Option Pricing

Contents

1 Introduction

The purpose of this small note (or mini-project) is to demonstrate derivations and notions regarding option pricing. Various investigations and experiments are provided here. The intention is to broadly follow some of the exercises at the end of Mark Joshi's book on 'The Concepts and Practice of Mathematical Finance' whereby the coded projects in this file are heavily based on Mark Joshi's book on 'C++ design patterns and derivatives pricing'. In this pdf, the hope is to gather the theoretical background calculations used in the coded projects, i.e. the derivations of various analytic formulae for options of various types. These calculations are broadly based and adapted from Steven Shreve's second volume book on 'Stochastic Calculus for finance II'.

The main intention for this mini-project and all the code is that of self-education. As time (and my knowledge) moves on I hope to change/add to these.

2 Risk-Neutral Measures

In the cases that we consider, the only source of uncertainty comes from the stochasticity of the model of stock evolution over time. The differential of which is taken to be:

$$dS_t = S_t\left(\alpha_t dt + \sigma_t dW_t\right), \tag{1}$$

where dW_t is taken to be brownian motion which drives the stochasticity of the model. The solution is given by

$$S_t = S_0 e^{\int_0^t \sigma_s dW_s + \int_0^t \left(\alpha_s - \frac{1}{2}\sigma_s^2\right) ds}. \tag{2}$$

One then considers a risk neutral adaptive interest rate process:

$$D_t = e^{-\int_0^t R_s ds}, \tag{3}$$

in which it follows that

$$d(D_t S_t) = \sigma_t D_t S_t \left[\left(\frac{\alpha_t - R_t}{\sigma_t}\right) dt + dW_t\right]. \tag{4}$$

In accordance to Girsanovs theorem, we may write the l.h.s as a brownian motion in its own right: at the expense of changing probability space to $\tilde{\mathbb{P}}$, via an associated (but not important) Radon Nikodym factor. We thus get

$$dS_t = S_t \left(R_t dt + \sigma_t d\tilde{W}_t \right), \tag{5}$$

and $D_t S_t$ is now a martingale (which is the main point, as to guarantee that eq.4 is a martingale, we need to get rid of the time integral to leave an Ito integral).

We can solve and rewrite the stock evolution under this new probability measure.

$$S_{t_2} = S_{t_1} e^{\int_{t_1}^{t_2} \sigma_s d\tilde{W}_s + \int_{t_1}^{t_2} \left(R_s - \frac{1}{2}\sigma_s^2 \right) ds}. \tag{6}$$

In the case where the rate and volatility is constant, $\int_{t_1}^{t_2} \sigma_s d\tilde{W}_s = \sigma \left(\tilde{W_{t_2}} - \tilde{W_{t_1}} \right)$. Also, $\int_{t_1}^{t_2} \left(R_s - \frac{1}{2}\sigma_s^2 \right) ds = \left(R - \frac{1}{2}\sigma^2 \right)\tau$, where $\tau = t_2 - t_1$. It is useful to use the notation: $\sigma \left(\tilde{W_{t_2}} - \tilde{W_{t_1}} \right) = -\sigma\sqrt{\tau}y$, where $y = -\frac{(\tilde{W_{t_2}} - \tilde{W_{t_1}})}{\sqrt{\tau}}$. Thus in the case that we take volatility and the rate to be constants, we may write:

$$S_{t_2} = S_{t_1} e^{-\sigma\sqrt{\tau}y + \left(R - \frac{1}{2}\sigma^2 \right)\tau}. \tag{7}$$

2.0.1 The short hedge portfolio

We now consider a portfolio: X_0, whereby the investor buys Δ_0 shares whilst investing the rest into the money market: namely:

$$dX_t = \Delta_t dS_t + R_t (X_t - \Delta_t S_t) dt, \tag{8}$$

as a result we have $d(D_t X_t) = \Delta_t d(D_t S_t)$, thus $D_t X_t$ is a martingale.

3 Vanilla Option

We define a vanilla option to be one that has a fixed expiration time T and at that point has a pay off which is dependent on the value of the underlying asset and the proposed 'strike' K. If we defined the price of the contract to be V_t, where t refers to time, then a boundary condition at expiration is given by V_T which is determined by the principle condition of the option contract.

Following [3], we have our own wealth given by X_t, a risk-less rate discount $D_t = e^{-\int_0^t R_s ds}$, and the so-called risk-neutral expectation value $\tilde{\mathbb{E}}$, which is the usual map whose domain is in $\mathcal{F}$. We demand that at expiration we have enough to pay off the contract (a short hedge):

$$X_T = V_T, \tag{9}$$

from [3], it is known that $D_t X_t$ is a martingale under $\tilde{\mathbb{P}}$ associated to the filtration $\mathcal{F}_t$. As a result, we can write:

$$D_t X_t = \tilde{\mathbb{E}} \left[D_T X_T | \mathcal{F}_t \right]. \tag{10}$$

Since, $X_T = V_T$ and the fact that we consistently re-hedge $X_t = V_t$ for all t, we can now write:

$$D_t V_t = \tilde{\mathbb{E}} \left[D_T V_T | \mathcal{F}_t \right], \tag{11}$$

which implies that

$$V_t = \tilde{\mathbb{E}} \left[e^{-\int_t^T R_s ds} V_T | \mathcal{F}_t \right], \tag{12}$$

which is what we want to evaluate given the boundary conditions V_T.

3.1 European Options

European options are defined by a pay off attained depending on whether the underlying asset is more than or less than the strike price predefined in option contract. Simply:

$$V_T = \text{call option: } (S_T - K)^+ \text{or put option: } (K - S_T)^+ \tag{13}$$

3.1.1 Call Option

We wish to evaluate:

$$c_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(S_T - K)^+ | \mathcal{F}_t\right], \tag{14}$$

since τ is a time scale ahead of t, we can write this as an expectation value without the filtration. The condition correspondence is:

$$(S_T - K)^+ \implies y < d_-(K), \tag{15}$$

where

$$d_\pm(K) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{S_t}{K}\right) + \left(R \pm \frac{1}{2}\sigma^2\right)\tau\right]. \tag{16}$$

Thus, we wish to evaluate:

$$c_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(S_T - K)^+ | \mathcal{F}_t\right] = \tilde{\mathbb{E}}\left[e^{-R\tau}(S_T - K)^+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(K)} e^{-R\tau}\left(S_t e^{-\sigma\sqrt{\tau}y + \left(R - \frac{1}{2}\sigma^2\right)\tau} - K\right) e^{-\frac{y^2}{2}} dy. \tag{17}$$

This is a simple calculation which yields:

$$c_t = S_t N(d_+(K)) - e^{-R\tau} K N(d_- K), \tag{18}$$

where $N(x)$ is the cumulative distribution function for the normal distribution.

3.1.2 Put Option

We wish to evaluate:

$$p_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(K - S_T)^+ | \mathcal{F}_t\right], \tag{19}$$

the condition yields:

$$(K - S_T)^+ \implies y > d_-(K), \tag{20}$$

which leads to a similar result to the call option, only that we have an obvious global factor to (-1), in addition, since we have $y > d_-(K)$ as opposed an upper bound, we have:

$$p_t = K e^{-R\tau} N(-d_-(K)) - S_t N(-d_+(K)) \tag{21}$$

3.2 Digital Options

Digital Options (aka binary, all-or-nothing), are those which have a fixed value pay off depending on specific properties of the value of the underlying at expiration. In the case at hand we will just take the maximal pay off to be 1, and 0 is the option is never exercised. We will consider the pay off to be 1 is the value of the underlying sits between a two strike values, i.e. with $k_2 > k_1$:

$$\begin{aligned} V_T = \; & \text{call option: if } k_1 \leq S_T \leq k_2 \text{ then } V_T = 1, \text{ otherwise } 0, \\ & \text{put option: if } k_1 \leq S_T \leq k_2 \text{ then } V_T = 0, \text{ otherwise } 1. \end{aligned} \tag{22}$$

3.2.1 Call Option

Here we take

$$V_T = \mathcal{H}(S_T - k_1)\mathcal{H}(k_2 - S_T)\mathcal{H}(k_2 - k_1), \tag{23}$$

where $\mathcal{H}$ is the Heaviside step function. We thus wish to evaluate:

$$c_t = \tilde{\mathbb{E}}\left[e^{-R\tau}\mathcal{H}(S_T - k_1)\mathcal{H}(k_2 - S_T)\mathcal{H}(k_2 - k_1)|\mathcal{F}_t\right]. \tag{24}$$

The expiration conditions lead to $k_2 - S_T \geq 0 \implies y \geq d_-(k_2)$ and $S_T - k_1 \geq 0 \implies y \leq d_-(k_1)$, we therefore have that

$$\tilde{\mathbb{E}}\left[e^{-R\tau}\mathcal{H}(S_T - k_1)\mathcal{H}(k_2 - S_T)\mathcal{H}(k_2 - k_1)\right] = \mathcal{H}(k_2 - k_1)e^{-R\tau}\int_{d_-(k_2)}^{d_-(k_1)} e^{-\frac{y^2}{2}}\,dy. \tag{25}$$

Note that if we take $d_-(k_1) \geq d_-(k_2)$ then since the strikes are within the log arguments of the corresponding functions, we should have $k_2 \geq k_1$, therefore the heaviside pre-factor is trivially satisfied, and we have

$$e^{-R\tau}\int_{d_-(k_2)}^{d_-(k_1)} e^{-\frac{y^2}{2}}\,dy = e^{-R\tau}\left(N(d_-(k_1)) - N(d_-(k_2))\right), \tag{26}$$

and so

$$c_t = e^{-R\tau}\left(N(d_-(k_1)) - N(d_-(k_2))\right). \tag{27}$$

3.2.2 Put Option

For the put, we wish to evaluate:

$$p_t = \tilde{\mathbb{E}}\left[e^{-R\tau}\mathcal{H}(S_T - k_2)\mathcal{H}(k_1 - S_T)\mathcal{H}(k_2 - k_1)|\mathcal{F}_t\right]. \tag{28}$$

The expiration conditions lead to $k_1 - S_T \geq 0 \implies y \geq d_-(k_1)$ and $S_T - k_2 \geq 0 \implies y \leq d_-(k_2)$, we therefore have that

$$\tilde{\mathbb{E}}\left[e^{-R\tau}\mathcal{H}(S_T - k_2)\mathcal{H}(k_1 - S_T)\mathcal{H}(k_2 - k_1)\right] = \mathcal{H}(k_2 - k_1)e^{-R\tau}\int_{d_-(k_1)}^{d_-(k_2)} e^{-\frac{y^2}{2}}\,dy. \tag{29}$$

Contrary to the call option, the heaviside step function is not trivially satisfied. The way out is that the integral in the call option case was well defined window in normally distributed random variables, the complement of this space (which is where the put option is valid) is nothing more than the window taken away from the entire probability space. We can therefore write:

$$\mathcal{H}(k_2 - k_1)e^{-R\tau}\int_{d_-(k_1)}^{d_-(k_2)} e^{-\frac{y^2}{2}}\,dy = \mathcal{H}(k_2 - k_1)e^{-R\tau}\left(1 - \int_{d_-(k_2)}^{d_-(k_1)} e^{-\frac{y^2}{2}}\,dy\right) = e^{-R\tau}\left(1 - N(d_-(k_1)) + N(d_-(k_2))\right), \tag{30}$$

and so

$$p_t = e^{-R\tau}\left(1 - N(d_-(k_1)) + N(d_-(k_2))\right). \tag{31}$$

3.3 Power Options

Power options are similar to European options, only that the execution of the option depends non-linearly on the underlying. Simply:

$$V_T = \text{call option: } (S_T^p - K)^+ \text{or put option: } (K - S_T^p)^+, \tag{32}$$

where p is some power. For all results to come, we should arrive back at the European option when we set p to one.

3.3.1 Call Option

We wish to evaluate:

$$c_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(S_T^p - K)^+ | \mathcal{F}_t\right], \tag{33}$$

The condition correspondence is:

$$(S_T^p - K)^+ \implies y < d_-(K) - \frac{1-p}{p\sigma\sqrt{\tau}}\log(K), \tag{34}$$

Thus, we wish to evaluate:

$$c_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(S_T^p - K)^+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(K)+\frac{p-1}{p\sigma\sqrt{\tau}}\log(K)} e^{-R\tau}\left(S_t^p e^{-p\sigma\sqrt{\tau}y+p\left(R-\frac{1}{2}\sigma^2\right)\tau} - K\right)e^{-\frac{y^2}{2}}dy, \tag{35}$$

which can be evaluated with relatively little trouble to give:

$$c_t = S_t^p e^{\frac{1}{2}\tau(p-1)(2R+p\sigma^2)} N\left(d_-(K) + \frac{p-1}{p\sigma\sqrt{\tau}}\log(K) + p\sigma\sqrt{\tau}\right) - e^{-R\tau}KN\left(d_-(K) + \frac{p-1}{p\sigma\sqrt{\tau}}\log(K)\right). \tag{36}$$

Note that when we take p to 1, we recover our original European call option.

3.3.2 Put Option

We wish to evaluate:

$$p_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(K - S_T^p)^+ | \mathcal{F}_t\right], \tag{37}$$

The condition correspondence is:

$$(K - S_T^p)^+ \implies y > d_-(K) - \frac{1-p}{p\sigma\sqrt{\tau}}\log(K), \tag{38}$$

Thus, we wish to evaluate:

$$p_t = \tilde{\mathbb{E}}\left[e^{-R\tau}(K - S_T^p)^+\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\left(d_-(K)+\frac{p-1}{p\sigma\sqrt{\tau}}\log(K)\right)} e^{-R\tau}\left(K - S_t^p e^{-p\sigma\sqrt{\tau}y+p\left(R-\frac{1}{2}\sigma^2\right)\tau}\right)e^{-\frac{y^2}{2}}dy, \tag{39}$$

which gives

$$p_t = e^{-R\tau}KN\left(-\left(d_-(K) + \frac{p-1}{p\sigma\sqrt{\tau}}\log(K)\right)\right) - S_t^p e^{\frac{1}{2}\tau(p-1)(2R+p\sigma^2)} N\left(-\left(d_-(K) + \frac{p-1}{p\sigma\sqrt{\tau}}\log(K) + p\sigma\sqrt{\tau}\right)\right). \tag{40}$$

4 Investigation 1: Vega in various situations

Vega is a greek metric, defined as

$$\nu = \frac{\partial V}{\partial \sigma}. \tag{41}$$

In the Option Pricing c++ project in the attached files, we have two pricing tools. The first is the analytic forms that have been derived in this document, and we also have a monte carlo pricer which is built by finding the numerical expectations values of various options. Using these two pricers, we probe their abilities to find vega for various expirations. These allows us to understand ν, but also compare how the monte carlo works against the true analytic result.

Firstly we consider the european call options, with spot 100, strike 110 and interest rate 0.01. ν is plotted in fig.1. This can be compared with the same result coming from the Monte Carlo pricer, which is found in in fig.2. The main distinguishing feature is that apparent failure of the Monte Carlo pricer for large expiration times and volatilities.

Next we consider ν for the digital call option for various expirations. Similarly to the previous case, we take the interest rate to be 0.01, whilst the option is only exercised if the the spot (initialised at 100) evolves between 95 and 110. fig.3 shows various expirations against volatility and price.

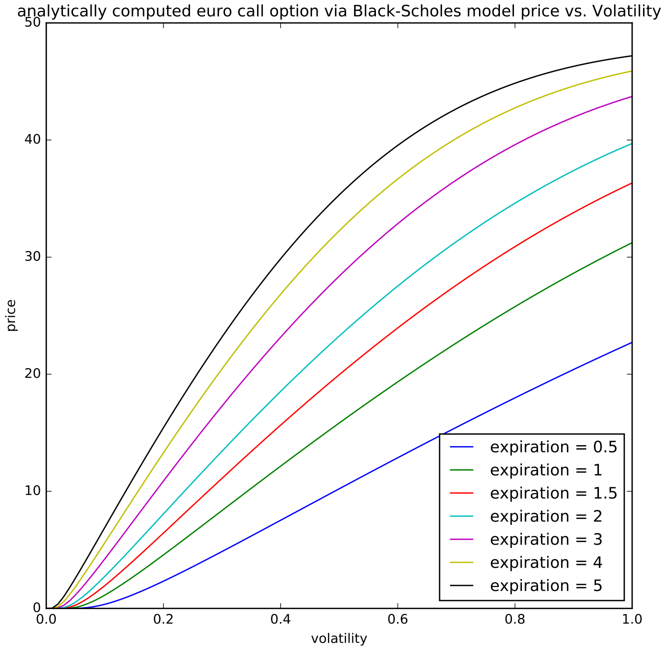

Figure 1: Vega for European Call option for various expirations via the analytic form.

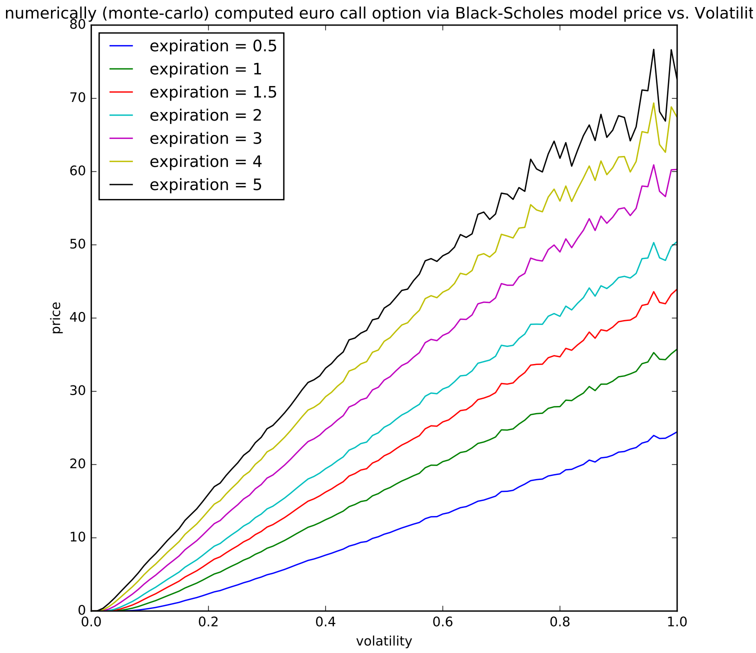

Figure 2: Vega for European Call option for various expirations via the MC pricer.

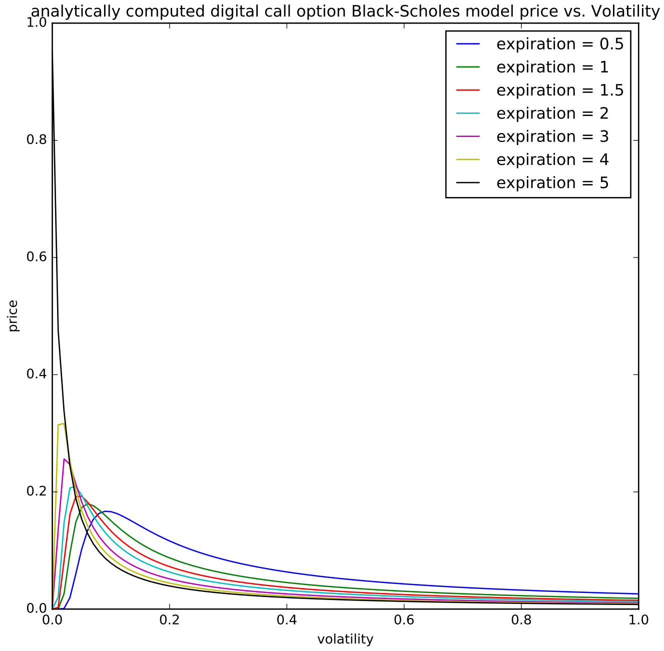

Figure 3: Vega for Digital Call option for various expirations via the MC pricer.

References

[1] Mark Joshi, “The Concepts and Practice of Mathematical Finance,”

[2] Mark Joshi “C++ design patterns and derivatives pricing,”

[3] Steven Shreve, “Stochastic Calculus for finance 2: Continuous-Time Methods,”