Coulomb Scattering *

Edson Israel Ríos Coronado **Asesor:***Jorge Martínez Ortega*

[1] *Departamento de Física, CUCEI, Universidad de Guadalajara*
Blvd. Marcelino García Barragán 1421, Col. Olímpica, Guadalajara Jal., C. P. 44430, México

Abstract

When it comes to analyzing tiny particles, bouncing high energy electrons off the nucleus and using large detector systems is a good way to get some information of the nucleus. Performance of an experimental scattering method (in general the scattering of charged particles by other charged particles) is made in many high tech laboratories for this purpose. In the current article, Coulomb scattering is inspected, focusing in obtaining and comparing the differential cross-section in Classical, Quantum and Quantum Relativistic cases, leading to an understanding of the phenomenon and the interaction between charged particles with different energies with a massive charged particle. To support the theory and analyze the equations obtained in each case, a simulation[1] which can reproduce different features of this process has been made. Furthermore, it was found that from the experimental point of view, the Classical and Quantum cases are indistinguishable. Nevertheless, the Quantum Relativistic case makes different predictions for relativistic velocities.

Introduction

From ancient times, humans have wondered about what matter is made up of. Since 400 years B.C. Democritus considered that matter was constituted by little tiny indivisible particles; he called them atoms because of its meaning in Greek. However, Democritus' ideas about matter weren't accepted by the philosophers of its epoch. The idea of indivisible particles was reconsidered when John Dalton proposed his theory of the atomic model of matter. Later, in 1897, J.J. Thompson demonstrated that smaller negatively charged particles were contained in the atom. Rutherford, basing his model on his experimental results obtained by projecting alpha particles (Helium nuclei) into a thin gold foil (see fig.1), found out that atoms were mostly empty and contained a tiny positively charged nuclei at its center and electrons at its cortex. Nowadays, humans have discovered smaller particles that make up atoms (leading to a better knowledge of matter). Furthermore, a new theoretical model that explains the behavior of the most elementary particles (and their interactions) of universe, the Standard Model of Particle physics, has been developed. Further models where proposed, however Rutherford's experiment provided a big change in studying the nucleus structure.

*Thanks to Gustavo López and Alexander I. Nesterov for providing help and bibliography

[1] The simulation script is not present in this text because it is still in process and the units also need re-scaling.

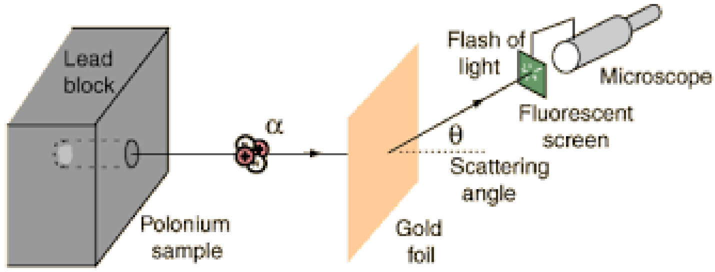


Figure 1: Rutherford experiment, retrieved from [7]

In general, determining motion of a particle interacting with an external field which potential energy only depends on distance "r" from a fixed point in space is essential for describing these phenomena; the external field is called *central*. The central force interacting with the particle is :

$$\mathbf{F} = -\frac{\partial U(r)}{\partial \mathbf{r}}\hat{\mathbf{r}}$$

As we may know, spherical coordinates Lagrangian of a particle moving inside an external central force is of the following form:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

And we can see that this function doesn't contain explicitly the coordinate ϕ; thus the generalized momentum due to this coordinate is conserved after ϕ being cyclic.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}P_\phi = 0$$

where $P_\phi = M = mr^2\dot{\phi}$. Since $\mathbf{M} = \mathbf{r} \times \mathbf{P}$ and $\mathbf{M}$ is perpendicular to vector $\mathbf{r}$, the conservation of $\mathbf{M}$ means that the motion of the particle remains in one plane (perpendicular to $\mathbf{M}$).
To obtain a complete solution of a central force problem, we analyze the conserved quantities, such as energy ($E = \frac{1}{2}mv_\infty^2$) and angular momentum magnitude ($M = mbv_\infty = b\sqrt{2mE}$) taking into account that v_∞ is the incident velocity. The energy of the particle expressing $\dot{\phi}$ in terms of M^2 is:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{M^2}{mr^2} + U(r)$$

Thus

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left[E - U(r)\right] - \frac{M^2}{m^2r^2}} \tag{1}$$

Time can also be obtained by separating and integrating equation (1).Furthermore, we can separate variables from angular momentum and obtain the trajectory of the particle:

$$P_\phi = M = mr^2\dot{\phi} \longrightarrow d\phi = \frac{M}{mr^2}dt$$

[2] Whenever I write M without bold notation, I'll be referring to its magnitude

and substituting dt we get:

$$\phi = \int \frac{M}{r^2\sqrt{2m[E-U(r)]-\frac{M^2}{r^2}}}dr + C_2 \tag{2}$$

Equations (1) and (2) are useful in describing Coulomb scattering. This phenomena is an elastic scattering of charged particles by the Coulomb interaction in which charged particles are involved and its potential is:

$$U = \frac{zz'e^2}{4\pi\epsilon_0 r} \tag{3}$$

For the Coulomb scattering problem imagine a positively charged particle going towards a heavy nucleus (also with positive charge). The particle comes with energy E, an impact parameter b and it is scattered by some scattering angle θ, as shown in the next figure:

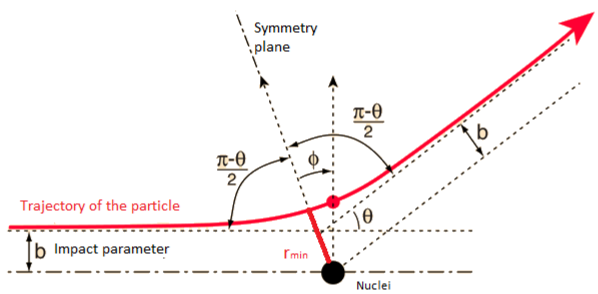


Figure 2: Coulomb scattering geometry, retrieved from [7]

Since Coulomb force is actually a central force the trajectory remains in one plane. The nucleus' recoil is negligible after its weigh. We can also see from equation (1) that dr can be either negative (if the particle is approaching to the nuclei, $r_2 < r_1$) or positive (if the particle recede, $r_1 < r_2$). If we take the returning point as reference and we go either way, we will notice that the angle ϕ only differs by a sign; this means that the trajectory is symmetric with respect to the line ϕ_0 (line pointing toward the returning point). The returning point is in fact the minimal distance between the particle an the nuclei (r_{min}).

Using the expression for energy mentioned before for a particle moving inside an external central force with $U = \frac{\alpha}{r}$ (with $\alpha > 0$) we obtain:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{M^2}{mr^2} + \frac{\alpha}{r}$$

Let the second and the third element be U_{eff} and we notice that it is always positive.

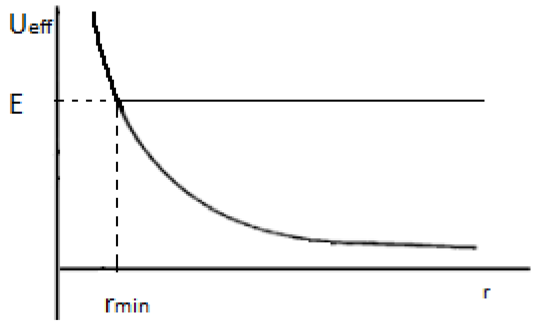


Figure 3: U_{eff} vs r

The potential barrier won't let the particle approach more than r_{min}. The final direction of the motion doesn't agree with the incident one that's why it is said that the particle has been scattered. In defining this phenomena it is useful to introduce some other terms. Let $\sigma(\Omega)$ be the scattering cross section which is defined as the number of scattered particles per unit time into a solid angle divided by the incident intensity.

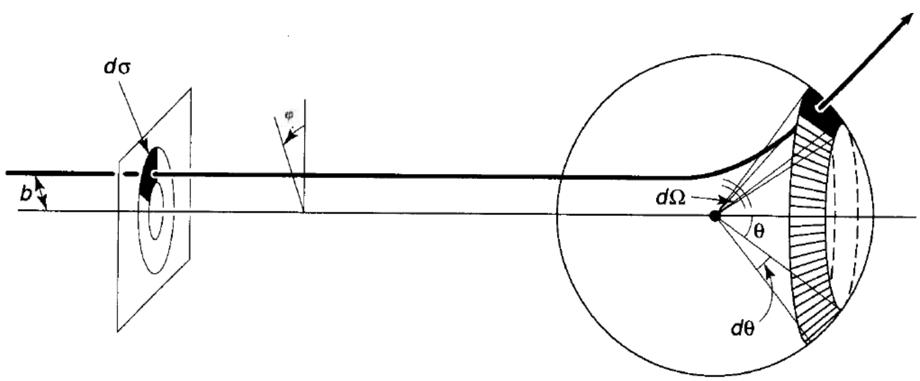


Figure 4: Particle incident in the area $d\sigma$ scatter into the solid angle $d\Omega$ (taken from David J. Griffiths [3])

In terms of the impact parameter and the scattering angle, $d\sigma = bdbd\varphi$ and $d\Omega = sin(\theta)d\theta d\varphi$, the "differential cross-section" is:

$$\frac{d\sigma}{d\Omega} = \frac{b}{sin(\theta)}\frac{db}{d\theta} \tag{4}$$

We will focus in obtaining the differential cross-section for the different cases separately and comparing them all. A simulation was made for a better analysis of the problem.

Classical Coulomb scattering

For solving the classical scattering, we take figure (2) and equation (2) and we notice that half of the trajectory is $\Psi = \frac{\pi - \theta}{2}$ and in terms of r goes from r_{min} to ∞.

$$\Psi = \int_{r_{min}}^{\infty} \frac{M}{r^2\sqrt{2m[E - U(r)] - \frac{M^2}{r^2}}} dr$$

We get a hyperbolic trajectory and for a Coulomb potential (3) we know that $\alpha = zz'e^2/4\pi\epsilon_0$; where z and z' are the number of charged particles in the scattered particle and in the massive nuclei respectively. Thus, we can write the next relation:

$$\frac{1}{r} = \frac{mzz'e^2}{4\pi\epsilon_0 M^2}(\epsilon cos(\Psi) - 1) \tag{5}$$

Where eccentricity is defined as $\epsilon = \sqrt{1 + \left(\frac{2Eb(4\pi\epsilon_0)}{zz'e^2}\right)^2}$. We are interested in the scattered particle and that occurs when $r \to \infty$. Furthermore, we can establish a relation between the scattering angle and the other parameters. Using equation (4) we get the scattering cross-section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{kzz'e^2}{4^2\pi\epsilon_0 Esen^2\left(\frac{\theta}{2}\right)}\right)^2 \tag{6}$$

Which is *Rutherford scattering cross-section*[3].

Quantum Coulomb Scattering

To analyze this section, we need to solve Schrödinger equation. In general I am not interested in an interaction with the magnetic field. If I were, I would've taken the Schrödringer-Pauli's equation that includes the interaction of a magnetic field[4]. Spin can be postulated in non-relativistic quantum mechanics, but it doesn't come as a solution of Schrödinger's equation. Furthermore, Spin doesn't literally mean that the particle is spinning, however the best we can do to introduce spin in a non-relativistic case is finding the angular momentum that allows half-integral values. Keep in mind that the spin of a particle is a consequence of a Lorentz invariance, Schrödinger's equation is not covariant whereas Dirac's[5] is covariant. For Quantum scattering problem, we have an incident plane wave traveling in some direction that encounters a scattering potential that produces an outgoing spherical wave. We expect the solution to be as follows:

$$\psi(r, \theta, \phi) = \psi(r, \theta) = A\left[e^{i\kappa x_i} + f(\theta)\frac{e^{i\kappa r}}{r}\right] \tag{7}$$

[3]for a deeper understanding I would recommend to check [2]

[4]Stern-Gerlach term arises when the vector potential is different to $0 \to \mathbf{A} \neq 0$

[5]You'll see in the Quantum relativistic case how we take into account the spin and give the same probabilities to each one without taking the interaction with a magnetic field.

where $\kappa = \sqrt{\frac{2mE}{\hbar^2}}$ and $f(\theta)$ is called the scattering amplitude. In the Quantum case, we can match the probability of the incident particle (*with speed* v_∞) going through the infinitesimal area $d\sigma$ for a given lapse dt with the probability of the scattered particle (*with the same speed*) going through the infinitesimal solid angle $d\Omega$. According to the expectation (7), we have:

$$dP = |\psi_{incident}|dV = |A|^2(v_\infty dt)d\sigma$$

$$dP = |\psi_{scattered}|dV = |A|^2|f(\theta)|^2\frac{(v_\infty dt)}{r^2}r^2 d\Omega$$

And the differential cross-section in terms of the scattering amplitude $f(\theta)$ is:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{8}$$

The Schrödinger equation in cartesian coordinates is:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = E\psi$$

to simplify we can write it in the following way:

$$(\nabla^2 + \kappa^2)\psi = Q \tag{9}$$

We get the inhomogeneous Helmholtz equation (with Q depending on ψ). First of all, we need to find a function G($\mathbf{r}$) (*Green's function*) that solves the Helmholtz equation with a delta function source:

$$(\nabla^2 + \kappa^2)G(\mathbf{r}) = \delta^3(\mathbf{r}) \tag{10}$$

We know that the delta can be expressed as a Fourier transform:

$$\delta(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\kappa r} d\kappa$$

And we take G($\mathbf{r}$) as the Fourier transform of g(s).

$$G(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{s}\cdot\mathbf{r}} g(\mathbf{s}) d^3\mathbf{s}$$

We first have to find the function $g(\mathbf{s})$, that can be easily achieved by substituting the delta representation and the Fourier transform into equation (10):

$$g(\mathbf{s}) = \frac{1}{(2\pi)^{3/2}(\kappa^2 - \mathbf{s}^2)}$$

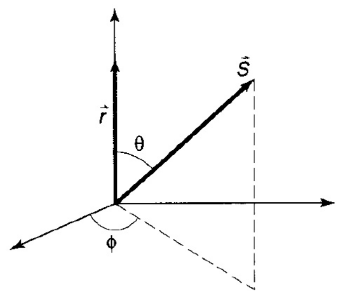


Figure 5: Coordinates chosen to solve G($\mathbf{r}$)

For solving for G($\mathbf{r}$) I followed the coordinates chosen by David J. Griffith's [3] that can be seen in figure (5). Using $\mathbf{r}\cdot\mathbf{s} = |\mathbf{s}||\mathbf{r}|cos(\theta)$ and integrating over $d^3\mathbf{s}$ we get:

$$G(\mathbf{r}) = \frac{i}{8\pi^2|\mathbf{r}|}\int_{-\infty}^{\infty} s\left[\frac{e^{i|\mathbf{s}||\mathbf{r}|} - e^{-i|\mathbf{s}||\mathbf{r}|}}{(|\mathbf{s}|^2-\kappa^2)}\right]d|\mathbf{s}|$$

Factoring and distributing the denominator we notice that the solution can be found using Cauchy's integral formula:

$$\oint \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z_0) \tag{11}$$

where z_0 is a singular point. The contours are chosen in such a way that they don't contribute at infinity; that is $e^{i|\mathbf{s}||\mathbf{r}|} \to 0$ when s has a large imaginary part from contour I_1 enclosing $s = +k$ and so for I_2.

$$I_1 = \oint\left[\frac{|\mathbf{s}|e^{i|\mathbf{s}||\mathbf{r}|}}{|\mathbf{s}|+\kappa}\right]\frac{1}{|\mathbf{s}|-\kappa}d|\mathbf{s}| = 2\pi i\left[\frac{|\mathbf{s}|e^{i|\mathbf{s}||\mathbf{r}|}}{|\mathbf{s}|+\kappa}\right]_{|\mathbf{s}|=\kappa} = i\pi e^{ik|\mathbf{r}|}$$

The Green's function becomes:

$$G(\mathbf{r}) = -\frac{e^{i\kappa|\mathbf{r}|}}{4\pi|\mathbf{r}|}$$

This function solves Helmholtz equation with a delta-function source. And we can express ψ from equation (9) as an integral:

$$\psi = \int G(\mathbf{r}-\mathbf{r}_0)Q(\mathbf{r}_0)d^3\mathbf{r}_0$$

to this solution we must add the solution for the homogeneous equation $(\nabla^2+\kappa^2)G(\mathbf{r}_0) = 0$ which solution satisfies a free particle $\psi_0 = Ae^{ikr}$ and hence:

$$\psi = Ae^{i\kappa r} - \frac{m}{2\pi\hbar^2}\int\frac{e^{i\kappa|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0$$

We are mainly interested in the radiation zone, therefore we take $|\mathbf{r}| \gg |\mathbf{r}_0|$. In this region, we can approximate $|\mathbf{r}-\mathbf{r}_o| \simeq r - \hat{r}\cdot\mathbf{r}_0$ and therefore $\frac{e^{i\kappa|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} \simeq \frac{e^{i\kappa r}}{r}e^{-i\kappa\hat{r}\cdot\mathbf{r}_0}$. After doing approximations we can notice that, according to the equation (7), the scattering amplitude would be:

$$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2 A}\int e^{-i\kappa\cdot\mathbf{r}_0}V(\mathbf{r}_0)\psi(\mathbf{r}_0)d^3\mathbf{r}_0$$

Here we introduce the so called "*Born Approximation*", that consists in supposing that we have a weak potential and it doesn't alter essentially the incoming plane wave (that was coming in the z-direction):

$$\psi(\mathbf{r}_0) \simeq \psi_0(\mathbf{r}_0) = Ae^{i\kappa z_0} = Ae^{i\kappa'\cdot\mathbf{r}_0}$$

where κ' is the incoming particle ($\kappa' = \kappa\hat{z}$).

$$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2}\int e^{i(\kappa'-\kappa)\cdot\mathbf{r}_0}V(\mathbf{r}_0)d^3\mathbf{r}_0$$

Integrating over the angular coordinates we get:

$$f(\theta,\phi) \simeq -\frac{2m}{\hbar^2\mathbf{K}}\int_0^\infty r_0V(r_0)sin(\kappa r_0)dr_0 \tag{12}$$

since $|\kappa| = |\kappa'|$, we have $|\kappa - \kappa'| = 2|\kappa|sen\frac{\theta}{2}$ which is the angular dependence of the scattering amplitude. Notice that this integral is undefined for Coulomb potential (3), for solving this problem we have to remember that the scattering occurs close to the nuclei, not at infinity; Yukawa potential (13) regulates the behavior at infinity because its range is finite. *Keep in mind that in the representation usage, integrals are supposed to be done first and then we take the limit. This order doesn't commute.*

$$V(r_0) = \beta\frac{e^{-\mu r}}{r} \tag{13}$$

Inserting the Yukawa potential (13) into the scattering amplitude (12) we get:

$$f(\theta) = -\frac{2m\beta}{\hbar^2(\mu^2 + \left(2|\kappa|sen\frac{\theta}{2}\right)^2)}$$

and if we put $\beta = zz'e^2/4\pi\epsilon_0$ and $\mu = 0$ the Yukawa potential becomes Coulomb potential. Substituting and using equation (8) we obtain what was obtained in the classical case , that is, the Rutherford differential cross-section (6).

Quantum Relativistic Coulomb Scattering

By applying the Propagator formalism we get the coefficients of the free-wave solutions and recognize them as the S-Matrix elements. This Matrix contain the rules for calculating the ordinary scattering process, all details within the interaction, coupling constants and angular dependence. We are interested in the transition matrix element:

$$S_{fi} = -ie\int d^4x\bar{\Psi}_f(x)\not{A}(x)\Psi_i(x) \tag{14}$$

Here, $\Psi_{i,f}$[6] represents the incident/scattered plane wave of a fermion with momentum $p_{i,f}$ and spin $s_{i,f}$ normalized to unit probability in a box of volume V:

$$\Psi_{i,f}(x) = \sqrt{\frac{mc^2}{E_{i,f}V}}u(p_{i,f},s_{i,f})e^{\mp ip_{i,f}\cdot x} \tag{15}$$

[6] Here and only here, I'm taking Ψ_f as $\bar{\Psi}_f$ and I do the same for the spinor $u(p_f,s_f)$ in equation (15)

The Coloumb potential $\not{A} = \gamma^\mu A_\mu$ is represented by:

$$A_0 = \frac{-Ze}{4\pi\epsilon_0|\mathbf{X}|} \qquad\qquad \mathbf{A} = 0$$

Computing this into the Transition Matrix Element we get:

$$S_{fi} = i\frac{-Ze^2}{4\pi\epsilon_0 V}\sqrt{\frac{m^2c^4}{E_iE_f}}\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\int d^4x\frac{e^{i(p_f-p_i)\cdot x}}{|\mathbf{X}|}$$

The integration over time yields $2\pi\delta(E_f - E_i)$ and the integration over space is the Fourier transform of the coulomb potential, calculated using $\bigtriangledown\frac{1}{|\mathbf{X}|} = -4\pi\delta^3(x)$ and integrating by parts. The number of final states in momentum interval d^3p_f is $\frac{Vd^3p_f}{(2\pi)^3}$ and thus the transition probability per particle into these states is:

$$|S_{fi}|^2\frac{Vd^3p_f}{(2\pi)^3} = \frac{Z^2(mc^2e^2)^2}{\epsilon_0^2 E_i V}\frac{|\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{p_f} - \mathbf{p_i}|^4}\left[2\pi\delta(E_f - E_i)\right]^2\frac{d^3p_f}{(2\pi)^3E_f}$$

The square of δ is not well defined mathematically. If we analyze the delta function and instead of integrating over an infinite interval we consider the transitions in a given (*large but finite*) time interval T $(T/2, -T/2)$. The integration over time coordinate in the Transition Matrix Element gives:

$$\int_{-T/2}^{T/2} dte^{i(E_f-E_i)t} = \frac{2sin\left(T/2(E_f - E_i)\right)}{E_f - E_i}$$

if we square it and consider it as a function of E_f, the area under the curve becomes $2\pi T$, therefore the square energy-preserving δ may take the following meaning (*applying the Fourier representation of the delta*):

$$\left[2\pi\delta(E_f - E_i)\right]^2 = \left[2\pi\delta(0)2\pi\delta(E_f - E_i)\right] = \left[2\pi T\delta(E_f - E_i)\right]$$

As we did in the other cases, we consider the differential cross-section as the particles going from $d\sigma$ into the solid angle $d\Omega$ per unit of time. Thus, dividing out the time we get the transition probability per particle per unit of time within the momentum interval d^3p_f and finally get the scattering cross-section if we also divide by the incoming current of particles (see equation 16).

$$J^a_{inc} = c\bar{\Psi}_i(x)\gamma^a\Psi_i(x) \tag{16}$$

Where we take the spinors (15) with spin polarization in the z-direction ($a = 3$) on the Dirac representation. The solutions for a free-particle of the Dirac equation can be obtained at rest and then applying a Lorentz transform to the solution [5]. Also the Dirac equation can be solved for a momentum $\mathbf{p}$ (leading to the same results) and the current of incoming particles can be calculated by taking the four solutions[7] and computing them into equation (16). The magnitude by the current

[7]The first two solutions describe the two spin degrees of freedom of a Schrödinger-Pauli electron; for this solutions the third and fourth components are "small-components" in a non relativistic approximation. The other 2 solutions correspond to a "negative-energy" particles, and for them the first and the second components are the small ones in the non-relativistic case.

coming in the z-direction taking into account the relativistic momentum and energy is then:

$$|\mathbf{J_{inc}}| = \frac{|\mathbf{p_i}|c^2}{E_i}\frac{1}{V} = \frac{|\mathbf{v_i}|}{V}$$

Then $d\sigma$ is:

$$d\sigma = \frac{|S_{fi}|^2}{T|\mathbf{J_{inc}}|}\frac{V d^3 p_f}{(2\pi)^3} = \frac{Z^2(mc^2e^2)^2}{T\frac{|\mathbf{v_i}|}{V}\epsilon_0^2 E_i V}\frac{|\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)|^2}{|\mathbf{p_f}-\mathbf{p_i}|^4}\left[2\pi T\delta(E_f-E_i)\right]\frac{d^3p_f}{(2\pi)^3 E_f}$$

Again, the scattered particle conserves the magnitude of its momentum since we are dealing with an elastic "collision" (our particle hasn't substantially changed so the mass is constant) that way we have $|\mathbf{p_f}| = |\mathbf{p_i}|$, and again $|\mathbf{p_f} - \mathbf{p_i}| = 2|\mathbf{p_f}|sen\frac{\theta}{2}$. Furthermore, if the momentum space volume is $d^3p_f = \mathbf{p_f}^2 d|\mathbf{p_f}|d\Omega_f$ and the energy of a relativistic particle is $E_f^2 = \mathbf{p_f}^2 + m^2$ (and therefore $E_f dE_f = |\mathbf{p_f}|d|\mathbf{p_f}|$), we can take this into account and reacommodate terms:

$$\frac{d\sigma}{d\Omega_f} = \frac{Z^2(me^2)^2}{4\pi^2\epsilon_0^2}\frac{|\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)|^2}{|\mathbf{p_f}-\mathbf{p_i}|^4}\int_{\Delta p_f}\frac{E_f|\mathbf{p_f}|c^2\delta(E_f-E_i)}{E_i|\mathbf{v_i}|E_f}dE_f$$

The integral is introduced since we have to integrate over a small interval Δp_f (Uncertainty of measurement) and helps on vanishing the singular behaviour of our δ. The integral is reduced to 1 and we get the differencial cross-section:

$$\frac{d\sigma}{d\Omega_f} = \frac{Z^2(me^2)^2}{4\pi^2\epsilon_0^2}\frac{|\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)|^2}{\left(2|\mathbf{p_f}|sen\left(\frac{\theta}{2}\right)\right)^4} \tag{17}$$

For the non-relativistic case [8] the differential cross-section is the same as in the Rutherford's(6). So far squared matrix element has to be calculated, for that we first reduced to calculating traces and then we apply some Trace theorems (check D. Bjorken [6]) that are derived from the commutation algebra of the $\gamma's$ (independently on which representation we take).
We are not interested in measuring all the spin states, that's why we make a sum over all of them. The cross-section will be a sum over the final spin states and an average over the initial states:

$$\frac{d\sigma}{d\Omega_f} = \frac{Z^2(me^2)^2}{4^3\pi^2\epsilon_0^2\left(|\mathbf{p_f}|sen\left(\frac{\theta}{2}\right)\right)^4}\frac{\Sigma_{\pm s_f,s_i}|\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)|^2}{2}$$

In which the spin sum can be written in the following way:

$$\Sigma_{s_f,s_i}\bar{u}_\alpha(p_f,s_f)\gamma^0_{\alpha\beta}u_\beta(p_i,s_i)\bar{u}_\delta(p_i,s_i)\gamma^0_{\delta\sigma}u_\sigma(p_f,s_f)$$

The initial spin sum, that is in the middle will become[9]:

$$\Sigma_{\pm s_i}u_\beta(p_i,s_i)\bar{u}_\delta(p_i,s_i) = \Sigma_{r=1}^4\epsilon_r w_\beta^r(p_i)\bar{w}_\delta^r(p_i)\left(\frac{\not{p}_i+mc}{2mc}\right) = \left(\frac{\not{p}_i+mc}{2mc}\right)_{\beta\delta} = \left[\Lambda_+(p_i)\right]_{\beta\delta}$$

[8] $\left[\bar{u}(p_f,s_f)\gamma^0 u(p_i,s_i)\right]^2 = \left[\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right]^2 = 1$

[9] keep in mind that we still have γ^0 in the spin sum

where $w^r(p_i)$ (for r=1,2,3,4) are the spinors[10] with momentum p_i and Λ is an operator that projects the positive or negative energy eigenstates for a given p_i. This operator may be found directly from $(\not{p} - \epsilon_r mc)w^r(\mathbf{p}) = 0$ and $\overline{w^r(\mathbf{p})}(\not{p} - \epsilon_r mc) = 0$. Continuing with the sum of final polarization states, we will get:

$$\Sigma_{\pm s_f, s_i} |\bar{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2 = \Sigma_{\alpha\sigma} \left(\gamma^0 \frac{\not{p}_i + mc}{2mc}\gamma^0\right)_{\alpha\sigma} \left(\frac{\not{p}_f + mc}{2mc}\right)_{\sigma\alpha} = Tr\left[\gamma^0 \frac{\not{p}_i + mc}{2mc}\gamma^0 \frac{\not{p}_f + mc}{2mc}\right]$$

If we insert this into (17) and we apply the trace properties we get:

$$\frac{d\sigma}{d\Omega_f} = \frac{Z^2(me^2)^2}{4^3\pi^2\epsilon_0^2\left(|\mathbf{p_f}|sen\left(\frac{\theta}{2}\right)\right)^4} \frac{1}{8m^2c^2}\left[Tr(\gamma^0\not{p}_i\gamma^0\not{p}_f) + m^2c^2 Tr(\gamma^0)^2\right]$$

Applying the Trace theorems the elements inside the brackets become $Tr(\gamma^0\not{p}_i\gamma^0\not{p}_f) = 4E_iE_f/c^2 + 4\mathbf{p_i}\cdot\mathbf{p_f}$ and $Tr(\gamma^0)^2 = 4$ and thus:

$$\frac{d\sigma}{d\Omega_f} = \left(\frac{Z(me^2)}{4\pi\epsilon_0 sen^2\left(\frac{\theta}{2}\right)}\right)^2 \frac{1}{4m^2c^2|\mathbf{p}|^2\beta^2}\left(1 - \beta^2 sen^2\left(\frac{\theta}{2}\right)\right) \tag{18}$$

Which is the ***Mott***[11] *differential cross-section.* This equation reduces to Rutherford's for $v << c$, making this theory consistent with the prior scattering cross-sections, but an extra term arises for the relativistic case. We notice also that when $v \to c$ the last term becomes $cos(\theta/2)$.

Results and discussion

The simulation made can reproduce the behavior of the scattered particle's trajectories varying parameters (Energy, charge or parameter impact), such trajectories can be seen in Figure (6).

[10]from the Dirac solutions
[11]N. F. Mott, Proc. Roy. Soc. (London), A124, 425 (1929)

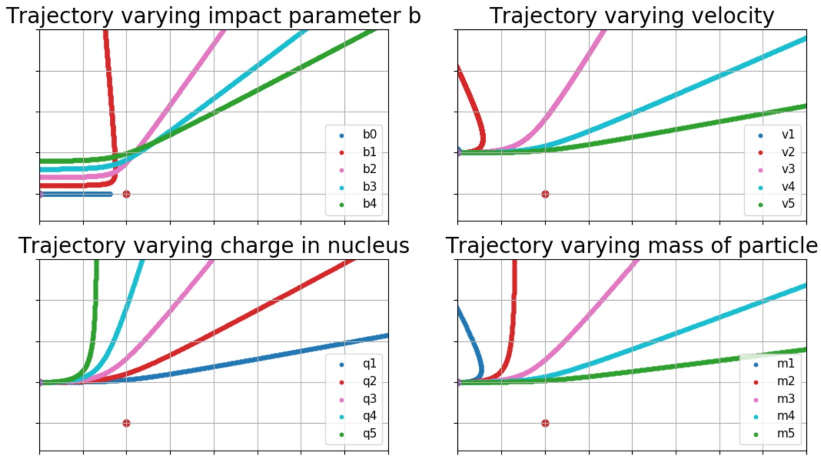


Figure 6: Trajectories of scattered particles for different parameters, the index of each value helps to understand which is bigger. A bigger number corresponds to a bigger magnitude $0 < 1 < 2 < 3 < 4 < 5$

For the non-relativistic case, from equation (5), we can obtain the scattering angle making $r >> 1$:

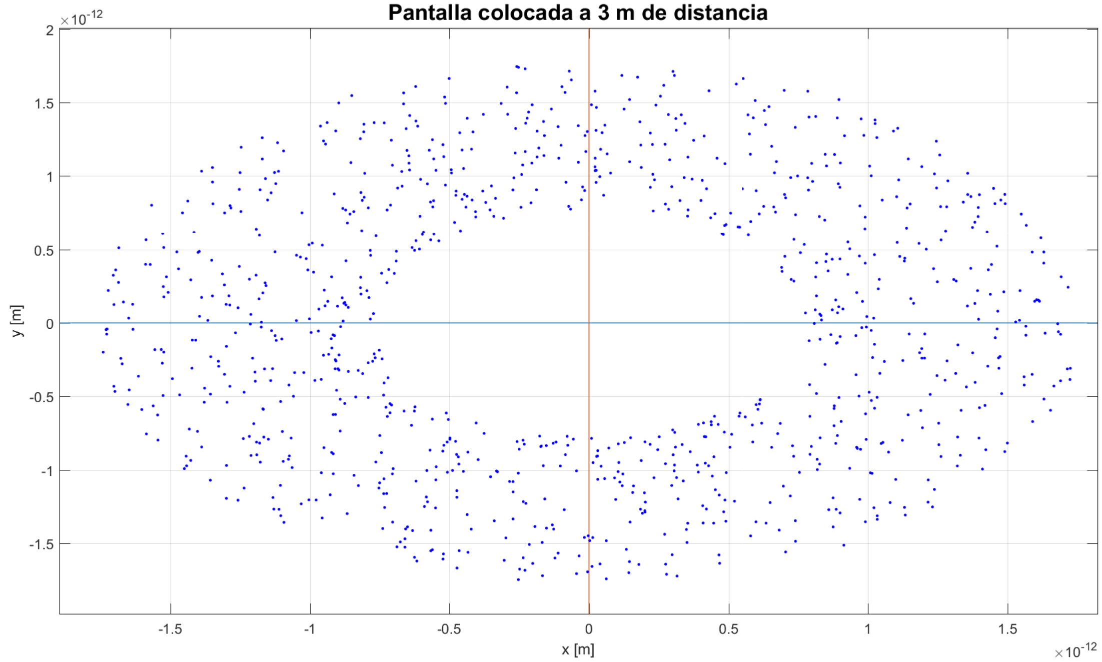
$$\theta = 2cot^{-1}\left(\frac{2Eb}{kzz'e^2}\right)$$

We can use the last equation to project some scattered particles into a screen placed at some distance d. For the relativistic case, using the Mott Scattering cross-section (18) and the equation (4), we can obtain a relation for the scattering angle and the other parameters. For defining the integrating limits, I solved the classic case through this process and notice that I could get the same equation (5) integrating $d\theta$ from π to θ and the parameter impact from 0 to b. Thus it is easy to obtain:

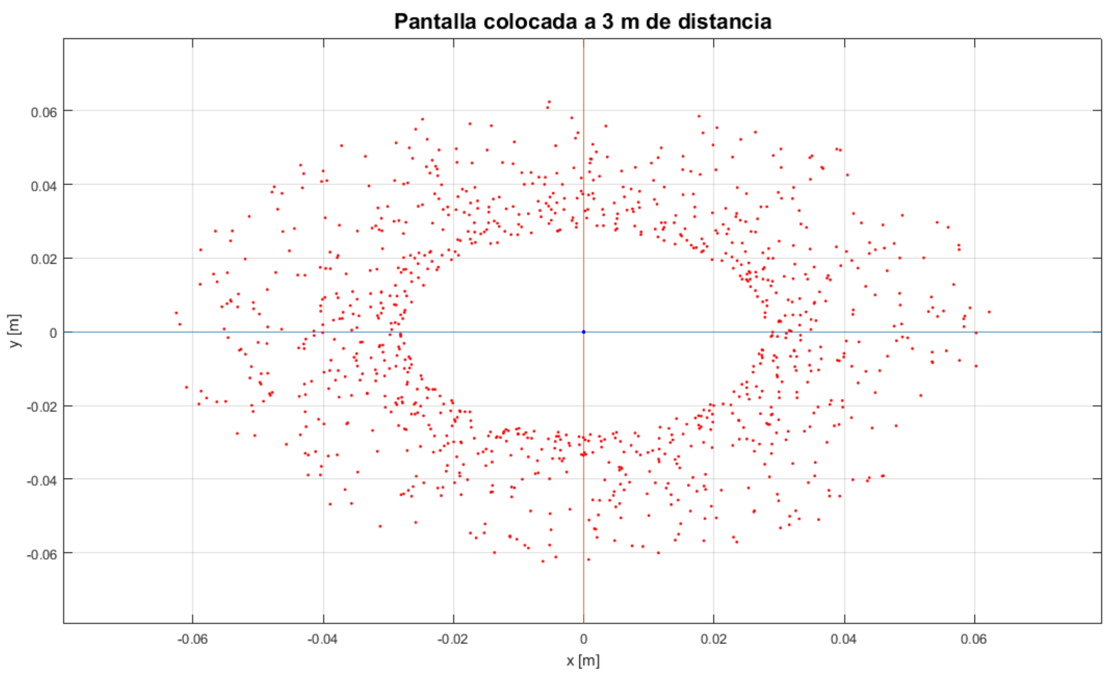
$$2\left(\frac{Ze^2}{4\pi\epsilon_0 m^2 v^4 \gamma^2}\right)^2\left[cot^2\left(\frac{\theta}{2}\right) - 2\beta^2 ln\left(sen\left(\frac{\theta}{2}\right)\right)\right] = b^2$$

Solving for the scattering angle, in order to project particles into a screen[12], is possible using a numerical method which can be useful in comparing the different scatterings variating the parameters. So far, the main interest would be comparing the relativistic and non-relativistic case and see if the Figure (6) is as expected for high velocities.

[12]placed behind some heavy, charged nuclei that is located in the center (0,0)



(a) Incident particles



(b) Scattered particles

Figure 7: Projections on a screen placed at 3 m behind the collision for some charged beam without or with a nucleus on their way. Notice that in figure (b) we can also see the incident particles, but they are very close to the center and we can't appreciate the distribution

For the last simulation, I took the example of the Rutherford's experiment, that is, alpha particles produced from decay (with velocity $v = 0.05c$) bouncing off a gold nucleus. I'm not taking into account interactions with many nuclei. The impact parameter goes from $b = 7.7 * 10^{-13}$ to $b + db$ where $db = 1 * 10^{-12}$ with a random uniform distribution. This was made just to show an overview of the scale of this interaction.

Conclutions

The scattering of a particle in a Coulomb potential was revised. Differential Cross-section was calculated and compared in different cases. We found that the Classical and Quantum scatterings are indistinguishable from an experimental point of view. However, the Quantum relativistic case provided a different scattering formula, in which it arises a second element:

$$\frac{d\sigma}{d\Omega}_{Mott} = \frac{d\sigma}{d\Omega}_{Rutherford} \left(1 - \underbrace{\beta^2 sen^2 \left(\frac{\theta}{2} \right)}_{\text{Relativistic term}} \right)$$

A formula was developed for calculating the scattering angle based on some parameters that can be controlled. Furthermore, the simulation made works but still need improvement and rescaling. So far, the structure of the atom (and spin) is not taken into account, and this may provide more information of the system. Comparisons with experimental data is remaining in order to get an approach with the real case.

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