Digital Image & Video Processing

Lecture 8
Karhunen Loève Transform

KHOA CÔNG NGHỆ THÔNG TIN
TRƯỜNG ĐẠI HỌC KHOA HỌC TỰ NHIÊN

8. Karhunen Loève Transform

8.1. Problem statement

8.2. Method

8.3. Applications

8.1. Problem statement

Let x^T be the vector of input samples:

$x^T=[x(0)\ x(1)\ \ldots x(N-1)]$

Find out the linear transformation **A** to:

-Reduce the feature space dimension but retain the principal components.

- Minimize the information loss.

8.2. Method

The linear transformation **A** maps x to y:

$$y = A^T x$$

$$\Rightarrow \quad x = Ay = \sum_{i=0}^{N-1} y_i a_i \; ; \; a_i^T a_i = 1, \; a_i^T a_j = 0 \; \forall i \neq j$$

8.2. Method

Approximate x by x':

$$x' = \sum_{i=0}^{m-1} y_i a_i \quad (m < n)$$

$$error = E[\| x - x' \|^2] = E[\| \sum_{i=m}^{N-1} y_i a_i \|^2] = \sum_{i=m}^{N-1} y_i^2$$

$$= \sum_{i=m}^{N-1} a_i^T E[xx^T] a_i$$

8.2. Method

Minimize error is optimization problem with constraints:

$$\text{Minimize} \quad \sum_{i=m}^{N-1} a_i^T E[xx^T] a_i$$

$$a_i^T a_i = 1,$$

$$a_i^T a_j = 0 \; \forall i \neq j$$

8.2. Method

Optimization based on Lagrange Multiplier Method:

$$\text{Let L} = \sum_{i=m}^{N-1} a_i^T E[xx^T]a_i + \sum_{i=m}^{N-1} \lambda_i (1 - a_i^T a_i) + \sum_{i=m}^{N-1} \gamma_i a_i^T a_j)$$

The necessary condition to L get extreme:

$$\frac{\partial L}{\partial a_i} = 0$$

$$\Rightarrow E[xx^T]a_i = \lambda_i a_i$$

8.2. Method

$$\text{error} = \sum_{i=m}^{N-1} a_i^T E[xx^T] a_i = \sum_{i=m}^{N-1} a_i^T \lambda_i a_i = \sum_{i=m}^{N-1} \lambda_i .$$

If select m eigenvectors with m maximum eigenvalues, the error is minimal.

Properties :

$$E[y_i^2] = a_i^T E[xx^T] a_i = a_i^T \lambda_i a_i = \lambda_i$$

$$E[yy^T] = E[A^T xx^T A] = A^T E[xx^T] A = \Lambda$$