拓课堂2025暑期牛剑营课程

Lesson 3: Algebra

3.1 不等式(Inequality)

Introduction

REMARK

In general, inequalities are used to make comparisons, and to make estimations. It is one of two simple tools that form the language of estimations, the other being absolute values for measuring size and distance.

Estimation

Estimation

If c is a number we are estimating, and $m < c < M$, we say that m is a lower estimate (lower bound) for c and M is an upper estimate (upper bound) for c.

If two sets of upper and lower bounds satisfy the inequalities

$$m < m' < c < M' < M$$

we say that m', M' are stronger or sharper estimates for c, while m, M are weaker estimates.

Example

EXAMPLE

Give upper and lower bounds for $\dfrac{1}{a^4 + 3a^2 + 1}$.

Exercise

EXERCISE

(a) Give upper and lower estimates for $\dfrac{1+\sin^2 n}{1+\cos^2 n}$, for $n \geq 0$.

(b) In Fourier Analysis, one uses trigonometric sums of the form

$$S_n = c_1 \cos t + c_2 \cos 2t + \ldots + c_n \cos nt$$

If $c_i = \dfrac{1}{2^i}$, give an upper bound for S_n.

Absolute value

REMEMBER remember

Two important and common uses of the absolute value are as follows:

(1) $|a| \geq 0$ *for all real values* a.

(2) $|a| \leq M \Leftrightarrow -M \leq a \leq M$. *The absolute value is also an efficient way to give symmetric bounds. In fact, the inequality on the left is often more convenient to use. If the bounds are not symmetric to start with, they can be made so by doing the following:*

$$K \leq a \leq L \Rightarrow |a| \leq M, \text{ where } M = \max(|K|, |L|).$$

In working with absolute values, we will make frequent use of the simple property $|ab| = |a||b|$ *as well as the triangle inequality* ($|a+b| \leq |a| + |b|$)*, a law that connects the absolute value with sums.*

Polynomial inequality

TIP

Step 1: Rewrite the inequality so there is a zero on the right side of the inequality. The expression on the left side designate as $f(x)$.

Step 2: Find the critical numbers. Critical numbers for polynomial functions are the real number solutions to $f(x) = 0$. Draw a number line with the critical numbers labelled. Draw an open circle at each critical number if the inequality uses "<" or ">"; draw a filled in circle at each critical number if the inequality uses "$\leq$" or "$\geq$".

Step 3: Create a sign chart. The critical numbers partition the number line into regions. Choose a test value for each region, including one to the left of all the critical values and one to the right of all the critical values. For each test value, v, determine if $f(v)$ is positive or negative and record the result on the sign chart for each region.

Step 4: Use the sign chart to find the set of all values of x for which the inequality is true. Write the answer in interval notation.

Example 24

EXAMPLE

Solve $-3x^4 + 12x^3 - 9x^2 > 0$.

Rational inequality

TIP

Step 1: Rewrite the inequality so there is a zero on the right side of the inequality. Write the expression on the left as a single algebraic fraction. Designate this fraction as $f(x)$.

Step 2: Find the critical numbers. For rational inequalities these critical numbers come from two sources.

(a) Solve $f(x) = 0$. Find values that make the numerator of $f(x)$ zero.

(b) Solve $f(x) =$ undefined. Find values that make the denominator of $f(x)$ zero. These critical points are where the vertical asymptotes are. These values are not part of the domain of $f(x)$. Therefore these critical values are never part of the solution set.

Step 3: Create a sign chart. The critical numbers partition the number line into regions. Choose a test value for each region. For each test value, v, determine if $f(v)$ is positive or negative and record the result on the sign chart for each region.

Step 4: Use the sign chart to find the set of all values of x for which the inequality is true. Write the answer in interval notation.

Example 25

EXAMPLE

Solve $\frac{1}{x^2-4} \leq \frac{1}{2-x}$.

AM-GM inequality

THEOREM

AM-GM inequality

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i\right)^{1/n}$$

The equality holds if and only if $x_1 = x_2 = \cdots = x_n$ and they are positive numbers.

There are several ways to prove the AM-GM inequality. The most famous one is through induction by Cauchy.

Example 26

EXAMPLE

Let $a_0 > a_1 > \ldots > a_n$ be real numbers. By rewriting $a_0 - a_n$ as $(a_0 - a_1) + (a_1 - a_2) + \ldots + (a_{n-1} - a_n)$, prove that

$$a_0 + \frac{1}{a_0 - a_1} + \frac{1}{a_1 - a_2} + \ldots + \frac{1}{a_{n-1} - a_n} \geq a_n + 2n$$

Exercise 26

EXERCISE

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Show that

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n$$

Example 27

EXAMPLE

If $s > 0, t > 0$.

What is the minimum of $\dfrac{4s^2 + 3t^2}{2st}$?

Rearrangement inequality

THEOREM

Rearrangement inequality

In mathematics, the rearrangement inequality states that

$$x_1 y_n + \cdots + x_n y_1 \leq x_1 y_{\sigma(1)} + \cdots + x_n y_{\sigma(n)} \leq x_1 y_1 + \cdots + x_n y_n$$

for every choice of real numbers

$$x_1 \leq \cdots \leq x_n \quad \text{and} \quad y_1 \leq \cdots \leq y_n$$

and every permutation $y_{\sigma(1)}, \ldots, y_{\sigma(n)}$ of $y_1, \ldots, y_n$

Proof

TIP

Key observations:

- $a > b, c > d \Rightarrow (a-b)(c-d) > 0 \Rightarrow ac + bd > ad + bc$
- It suffices to prove the upper bound and apply to $-y_n \leq \cdots \leq -y_1$ to prove the lower bound
- Use proof by contradiction (suppose c_n is paired with $b_m (m \neq n)$; b_n is paired with $c_l (l \neq n)$ in the maximum case)
- Then use "induction"

Example 28

EXAMPLE

Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

for $a, b, c > 0$

Exercise 28

EXERCISE

Use rearrangement inequality to prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Example 29

EXAMPLE

Think about it!

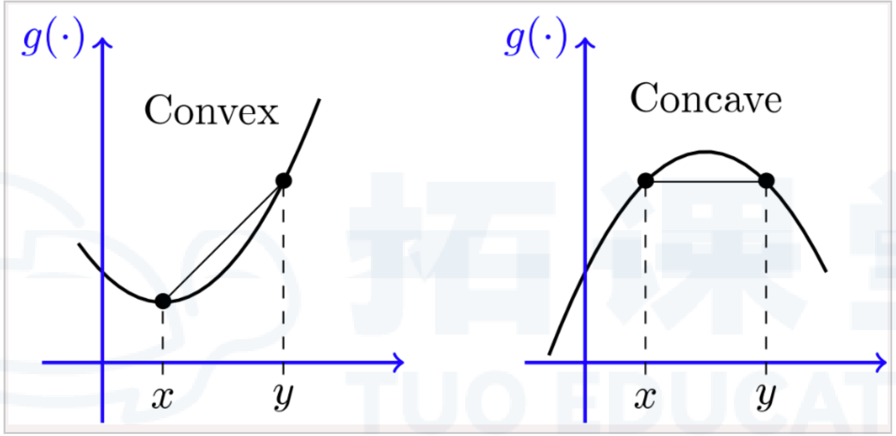
For fixed perimeter, a shape achieves its maximum area when it is a circle

Concave or convex

Concave or convex

A twice-differentiable function $g : I \mapsto \mathbb{R}$ is **convex** if and only if $g''(x) \geq 0$ for all $x \in I$.

A twice-differentiable function $g : I \mapsto \mathbb{R}$ is **concave** if and only if $g''(x) \leq 0$ for all $x \in I$.



Jensen’ s inequality

THEOREM

Jensen's inequality

For $\left(\sum_{i=1}^{n} p_i = 1\right)$

If $f(x)$ is convex

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)$$

If $f(x)$ is concave

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f(x_i)$$

Example 30

EXAMPLE

By considering the function $f(x) = \ln x$, use Jensen’ s inequality to prove AM-GM inequality.

Exercise 30

EXERCISE

For a circle with fixed radius R. What is the maximum area of its inscribed triangle?

Cauchy-Schwarz inequality

THEOREM

Cauchy-Schwarz inequality

There are several ways to write the Cauchy-Schwarz inequality:

- $(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2) \geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2$
- $\left|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right| \leq \left|\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right| \left|\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right|$
- T2 lemma: $\dfrac{\left(\sum_{i=1}^n u_i\right)^2}{\sum_{i=1}^n v_i} \leq \displaystyle\sum_{i=1}^n \frac{u_i^2}{v_i}$

Example 31

EXAMPLE

Minimize $\frac{1}{x^2} + \frac{9}{y^2}$ for points on the circle $x^2 + y^2 = 4$.

Exercise 31

EXERCISE

Prove the T2 lemma

Inequalities involving monotonic functions

TIP

We can make use of strictly increasing/decreasing functions to solve inequalities

Example 32

EXAMPLE

(a) Compare $\sqrt{2}^{\sqrt{3}}$ and $\sqrt{3}^{\sqrt{2}}$

(b) $ae^b = be^a, \quad a > 0$ and $b > 0, a \neq b$ what can we say about $a + b$

Exercise 32

EXERCISE

Compare $\sqrt{2022}-\sqrt{2021}$ and $\sqrt{2021}-\sqrt{2020}$

3.2 数列与级数(Sequences and series)

Sequence definition

Sequence

In mathematics, a sequence is an enumerated collection of objects in which repetitions are allowed and order matters.

For example, (M, A, R, Y) is a sequence of letters with the letter 'M' first and 'Y' last. This sequence differs from (A, R, M, Y). Also, the sequence (1, 1, 2, 3, 5, 8), which contains the number 1 at two different positions, is a valid sequence. Sequences can be finite, as in these examples, or infinite, such as the sequence of all even positive integers $(2, 4, 6, \ldots)$.

The notations we often use are as follows:

$$
\begin{aligned}
a_1 &= \text{1st element of } (a_n)_{n\in\mathbb{N}} \\
a_2 &= \text{2nd element} \\
&\vdots \\
a_{n-1} &= (n-1)\text{th element} \\
a_n &= n\text{th element} \\
a_{n+1} &= (n+1)\text{th element}
\end{aligned}
$$

Recurrence relation

Recurrence relation

Sequences whose elements are related to the previous elements in a straightforward way are often defined using recursion. This is in contrast to the definition of sequences of elements as functions of their positions.

To define a sequence by recursion, one needs a rule, called recurrence relation to construct each element in terms of the ones before it. In addition, enough initial elements must be provided so that all subsequent elements of the sequence can be computed by successive applications of the recurrence relation.

Fibonacci sequence

EXAMPLE

The Fibonacci sequence is a simple classical example, defined by the recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

with initial terms $a_0 = 0$ and $a_1 = 1$. From this, a simple computation shows that the first ten terms of this sequence are $0, 1, 1, 2, 3, 5, 8, 13, 21$, and 34.

A typical type of sequence question will be about finding the general solution ($a_n = f(n)$) from the recurrence relation.

Increasing/decreasing sequence

Increasing/decreasing sequence

A sequence is **increasing** if $x_{n+1} \geq x_n$ for all n.

Strictly increasing: $x_{n+1} > x_n$ for all n.

A sequence is **decreasing** if $x_{n+1} \leq x_n$ for all n.

Strictly decreasing: $x_{n+1} < x_n$ for all n.

Bounded/Unbounded

Bounded/Unbounded

A sequence is **bounded** if there is a positive number M such that $|x_n| \leq M$ for all n.

e.g.

$$1, -1, 1, -1, \cdots$$

Otherwise, it is said to be **unbounded**.

e.g.

$$1^2, 2^2, 3^2, \cdots$$

Arithmetic progression

Arithmetic progression

One of the most common sequences is the arithmetic progression/sequence.

The recurrence relation is given by

$$a_n = a_{n-1} + d, \quad n \geq 0, \quad d \text{ is a constant}$$

One can also show that

$$a_n = a_1 + (n-1)d$$
$$S_n = \sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

Note: AP is not bounded if $d \neq 0$

Geometric progression

Geometric progression

Geometric Sequence: The recurrence relation is given by

$$a_n = ra_{n-1}, \quad n \geq 1, \quad r \text{ is a non-zero constant}$$

One can also show that

$$S_n = \sum_{i=1}^{n} a_i = \frac{a_1(1-r^n)}{1-r}$$

As n goes to infinity, if $|r| < 1, r \neq 0$

$$\lim_{n\to\infty} S_n = \frac{a_1}{1-r}$$

Note: GP is bounded if common ratio $|r| < 1, r \neq 0$.

Example 33

EXAMPLE

Find the general solution to the recurrence relation $a_n = ka_{n-1} + b$ in terms of k, b, and a_1.

Example 34

✏️ **EXAMPLE**

Induction can be a powerful tool in solving sequences and series questions.

Fermat’ s number:

$$F_n = 2^{2^n} + 1$$

Claim: $F_{n+1} - 2 = F_n F_{n-1} F_{n-2} \cdots F_1 F_0$

Example 35

EXAMPLE

$\{y_n\}$ satisfies $y_{n+1} = 5 - \dfrac{6}{y_n} \quad \forall n \geq 0.$

And $2 < y_0 < 3.$

Prove: $\{y_n\}$ is increasing and bounded.

Find $\lim\limits_{n\to\infty} y_n$

Exercise 35

EXERCISE

$\{x_n\}$ satisfies $x_{n+1} = \dfrac{x_n^2 + 6}{5}$ and $x_0 > 3$.

Prove: $\{x_n\}$ is increasing

Series definition

Series definition

In modern terminology, any (ordered) infinite sequence $(a_1, a_2, a_3, \ldots)$ of terms (that is, numbers, functions, or anything that can be added) defines a series, which is the operation of adding the a_i one after the other. To emphasize that there are an infinite number of terms, a series may be called an infinite series. Such a series is represented (or denoted) by an expression like

$$a_1 + a_2 + a_3 + \cdots$$

or, using the summation sign,

$$\sum_{i=1}^{\infty} a_i$$

Series definition (continued)

Series definition (continued)

The infinite sequence of additions implied by a series cannot be effectively carried on (at least in a finite amount of time). However, if the set to which the terms and their finite sums belong has a notion of limit, it is sometimes possible to assign a value to a series, called the sum of the series. This value is the limit as n tends to infinity (if the limit exists) of the finite sums of the n first terms of the series, which are called the nth partial sums of the series. That is,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

When this limit exists, one says that the series is convergent, or that the sequence $(a_1, a_2, a_3, \ldots)$ is summable. In this case, the limit is called the sum of the series. Otherwise, the series is said to be divergent.

Convergence

TIP

To test whether a series is convergent, we often make comparison with common convergent series (sum of GP with a common ratio less than 1).

There are other tests available such as the integral test, the root test, or limit comparison test…

Example 36

EXAMPLE

Prove that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

is divergent.

Prove also that the sum of reciprocals of primes also diverges

Sums of powers

REMEMBER remember

The n-th partial sum of sequences of powers of natural numbers is quite common in MAT questions.

Some important results are as follows:

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example 37

EXAMPLE

Compute: $1^2 - 2^2 + 3^2 - \cdots - 99^2 + 100^2$

Method of differences

TIP

The method of differences can be applied to find the sum of a series so long as the general term in the series can be expressed in the form $f(k+1) - f(k)$.

Example 38

EXAMPLE

Find the infinite product $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$

Exercise 38

EXERCISE

Use method of differences to find $\sum_{k=1}^{n} k^2, \quad \sum_{k=1}^{n} k^3, \quad \text{and} \quad \sum_{k=1}^{n} k^4$

Example 39

EXAMPLE

(i) Find positive integers, x_1 and y_1, satisfying the equation $(x_1)^2 - 2(y_1)^2 = 1$

(ii) Given integers a, b, we define two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ by setting

$$x_{n+1} = 3x_n + 4y_n, \quad y_{n+1} = ax_n + by_n$$

for $n \geq 1$.

Find positive values for a, b such that $(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2$. Deduce that there are infinitely many pairs of integer solutions x, y which solve the equation $x^2 - 2y^2 = 1$.

(iii) Find a pair of integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$.

(iv) Using the values of a and b found in part (ii), what is the approximate value of x_n/y_n as n increases?

Thank you!

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