拓课堂2025暑期牛剑营课程

Lesson 4: Logic and Proof

4.1 集合术语(Set Notations)

Introduction

REMARK

Mathematics needs a particular precision, and within each of these languages, most of mathematics, and all the mathematics that we shall do, is written in the language of sets, using statements and arguments that are based on the grammar and logic of the predicate calculus.

– A Course in Mathematical Analysis

Definition of a set

Set

A **set** is a collection of elements, without regard to order. Elements in a set are only counted once. For example, if $a = 2, b = c = 1$, then $A = \{a, b, c\}$ has only two members. We write $x \in X$ if x is a member of the set X.

Note: The elements of a set can be any stuff (numbers, names, functions, SETS!)

Example 1

EXAMPLE

In Zermelo-Fraenkel (ZF) set theory, the natural numbers are defined recursively by letting $0 = \{\}$ be the empty set and $n + 1 = n \cup \{n\}$ for each n. In this way $n = \{0, 1, \ldots, n - 1\}$ for each natural number n. This definition has the property that n is a set with n elements. The first few numbers defined this way are:

$$
\begin{aligned}
0 &= \{\} &&= \varnothing, \\
1 &= \{0\} &&= \{\varnothing\} \\
2 &= \{0, 1\} &&= \{\varnothing, \{\varnothing\}\}, \\
3 &= \{0, 1, 2\} &&= \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}.
\end{aligned}
$$

Russell’ s paradox

WARNING

Consider a set A, $A = \{x \mid x \notin x\}$, does $A \in A$?

If $A \in A$, it does not satisfy the definition: $A \notin A$

If $A \notin A$, it satisfies the definition: $A \in A$

ZF set theory

TIP

In 1908, Zermelo introduced a system of axioms; these were modified in 1922 by Fraenkel and Skolem. The resulting system, known as the Zermelo–Fraenkel axiom system ZF, has stood the test of time, and it is the one that we shall describe and use.

The sets and the relation $\in$ are required to satisfy certain axioms, some of the axioms are as follows:

- The extension axiom
- The empty set axiom
- The pairing axiom
- The union axiom
- The power set axiom

The extension axiom

The extension axiom

This states that two sets are equal if and only if they have the same elements. Thus the set with members 1,2 and 3 and the set with members $1, 3, 2$ and 1 are the same; the order in which they are listed is unimportant, as is the fact that repetition can occur. Set theory is all about membership, and about nothing else.

If a and b are sets, and every member of a is a member of b, then we say that a is a subset of b, or that b contains a, and write $a \subseteq b$ or $b \supseteq a$. Thus the extension axiom says that $a = b$ if and only if $a \subseteq b$ and $b \subseteq a$. If $a \subseteq b$ and $a \neq b$, we say that a is a proper subset of b, or that a is properly contained in b, and write $a \subset b$ or $b \supset a$.

The empty set axiom

📚 The empty set axiom

This states that there is a set with no members. The extension axiom then implies that there is only one such set: we denote it by $\varnothing$ and call it the empty set. It also has a rather paradoxical nature, since it is a subset of every set a (if not, there is a member b of $\varnothing$ which is not in a; but $\varnothing$ has no members).

The pairing axiom

The pairing axiom

This says that if a and b are sets, then there exists a set whose members are a and b. The extension axiom again says that there is only one such set: we denote it by $\{a, b\}$. Note that $\{a, b\} = \{b, a\}$: we have an unordered pair. We can take $a = b$: then the set $\{a, a\}$ has only one element a. We write this set as $\{a\}$ and call it a singleton set.

We can use the pairing axiom to define ordered pairs. If a and b are sets, the ordered pair (a, b) is represented by the set containing two elements: the singleton set $\{a\}$ and the pair $\{a, b\}$. That is, $\{\ \{a\},\ \{a, b\}\ \}$

Example 2

EXAMPLE

Prove that if (a, b) and (c, d) are ordered pairs and $(a, b) = (c, d)$, then $a = c$ and $b = d$.

The union axiom

The union axiom

This says that there is a set whose elements are exactly the sets which are members of members of A. We denote this set by $\cup_{a\in A}a$ (here a is a variable, so we could as well write $\cup_{x\in A}x$) and call it the union of the members of A. The essential feature of this axiom is that the sets whose members make up the union must all be members of a single set; we cannot form the union of all sets since there is no set to which all sets belong. If A and B are sets, we can consider the set $\cup_{C\in\{A,B\}}C$. This is the set whose elements are either in A or in B: we write this as $A\cup B$.

The power set axiom

The power set axiom

There is an essential difference between the statements $b \in A$ (b is a member of A) and $b \subseteq A$ (b is a subset of A). The power set axiom states that if A is a set, then there exists a set, the power set $P(A)$ of A, whose elements are the subsets of A. Thus $b \in P(A)$ if and only if $b \subseteq A$. For example, the elements of $P(\{a, b\})$ are $\varnothing$, $\{a\}$, $\{b\}$ and $\{a, b\}$, and the ordered pair $(a, b) = \{\{a\}, \{a, b\}\}$ is an element of $P(P(\{a, b\}))$.

Set symbols

REMEMBER remember

- $\varnothing$ *empty set: contains no elements*
- $\mathbb{N} = \{1, 2, 3, \cdots\}$ *is the set of natural numbers*
- $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ *is the set of natural numbers with 0*
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ *is the set of integers*
- $\mathbb{Q} = \left\{\frac{a}{b}; a, b \in \mathbb{Z}, b \neq 0\right\}$ *is the set of rational numbers*
- $\mathbb{R}$ *is the set of real numbers*
- $\mathbb{C}$ *is the set of complex numbers*

Set operations

REMEMBER remember

- **intersection**: $A \cap B = \{x : x \in A$ *and* $x \in B\}$
- **union**: $A \cup B = \{x : x \in A$ *or* $x \in B\}$
- **complementary** *(of A in C)*: $\bar{A} \cup A = C$ *and* $\bar{A} \cap A = \varnothing$
- **set difference**: $A \backslash B = \{x : x \in A$ *and* $x \notin B\}$
- **symmetric difference**: $A \Delta B = \{x : x \in A$ *xor* $x \in B\}$ *elements in exactly one of the two sets.*
- **the order/cardinality of a set**: $|A| =$ *the number of elements in a set (to be more specific, it is finite).*

Example 3

EXAMPLE

Prove that $|P(A)| = 2^{|A|}$

Solution to Example 3

SOLUTION

Proof:

We want to find the number of subsets of a set A.

Consider: for each element $x \in A$. Then a subset can contain x or not. And we have $|A|$ different x' s.

Thus, the total number of subsets is $2^{|A|}$.

Example:

$$\begin{array}{ll} A = \{1,2,3\} & P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \\ |A| = 3 & |P(A)| = 8 = 2^3. \end{array}$$

Example 4

EXAMPLE

Prove the followings:

(a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

(b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Note: Do not use the Venn Diagram to prove set relations

To prove these results, we need to be familiar with formal logic first.

Exercise 4

EXERCISE

Use the Venn Diagram to visualize the following results:

$(a)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(b)\ A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$

$(c)\ A \Delta B = (A \cup B) \setminus (A \cap B)$

4.2 逻辑(Logic)

Statement

Statement

A **statement**, or a **proposition**, is a sentence which is either true or false, but not both.

We say that a statement can have a **truth value**.

Examples of statements:

- $1 + 1 = 2$
- There are infinitely many primes of the form $n^2 + 1$.
- There is always a prime number between n and $2n$.
- Today is Friday.

Examples of non-statements:

- John is a smart guy.
- x is a very large number.

Negation

Negation

Given a statement p, $\sim p$ or $\neg p$ (read as "not p") is the **negation** of the statement p.

For example, consider the statement p: "I have a headache."

The statement "I do not have a headache" is the negation of p.

Note: p and $\neg p$ always have different truth values.

Example 5

EXAMPLE

Write down the negation of the following statements:

(a) There are integers x and y such that $2x + 3y = 0$.

(b) If x and y are any odd integers, then xy is an odd integer.

(c) For all real values x, $x^2 > 0$ and $\sin x \leq 1$.

Quantifiers

Quantifiers

- $\forall$ symbolizes "for all"
- $\exists$ symbolizes "there exists"

Forms of quantified statements:

1. $\forall x \in \mathbb{R}, x^2 \geq 0$
   - For any real number x, $x^2 \geq 0$.
2. $\exists x \in \mathbb{R}, x^2 = 2$
   - There exists a real number such that $x^2 = 2$.

Negation with Quantifiers

EXAMPLE

Consider the statement p: "The height of everyone in the room is at least 150cm."

- $p : \forall i \in \{\text{Alan, Beth, ..., Zach}\}, h_i \geq 150$
- $\neg p : \exists i \in \{\text{Alan, Beth, ..., Zach}\}, h_i < 150$

For p to be true, every single person must be at least 150cm tall. For p to be false, it suffices for at least one person to be less than 150cm tall.

Negation with Quantifiers

TIP

Key rule for negating quantifiers:

$$\neg(\forall x, p(x)) \equiv (\exists x, \neg p(x))$$

Exercise 5

🤔 **EXERCISE 5**

Write the negation of the following statements:

1. $\exists x \in \mathbb{R}, -2 < x < 3$
2. $\forall x \in \mathbb{Z}$, if x is odd, then x^2 is odd.

Conditional Statements

Conditional Statement

An important connective is the **conditional statement** $(p \Rightarrow q)$.

The conditional statement "p implies q" $(p \Rightarrow q)$ means that if p is a true statement then q is also a true statement.

Key point: $p \Rightarrow q$ fails to hold only when p is true and q is false!

If p is false then the statement $p \Rightarrow q$ is said to be **vacuously true**.

Ways to Express Conditional Statements

TIP

There are several ways in which $p \Rightarrow q$ can be expressed in Mathematics:

- p is the **hypothesis** of the conditional statement
- q is the **conclusion**

Equivalent expressions:

1. "if p then q"
2. "p only if q"
3. "p is a sufficient condition for q"
4. "q is a necessary condition for p"

Truth Table

REMEMBER remember

The relations between P, Q, *and* $P \Rightarrow Q$:

P	Q	$P \Rightarrow Q$
T	*T*	*T*
F	*F*	*T*
F	*T*	*T*
T	*F*	*F*

Truth Table

REMEMBER remember

A	B	$A \Rightarrow B$	$\neg B$	$\neg A$	$\neg B \Rightarrow \neg A$
T	T	T	F	F	T
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F

From this table, we can see that $A \Rightarrow B$ *always has the same true value as* $\neg B \Rightarrow \neg A$.

Contrapositive

Contrapositive

$\neg B \Rightarrow \neg A$ is called the **contrapositive** of $A \Rightarrow B$.

Important: $A \Rightarrow B$ always has the same truth value as $\neg B \Rightarrow \neg A$.

More precisely, $A \Rightarrow B$ and $\neg B \Rightarrow \neg A$ are **equivalent statements**.

Necessary and Sufficient Conditions

Biconditional

"p is a necessary and sufficient condition for q" means "p if and only if q" (p iff q) or symbolically, $p \Leftrightarrow q$.

From the contrapositive property:

$$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$$

Example 6

EXAMPLE

Write the contrapositive of the following statements:

(a) If it is raining, I wear my coat.

(b) $x^2 = 4 \Rightarrow x = \pm 2$

Exercise 6

EXERCISE 6

Write the contrapositive of the following statement:

$x^2 = 4$ and $x > 0 \Rightarrow x = 2$

Example 7

✏️ **EXAMPLE**

What is the relation between statement p and statement q in the following contexts?

(a) $p : y = x^2; q : \dfrac{dy}{dx} = 2x$

(b) $p : (x-1)(x-2) > 0; q : x > 2$

Exercise 7

EXERCISE 7

Insert the correct conditional symbol between the given statements. That is, $p \Rightarrow q, p \Leftarrow q$, or $p \Leftrightarrow q$:

1. $p : x^2 = 4; q : x = 2$
2. $p : \dfrac{x^2}{(x-1)} \leq 0; q : x < 1$
3. $p : \sin\theta = 0; q : \tan\theta = 0$
4. $p : \sin\theta = 0; q : \cos\theta = 1$

Extended Truth Table

REMEMBER

remember

Complete Truth Table:

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$	$\neg P$	$\neg P \vee Q$
T	*T*	*T*	*T*	*T*	*T*	*F*	*T*
T	*F*	*F*	*T*	*F*	*F*	*F*	*F*
F	*T*	*F*	*T*	*T*	*F*	*T*	*T*
F	*F*	*F*	*F*	*T*	*T*	*T*	*T*

Negation of "P implies Q"

TIP

Key results:

1. The negation of "P implies Q" is "P and not Q"
2. "P implies Q" always has the same truth value as "not P or Q"

Proof:

$$\begin{aligned} \neg(P \Rightarrow Q) &= \neg(\neg P \vee Q) \\ &= P \wedge \neg Q \end{aligned}$$

Example 8

EXAMPLE

Write the negation of the following statement:

For any integer x, there exists an integer y such that for any integer z, the inequality $z < x$ implies $z < x + y$.

4.3 证明方法(Methods of Proof)

What is a Proof?

REMARK

A proof always has the following format:

Start	Middle	End
Conditions given in the question	$\Rightarrow$	*What you want to prove*

Direct Proof

Direct Proof

Definitions play an important role in Mathematics. A **direct proof** of a proposition is often a demonstration that the proposition follows logically from certain definitions.

Example 9

EXAMPLE

Prove that if x and y are odd integers, then xy is an odd integer.

Example 4 Revisited

EXAMPLE

Prove the following:

$(a)\ \overline{A \cup B} = \bar{A} \cap \bar{B}$

$(b)\ \overline{A \cap B} = \bar{A} \cup \bar{B}$

Note: Do not use the Venn Diagram to prove set relations.

Hint: To prove $A = B$, prove $A \subseteq B$ and $B \subseteq A$.

Example 10

EXAMPLE

Prove that, if $a > 0$, the equation $x^3 + ax + b = 0$ has only one real root.

Use of Contrapositive

Proof by Contrapositive

Sometimes, when it is difficult to prove a statement in the form $p \Rightarrow q$, a direct proof of the contrapositive statement, $\neg q \Rightarrow \neg p$, can be used.

Example 11

EXAMPLE

Prove that if n is a natural number such that n^2 is even, then n is even.

Exercise 11

EXERCISE 11

Prove that if n is a perfect number, then n is not a prime number.

Note: A perfect number is equal to the sum of its positive divisors excluding itself, e.g., $6 = 3 + 2 + 1$.

Use of Counter Examples

📚 Proof by Counter Example

If a statement is suspected of being false, then one single **counter example** is sufficient to prove this fact.

Example 12

EXAMPLE

Determine if the following statements are true or false, justifying your answer:

(a) If $f''(x) = 0$ when $x = a$, then $f(x)$ has a point of inflection when $x = a$.

(b) If a function f defined on $[0, 1]$ attains its maximum at $c \in [0, 1]$, then $f'(c) = 0$.

Exercise 12

EXERCISE 12

Determine if the following statements are true or false, justifying your answer:

(a) If $\lim_{n\to\infty} u_n = 0$, then the series $u_1 + u_2 + \ldots$ converges.

(b) If $u_1, u_2, \ldots$ is bounded, then the sequence $u_1, u_2, \ldots$ converges.

Mathematical Induction

Mathematical Induction

A proof by **induction** consists of two cases:

1. **Base case:** Proves the statement for $n = 0$ (or $n = 1$, or some fixed $n = N$) without assuming any knowledge of other cases.
2. **Induction step:** Proves that if the statement holds for any given case $n = k$, then it must also hold for the next case $n = k + 1$.

These two steps establish that the statement holds for every natural number $n \geq N$.

Strong Induction

Strong Induction

Strong induction is similar to standard induction, but with a difference in the inductive hypothesis:

- **Standard induction:** Assume $P(k)$ is true to prove $P(k+1)$
- **Strong induction:** Assume all of $P(1), P(2), \ldots, P(k)$ are true to prove $P(k+1)$

Example 13

EXAMPLE

Prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all positive integers n.

Example 14

EXAMPLE

Show that an equilateral triangle can be dissected into n equilateral triangles for $n \geq 6$.

Thank you!

🎓 Thank you for choosing TuoEdu! We wish you a joyful and fruitful learning experience.

感谢您选择拓课堂教育！我们祝您学习愉快、学业进步！