Midterm

Name: ______________________________

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Problem	Score	Max
1		5
2		5
3		5
Total		

Notes:

- Open notes exam. But no electronic equipments to be used (this includes cellphones, calculators, computers, tablets etc.)
- Precision and thoroughness are both appreciated. **Reaching the correct answer without the appropriate justification or incorrect reasoning will be penalized.**
- We have proof-read the problem set multiple times to ensure there are no bugs. Some details are however left out intentionally and are for you to figure out. So, if you really think some piece of information is missing and need to make an assumption to solve the problem — please go ahead; there is no need to run it by the instructors. Do not forget to mention the assumption made for solving the problem. Needless to say, unreasonable assumptions are, by definition, unreasonable.
- If you do not have it handy, the DTFS and the inverse DTFS relationship are as follows

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \qquad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

1. **(5 pts)** Let the radon transform of an image $i_1(x, y)$ be $r_1(\alpha, \theta)$.

   Derive an expression for $r_f(\alpha, \theta)$, the radon transform of $f(x, y)$ that is defined as follows:

$$f(x, y) = \frac{d}{dx} i_1(x, y).$$

2. **(5 pts)** We derived in class that the solution to the regularized least squares problem:

$$\arg\min_{\mathbf{x}} \|\mathbf{y} - A\mathbf{x}\|^2 + \lambda\|\mathbf{x}\|^2$$

is

$$\widehat{\mathbf{x}} = (A^\top A + \lambda I)^{-1} A^\top \mathbf{y}.$$

Let A be an $M \times N$ matrix. Suppose that we are given the singular value decomposition of A, i.e, we are given the left and right singular vectors $U \in \mathbb{R}^{M\times M}$ and $V \in \mathbb{R}^{N\times N}$, and the diagonal matrix of singular values $S \in \mathbb{R}^{M\times N}$ such that

$$A = USV^\top.$$

Derive (and simplify) an expression for $\widehat{\mathbf{x}}$ in terms of $\mathbf{y}, U, S, V$ and λ.

3. **(5 pts)** Let $x[n]$ be a N_0-length DT signal. Let $d[k]$ be its DCT-II transform defined as

$$d[k] = 2\sum_{n=0}^{N_0-1} x[n]\cos\left(\frac{\pi k}{N_0}\left(n+\frac{1}{2}\right)\right).$$

Derive an expression for the energy of $x[n]$

$$\sum_{n=0}^{N_0-1} |x[n]|^2$$

in terms of its DCT-II coefficients $d[k]$.

4. **(5 pts)** Images are generally positive and this implies that their DC term is the most dominant Fourier coefficient. We will explore this in this problem.

   Let $x[n]$ be a real positive signal of length N. That is, $x[n] \geq 0$ and is real-valued.

   Let $X[k]$ be the DTFS of the signal $x[n]$. Show that

   $$X[0] \geq |X[k]|.$$

   That is, for real-valued positive signal (ahem, images), the DC component is the largest in magnitude among all Fourier coefficients.