拓课堂2025暑期牛剑营课程

Lesson 2: Algebra

2.1 多项式(Polynomials)

Quadratics

Quadratics

In algebra, a quadratic equation is any equation that can be rearranged in standard form as

$$ax^2 + bx + c = 0$$

where x represents an unknown value, and a, b, and c represent known numbers, where $a \neq 0$. (If $a = 0$ and $b \neq 0$ then the equation is linear, not quadratic)

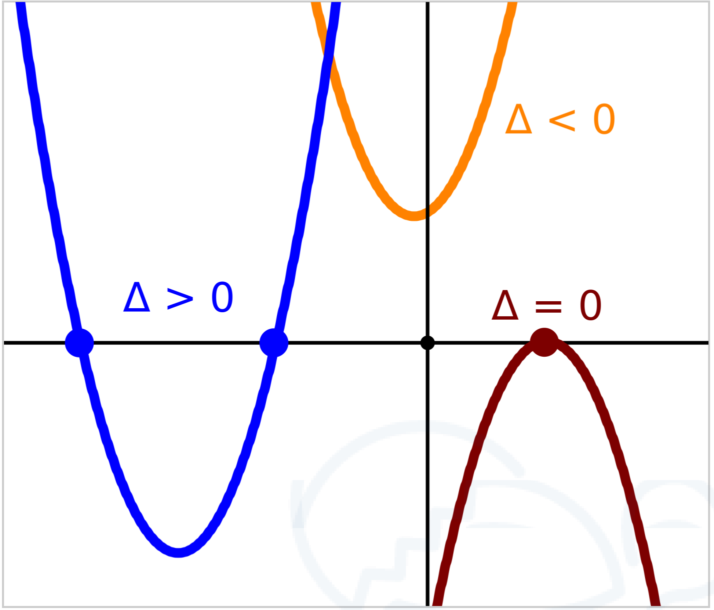
Quadratic formula

THEOREM

Quadratic formula

$$
\begin{aligned}
ax^2+bx+c=0,\quad a\neq 0 \\
\Leftrightarrow x^2+\frac{b}{a}x=-\frac{c}{a} \\
\Leftrightarrow x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2 \\
\Leftrightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2} \\
\Leftrightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \\
\Leftrightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

Discriminant



REMEMBER remember

A quadratic equation with real coefficients can have either one or two distinct real roots, or two distinct complex roots. In this case the discriminant $\Delta = b^2 - 4ac$ determines the number and nature of the roots. There are three cases:

- *If the discriminant is positive, then there are two distinct roots*
- *If the discriminant is zero, then there is exactly one real root, sometimes called a repeated or double root or two equal roots*
- *If the discriminant is negative, then there are no real roots. Rather, there are two distinct (non-real) complex roots which are complex conjugates of each other.*

Example 1

EXAMPLE

$$(*) \quad ax^2 + bx + c = 0 \quad a > 0$$

What is the necessary and sufficient condition for equation (*) to have at least one positive root?

Solution?

QUESTION

? Is this the correct solution

Consider the following "solution"

Let α, β be the two roots of $(*)$

$\alpha + \beta = -\dfrac{b}{a}$

$\alpha \cdot \beta = \dfrac{c}{a}$

Wlog (without loss of generality), assume that $\alpha \leq \beta$. We just need $\beta > 0$

$$\begin{cases} (1) c < 0 \\ (2) c > 0, \quad b < 0. \end{cases}$$

Solution

SOLUTION

$$
\begin{aligned}
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad a &> 0 \\
\sqrt{b^2 - 4ac} - b &> 0 \\
\sqrt{b^2 - 4ac} &> b
\end{aligned}
$$

Example 2

EXAMPLE

Solve the following questions on quadratics:

$(a)\ \sqrt{90+\sqrt{90+\sqrt{90+\cdots}}}$

$(b)\ \sqrt{61-28\sqrt{3}}=$

Polynomials

Polynomials

A polynomial in a single variable x can always be written (or rewritten) in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, \ldots, a_n$ are constants that are called the coefficients of the polynomial.

This can be expressed more concisely by using summation notation:

$$\sum_{k=0}^{n} a_k x^k$$

In mathematics, the degree of a polynomial is the highest of the degrees of the polynomial’ s monomials (individual terms) with non-zero coefficients. The degree of a term is the sum of the exponents of the variables that appear in it, and thus is a non-negative integer.

Example 3

EXAMPLE

Consider polynomial $P(x)$ with degree m and polynomial $Q(x)$ with degree n. Find, in different cases, the degree of

(1) $P(x)+Q(x)$

(2) $P(x)-Q(x)$

(3) $P(x)Q(x)$

(4) $P(Q(x))$

Factorization

TIP

In elementary algebra, factoring a polynomial reduces the problem of finding its roots to finding the roots of the factors.

For example, $P(x) = Q(x)R(x) = 0 \Leftrightarrow Q(x) = 0$ or $R(x) = 0$.

Polynomial identities

REMEMBER remember

There are some identities we often use in factorization:

- $(x \pm y)^2 = x^2 + y^2 \pm 2xy$
- $a^2 - b^2 = (a+b)(a-b)$
- $x^2 + (a+b)x + ab = (x+a)(x+b)$
- $x^3 \pm y^3 = (x+y)(x^2 + y^2 \mp xy)$
- $(x^n - y^n) = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$
- $(x^n + y^n) = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ *for odd* n
- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$

Example 4

EXAMPLE

Simplify $(1-x+x^2)(1-x^2+x^4)(1-x^4+x^8)\cdots(1-x^n+x^{2n})$

Exercise 4

EXERCISE

(a) Factorize $a^n + 1$ when n is odd.

(b) Let a and n be integers greater than 1. Suppose that $a^n - 1$ is prime. Show that $a = 2$ and n is prime.

Sophie Germain’ s identity

THEOREM

Sophie Germain's identity

In mathematics, Sophie Germain’ s identity is a polynomial factorization named after Sophie Germain stating that

$$\begin{aligned} x^4 + 4y^4 &= ((x+y)^2 + y^2) \cdot ((x-y)^2 + y^2) \\ &= (x^2 + 2xy + 2y^2) \cdot (x^2 - 2xy + 2y^2). \end{aligned}$$

Derivation

NOTE

$$\begin{aligned} x^4 + y^4 &= (x^2 - y^2)^2 + 2(xy)^2 \\ &= (x^2 + y^2)^2 - 2(xy)^2. \end{aligned}$$

Modifying this equation by multiplying y by $\sqrt{2}$ gives

$$x^4 + 4y^4 = (x^2 + 2y^2)^2 - 4(xy)^2$$

Example 5

EXAMPLE

Find the prime decomposition of $2^{22}+1$

Exercise 5

EXERCISE

Prove that $2^{1984} + 1$ is not prime

Factor theorem and remainder theorem

THEOREM

Factor theorem and remainder theorem

Factor theorem: A polynomial $p(x)$ has a factor $(x-k)$ if and only if $p(k)=0$.

Remainder theorem: A polynomial $p(x)$ divides $(x-k)$ has remainder $p(k)$.

Note: to prove the above theorems, consider $p(x)=(x-k)q(k)+$ remainder.

Example 6

EXAMPLE

Factorize the following results:

$$x^3 + 7x^2 + 8x + 2$$
$$x^3 - 9x + 8$$
$$x^9 + x^6 + x^3 - 3$$

Rational root test

THEOREM

Rational root test

Let

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

be an equation with integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$. The theorem states that each rational solution $x = p/q$, written in lowest terms so that p and q are relatively prime, satisfies:

- p is an integer factor of the constant term a_0, and
- q is an integer factor of the leading coefficient a_n.

Example 7

EXAMPLE

Factorize $x^3 + 2x^2 - 3x - 10$

Example 8

EXAMPLE

Use de Moivre’ s theorem to show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

Without using a calculator, verify that $\cos 4\theta = -\cos 3\theta$ for each of the values $\theta = \frac{1}{7}\pi, \frac{3}{7}\pi, \frac{5}{7}\pi, \pi$.

Using the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, show that the roots of the equation

$$8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0$$

are $\cos\frac{1}{7}\pi, \cos\frac{3}{7}\pi, \cos\frac{5}{7}\pi, -1$

Hence prove that $\cos\frac{1}{7}\pi$ is irrational

Exercise 8

EXERCISE

Prove that $\sqrt{2}$ is irrational

Fundamental theorem of algebra

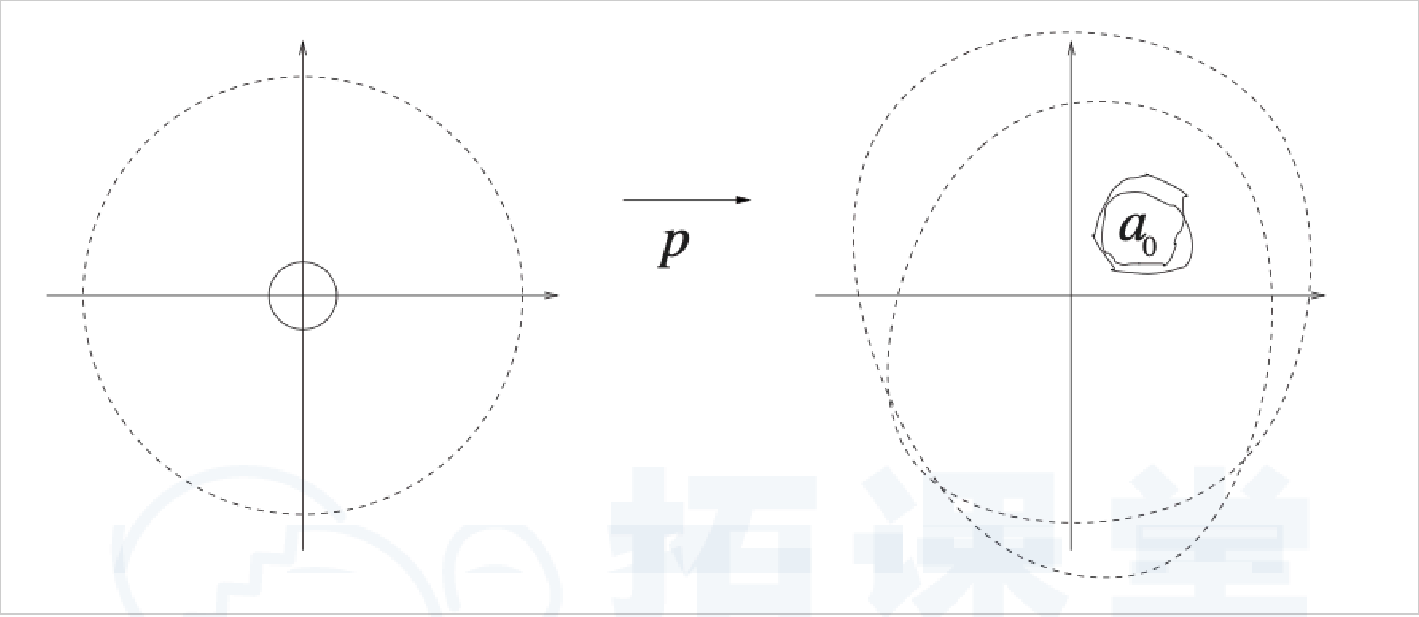
THEOREM

Fundamental theorem of algebra

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n, where $n \geq 1$ and $a_n \neq 0$. Then there are complex numbers $z_1, \ldots, z_n$ such that, for all z

$$p(z) = a_n(z - z_1) \cdots (z - z_n).$$

Sketch proof



Corollary

THEOREM

Corollary

Let p and q be polynomials of degree at most n. If $p(z) = q(z)$ at $n+1$ distinct points then $p(z) = q(z)$ for all z.

Vieta's formulas

THEOREM

Vieta's formulas

For any general polynomial of degree n

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Vieta's formulas relate the polynomial's coefficients to signed sums of products of the roots $r_1, r_2, \ldots, r_n$ as follows:

$$\begin{cases} r_1 + r_2 + \cdots + r_{n-1} + r_n = -\frac{a_{n-1}}{a_n} \\ (r_1r_2 + r_1r_3 + \cdots + r_1r_n) + (r_2r_3 + r_2r_4 + \cdots + r_2r_n) + \cdots + r_{n-1}r_n = \frac{a_{n-2}}{a_n} \\ \quad \vdots \\ r_1r_2\cdots r_n = (-1)^n \frac{a_0}{a_n}. \end{cases}$$

Newton’s sum of roots formula

THEOREM

Newton's sum of roots formula

Consider the cubic equation $ax^3 + bx^2 + cx + d = 0$,

let $S_n = \sum x_i^n$ where x_i are the roots:

$$aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$$

Example 9

EXAMPLE

The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$,

(ii) $\alpha^3 + \beta^3 + \gamma^3$.

Exercise 9

EXERCISE

The roots of the equation $x^4 - 4x^2 + 3x - 2 = 0$ are α, β, γ and δ; the sum $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. By using the relation $y = x^2$, or otherwise, show that $\alpha^2, \beta^2, \gamma^2$ and δ^2 are the roots of the equation

$$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0$$

State the value of S_2 and hence show that

$$S_8 = 8S_6 - 12S_4 - 72$$

Binomial theorem

THEOREM

Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 10

EXAMPLE

(a) What is the coefficient of x^2y^3 in the expansion $(x+y)^5$

(b) What is the coefficient of x^2y^3 in the expansion $(x+y)^6$

Example 11

EXAMPLE

$$\left(\frac{1}{2}+x+2y\right)^{10}$$

What is the coefficient of x^2y^3?

Exercise 11

EXERCISE

How about the coefficients of x^3y^3, x^4y^2?

Multinomial theorem

THEOREM

Multinomial theorem

For any positive integer m and any non-negative integer n, the multinomial formula describes how a sum with m terms expands when raised to an arbitrary power n:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n; k_1,k_2,\cdots,k_m \geq 0} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is a multinomial coefficient.

2.2 函数(Functions)

Functions

Functions

Definition: A **function** (or map) $f : A \mapsto B$ is a 'rule' that assigns, for each $a \in A$, precisely one element $f(a) \in B$. We can write $a \mapsto f(a)$. A and B are called **domain** and **co-domain** respectively.

Injective

Injective

A function $f : X \mapsto Y$ is **injective** if it hits everything at most once.

$$(\forall x, y \in X) \quad f(x) = f(y) \Rightarrow x = y$$

Surjective

Surjective

A function $f : X \mapsto Y$ is **surjective** if it hits everything at least once.

$$(\forall y \in Y)(\exists x \in X) \quad f(x) = y$$

Bijective

Bijective

A function is **bijective** if it is both injective and surjective.

Example 12

EXAMPLE

Consider the function $f : \mathbb{R} \mapsto \mathbb{R}$, $f(x) = x^2$.

Is $f(x)$ surjective? injective? bijective?

Exercise 12

EXERCISE

Consider the function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+, f(x) = x^2$.

Is $f(x)$ surjective? injective? bijective?

Example 13

EXAMPLE

Consider the (infinite) set $\mathbb{Z}$ of integers. Show that there is a function $f : \mathbb{Z} \mapsto \mathbb{Z}$ that is injective but not surjective, and a function $g : \mathbb{Z} \mapsto \mathbb{Z}$ that is surjective but not injective. Now let X be any finite set, and let $f : X \mapsto X$ be any function. Show that the following statements are equivalent:

(a) $f : X \mapsto X$ is injective;

(b) $f : X \mapsto X$ is surjective;

(c) $f : X \mapsto X$ is bijective.

Inverse function

THEOREM

Inverse function

We now discuss conditions under which a function $f : X \mapsto Y$ has an inverse function $f^{-1} : Y \mapsto X$ which 'reverses' the action of f. Explicitly, we want to find a function $g : Y \mapsto X$ such that for all x in X, $gf(x) = x$ and for all y in Y, $fg(y) = y$. These two conditions are equivalent to $gf = I_X$ and $fg = I_Y$.

A function $f : X \mapsto Y$ is invertible if and only if it is a bijection. If this is so then $f^{-1} : Y \mapsto X$ is also a bijection.

Monotonic functions

Monotonic functions

A function is called **monotonically increasing** (also non-decreasing) if for all x and y such that $x \leq y$ one has $f(x) \leq f(y)$, so f preserves the order. Likewise, a function is called **monotonically decreasing** (also non-increasing) if, whenever $x \leq y$, then $f(x) \geq f(y)$, so it reverses the order.

If the order $\leq$ in the definition of monotonicity is replaced by the strict order $<$, one obtains a stronger requirement. A function with this property is called **strictly increasing** (also increasing). Again, by inverting the order symbol, one finds a corresponding concept called **strictly decreasing** (also decreasing). A function with either property is called **strictly monotone**. Functions that are strictly monotone are one-to-one/bijective.

Monotonic functions

TIP

All strictly monotonic functions are invertible.

$f(x)$ is strictly monotonic $\Rightarrow f^{-1}(x)$ exists

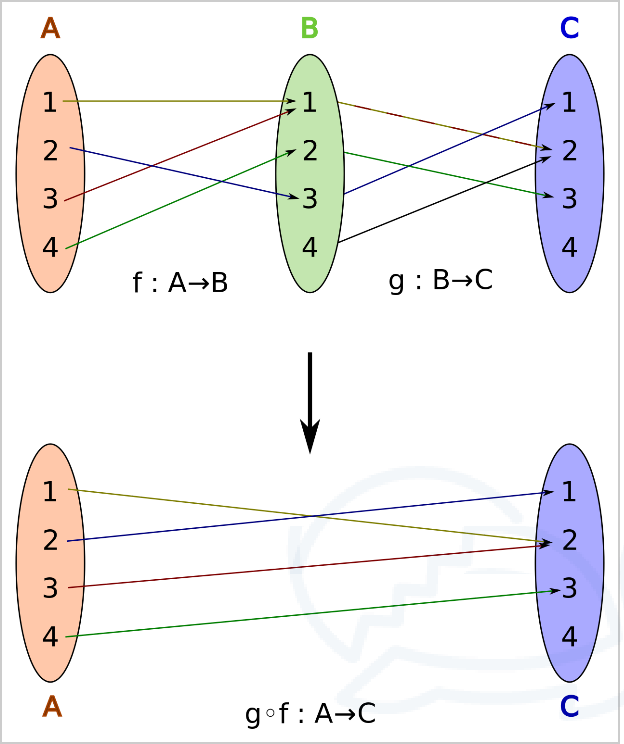
Example 14

EXAMPLE

Show that the map $f : \mathbb{R} \mapsto \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ when $x \neq 0$, and $f(0) = 0$, is a bijection.

What is the inverse of this function?

Composite functions



Composite functions

In mathematics, function composition is an operation $\circ$ that takes two functions f and g, and produces a function $h = g \circ f$ such that $h(x) = g(f(x))$. In this operation, the function g is applied to the result of applying the function f to x (f first, followed by g). That is, the functions $f : X \mapsto Y$ and $g : Y \mapsto Z$ are composed to yield a function that maps x in domain X to $g(f(x))$ in codomain Z.

Properties of composite functions

REMEMBER remember

The composition of functions is always associative, that is, $hgf(x) = (hg)f(x) = h(gf)(x)$

The composition of functions is generally not commutative, that is, $gf(x) \neq fg(x)$

The composition of one-to-one (injective) functions is always one-to-one. Similarly, the composition of onto (surjective) functions is always onto. It follows that the composition of two bijections is also a bijection. The inverse function of a composition (assumed invertible) has the property that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

Example 15

EXAMPLE

Let

$$f(x) = x + 1 \quad \text{and} \quad g(x) = 2x.$$

We will, for example, write fg to denote the function "perform g then perform f" so that

$$fg(x) = f(g(x)) = 2x + 1$$

If $i \geq 0$ is an integer we will, for example, write f^i to denote the function which performs f i times, so that

$$f^i(x) = \underbrace{ff\cdots f}_{i \text{ times}}(x) = x + i.$$

(i) Show that

$$f^2g(x) = gf(x)$$

Example 15 (continued)

EXAMPLE

(ii) Note that

$$gf^2g(x) = 4x + 4$$

Find all the other ways of combining f and g that result in the function $4x + 4$.

(iii) Let $i, j, k \geq 0$ be integers. Determine the function

$$f^i g f^j g f^k(x)$$

(iv) Let $m \geq 0$ be an integer. How many different ways of combining the functions f and g are there that result in the function $4x + 4m$?

Parity of a function

Parity

A function $f(x)$ (given that its domain is symmetrical about 0) is

- **even** if $f(-x) = f(x)$
- **odd** if $f(-x) = -f(x)$

Example 16

EXAMPLE

What are some examples of odd and even functions?

Show that every function $f : \mathbb{R} \mapsto \mathbb{R}$ can be written as a sum of an even function and an odd function

Parity of polynomial functions

TIP

In general, any polynomial function with only even powers of x is an even function.

Any polynomial function with only odd powers of x is an odd function.

Basic Properties - Addition and subtraction

REMEMBER remember

- *The sum of two even functions is even.*
- *The sum of two odd functions is odd.*
- *The difference between two odd functions is odd.*
- *The difference between two even functions is even.*

Basic Properties - Multiplication and division

REMEMBER remember

- *The product of two even functions is an even function.*
- *The product of two odd functions is an even function.*
- *The product of an even function and an odd function is an odd function.*
- *The quotient of two even functions is an even function.*
- *The quotient of two odd functions is an even function.*
- *The quotient of an even function and an odd function is an odd function.*

Basic Properties - Composition

REMEMBER remember

- *The composition of two even functions is even.*
- *The composition of two odd functions is odd.*
- *The composition of an even function and an odd function is even.*
- *The composition of any function with an even function is even (but not vice versa).*

Periodic functions

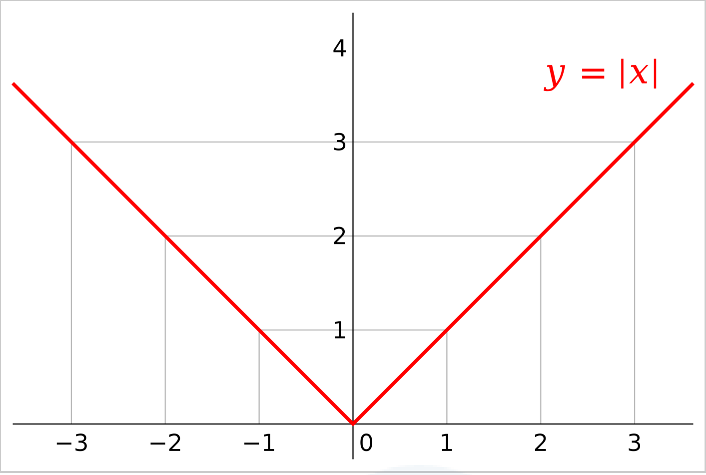
Periodic functions

A function f is said to be **periodic** if, for some nonzero constant P, it is the case that

$$f(x+P)=f(x)$$

for all values of x in the domain. A nonzero constant P for which this is the case is called a **period** of the function. If there exists a least positive constant P with this property, it is called the **fundamental period** (also primitive period, basic period, or prime period).

Modulus functions



TIP

$y = |x|$ is continuous everywhere. It is differentiable everywhere except for $x = 0$. It is monotonically decreasing on the interval $(-\infty, 0]$ and monotonically increasing on the interval $[0, +\infty)$. Since a real number and its opposite have the same absolute value, it is an even function, and is hence not invertible.

Continuity

Continuity

Continuity is a local property!

$f : \mathbb{R} \mapsto \mathbb{R}$ is said to be **continuous at** $x = a$ if and only if

For each $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$

Continuity over closed interval

Continuity over closed interval

(1) $f(x)$ is said to be **left-continuous** at q if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(q)| < \epsilon$ whenever x is in the interval $(q - \delta, q]$.

(2) f is said to be **right-continuous** at p if for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \epsilon$ whenever x is in the interval $(p, p + \delta)$.

(3) $f : [p, q] \mapsto \mathbb{R}$ is **continuous on** $[p, q]$ if f is right-continuous at p, left-continuous at q and continuous at every point strictly between p and q.

Boundedness Theorem

THEOREM

Boundedness Theorem

If f is defined and continuous on a finite closed interval $[a, b]$, then it is bounded and there exists values x_1, x_2 for which $f(x_1) =$ greatest lower bound of $f(x)$ and $f(x_2) =$ least upper bound of $f(x)$.

We say that f is bounded and attains its bounds.

Intermediate value theorem

THEOREM

Intermediate value theorem

- If $f(a) \leq 0$ and $f(b) \geq 0$ and f is continuous on $[a, b]$, then there is some α with $a \leq \alpha \leq b$ and $f(\alpha) = 0$
- If f is continuous on $[a, b]$ and z is any real number between $f(a)$ and $f(b)$, then there is some number α with $a \leq \alpha \leq b$ and $f(\alpha) = z$

Example 17

EXAMPLE

$f : [0, 1] \mapsto [0, 1]$ is continuous, show that $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

(x_0 is a fixed point of $f(x)$)

2.3 曲线绘制(Curve Sketching)

Key features

REMEMBER remember

These are the key features to look out for when you do curve sketching:

- *Domain and Range*
- *x-intercepts/y-intercepts*
- *Symmetry and Asymptotes*
- *Local Max/Min*
- *Concavity and Points of Inflection*
- *Continuity and Differentiability*
- *Periodic behaviour*

Example 18

✏️ **EXAMPLE**

Plot the graph of $\dfrac{x^2}{x-1}$ (rational function)

Example 19

EXAMPLE

Plot the graph of $e^{\sin x}$ (exponential and trigo function)

Exercise 19

EXERCISE

Plot the graph of $y = x \cdot \sin\left(\frac{1}{x}\right)$

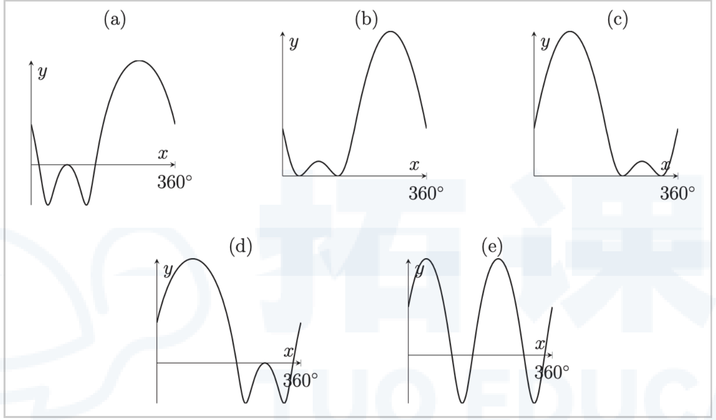
Example 20

EXAMPLE

Which of the following graphs shows

$$y = \log_2(9 - 8\sin x - 6\cos^2 x)$$

in the range $0 \leqslant x \leqslant 360°$? (logarithm and trigo function)



Graph relations

QUESTION

? Graph relations

What are some special features when plotting $y = \dfrac{1}{f(x)}$?

- increasing? decreasing?
- x-intercepts? y-intercepts?
- asymptotes
- Max? Min?

Graph relations

QUESTION

? Graph relations

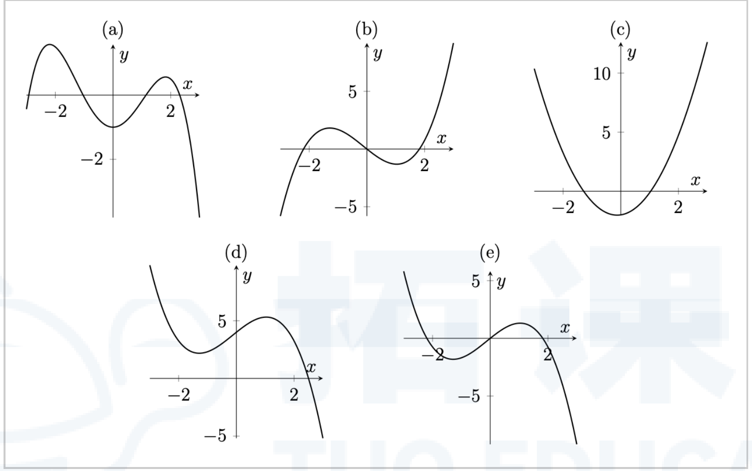
What are some special features when plotting $y = f'(x)$?

- increasing? decreasing?
- x-intercepts? y-intercepts?
- asymptotes
- Max? Min?

Example 21

EXAMPLE

The following five graphs are, in some order, plots of $y = f(x)$, $y = g(x)$, $y = h(x)$, $y = \frac{\mathrm{d}f}{\mathrm{d}x}$ and $y = \frac{\mathrm{d}g}{\mathrm{d}x}$; that is, three unknown functions and the derivatives of the first two of those functions. Which graph is a plot of $h(x)$?



Curve transformations

TIP

There are mainly two types of transformations: translation and stretching.

For translation,

$$y = f(x) \xrightarrow{a \text{ to left}} f(x+a)$$

$$y = f(x) \xrightarrow{a \text{ to right}} f(x-a)$$

$$y = f(x) \xrightarrow{a \text{ upwards}} f(x)+a$$

$$y = f(x) \xrightarrow{a \text{ downwards}} f(x)-a$$

Curve transformation

TIP

For stretching,

$$y = f(x) \xrightarrow{\text{factor a along x-axis}} f\left(\frac{x}{a}\right)$$

$$y = f(x) \xrightarrow{\text{factor a along y-axis}} \frac{f(x)}{a}$$

Order

TIP

When there are multiple transformations, apply the "AMMA" rule,

that is, apply 'addition/subtraction on x,' 'multiplication/division on x,' 'multiplication/division on y,' 'addition/subtraction on y' in this order.

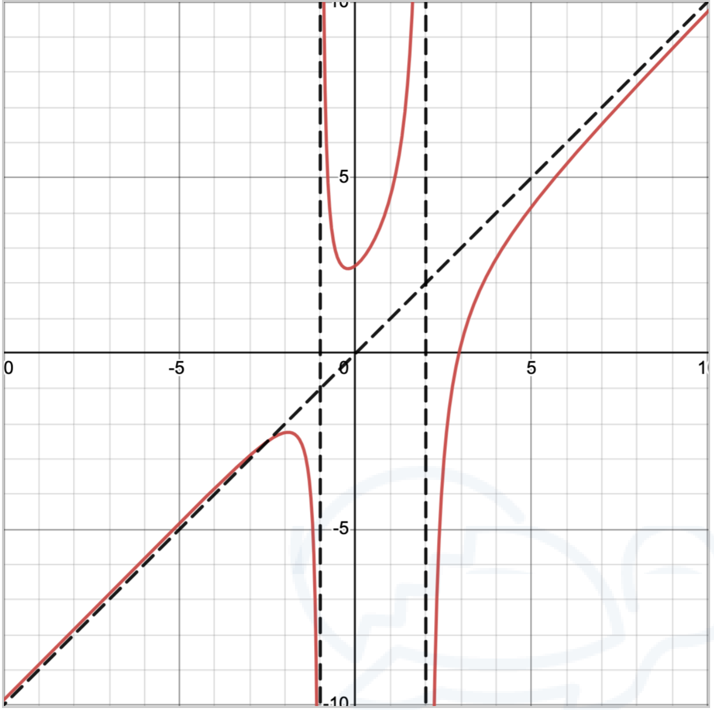
Modulus

TIP

$f(|x|) \longrightarrow$ flip what's on the right (of the y-axis) to the left

$|f(x)| \longrightarrow$ flip what's below (the x-axis) to above

Example 22



EXAMPLE

Graph of $y = f(x)$ is on the left.

Plot, on separate diagrams,

(a) $y = \dfrac{f(x+1)}{2} + 1$

(b) $y = \dfrac{1}{f(x)}$

(c) $y = f'(x)$

(d) $y = |f(x)|$

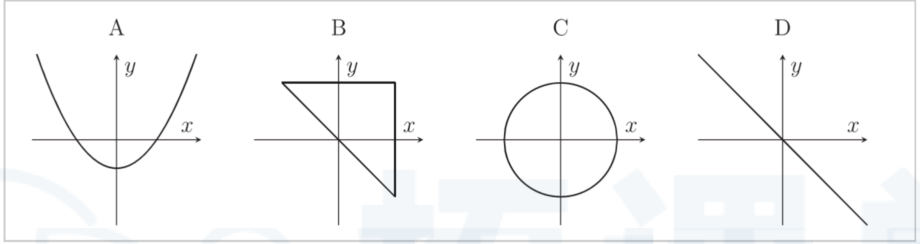
Example 23

EXAMPLE

Which of the following could be the sketch of a curve

$$p(x) + p(y) = 0$$

for some polynomial p?



(a) A and D, but not B or C; (b) A and B, but not C or D; (c) C and D, but not A or B;

(d) A, C and D, but not B; (e) A, B and C, but not D.

Thank you!

🎓 Thank you for choosing TuoEdu! We wish you a joyful and fruitful learning experience.

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