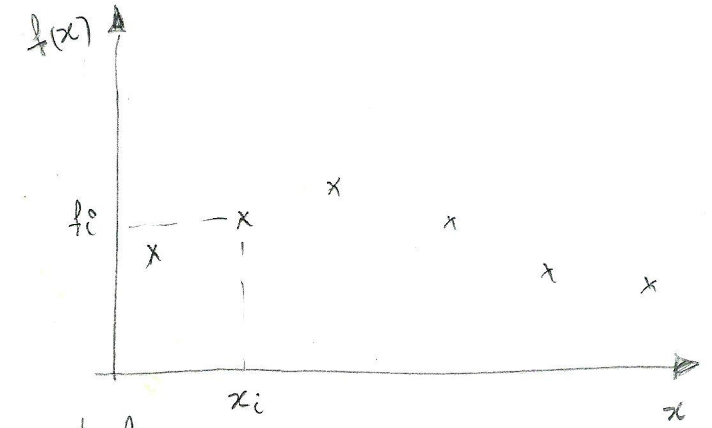
Interpolation

Polynomial interpolation



$f(x)$?

x	f
x_1	f_1
$\vdots$	$\vdots$
x_N	f_N
x_{N+1}	f_{N+1} $\rightarrow$ N+1 points

Consider that $f(x)$ can be properly described by a polynomial $P_N(x)$

$$f(x) \approx P_N(x)$$

$\rightarrow$ polynomial of degree N

Determine $f(x)$ at points x that are different from x_i, $i = 1, \ldots, N+1$

Lagrange's method

$$P_N(x) = l_1 f_1 + l_2 f_2 + \cdots + l_{N+1} f_{N+1}$$

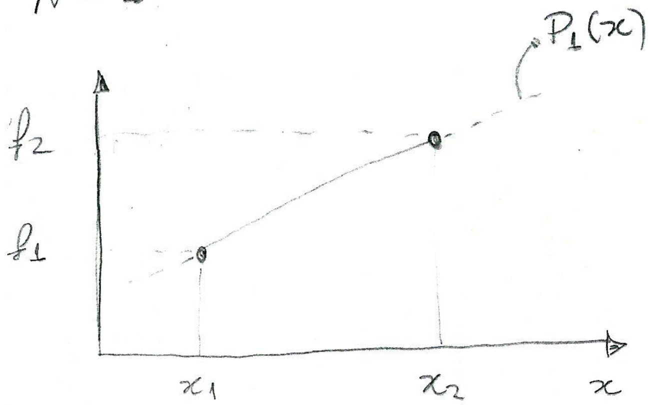
$$= \sum_{i=1}^{N+1} l_i(x) f_i$$

cardinal functions

$$l_i(x) = \frac{x - x_1}{x_i - x_1} \frac{x - x_2}{x_i - x_2} \cdots \frac{x - x_{i-1}}{x_i - x_{i-1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_{N+1}}{x_i - x_{N+1}}$$

$$= \prod_{\substack{j=1 \\ j \neq i}}^{N+1} \frac{x - x_j}{x_i - x_j} \quad , \quad i = 1, \ldots, N+1$$

$N = 1$



$$P_1(x) = l_1(x) f_1 + l_2(x) f_2$$

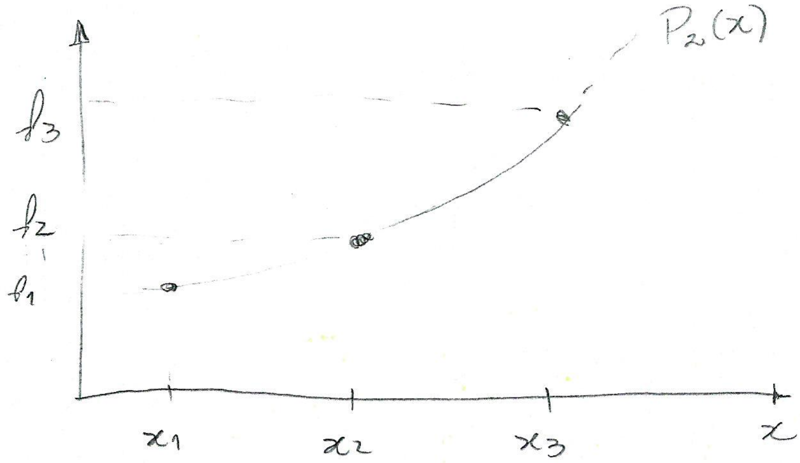
$$= \frac{x - x_2}{x_1 - x_2} f_1 + \frac{x - x_1}{x_2 - x_1} f_2$$

$$P_1(x) \equiv P_1[x_1, x_2], \quad f_1 \equiv P_0[x_1], \quad f_2 \equiv P_0[x_2]$$

$$P_1[x_1, x_2] = \frac{(x - x_2) P_0[x_1] + (x_1 - x) P_0[x_2]}{x_1 - x_2}$$

$$P_1[x_i, x_{i+N}] = \frac{(x - x_{i+N}) P_0[x_i] + (x_i - x) P_0[x_{i+N}]}{x_i - x_{i+N}}$$

$N=2$



$$P_2(x) = l_1(x)\,f_1 + l_2(x)\,f_2 + l_3(x)\,f_3$$

$$= \frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\,f_1 + \frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\,f_2 +$$

$$+ \frac{x-x_1}{x_3-x_1}\,\frac{x-x_2}{x_3-x_2}\,f_3 =$$

$$= \frac{x-x_3}{x_1-x_3}\,\frac{x-x_2}{x_1-x_2}\,f_1 + \frac{(x_1-x_2)+(x_2-x_3)}{x_1-x_3}\,\frac{x-x_3}{x_2-x_3}\,\frac{x-x_1}{x_2-x_1}\,f_2 +$$

$$+ \frac{x_1-x}{x_1-x_3}\,\frac{x-x_2}{x_3-x_2}\,f_3 =$$

$$= \frac{x-x_3}{x_1-x_3}\,\frac{x-x_2}{x_1-x_2}\,f_1 + \frac{x-x_3}{x_1-x_3}\,\cancel{\frac{x_2-x_3}{x_2-x_3}}\,\frac{x-x_1}{x_2-x_1}\,f_2 +$$

$$+ \frac{x_1-x}{x_1-x_3}\,\cancel{\frac{x_1-x_2}{x_1-x_2}}\,\frac{x-x_3}{x_2-x_3}\,f_2 + \frac{x_1-x}{x_1-x_3}\,\frac{x-x_2}{x_3-x_2}\,f_3$$

$$= \frac{x-x_3}{x_1-x_3}\left(\frac{x-x_2}{x_1-x_2}\,f_1 + \frac{x-x_1}{x_2-x_1}\,f_2\right) +$$

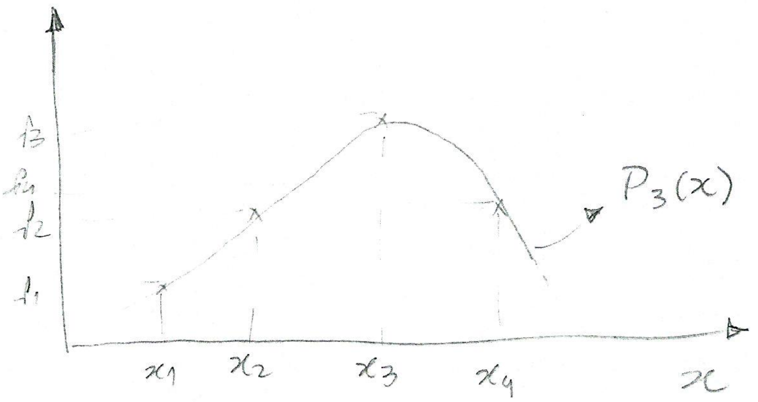
$$+ \frac{x_1-x}{x_1-x_3}\left(\frac{x-x_3}{x_2-x_3}\,f_2 + \frac{x-x_2}{x_3-x_2}\,f_3\right)$$

$$\overbrace{P_2[x_1, x_2, x_3]}^{P_2(x)} = \frac{(x - x_3)\, P_1[x_1, x_2] + (x_1 - x)\, P_1[x_2, x_3]}{x_1 - x_3}$$

$$P_2[x_i, x_{i+1}, x_{i+N}] = \frac{(x - x_{i+\overset{N}{\downarrow}2})\, P_1[x_i, x_{i+1}] + (x_i - x)\, P_1[x_{i+1}, x_{i+\overset{N}{\downarrow}2}]}{x_i - x_{i+2 \leftarrow N}}$$

$$P_1[x_i, x_{i+1}] = \frac{(x - x_{i+1})\, P_0[x_i] + (x_i - x)\, P_0[x_{i+1}]}{x_i - x_{i+1}}$$ comparison

$N = 3$



$$P_3(x) = l_1(x) f_1 + l_2(x) f_2 + l_3(x) f_3 + l_4(x) f_4$$

$$= \frac{x - x_2}{x_1 - x_2} \frac{x - x_3}{x_1 - x_3} \frac{x - x_4}{x_1 - x_4} f_1 +$$

$$+ \frac{x - x_1}{x_2 - x_1} \frac{x - x_3}{x_2 - x_3} \frac{x - x_4}{x_2 - x_4} f_2 +$$

$$+ \frac{x - x_1}{x_3 - x_1} \frac{x - x_2}{x_3 - x_2} \frac{x - x_4}{x_3 - x_4} f_3 +$$

$$+ \frac{x - x_1}{x_4 - x_1} \frac{x - x_2}{x_4 - x_2} \frac{x - x_3}{x_4 - x_3} f_4$$

$$= \frac{x-x_4}{x_1-x_4}\left(\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\,f_1\right) +$$

$$+ \frac{(\cancel{x_1}-\cancel{x_2})+(\cancel{x_2}-x_4)}{x_1-x_4}\,\frac{x-x_1}{\cancel{x_2-x_1}}\,\frac{x-x_3}{x_2-x_3}\,\frac{1}{\cancel{x_2-x_4}}\,(x-x_4)\,f_2 +$$

$$+ \frac{(\cancel{x_1}-\cancel{x_3})+(\cancel{x_3}-x_4)}{x_1-x_4}\,\frac{x-x_1}{\cancel{x_3-x_1}}\,\frac{x-x_2}{x_3-x_2}\,\frac{1}{\cancel{x_3-x_4}}\,(x-x_4)\,f_3 +$$

$$+ \frac{x_1-x}{x_1-x_4}\left(\frac{x-x_2}{x_4-x_2}\,\frac{x-x_3}{x_4-x_3}\,f_4\right) =$$

$$= \frac{x-x_4}{x_1-x_4}\left(\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\,f_1\right) +$$

$$+ \frac{1}{x_1-x_4}\left[\cancel{(x_1-x_2)}\,\frac{(x-x_1)}{\cancel{(x_2-x_1)}}\,\frac{(x-x_3)}{(x_2-x_3)}\,\frac{(x-x_4)}{(x_2-x_4)} + \cancel{(x_2-x_4)}\,\frac{(x-x_1)}{(x_2-x_1)}\,\frac{(x-x_3)}{(x_2-x_3)}\,\frac{(x-x_4)}{\cancel{(x_2-x_4)}}\right] f_2 +$$

$$+ \frac{1}{x_1-x_4}\left[\cancel{(x_1-x_3)}\,\frac{(x-x_1)}{\cancel{(x_3-x_1)}}\,\frac{(x-x_2)}{(x_3-x_2)}\,\frac{(x-x_4)}{(x_3-x_4)} + \cancel{(x_3-x_4)}\,\frac{(x-x_1)}{(x_3-x_1)}\,\frac{(x-x_2)}{(x_3-x_2)}\,\frac{(x-x_4)}{\cancel{(x_3-x_4)}}\right] f_3 +$$

$$+ \frac{x_1-x}{x_1-x_4}\left(\frac{x-x_2}{x_4-x_2}\,\frac{x-x_3}{x_4-x_3}\,f_4\right) =$$

$$= \frac{x-x_4}{x_1-x_4}\overbrace{\left(\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\,f_1 + \frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\,f_2 + \frac{x-x_1}{x_3-x_1}\,\frac{x-x_2}{x_3-x_2}\,f_3\right)}^{P_2[x_1,x_2,x_3]} +$$

$$\frac{x_1-x}{x_1-x_4}\overbrace{\left(\frac{x-x_3}{x_2-x_3}\,\frac{x-x_4}{x_2-x_4}\,f_2 + \frac{x-x_2}{x_3-x_2}\,\frac{x-x_4}{x_3-x_4}\,f_3 + \frac{x-x_2}{x_4-x_2}\,\frac{x-x_3}{x_4-x_3}\,f_4\right)}^{P_2[x_2,x_3,x_4]}$$

$$P_3[x_1,x_2,x_3,x_4] = \frac{(x-x_4)\,P_2[x_1,x_2,x_3] + (x_1-x)\,P_2[x_2,x_3,x_4]}{x_1-x_4}$$

$$P_3[x_i,x_{i+1},x_{i+2},x_{i+3}] = \frac{(x-x_{i+3})\,P_2[x_i,x_{i+1},x_{i+2}] + (x_i-x)\,P_2[x_{i+1},x_{i+2},x_{i+3}]}{x_i-x_{i+3}}$$

$$P_N[x_i, x_{i+1}, \dots, x_{i+N}] =$$

$$= \frac{(x - x_{i+N}) P_{N-1}[x_i, x_{i+1}, \dots, x_{i+N-1}] + (x_i - x) P_{N-1}[x_{i+1}, x_{i+2}, \dots, x_{i+N}]}{x_i - x_{i+N}}$$

Neville's method

x_1	y_1	$P_0[x_1]$	$P_1[x_1, x_2]$	$P_2[x_1, x_2, x_3]$
x_2	y_2	$P_0[x_2]$	$P_1[x_2, x_3]$	$P_2[x_2, x_3, x_4]$
x_3	y_3	$P_0[x_3]$	$P_1[x_3, x_4]$	
x_4	y_4	$P_0[x_4]$		

$P_3[x_1, x_2, x_3, x_4]$

Lagrange's method (computational implementation)

$$l_i(x_c) = \frac{x_c - x_1}{x_i - x_1} \frac{x_c - x_2}{x_i - x_2} \cdots \frac{x_c - x_{i-1}}{x_i - x_{i-1}} \frac{x_c - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x_c - x_{N+1}}{x_i - x_{N+1}}$$

$$P_N(x_c) = \underbrace{l_1(x_c) f_1}_{j=1} + \cdots + \underbrace{l_{N+1}(x_c) f_{N+1}}_{j=N+1}$$

$j = N+1$ → number of data

Consider a data set formed by 4 points $(N=3)$

$x = [x_1, x_2, x_3, x_4]$

$y = [y_1, y_2, y_3, y_4]$

x_c interpolation point

mask = np.zeros(4, dtype = bool) → [False, False, False, False]

mask[0] = True

$j = 1$

$$l_1 = \frac{(x_c - x_2) * (x_c - x_3) * (x_c - x_4)}{(x_1 - x_2) * (x_1 - x_3) * (x_1 - x_4)}$$

~~mask = [False, True, True, True]~~

x[mask] = $[x_2, x_3, x_4]$

$$l_1 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_1 - x[\text{mask}])}$$

$$\boxed{\begin{array}{l} a = [a_1, a_2, a_3] \\ \text{np.prod}(a) = a_1 * a_2 * a_3 \end{array}}$$

$a = [a_1, a_2, a_3]$

$\text{np.roll}(a, 1) = [a_3, a_1, a_2]$

$j = 2$

mask = np.roll(mask, 1)

mask = [~~True, False~~, True, True]

$$l_2 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_2 - x[\text{mask}])}$$

$j = 3$

np.roll(mask, 1)

mask = [~~True, True~~, False, True]

$$l_3 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_3 - x[\text{mask}])}$$

$j = 4$

np.roll(mask, 1)

mask = [~~True, True~~, True, False]

$$l_4 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_4 - x[\text{mask}])}$$

Neville's method (computational implementation)

$$P_N[x_i, \ldots, x_{i+N}] = \frac{(x - x_{i+N})\,P_{N-1}[x_i, \ldots, x_{i+N-1}] + (x_i - x)\,P_{N-1}[x_{i+1}, \ldots, x_{i+N}]}{x_i - x_{i+N}}$$

$$\begin{array}{llllll}
x_1 & — & f_1 = P_0[x_1] & P_1[x_1, x_2] & P_2[x_1, \ldots, x_3] & P_3[x_1, \ldots, x_4] \\
x_2 & — & f_2 = P_0[x_2] & P_1[x_2, x_3] & P_2[x_2, \ldots, x_4] & \\
x_3 & — & f_3 = P_0[x_3] & P_1(x_3, x_4) & & \\
\underbrace{x_4}_{x} & — & \underbrace{f_4}_{f} = P_0[x_4] & & &
\end{array}$$

$L = 4$ points

$j = 1$	$j = 2$	$j = 3$	$j = 4$
$k = 1$	$k = 1$	$k = 1$	$k = 1$
$\vdots$	$\vdots$	$k = 2$	
	$k = 3$		
$k = 4$			

$L - j + 1$

x_c interpolating point

$j = 1$

$AUX_1 = P_0[x_1]$

$\vdots$

$AUX_4 = P_0[x_4]$

$j = 2$

$$AUX_1 = \frac{(x_c - x_2)\,P_0[x_1] + (x_1 - x_c)\,P_0[x_2]}{x_1 - x_2} = P_1[x_1, x_2]$$

$\vdots$

$$AUX_3 = \frac{(x_c - x_4)\,P_0[x_3] + (x_3 - x_c)\,P_0[x_4]}{x_3 - x_4} = P_1[x_3, x_4]$$

$j = 3$

$$AUX_1 = \frac{(x_c - x_3) P_1[x_1, x_2] + (x_1 - x_c) P_1[x_2, x_3]}{x_1 - x_3} = P_2[x_1, \dots, x_3]$$

$$AUX_2 = \frac{(x_c - x_4) P_1[x_2, x_3] + (x_2 - x_c) P_1[x_3, x_4]}{x_2 - x_4} = P_2[x_2, \dots, x_4]$$

$j = 4$

$$AUX_1 = \frac{(x_c - x_4) P_2[x_1, \dots, x_3] + (x_1 - x_c) P_2[x_2, \dots, x_4]}{x_1 - x_4} =$$

$$= P_3[x_1, \dots, x_4]$$