# Digital Communications - HW3

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## Problem

We have to implement six different versions of the receiver structure in a QPSK modulation scheme. First we present the setup of the transmitter and the channel as given, the we analyze the different configurations one by one and give a brief discussions of the resulting probabilities of symbol error obtained from simulation over different values of the SNR at the channel output, $\Gamma$.

## Transmitter and Channel

The system takes a sequence of input symbols $a_k$ at sampling time $T = 1$ and applies an upsampling of factor 4, obtaining $a'_k$ at $T/4$. This new sequence is then filtered by $q_c$ as described by the following difference equation:

$$s_c(nT/4) = 0.67 s_c((n-1)T/4) + 0.7424 a_{n-5} \tag{1}$$

After the filtering white noise is added. The SNR at the channel output for all the configurations in this first phase is $\Gamma = 10$ dB, so from the following relations we can derive $\sigma_w^2$, the variance of the complex valued Gaussian noise:

$$\Gamma = \frac{M_{s_c}}{N_0 \frac{1}{T}} = \frac{\sigma_a^2 E_{q_c}}{\sigma_w^2} \longrightarrow \sigma_w^2 = \frac{\sigma_a^2 E_{q_c}}{\Gamma} = 2\sigma_I^2 \tag{2}$$

where $\sigma_I^2$ is the variance per component. In addition we can also compute the PSD as $N_0 = \sigma_w^2 T_c = \sigma_w^2/4$, because the sampling time $T_c$ at which we add the noise is $T/4$. In figure 1 we plot the impulse response and the frequency response of the filter $q_c$. This implementation of the transmitter is the same for all the following discussion.

## Point A

In point A at the receiver we have a matched filter $g_M$ (see figure 2), obtained from $q_c$ as $g_M = q_c^*(t_0 - t)$. For simplicity in the last formula we have denoted the filters as is they were defined on continuous time while in the actual simulation they are at $T/4$.
The output of the matched filter is then sampled at $T$ starting from an initial offset called *timing phase* $t_0$. In our case the choice of $t_0$ is made easy by the presence of the matched filter, as we can just choose the value $\bar{t}_0$, multiple of

$T/4$, that is the index of the peak of the correlation between $q_c$ and $g_M$, then $t_0$ will be equal to $\bar{t}_0 T/4$. Following this reasoning we chose $\bar{t}_0 = 17$, equal also to the length of $g_M$ (see figure 2).
The signal is then passed to a linear equalizer (LE) derived by a particular case of a Decision Feedback Equalizer (DFE) where we only have the feedforward filter $c$ (see point B for the detailed analysis of the DFE). The signal at this point in the receiver system is called $x_k$ and is the result of the convolution of the input sequence $a_k$ with the overall impulse response $h_i = q_c * g_M$ that goes from $-N1$ to $N2$. We will call precursors the taps of $h$ that go from $-N1$ to $-1$ and postcursors the taps from 1 to $N2$. To obtain the coefficients of $c$ we used the Wiener approach on the input random process and solved the Wiener-Hopf equation $\boldsymbol{c}_{opt} = \boldsymbol{R}^{-1}\boldsymbol{p}$ using the matrix $\boldsymbol{R}$ and vector $\boldsymbol{p}$ as in equations 5 and 6 with the parameter $M_2$ (the order of the feedback filter) set to 0 because we have no feedback filter in this case. The free parameters that we have to choose are $M1$, the order of filter $c$ and $D$ the delay that it will introduce on the input sequence. We carried out this choice looking at the value of the cost function $J_{min}$ (see equation 7) for each combination of the two parameters, preferring low values if possible to avoid increasing the complexity. The best choice in this case was $M1 = 5$ and $D = 2$. In figure 3 we plot the impulse response $c_i$ at sampling time $T$ as obtained from the Wiener solution.
The aim of filtering with $c$ is to obtain an overall impulse response of the system that satisfies the Nyquist conditions for the absence of ISI at time $T$. This implies that in the ideal case $\psi = h * c$ is a delayed impulse centered on $D$ that is the delay. In our case the $\psi$ obtained is shown in figure 4. We can see the result is pretty good as all the precursors and postcursors are almost canceled by the equalizer.
The detected symbols $a_{k-D}$ are chosen using a threshold detector that analyzes the sign of the imaginary and real part of the input complex value. Not that the same detector is also used at point B, C and D.

# Point B

For point B the system is the same as in point A up to the equalizer, so the choice of $\bar{t}_0 = 17$ is the same. The matched filter is the same as in point A, see figure 5. However in this case we equalize with a DFE, that is made of two filters called feedforward and feedback filter denoted by $c$ and $b$. The feedforward filter has the role of equalizing only the precursors of the overall impulse response, while the ISI due to postcursors will be canceled by filter $b$ positioned on a feedback loop between the output of the threshold detector and its input.
The computation of the optimal filters $c$ and $b$ is carried out using the Wiener filter approach. The relation between the input random process and the output is:

$$\begin{aligned} y_k &= x_{FF,k} + x_{FB,k} \\ &= \sum_{i=0}^{M_1-1} c_i x_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j} \end{aligned} \tag{3}$$

where $M1$ is the order of the feedforward filter, $M2$ is the order of the feedback filter and $a_{k-D}$ are the already detected past symbols fed back through $b$.

Defining postcursors and precursors as in point A, we have that we can apply the Wiener-Hopf equations on the process:

$$y_k = \sum_{i=0}^{M_1-1} c_i \left( x_{k-i} - \sum_{j=1}^{M2} h_{j+D-i} a_{k-j-D} \right) \tag{4}$$

The result can be easily computed as $c_{opt} = \boldsymbol{R}^{-1}\boldsymbol{p}$ once we find the atocorrelation matrix $\boldsymbol{R}$ and the correlation vector $\boldsymbol{p}$, expressed as [1]:

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left( \sum_{j=-N_1}^{N_2} h_j h^*_{j-(p-q)} - \sum_{j=1}^{M_2} h_{j+D-q} h^*_{j+d-p} \right) + r_{\tilde{w}}(p-q) \tag{5}$$

$$[\mathbf{p}]_p = \sigma_a^2 h^*_{D-p} \qquad\qquad p = 0, 1, \ldots, M_1 - 1 \tag{6}$$

where for a QPSK scheme $\sigma_a^2 = 2$ because it is the sum of two orthogonal components each with power 1. The values of $r_{\tilde{w}}$ are the result of the autocorrelation of the noise after being filtered by $g_M$, so being the noise white we have $r_{\tilde{w}}(n) = N_0 r_{g_M}(nT)$. At this point we can define the overall impulse response up to the threshold detector $\psi = h * c_{opt}$ and derive the optimal coefficients for filter $b$ as $b_i = -\psi_{i+D}$ for $i = 1, \ldots, M2$.
The value of the cost function $J_{min}$ obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left( 1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{D-l} \right) \tag{7}$$

Again the parameters to choose are the order of filter $c$, $M1$, and the delay introduced $D$. This is because the order of $b$ can be chosen in such a way that all the postcursors are canceled by the feedback: $M2 = N2 + M1 - D - 1$, and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional $J_{min}$, this time being $M1 = 5$ and $D = 4$, and consequently $M2 = 2$ because $N2 = 2$. In figure 6, 7 and 9 we plot the resulting filters $c$, $\psi$ and $b$ at sampling time $T$.

## Point C

In the receiver at point C we use a different type of approach. Before sampling the received signal we use an anti-aliasing filter instead of a matched filter. This is not an optimal solution but can be useful in some situation where the channel is not known or varies in time. Also the downsampling is at $T/2$ instead of $T$ and this gives more degrees of freedom int the equalization and more robustness with respect to the choice of the timing phase. The anti-aliasing filter has to avoid overlapping in frequency when we downsample the signal at the output of the channel. Our signal has frequency content that is periodic of period $4/T$, and has the shape of the Fourier transform of filter $q_c$. Therefore its bandwidth is not limited and our anti-aliasing filter $g_{AA}$ will remove useful information. The sampling at $T/2$ causes the frequency content of the signal to be repeated every $2/T$, causing an overlap. To avoid this, the cutting frequency of $g_{AA}$ has to be around $1/T$. In figure **??** we show the magnitude of the frequency response $|G_{AA}|$. The sampling then has to start after an offset $\bar{t}_0$ equal to the peak of the overall impulse response $q_c * g_{AA}$ at time $T/4$ that in this case was equal to 21. In this kind of configuration we also add a digital matched filter after the sampling that is different from the previous points, now the matched filter is

$$g_M = \{q_c * g_{AA}\}^* \left(t_0 + i\frac{T}{2}\right) \qquad i = 0, 1, \ldots, N \tag{8}$$

And the inpulse response is depicted in figure **??**.
For find out the equalizer we still use the Wiener problem. Now the $c$ filter work at $T/2$ so we must take care about it and the equation used for the Wiener solution aren't the same used for the points A and B. The equations 4, 5 and 6 became

$$y_k = \sum_{i=0}^{M_1-1} c_i \left( x_{2k-i} - \sum_{j=1}^{M2} h_{2(j+D)-i} a_{k-j-D} \right) \tag{9}$$

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left( \sum_{n=-\infty}^{\infty} h_{2n-q} h_{2n-p}^* - \sum_{j=1}^{M_2} h_{2(j+D)-q} h_{2(j+D)-p}^* \right) + r_{\tilde{w}}(p-q) \tag{10}$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{2D-p}^* \qquad\qquad p, q = 0, 1, \ldots, M_1 - 1 \tag{11}$$

where for a QPSK scheme $\sigma_a^2 = 2$ because it is the sum of two orthogonal components each with power 1. The values of $r_{\tilde{w}}$ are the result of the autocorrelation of the noise after being filtered by $g_{AA}$ and $g_M$, so being the noise white we have $r_{\tilde{w}}(n) = N_0 r_{g_M * g_{AA}}(nT/2)$. At this point we can define the overall impulse response up to the threshold detector $\psi = h * c_{opt}$ and derive the optimal coefficients for filter $b$ as $b_i = -\psi_{2(i+D)}$ for $i = 1, \ldots, M2$.
The value of the cost function $J_{min}$ obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left( 1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{2D-l} \right) \tag{12}$$

Again the parameters to choose are the order of filter $c$, $M1$, and the delay introduced $D$. This is because the order of $b$ can be chosen in such a way that all the postcursors are canceled by the feedback: $M2 = N1 + M1 - D - 1$, and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional $J_{min}$, this time being $M1 = 9$ and $D = 4$, and consequently $M2 = 4$ because $N2 = 2$ (Siamo sicuri che sia N1 e non N2 come scritto nel codice ? a parte che sono uguali perché h è simmetrica, bisogna stare attenti che nel punto D non sia stato fatto lo stesso errore che li NON è simmetrica). In figure **??**, **??** and **??** we plot the resulting filters $c$, $\psi$ at sampling time $T/2$ and $b$ at sampling time $T$.

| | $N1$ | $N2$ | $M1$ | $M2$ | $D$ | $\bar{t}_0$ |
|---|---|---|---|---|---|---|
| A | 4 | 4 | 5 | 0 | 2 | 17 |
| B | 2 | 2 | 5 | 0 | 4 | 17 |
| C | 10 | 12 | 9 | 16 | 4 | 21 |
| D | 10 | 12 | 9 | 16 | 4 | 21 |
| | Real | Corr | LS | | | |
| $\sigma_w^2$ [dB] | -8 | -7.9776 | -8.0404 | | | |

Table 1: .

# Point D

# Point E

# Point F

# Simulation results

# References


[1] Nevio Benvenuto, Giovanni Cherubini, *Algorithms for Communication Systems and their Applications*. Wiley, 2002.

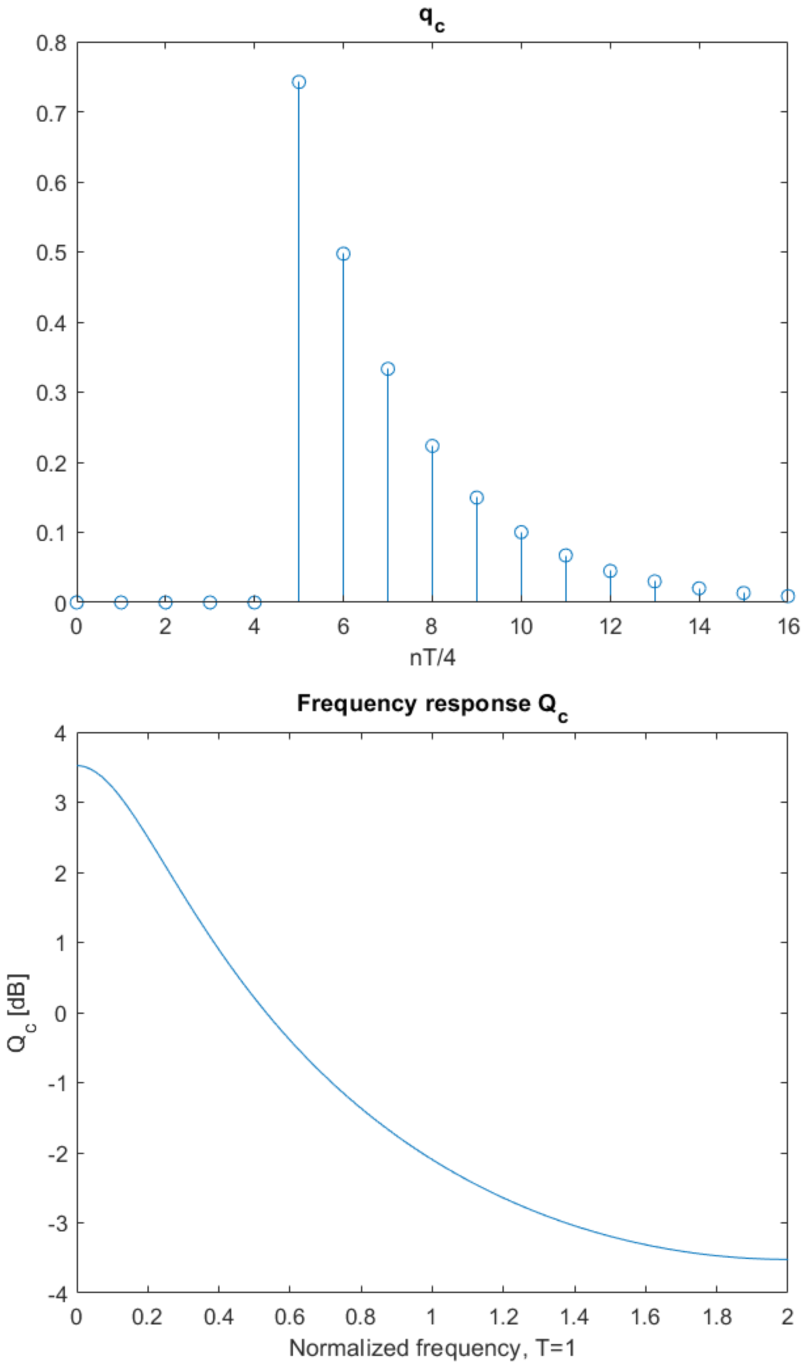


Figure 1: Impulse and frequency response of the filter $q_c$ at $T/4$.

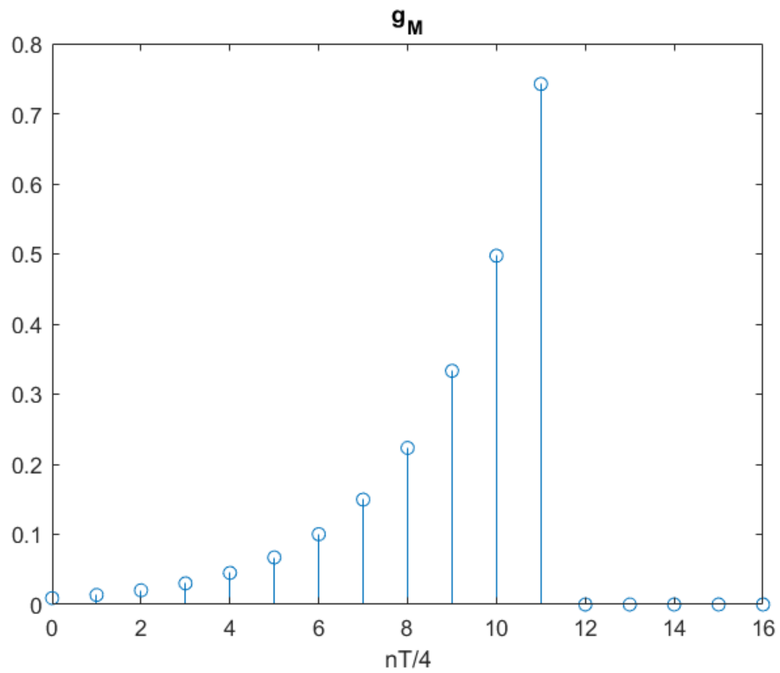


Figure 2: Impulse response of the matched filter $g_M$ for the receiver in point A.

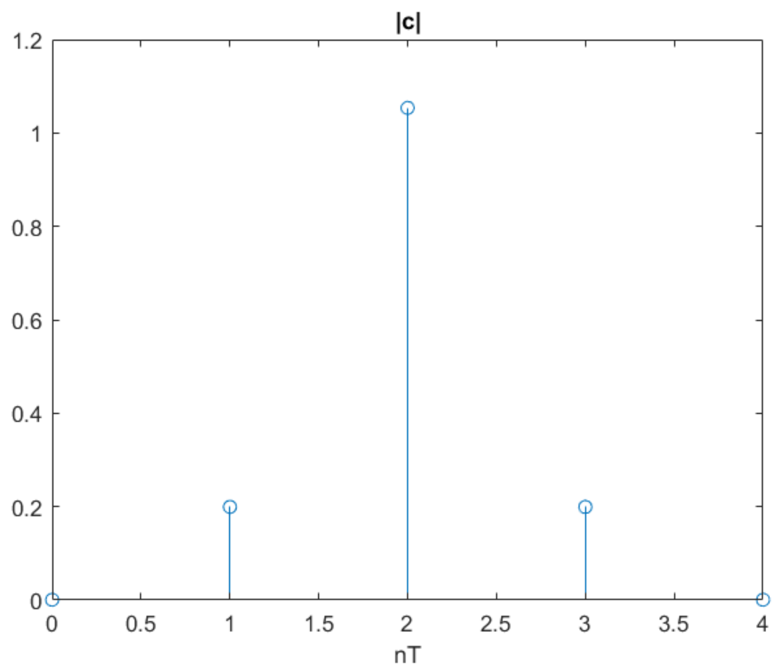


Figure 3: Magnitude of the impulse response of filter $c$ in point A.

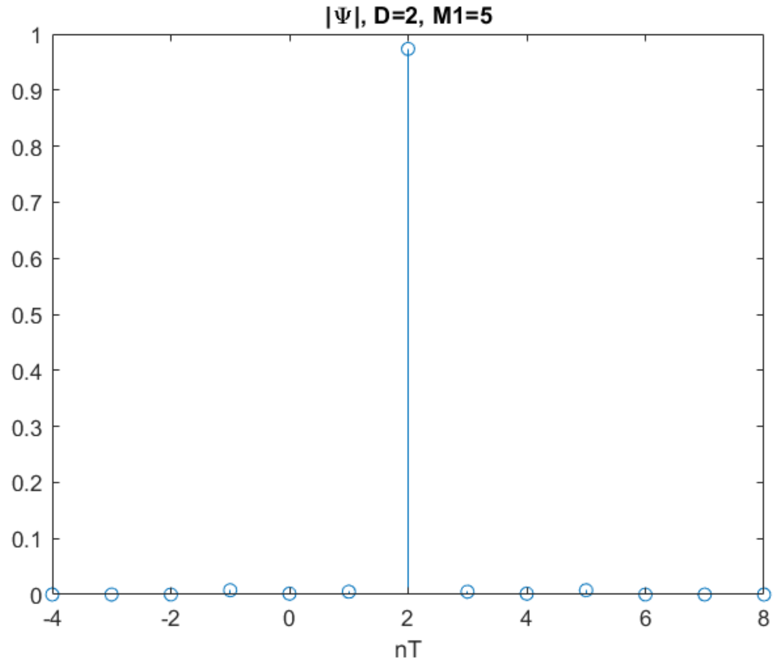


Figure 4: Magnitude of the impulse response of the system $\psi$ in point A.

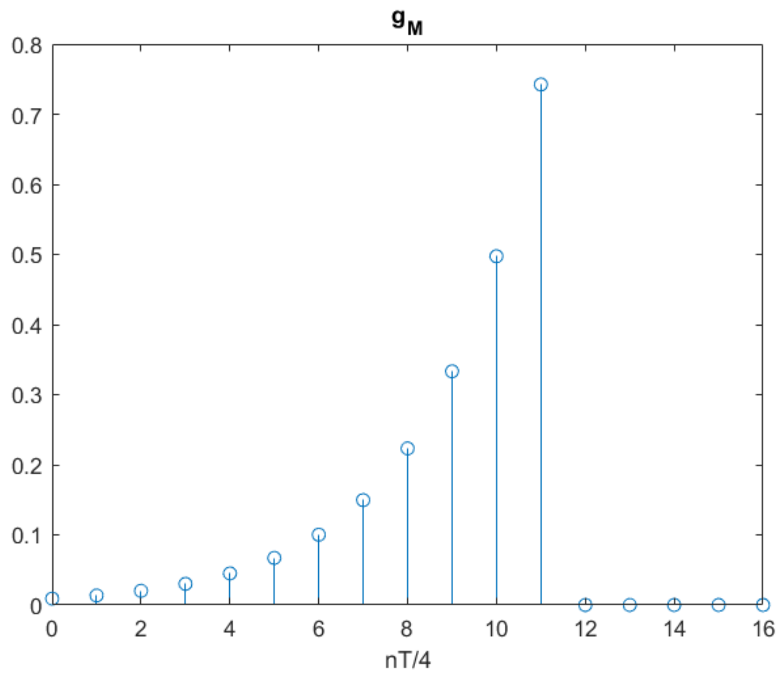


Figure 5: Impulse response of the matched filter $g_M$ for the receiver in point B.

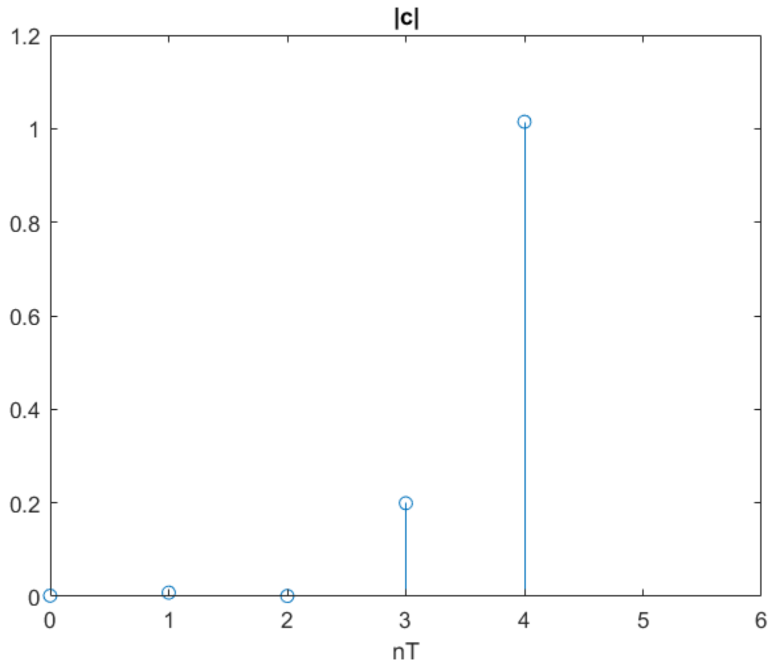


Figure 6: Magnitude of the impulse response of the filter $c$ (feedforward filter) for the receiver in point B.

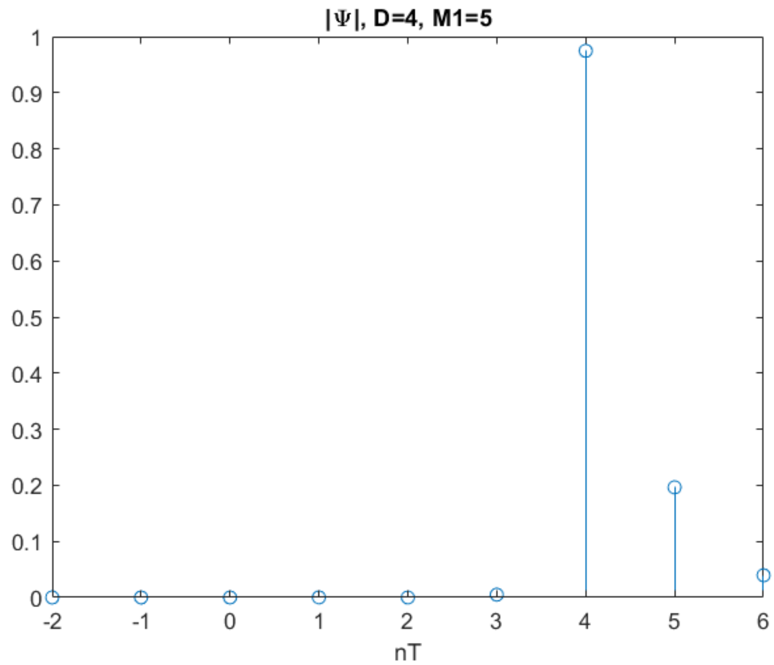


Figure 7: Magnitude of the impulse response of the system $\psi$ in point B.

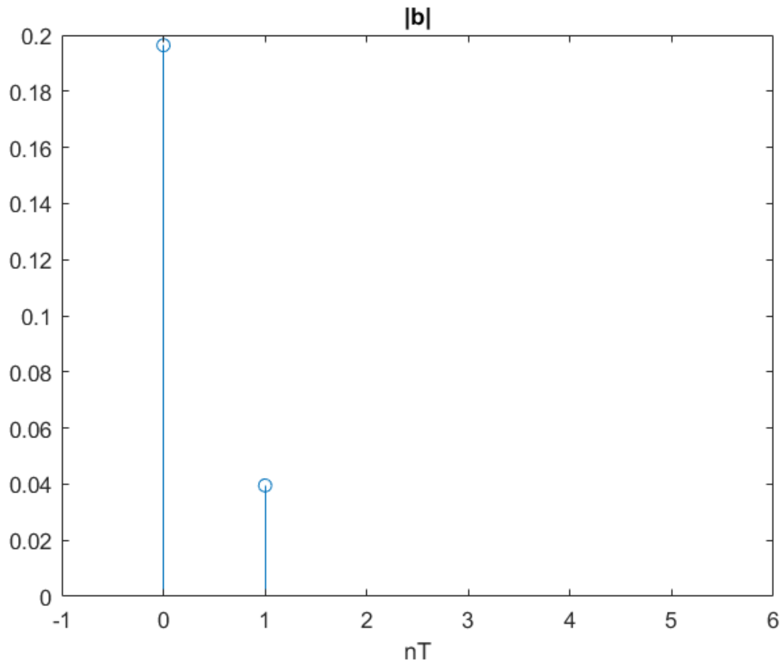


Figure 8: Magnitude of the impulse response of the filter $b$ (feedback filter) in point B.

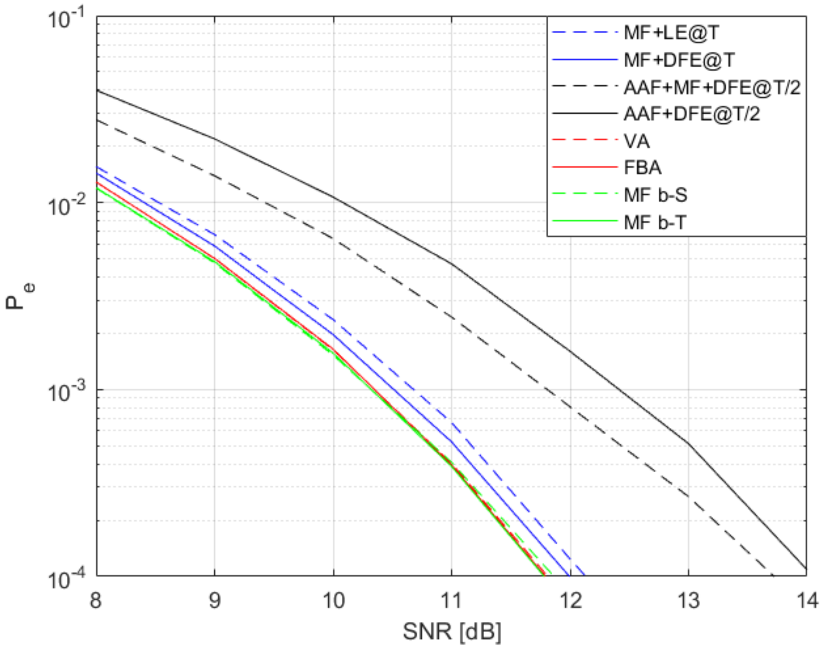


Figure 9: Results of the simulation over values of the SNR at the channel output from 8 dB to 14 dB.