Digital Image & Video Processing

Lecture 7

Fourier Transform

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7. Fourier Transform

7.1. Spatial frequency

7.2. Fourier Theory

7.3. Discrete Fourier Transform

7.4. Filtering of Images

7.1. Spatial frequency

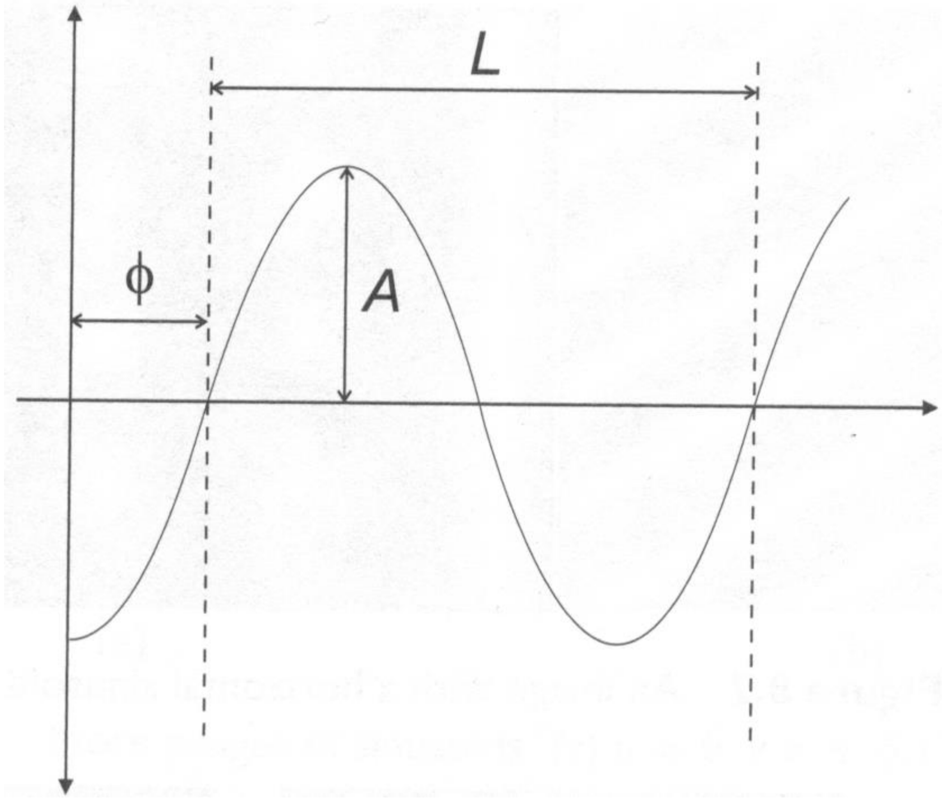
A sinusoidal function, characterised by

A period: L

Spatial Frequency: 1/L

Amplitude: A

Phase: ϕ



7.1. Spatial frequency

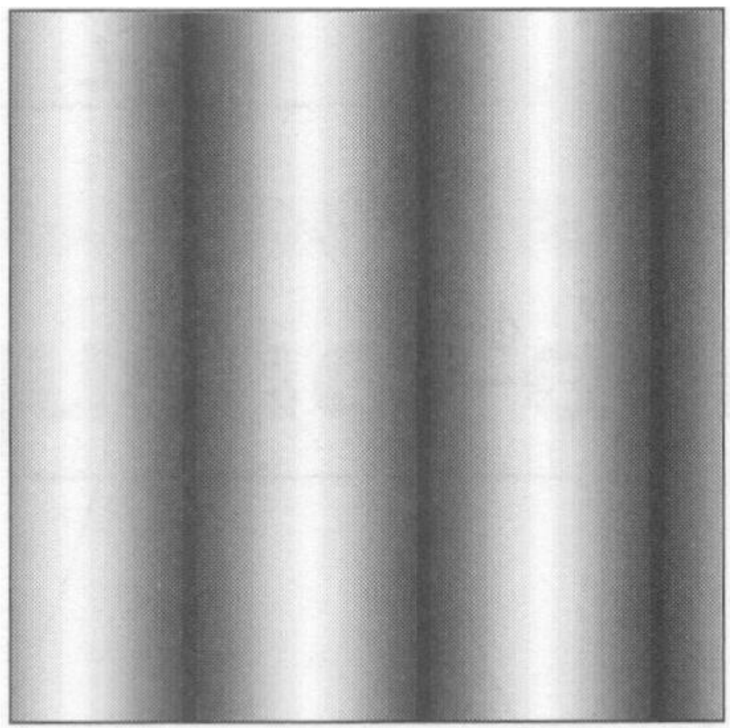
Defining a sinusoidal function and rendering it as an image:

$$f(x,y) = 128 + ASin\left(\frac{2\pi ux}{N-1} + \phi\right)$$

7.1. Spatial frequency

Ex: A=127, N=100, u=3, φ=0

$$f(x,y)=128+ASin\left(\frac{2\pi ux}{N-1}+\phi\right)$$



7.1. Spatial frequency

Ex:

Hình (a)

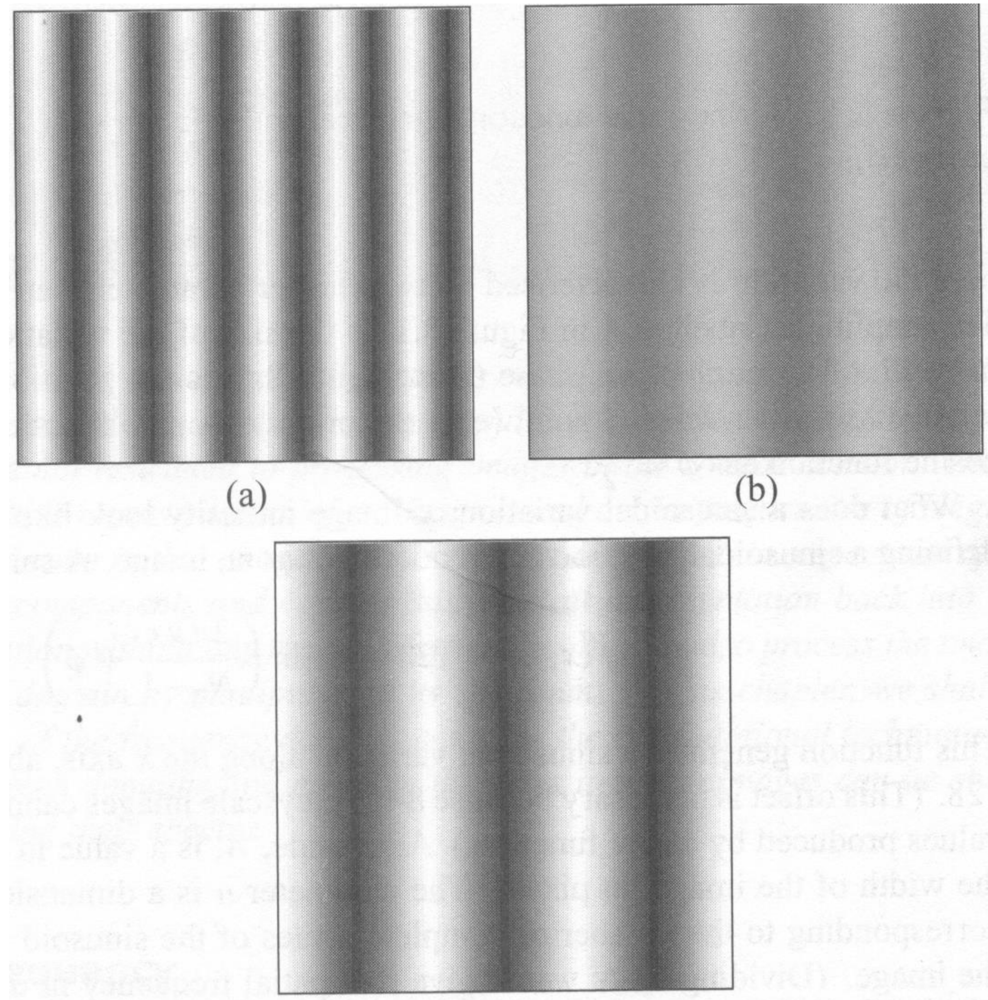
u=6

Hình (b)

Giảm A khoảng 60%

Hình (c)

▯=▯/2



(a) (b) (c)

7.1. Spatial frequency

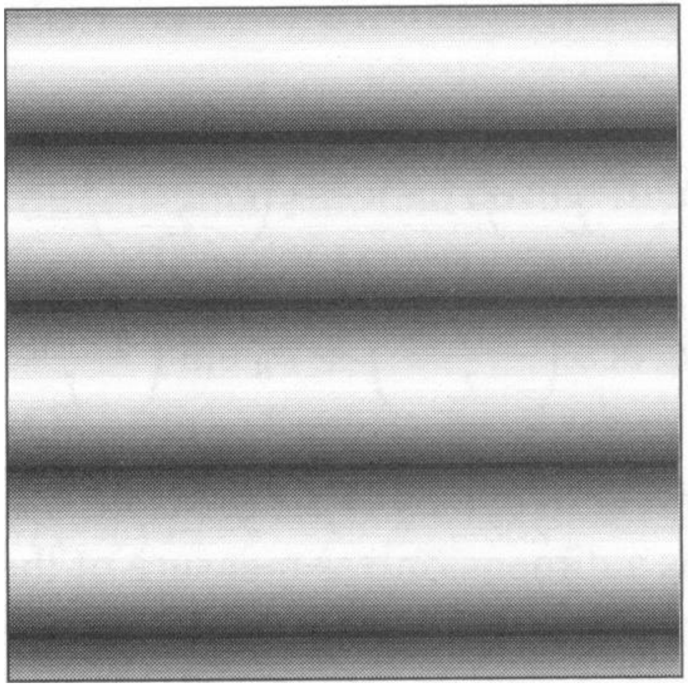
Ex:

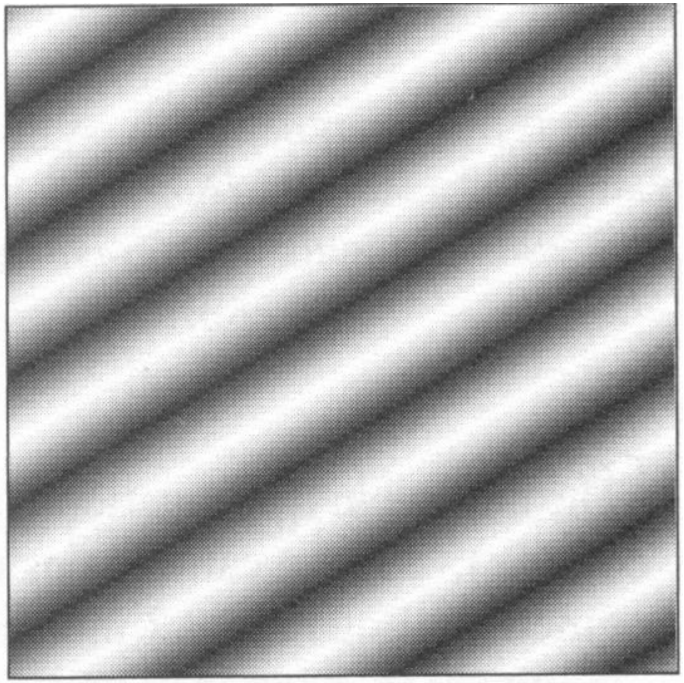
Hình (a)

u=0, v=4

Hình (b)

u=3, v=5





7.2. Fourier Theory

7.2.1. Fourier series (Jean Baptiste Joseph Fourier)

Any periodic function can be represented as a sum of these simpler sinusoids.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)$$

Period: L, Frequency: n

7.2. Fourier Theory

7.2.1. Fourier series (Jean Baptiste Joseph Fourier)

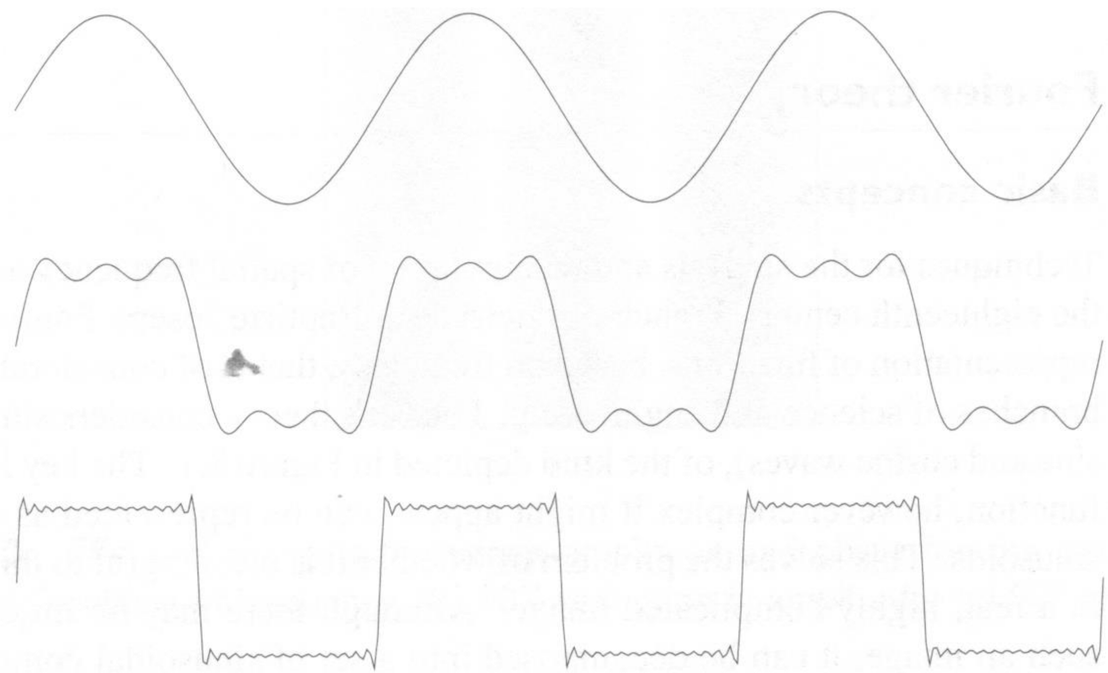


Fig 7.1. Fourier series consisting only of sine functions.

Top: 1 term. **Middle:** 3 terms. **Bottom:** 15 terms.

7.2. Fourier Theory

7.2.1. Fourier series (Jean Baptiste Joseph Fourier)

A fourier series representation of a two-dimensional function, $f(x,y)$, having a period L in both x and y directions, can be written:

$$f(x,y)=\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}a_{u,v}\cos\left(\frac{2\pi(ux+vy)}{L}\right)+b_{u,v}\sin\left(\frac{2\pi(ux+vy)}{L}\right)$$

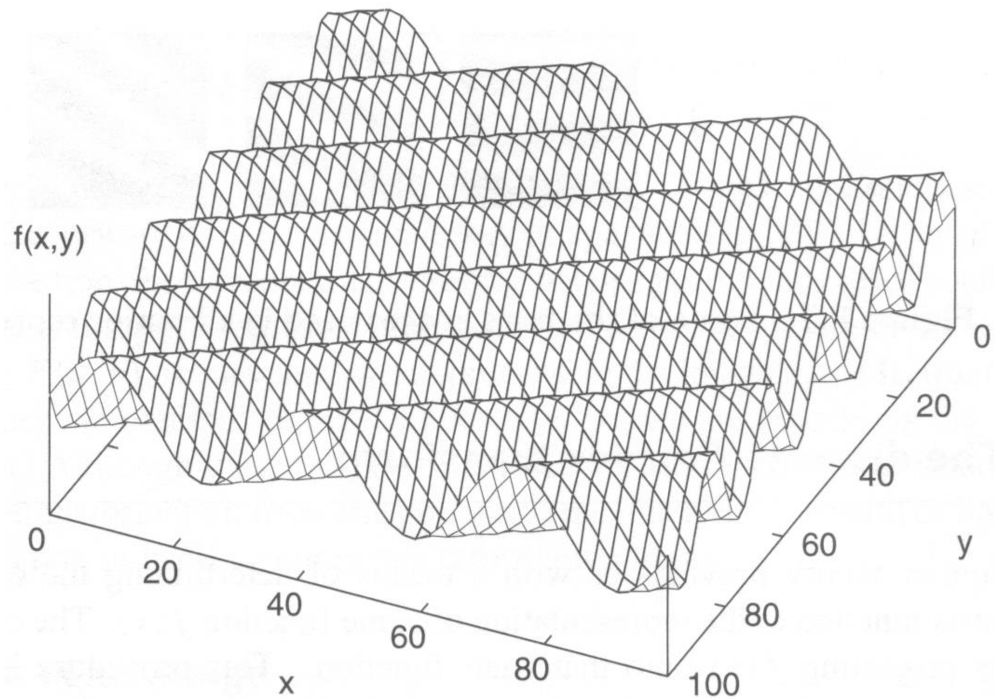
u,v: number of cycles fitting into one horizontal and vertical period of $f(x,y)$.

7.2. Fourier Theory

7.2.1. Fourier series (Jean Baptiste Joseph Fourier)

$$f(x,y)=\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}a_{u,v}\cos\left(\frac{2\pi(ux+vy)}{L}\right)+b_{u,v}\sin\left(\frac{2\pi(ux+vy)}{L}\right)$$

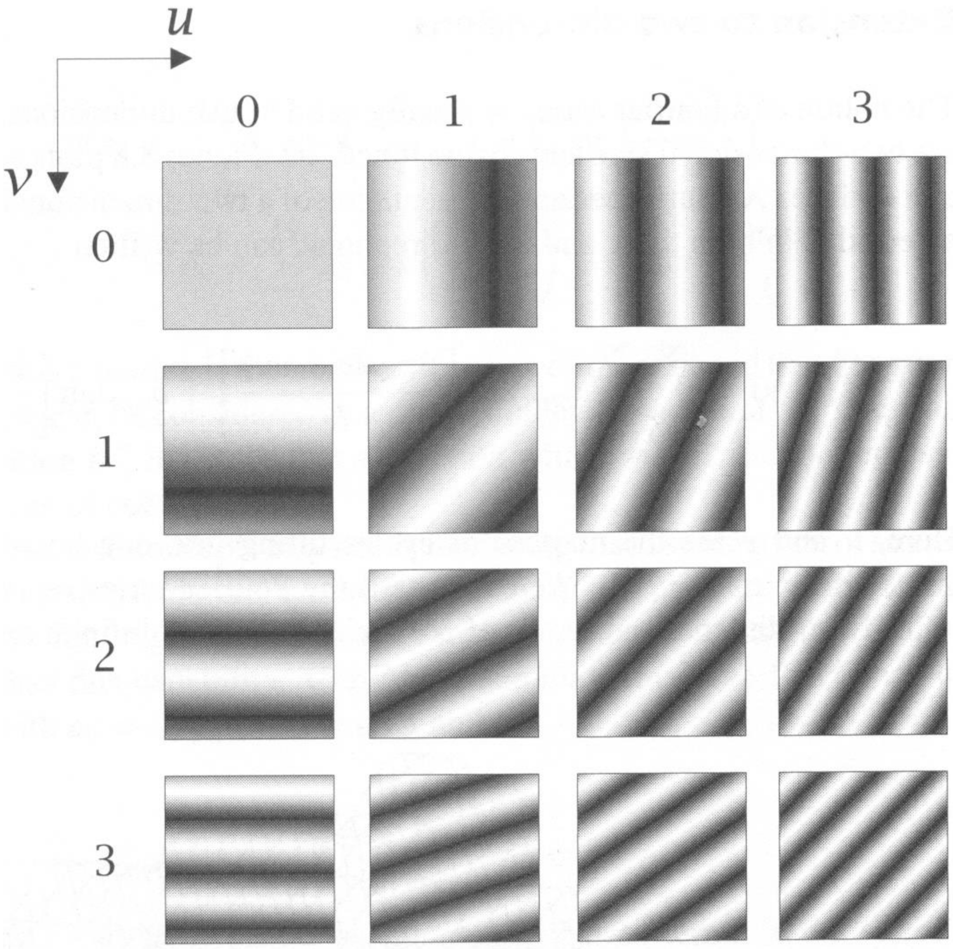
u=3, v=5



7.2. Fourier Theory

7.2.1. Fourier series (Jean Baptiste Joseph Fourier)

Fig. 7.2. Some of basic images used in a Fourier representation of an image



7.2.2. Fourier Transform

Fourier Transform

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(ux+vy)}dxdy$$

Inverse Fourier Transform

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{2\pi i(xu+yv)}dudv$$

7.2. Fourier Theory

7.2.2. Fourier Transform

Fourier Transform

$$F(u,v) = R(u,v) + i.I(u,v)$$

Magnitude or **Spectrum of Fourier Transform**

$$|F(u,v)| = (R^2(u,v) + I^2(u,v))^{1/2}$$

Phase angle or **Phase spectrum of Fourier Transform**

$$\phi(u,v) = \arctan[(I(u,v)/R(u,v))]$$

7.2.2. Fourier Transform

Convolution theorem

$$\zeta\{(f*h)(x,y)\} = F(u,v).H(u,v)$$

$$\zeta\{f(x,y).h(x,y)\} = (F*H)(u,v)$$

Discrete Fourier Transform

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-2\pi i \left(\frac{xu}{M} + \frac{yv}{N} \right)}$$

$$u = 0,1,\ldots,M-1 \qquad \mathrm{v} = 0,1,\ldots,N-1$$

Inverse Discrete Fourier Transform

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{2\pi i \left(\frac{xu}{M} + \frac{yv}{N} \right)}$$

$$x = 0,1,\ldots,M-1 \qquad \mathrm{y} = 0,1,\ldots,N-1$$

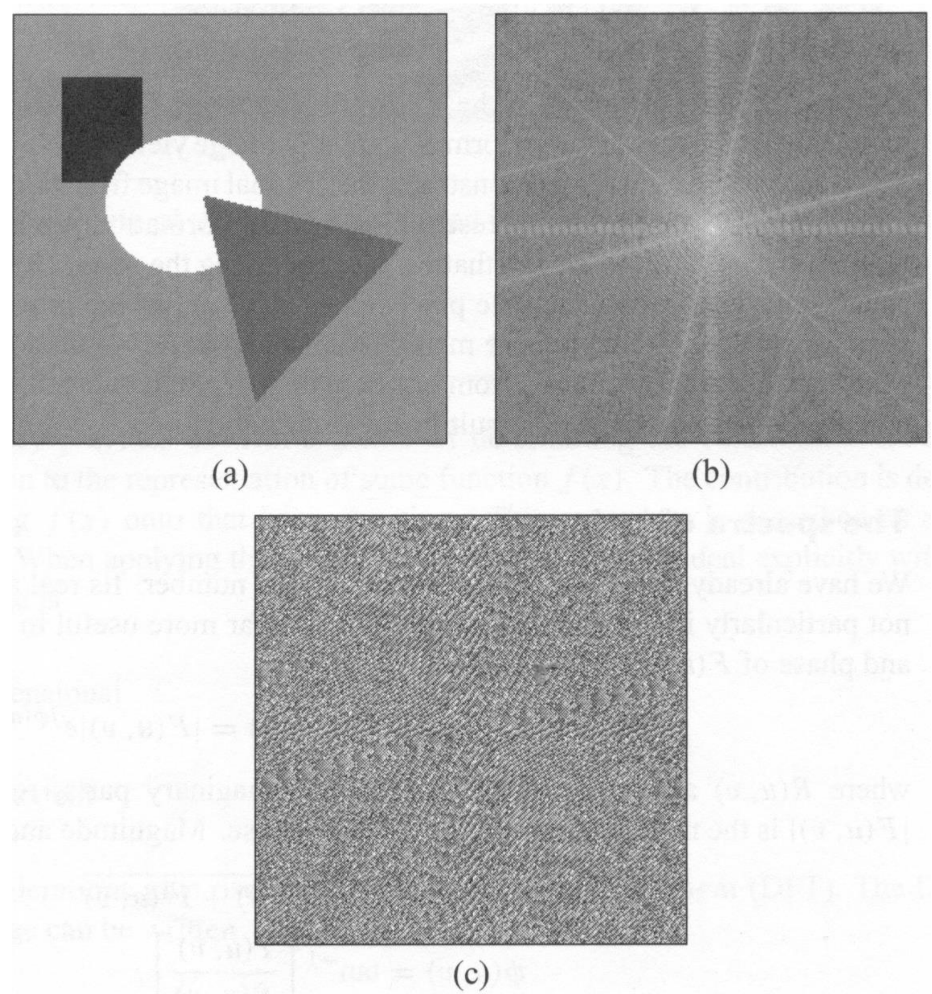
7.3. Discrete FourierTransform

The basic quantities

(a) Image

(b) Amplitude spectrum

(c) Phase spectrum



Properties of the Fourier Transform

DC component of spectrum

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

Symmetric

$$|F(u,v)| = |F(-u,-v)|$$

Properties of the Fourier Transform

Convolution theorem

$$\zeta\{(f*h)(x,y)\} = F(u,v).H(u,v)$$

$$\zeta\{f(x,y).h(x,y)\} = (F*H)(u,v)$$

7.3. Discrete FourierTransform

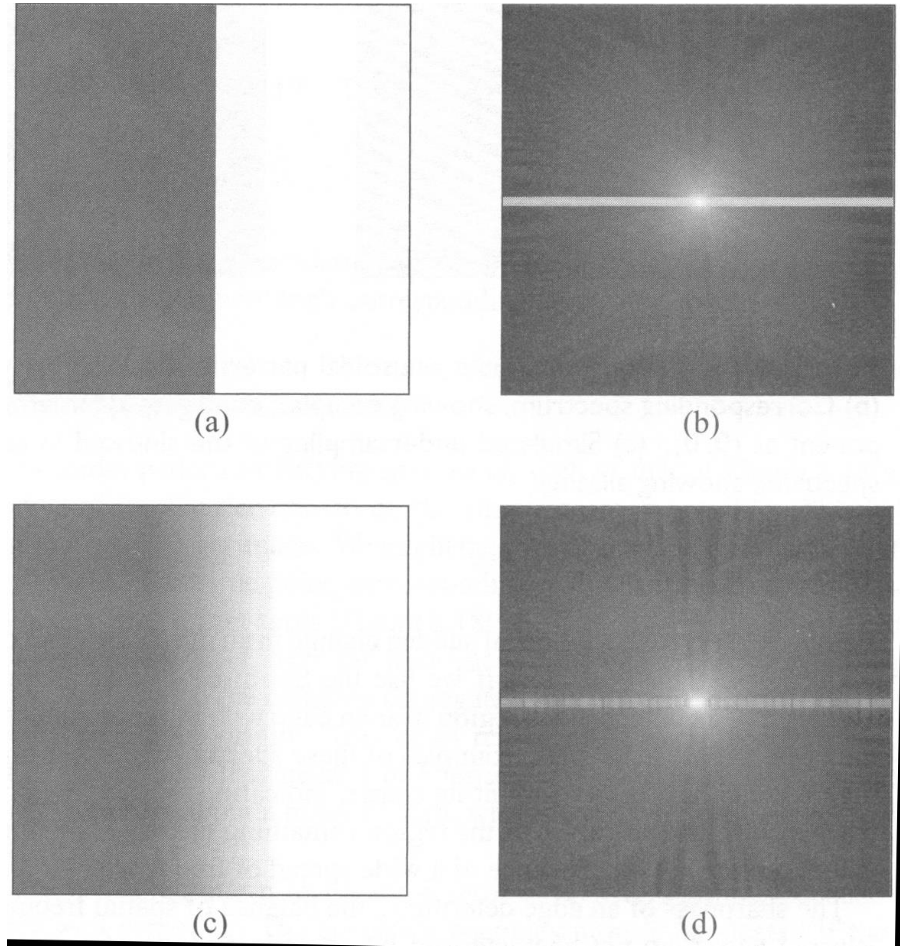
- Ex:

Spectral characteristics

Of sharp and blurred edge

b) Spectrum of a sharp edge

c) Spectrum of a blurred edge

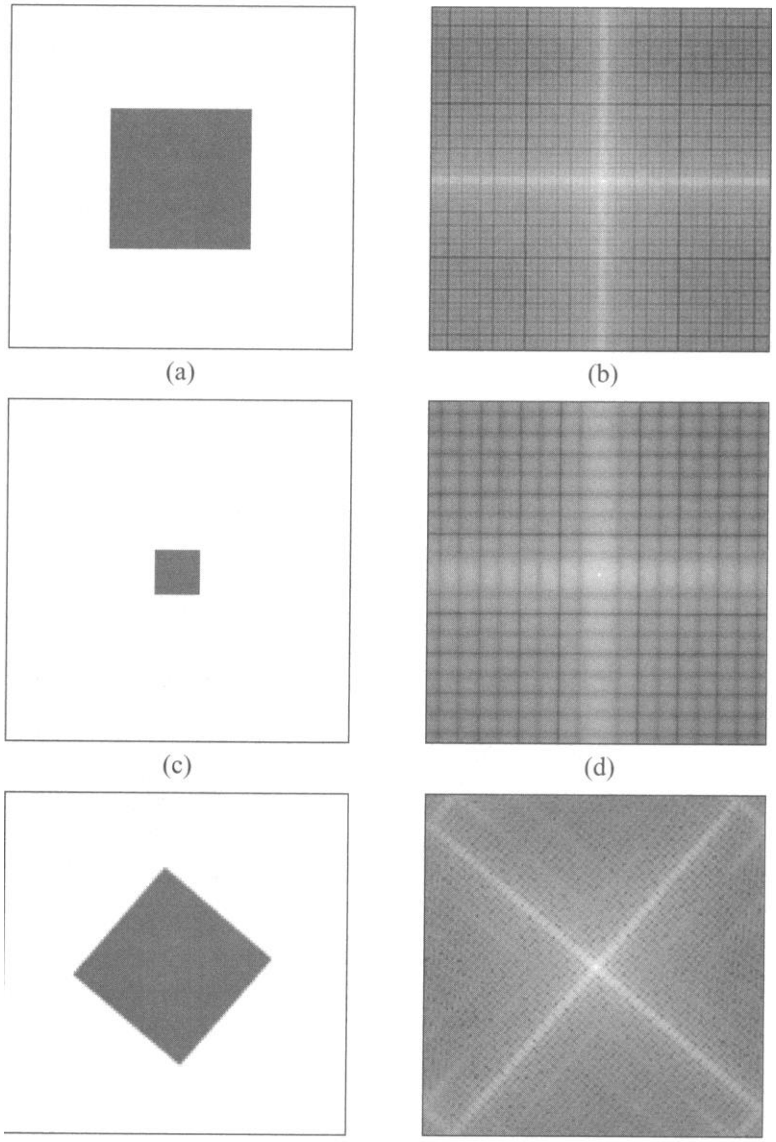


7.3. Discrete FourierTransform

- Ex:

Spectral characteristics

A square, transformed versions of that square and their corresponding spectra



7.3. Discrete FourierTransform

- Ex: **Spectral characteristics**

The spectrumviewer application

7.4. Filtering of Images

7.4.1. Principle of Frequency-Domail Filters

7.4.2. Smoothing Frequency Domain Filters

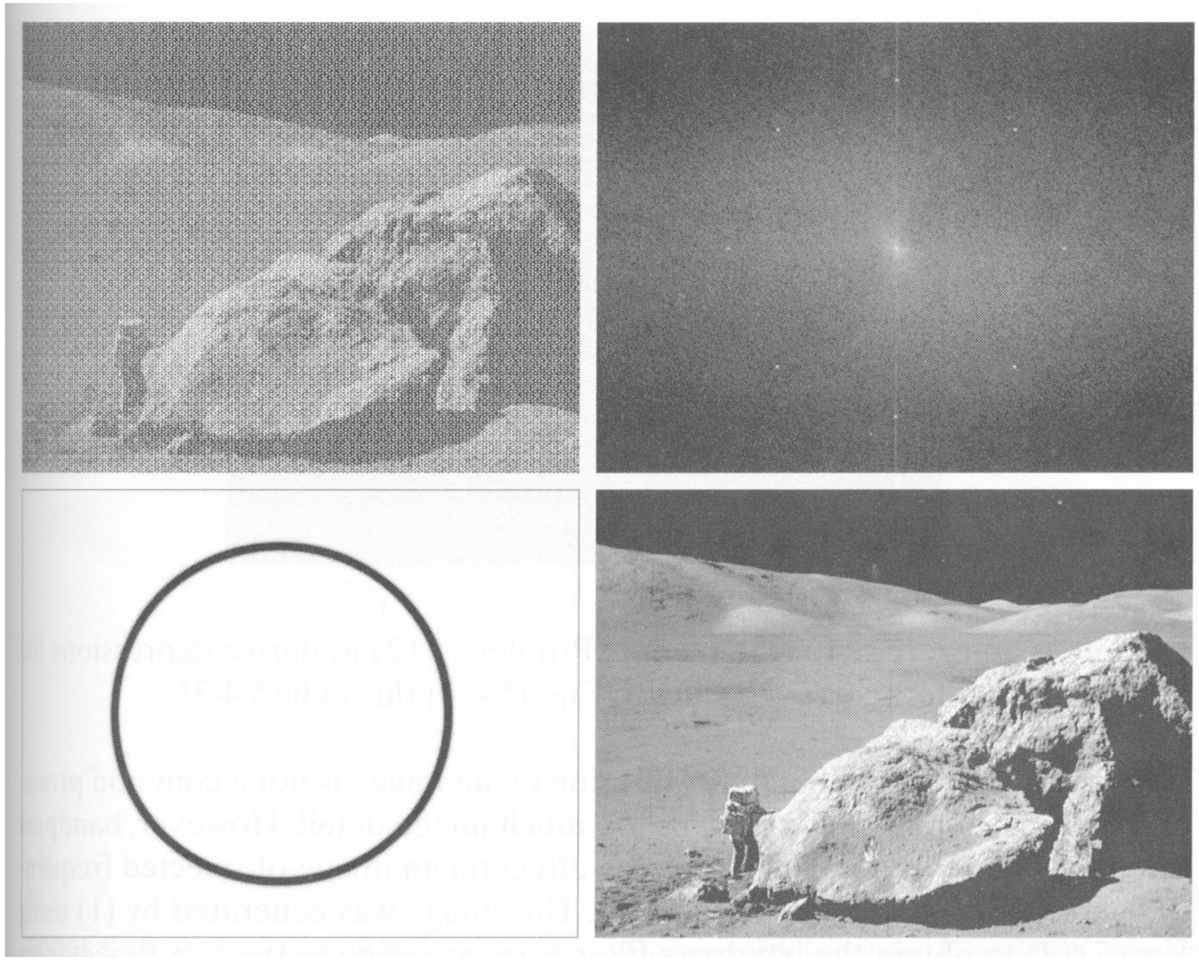
(Low pass filtering)

7.4.3. Sharpening Frequency Domain Filters

(High pass filtering)

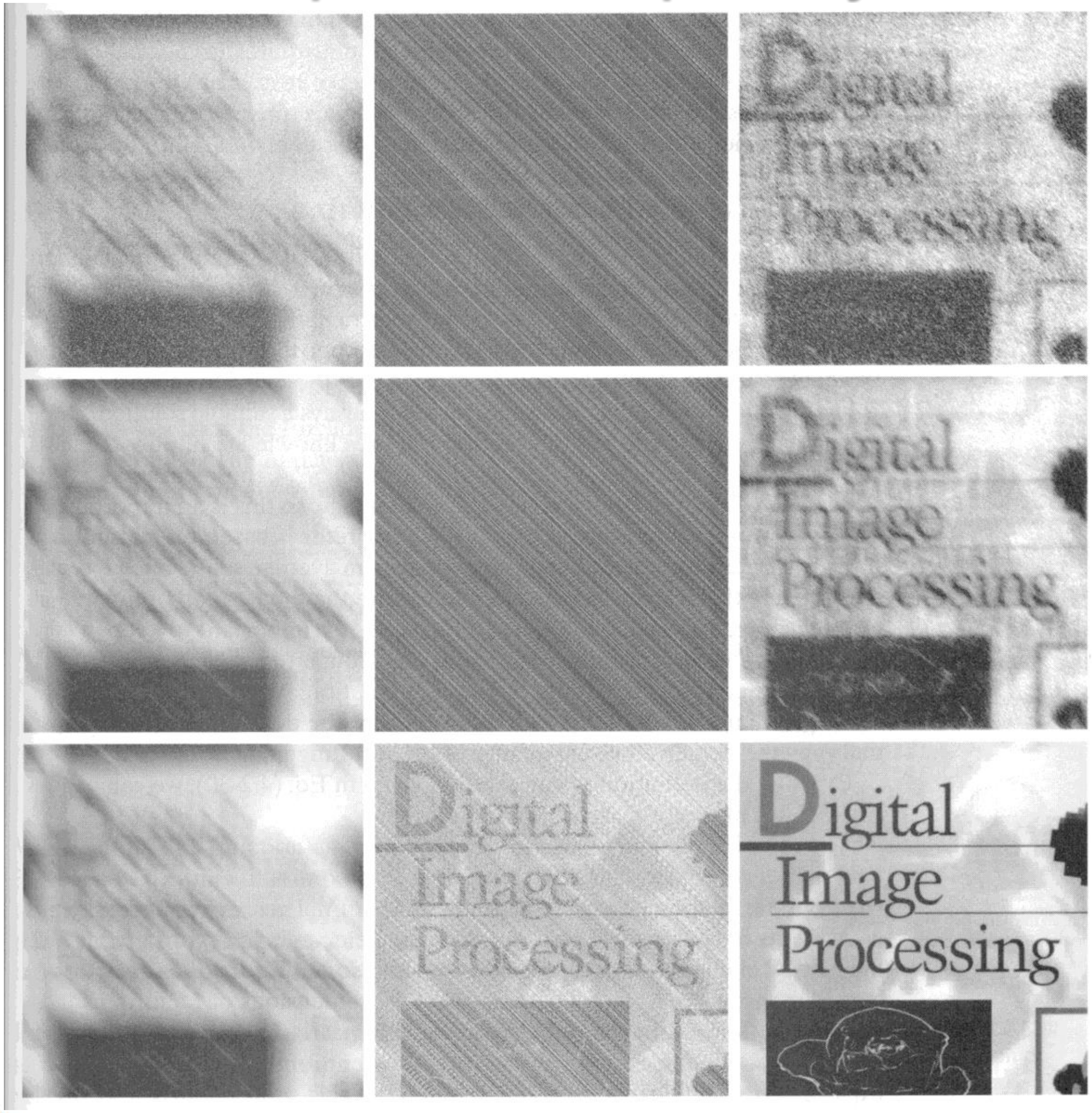
7.4.1. Principle of Frequency Domain Filters

Principle



7.4.1. Principle of Frequency Domain Filters

Principle



7.4.1. Principle of Frequency Domain Filters

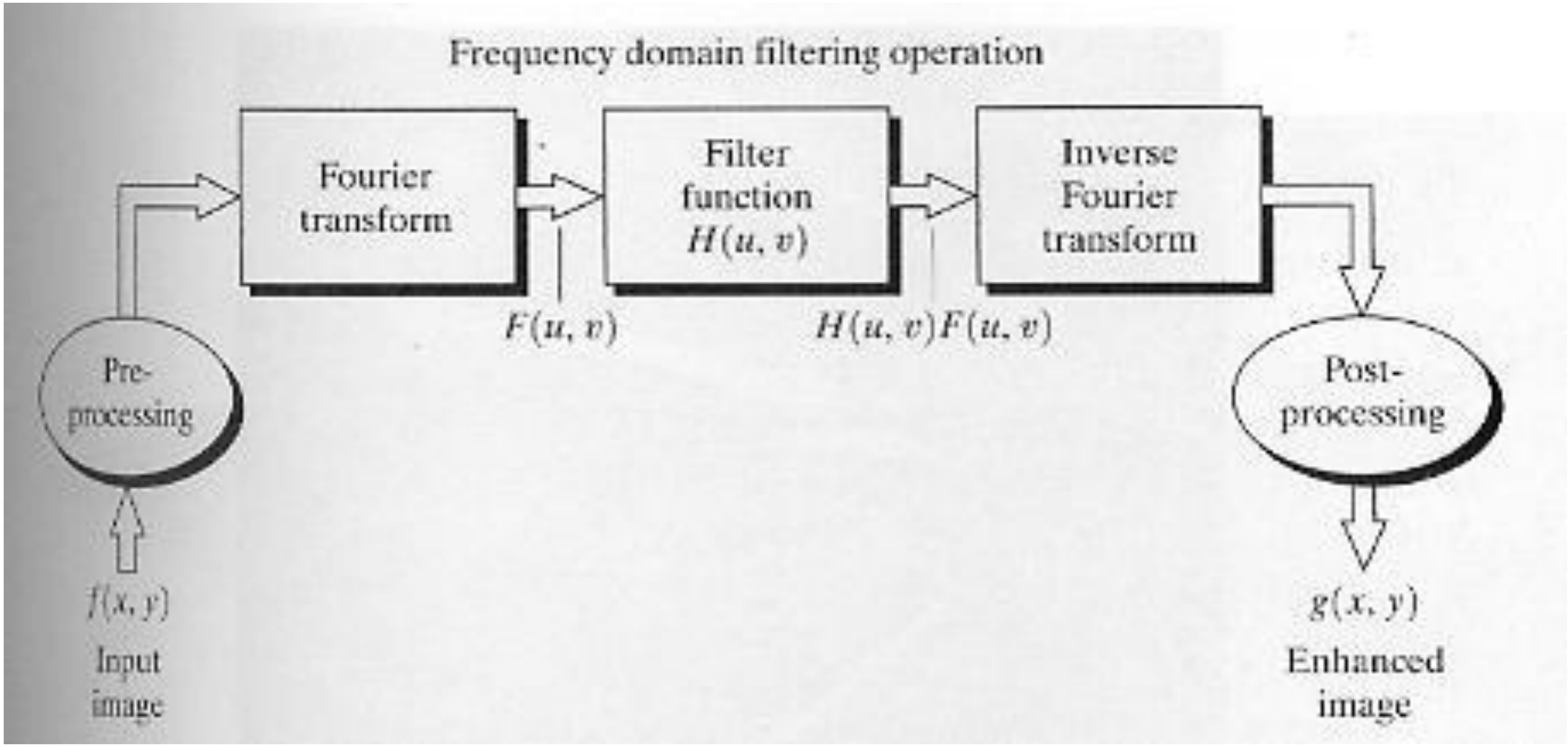
Principle

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(x-m,y-n)h(m,n)$$

$$f(x,y)*h(x,y)\Leftrightarrow F(u,v)H(u,v)$$

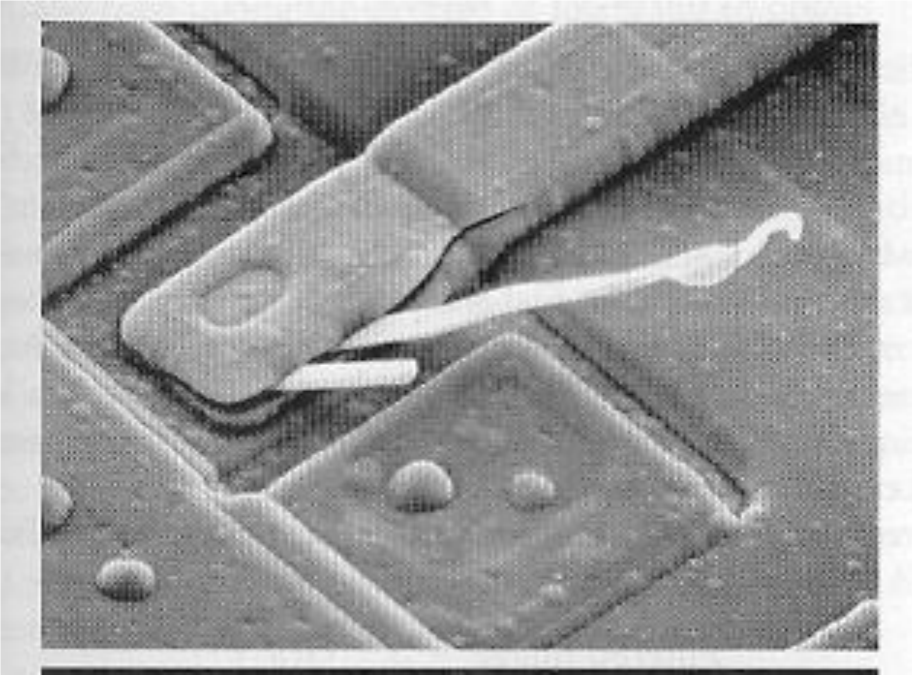
7.4.1. Principle of Frequency Domain Filters

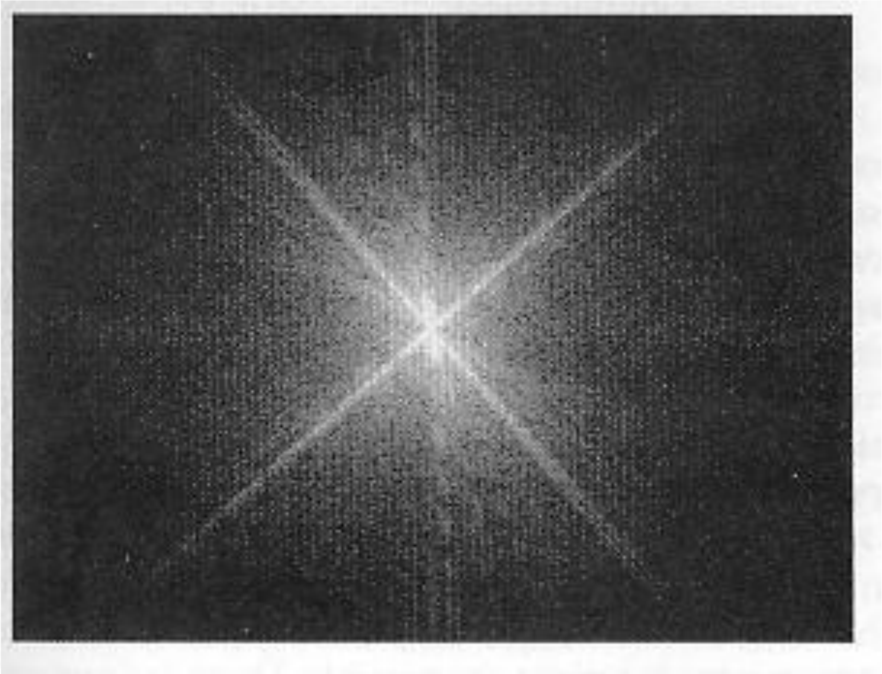
Principle



7.4.1. Principle of Frequency Domain Filters

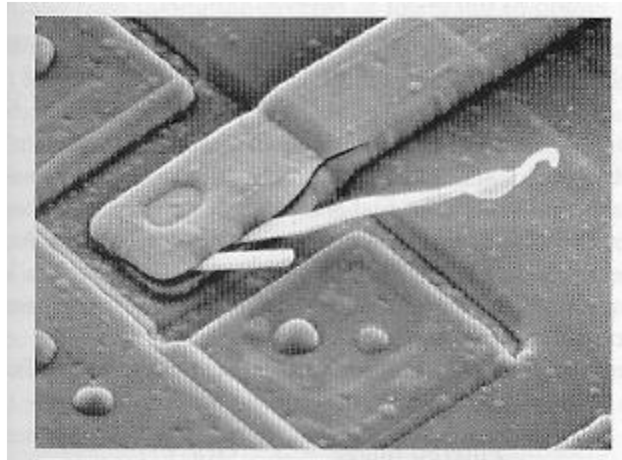
Principle

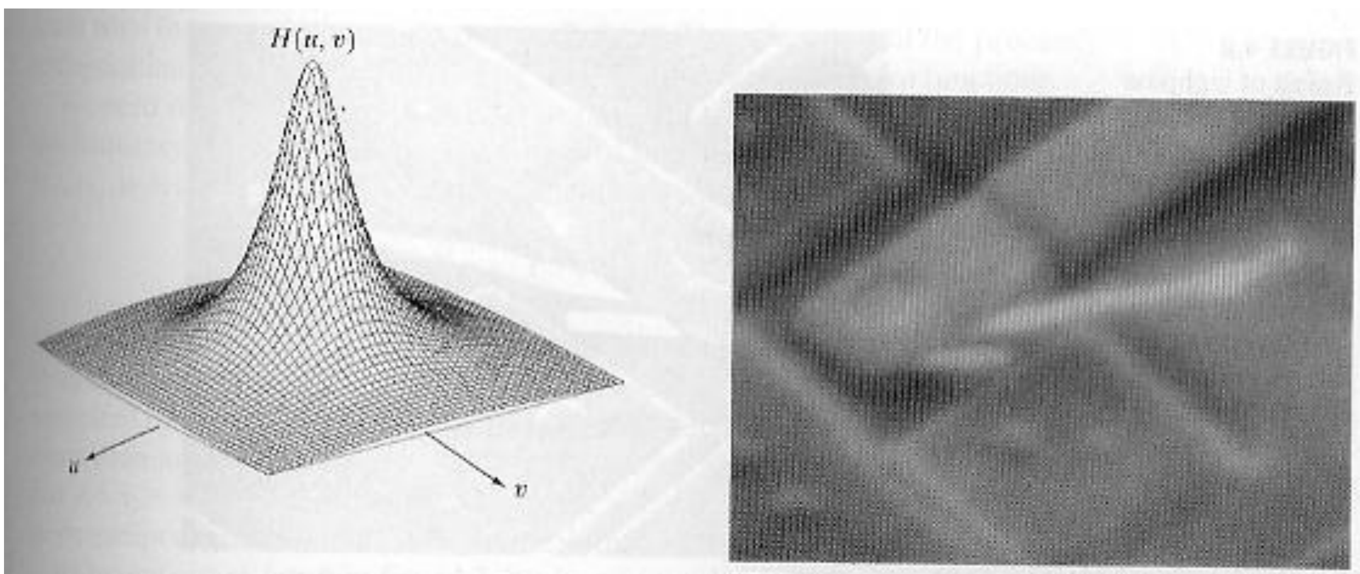




7.4.1. Principle of Frequency Domain Filters

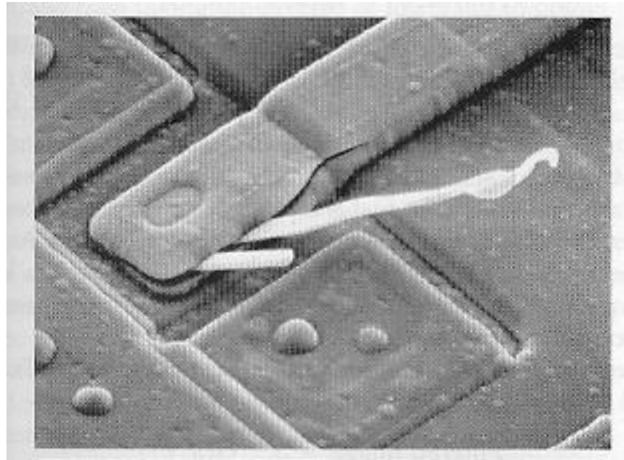
Principle

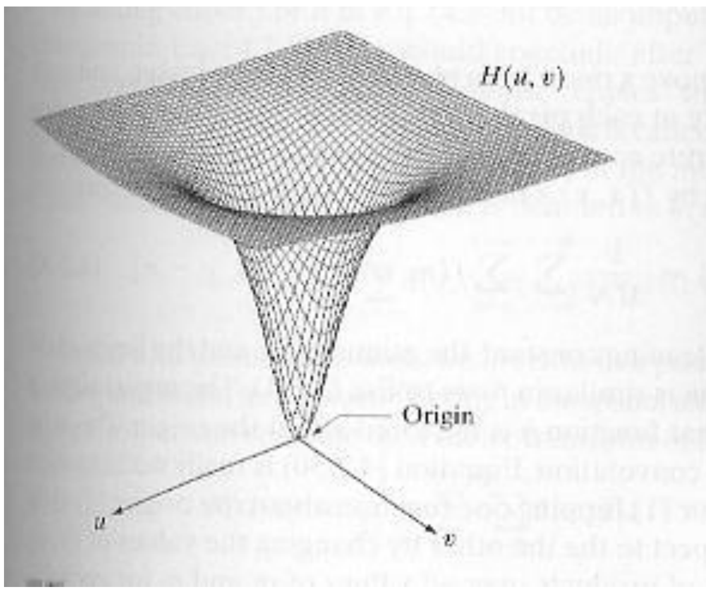


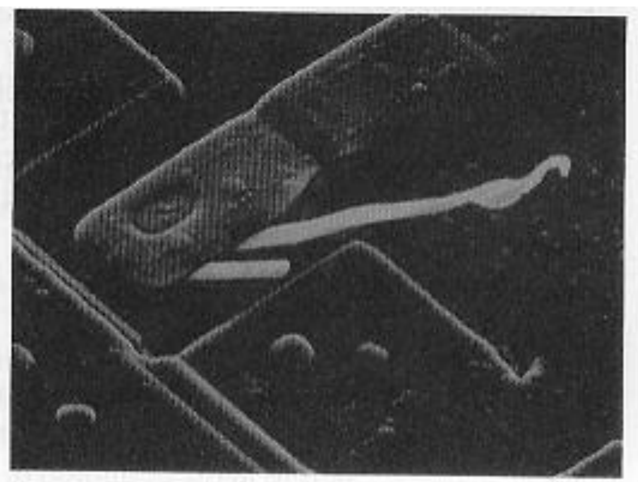


7.4.1. Principle of Frequency Domain Filters

Principle







7.4.2. Smoothing Frequency Domain Filters

- Ideal Lowpass Filters
- Butterworth Lowpass Filters
- Gaussian Lowpass Filters

7.4.2. Smoothing Frequency Domain Filters

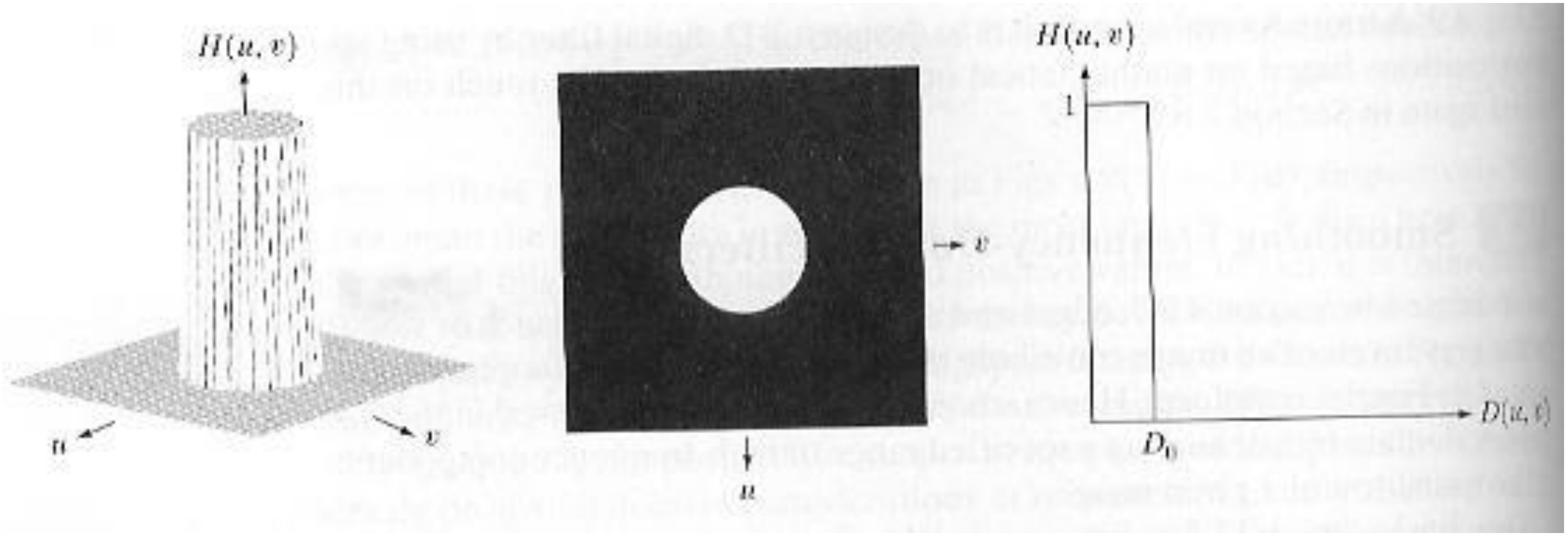
- Ideal Lowpass Filters

$$H(u,v) = \begin{cases} 1 & if \ D(u,v) \le D_0 \\ 0 & if \ D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2 \right]^{1/2}$$

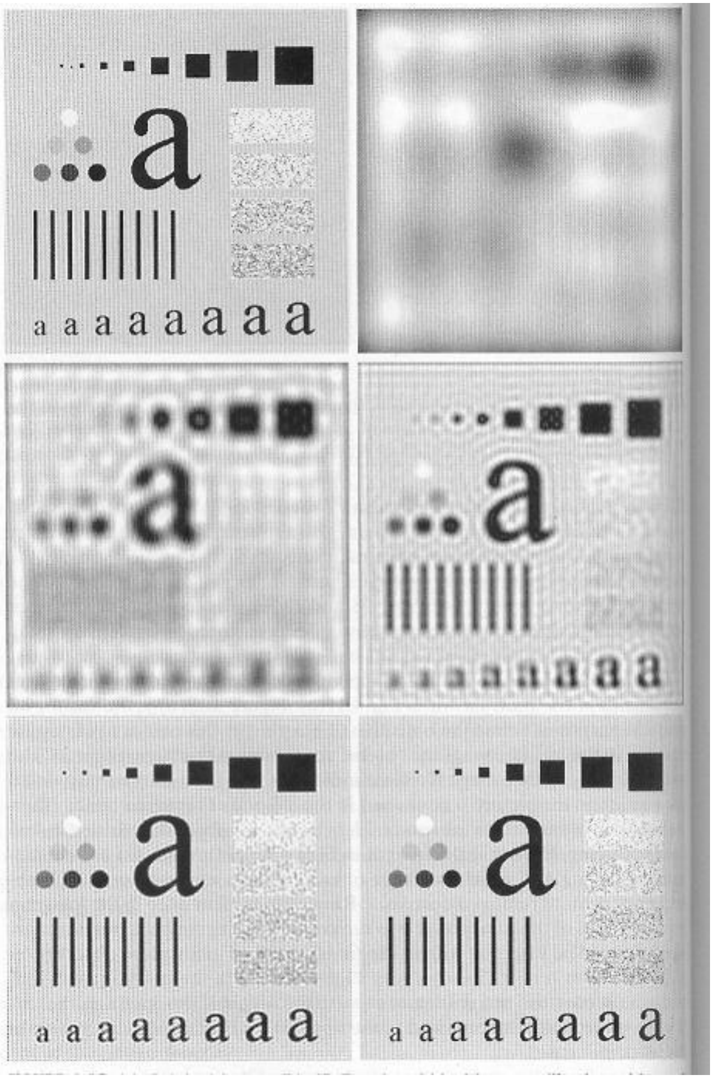
7.4.2. Smoothing Frequency Domain Filters

- Ideal Lowpass Filters



7.4.2. Smoothing Frequency Domain Filters

- Ideal Lowpass Filters



7.4.2. Smoothing Frequency Domain Filters

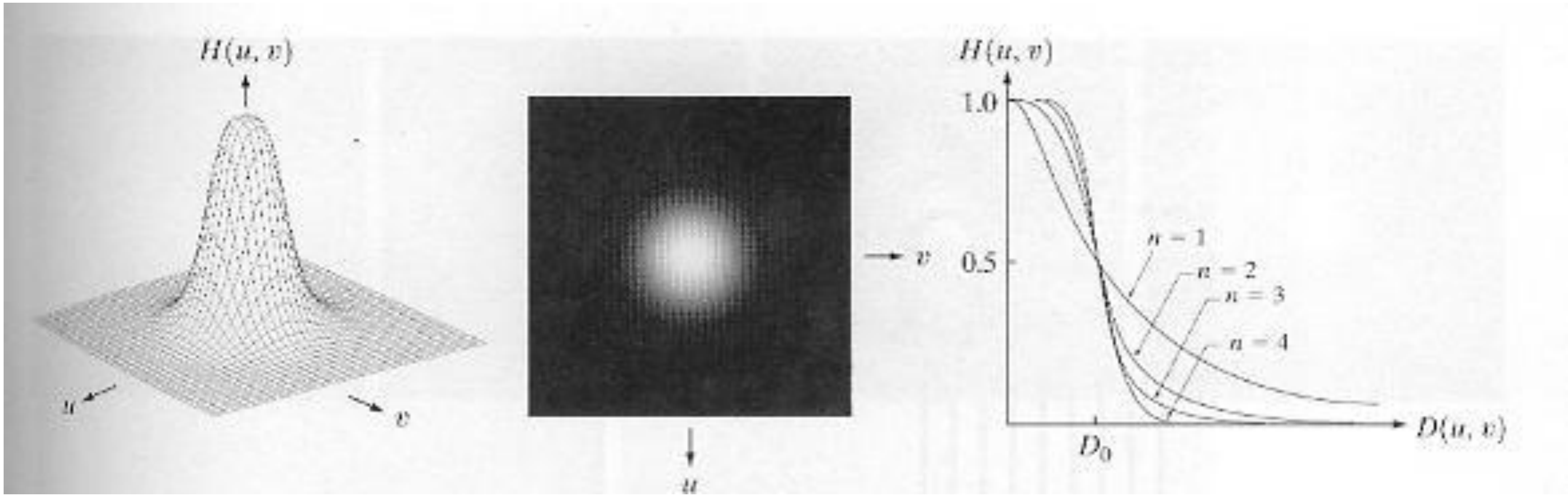
- Butterworth Lowpass Filters

$$H(u,v)=\frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

$$D(u,v)=\left[(u-M/2)^2+(v-N/2)^2\right]^{1/2}$$

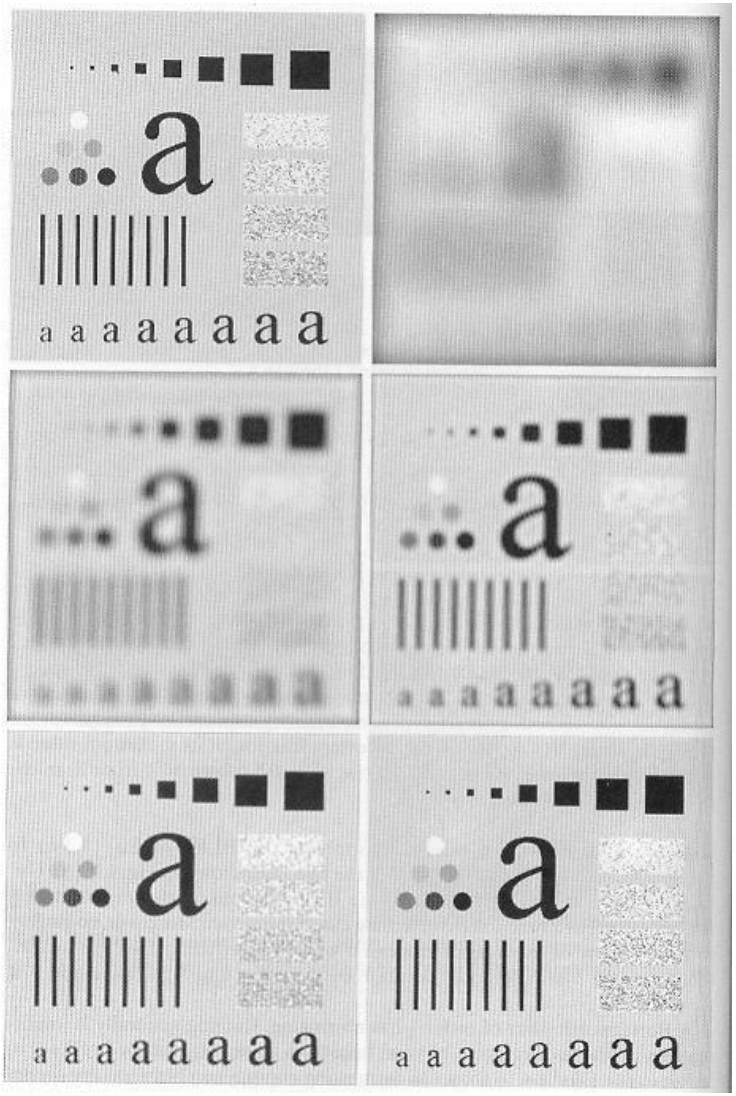
7.4.2. Smoothing Frequency Domain Filters

- Butterworth Lowpass Filters



7.4.2. Smoothing Frequency Domain Filters

- Butterworth Lowpass Filters



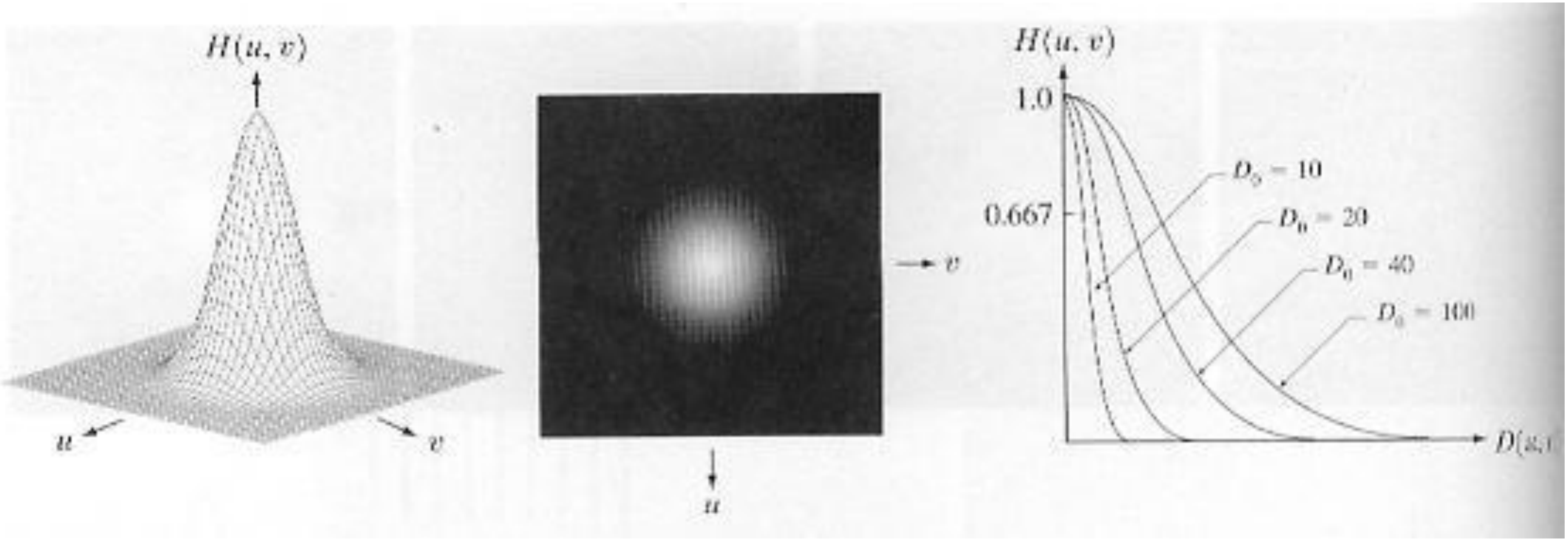
7.4.2. Smoothing Frequency Domain Filters

- Gaussian Lowpass Filters

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

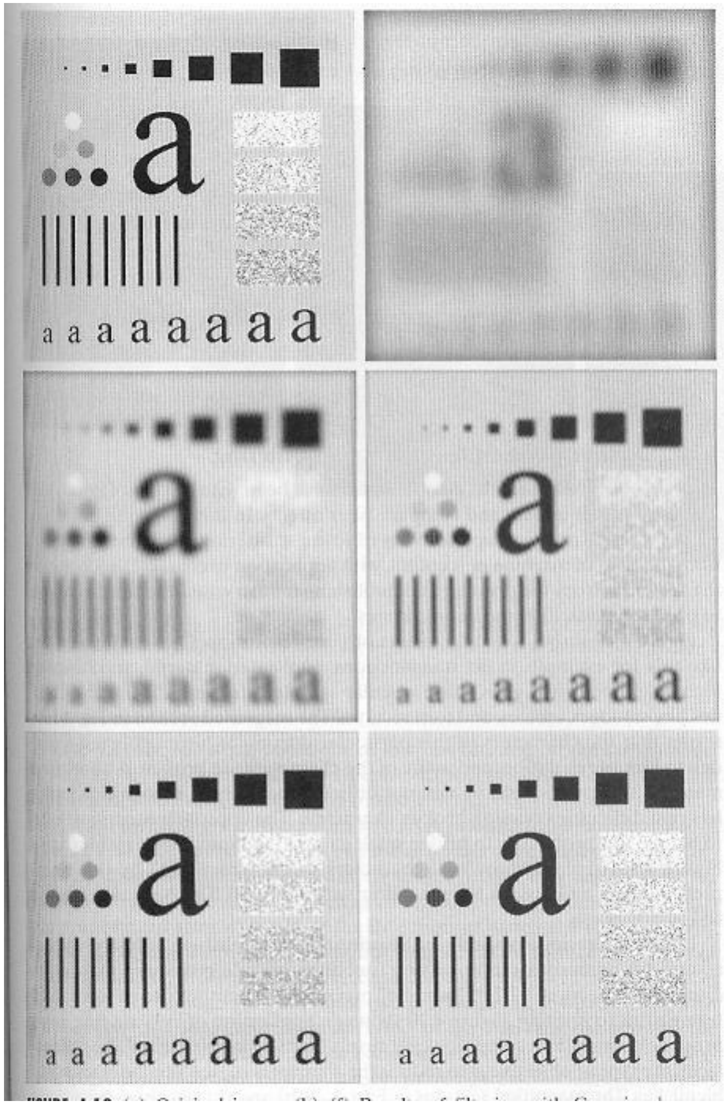
$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$

- Gaussian Lowpass Filters



7.4.2. Smoothing Frequency Domain Filters

- Gaussian Lowpass Filters



7.4.3. Sharpen Frequency Domain Filters

- Ideal Highpass Filters
- Butterworth Highpass Filters
- Gaussian Highpass Filters
- Laplacian in Frequency Domain

7.4.3. Sharpen Frequency Domain Filters

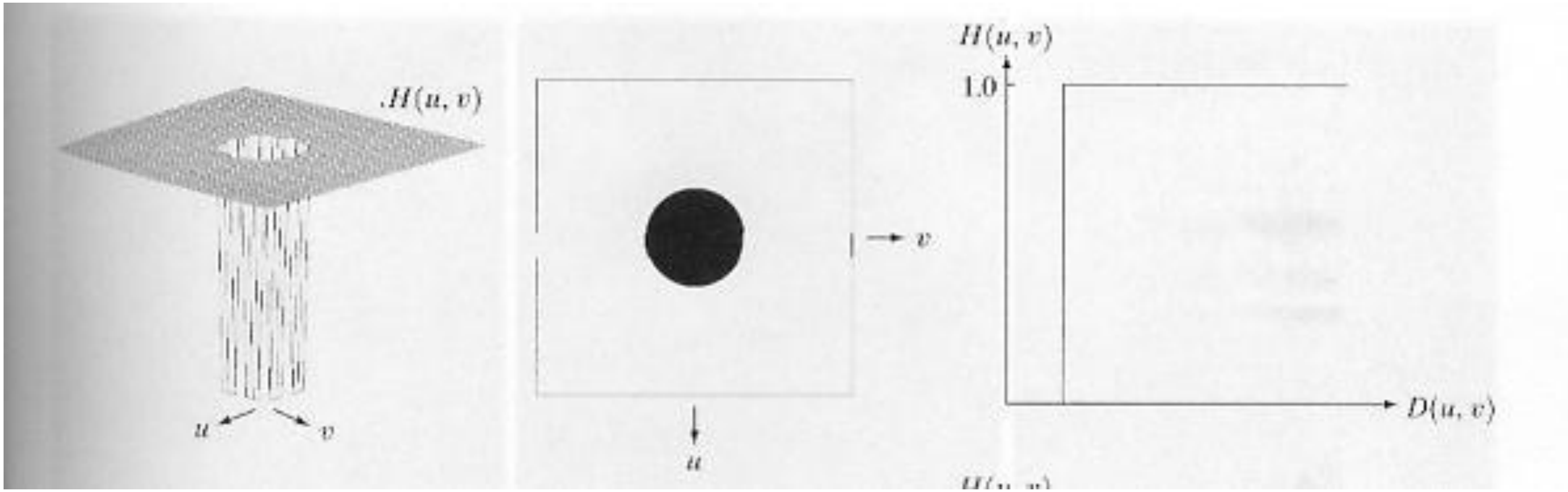
- Ideal Highpass Filters

$$H(u,v)=\begin{cases}0 & if\ D(u,v)\le D_0\\ 1 & if\ D(u,v)>D_0\end{cases}$$

$$D(u,v)=\left[(u-M/2)^2+(v-N/2)^2\right]^{1/2}$$

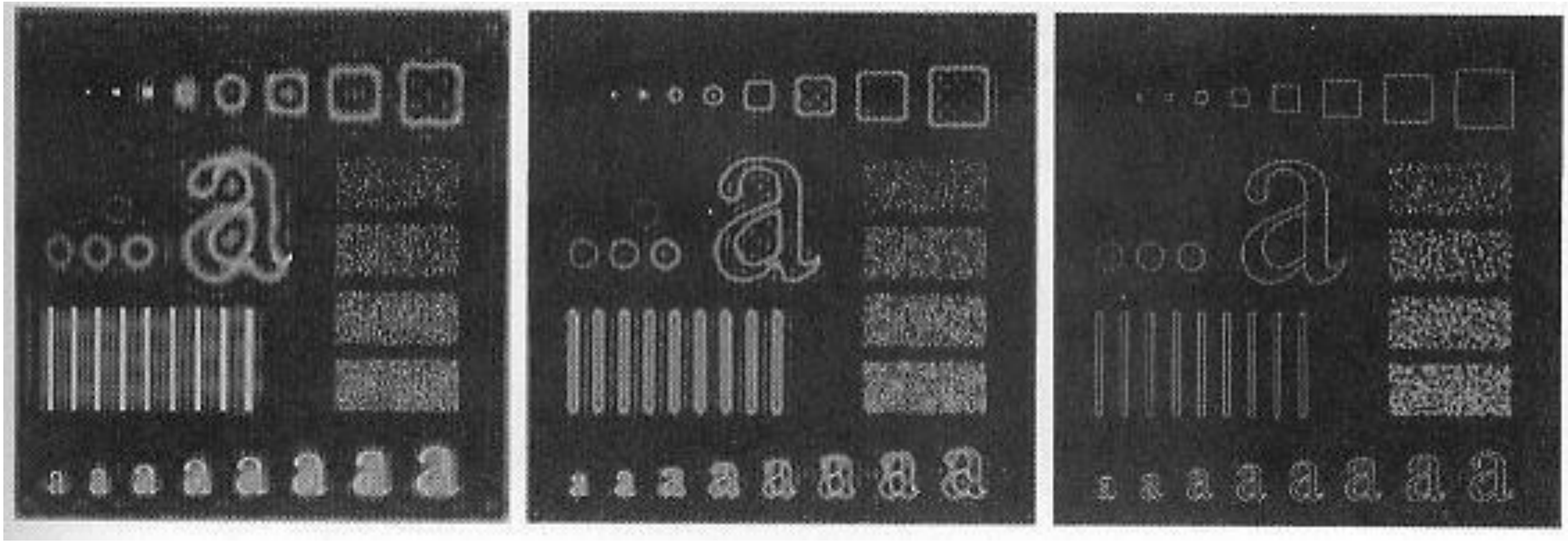
7.4.3. Sharpen Frequency Domain Filters

- Ideal Highpass Filters



7.4.3. Sharpen Frequency Domain Filters

- Ideal Highpass Filters



7.4.3. Sharpen Frequency Domain Filters

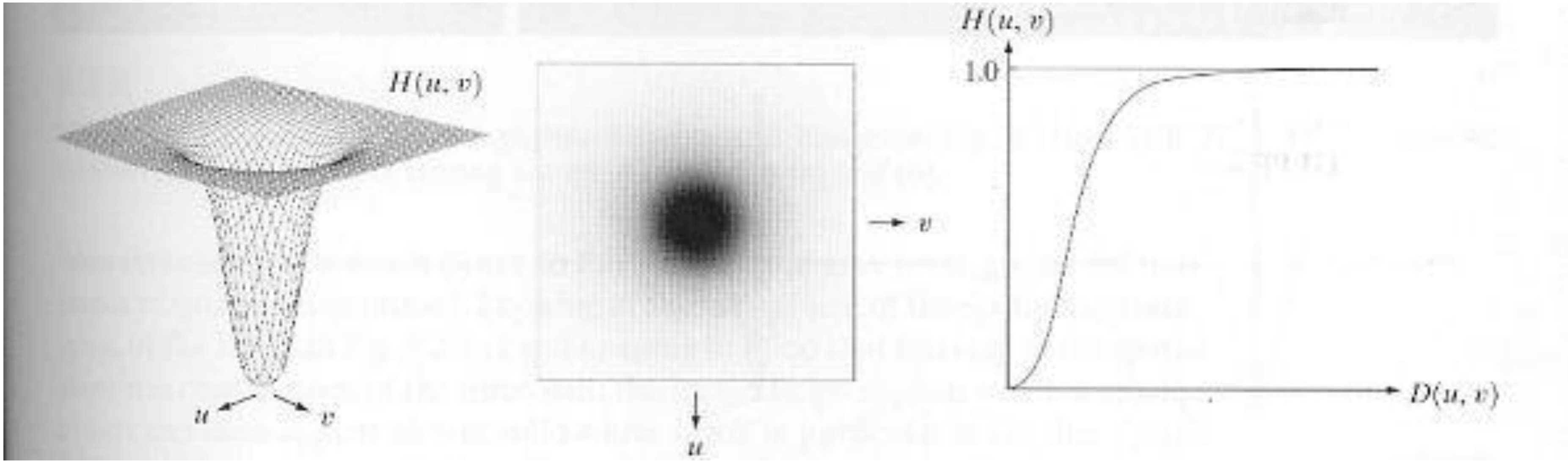
- Butterworth Highpass Filters

$$H(u,v) = \frac{1}{1 + \left[D_0 / D(u,v) \right]^{2n}}$$

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2 \right]^{1/2}$$

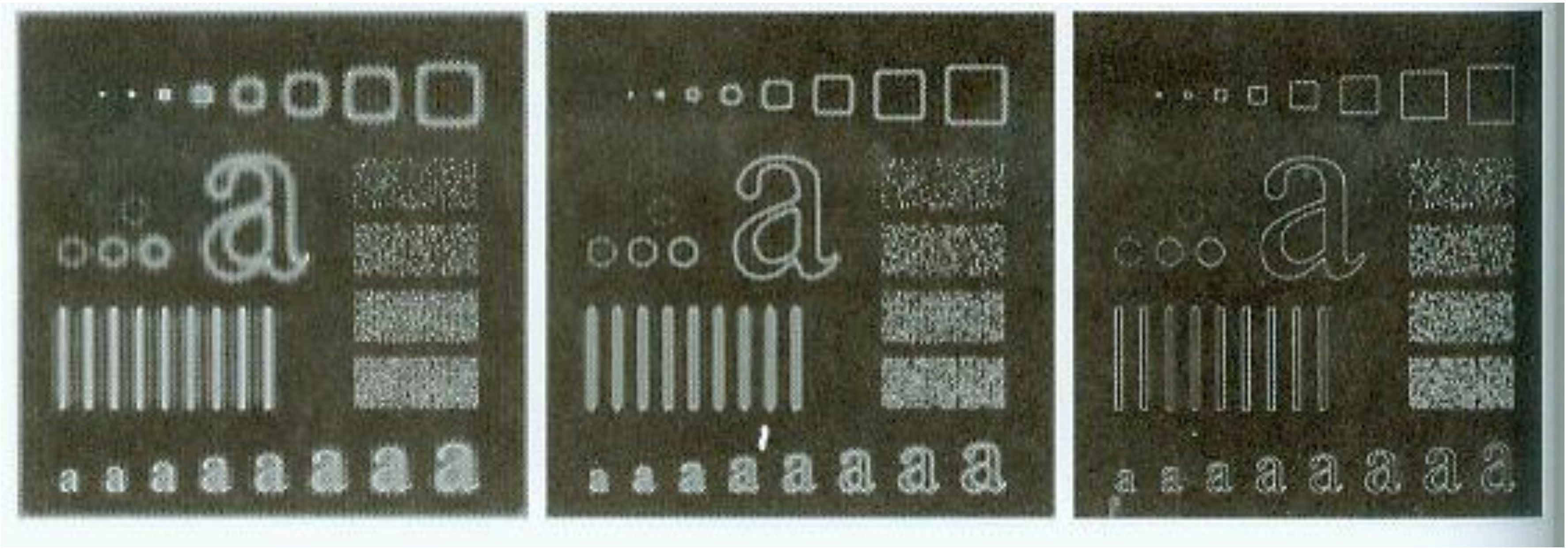
7.4.3. Sharpen Frequency Domain Filters

- Butterworth Highpass Filters



7.4.3. Sharpen Frequency Domain Filters

- Butterworth Highpass Filters



7.4.3. Sharpen Frequency Domain Filters

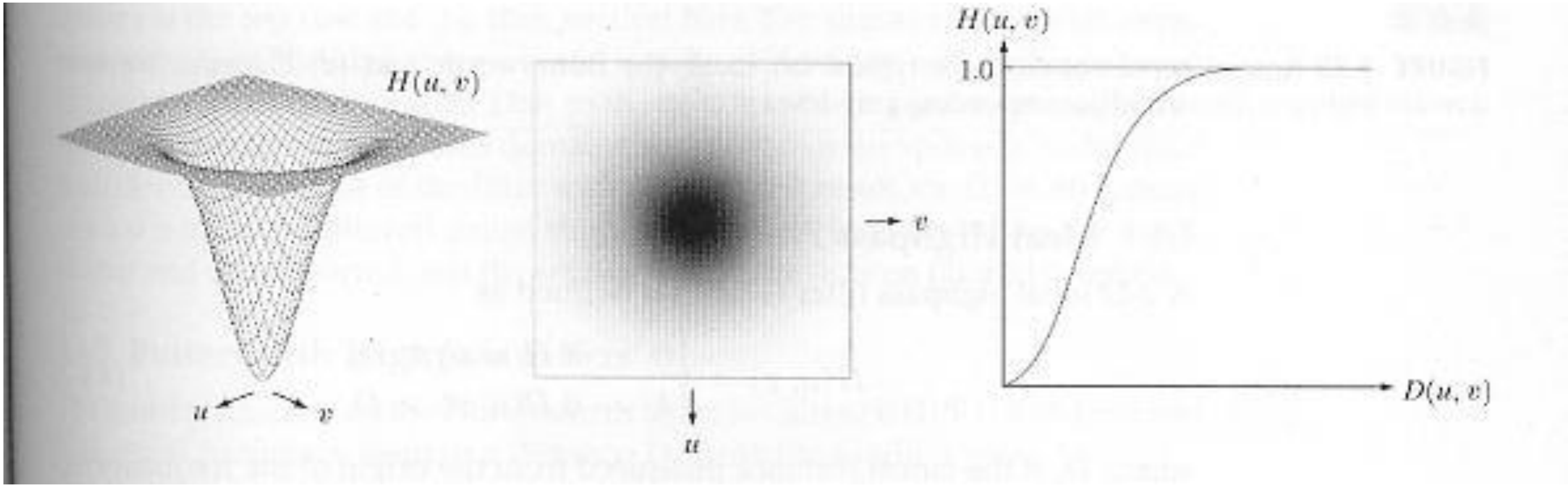
- Gaussian Highpass Filters

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^{\,2}}$$

$$D(u,v) = \left[(u - M/2)^2 + (v - N/2)^2\right]^{1/2}$$

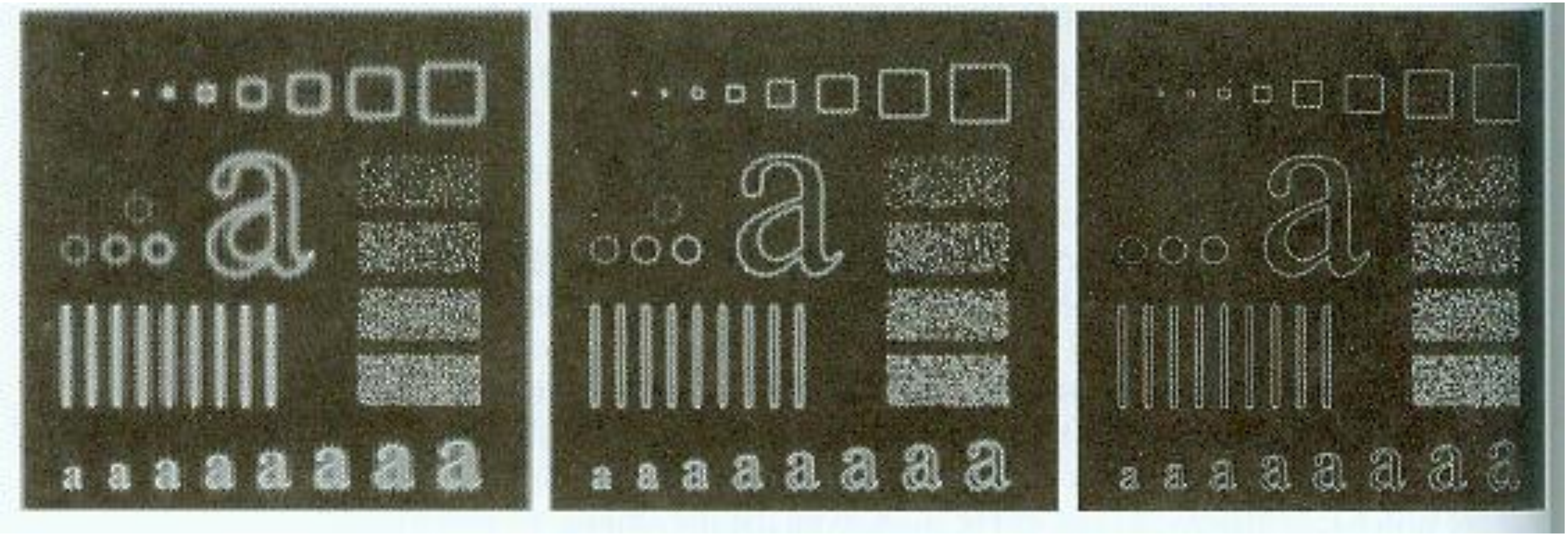
7.4.3. Sharpen Frequency Domain Filters

- Gaussian Highpass Filters



7.4.3. Sharpen Frequency Domain Filters

- Gaussian Highpass Filters



7.4.3. Sharpen Frequency Domain Filters

- Laplacian in Frequency Domain

$$\zeta\left[\nabla^2 f(x,y)\right] = -(u^2+v^2)F(u,v)$$

$$\nabla^2 f(x,y) = \zeta^{-1}\left[-(u^2+v^2)F(u,v)\right]$$