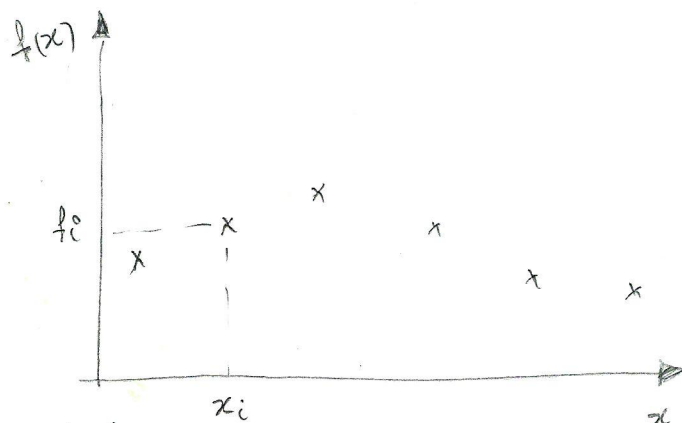


Interpolation

Polynomial interpolation



x	f
x_1	f_1
\vdots	\vdots
x_N	f_N
x_{N+1}	f_{N+1}

Consider that $f(x)$ can be properly described by a polynomial $P_N(x)$

$$f(x) \approx P_N(x)$$

→ polynomial of degree N

Determine $f(x)$ at points x that are different from $x_i, i=1, \dots, N+1$

LAGRANGE'S method

$$P_N(x) = l_1 f_1 + l_2 f_2 + \dots + l_{N+1} f_{N+1}$$

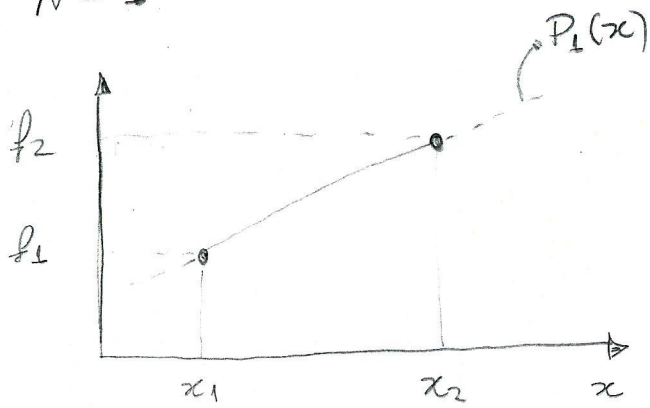
$$= \sum_{i=1}^{N+1} l_i(x) f_i$$

cardinal functions

$$l_i(x) = \frac{x-x_1}{x_i-x_1} \frac{x-x_2}{x_i-x_2} \dots \frac{x-x_{i-1}}{x_i-x_{i-1}} \frac{x-x_{i+1}}{x_i-x_{i+1}} \dots \frac{x-x_{N+1}}{x_i-x_{N+1}}$$

$$= \prod_{\substack{j=1 \\ j \neq i}}^{N+1} \frac{x-x_j}{x_i-x_j}, \quad i=1, \dots, N+1$$

$$N = 1$$



$$P_1(x) = l_1(x) f_1 + l_2(x) f_2$$

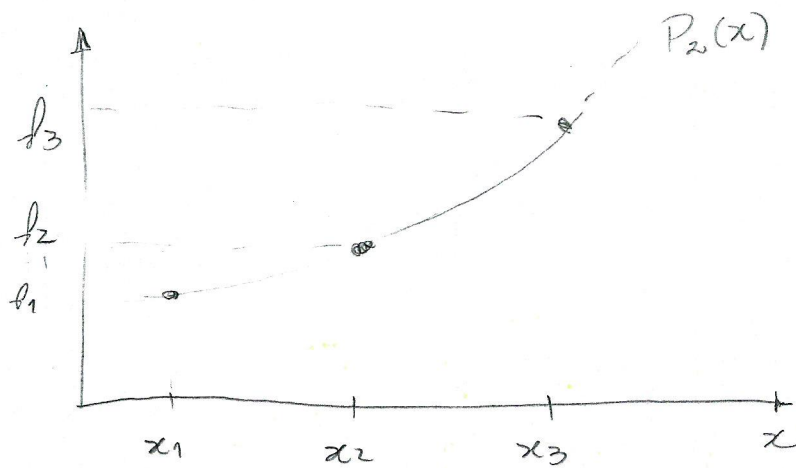
$$= \frac{x - x_2}{x_1 - x_2} f_1 + \frac{x - x_1}{x_2 - x_1} f_2$$

$$P_1(x) \equiv P_1[x_1, x_2], \quad f_1 \equiv P_0[x_1], \quad f_2 \equiv P_0[x_2]$$

$$P_1[x_1, x_2] = \frac{(x - x_2) P_0[x_1] + (x_1 - x) P_0[x_2]}{x_1 - x_2}$$

$$P_1[x_i, x_{i+N}] = \frac{(x - x_{i+N}) P_0[x_i] + (x_i - x) P_0[x_{i+N}]}{x_i - x_{i+N}}$$

$$N=2$$



$$P_2(x) = l_1(x)f_1 + l_2(x)f_2 + l_3(x)f_3$$

$$= \frac{x-x_2}{x_1-x_2} \frac{x-x_3}{x_1-x_3} f_1 + \frac{x-x_1}{x_2-x_1} \frac{x-x_3}{x_2-x_3} f_2 +$$

$$+ \frac{\overbrace{x-x_1}^{x_3-x_1}}{x_3-x_1} \frac{x-x_2}{x_3-x_2} f_3 =$$

$$= \frac{x-x_3}{x_1-x_3} \frac{x-x_2}{x_1-x_2} f_1 + \frac{(x_1-x_2)+(x_2-x_3)}{x_1-x_3} \frac{x-x_3}{x_2-x_3} \frac{\overbrace{x-x_1}^{x_2-x_1}}{x_2-x_1} f_2 +$$

$$+ \frac{x_1-x}{x_1-x_3} \frac{x-x_2}{x_3-x_2} f_3 =$$

$$= \frac{x-x_3}{x_1-x_3} \frac{x-x_2}{x_1-x_2} f_1 + \frac{x-x_3}{x_1-x_3} \frac{\cancel{x_2-x_3}}{\cancel{x_2-x_3}} \frac{x-x_1}{x_2-x_1} f_2 +$$

$$+ \frac{x_1-x}{x_1-x_3} \frac{\cancel{x_1-x_2}}{\cancel{x_1-x_2}} \frac{x-x_3}{x_2-x_3} f_2 + \frac{x_1-x}{x_1-x_3} \frac{x-x_2}{x_3-x_2} f_3$$

$$= \frac{x-x_3}{x_1-x_3} \left(\frac{x-x_2}{x_1-x_2} f_1 + \frac{x-x_1}{x_2-x_1} f_2 \right) +$$

$$+ \frac{x_1-x}{x_1-x_3} \left(\frac{x-x_3}{x_2-x_3} f_2 + \frac{x-x_2}{x_3-x_2} f_3 \right)$$

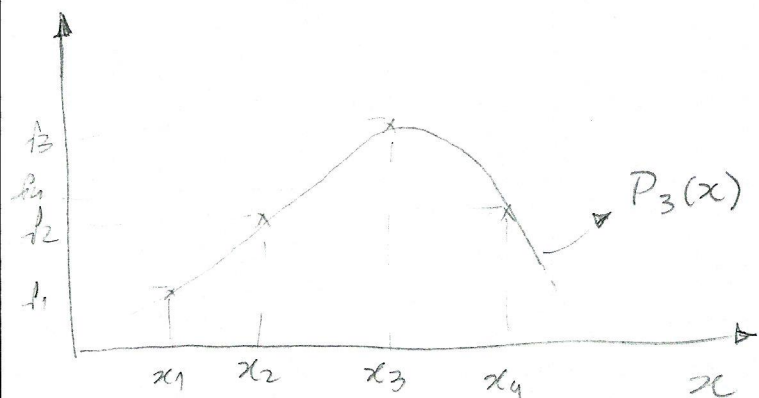
$$P_2(x)$$

$$P_2[x_1, x_2, x_3] = \frac{(x-x_3) P_1[x_1, x_2] + (x_1-x) P_1[x_2, x_3]}{x_1-x_3}$$

$$P_2[x_i, x_{i+1}, x_{i+N}] = \frac{(x-x_{i+2}) P_1[x_i, x_{i+1}] + (x_i-x) P_1[x_{i+1}, x_{i+N}]}{x_i-x_{i+2}}$$

$$P_1[x_i, x_{i+1}] = \frac{(x-x_{i+1}) P_0[x_i] + (x_i-x) P_0[x_{i+1}]}{x_i-x_{i+1}} \quad \text{COMPARISON}$$

$$N=3$$



$$P_3(x) = l_1(x) f_1 + l_2(x) f_2 + l_3(x) f_3 + l_4(x) f_4$$

$$= \frac{x-x_2}{x_1-x_2} \frac{x-x_3}{x_1-x_3} \frac{x-x_4}{x_1-x_4} f_1 +$$

$$+ \frac{x-x_1}{x_2-x_1} \frac{x-x_3}{x_2-x_3} \frac{x-x_4}{x_2-x_4} f_2 +$$

$$+ \frac{x-x_1}{x_3-x_1} \frac{x-x_2}{x_3-x_2} \frac{x-x_4}{x_3-x_4} f_3 +$$

$$+ \frac{x-x_1}{x_4-x_1} \frac{x-x_2}{x_4-x_2} \frac{x-x_3}{x_4-x_3} f_4$$

$$= \frac{x-x_4}{x_1-x_4} \left(\frac{x-x_2}{x_1-x_2} \frac{x-x_3}{x_1-x_3} f_1 \right) +$$

$$+ \frac{(\cancel{x_1-x_2}) + (\cancel{x_2-x_4})}{x_1-x_4} \frac{x-x_1}{\cancel{x_2-x_2}} \frac{x-x_3}{x_2-x_3} \frac{1}{\cancel{x_2-x_4}} (x-x_4) f_2 +$$

$$+ \frac{(\cancel{x_1-x_3}) + (\cancel{x_3-x_4})}{x_1-x_4} \frac{x-x_1}{\cancel{x_3-x_1}} \frac{x-x_2}{x_3-x_2} \frac{1}{\cancel{x_3-x_4}} (x-x_4) f_3 +$$

$$+ \frac{x_1-x}{x_1-x_4} \left(\frac{x-x_2}{x_4-x_2} \frac{x-x_3}{x_4-x_3} f_4 \right) =$$

$$= \frac{x-x_4}{x_1-x_4} \left(\frac{x-x_2}{x_1-x_2} \frac{x-x_3}{x_1-x_3} f_1 \right) + \frac{1}{x_1-x_4} \left(\frac{x-x_1}{x_2-x_1} \frac{x-x_3}{x_2-x_3} \frac{x-x_4}{x_2-x_4} f_2 + \right.$$

$$+ \frac{1}{x_1-x_4} \left[\frac{(\cancel{x_1-x_2})}{(\cancel{x_2-x_1})} \frac{(\cancel{x-x_1})}{(x_2-x_1)} \frac{(x-x_3)}{(x_2-x_3)} \frac{(x-x_4)}{(x_2-x_4)} + \frac{(\cancel{x_2-x_4})}{(x_2-x_1)} \frac{(x-x_1)}{(x_2-x_3)} \frac{(x-x_4)}{(x_2-x_4)} \right] f_2 +$$

$$+ \frac{1}{x_1-x_4} \left[\frac{(\cancel{x_1-x_3})}{(\cancel{x_3-x_1})} \frac{(\cancel{x-x_1})}{(x_3-x_1)} \frac{(x-x_2)}{(x_3-x_2)} \frac{(x-x_4)}{(x_3-x_4)} + \frac{(\cancel{x_3-x_4})}{(x_3-x_1)} \frac{(x-x_1)}{(x_3-x_2)} \frac{(x-x_4)}{(x_3-x_4)} \right] f_3 +$$

$$+ \frac{x_1-x}{x_1-x_4} \left(\frac{x-x_2}{x_4-x_2} \frac{x-x_3}{x_4-x_3} f_4 \right) = P_2[x_1, x_2, x_3]$$

$$= \frac{x-x_4}{x_1-x_4} \left(\frac{x-x_2}{x_1-x_2} \frac{x-x_3}{x_1-x_3} f_1 + \frac{x-x_1}{x_2-x_1} \frac{x-x_3}{x_2-x_3} f_2 + \frac{x-x_1}{x_3-x_1} \frac{x-x_2}{x_3-x_2} f_3 \right) +$$

$$\frac{x_1-x}{x_1-x_4} \left(\frac{x-x_3}{x_2-x_3} \frac{x-x_4}{x_2-x_4} f_2 + \frac{x-x_2}{x_3-x_2} \frac{x-x_4}{x_3-x_4} f_3 + \frac{x-x_2}{x_4-x_2} \frac{x-x_3}{x_4-x_3} f_4 \right)$$

$$P_3[x_1, x_2, x_3, x_4] = \frac{(x-x_4) P_2[x_1, x_2, x_3] + (x_1-x) P_2[x_2, x_3, x_4]}{x_1-x_4}$$

$$P_3[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{(x-x_{i+3}) P_2[x_i, x_{i+1}, x_{i+2}] + (x_i-x) P_2[x_{i+1}, x_{i+2}, x_{i+3}]}{x_i-x_{i+3}}$$

$$P_N[x_i, x_{i+1}, \dots, x_{i+N}] =$$

$$= \frac{(x - x_{i+N}) P_{N-1}[x_i, x_{i+1}, \dots, x_{i+N-1}] + (x_i - x) P_{N-1}[x_{i+1}, x_{i+2}, \dots, x_{i+N}]}{x_i - x_{i+N}}$$

Neville's method

x_1	y_1	$P_0[x_1]$	$P_1[x_1, x_2]$	$P_2[x_1, x_2, x_3]$
x_2	y_2	$P_0[x_2]$	$P_1[x_2, x_3]$	$P_2[x_2, x_3, x_4]$
x_3	y_3	$P_0[x_3]$	$P_1[x_3, x_4]$	
x_4	y_4	$P_0[x_4]$		$P_3[x_1, x_2, x_3, x_4]$

LAGRANGE'S method (computational implementation)

$$l_i(x_c) = \frac{x_c - x_1}{x_i - x_1} \frac{x_c - x_2}{x_i - x_2} \dots \frac{x_c - x_{i-1}}{x_i - x_{i-1}} \frac{x_c - x_{i+1}}{x_i - x_{i+1}} \dots \frac{x_c - x_{N+1}}{x_i - x_{N+1}}$$

$$P_N(x_c) = \underbrace{l_1(x_c)}_{j=1} f_1 + \dots + \underbrace{l_{N+1}(x_c)}_{j=N+1} f_{N+1}$$

→ number of data

Consider a data set formed by 4 points ($N=3$)

$$x = [x_1, x_2, x_3, x_4]$$

$$y = [y_1, y_2, y_3, y_4]$$

x_c interpolation point

$$j = 1$$

$$l_1 = \frac{(x_c - x_2) * (x_c - x_3) * (x_c - x_4)}{(x_1 - x_2) * (x_1 - x_3) * (x_1 - x_4)}$$

mask = np.ones(4, dtype = bool)
mask[0] = False

$$\text{mask} = [\text{False}, \text{True}, \text{True}, \text{True}]$$

$$x[\text{mask}] = [x_2, x_3, x_4]$$

$$l_1 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_1 - x[\text{mask}])}$$

$$a = [a_1, a_2, a_3]$$

$$\text{np.prod}(a) = a_1 * a_2 * a_3$$

$$a = [a_1, a_2, a_3]$$

$$\text{np.roll}(a, 1) = [a_3, a_1, a_2]$$

$$j = 2$$

$$\text{mask} = \text{np.roll}(\text{mask}, 1)$$

$$\text{mask} = [\text{True}, \text{False}, \text{True}, \text{True}]$$

$$l_2 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_2 - x[\text{mask}])}$$

$$j = 3$$

$$\text{np.roll}(\text{mask}, 1)$$

$$\text{mask} = [\text{True}, \text{True}, \text{False}, \text{True}]$$

$$l_3 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_3 - x[\text{mask}])}$$

$$j = 4$$

$$\text{np.roll}(\text{mask}, 1)$$

$$\text{mask} = [\text{True}, \text{True}, \text{True}, \text{False}]$$

$$l_4 = \frac{\text{np.prod}(x_c - x[\text{mask}])}{\text{np.prod}(x_4 - x[\text{mask}])}$$

Neville's method (computational implementation)

$$P_N[x_i, \dots, x_{i+N}] = \frac{(x - x_{i+N})P_{N-1}[x_i, \dots, x_{i+N-1}] + (x_i - x)P_{N-1}[x_{i+1}, \dots, x_{i+N}]}{x_i - x_{i+N}}$$

$$\begin{array}{lcl} x_1 & \text{---} & f_1 = P_0[x_1] \rightarrow P_1[x_1, x_2] \rightarrow P_2[x_1, \dots, x_3] \rightarrow P_3[x_1, \dots, x_4] \\ x_2 & \text{---} & f_2 = P_0[x_2] \rightarrow P_1[x_2, x_3] \rightarrow P_2[x_2, \dots, x_4] \\ x_3 & \text{---} & f_3 = P_0[x_3] \rightarrow P_1[x_3, x_4] \\ x_4 & \text{---} & f_4 = P_0[x_4] \end{array}$$

$N = 4$
 $L = 4$ points

$j = 1$

$k = 1$

\vdots

$k = 4$

$j = 2$

$k = 1$

\vdots

$k = 3$

$j = 3$

$k = 1$

$k = 2$

$j = 4$

$k = 1$

$L - j + 1$

x_c interpolating point

$j = 1$

$$AUX_1 = P_0[x_1]$$

$$\vdots$$

$$AUX_4 = P_0[x_4]$$

$j = 2$

$$AUX_1 = \frac{(x_c - x_2)P_0[x_1] + (x_1 - x_c)P_0[x_2]}{x_1 - x_2} = P_1[x_1, x_2]$$

\vdots

$$AUX_3 = \frac{(x_c - x_4)P_0[x_3] + (x_3 - x_c)P_0[x_4]}{x_3 - x_4} = P_1[x_3, x_4]$$

$$\hat{j} = 3$$

$$AUX_1 = \frac{(x_c - x_3) P_1[x_1, x_2] + (x_1 - x_c) P_1[x_2, x_3]}{x_1 - x_3} = P_2[x_1, \dots, x_3]$$

$$AUX_2 = \frac{(x_c - x_4) P_1[x_2, x_3] + (x_2 - x_c) P_1[x_3, x_4]}{x_2 - x_4} = P_2[x_2, \dots, x_4]$$

$$\hat{j} = 4$$

$$AUX_1 = \frac{(x_c - x_4) P_2[x_1, \dots, x_3] + (x_1 - x_c) P_2[x_2, \dots, x_4]}{x_1 - x_4} =$$

$$= P_3[x_1, \dots, x_4]$$