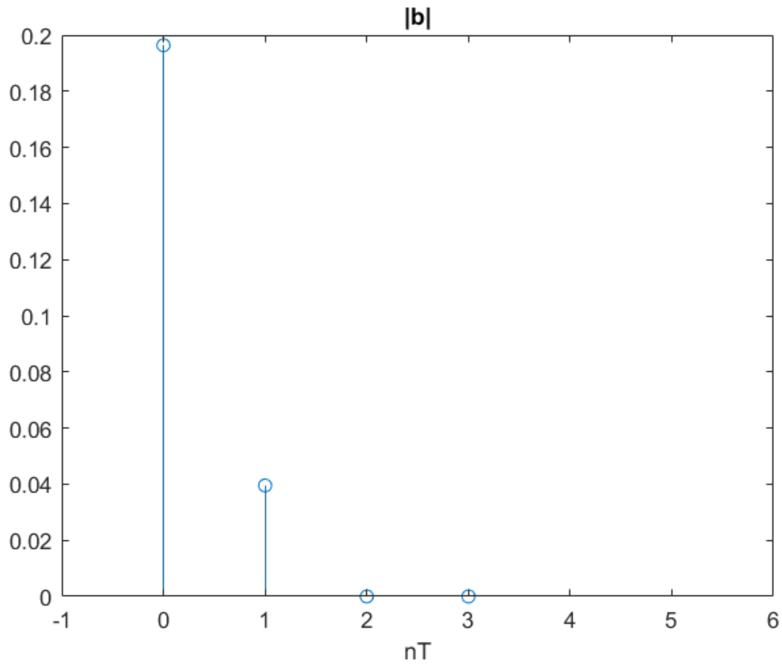

Figure 8: Magnitude of the impulse response of the filter b (feedback filter) in point B.

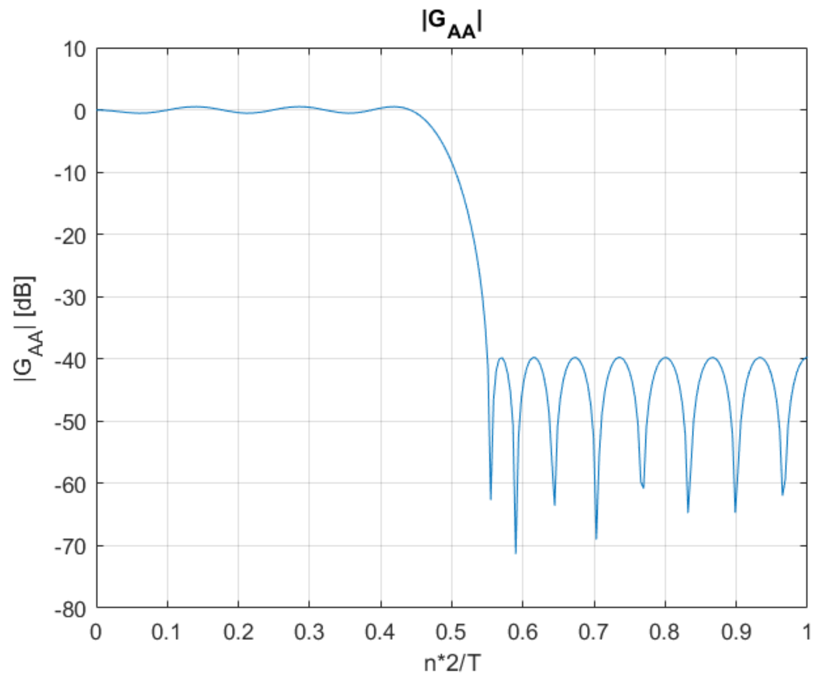

Figure 9: Magnitude of the frequency response of the anti-aliasing filter in point C.

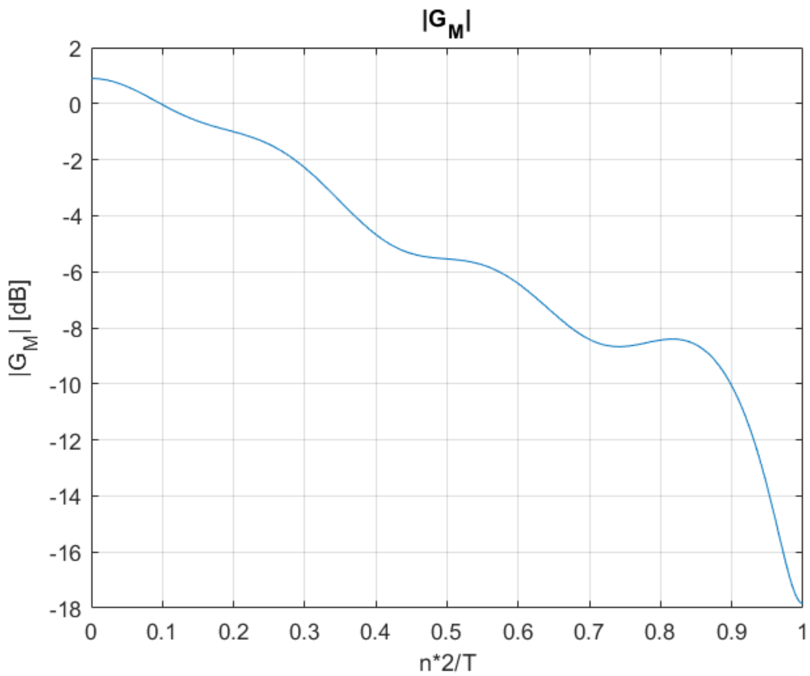

Figure 10: Magnitude of the frequency response of the matched filter in point C.

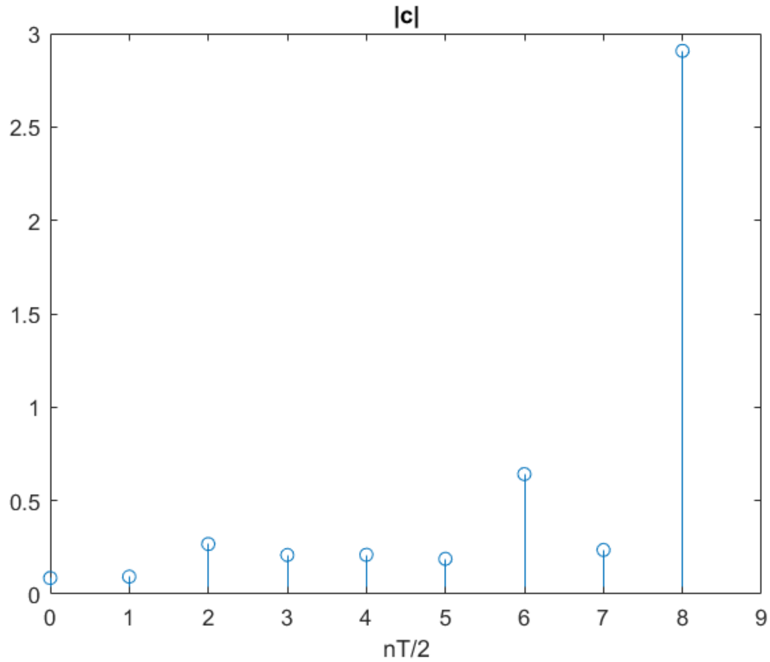

Figure 11: Magnitude of the impulse response of the filter c (feedforward filter) for the receiver in point C.

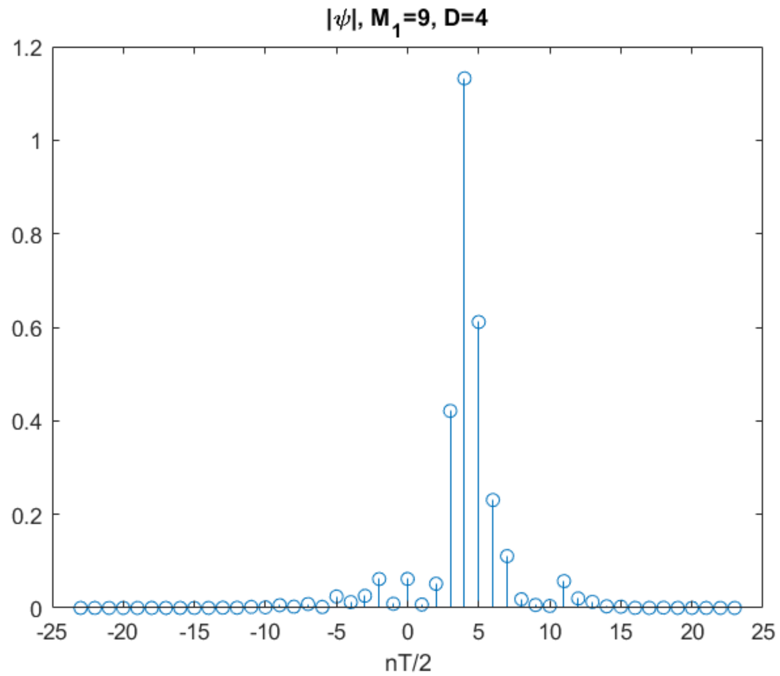

Figure 12: Magnitude of the impulse response of the system ψ in point C.

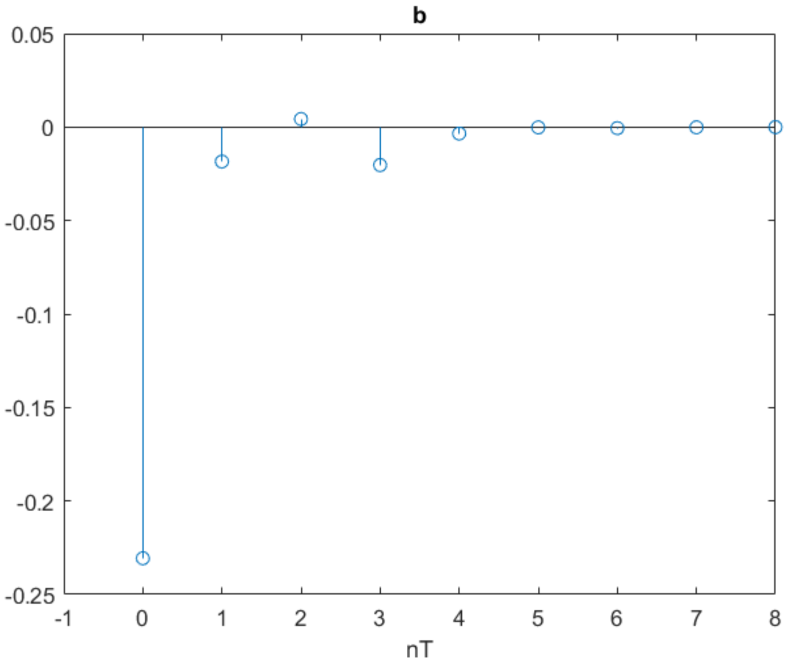

Figure 13: Magnitude of the impulse response of the filter b (feedback filter) in point C.

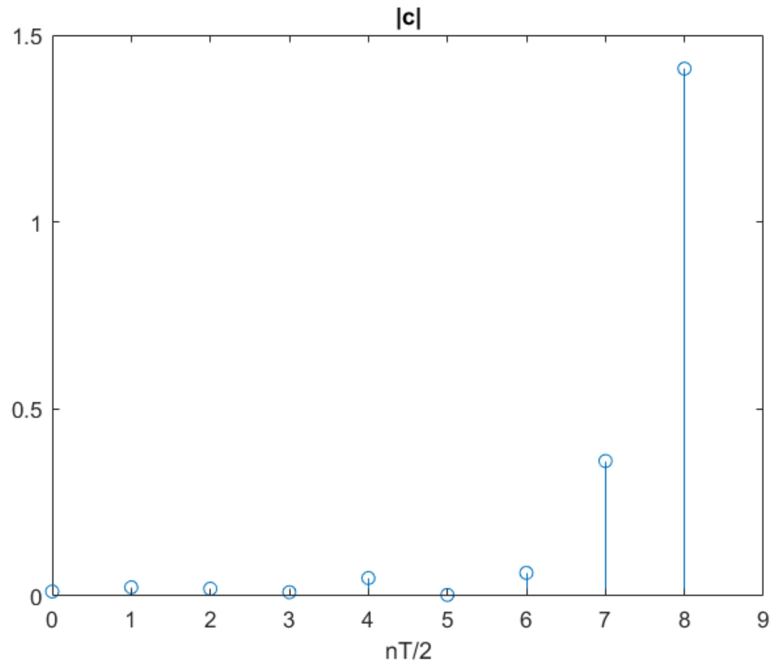

Figure 14: Magnitude of the impulse response of the filter c (feedforward filter) for the receiver in point D.

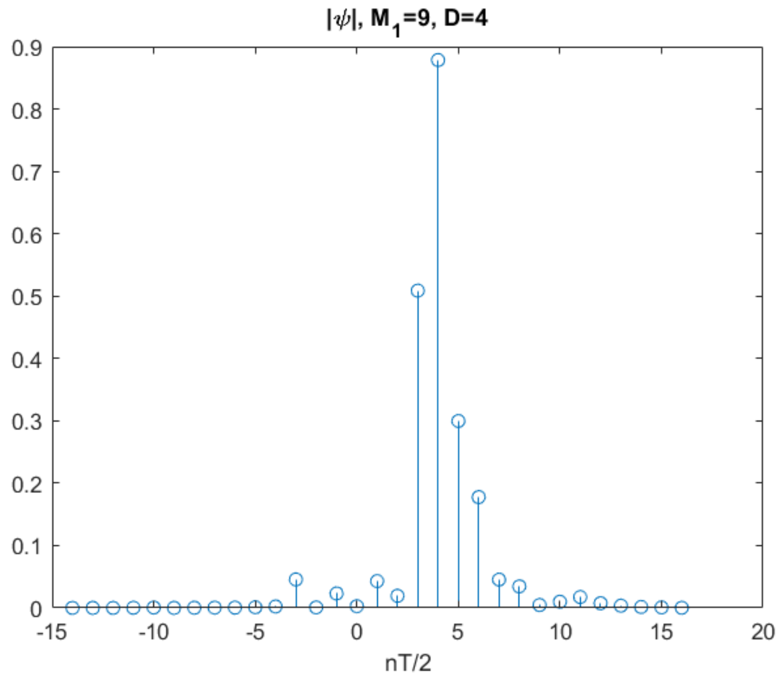

Figure 15: Magnitude of the impulse response of the system ψ in point D.

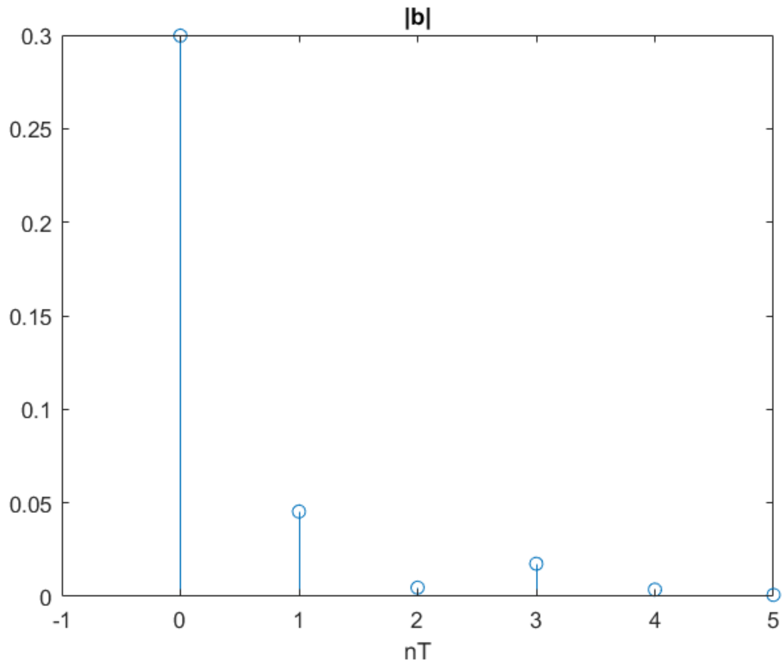

Figure 16: Magnitude of the impulse response of the filter b (feedback filter) in point D.

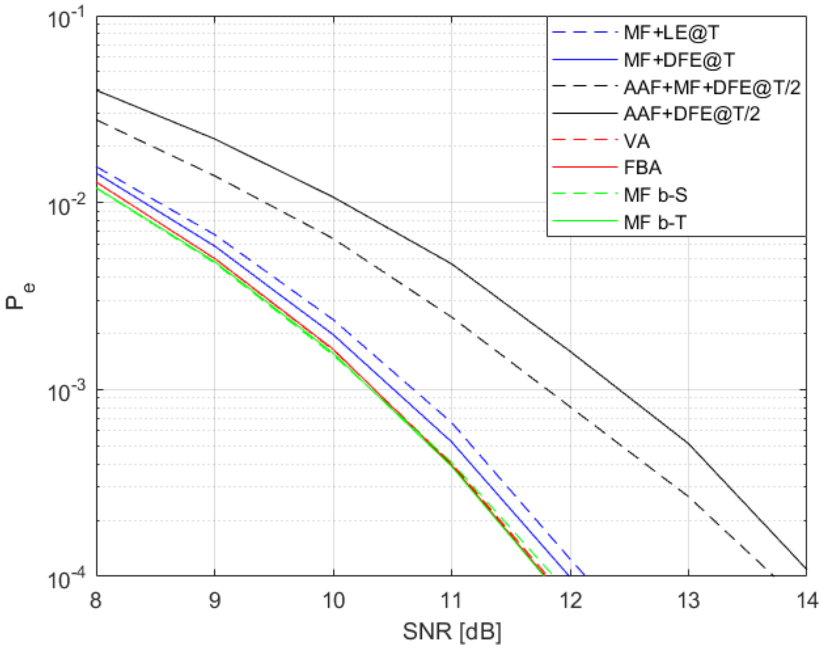

Figure 17: Results of the simulation over values of the SNR at the channel output from 8 dB to 14 dB.