# Topics on nonlinear optimization

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  - Levenberg-Marquardt's method

# Introduction

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} \qquad \mathbf{g}(\mathbf{p}) = \begin{bmatrix} g_1(\mathbf{p}) \\ g_2(\mathbf{p}) \\ \vdots \\ g_N(\mathbf{p}) \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{bmatrix}$$

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$$\Phi(\mathbf{p}) = [\mathbf{d} - \mathbf{g}(\mathbf{p})]^\top [\mathbf{d} - \mathbf{g}(\mathbf{p})]$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{bmatrix}$$

$$\nabla \Phi(\mathbf{p}^*) = \mathbf{0}_{M \times 1}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} \qquad \mathbf{g}(\mathbf{p}) = \begin{bmatrix} g_1(\mathbf{p}) \\ g_2(\mathbf{p}) \\ \vdots \\ g_N(\mathbf{p}) \end{bmatrix}$$

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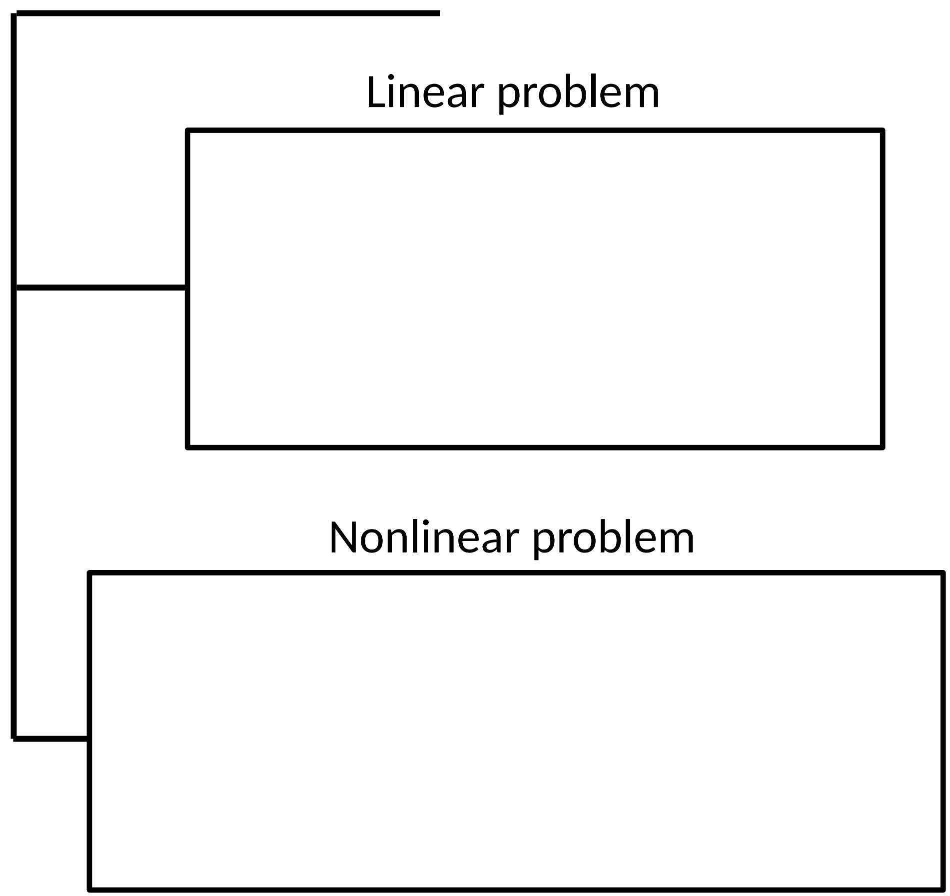

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$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} \qquad \boxed{\mathbf{g}(\mathbf{p}) = \begin{bmatrix} g_1(\mathbf{p}) \\ g_2(\mathbf{p}) \\ \vdots \\ g_N(\mathbf{p}) \end{bmatrix}}$$

Linear problem

Nonlinear problem

$$\Phi(\mathbf{p}) = [\mathbf{d} - \mathbf{g}(\mathbf{p})]^\top [\mathbf{d} - \mathbf{g}(\mathbf{p})]$$

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$$\nabla\Phi(\mathbf{p}^*) = \mathbf{0}_{M \times 1}$$

Linear problem

$$\mathbf{g}(\mathbf{p}) = \mathbf{B}\mathbf{p} + \mathbf{b}$$

$$\mathbf{p}^* = \left(\mathbf{B}^\top\mathbf{B}\right)^{-1}\mathbf{B}^\top(\mathbf{d} - \mathbf{b})$$

Nonlinear problem

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{bmatrix}$$

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$$\Phi(\mathbf{p}) = [\mathbf{d} - \mathbf{g}(\mathbf{p})]^\top [\mathbf{d} - \mathbf{g}(\mathbf{p})]$$

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Linear problem

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$$\mathbf{p}^* = \left(\mathbf{B}^\top\mathbf{B}\right)^{-1}\mathbf{B}^\top(\mathbf{d} - \mathbf{b})$$

Nonlinear problem

$$\mathbf{g}(\mathbf{p}) \neq \mathbf{B}\mathbf{p} + \mathbf{b}$$

$$\mathbf{p}_k = \mathbf{p}_{k-1} + \Delta\mathbf{p}_k$$

$$\mathbf{p}^* \approx \mathbf{p}_0 + \Delta\mathbf{p}_1 + \cdots + \Delta\mathbf{p}_L$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_M \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} \qquad \boxed{\mathbf{g}(\mathbf{p}) = \begin{bmatrix} g_1(\mathbf{p}) \\ g_2(\mathbf{p}) \\ \vdots \\ g_N(\mathbf{p}) \end{bmatrix}}$$

$$\Phi(\mathbf{p}) = [\mathbf{d} - \mathbf{g}(\mathbf{p})]^\top [\mathbf{d} - \mathbf{g}(\mathbf{p})]$$

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Linear problem

$$\mathbf{g}(\mathbf{p}) = \mathbf{B}\mathbf{p} + \mathbf{b}$$

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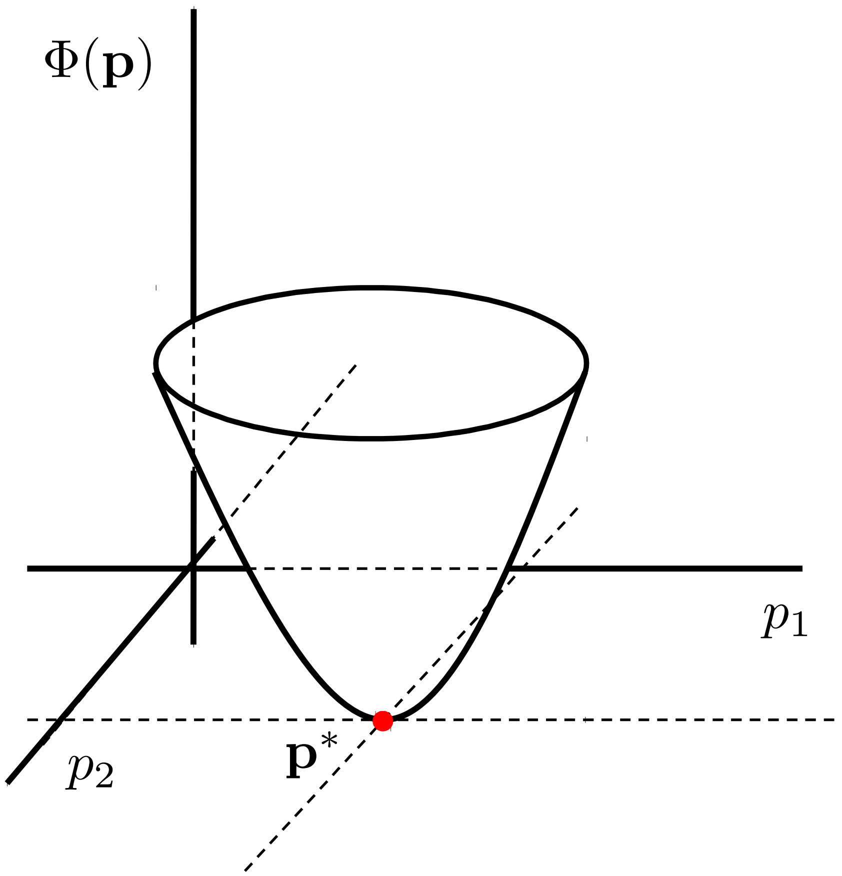
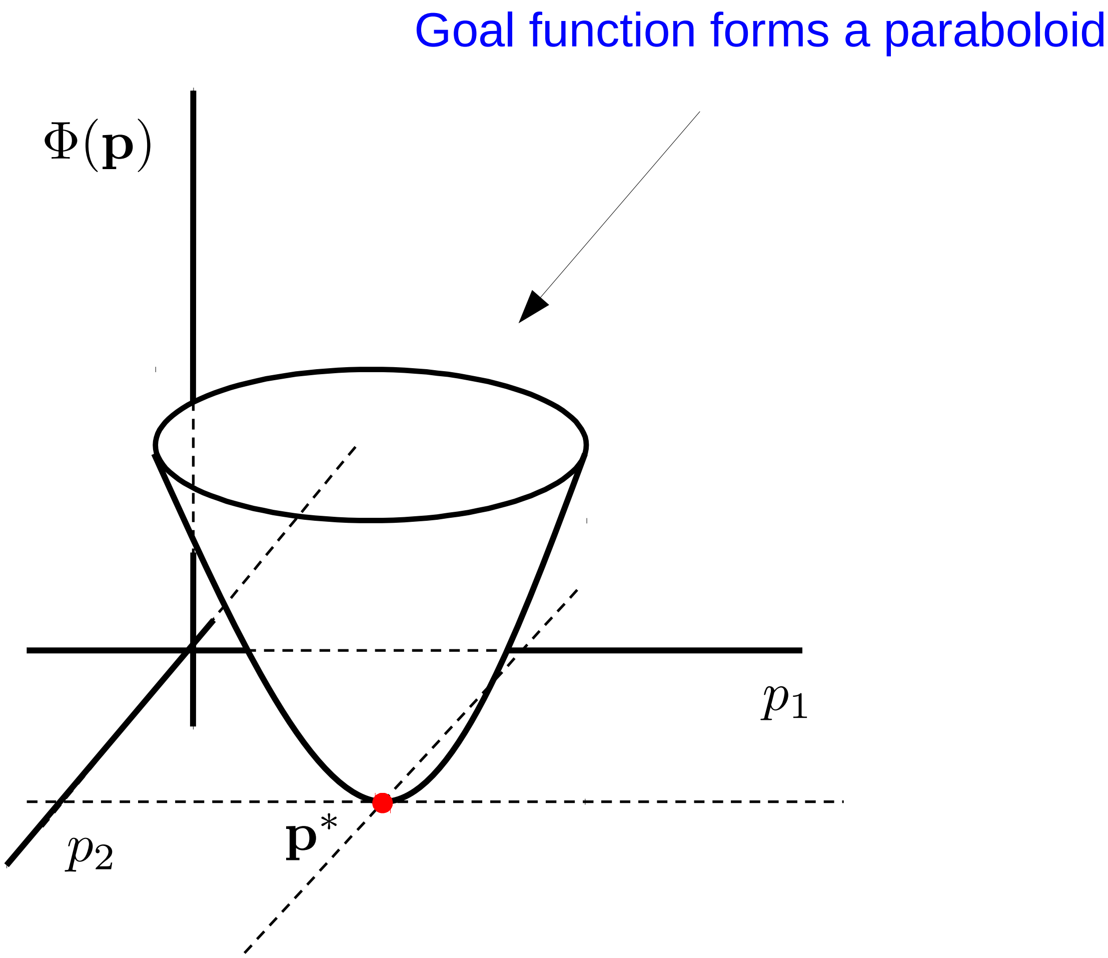
$$\mathbf{g}(\mathbf{p}) \neq \mathbf{B}\mathbf{p} + \mathbf{b}$$

$$\mathbf{p}_k = \mathbf{p}_{k-1} + \Delta\mathbf{p}_k$$

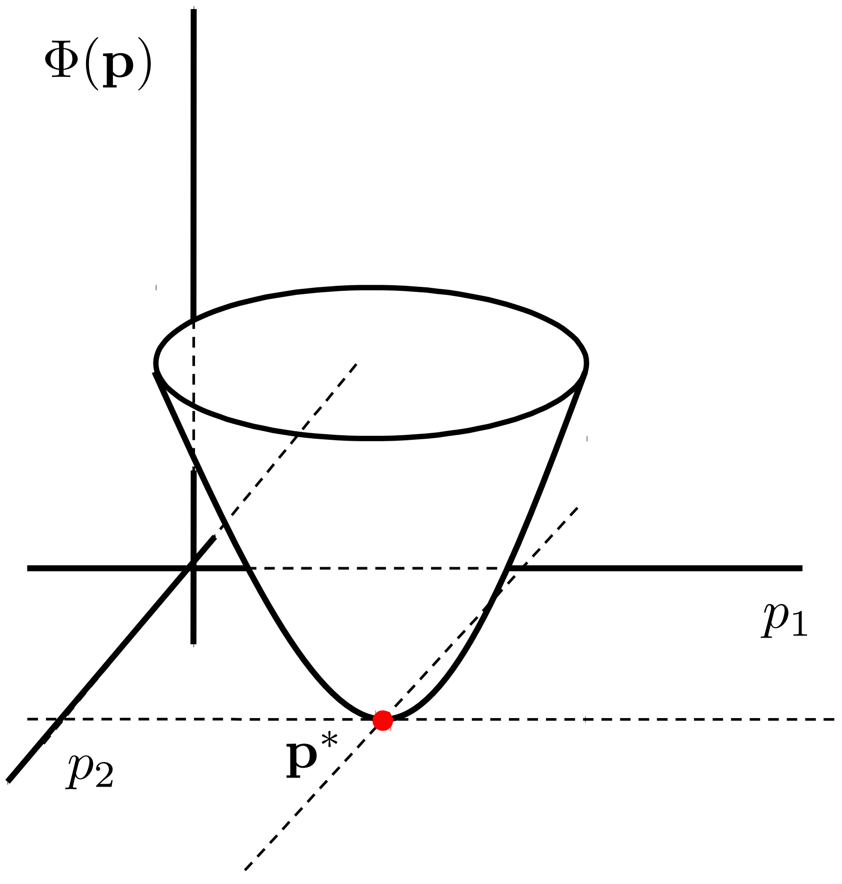
$$\mathbf{p}^* \approx \mathbf{p}_0 + \Delta\mathbf{p}_1 + \cdots + \Delta\mathbf{p}_L$$



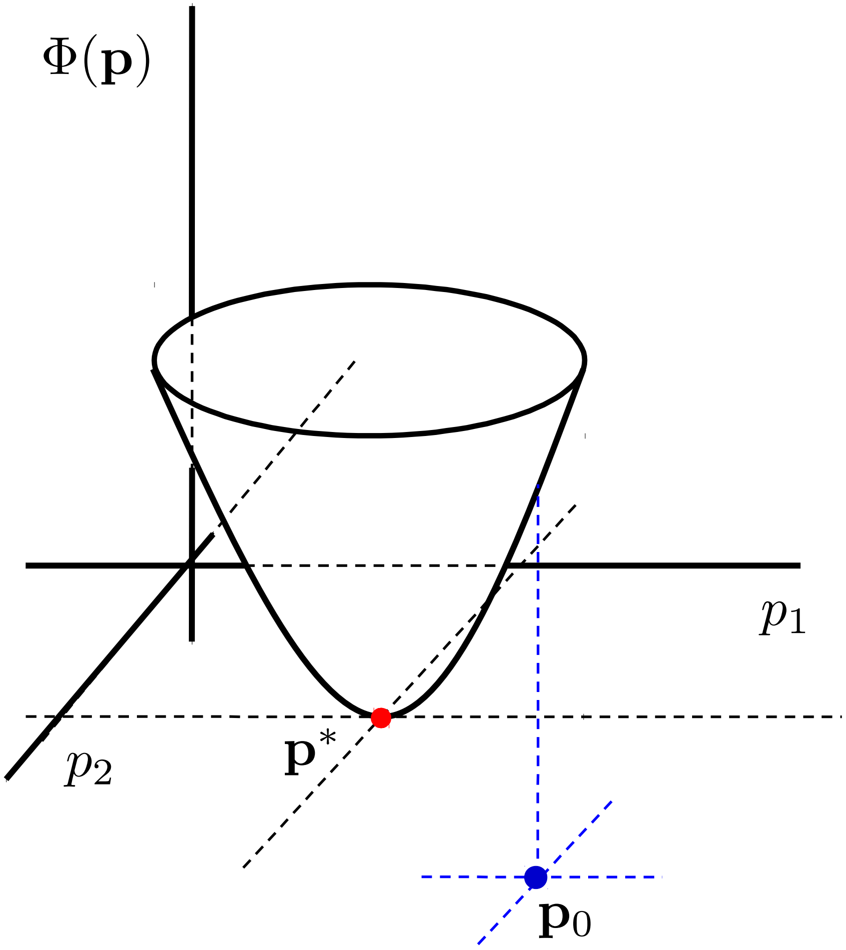

Linear
problem



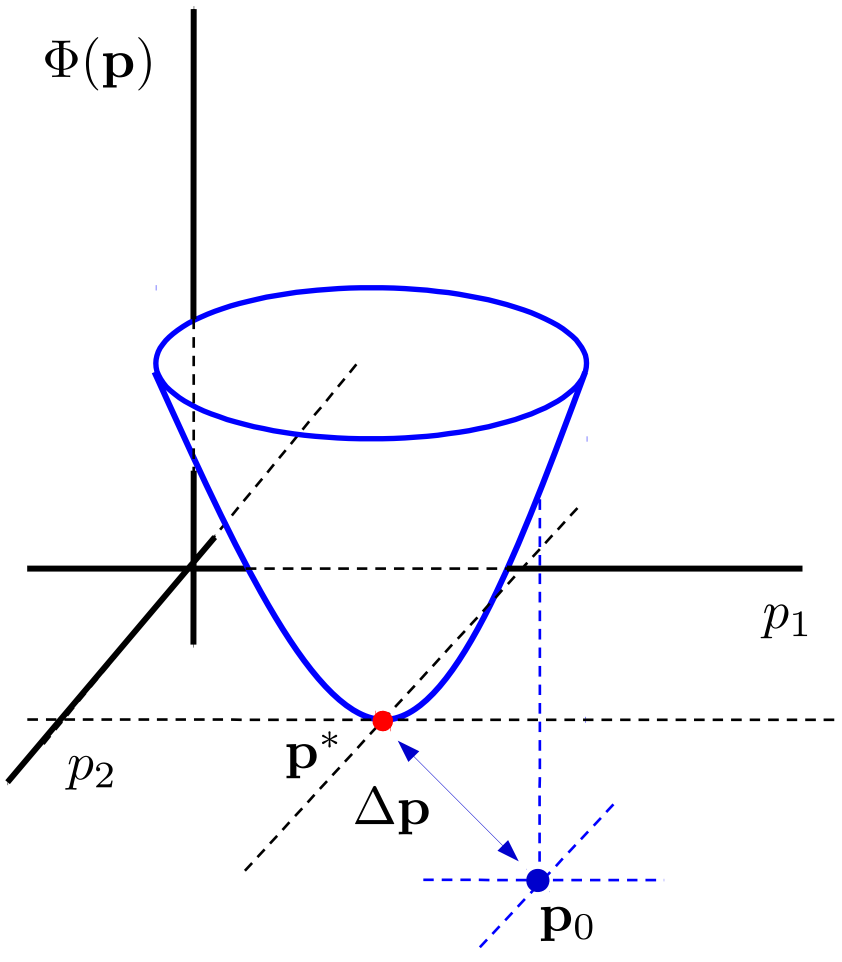

Linear problem



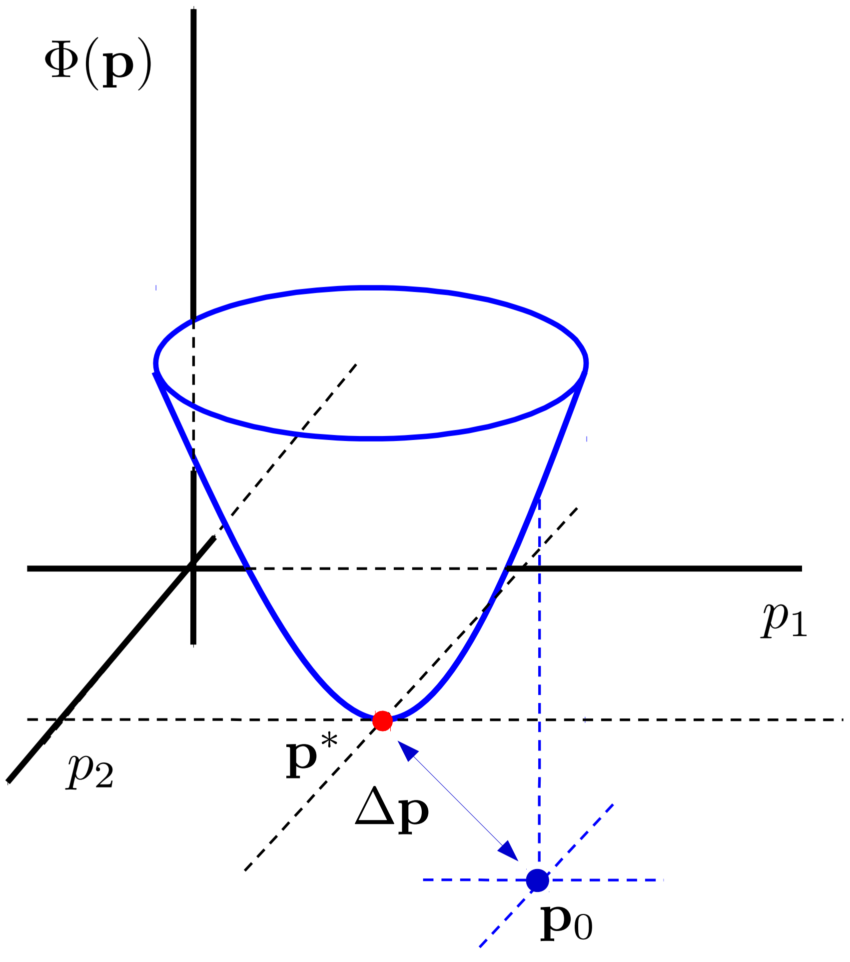

The minimum can be computed in a single step




Or iteratively, from a given initial approximation

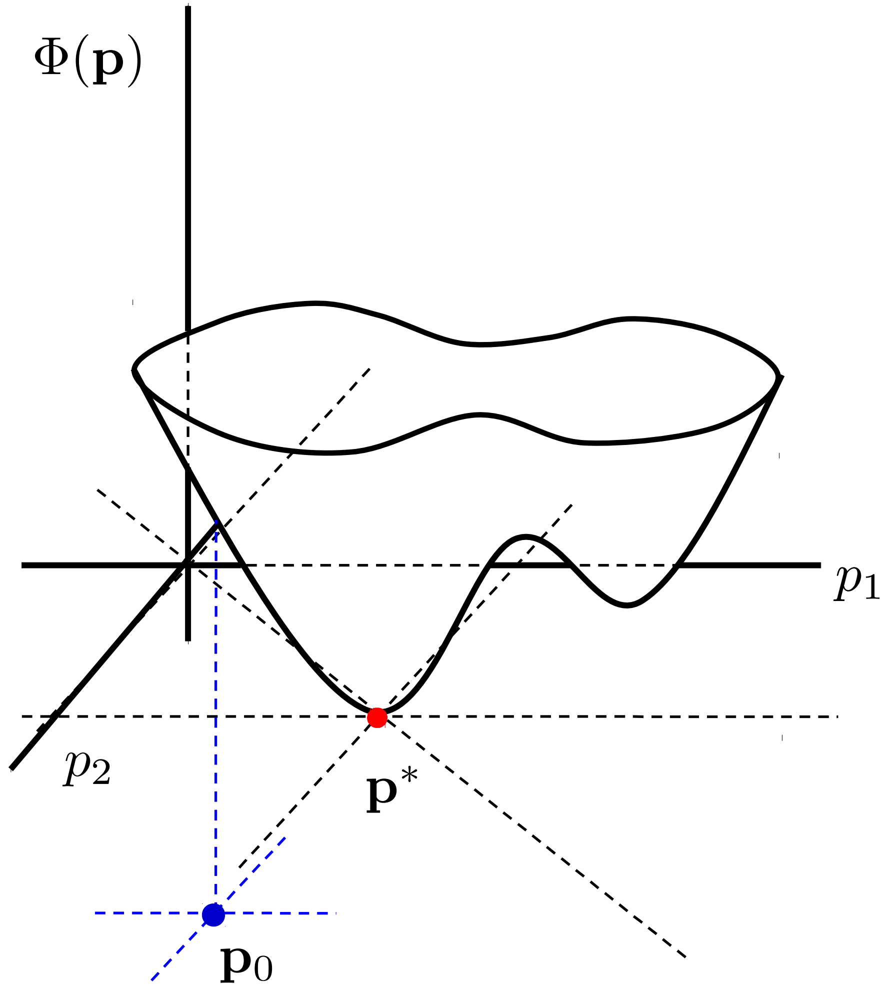


$\Phi(\mathbf{p})$
$p_1$
$p_2$
$\mathbf{p}^*$
$\Delta\mathbf{p}$
$\mathbf{p}_0$

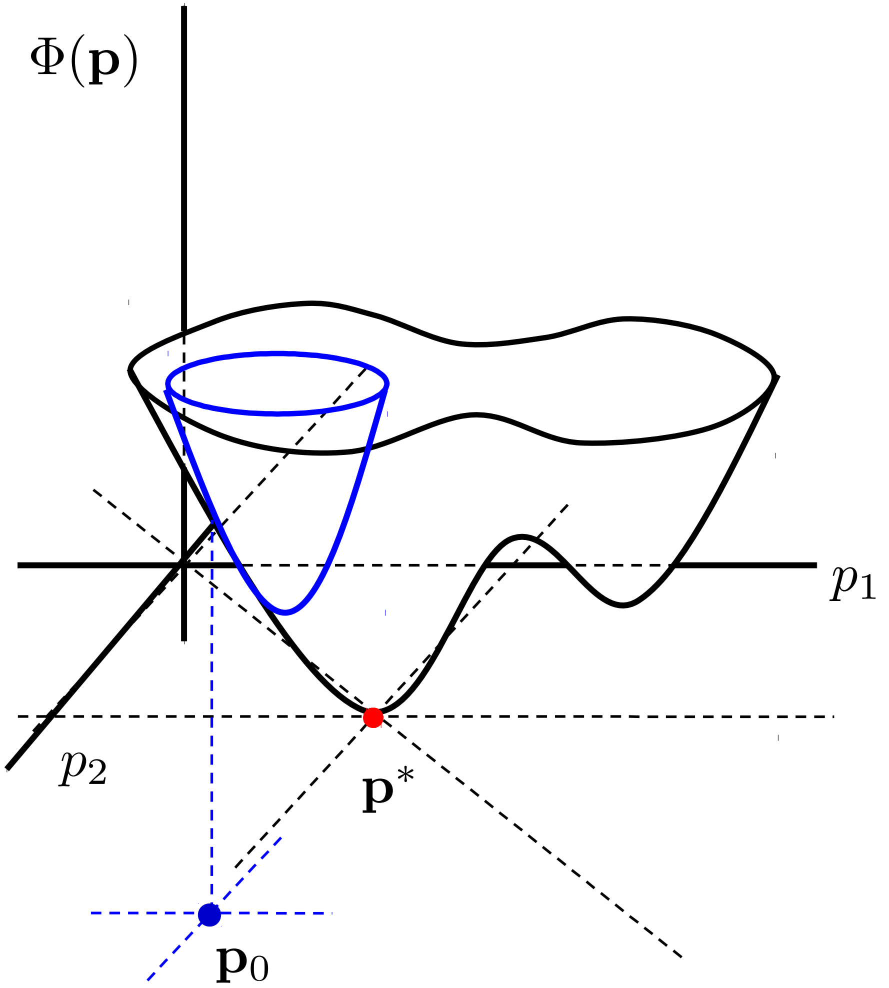



In this case, the minimum is estimated in a single step from the given initial approximation

On the other hand, in a nonlinear problem, the minimum is estimated after several steps from the initial approximation

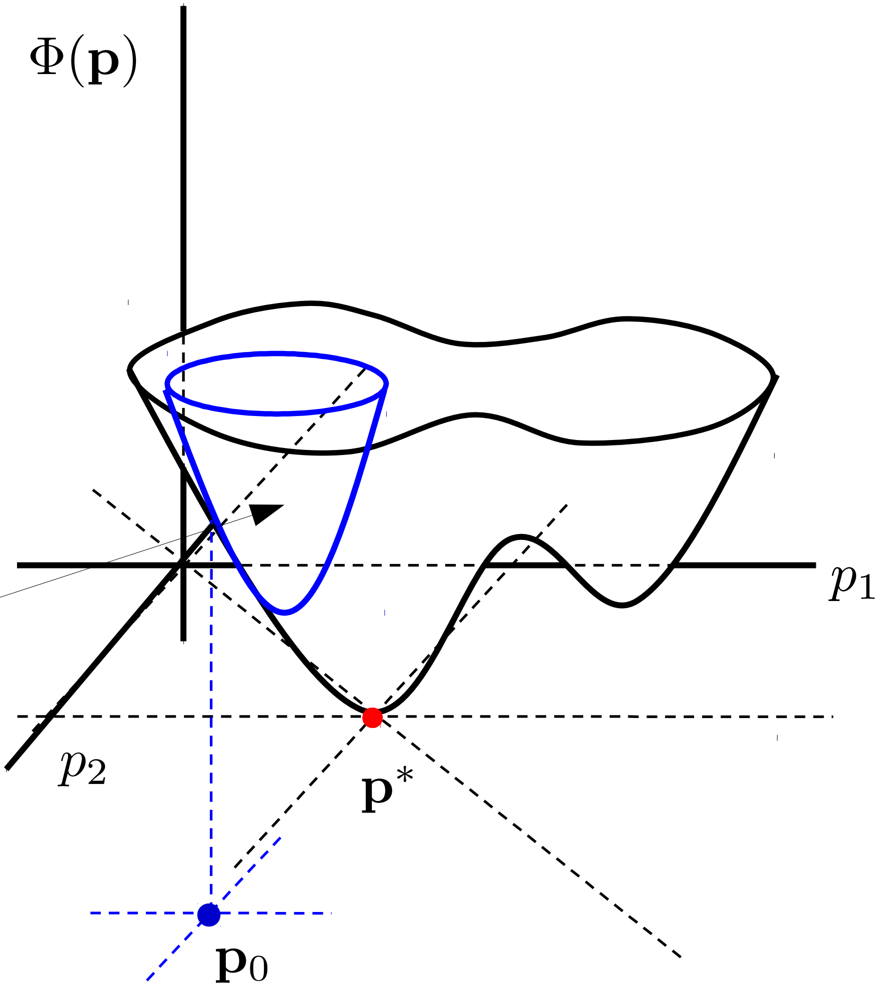


Approximate the nonlinear function around the initial approximation

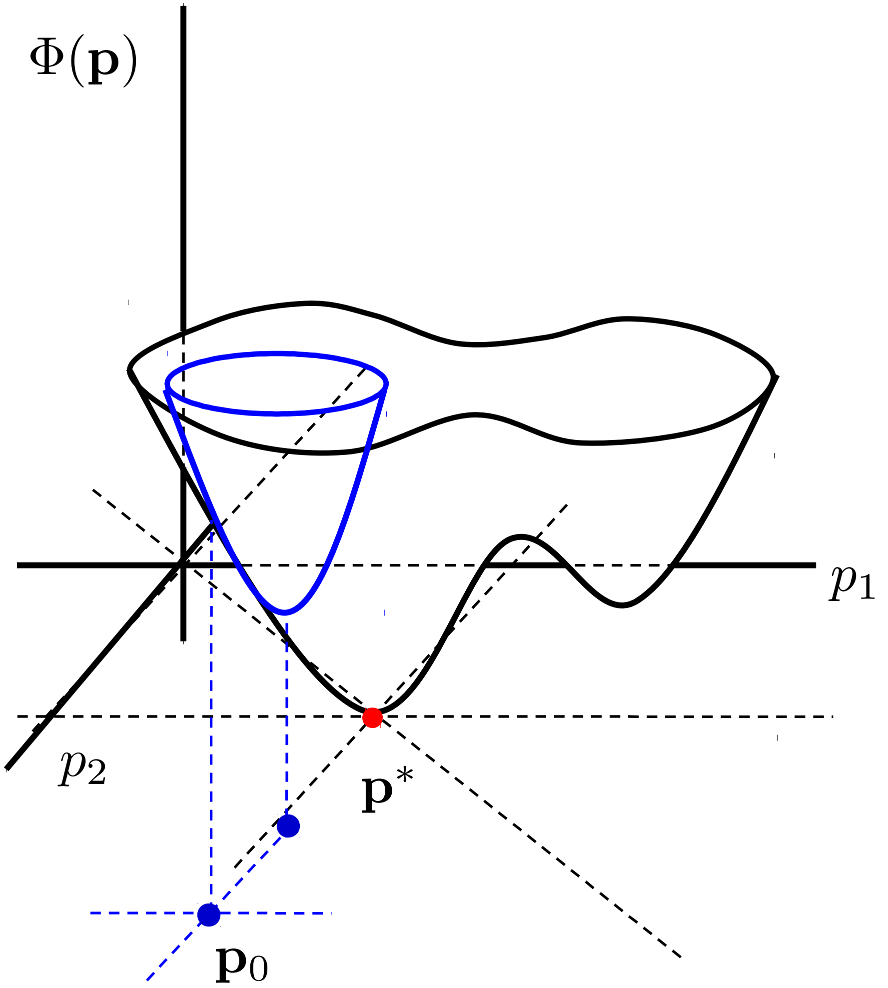


# Approximate the nonlinear function around the initial approximation

Approximating paraboloid

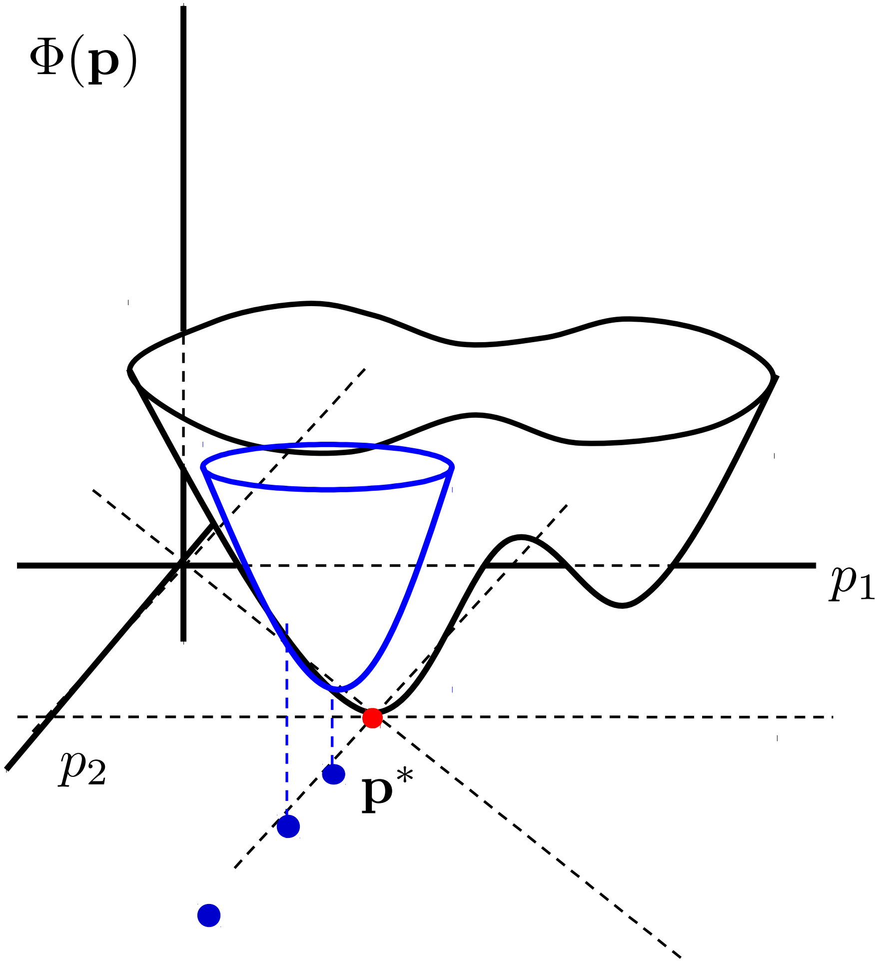


Approximate the nonlinear function around the initial approximation

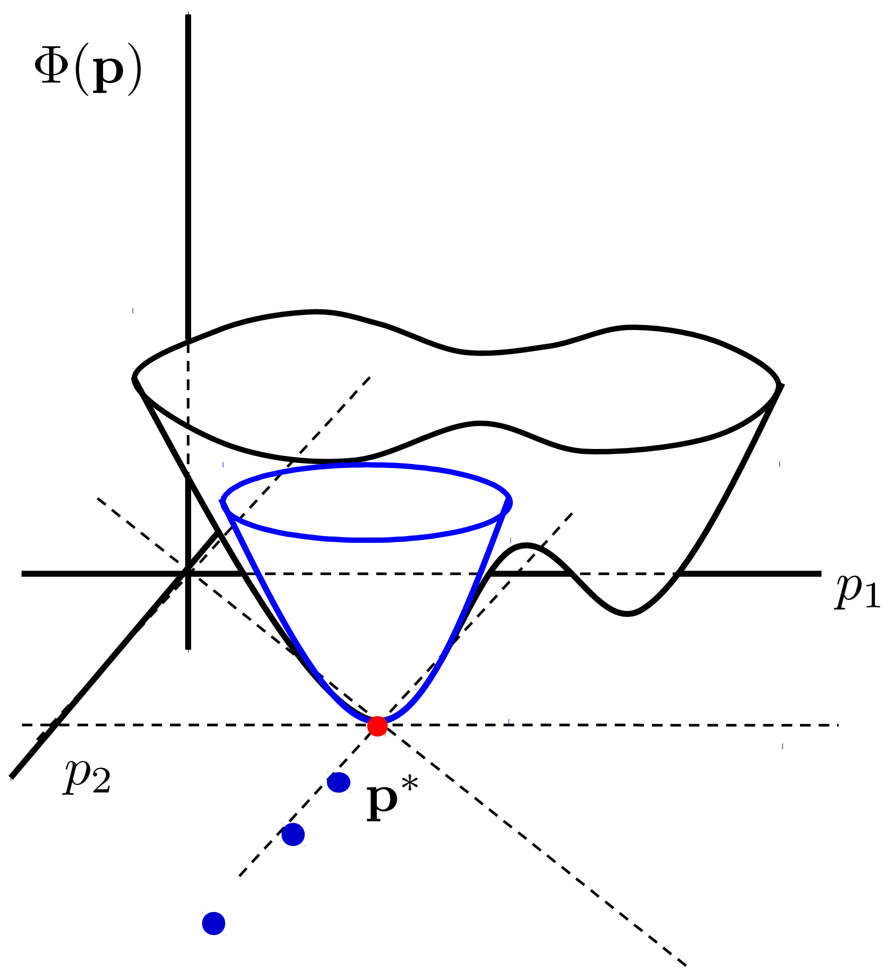


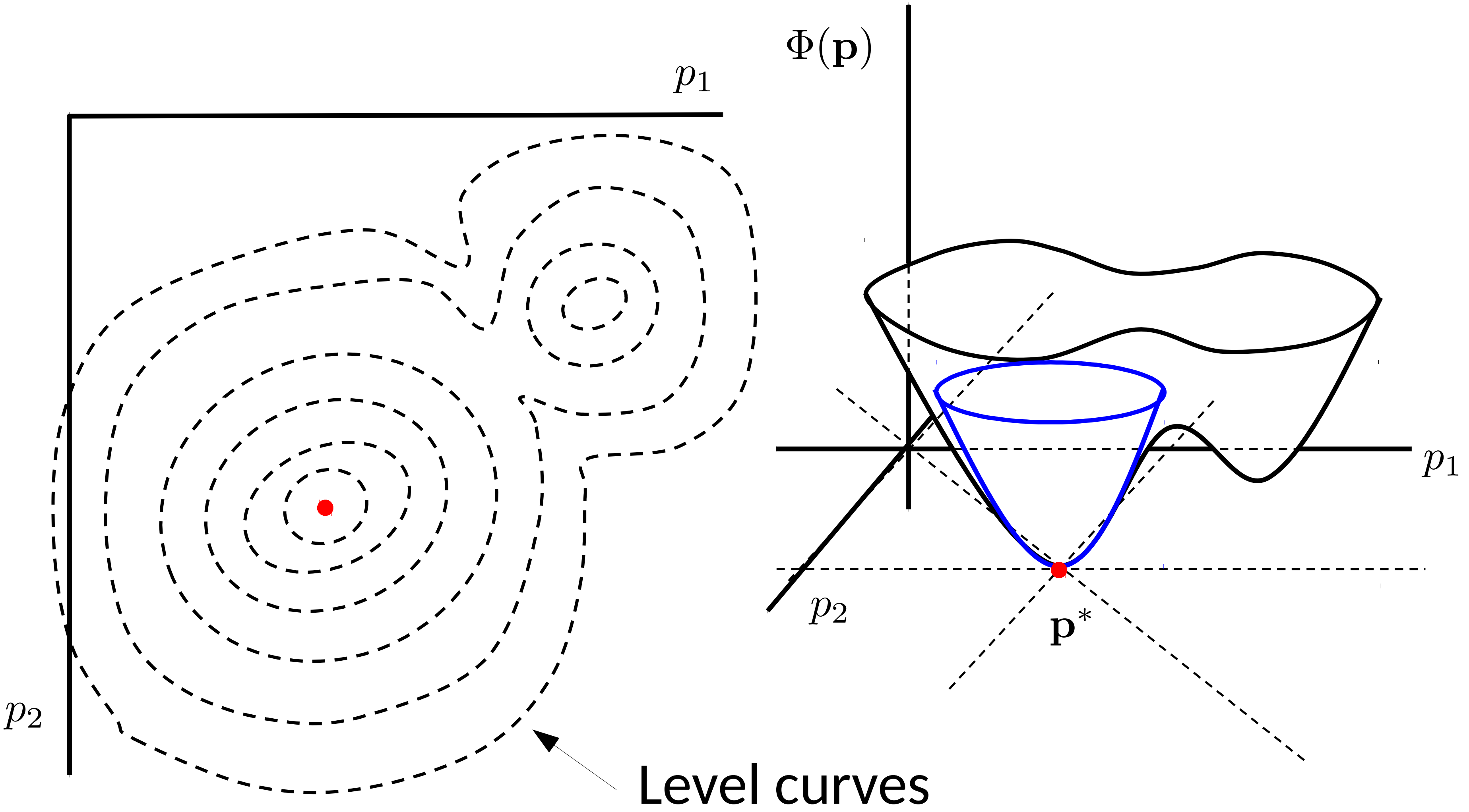

This approximation also has a minimum

Compute a new approximation around this minimum

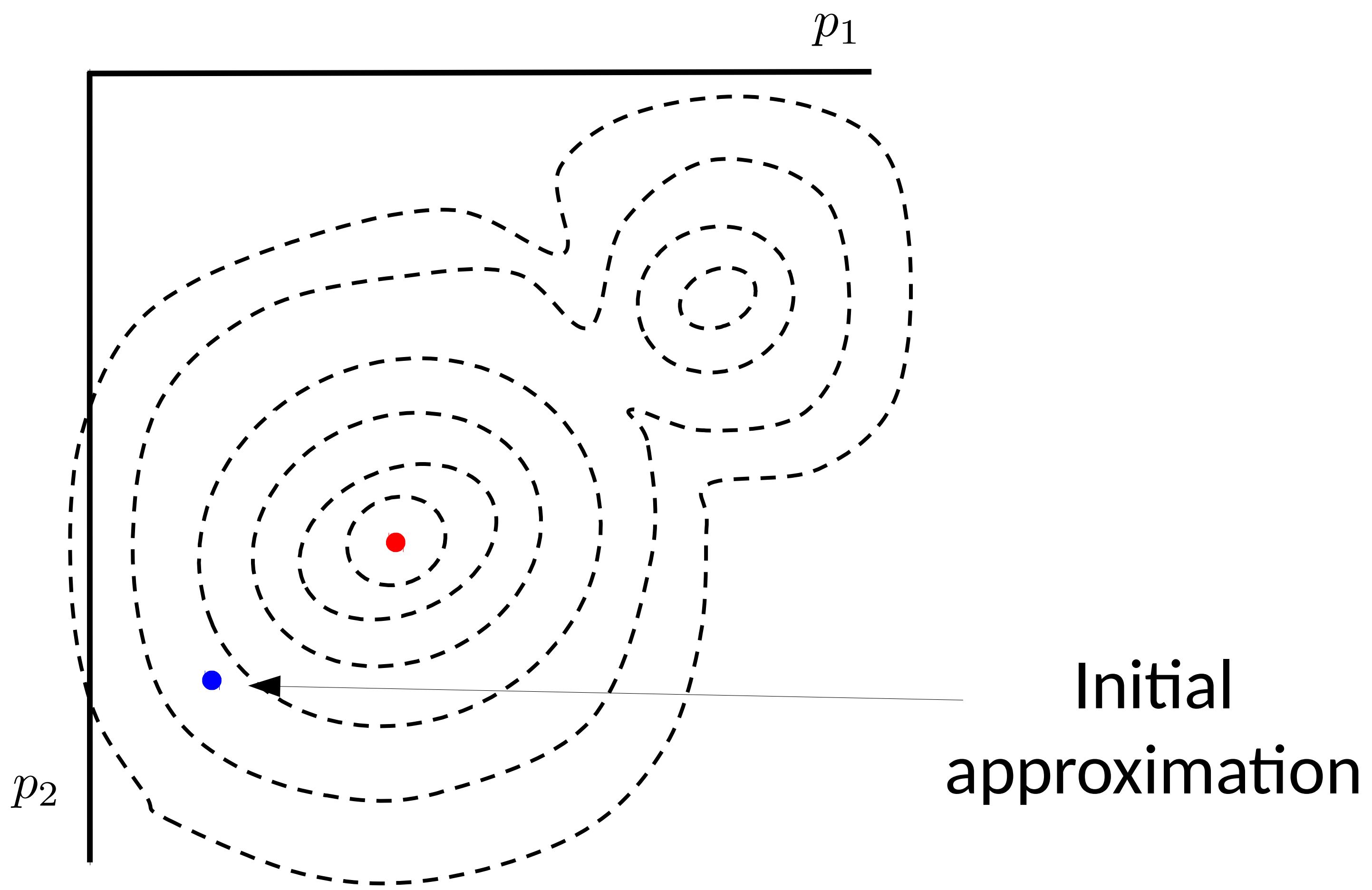


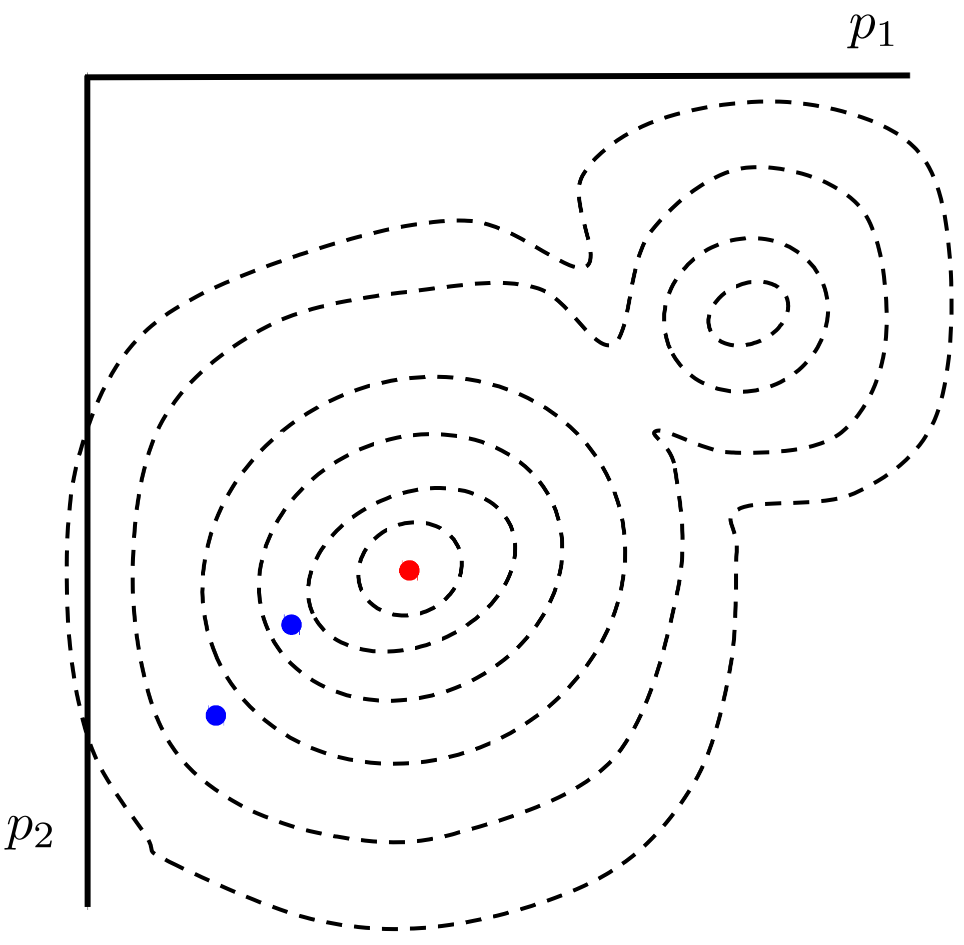
And so on ...



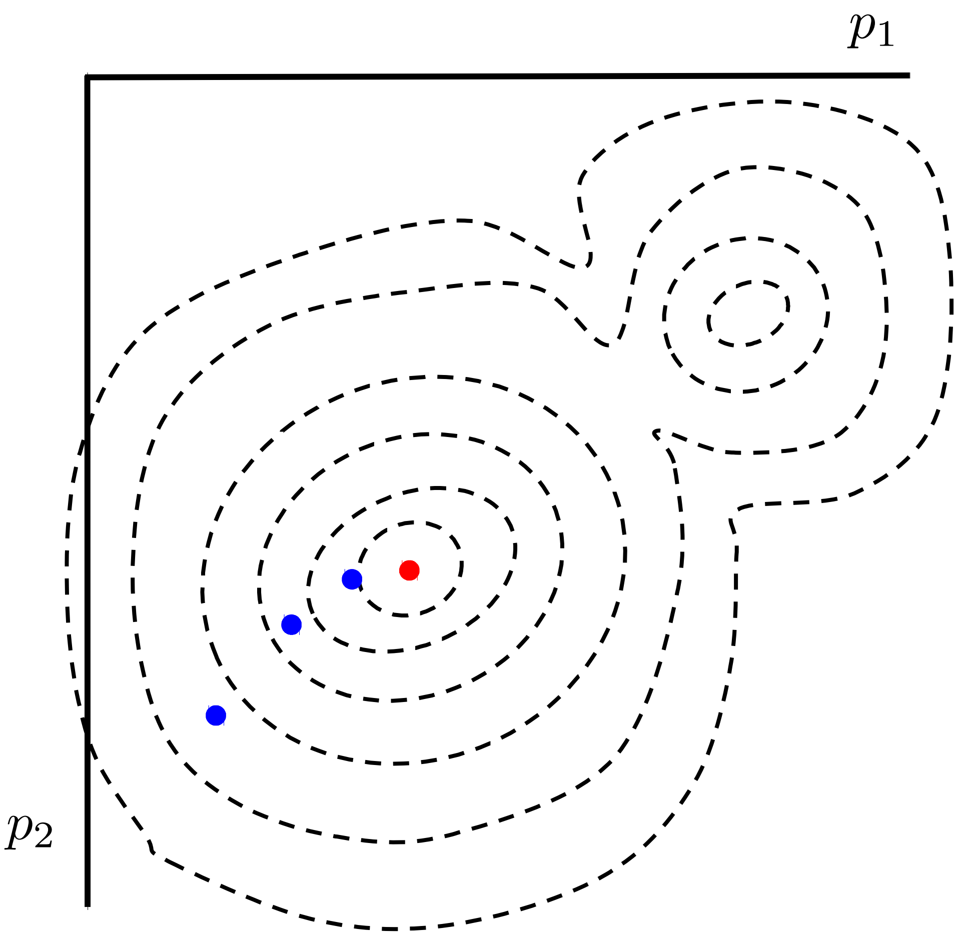


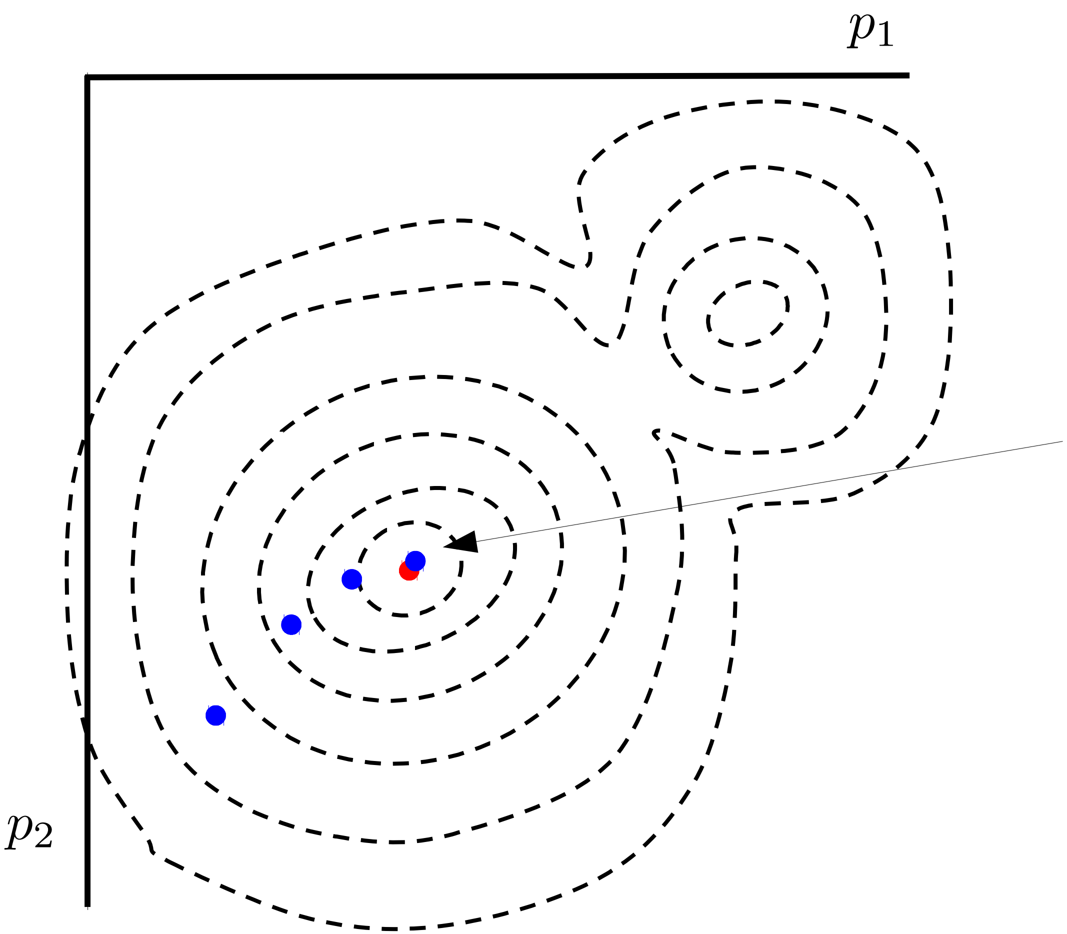
$p_1$
$p_2$
$\Phi(\mathbf{p})$
$p_1$
$p_2$
$\mathbf{p}^*$
Level curves


$p_1$
$p_2$
Initial approximation

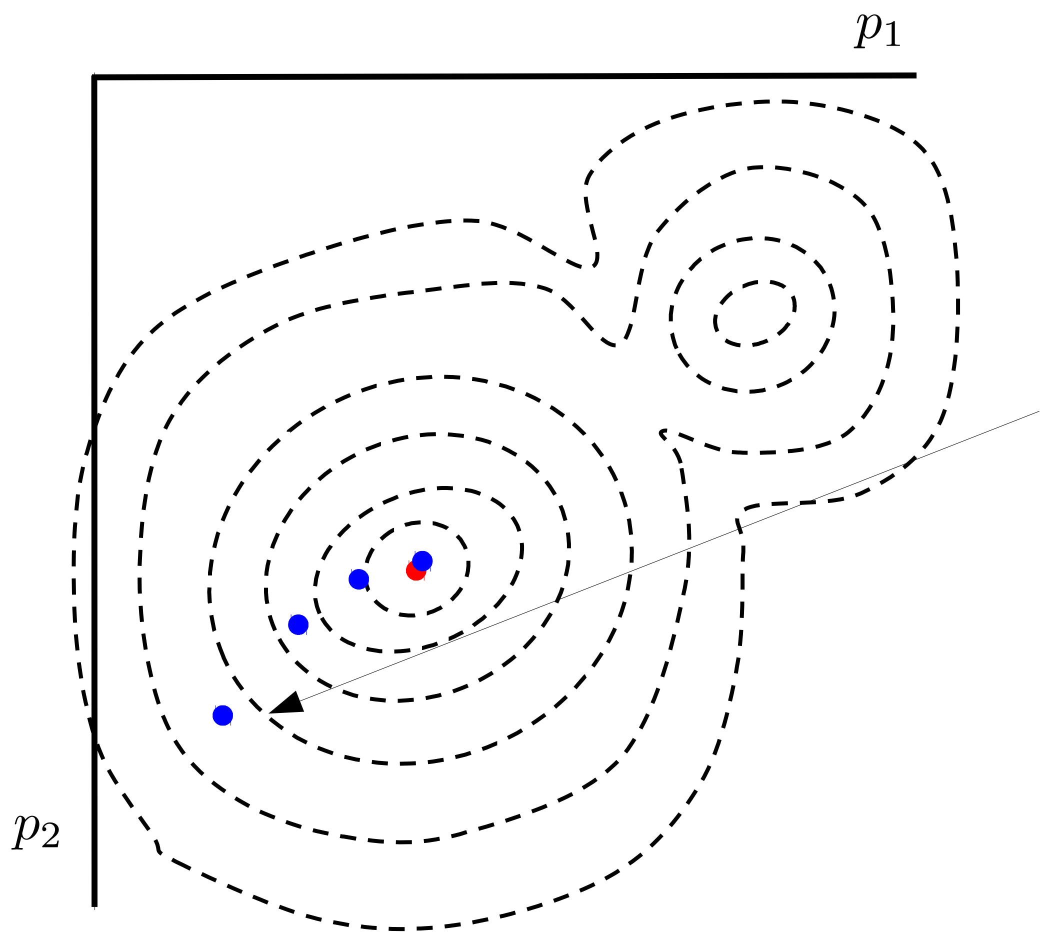


$p_1$
$p_2$

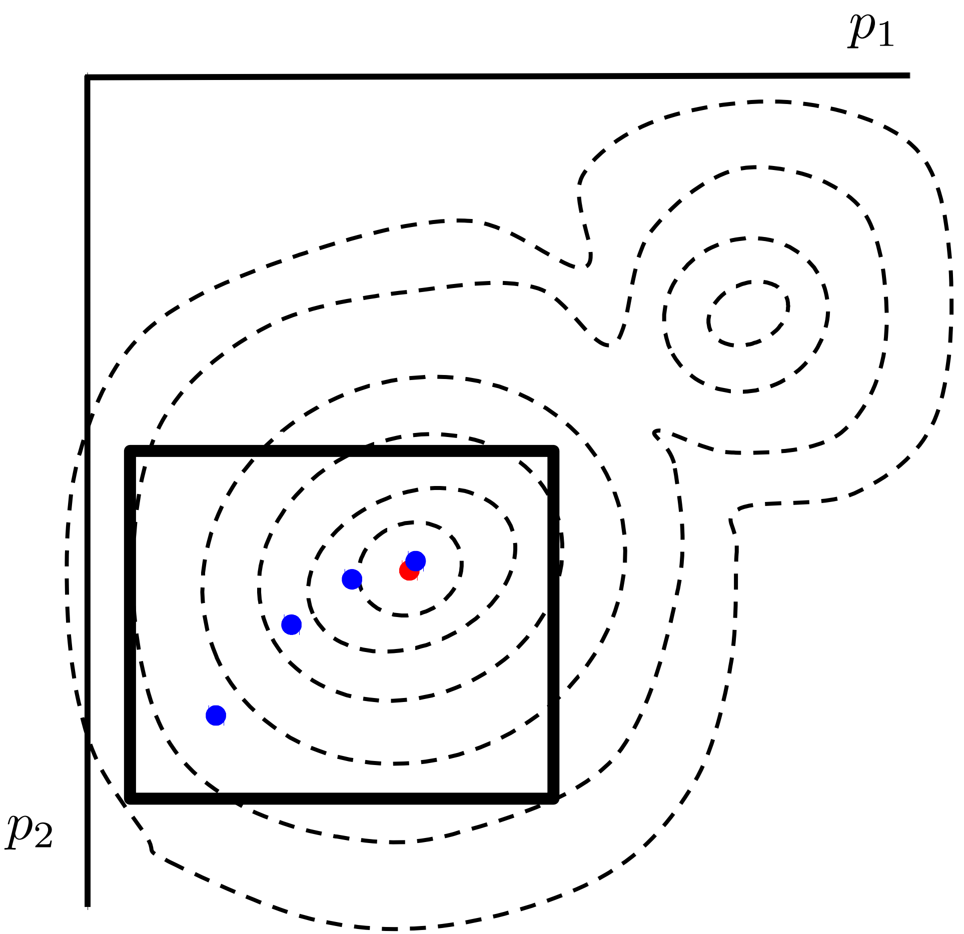

$p_1$
$p_2$

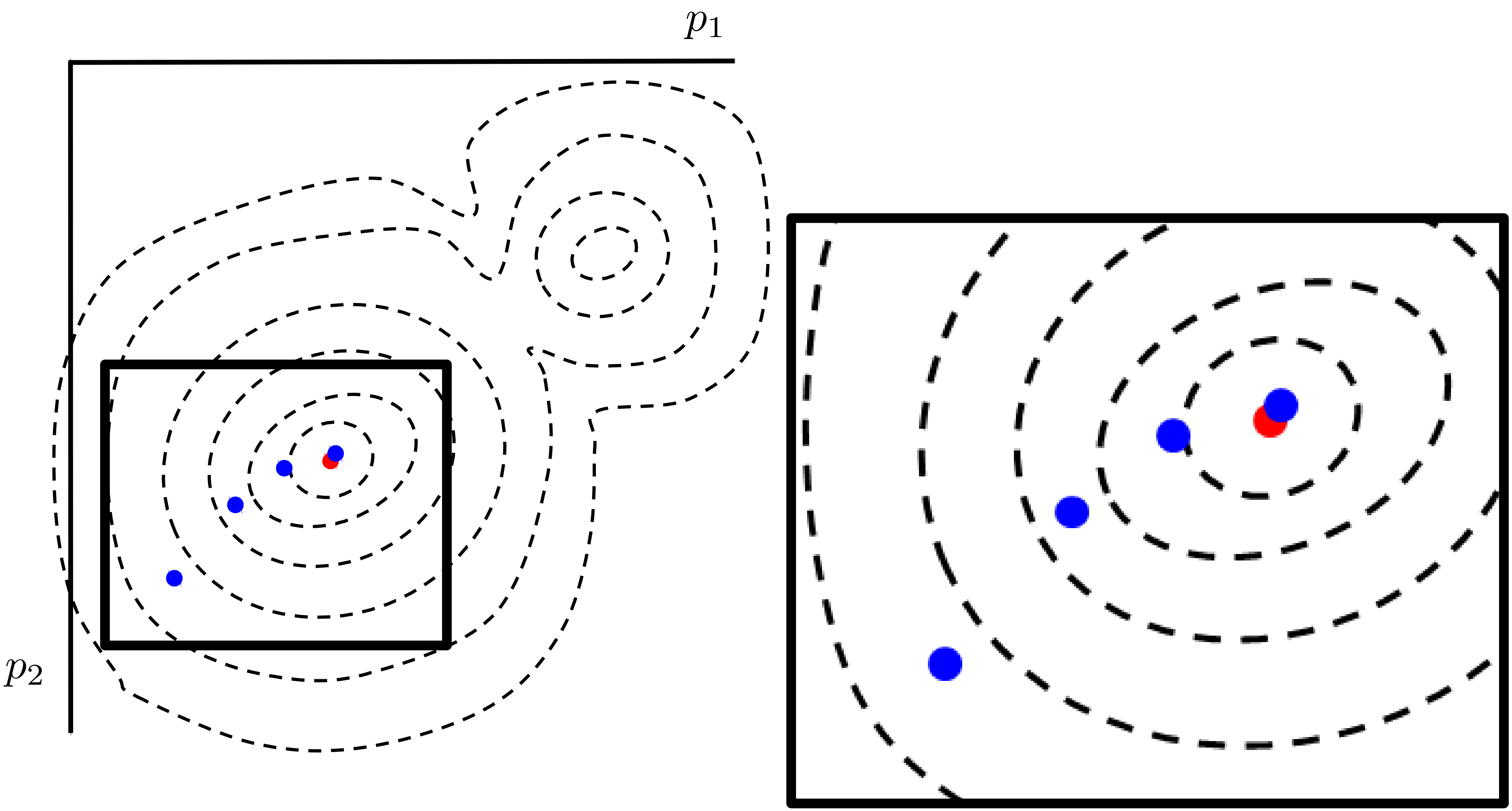



Estimated minimum

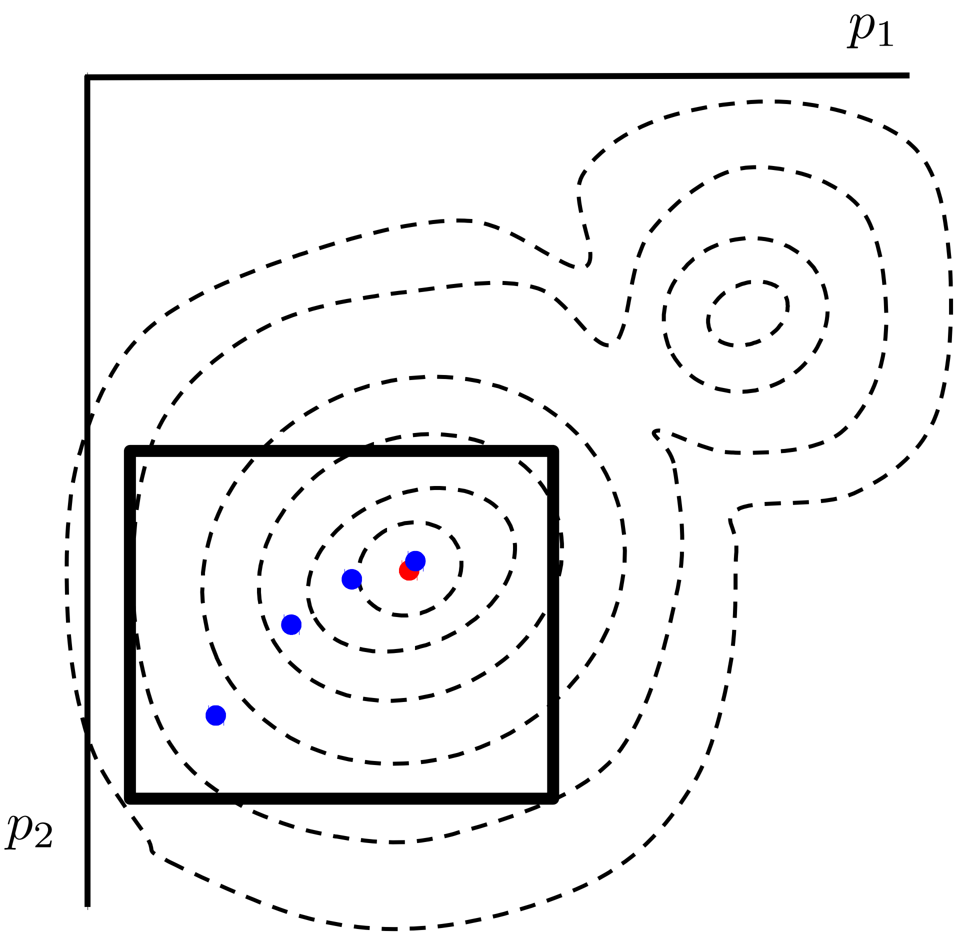



Given a point, it is needed to define a direction and the step length

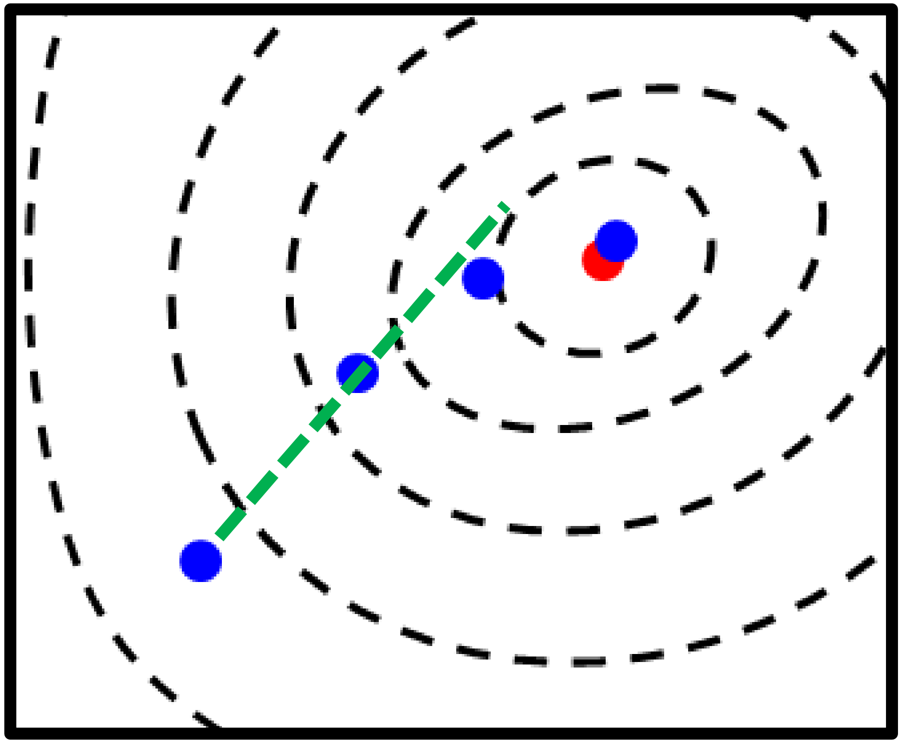

$p_1$
$p_2$

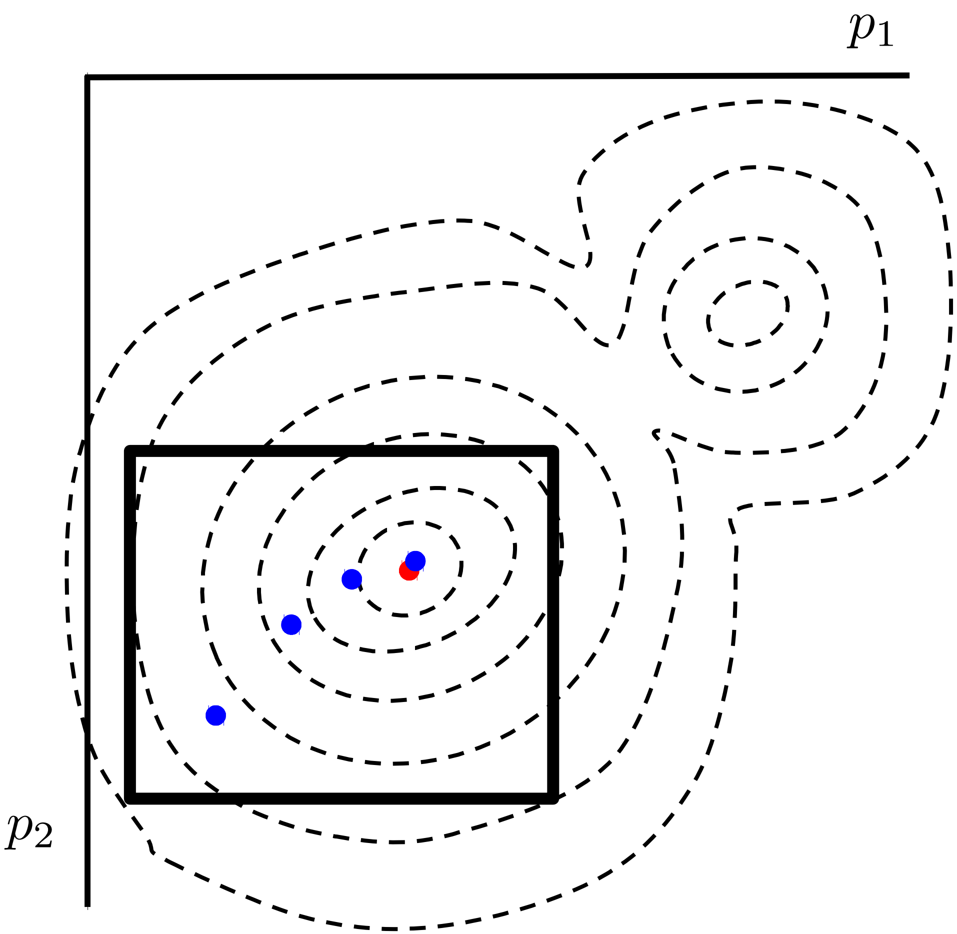


$p_1$
$p_2$

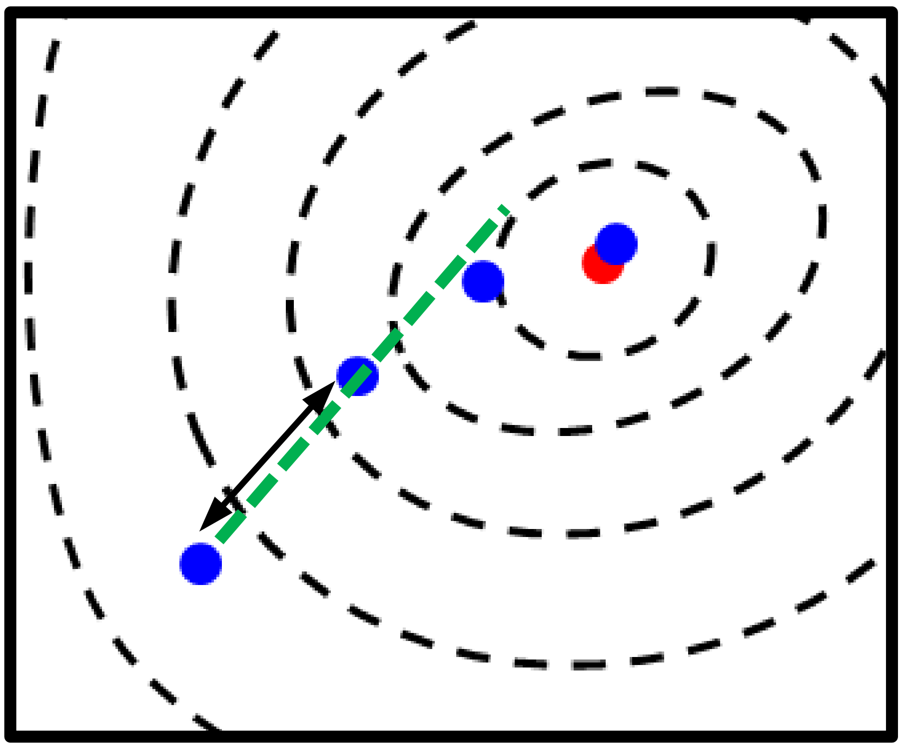



Direction

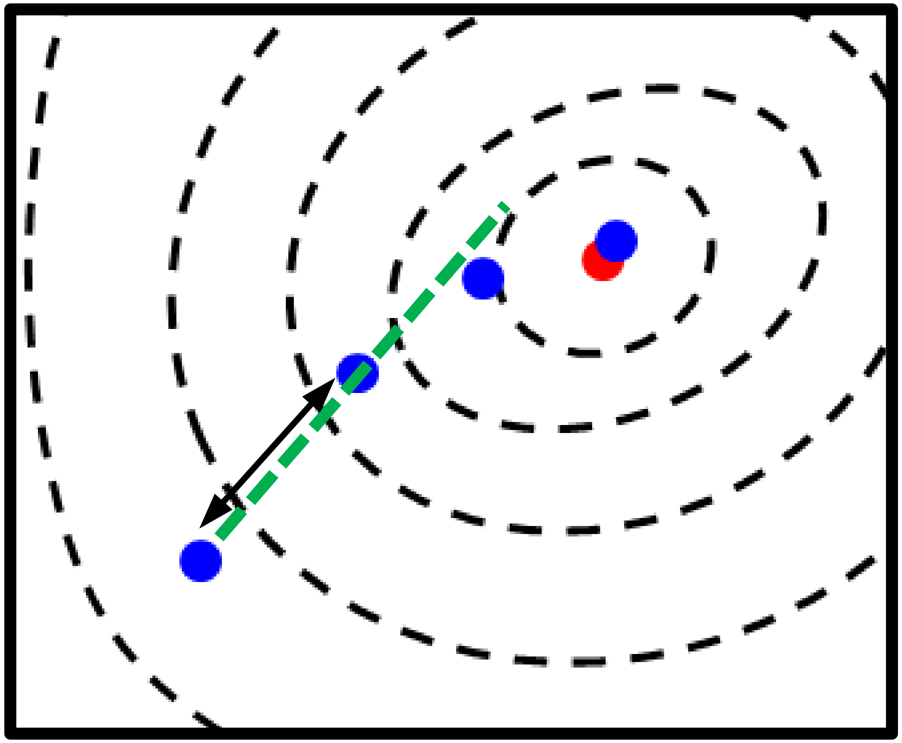





# Step length



The direction and step length may be defined by using the gradient



$$\Phi(\mathbf{p})$$

$$\Phi(\mathbf{p})$$

$$\Phi(\mathbf{p}_0 + \Delta\mathbf{p}) \approx \Phi(\mathbf{p}_0) + \nabla\Phi(\mathbf{p}_0)^\top \Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^\top \mathbf{H}(\mathbf{p}_0)\Delta\mathbf{p}$$

$$\Phi(\mathbf{p})$$

$$\Phi(\mathbf{p}_0 + \Delta\mathbf{p}) \approx \boxed{\Phi(\mathbf{p}_0) + \nabla\Phi(\mathbf{p}_0)^\top \Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^\top \mathbf{H}(\mathbf{p}_0)\Delta\mathbf{p}}$$

Approximating paraboloid

$$\Phi(\mathbf{p})$$

$$\Phi(\mathbf{p}_0 + \Delta\mathbf{p}) \approx \Phi(\mathbf{p}_0) + \nabla\Phi(\mathbf{p}_0)^\top \Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^\top \mathbf{H}(\mathbf{p}_0)\Delta\mathbf{p}$$

$$\mathbf{H}(\mathbf{p}_0)\Delta\mathbf{p} = -\nabla\Phi(\mathbf{p}_0)$$

$$\Phi(\mathbf{p})$$

$$\Phi(\mathbf{p}_0 + \Delta\mathbf{p}) \approx \Phi(\mathbf{p}_0) + \nabla\Phi(\mathbf{p}_0)^\top \Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^\top \boxed{\mathbf{H}(\mathbf{p}_0)} \Delta\mathbf{p}$$

$$\boxed{\mathbf{H}(\mathbf{p}_0)} \Delta\mathbf{p} = -\nabla\Phi(\mathbf{p}_0)$$

Difference between methods

$$\Phi(\mathbf{p})$$

$$\Phi(\mathbf{p}_0 + \Delta\mathbf{p}) \approx \Phi(\mathbf{p}_0) + \nabla\Phi(\mathbf{p}_0)^\top \Delta\mathbf{p} + \frac{1}{2}\Delta\mathbf{p}^\top \boxed{\mathbf{H}(\mathbf{p}_0)} \Delta\mathbf{p}$$

$$\boxed{\mathbf{H}(\mathbf{p}_0)} \Delta\mathbf{p} = -\nabla\Phi(\mathbf{p}_0)$$

| | |
|---|---|
| Newton | $\mathbf{H}(\mathbf{p}_0)$ |
| Gauss - Newton | $\tilde{\mathbf{H}}(\mathbf{p}_0)$ |
| Steepest decent | $\lambda\mathbf{I}$ |
| Levenberg - Marquardt | $\tilde{\mathbf{H}}(\mathbf{p}_0) + \lambda\mathbf{I}$ |

Difference between methods