GEOPHYSICS, VOL. 39, NO. 1 (FEBRUARY 1974), P. 39–48, 3 FIGS., 3 TABLES

MACHINE CONTOURING USING MINIMUM CURVATURE†

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Machine contouring must not introduce information which is not present in the data. The one-dimensional spline fit has well defined smoothness properties. These are duplicated for two-dimensional interpolation in this paper, by solving the corresponding differential equation. Finite difference equations are deduced from a principle of minimum total curvature, and an iterative method of solution is outlined. Observations do not have to lie on a regular grid. Gravity and aeromagnetic surveys provide examples which compare favorably with the work of draftsmen.

INTRODUCTION

Contour maps are useful in the evaluation and interpretation of geophysical data. With the rapid increase in the rate of acquisition of data, a computer is an attractive means of producing contour maps.

Although errors occur in most geophysical observations, contour maps are usually drawn so that the imaginary surface on which the contours lie passes exactly through the observations. The problem of interpolation is then either: (a) to define a continuous function of the two space variables, which takes the values of the observations at the required, perhaps random, positions; or (b) to define a set of values at the points of a regular grid, so that a grid point value tends to an observational value if the position of the observation tends to the grid point. A solution to (a) gives a solution to (b), but a solution to (b) may not give a solution to (a). The solution to (b) is the one most commonly used as an input to a program which draws contour lines.

Methods for the production of contour maps have been published by Crain & Bhattacharyya (1967), Smith (1968), Cole (1968), Pelto et al (1968), and McIntyre et al (1968). These methods are variations of either weighting or function fitting or both, and give a solution to problem (a) and, hence, (b). Crain (1970) has provided a review of these methods.

This article describes a method for finding a solution to problem (b) without first finding a solution to problem (a). The solution also happens to be the smoothest. This attribute gives confidence in the use of the method and explains the quality of the resulting contour maps.

The problem of interpolation in one dimension has led to the piecewise polynomial fit, or spline (Ahlberg et al, 1967). A continuous function is found for all values of the independent variable. This method has been extended to two dimensions (De Boor, 1962), and used by Bhattachryya (1969) to give a solution to problem (a).

However, if the observation points in two dimensions are randomly situated, the fitting of piecewise two-dimensional polynomials to polygons seems difficult, although it is possible if the set of polygons are topologically equivalent to a rectangular grid (Hessing et al, 1972).

The optimum properties of the spline fit can be obtained in both one and two dimensions by solving the differential equation equivalent to a third-order spline. This is the equation which describes the displacement of a thin sheet in one or two dimensions under the influence of point forces. The 'boundary conditions' are not only at the ends or boundary, but within the region of interest. The solution is forced to take up the value of the observation at the point of observation, in one or two dimensions. The equation is

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solved numerically, and thus gives a solution to problem (b).

The smoothness properties follow from the method of deducing the difference equations, and the quality of the resulting contour map is thus determined. The solution of the set of difference equations is a time-consuming process, but the iteration times on the computer have been reduced and can be reduced still further.

THE METHOD

The differential equation

The thin metallic strip or sheet is bent by forces acting at points so that the displacement at these points is equal to the observation to be satisfied. Let u be the displacement, x, y the space variables, and let forces f_n act at (x_n, y_n), $n=1, \cdots, N$, where the observations are w_n, then (Love, 1926)

$$\begin{aligned} \frac{d^4u}{dx^4} &= f_n,\ x = x_n, \\ &= 0 \text{ otherwise,} \end{aligned} \tag{1}$$

in one dimension and

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} \\ = f_n,\ x = x_n,\ y = y_n, \\ = 0 \text{ otherwise,} \end{aligned} \tag{2}$$

in two dimensions. The units are dimensionless. A condition on the solution is that $u(x_n)=w_n$ or $u(x_n, y_n)=w_n$. In one dimension, u, $\partial u/\partial x$, and $\partial^2 u/\partial x^2$, the curvature, are continuous across the point where the force is acting, but $\partial^3 u/\partial x^3$ is discontinuous across such a point and the value of the discontinuity is equal to the force acting at that point (Love, 1926). A solution in one dimension is given by a third-order polynomial

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

for each segment between the points where the forces are acting. The coefficients $a_0, \cdots, a_3$ are found by using the continuity conditions above. This solution is a cubic spline.

In two dimensions the solution of equation (2) is to be used in place of the two-dimensional, third-order piecewise polynomial fit.

Boundary conditions

The most suitable condition for the ends of the strip or edge of the thin sheet is that of freedom. For a strip, the region between the end and the extreme observation will have a linear form, and for a sheet, the area between the edge and the observations will tend to a plane as the sheet becomes larger.

For one and two dimensions, at the ends or edge, the force is zero, and the bending moment about a tangential line is zero. For one dimension, these conditions give

$$\frac{\partial^3 u}{\partial x^3} = 0, \tag{3}$$

and

$$\frac{\partial^2 u}{\partial x^2} = 0, \text{ respectively.} \tag{4}$$

For two dimensions, they give

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \tag{5}$$

where the normal to the edge is in the x-direction, and give (4) also. The condition that

$$u(x_n) = w_n,$$

or

$$u(x_n, y_n) = w_n, \tag{6}$$

is also a "boundary" condition.

Equation (1) with boundary conditions (3), (4), and (6) or equation (2) with boundary conditions (5), (4), and (6) are solved numerically.

Finite difference equations

Equation (2) can be derived from the principle of minimum curvature. Difference equations can be formed from equation (2) using Taylor's theorem (Young, 1962) or directly from the principle of minimum curvature. The boundary equations are more easily deduced by the latter means. Equations to be used when an observation does not lie on a grid point are more easily deduced by the former.

Consider the total squared curvature

$$C(u) = \iint \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)^2 dxdy. \tag{7}$$

It must be shown that if a function $u(x, y)$ makes C an extremum, then it obeys equation (2), and also that if a function u obeys equation (2) then it minimizes C. Let $u(x, y)$ be a function on a region in R^2 with boundary B. Let u make C an extremum. Let $z(x, y)$ be a function of $u(x, y)$ and an arbitrary function g, with

$$g = 0 \quad \text{and} \quad \frac{\partial g}{\partial n} = 0 \text{ on } B,$$

where $\partial/\partial n$ denotes a derivative along the normal to B,

$$z(x, y) = u(x, y) + \epsilon g(x, y),$$

where ϵ is a real number.

Then

$$\left.\frac{\partial C(z)}{\partial \epsilon}\right|_{\epsilon=0} = 0,$$

and this must hold for all functions $g(x, y)$. Writing ∇^2 for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$C(z) = \iint (\nabla^2 u)^2 dxdy + 2\epsilon \iint \nabla^2 u \nabla^2 g dxdy + \epsilon^2 \iint \nabla^2 g dxdy,$$

and

$$\left.\frac{\partial C(z)}{\partial \epsilon}\right|_{\epsilon=0} = 2 \iint \nabla^2 u \nabla^2 g dxdy.$$

Using Green's theorem, (Courant and Hilbert, 1953) the right-hand side gives

$$2\left(\iint g\nabla^2(\nabla^2 u) dxdy + \int_B \nabla^2 u \frac{\partial g}{\partial n} dl - \int_B g \frac{\partial}{\partial n} (\nabla^2 u) dl\right).$$

The last two integrals vanish and leave

$$\iint g\nabla^2(\nabla^2 u) dxdy = 0.$$

Since this must hold for all g, $\nabla^2(\nabla^2 u) = 0$. Conversely, if u obeys equation (2), and if z is any other function on R^2, with $z = u$, and $\partial z/\partial n = \partial u/\partial n$ on B, we can show that $C(u) \leq C(z)$. Consider

$$C(z) - C(u) = \iint \left[(\nabla^2 z)^2 - (\nabla^2 u)^2\right] dxdy.$$

The right-hand side gives

$$\iint (\nabla^2 z - \nabla^2 u)^2 dxdy + 2 \iint \nabla^2 u (\nabla^2 z - \nabla^2 u) dxdy.$$

The last term gives upon the use of Green's theorem

$$2\left(\iint (z - u)\nabla^2(\nabla^2 u) dxdy + \int_B \nabla^2 u \frac{\partial}{\partial n} (z - u) dl - \int_B (z - u) \frac{\partial}{\partial n} (\nabla^2 u) dl\right).$$

The integrals are zero; $C(u)$ is then always less than or equal to $C(z)$.

The principle of minimum curvature is used to deduce the normal difference equations. The total squared curvature (7) is constructed directly in terms of elements of the set of grid point values

$$u_{i,j} \equiv u(x_i, y_j),$$
$$x_i = (i - 1)h,\ y_j = (j - 1)h,$$
$$i = 1, \cdots, I,\ j = 1, \cdots, J,$$

where h is the grid spacing. The discrete total squared curvature is

$$C = \sum_{i=1}^{I} \sum_{j=1}^{J} (C_{i,j})^2, \tag{8}$$

where $C_{i,j}$ is the curvature at (x_i, y_j). $C_{i,j}$ is a function of $u_{i,j}$ and some neighboring grid values; the exact set depends on the accuracy with which the curvature is to be represented.

To minimize the sum C, the functions

$$\frac{\partial C}{\partial u_{i,j}}, \quad i = 1, \cdots, I;\ j = 1, \cdots, J, \tag{9}$$

are set equal to zero (Stiefel, 1963). The resulting equations determine a set of relations between neighboring grid-point values, one relation for each grid point.

In one dimension the simplest approximation to the curvature at x_i is

$$(u_{i+1} + u_{i-1} - 2u_i)/h^2,$$

and in two dimensions at (x_i, y_i), it is

$$C_{i,j} = (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j})/h^2. \tag{10}$$

Along edges and rows one from the edge, and near corners, different expressions for the curvature are used. For example, at an edge $j=1$,

$$C_{i,j} = (u_{i+1,j} + u_{i-1,j} - 2u_{i,j})/h^2. \tag{11}$$

These special cases are also included in the total for C. Away from the edges, (10) shows that a grid point value $u_{i,j}$ occurs in the expressions for,

$$C_{i,j},\ C_{i+1,j},\ C_{i-1,j},\ C_{i,j+1} \quad \text{and} \quad C_{i,j-1}.$$

Thus, only these need to be considered when equation (10) is used. Using (8), (9), and (10) the common difference equation for the biharmonic equation results:

$$\begin{aligned} &u_{i+2,j} + u_{i,j+2} + u_{i-2,j} + u_{i,j-2} \\ &+ 2(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}) \\ &- 8(u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1}) \\ &+ 20u_{i,j} = 0. \end{aligned} \tag{12}$$

For the edge $j=1$, the difference equation is

$$\begin{aligned} u_{i-2,j} + u_{i+2,j} + u_{i,j+2} + u_{i-1,j+1} + u_{i+1,j+1} \\ - 4(u_{i-1,j} + u_{i,j+1} + u_{i+1,j}) \\ + 7u_{i,j} = 0. \end{aligned} \tag{13}$$

A complete set is given in Appendix B.

The point boundary conditions (6) are used by setting $u_{i,j} = w_n$ wherever $u_{i,j}$ occurs in the set of linear equations, and by removing those equations which correspond to these fixed grid points.

Observation not on a grid point

If an observation does not coincide with a grid point another difference equation is required for grid points which are the vertices of the grid square in which the observation falls. The observation point becomes part of the grid.

The equation used is a special case of a general method for using a random grid for the numerical solution of differential equations. The general method is used by letting one grid point, the observation, be on an irregular grid; the remaining neighbors are on the regular grid in the difference equation relating a grid-point value to its neighbors.

Equation (2) is equivalent to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

If

$$C_{ij} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{at } (x_i, y_j), \tag{14}$$

equation (14) gives the difference equation (Young, 1962),

$$C_{i+1,j} + C_{i-1,j} + C_{i,j+1} + C_{i,j-1} - 4C_{i,j} = 0. \tag{15}$$

If equation (10) is used in equation (15), equation (12) results. However, we need an expression for $C_{i,j}$ which uses values of u at discrete points not lying on a regular grid.

Let u be a continuous function on the real two-dimensional space R^2 and let (x_0, y_0) be in R^2. If the set of points

$$\{x_0 + \xi_k,\ y_0 + \eta_k\}, \qquad k = 1, \cdots, 5$$

are also in R^2, then for sufficiently small ξ_k, η_k and if u has sufficiently many derivatives,

$$u_k \equiv u(x_0 + \xi_k,\ y_0 + \eta_k), \qquad k = 1, \cdots, 5,$$

is approximated by

$$u_0 + \xi_k \left.\frac{\partial u}{\partial x}\right|_0 + \eta_k \left.\frac{\partial u}{\partial y}\right|_0 + \tfrac{1}{2}\xi_k^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_0 + \xi_k\eta_k \left.\frac{\partial^2 u}{\partial x \partial y}\right|_0 + \tfrac{1}{2}\eta_k^2 \frac{\partial^2 u}{\partial y^2} \cdot \tag{16}$$

To find an expression for

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{at } (x_0, y_0),$$

both sides of equations (16) are multiplied by a real number b_k and a sum is made over k, so that

$$\sum_{k=1}^{5} b_k u_k = u_0 \sum b_k + \left.\frac{\partial u}{\partial x}\right|_0 \sum b_k \xi_k + \left.\frac{\partial u}{\partial y}\right|_0 \sum b_k \eta_k + \frac{1}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_0 \sum b_k \xi_k^2 + \left.\frac{\partial^2 u}{\partial x \partial y}\right|_0 \sum b_k \xi_k \eta_k + \frac{1}{2} \left.\frac{\partial^2 u}{\partial y^2}\right|_0 \sum b_k \eta_k^2. \quad (17)$$

If the b_k are chosen such that

$$\sum b_k \xi_k = 0, \qquad \sum b_k \eta_k = 0, \qquad \sum b_k \xi_k^2 = 2, \qquad \sum b_k \xi_k \eta_k = 0, \qquad \sum b_k \eta_k^2 = 2, \quad (18)$$

then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{at } (x_0, y_0)$$

is approximated by

$$\sum_{k=1}^{5} b_k u_k - u_0 \sum_{k=1}^{5} b_k. \quad (19)$$

The matrix

$$\begin{Bmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 & \xi_5^2 \\ \xi_1\eta_1 & \xi_2\eta_2 & \xi_3\eta_3 & \xi_4\eta_4 & \xi_5\eta_5 \\ \eta_1^2 & \eta_2^2 & \eta_3^2 & \eta_4^2 & \eta_5^2 \end{Bmatrix} \quad (20)$$

must be nonsingular, for the b_k to exist. For the present purpose, where one u_k lies off the regular grid, and the remaining four lie on the regular grid, with

$$\xi, \eta = h, 0, -h,$$

a suitable set is

$$(h, -h), (0, -h), (-h, 0), (-h, h), (\xi_5, \eta_5),$$

with $\xi_5 > 0$ and $\eta_5 > 0$. Thus, for an expression for the curvature at (x_i, y_j), we can use

$$C_{i,j} = \sum_{k=1}^{4} b_k u_k - u_{i,j} \sum_{k=1}^{5} b_k + b_5 w_n, \quad (21)$$

where $\{u_k\}$ is

$$u_{i+1,j-1},\ u_{i,j-1},\ u_{i-1,j},\ u_{i-1,j+1},$$

and w_n is the nearby observation value.

Equation (21) can be used in (15) to give a linear equation relating a grid point to neighboring grid points and an observation. This is used in place of equation (12).

Iteration matrix

The set of linear algebraic equations (12), (13), and others are best solved iteratively (Young, 1962). Given an approximate set of $u_{i,j}$, a new set is obtained by making $u_{i,j}$ the subject of equations (12) and (13) and others. For example, (13) gives

$$\overset{p+1}{u_{i,j}} = [4(\overset{p}{u}_{i-1,j} + \overset{p}{u}_{i,j+1} + \overset{p}{u}_{i+1,j}) - (\overset{p}{u}_{i-2,j} + \overset{p}{u}_{i+2,j} + \overset{p}{u}_{i,j+2} + \overset{p}{u}_{i-1,j+1} + \overset{p}{u}_{i+1,j+1})]/7, \quad (22)$$

where the index p indicates the pth iteration. Starting values must be given, and one suitable method is to use the value of the nearest observation or a weighted sum of neighboring observations.

Iteration matrices which give faster rates of convergence than that defined by (22) are known (Young, 1962; Parter, 1959), but are not described here. The proof of the existence of a solution to the linear equations is omitted (Stiefel, 1963).

Smoothness properties

The measure of smoothness, $C = \sum (c_{i,j})^2$ is a function of h and the precision of the approximation for $C_{i,j}$. Because the linear equations are deduced from the principle of minimum C, for a given h and for a given definition of curvature, the resulting grid-point surface is smoother than, or as smooth as, any other grid-point surface. Two contour maps produced by different means but using the same data, can be compared for smoothness by digitizing the map, if necessary, and calculating the total curvature C. The map with the lower value of C is usually the more acceptable, and delineates trends more clearly.

Nothing will be said here about the convergence of the grid point values or of C, as the grid spacing tends to zero. However, for a given grid spacing, the method gives the smoothest possible contour map, and it can be used with some confidence as a representation of the given data.

Drawing lines

There are many different methods of drawing the contours once the grid (Crain, 1970) surface has been found. The method used in the examples involves a four-point cubic interpolation between grid points to find contour cuts, and then a cubic spline to join the cuts. The observations art not used. This is the weak link in the present scheme. Improvements can be made by using the observations or by using two-dimensional cubic interpolation over a grid square. The overall success of the application of minimum total curvature warrants the undertaking of further work in the improvement of details.

Table 1. The smoothest set of discrete values fixed at i=3, 5, 8.

i	V_i
1	-5.62
2	1.69
3	9.00
4	16.31
5	25.00
6	36.46
7	49.77
8	64.00
9	78.23
10	92.46

EXAMPLES

For each map the time for one iteration for one grid point was approximately 0.4 msec using a CDC 3600 computer. Up to 260,000 grid points have been used to contour 60,000 observations at one time. To provide edge matching when an entire survey cannot be contoured at once, data beyond the area to be contoured are used.

The iterations were discontinued when all significant relocation of contour lines had taken place.

Test cases

Two simple test examples are: (1) a one-dimensional set of data taken to lie on a straight line; and (2) a set of data points (at least four are necessary) taken to lie on a plane. In case (1) the free grid points tend to values lying on the same straight line, and in case (2) the free grid points tend to values lying in the same plane.

Table 1 gives the values of a one-dimensional set of grid points which minimize the total curvature. Grid points at $i=3$, 5, 8 are fixed and the imaginary forces required to bend the spline act at these points. The difference equations used are given in Appendix A.

Table 2 gives the values of a two-dimensional set of grid points fixed at (7, 3), (8, 5), (5, 5), (8, 8), and (4, 8). This set of grid points minimizes the sum of the point curvatures defined by equation (10).

Table 3 gives the values of a two-dimensional set of grid points fixed at infinity and at $x=.2$, $y=.3$ where the observation is $(.2)^2+(.3)^2=.13$.

Table 2. The smoothest set of grid valves fixed at (7, 3), (8, 5), (5, 5), (8, 8), (4, 8).

i/j	1	2	3	4	5	6	7	8	9	10
1	-99.34	-89.96	-80.30	-70.10	-59.19	-47.48	-35.01	-21.93	-8.44	5.25
2	-84.07	-75.42	-66.30	-56.53	-45.95	-34.46	-22.12	-9.12	4.35	18.14
3	-69.07	-61.31	-52.89	-43.67	-33.48	-22.17	-9.86	3.21	16.80	30.84
4	-54.66	-47.83	-40.14	-31.56	-21.83	-10.64	1.74	15.00	28.79	43.14
5	-41.19	-35.18	-28.14	-20.19	-11.00	0.13	12.61	26.05	40.14	54.87
6	-29.03	-23.59	-16.97	-9.55	-0.68	10.25	22.80	36.46	50.85	65.94
7	-18.57	-13.42	-7.00	-0.14	8.40	19.37	32.15	46.16	60.85	76.25
8	-9.89	-5.04	0.86	7.61	16.00	27.31	40.50	55.00	70.01	85.74
9	-2.59	2.03	7.55	14.29	22.95	34.23	47.63	62.51	78.20	94.48
10	4.00	8.15	13.01	19.37	28.03	39.44	53.34	69.00	85.67	102.78

Table 3. The smoothest set of grid valves fixed at infinity and $x=.2$, $y=.3$, with $x=0, y=0$ at (3, 3).

i \ j	1	2	3	4	5
1	8.00	5.00	4.00	5.00	8.00
2	5.00	2.00	1.00	2.00	5.00
3	4.00	1.00	0.00	1.00	4.00
4	5.00	2.00	1.00	2.00	5.00
5	8.00	5.00	4.00	5.00	8.00

The condition at infinity is simulated by setting grid values at (x, y) to x^2+y^2 beyond a limit, and by not using the boundary difference equations. This table shows the results of using equation (21) for the case where an observation does not fall on a grid point. The grid points used are $(-1, 1)$, $(-1, 0)$, $(0, -1)$, $(1, -1)$, and the observation at $(.2, .3)$. The matrix (20) is

$$\begin{bmatrix} -1 & -1 & 0 & 1 & .2 \\ 1 & 0 & -1 & -1 & .3 \\ 1 & 1 & 0 & 1 & .04 \\ -1 & 0 & 0 & -1 & .06 \\ 1 & 0 & 1 & 1 & .09 \end{bmatrix}$$

and the resulting coefficients b_k, $k=1, \cdots, 5$ are .68, .60, .73, .48, and 2.67.

These are used in equations (21) and (15) to give a value for the grid point at $x=0$, $y=0$.

These and other higher-order surfaces test the method in general and the difference equations in particular. The illustrated examples use real data.

Almost uniform data

Figure 1 is the resulting contour map for gravimetric data sampled in mgal on a nominal 11 km

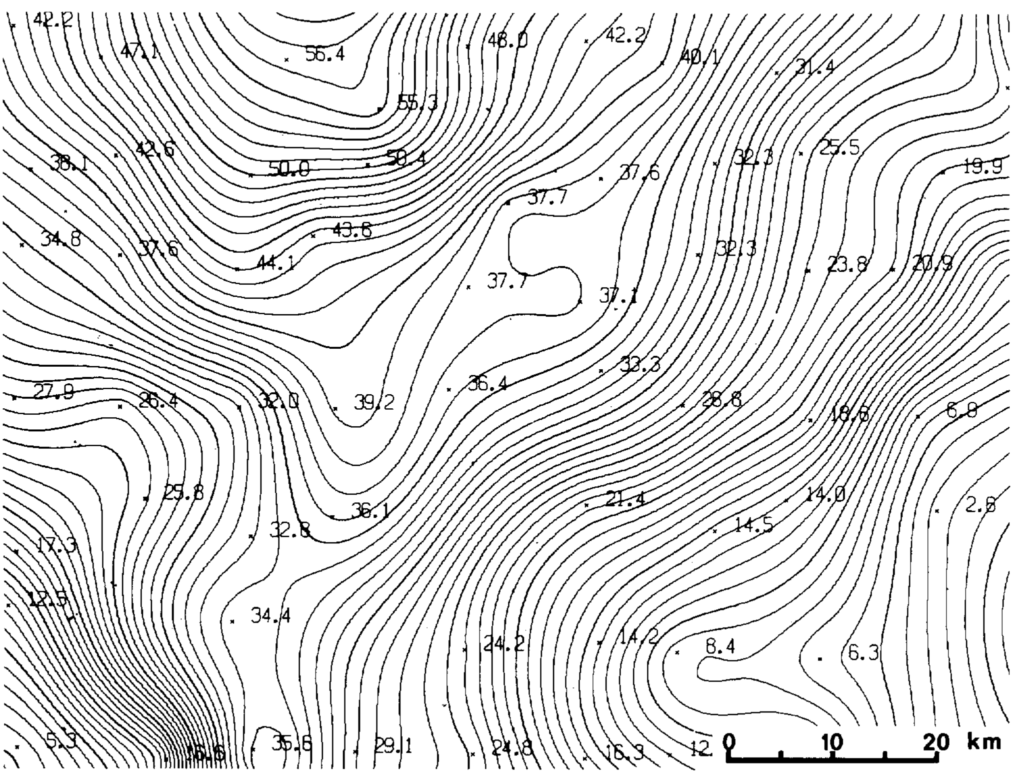

FIG. 1. Gravity data contoured at 1-mgal intervals using a grid spacing of 1.85 km.

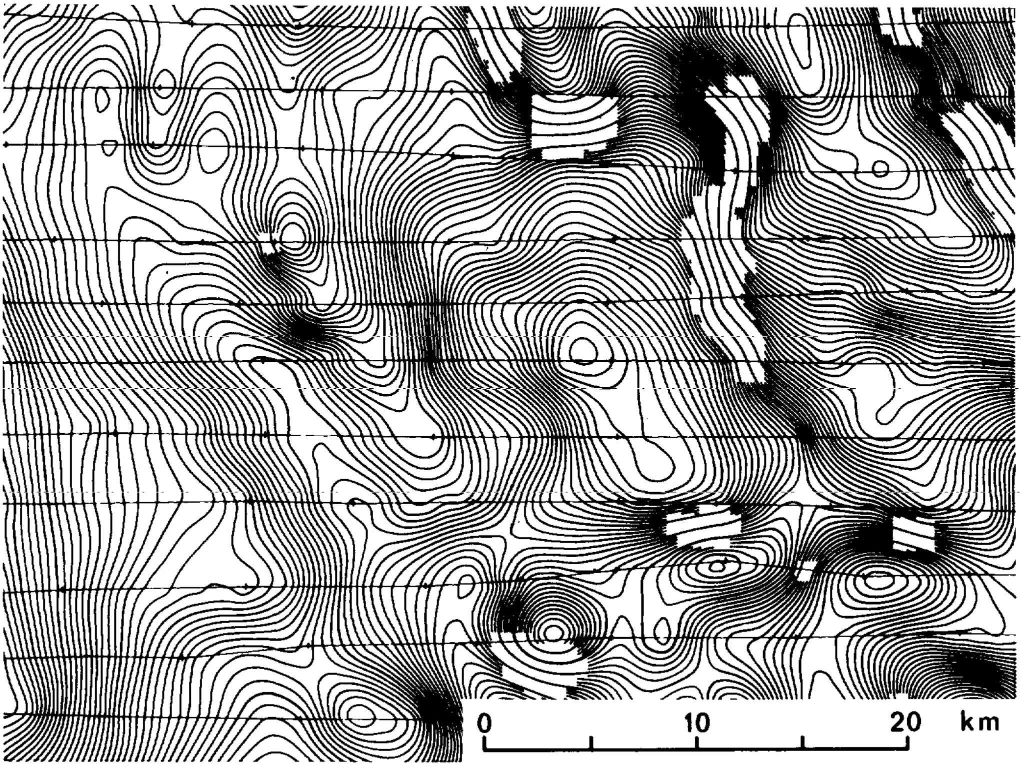

FIG. 2. Aeromagnetic data contoured at 10-gamma intervals using a grid spacing of 0.5 km.

network. The grid spacing was 1.85 km and the number of iterations was 90. The number of grid points was 1500. The smoothness of the interpolating grid surface is apparent. The shapes of contours for data for this type generally agree with those of draftsmen. Differences occur when the interpolating grid surface lies outside the range of a closed group of observations.

The deficiency in the line-drawing routine shows itself when the 50-mgal contour does not pass exactly through a 50-mgal observation.

Line data

A more difficult set of data to contour is one whose density of sampling is not isotropic. The data for the total intensity aeromagnetic maps of Figure 2 and Figure 3 were taken at 0.8 km intervals along flight-lines nominally 3.2 km apart and at a height of 650 m above ground. The grid spacing used in the contouring was 0.5 km and the number of iterations was 60. The number of grid points is 6000 in Figure 2 and 16,500 in Figure 3.

The general smoothness is satisfactory although a relatively rough profile along a line has an effect on the contours close to the flight-line, and this may not be desirable. The interpolation is not a product of one-dimensional splines. A section across the flight-lines is not the result of a one-dimensional spline. This may be a drawback in some cases, but enables trends not lying at right-angles to the flight path to be displayed.

CONCLUSION

The principle of minimum total curvature provides a method of two-dimensional interpolation which allows a computer to draw reasonable maps of geophysical data. The results are not always as a draftsman would have them, but are an adequate substitute in most cases.

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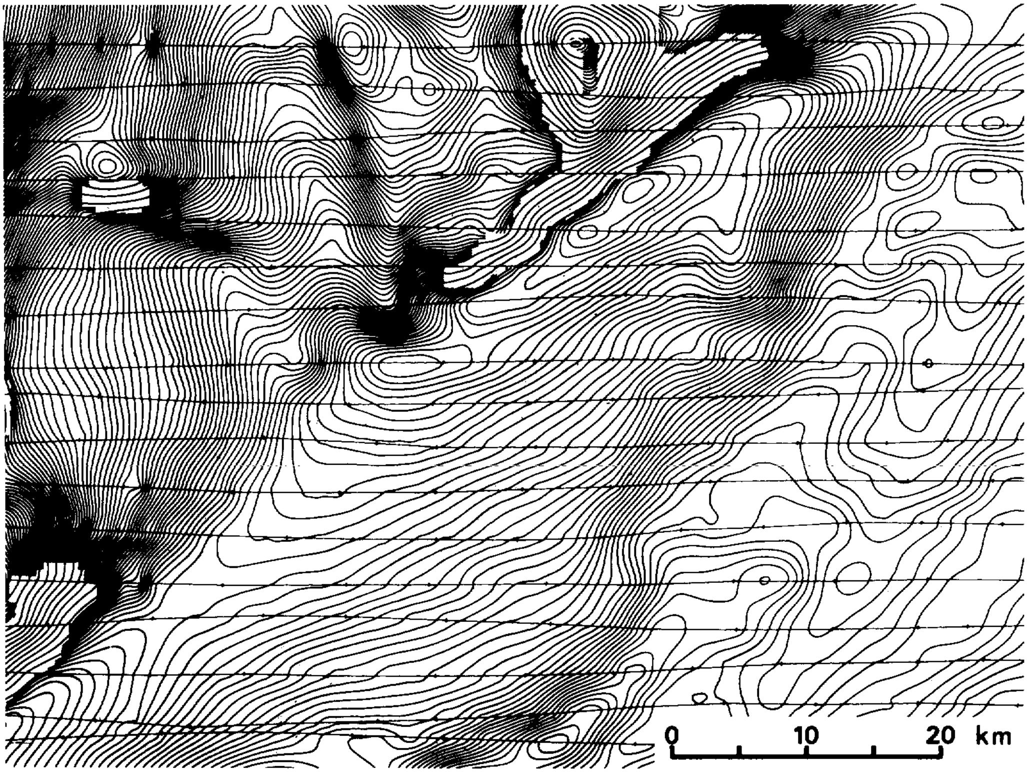

FIG. 3. Aeromagnetic data contoured at 10-gamma intervals using a grid spacing of 0.5 km.

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APPENDIX A

A set of difference equations for one-dimensional interpolation is given. The curvature used is

$$C_i = (u_{i+1} + u_{i-1} - 2u_i)/h^2.$$

Normal

Away from the ends, use

$$u_{i-2} + u_{i+2} - 4(u_{i-1} + u_{i+1}) + 6u_i = 0.$$

End

At the end $i=1$, use

$$u_i + u_{i+2} - 2u_{i+1} = 0.$$

Next to end

At the point next to the end, $i=2$, use

$$u_{i+2} - 2u_{i-1} - 4u_{i+1} + 5u_i = 0.$$

APPENDIX B

A complete set of difference equations for two-dimensional interpolation is given. The curvature used is that given by equation (10).

Normal

Away from edges and observations the difference equation is:

$$u_{i+2,j} + u_{i,j+2} + u_{i-2,j} + u_{i,j-2}$$
$$+ 2(u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1})$$
$$- 8(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$
$$+ 20u_{i,j} = 0.$$

Edge

For the edge $j=1$ and away from corners:

$$u_{i-2,j} + u_{i+2,j} + u_{i,j+2} + u_{i-1,j+1} + u_{i+1,j+1}$$
$$-4(u_{i-1,j} + u_{i,j+1} + u_{i+1,j}) + 7u_{i,j} = 0.$$

One row from edge

For the row $j=2$, and away from corners, use

$$u_{i-2,j} + u_{i+2,j} + u_{i,j+2} + 2(u_{i-1,j+1} + u_{i+1,j+1})$$
$$+ u_{i-1,j-1} + u_{i+1,j-1}$$
$$- 8(u_{i-1,j} + u_{i,j+1} + u_{i+1,j})$$
$$- 4u_{i,j-1} + 19u_{i,j} = 0.$$

The corners

For the corner $i=1, j=1$, use

$$2u_{i,j} + u_{i,j+2} + u_{i+2,j}$$
$$- 2(u_{i,j+1} + u_{i+1,j}) = 0.$$

Next to corner

For the grid point next to the corner and lying on a diagonal $i=2, j=2$, use

$$u_{i,j+2} + u_{i+2,j} + u_{i-1,j+1} + u_{i+1,j-1} + 2u_{i+1,j+1}$$
$$- 8(u_{i,j+1} + u_{i+1,j}) - 4(u_{i,j-1} + u_{i-1,j})$$
$$+ 18u_{i,j} = 0.$$

Edges next to corner

For the grid point next to the corner point, which lies on the edge, $i=2, j=1$, use

$$u_{i,j+2} + u_{i+1,j+1} + u_{i-1,j+1} + u_{i+2,j} - 2u_{i-1,j}$$
$$- 4(u_{i+1,j} + u_{i,j+1}) + 6u_{i,j} = 0.$$