

# Digital Communications - HW3

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## Problem

We have to implement six different versions of the receiver structure in a QPSK modulation scheme. First we present the setup of the transmitter and the channel as given, then we analyze the different configurations one by one and give a brief discussion of the resulting probabilities of symbol error obtained from simulation over different values of the SNR at the channel output,  $\Gamma$ .

## Transmitter and Channel

The system takes a sequence of input symbols  $a_k$  at sampling time  $T = 1$  and applies an upsampling of factor 4, obtaining  $a'_k$  at  $T/4$ . This new sequence is then filtered by  $q_c$  as described by the following difference equation:

$$s_c(nT/4) = 0.67s_c((n-1)T/4) + 0.7424a_{n-5} \quad (1)$$

After the filtering white noise is added. The SNR at the channel output for all the configurations in this first phase is  $\Gamma = 10$  dB, so from the following relations we can derive  $\sigma_w^2$ , the variance of the complex valued Gaussian noise:

$$\Gamma = \frac{M_{s_c}}{N_0 \frac{1}{T}} = \frac{\sigma_a^2 E_{q_c}}{\sigma_w^2} \longrightarrow \sigma_w^2 = \frac{\sigma_a^2 E_{q_c}}{\Gamma} = 2\sigma_I^2 \quad (2)$$

where  $\sigma_I^2$  is the variance per component. In addition we can also compute the PSD as  $N_0 = \sigma_w^2 T_c = \sigma_w^2/4$ , because the sampling time  $T_c$  at which we add the noise is  $T/4$ . In figure 1 we plot the impulse response and the frequency response of the filter  $q_c$ . This implementation of the transmitter is the same for all the following discussion.

## Point A

In point A at the receiver we have a matched filter  $g_M$  (see figure 2), obtained from  $q_c$  as  $g_M = q_c^*(t_0 - t)$ . For simplicity in the last formula we have denoted the filters as if they were defined on continuous time while in the actual simulation they are at  $T/4$ .

The output of the matched filter is then sampled at  $T$  starting from an initial offset called *timing phase*  $t_0$ . In our case the choice of  $t_0$  is made easy by the presence of the matched filter, as we can just choose the value  $\bar{t}_0$ , multiple of

$T/4$ , that is the index of the peak of the correlation between  $q_c$  and  $g_M$ , then  $t_0$  will be equal to  $\bar{t}_0 T/4$ . Following this reasoning we chose  $\bar{t}_0 = 17$ , equal also to the length of  $g_M$  (see figure 2).

The signal is then passed to a linear equalizer (LE) derived by a particular case of a Decision Feedback Equalizer (DFE) where we only have the feedforward filter  $c$  (see point B for the detailed analysis of the DFE). The signal at this point in the receiver system is called  $x_k$  and is the result of the convolution of the input sequence  $a_k$  with the overall impulse response  $h_i = q_c * g_M$  that goes from  $-N1$  to  $N2$ . We will call precursors the taps of  $h$  that go from  $-N1$  to  $-1$  and postcursors the taps from  $1$  to  $N2$ . To obtain the coefficients of  $c$  we used the Wiener approach on the input random process and solved the Wiener-Hopf equation  $\mathbf{c}_{opt} = \mathbf{R}^{-1} \mathbf{p}$  using the matrix  $\mathbf{R}$  and vector  $\mathbf{p}$  as in equations 5 and 6 with the parameter  $M_2$  (the order of the feedback filter) set to 0 because we have no feedback filter in this case. The free parameters that we have to choose are  $M1$ , the order of filter  $c$  and  $D$  the delay that it will introduce on the input sequence. We carried out this choice looking at the value of the cost function  $J_{min}$  (see equation 7) for each combination of the two parameters, preferring low values if possible to avoid increasing the complexity. The best choice in this case was  $M1 = 5$  and  $D = 2$ . In figure 3 we plot the impulse response  $c_i$  at sampling time  $T$  as obtained from the Wiener solution.

The aim of filtering with  $c$  is to obtain an overall impulse response of the system that satisfies the Nyquist conditions for the absence of ISI at time  $T$ . This implies that in the ideal case  $\psi = h * c$  is a delayed impulse centered on  $D$  that is the delay. In our case the  $\psi$  obtained is shown in figure 4. We can see the result is pretty good as all the precursors and postcursors are almost canceled by the equalizer.

The detected symbols  $a_{k-D}$  are chosen using a threshold detector that analyzes the sign of the imaginary and real part of the input complex value. Not that the same detector is also used at point B, C and D.

## Point B

For point B the system is the same as in point A up to the equalizer, so the choice of  $\bar{t}_0 = 17$  is the same. The matched filter is the same as in point A, see figure 5. However in this case we equalize with a DFE, that is made of two filters called feedforward and feedback filter denoted by  $c$  and  $b$ . The feedforward filter has the role of equalizing only the precursors of the overall impulse response, while the ISI due to postcursors will be canceled by filter  $b$  positioned on a feedback loop between the output of the threshold detector and its input.

The computation of the optimal filters  $c$  and  $b$  is carried out using the Wiener filter approach. The relation between the input random process and the output is:

$$\begin{aligned} y_k &= x_{FF,k} + x_{FB,k} \\ &= \sum_{i=0}^{M1-1} c_i x_{k-i} + \sum_{j=1}^{M2} b_j a_{k-D-j} \end{aligned} \quad (3)$$

where  $M1$  is the order of the feedforward filter,  $M2$  is the order of the feedback filter and  $a_{k-D}$  are the already detected past symbols fed back through  $b$ .

Defining postcursors and precursors as in point A, we have that we can apply the Wiener-Hopf equations on the process:

$$y_k = \sum_{i=0}^{M_1-1} c_i \left( x_{k-i} - \sum_{j=1}^{M_2} h_{j+D-i} a_{k-j-D} \right) \quad (4)$$

The result can be easily computed as  $c_{opt} = \mathbf{R}^{-1} \mathbf{p}$  once we find the autocorrelation matrix  $\mathbf{R}$  and the correlation vector  $\mathbf{p}$ , expressed as [1]:

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left( \sum_{j=-N_1}^{N_2} h_j h_{j-(p-q)}^* - \sum_{j=1}^{M_2} h_{j+D-q} h_{j+D-p}^* \right) + r_{\tilde{w}}(p-q) \quad (5)$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{D-p}^* \quad p, q = 0, 1, \dots, M_1 - 1 \quad (6)$$

where for a QPSK scheme  $\sigma_a^2 = 2$  because it is the sum of two orthogonal components each with power 1. The values of  $r_{\tilde{w}}$  are the result of the autocorrelation of the noise after being filtered by  $g_M$ , so being the noise white we have  $r_{\tilde{w}}(n) = N_0 r_{g_M}(nT)$ . At this point we can define the overall impulse response up to the threshold detector  $\psi = h * c_{opt}$  and derive the optimal coefficients for filter  $b$  as  $b_i = -\psi_{i+D}$  for  $i = 1, \dots, M_2$ .

The value of the cost function  $J_{min}$  obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left( 1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{D-l} \right) \quad (7)$$

Again the parameters to choose are the order of filter  $c$ ,  $M_1$ , and the delay introduced  $D$ . This is because the order of  $b$  can be chosen in such a way that all the postcursors are canceled by the feedback:  $M_2 = N_1 + M_1 - D - 1$ , and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional  $J_{min}$ , this time being  $M_1 = 5$  and  $D = 4$ , and consequently  $M_2 = 2$  because  $N_1 = 2$ . In figure 6, 7 and 9 we plot the resulting filters  $c$ ,  $\psi$  and  $b$  at sampling time  $T$ .

## Point C

In the receiver at point C we use a different type of approach. Before sampling the received signal we use an anti-aliasing filter instead of a matched filter. This is not an optimal solution but can be useful in some situation where the channel is not known or varies in time. Also the downsampling is at  $T/2$  instead of  $T$  and this gives more degrees of freedom in the equalization and more robustness with respect to the choice of the timing phase. The anti-aliasing filter has to avoid overlapping in frequency when we downsample the signal at the output of the channel. Our signal has frequency content that is periodic of period  $4/T$ , and has the shape of the Fourier transform of filter  $q_c$ . Therefore its bandwidth is not limited and our anti-aliasing filter  $g_{AA}$  will remove useful information. The sampling at  $T/2$  causes the frequency content of the signal to be repeated every  $2/T$ , causing an overlap. To avoid this, the cutting frequency of  $g_{AA}$  has to be around  $1/T$ . In figure ?? we show the magnitude of the frequency response  $|G_{AA}|$ . The sampling then has to start after an offset  $\bar{t}_0$  equal to the peak of

the overall impulse response  $q_c * g_{AA}$  at time  $T/4$  that in this case was equal to 21. In this kind of configuration we also add a matched filter after the sampling

**Point D**

**Point E**

**Point F**

## Simulation results

$nT_y$	$h$	$\hat{h}_{corr}$	$\hat{h}_{ls}$
0	1.0000	1.0054	0.9991
1	0.9635	0.9247	0.9613
2	0.4641	0.5002	0.5062
3	-0.0001	0.0202	0.0255
4	-0.2155	-0.2549	-0.2253
	Real	Corr	LS
$\sigma_w^2$ [dB]	-8	-7.9776	-8.0404

Table 1: .

## References

- [1] Nevio Benvenuto, Giovanni Cherubini, *Algorithms for Communication Systems and their Applications*. Wiley, 2002.

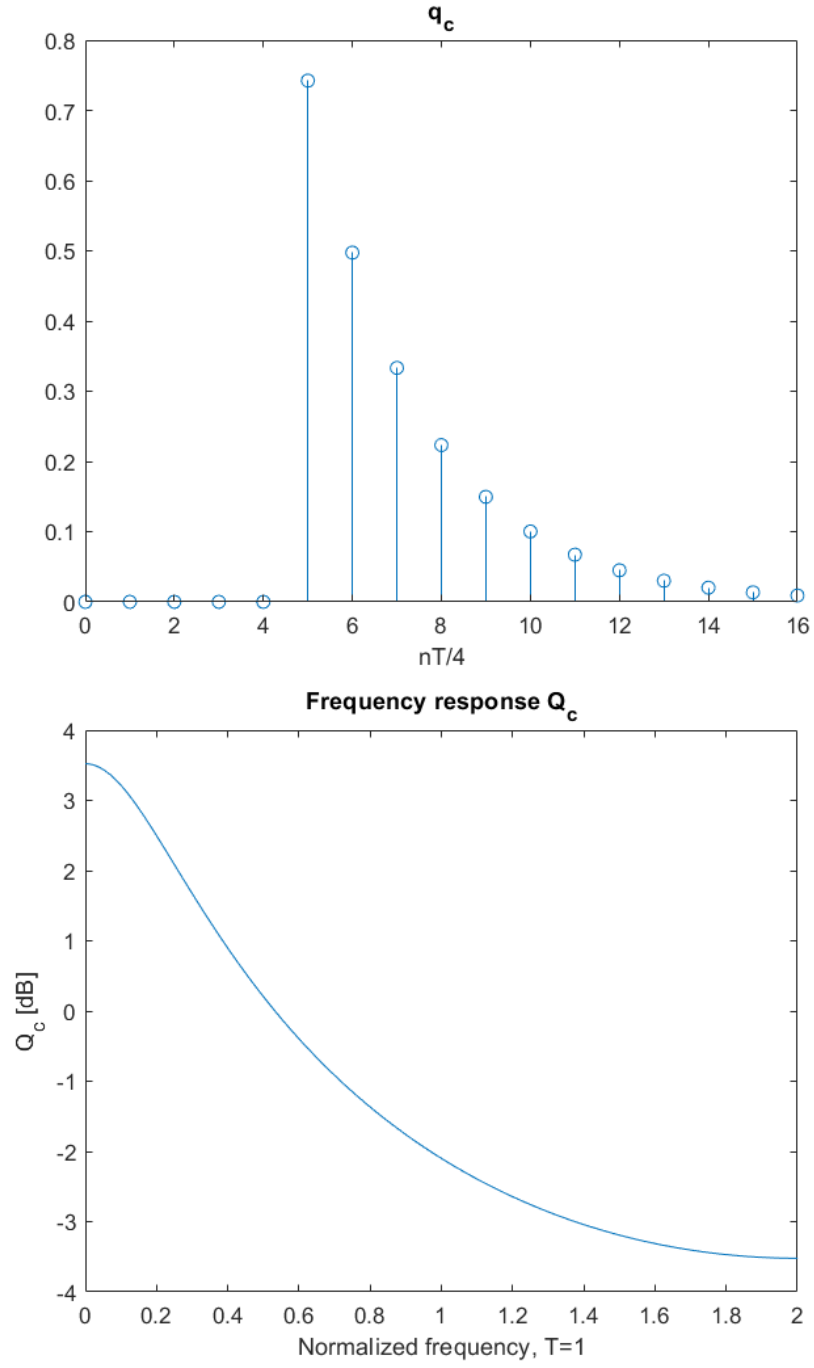


Figure 1: Impulse and frequency response of the filter  $q_c$  at  $T/4$ .

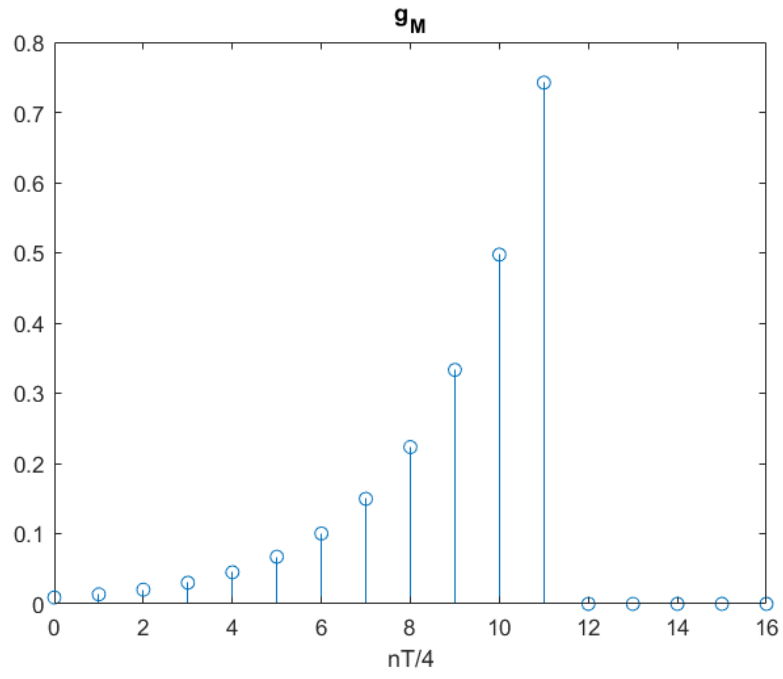


Figure 2: Impulse response of the matched filter  $g_M$  for the receiver in point A.

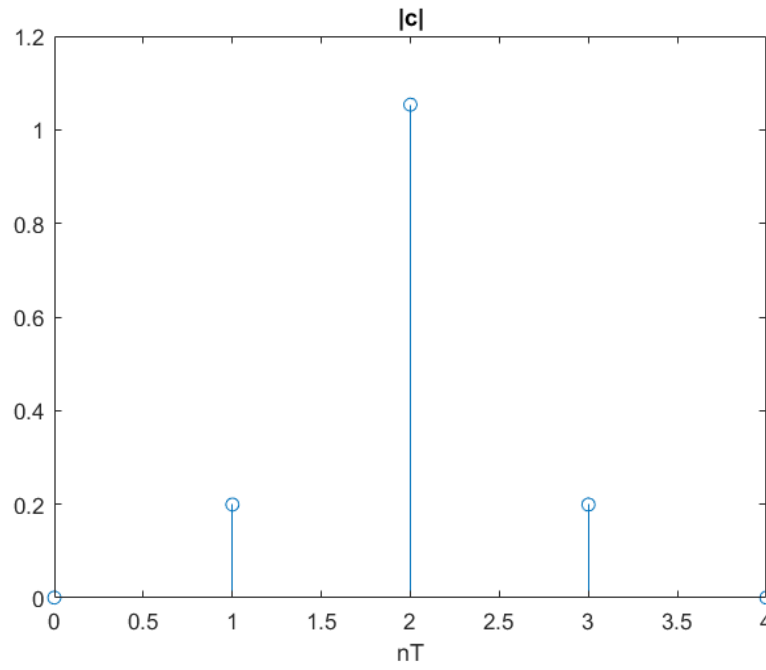


Figure 3: Magnitude of the impulse response of filter  $c$  in point A.

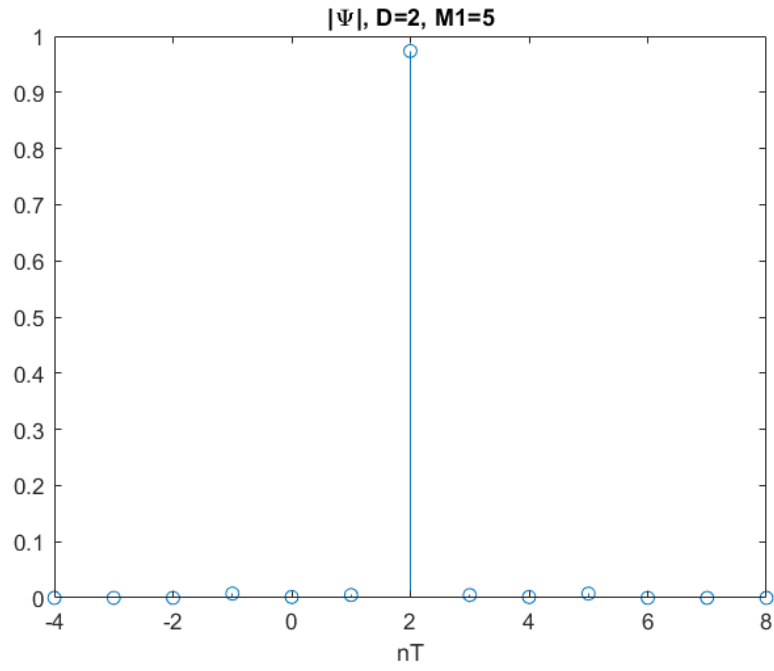


Figure 4: Magnitude of the impulse response of the system  $\psi$  in point A.

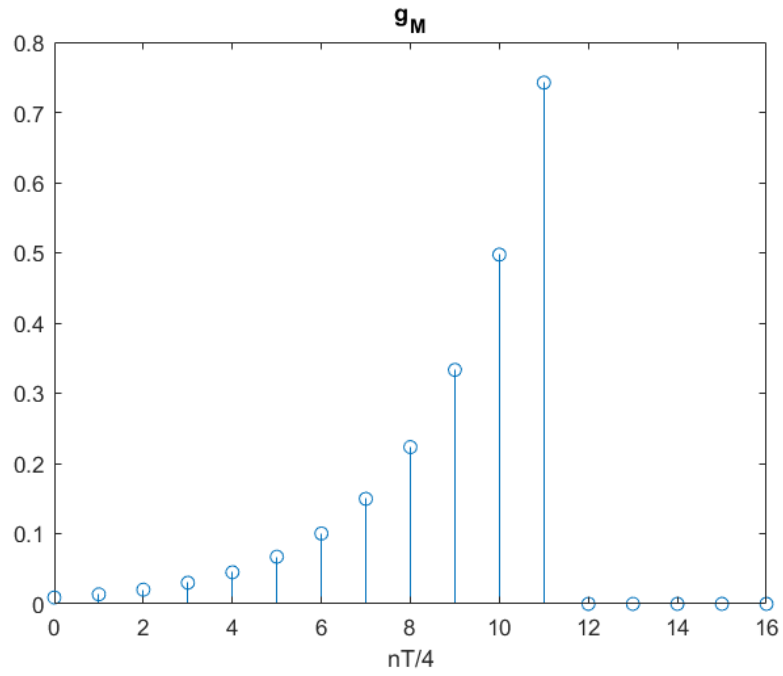


Figure 5: Impulse response of the matched filter  $g_M$  for the receiver in point B.

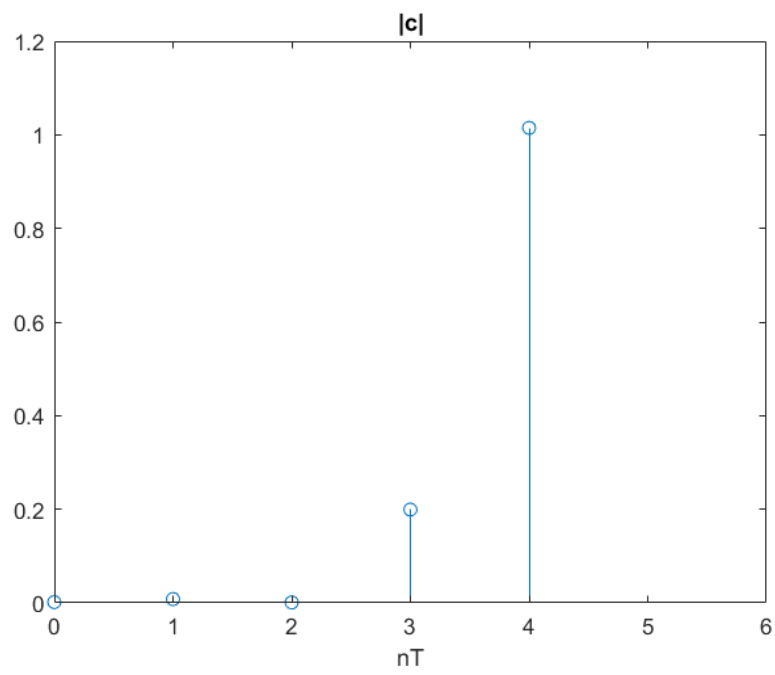


Figure 6: Magnitude of the impulse response of the filter  $c$  (feedforward filter) for the receiver in point B.



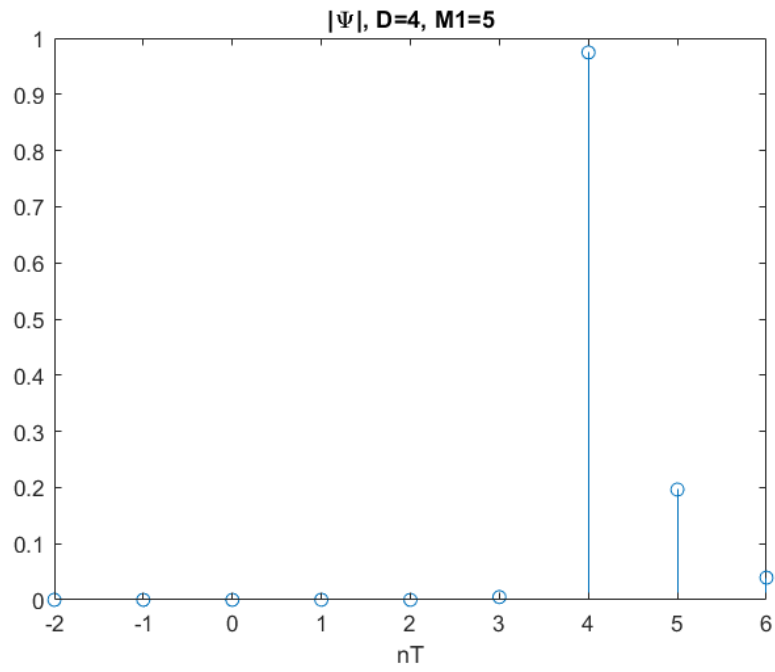


Figure 7: Magnitude of the impulse response of the system  $\psi$  in point B.

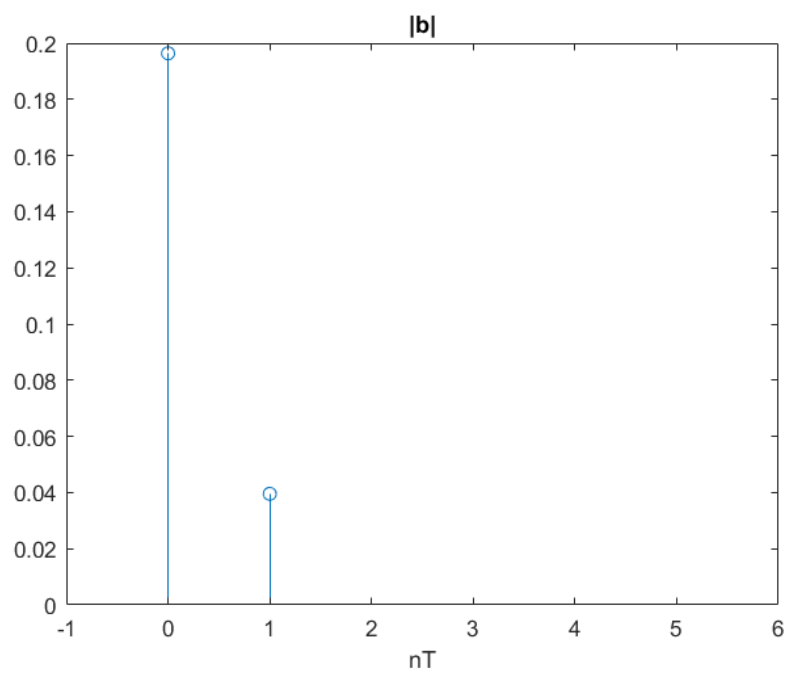


Figure 8: Magnitude of the impulse response of the filter  $b$  (feedback filter) in point B.

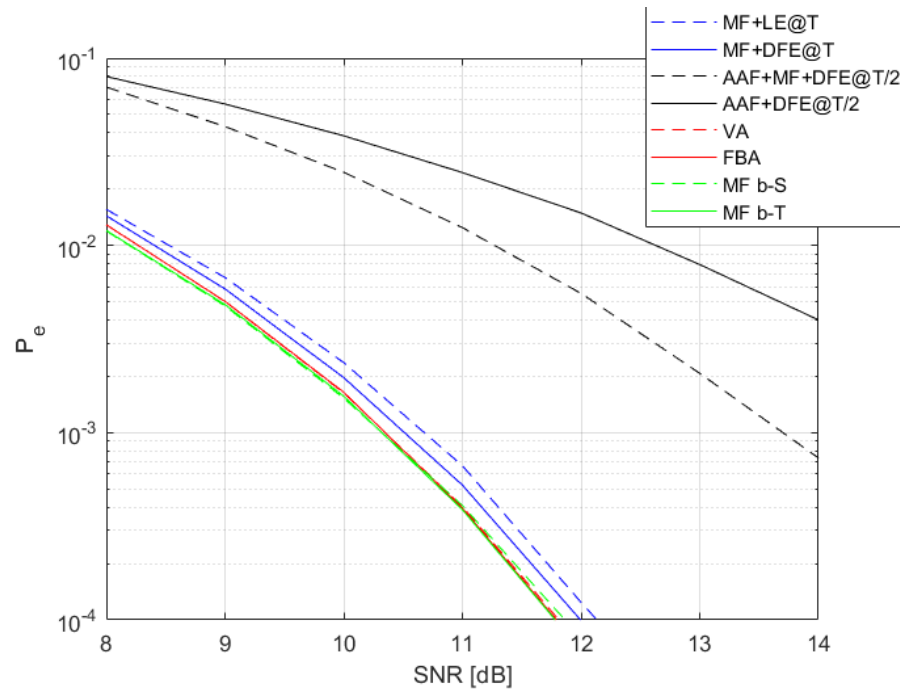


Figure 9: Results of the simulation over values of the SNR at the channel output from 8 dB to 14 dB.