Digital Communications - HW3

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21/05/2018

We have to implement six different versions of the receiver structure in a QPSK modulation scheme. First we present the setup of the transmitter and the channel as given, then we analyze the different configurations one by one and give a brief discussions of the resulting probabilities of symbol error obtained from simulation over different values of the SNR at the channel output, Γ.

Transmitter and Channel

The system takes a sequence of input symbols a_k at sampling time $T = 1$ and applies an upsampling of factor 4, obtaining a'_k at $T/4$. This new sequence is then filtered by q_c as described by the following difference equation:

$$s_c(nT/4) = 0.67s_c((n-1)T/4) + 0.7424a_{n-5} \tag{1}$$

After the filtering white noise is added. The SNR at the channel output for all the configurations in this first phase is $\Gamma = 10$ dB, so from the following relations we can derive σ_w^2, the variance of the complex valued Gaussian noise:

$$\Gamma = \frac{M_{s_c}}{N_0 \frac{1}{T}} = \frac{\sigma_a^2 E_{q_c}}{\sigma_w^2} \longrightarrow \sigma_w^2 = \frac{\sigma_a^2 E_{q_c}}{\Gamma} = 2\sigma_I^2 \tag{2}$$

where σ_I^2 is the variance per component. In addition we can also compute the PSD as $N_0 = \sigma_w^2 T_c = \sigma_w^2/4$, because the sampling time T_c at which we add the noise is $T/4$. In figure 1 we plot the impulse response and the frequency response of the filter q_c. This implementation of the transmitter is the same for all the following discussion.

Point A

In point A at the receiver we have a matched filter g_M (see figure 2), obtained from q_c as $g_M = q_c^*(t_0 - t)$. For simplicity in the last formula we have denoted the filters as is they were defined on continuous time while in the actual simulation they are at $T/4$.

The output of the matched filter is then sampled at T starting from an initial offset called *timing phase* t_0. In our case the choice of t_0 is made easy by the presence of the matched filter, as we can just choose the value $\bar{t}_0$, multiple of $T/4$, that is the index of the peak of the correlation between q_c and g_M, then t_0 will be equal to $\bar{t}_0 T/4$. Following this reasoning we chose $\bar{t}_0 = 17$, equal also to the length of g_M (see figure 2).

The signal is then passed to a linear equalizer (LE) derived by a particular case of a Decision Feedback Equalizer (DFE) where we only have the feedforward filter c (see point B for the detailed analysis of the DFE). The signal at this point in the receiver system is called x_k and is the result of the convolution of the input sequence a_k with the overall impulse response $h_i = q_c * g_M$ that goes from $-N_1$ to N_2. We will call precursors the taps of h that go from $-N1$ to -1 and postcursors the taps from 1 to N_2. To obtain the coefficients of c we used the Wiener approach on the input random process and solved the Wiener-Hopf equation $\boldsymbol{c}_{opt} = \boldsymbol{R}^{-1}\boldsymbol{p}$ using the matrix $\boldsymbol{R}$ and vector $\boldsymbol{p}$ as in equations 5 and 6 with the parameter M_2 (the order of the feedback filter) set to 0 because we have no feedback filter in this case. The free parameters that we have to choose are M_1, the order of filter c and D the delay that it will introduce on the input sequence. We carried out this choice looking at the value of the cost function J_{min} (see equation 7) for each combination of the two parameters, preferring low values if possible to avoid increasing the complexity. The best choice in this case was $M_1 = 5$ and $D = 2$. In figure 3 we plot the impulse response c_i at sampling time T as obtained from the Wiener solution.

The aim of filtering with c is to obtain an overall impulse response of the system that satisfies the Nyquist conditions for the absence of ISI at time T. This implies that in the ideal case $\psi = h * c$ is a delayed impulse centered on D that is the delay. In our case the ψ obtained is shown in figure 4. We can see the result is pretty good as all the precursors and postcursors are almost canceled by the equalizer.

The detected symbols a_{k-D} are chosen using a threshold detector that analyzes the sign of the imaginary and real part of the input complex value. Not that the same detector is also used at point B, C and D.

Point B

For point B the system is the same as in point A up to the equalizer, so the choice of $\bar{t}_0 = 17$ is the same. The matched filter is the same as in point A, see figure 5. However in this case we equalize with a DFE, that is made of two filters called feedforward and feedback filter denoted by c and b. The feedforward filter has the role of equalizing only the precursors of the overall impulse response, while the ISI due to postcursors will be canceled by filter b positioned on a feedback loop between the output of the threshold detector and its input.

The computation of the optimal filters c and b is carried out using the Wiener filter approach. The relation between the input random process and the output is:

$$
\begin{aligned}
y_k &= x_{FF,k} + x_{FB,k} \\
&= \sum_{i=0}^{M_1-1} c_i x_{k-i} + \sum_{j=1}^{M_2} b_j a_{k-D-j}
\end{aligned}
\tag{3}
$$

where M_1 is the order of the feedforward filter, M_2 is the order of the feedback filter and a_{k-D} are the already detected past symbols fed back through b. Defining postcursors and precursors as in point A, we have that we can apply

the Wiener-Hopf equations on the process:

$$y_k = \sum_{i=0}^{M_1-1} c_i \left(x_{k-i} - \sum_{j=1}^{M_2} h_{j+D-i} a_{k-j-D} \right) \tag{4}$$

The result can be easily computed as $c_{opt} = \boldsymbol{R}^{-1} \boldsymbol{p}$ once we find the autocorrelation matrix $\boldsymbol{R}$ and the correlation vector $\boldsymbol{p}$, expressed as [1]:

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left(\sum_{j=-N_1}^{N_2} h_j h_{j-(p-q)}^* - \sum_{j=1}^{M_2} h_{j+D-q} h_{j+d-p}^* \right) + r_{\tilde{w}}(p-q) \tag{5}$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{D-p}^* \qquad\qquad p = 0, 1, \ldots, M_1 - 1 \tag{6}$$

where for a QPSK scheme $\sigma_a^2 = 2$ because it is the sum of two orthogonal components each with power 1. The values of $r_{\tilde{w}}$ are the result of the autocorrelation of the noise after being filtered by g_M, so being the noise white we have $r_{\tilde{w}}(n) = N_0 r_{g_M}(nT)$. At this point we can define the overall impulse response up to the threshold detector $\psi = h * c_{opt}$ and derive the optimal coefficients for filter b as $b_i = -\psi_{i+D}$ for $i = 1, \ldots, M_2$.

The value of the cost function J_{min} obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left(1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{D-l} \right) \tag{7}$$

Again the parameters to choose are the order of filter c, M_1, and the delay introduced D. This is because the order of b can be chosen in such a way that all the postcursors are canceled by the feedback: $M_2 = N_2 + M_1 - D - 1$, and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional J_{min}, this time being $M_1 = 5$ and $D = 4$, and consequently $M_2 = 4$ because $N_2 = 4$. In figure 6, 7 and 8 we plot the resulting filters c, ψ and b at sampling time T.

Point C

In the receiver at point C we use a different type of approach. Before sampling the received signal we use an anti-aliasing filter instead of a matched filter. This is not an optimal solution but can be useful in some situation where the channel is not known or varies in time. Also the downsampling is at $T/2$ instead of T and this gives more degrees of freedom int the equalization and more robustness with respect to the choice of the timing phase. The anti-aliasing filter has to avoid overlapping in frequency when we downsample the signal at the output of the channel. Our signal has frequency content that is periodic of period $4/T$, and has the shape of the Fourier transform of filter q_c. Therefore its bandwidth is not limited and our anti-aliasing filter g_{AA} will remove useful information. The sampling at $T/2$ causes the frequency content of the signal to be repeated every $2/T$, causing an overlap. To avoid this, the cutting frequency of g_{AA} has to be around $1/T$. In figure 9 we show the magnitude of the frequency response $|G_{AA}|$, where we chose the passband at $0.45 \cdot 2/T$ and the stopband at $0.55 \cdot 2/T$.

The sampling then has to start after an offset $\bar{t}_0$ equal to the peak of the overall impulse response $q_c * g_{AA}$ at time $T/4$ that in this case was equal to 21. In this kind of configuration we also add a digital matched filter after the sampling that is different from the previous points, now the matched filter is given by flipping and taking the conjugate of the whole impulse response $q_c * g_{AA}$ at sampling time $T/2$:

$$g_M = \{q_c * g_{AA}\}^* \left(t_0 + i\frac{T}{2}\right) \tag{8}$$

And the impulse response is depicted in figure 10. In this case we use again a DFE. To derive the coefficients of the equalizer. Now the c filter works at $T/2$ so the equations used for the Wiener solution aren't the same used for the points A and B. The equations 4, 5 and 6 become

$$y_k = \sum_{i=0}^{M_1-1} c_i \left(x_{2k-i} - \sum_{j=1}^{M_2} h_{2(j+D)-i} a_{k-j-D}\right) \tag{9}$$

$$[\mathbf{R}]_{p,q} = \sigma_a^2 \left(\sum_{n=-\infty}^{\infty} h_{2n-q} h_{2n-p}^* - \sum_{j=1}^{M_2} h_{2(j+D)-q} h_{2(j+D)-p}^*\right) + r_{\tilde{w}}(p-q) \tag{10}$$

$$[\mathbf{p}]_p = \sigma_a^2 h_{2D-p}^* \qquad p,q = 0, 1, \ldots, M_1 - 1 \tag{11}$$

where for a QPSK scheme $\sigma_a^2 = 2$ because it is the sum of two orthogonal components each with power 1. The values of $r_{\tilde{w}}$ are the result of the autocorrelation of the noise after being filtered by g_{AA} and g_M, so being the noise white we have $r_{\tilde{w}}(n) = N_0 r_{g_M * g_{AA}}(nT/2)$. At this point we can define the overall impulse response up to the threshold detector $\psi = h * c_{opt}$ and derive the optimal coefficients for filter b as $b_i = -\psi_{2(i+D)}$ for $i = 1, \ldots, M_2$ because filter b works at T.

The value of the cost function J_{min} obtained using these the optimal filters is :

$$J_{min} = \sigma_a^2 \left(1 - \sum_{l=0}^{M_1-1} c_{opt,l} h_{2D-l}\right) \tag{12}$$

Again the parameters to choose are the order of filter c, M_1, and the delay introduced D. This is because the order of b can be chosen in such a way that all the postcursors are canceled by the feedback: $M_2 = N_2 + M_1 - D - 1$, and also the expression of the autocorrelation matrix significantly simplifies. The choice is carried out by selecting the values that minimize the functional J_{min}, this time being $M_1 = 9$ and $D = 4$, and consequently $M2 = 23$ because $N_1 = N_2 = 19$ In figure 11, 12 and 13 we plot the resulting filters c, ψ at sampling time $T/2$ and b at sampling time T. In the fractionally spaced version of the DFE the aim is again to obtain an overall pulse ψ that satisfies the Nyquist conditions at time T so the samples at even multiples of $T/2$ are not used in the equalizer and are therefore a degree of freedom of the system.

Point D

In this configuration the setup is similar to point C but without the matched filter after the downsampling at $T/2$. This means that all the burden of equalization and reduction of the noise is on filter c. The parameter $\bar{t}_0$ is 21 as

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before, because the impulse response of the anti-aliasing filter and of q_c have not changed. The equations for the derivation of the DFE filters with a Wiener approach are 10 and 11 and the parameters, chosen again by minimizing the value of J_{min} are $M_1 = 9$, $D = 4$ and consequently $M_2 = N_2 + M_1 - D - 1 = 16$ because this time h is not symmetric and $N_2 = 12$. In figure 14, 15 and 16 we show the magnitude of the coefficients of filter c, ψ and b.

The following table sums up the final parameters for the 4 configurations A, B, C, D.

Point	Length of h		Parameters			
	N_1	N_2	M_1	M_2	D	$\bar{t}_0$
A	4	4	5	0	2	17
B	4	4	5	4	4	17
C	19	19	9	23	4	21
D	10	12	9	16	4	21

Table 1: Choices for the various parameters in the different receiver implementations with respect to the length of h.

Point E

In this point we use the Viterbi algorithm (VA) to implement the Maximum Likelihood criterion for data detection. For the execution of the algorithm we consider the signal y_k at the output of filter c in the schema at point B:

$$y_k = \psi_D a_{k-D} + \psi_{D+1} a_{k-D-1} + \cdots + \psi_{D+M_2} a_{k-D-M_2} + w_k \qquad (13)$$

where w_k includes the residual ISI and the noise. The useful signal u_k can then be taken as $u_k = \psi_D a_{k-D} + \psi_{D+1} a_{k-D-1} + \cdots + \psi_{D+M_2} a_{k-D-M_2}$ and if we call $y_k = z_k$ we have the usual setup for VA: $z_k = u_k + w_k$. We will have then that the number of states is $N_s = M^{M_2}$ where M is 4, the cardinality of the QPSK constellation. In the specific case of point B $M_2 = 4$ so $N_s = 256$. We define the state basing ourselves of the choice of u_k so:

$$\begin{aligned} \boldsymbol{s}_k &= \left\{ a_{k-D}, a_{k-D-1}, a_{k-D-2}, \ldots, a_{k-(M_2-1)} \right\} = \\ &= \left\{ a_{k-D}, a_{k-D-1}, a_{k-D-2}, a_{k-D-3} \right\} \end{aligned} \qquad (14)$$

Point F

Here we apply the forward-backward algorithm to implement the MAP criterion for detection. Actually, we use an approximation of the algorithm called max-log MAP that is computationally more simple. The state definition we used is the same as in point E, due to the fact that the system is the same up to the decision point:

$$u_k = \psi_D a_{k-D} + \psi_{D+1} a_{k-D-1} + \cdots + \psi_{D+M_2} a_{k-D-M_2} \qquad (15)$$

and

$$\boldsymbol{s}_k = \left\{ a_{k-D}, a_{k-D-1}, a_{k-D-2}, \ldots, a_{k-(M_2-1)} \right\} = \\ = \left\{ a_{k-D}, a_{k-D-1}, a_{k-D-2}, a_{k-D-3} \right\} \tag{16}$$

Again the number of states is $N_s = M^{M_2}$.

Simulation results

The simulation of the six different receivers has been carried out using as input the result of applying a QPSK bitmap on a PN sequence with $r = 20$ and length $L = 2^r - 1$. The length of the resulting input in terms of symbols is 524287, because each symbol requires 2 bits to be determined. The obtained symbols are assumed to be distributed approximately as random. Given that we have to estimate the probability of error over 7 different values of the SNR (from 8 dB to 14 dB), for a range of values going from 10^{-1} to 10^{-4} we need to be sure that our estimate is not affected by a high variance. The estimator we chose is:

$$\hat{P}_e = \frac{\#_{wrong}}{\#_{tx}} \tag{17}$$

where $\#_{wrong}$ is the number of erroneously detected symbols and $\#_{tx}$ is the total number of transmitted symbols. As $\#_{tx}$ increases the estimator becomes approximately a Gaussian random variable, unbiased and with variance: $\frac{P_e(1-P_e)}{\#_{tx}}$. This confirms that our estimate is reliable because the resulting variance will be very small, according to the empirical rule for which if we want to estimate a P_e of 10^{-k}, we need to measure it transmitting at least $30 \cdot 10^k$ symbols. Indeed in our case the smallest P_e we need to estimate is 10^{-4} and so the required number of symbols would be $30 \cdot 10^4 = 300000$ that is much less than our $\#_{tx} = 524287$. The results obtained are showed in figure 17. In green we plotted the values of the $P - e$ according to the matched filter bound, as in the theoretical formula and from simulation[1]. In blue we have the LE (solid) and DFE (dashed) of points A and B. Their performance is very good, as we could guess by seeing the result of the overall pulse ψ (see figure 4 and 7, the ISI is almost completely canceled in both cases). In case of receivers C and D (black dashed and black solid lines) the performances are worse, because the anti aliasing filter cuts a components of the useful signal and causes a permanent loss of information. Nevertheless the result is acceptable and could be considered as an alternative in those cases where a matched filter is not available or impracticable.

References

[1] Nevio Benvenuto, Giovanni Cherubini, *Algorithms for Communication Systems and their Applications*. Wiley, 2002.

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[1]To simulate the matched filter bound we gave as input of a threshold detector the QPSK symbols a_k disturbed only by noise, so neglecting the effect os ISI as we would have with an ideal non dispersive channel.
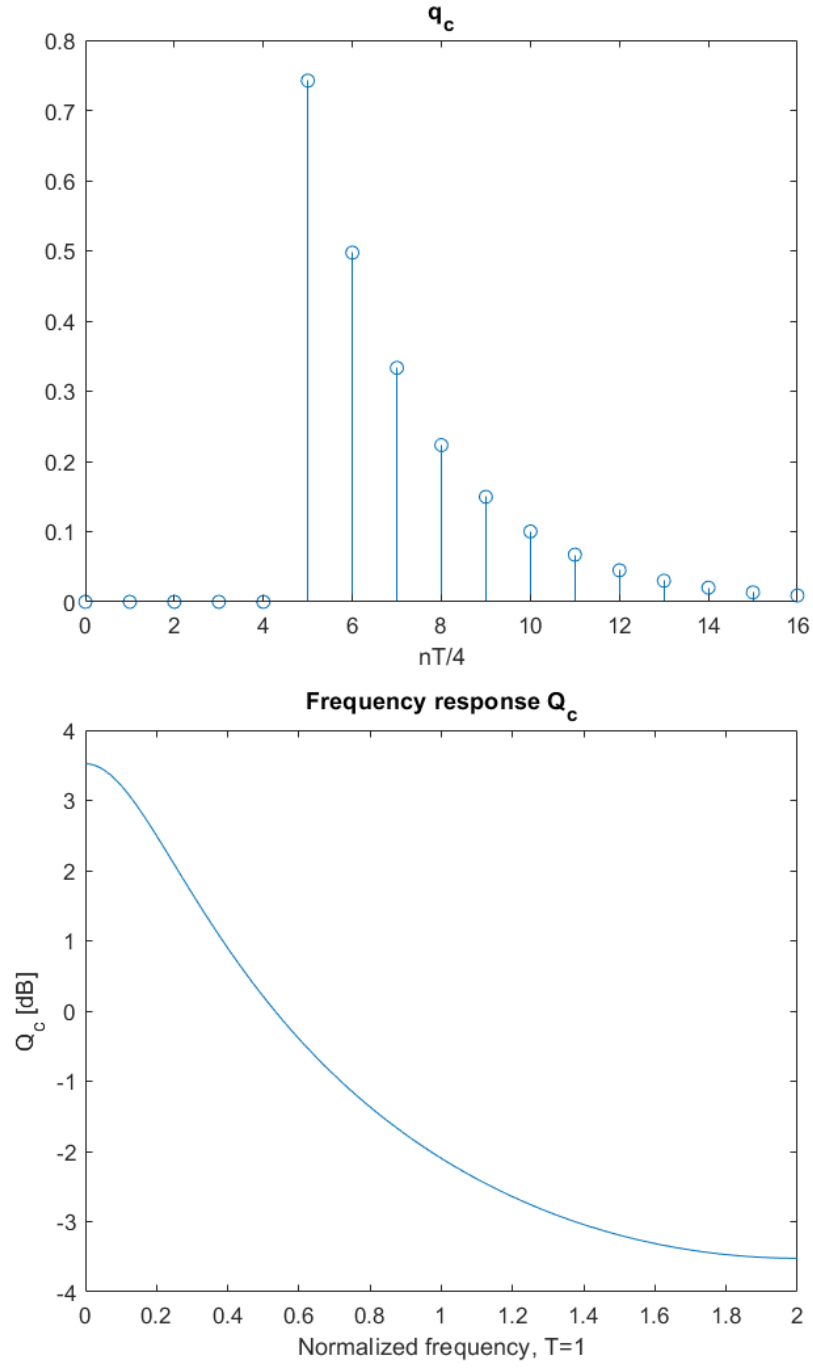
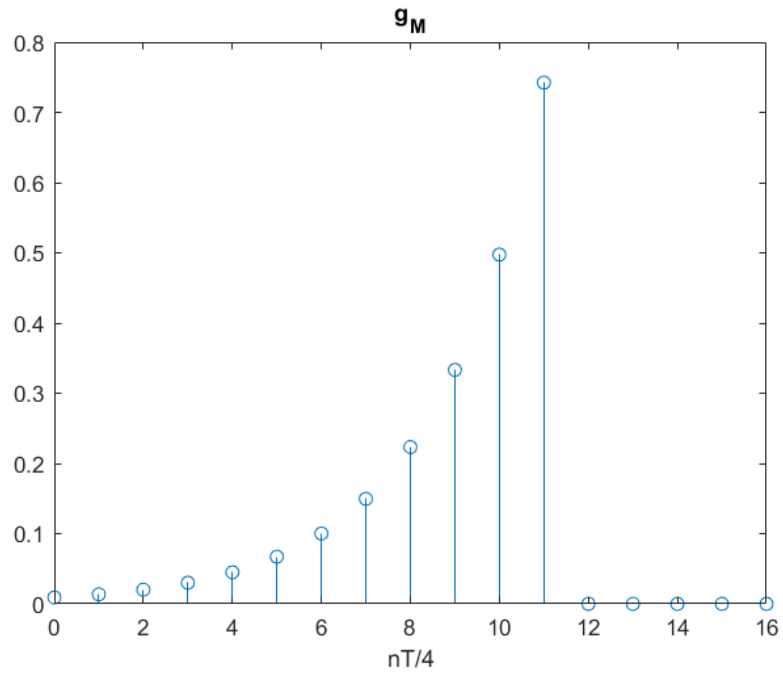Figure 1: Impulse and frequency response of the filter q_c at $T/4$.

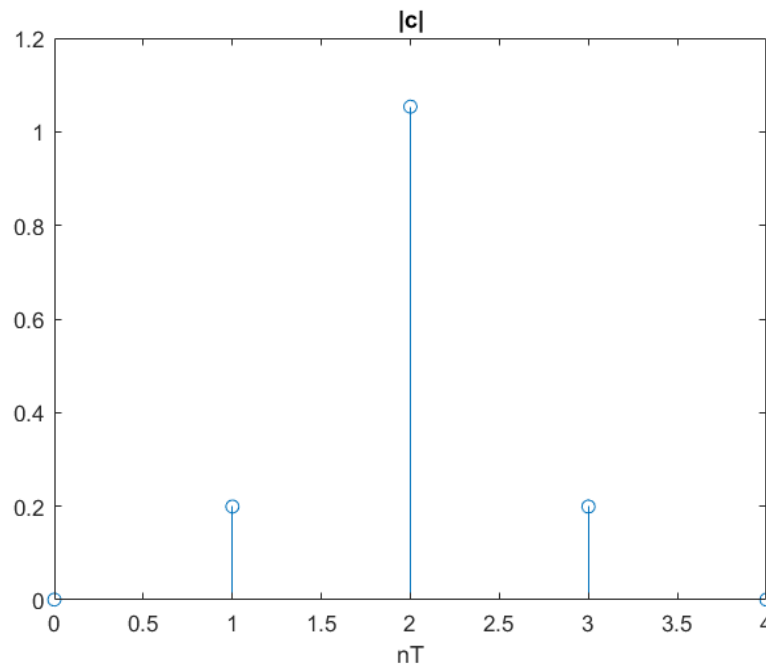Figure 2: Impulse response of the matched filter g_M for the receiver in point A.

Figure 3: Magnitude of the impulse response of filter c in point A.
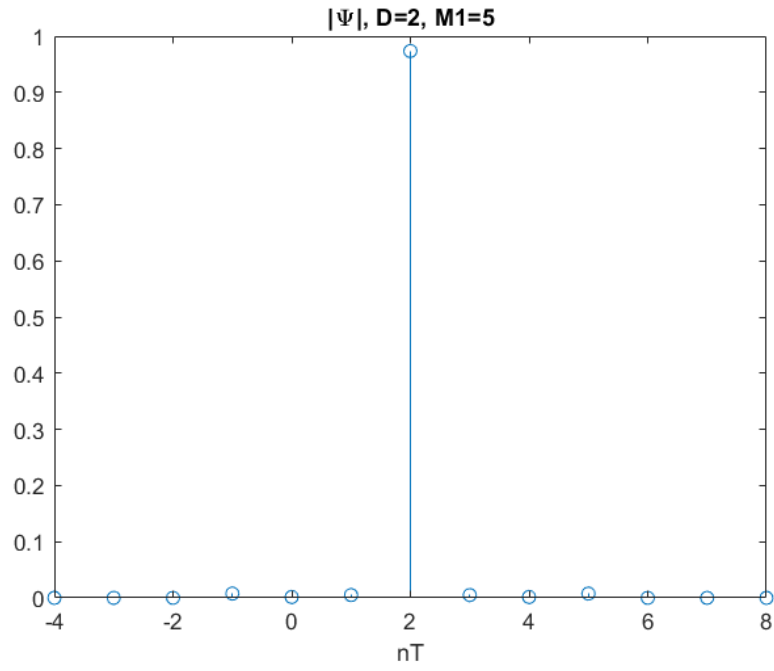
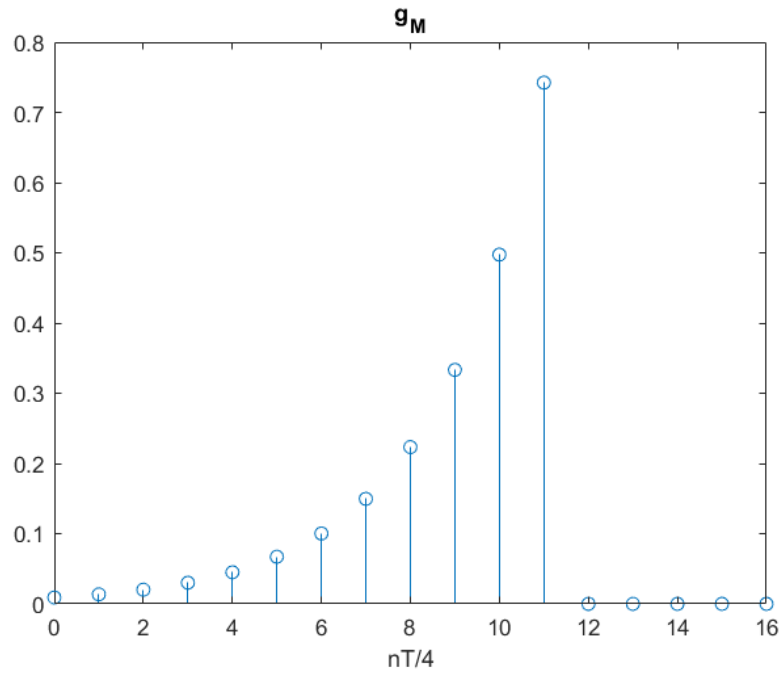Figure 4: Magnitude of the impulse response of the system ψ in point A.

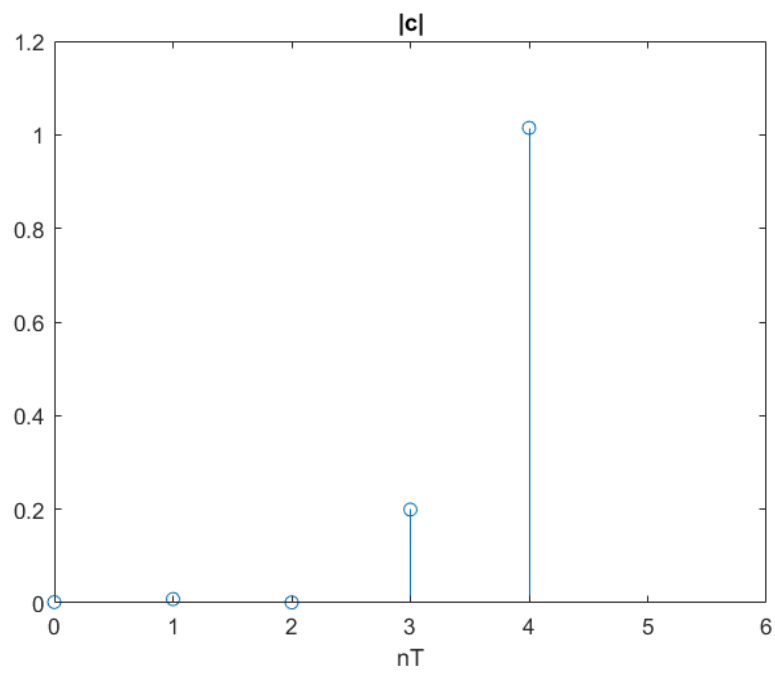Figure 5: Impulse response of the matched filter g_M for the receiver in point B.

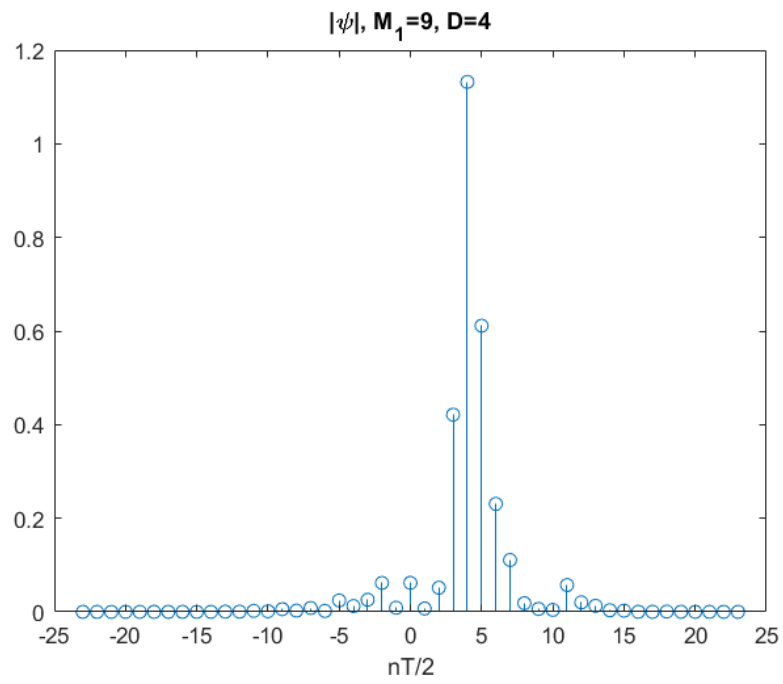Figure 6: Magnitude of the impulse response of the filter c (feedforward filter) for the receiver in point B.
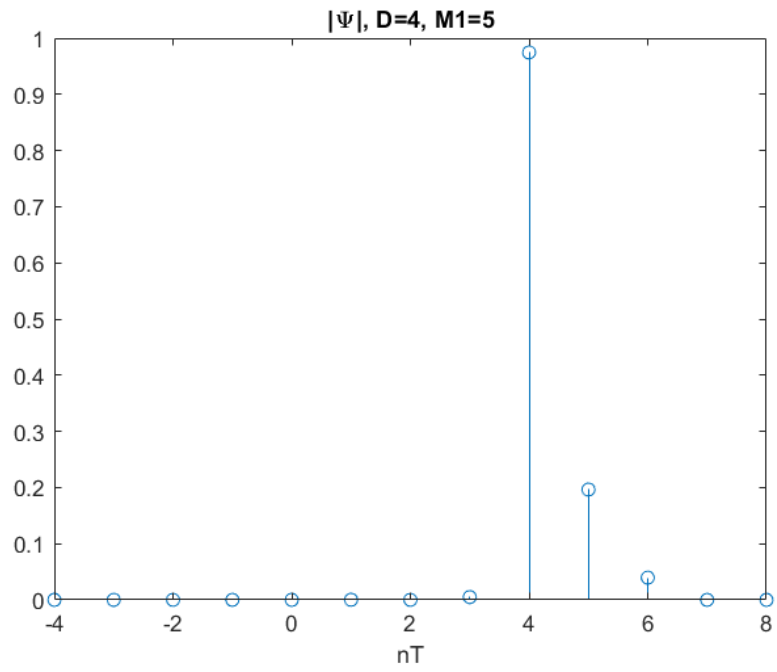
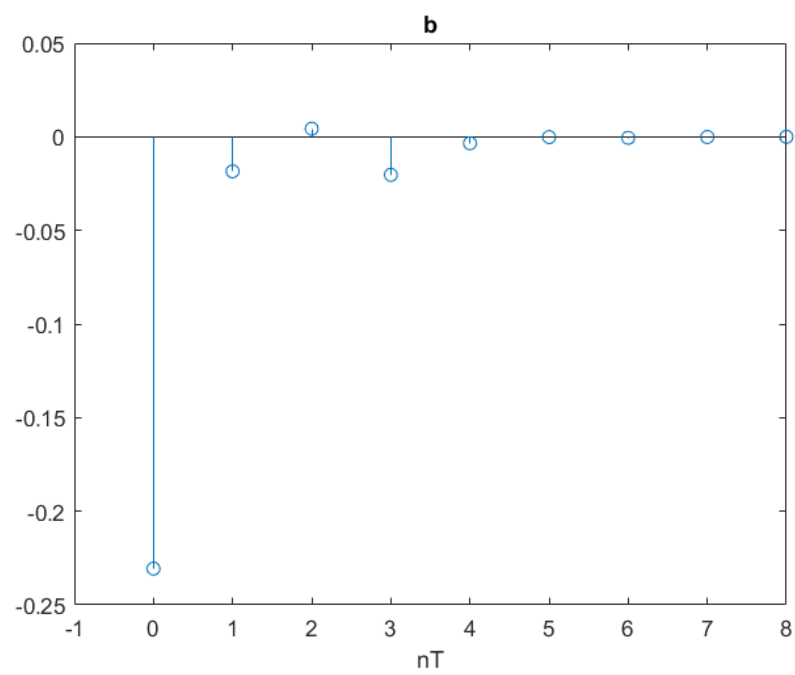Figure 7: Magnitude of the impulse response of the system ψ in point B.
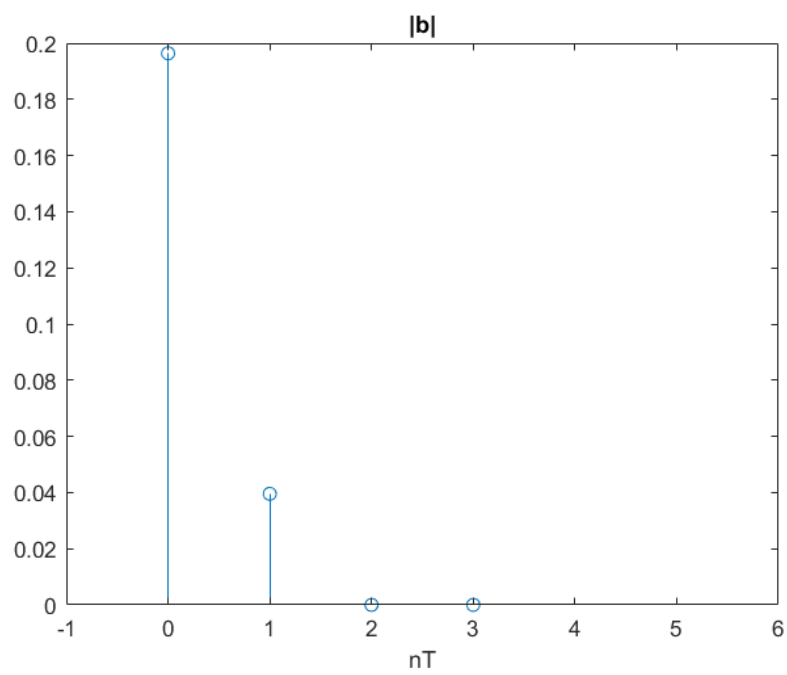
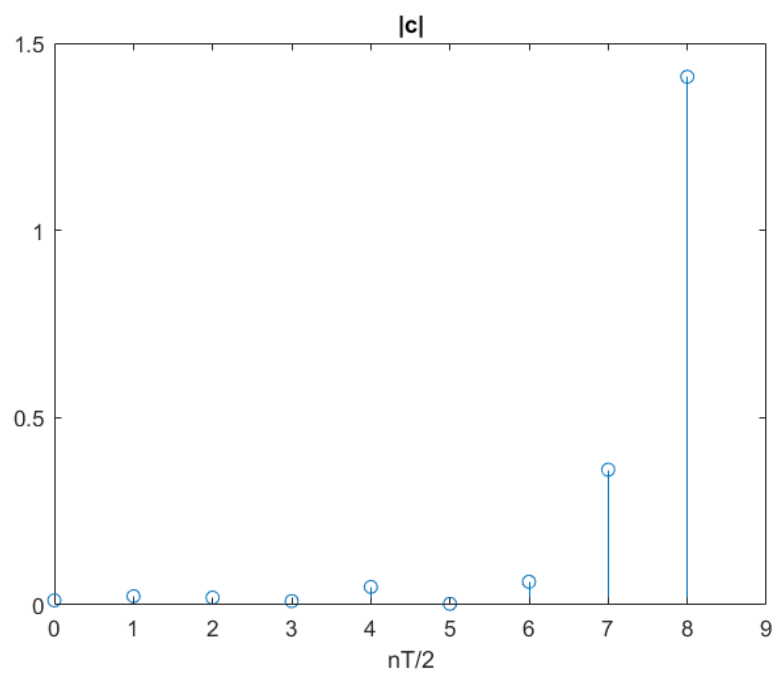Figure 8: Magnitude of the impulse response of the filter b (feedback filter) in point B.
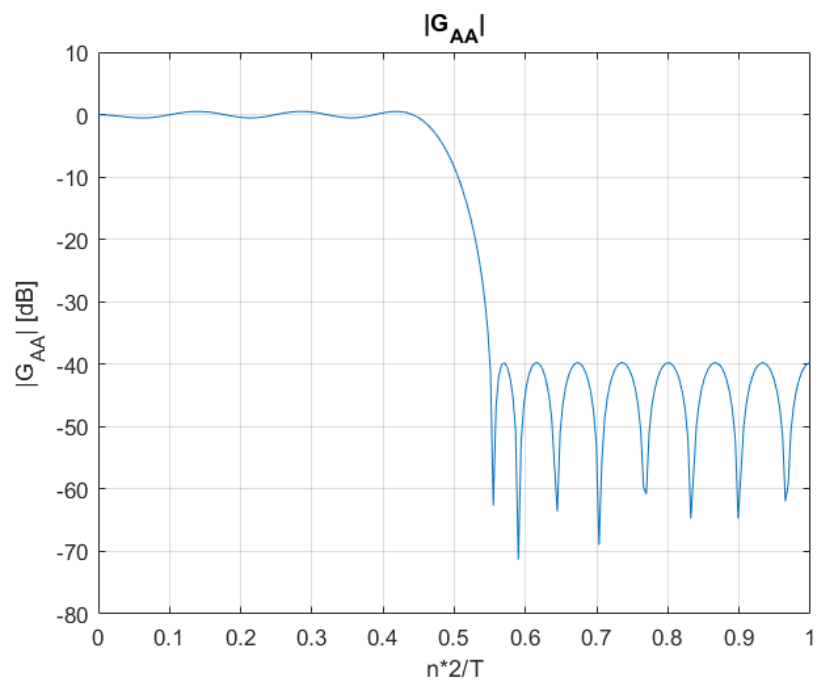
Figure 9: Magnitude of the frequency response of the anti-aliasing filter in point C.
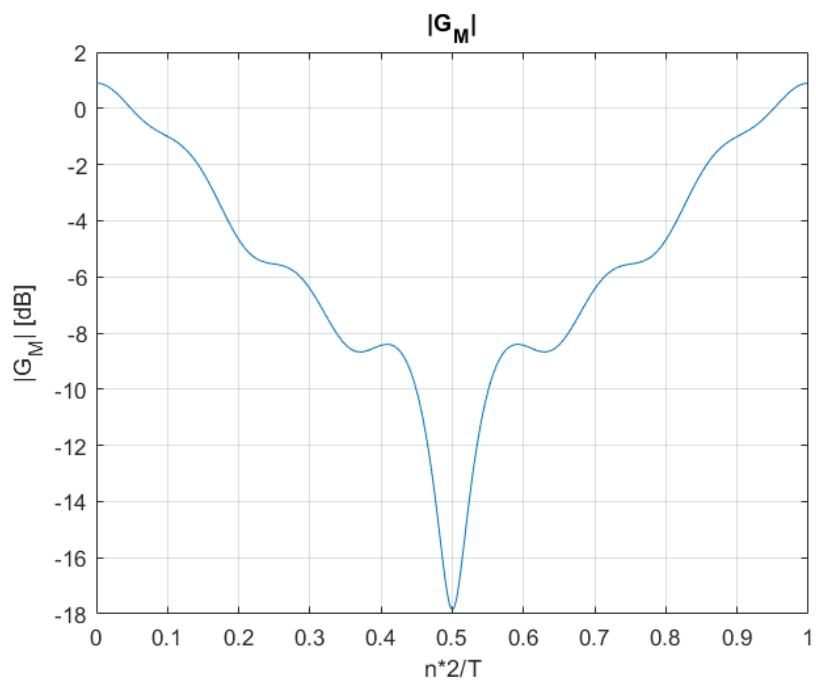
Figure 10: Magnitude of the frequency response of the matched filter in point C.
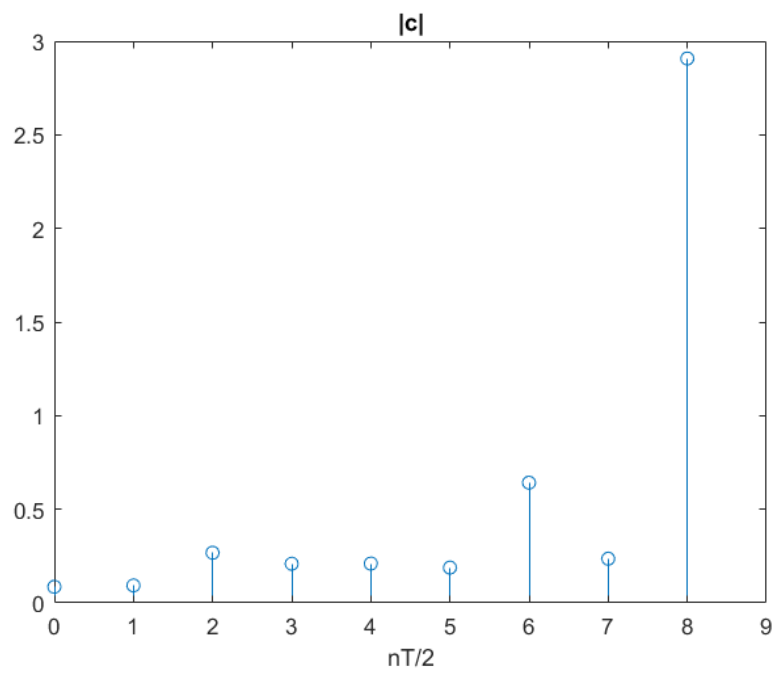
Figure 11: Magnitude of the impulse response of the filter c (feedforward filter) for the receiver in point C.

Figure 12: Magnitude of the impulse response of the system ψ in point C.

Figure 13: Magnitude of the impulse response of the filter b (feedback filter) in point C.

Figure 14: Magnitude of the impulse response of the filter c (feedforward filter) for the receiver in point D.
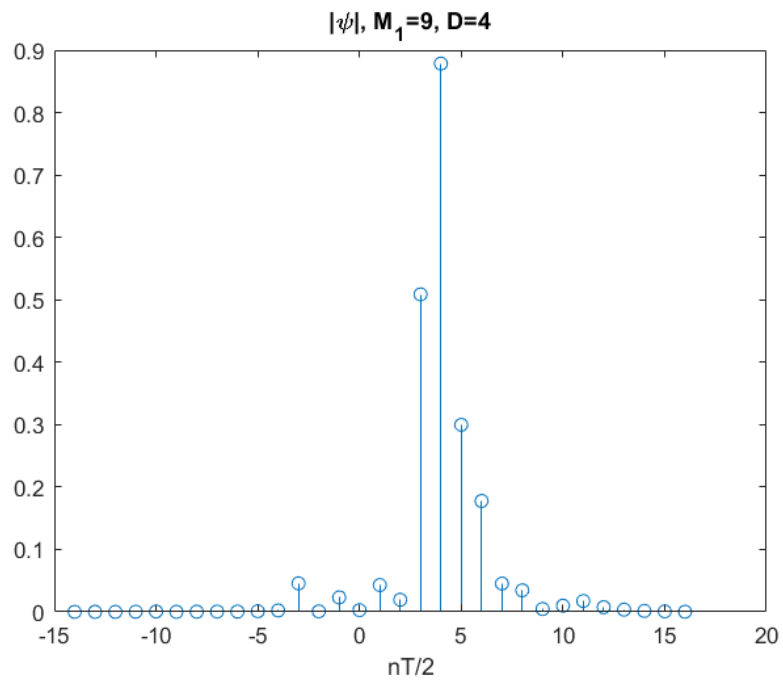
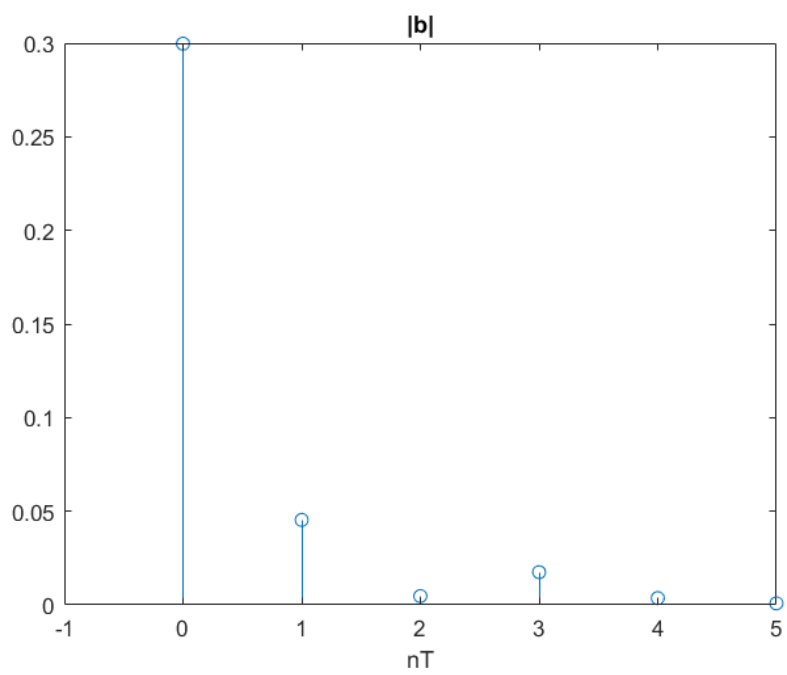Figure 15: Magnitude of the impulse response of the system ψ in point D.

Figure 16: Magnitude of the impulse response of the filter b (feedback filter) in point D.
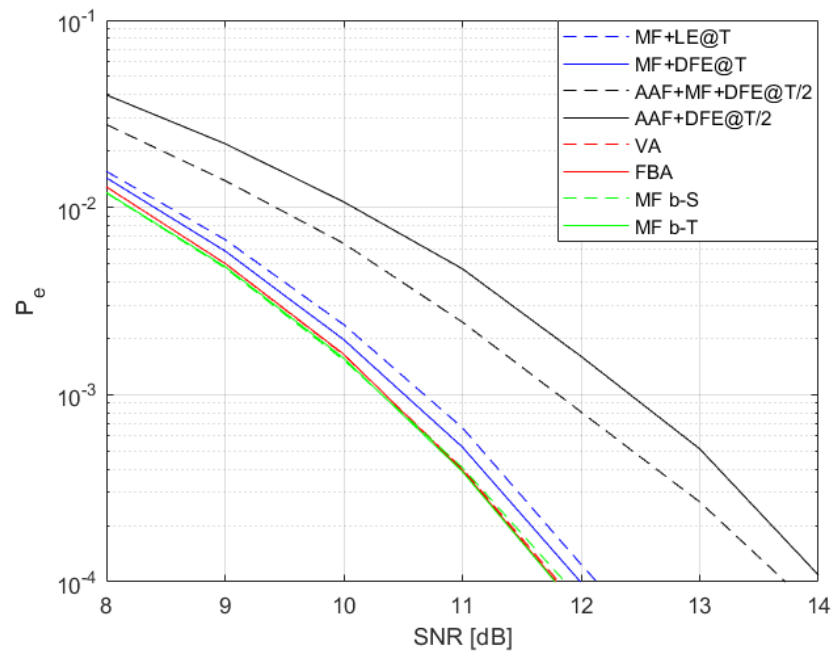
Figure 17: Results of the simulation over values of the SNR at the channel output from 8 dB to 14 dB.