Statistical Machine Learning

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(Many figures from C. M. Bishop, "Pattern Recognition and Machine Learning")

Outlines

Part X

Number of layers

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- expression of a function is compact when it has few computational elements, i.e. few degrees of freedom that need to be tuned by learning
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  co
  bett
- Exa
  - affine operations, sigmoid $\Rightarrow$ logistic re
    1, fixed number of units (a.k.a. neurons)
  - fixed kernel, affine operations $\Rightarrow$ kern
    has two levels, with as many units as data points
  - stacked neural network of multiple "linear transformation followed by a non-linearity" $\Rightarrow$ deep neural network has arbitrary depth with arbitrary number of units per layer

Easier to represent with more layers

An old result:

- functions that can be compactly represented by a depth k architecture might require an exponential number of computational elements to be represented by a depth $k-1$ architecture
- Exa

pa 1 d 0 o

- Theorem: d-bit parity circuits of depth 2 have e size

Analogous in modern deep learning:

- "Shallow networks require exponentially more parameters for the same number of modes" — Canadian deep learning mafia.

Recall: Multi-layer Neural Network Architecture

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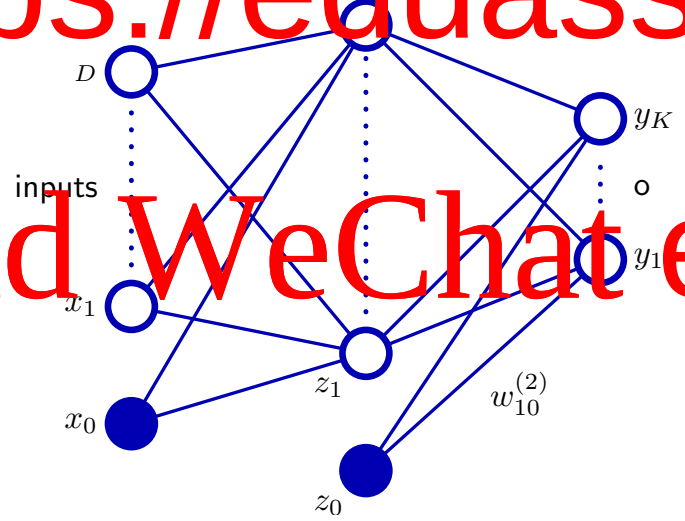
$$y_k(\mathbf{x}, \mathbf{w}) = g\left(\sum_{j=0}^{M} w_{kj}^{(2)} h\left(\sum_{i=0}^{D} w_{ji}^{(1)} x_i\right)\right)$$

where $\mathbf{w}$



We could add more hidden layers

Empirical observations - pre 2006

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- Deep architectures get stuck in local minima or plateaus
- As a gen
- Har
- 1 or 2 hidden layers seem to perform better
- 2006: Unsupervised pre-training, find distrib representation

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Deep representation - intuition

Bengio, "Learning Deep Architectures for AI", 2009

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AlexNet / VGG-F network visualized by mNeuron.

Recall: PCA

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- Idea: Linearly project the data points onto a lower dimensional subspace such that
  - the variance of the projected data is maximised, or
  - the distortion error from the projection is minimised
- Both formulation lead to the same result.
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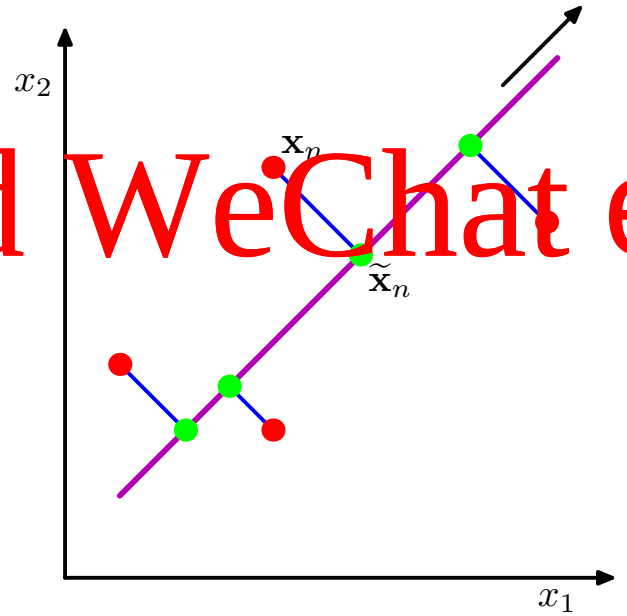

Multiple PCA layers? - linear transforms

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- Principle Component Analysis is a linear transformation (because it is a projection)
  - The
  - Lin
  - Let that

$$ST(X + X') = ST(X) + S$$

  - Similarly for multiplication with a scalar

$\Rightarrow$ multiple PCA layers pointless

Multiple PCA layers? - projection

- Let $X^T X = U \Lambda U^T$ be the eigenvalue decomposition of the covariance matrix (what is assumed about the mean?).
- Define U_k to be the matrix of the first k columns of U, for the k largest eigenvalues. Define Λ_k similarly
- Co
prin

- We perform PCA a second time, $Z^T Z$
- By the definition of Z and $X^T X$ and the ort

$$Z^T Z = (XU_k)^T (XU_k) = U_k^T X^T X U_k = U_k \, U \Lambda U \;\; U_k = \Lambda_k$$

- Hence $\Lambda_Z = \Lambda_k$ and V is the identity, therefore the second PCA has no effect

$\Rightarrow$ again, multiple PCA layers pointless

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Autoencoder

- An autoencoder is trained to encode the input x into some representation $c(x)$ so that the input can be reconstructed from that representation
- the target output of the autoencoder is the autoencoder input itself
- Wit
  crit
  spa
- If the hidden layer is nonlinear, the autoencode differently from PCA, with the ability to capture m aspects of the input distribution
- Let f be the decoder. We want to minimise the reconstruction error

$$\sum_{n=1}^{N} \ell\left(x_n, f(c(x_n))\right)$$

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Cost function

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- Recall: $f(c(x))$ is the reconstruction produced by the network
- Minimisation of the negative log likelihood of the reconstruction, given the encoding $c(x)$
- If $x|$
- If the inputs x_i are either binary or considere binomial probabilities, then the loss function is cross entropy

$$-\log P(x|c(x)) = -x_i \log f_i(c(x)) + (1 - x_i) \log(1 - f_i(c(x)))$$

where $f_i(\cdot)$ is the i^{th} component of the decoder

Undercomplete Autoencoder

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- Co
- $c(x)$
- Ca
  exa
- Hope code $c(x)$ is a distributed representa
  captures the main factors of variation in the data

Stacking autoencoders

- Let c_j and f_j be the encoder and corresponding decoder of the j^{th} layer
- Let z_j be the representation after the encoder c_j
- We can define multiple layers of autoencoders recursively.
- For the c
- Bec feature z_2 can capture more complex patter

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Pre-training supervised neural networks

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- Latent features z_j in layer j can capture high level patterns

$$z_j = c_j(c_{j-1}(\quad c_2(c_1(x))\quad))$$

- The tas
- In co constructed in an unsupervised fashion. j
- Discard the decoding layers, and directly use supervised training method, such as logistic re
- Various such pre-trained networks are available on-line, e.g VGG-19.

Xavier Initialisation / ReLU

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- Layer-wise unsupervised pre-training helps by extracting useful features for subsequent supervised backprop.
- Pre-training also avoids saturation (large magnitude arguments to, say, sigmoidal units).
- Simpler Xavier initialization can also avoid saturation.
- Let t
acti

$$\text{VAR}[z] = \mathbb{E}[(z - \mathbb{E}[z])^2] = \mathbb{E}[z^2] = \qquad ^2$$

$$= \sum_{i=1}^{m} \mathbb{E}[(x_i w_i)^2] = \sum_{i=1}^{m} \mathbb{E}[x_i$$

- So we set $\sigma = 1/\sqrt{m}$ to have "nice" activations.
- Glorot initialization takes care to have nice back-propagated signals — see the auto-encoder lab.
- ReLU activations $h(x) = \max(x, 0)$ also help in practice.

Higher dimensional hidden layer

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- if there is no other constraint, then an autoencoder with d-di
  d c
- Av

  - Early stopping of stochastic gradient descen
  - Add noise in the encoding
  - Sparsity constraint on code $z(x)$.

Denoising autoencoder

- Add noise to input, keeping perfect example as output
- Aut
  1.
  2.
- Re

$$-\log P(x|c(\hat{x}))$$

where x noise free, $\hat{x}$ corrupted

Image denoising

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- Images with Gaussian noise added.

- Denoised using Stacked Sparse Denoising A

Images from Xie et. al. NIPS 2012

Inpainting - 1

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Image from `http://cimg.eu/greycstoration/demonstration.shtml`

Inpainting - 2

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Undo text over image

Image from Bach et. al. ICCV tutorial 2009

Recall: Basis functions

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- For fixed basis functions $\phi(x)$, we use domain knowledge for encoding features
- Ne
  $\phi_i(x$
  lear
- The transformations $\phi_i(\cdot)$ for a particular d
  longer be orthogonal, and furthermore may be
  variations of each other.
- We collect all the transformed features into a ma

Sparse representations

- Idea: Have many hidden nodes, but only a few active for a particular code $c(x)$
- Student t prior on codes
- ℓ_1 penalty on coefficients α

$$\min_{\alpha} \sum_{n=1}^{N} \frac{1}{2} \|x_n - \Phi \alpha_n\|_2^2$$

- Φ is overcomplete, no longer orthogonal
- Sparse $\Rightarrow$ small number of non-zero α_i.
- Exact recovery under certain conditions (coherence): $\ell_1 \to \ell_0$.
- ℓ_1 regulariser $\sim$ Laplace prior $p(\alpha_i) = \frac{\lambda}{2} \exp(-\lambda |\alpha_i|)$.

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$$\underbrace{y}_{\text{measurements}} = \underbrace{x_{orig}}_{\text{original image}} + \underbrace{\quad}_{\text{noise}}$$

Sparsity assumption

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- Only have noisy measurements

$$y = x_{ori} + w$$

measurements original image noise

- Giv

where $\|\cdot\|_0$ is the number of non-zero eleme

- Φ is not necessarily features constructed fro data.

- Minimise reconstruction error

$$\min_{\alpha} \sum_{n=1}^{N} \frac{1}{2} \|x_n - \Phi\alpha_n\|_2^2 + \lambda\|\alpha\|_0$$

Convex relaxation

- Want to minimise number of components

$$\min_{\alpha} \sum_{n=1}^{N} \frac{1}{2}\|x_n - \Phi\alpha_n\|_2^2 + \lambda\|\alpha\|_0$$

- but
- Rel

$$\min_{\alpha} \sum_{n=1}^{N} \frac{1}{2}\|x_n - \Phi\alpha_n\|_2^2 +$$

where $\|\alpha\|_1 = \sum_n |\alpha_n|$.

- In some settings does minimisation with ℓ_1 regularisation give the same solution as minimisation with ℓ_0 regularisation (exact recovery)?

Mutual coherence

- Expect to be ok when columns of Φ "not too parallel"
- Assume columns of Φ are normalised to unit norm
- Let
  of th
- De

$$M = M(\Phi) = \max_{i \neq j} |K(i$$

- If we have an orthogonal basis, then Φ matrix, hence $K(i,j) = 0$ when $i \neq j$.
- However, if we have very similar columns, then $M \approx 1$.

Exact recovery conditions

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- If a minimiser of the true ℓ_0 problem α^* satisfies

$$\frac{1}{}$$

the

- If α

$$\|\alpha^*\|_0 < \frac{1}{2}\left(1 + \frac{1}{M}\right)$$

then the minimiser of the ℓ_1 relaxation h
sparsity pattern as α^*.

Motivation

Autoencoder

References

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