$N = 4$

$k = 0, 1, 2, 3$

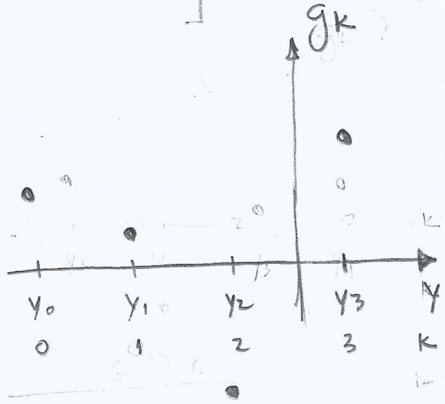
$n = -2, 1, 0, 1$

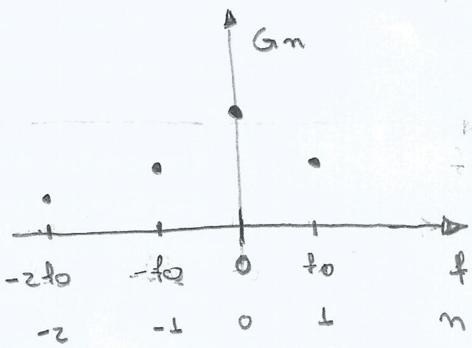
$$
\begin{array}{c c}
 & k \\
n & \begin{bmatrix} -20 & -21 & -22 & -23 \\ -10 & -11 & -12 & -13 \\ 00 & 01 & 02 & 03 \\ 10 & 11 & 12 & 13 \end{bmatrix}
\end{array}
$$

nk

eq 45

$$
\begin{array}{c}
G_n \\
\begin{matrix} f_n<0 \\ \\ \\ f_n>0 \end{matrix}
\begin{bmatrix} G_{-2} \\ G_{-1} \\ G_0 \\ G_1 \end{bmatrix}
\end{array}
=
\begin{array}{c}
\omega_N^{nk} \\
\begin{bmatrix} \omega_N^{0} & \omega_N^{-2} & \omega_N^{-4} & \omega_N^{-6} \\ \omega_N^{0} & \omega_N^{-1} & \omega_N^{-2} & \omega_N^{-3} \\ \omega_N^{0} & \omega_N^{0} & \omega_N^{0} & \omega_N^{0} \\ \omega_N^{0} & \omega_N^{1} & \omega_N^{2} & \omega_N^{3} \end{bmatrix}
\end{array}
\begin{array}{c}
g_k \\
\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix}
\end{array}
$$





$$\omega_N = e^{-i2\pi/N}$$

$$\omega_N^{(n+N)k} = \cos\left[2\pi(n+N)k/N\right] - i\sin\left[2\pi(n+N)k/N\right]$$

$$= \cos(2\pi nk/N)\underbrace{\cos(2\pi Nk/N)}_{1} - \sin(2\pi nk/N)\underbrace{\sin(2\pi Nk/N)}_{0} +$$

$$- i\sin(2\pi nk/N)\underbrace{\cos(2\pi Nk/N)}_{1} - i\underbrace{\sin(2\pi Nk/N)}_{0}\cos(2\pi nk/N)$$

$$= \cos(2\pi nk/N) - i\sin(2\pi nk/N)$$

$$= \underline{\omega_N^{nk}}$$

$n = -2, -1, 0, 1$

$\Downarrow$

$n = 2, 3, 0, 1$

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} 20 & 21 & 22 & 23 \\ 30 & 31 & 32 & 33 \\ 00 & 01 & 02 & 03 \\ 10 & 11 & 12 & 13 \end{bmatrix} \end{array}$$

$$\boxed{W_N^{nk} = W_N^{kn}} = \cos(2\pi nk/N) - i\sin(2\pi kn/N)$$

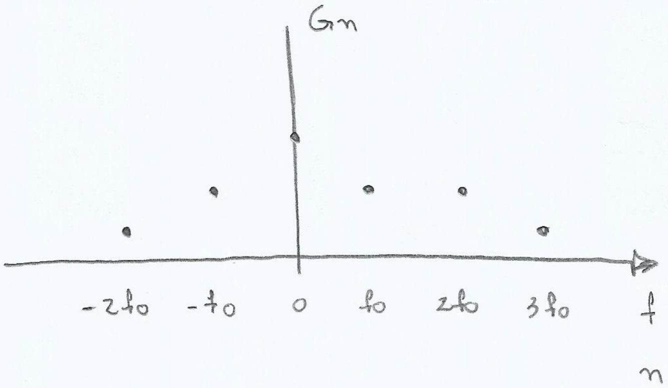
$$\overbrace{\begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \end{bmatrix}}^{\bar{G}} = \underbrace{\begin{bmatrix} \omega_N^0 & \omega_N^0 & \omega_N^0 & \omega_N^0 \\ \omega_N^0 & \omega_N^1 & \omega_N^2 & \omega_N^3 \\ \omega_N^0 & \omega_N^2 & \omega_N^4 & \omega_N^6 \\ \omega_N^0 & \omega_N^3 & \omega_N^6 & \omega_N^9 \end{bmatrix}}_{\bar{\bar{W}}_N} \underbrace{\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix}}_{\bar{g}}$$

$f_n > 0$: G_1

$f_n < 0$: G_2, G_3

$G_2 = G_{-2}$

$G_3 = G_{-1}$



2

eq 46a

$$
\begin{array}{cc}
 & n \\
k & \begin{bmatrix} 0(-2) & 0(-1) & 00 & 01 \\ 1(-2) & 1(-1) & 10 & 11 \\ 2(-2) & 2(-1) & 20 & 21 \\ 3(-2) & 3(-1) & 30 & 31 \end{bmatrix}
\end{array}
\quad kn
$$

g_k $\qquad$ $\tilde{\omega}_N^{nk}$ $\qquad$ G_n

$$
\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} = \frac{1}{N}
\begin{bmatrix}
\tilde{\omega}_N^{0} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{0} \\
\tilde{\omega}_N^{-2} & \tilde{\omega}_N^{-1} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{1} \\
\tilde{\omega}_N^{-4} & \tilde{\omega}_N^{-2} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{2} \\
\tilde{\omega}_N^{-6} & \tilde{\omega}_N^{-3} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{3}
\end{bmatrix}
\begin{bmatrix} G_{-2} \\ G_{-1} \\ G_0 \\ G_1 \end{bmatrix}
$$

$$\tilde{\omega}_N^{(n+N)k} = \tilde{\omega}_N^{nk}$$

$$
\underbrace{\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix}}_{\bar{g}} = \frac{1}{N}
\underbrace{\begin{bmatrix}
\tilde{\omega}_N^{0} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{0} & \tilde{\omega}_N^{0} \\
\tilde{\omega}_N^{0} & \tilde{\omega}_N^{1} & \tilde{\omega}_N^{2} & \tilde{\omega}_N^{3} \\
\tilde{\omega}_N^{0} & \tilde{\omega}_N^{2} & \tilde{\omega}_N^{4} & \tilde{\omega}_N^{6} \\
\tilde{\omega}_N^{0} & \tilde{\omega}_N^{3} & \tilde{\omega}_N^{6} & \tilde{\omega}_N^{9}
\end{bmatrix}}_{\tilde{\bar{\bar{W}}}_N}
\underbrace{\begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \end{bmatrix}}_{\bar{G}}
\begin{array}{l} \left.\begin{array}{l} \\ \end{array}\right\} f_n > 0 \\ \left.\begin{array}{l} \\ \end{array}\right\} f_n < 0 \end{array}
$$

eq 46b

$$x_j = j\,\Delta x \quad , \quad j = 0, 1, \dots, M-1$$

$$y_K = k\Delta y \quad , \quad k = 0, 1, \dots, N-1$$

$$u_m = \frac{m}{M\,\Delta x} \quad , \quad m = -\frac{M}{2}, \dots, \frac{M}{2} - 1$$

$$v_n = \frac{n}{N\,\Delta y} \quad , \quad n = -\frac{N}{2}, \dots, \frac{N}{2} - 1$$

$$G(u_m, v_n) \approx \sum_{j=0}^{M-1} \sum_{K=0}^{N-1} g(x_j, y_K)\, e^{-i2\pi(u_m x_j + v_n y_K)} \Delta x \Delta y$$

$$\approx \Delta x \Delta y \sum_{j=0}^{M-1} \sum_{K=0}^{N-1} g(x_j, y_K)\, e^{-i2\pi\left(\frac{m}{N_x \cancel{\Delta x}} j \cancel{\Delta x} + \frac{n}{N_y \cancel{\Delta y}} K \cancel{\Delta y}\right)}$$

$$\approx \Delta x \Delta y \sum_{j=0}^{M-1} \sum_{K=0}^{N-1} g(x_j, y_K)\, e^{-i2\pi m j/M}\; e^{-i2\pi n K/N}$$

$$G_{mn} \cong \sum_{j=0}^{M-1} \left(\sum_{K=0}^{N-1} g_{jK}\, e^{-i2\pi K n/N} \right) e^{-i2\pi m j/M}$$

$$\uparrow$$

$$\frac{G(u_m, v_n)}{\Delta x \Delta y}$$

eq 57b

4

$$g_{jk} \qquad g(x_j, y_k) \approx \sum_{m=-N_x/2}^{M/2-1} \sum_{n=-N_y/2}^{N/2-1} G(u_m, \vartheta_n)\, e^{i2\pi(u_m x_j + \vartheta_n y_k)} \qquad \frac{1}{M\Delta x} \; \frac{1}{N\Delta y}$$

$$\approx \frac{1}{MN} \sum_{m=-\frac{M}{2}}^{M/2-1} \sum_{n=-N/2}^{N/2-1} \frac{G(u_m, \vartheta_n)}{\Delta x \Delta y}\, e^{i2\pi mj/M}\, e^{i2\pi nk/N}$$

$$\approx \frac{1}{M\,N} \sum_{m=-\frac{M}{2}}^{M/2-1} \left(\sum_{n=-\frac{N}{2}}^{N/2-1} G_{mn}\, e^{i2\pi nk/N} \right) e^{i2\pi mj/M}$$

eq. 59a

$$\overline{\overline{G}} = \overline{\overline{W}}_M \, \overline{\overline{g}} \, \overline{\overline{W}}_N \qquad \text{eq 58}$$

$$\overline{\overline{g}} = \frac{1}{MN} \tilde{\overline{\overline{W}}}_M \, \overline{\overline{G}} \, \tilde{\overline{\overline{W}}}_N \qquad \text{eq 60}$$

$N=4$, $m=4$

$$\overline{\overline{g}} = \begin{array}{cc} \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} & j \\ k & \end{array} \qquad \overline{\overline{G}} = \begin{array}{cc} \begin{bmatrix} G_{-2(-2)} & G_{-2(-1)} & G_{-20} & G_{-21} \\ G_{-1(-2)} & G_{-1(-1)} & G_{-10} & G_{-11} \\ G_{0(-2)} & G_{0(-1)} & G_{00} & G_{01} \\ G_{1(-2)} & G_{1(-1)} & G_{10} & G_{11} \end{bmatrix} & m \\ n & \end{array}$$

$$G_{mn} = \sum_{j=0}^{M-1} \left(\sum_{k=0}^{N-1} g_{jk} e^{\overbrace{-i2\pi kn/N}^{\omega_N^{kn}}} \right) e^{-i2\pi mj/M} \quad \omega_M^{mj}$$

$$= \sum_{j=0}^{M-1} \left(\underbrace{\begin{bmatrix} g_{j0} & \cdots & g_{j(N-1)} \end{bmatrix}}_{\bar{g}_j} \underbrace{\begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)n} \end{bmatrix}}_{\omega_N} \right) \omega_M^{-mj}$$

$$= \omega_M^{m0} \begin{bmatrix} g_{00} & \cdots & g_{0(N-1)} \end{bmatrix} \begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)n} \end{bmatrix} +$$

$$+ \omega_M^{m1} \begin{bmatrix} g_{10} & \cdots & g_{1(N-1)} \end{bmatrix} \begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)n} \end{bmatrix} +$$

$$\vdots$$

$$+ \omega_M^{m(M-1)} \begin{bmatrix} g_{(M-1)0} & \cdots & g_{(M-1)(N-1)} \end{bmatrix} \begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)n} \end{bmatrix}$$

$$G_{mn} = \begin{bmatrix} \omega_M^{m0} & \cdots & \omega_M^{m(M-1)} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} g_{00} & \cdots & g_{0(N-1)} \end{bmatrix} \begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} g_{(M-1)0} & \cdots & g_{(M-1)(N-1)} \end{bmatrix} \begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)m} \end{bmatrix} \end{bmatrix}$$

$$G_{mn} = \left[\omega_M^{m0} \cdots \omega_M^{m(M-1)}\right] \bar{\bar{g}} \begin{bmatrix} \omega_N^{0n} \\ \vdots \\ \omega_N^{(N-1)n} \end{bmatrix}$$

$$G_{-2(-2)} = \left[\omega_M^{-20}\ \ \omega_M^{-21}\ \ \omega_M^{-20}\ \ \omega_M^{-21}\right] \bar{\bar{g}} \begin{bmatrix} \omega_N^{0(-2)} \\ \omega_N^{1(-2)} \\ \omega_N^{2(-2)} \\ \omega_N^{3(-2)} \end{bmatrix}$$

$$\omega_M^{(m+M)j} = \omega_M^{mj} \qquad \omega_N^{(k+N)n} = \omega_N^{kn}$$

$$G_{22} = \left[\omega_M^{20}\ \ \omega_M^{21}\ \ \omega_M^{20}\ \ \omega_M^{21}\right] \bar{\bar{g}} \begin{bmatrix} \omega_N^{02} \\ \omega_N^{12} \\ \omega_N^{22} \\ \omega_N^{32} \end{bmatrix}$$

$$\begin{array}{c} \ominus\ominus \qquad\qquad \ominus\oplus \\ \left[\begin{array}{cc|cc} G_{-2(-2)} & G_{-2(-1)} & G_{-20} & G_{-21} \\ G_{-1(-2)} & G_{-1(-1)} & G_{-10} & G_{-11} \\ \hline G_{0(-2)} & G_{0(-1)} & G_{00} & G_{01} \\ G_{1(-2)} & G_{1(-1)} & G_{10} & G_{11} \end{array}\right] \\ \oplus\ominus \qquad\qquad \oplus\oplus \end{array} =$$

$$= \left[\begin{array}{cc|cc} \omega_M^{-20} & \omega_M^{-21} & \omega_M^{-22} & \omega_M^{-23} \\ \omega_M^{-10} & \omega_M^{-11} & \omega_M^{-12} & \omega_M^{-13} \\ \hline \omega_M^{00} & \omega_M^{01} & \omega_M^{02} & \omega_M^{03} \\ \omega_M^{10} & \omega_M^{11} & \omega_M^{12} & \omega_M^{13} \end{array}\right] \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} \left[\begin{array}{cc|cc} \omega_N^{0(-2)} & \omega_N^{0(-1)} & \omega_N^{00} & \omega_N^{01} \\ \omega_N^{1(-2)} & \omega_N^{1(-1)} & \omega_N^{10} & \omega_N^{11} \\ \hline \omega_N^{2(-2)} & \omega_N^{2(-1)} & \omega_N^{20} & \omega_N^{21} \\ \omega_N^{3(-2)} & \omega_N^{3(-1)} & \omega_N^{30} & \omega_N^{31} \end{array}\right]$$

$$
\left[\begin{array}{cc|cc}
G_{22} & G_{23} & G_{20} & G_{21} \\
G_{32} & G_{33} & G_{30} & G_{31} \\
\hline
G_{02} & G_{03} & G_{00} & G_{01} \\
G_{12} & G_{13} & G_{10} & G_{11}
\end{array}\right] =
$$

$$
= \begin{bmatrix}
\omega_M^{20} & \omega_M^{21} & \omega_M^{22} & \omega_M^{23} \\
\omega_M^{30} & \omega_M^{31} & \omega_M^{32} & \omega_M^{33} \\
\omega_M^{00} & \omega_M^{01} & \omega_M^{02} & \omega_M^{03} \\
\omega_M^{10} & \omega_M^{11} & \omega_M^{12} & \omega_M^{13}
\end{bmatrix}
\begin{bmatrix}
g_{00} & & \\
& \cdots & \\
& &
\end{bmatrix}
\begin{bmatrix}
\omega_N^{02} & \omega_N^{03} & \omega_N^{00} & \omega_N^{01} \\
\omega_N^{12} & \omega_N^{13} & \omega_N^{10} & \omega_N^{11} \\
\omega_N^{22} & \omega_N^{23} & \omega_N^{20} & \omega_N^{21} \\
\omega_N^{32} & \omega_N^{33} & \omega_N^{30} & \omega_N^{31}
\end{bmatrix}
$$

$$
\bar{\bar{G}} = \bar{\bar{W}}_M \, \bar{\bar{g}} \, \bar{\bar{W}}_N
$$

$$
\bar{\bar{g}} = \frac{1}{MN}
\left[\begin{array}{cc|cc}
\tilde{\omega}_M^{0(-2)} & \tilde{\omega}_M^{0(-1)} & \tilde{\omega}_M^{00} & \tilde{\omega}_M^{01} \\
\tilde{\omega}_M^{1(-2)} & \tilde{\omega}_M^{1(-1)} & \tilde{\omega}_M^{10} & \tilde{\omega}_N^{11} \\
\hline
\tilde{\omega}_M^{2(-2)} & \tilde{\omega}_M^{2(-1)} & \tilde{\omega}_M^{20} & \tilde{\omega}_M^{21} \\
\tilde{\omega}_M^{3(-2)} & \tilde{\omega}_M^{3(-1)} & \tilde{\omega}_M^{30} & \tilde{\omega}_M^{31}
\end{array}\right]
\left[\begin{array}{cc|cc}
G_{(-2)(-2)} & G_{(-2)(-1)} & G_{(-2)0} & G_{(-2)1} \\
G_{(-1)(-2)} & G_{(-1)(-1)} & G_{(-1)0} & G_{(-1)1} \\
\hline
G_{0(-2)} & G_{0(-1)} & G_{00} & G_{01} \\
G_{1(-2)} & G_{1(-1)} & G_{10} & G_{11}
\end{array}\right]
\left[\begin{array}{cc|cc}
\tilde{\omega}_N^{(-2)0} & \tilde{\omega}_N^{(-2)1} & \tilde{\omega}_N^{(-2)2} & \tilde{\omega}_N^{(-2)3} \\
\tilde{\omega}_N^{(-1)0} & \tilde{\omega}_N^{(-1)1} & \tilde{\omega}_N^{(-1)2} & \tilde{\omega}_N^{(-1)3} \\
\hline
\tilde{\omega}_N^{00} & \tilde{\omega}_N^{01} & \tilde{\omega}_N^{02} & \tilde{\omega}_N^{03} \\
\tilde{\omega}_N^{10} & \tilde{\omega}_N^{11} & \tilde{\omega}_N^{12} & \tilde{\omega}_N^{13}
\end{array}\right]
$$

$$
= \frac{1}{MN}
\left[\begin{array}{cc|cc}
02 & 03 & 00 & 01 \\
12 & 13 & 10 & 11 \\
\hline
22 & 23 & 20 & 21 \\
32 & 33 & 30 & 31
\end{array}\right]
\left[\begin{array}{cc|cc}
22 & 23 & 20 & 21 \\
32 & 33 & 30 & 31 \\
\hline
02 & 03 & 00 & 01 \\
12 & 13 & 10 & 11
\end{array}\right]
\left[\begin{array}{cc|cc}
20 & 21 & 22 & 23 \\
30 & 31 & 32 & 33 \\
\hline
00 & 01 & 02 & 03 \\
10 & 11 & 12 & 13
\end{array}\right]
$$

$$
\bar{\bar{g}} = \frac{1}{MN} \, \bar{\bar{\tilde{W}}}_m \, \bar{\bar{G}} \, \bar{\bar{\tilde{W}}}_N
$$