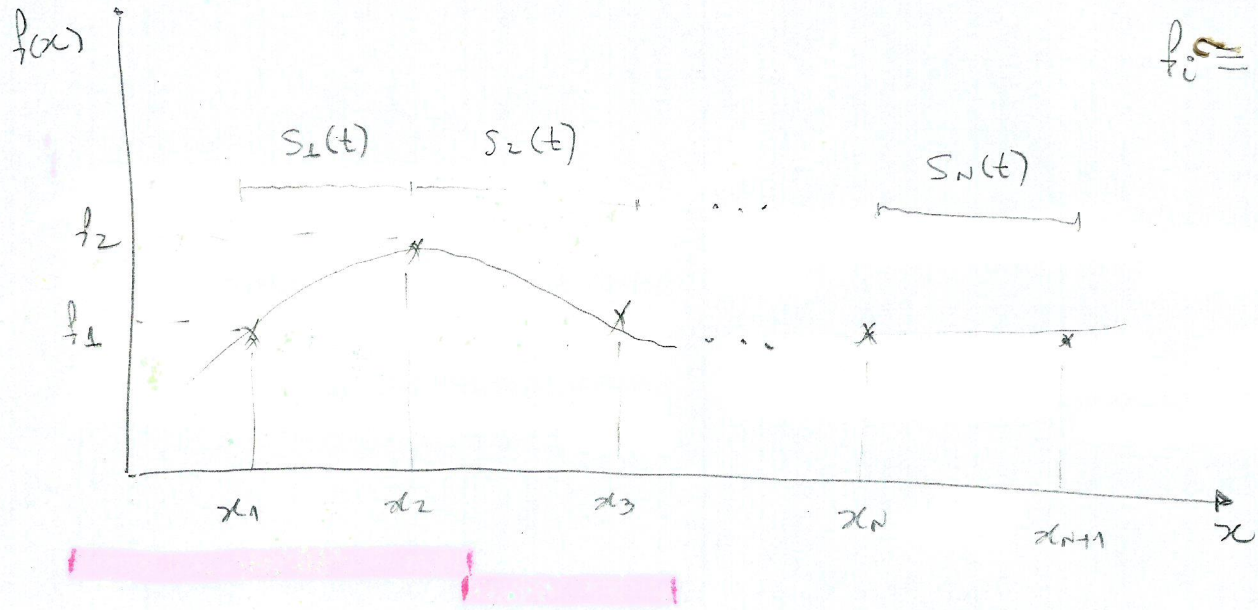
Cubic splines

$f_i = f(x_i)$



① $S_i(t) = a_i + b_i t + c_i t^2 + d_i t^3$ cubic polynomial in the parameter t

$t = \dfrac{x - x_i}{x_{i+1} - x_i}$ $\quad x_i \leq x \leq x_{i+1}$ $\quad 0 \leq t \leq 1$

② $S_i(0) = a_i = f_i$ → KNOWN VALUE AT $x = x_i$

③ $S_i(1) = a_i + b_i + c_i + d_i = f_{i+1}$ → KNOWN VALUE AT $x = x_{i+1}$

a_i, b_i, c_i, d_i ARE <u>UNKNOWNS</u>

First derivative of $S_i(t)$

④ $S_i'(t) = b_i + 2c_i t + 3d_i t^2$

⑤ $S_i'(0) = b_i = D_i$ NEW UNKNOWNS

⑥ $S_i'(1) = b_i + 2c_i + 3d_i = D_{i+1}$

From 2 and 5

(7) $a_i = f_i$

(8) $b_i = D_i$

From 3 and 6

$a_i + b_i + c_i + d_i = f_{i+1}$

$\boxed{d_i = f_{i+1} - f_i - D_i - c_i}$

$b_i + 2c_i + 3d_i = D_{i+1}$

$2c_i = D_{i+1} - D_i - 3(f_{i+1} - f_i - D_i - c_i)$

(9) $c_i = 3(f_{i+1} - f_i) - 2D_i - D_{i+1}$

$d_i = (f_{i+1} - f_i) - D_i - [3(f_{i+1} - f_i) - 2D_i - D_{i+1}]$

(10) $d_i = 2(f_i - f_{i+1}) + D_i + D_{i+1}$

According to 7 – 10, a_i, b_i, c_i, and d_i can be calculated in terms of D_i

Second derivative of $S_i(t)$

(11) $S_i''(t) = 2c_i + 6d_i t$

(12) $S_i''(0) = 2c_i$

(13) $S_i''(1) = 2c_i + 6d_i$

Now, impose the following condition:

$$S''_{i-1}(1) = S''_i(0) \quad , \quad i = 2, \dots, N$$

(14) $2c_{i-1} + 6d_{i-1} = 2c_i$

By substituting (9) and (10) in (14), we obtain

$$2\left[3(f_i - f_{i-1}) - 2D_{i-1} - D_i\right] + 6\left[2(f_{i-1} - f_i) + D_{i-1} + D_i\right] =$$

$$= 2\left[3(f_{i+1} - f_i) - 2D_i - D_{i+1}\right]$$

$$-4D_{i-1} - 2D_i + 6D_{i-1} + 6D_i + 4D_i + 2D_{i+1} =$$

$$= 6(f_{i+1} - f_i) - 12(f_{i-1} - f_i) - 6(f_i - f_{i-1})$$

$$2D_{i-1} + 8D_i + 2D_{i+1} = 6f_{i+1} - 6f_{i-1}$$

(15) $D_{i-1} + 4D_i + D_{i+1} = 3(f_{i+1} - f_{i-1})$

Finally, consider that

(16a) $S''_1(0) = 0$

(16b) $S''_N(1) = 0$

From (16a) and (9)

$2c_1 = 0$

$2\left[3(f_2 - f_1) - 2D_1 - D_2\right] = 0$

$2D_1 + D_2 = 3(f_2 - f_1)$ (17)

From (16b), (9), and (10)

$2c_N + 6d_N = 0$

$2\left[3(f_{N+1} - f_N) - 2D_N - D_{N+1}\right] +$

$+ 6\left[2(f_N - f_{N+1}) + D_N + D_{N+1}\right] = 0$

$2D_N + 4D_{N+1} = 6(f_{N+1} - f_N)$

$D_N + 2D_{N+1} = 3(f_{N+1} - f_N)$ (18)

Equations (15), (17), and (18) show that

$$\left.\begin{aligned} i=1 &: 2D_1 + D_2 = 3(f_2 - f_1) \end{aligned}\right\} \text{eq } 17$$

$$\left.\begin{aligned} i=2 &: D_1 + 4D_2 + D_3 = 3(f_3 - f_1) \\ i=3 &: D_2 + 4D_3 + D_4 = 3(f_4 - f_2) \\ &\ \vdots \\ i=N-1 &: D_{N-2} + 4D_{N-1} + D_N = 3(f_N - f_{N-2}) \\ i=N &: D_{N-1} + 4D_N + D_{N+1} = 3(f_{N+1} - f_{N-1}) \end{aligned}\right\} \text{eq } 15$$

$$\left.\begin{aligned} i=N+1 &: D_N + 2D_{N+1} = 3(f_{N+1} - f_N) \end{aligned}\right\} \text{eq } 18$$

$$\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 1 \\ & & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{N-1} \\ D_N \\ D_{N+1} \end{bmatrix} = \begin{bmatrix} 3(f_2 - f_1) \\ 3(f_3 - f_1) \\ 3(f_4 - f_2) \\ \vdots \\ 3(f_N - f_{N-2}) \\ 3(f_{N+1} - f_{N-1}) \\ 3(f_{N+1} - f_N) \end{bmatrix} \qquad (19)$$