More Terminology

Definition (Directed Graph)

A directed graph is a graph $G = (V, E)$ where V is a set (of objects) and E is a set of ordered pairs of elements of V.

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- In a directed graph each edge (u, v) has a direction
- Edges (u, v) and (v, u) can both exist, and have different weights
- An undirected graph can be seen as a special type of directed graph

Shortest Paths

With weighted edges a simple breadth-first search will not find the shortest paths

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- The "shortest" path from a to e is $\langle a,$

Questions

- What might a "brute force" algorithm do?
- How long would it take?

Bellman-Ford

The Bellman-Ford algorithm solves the general problem where edges may have negative weights

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- A `distance` array is used again
- `distance[v]` is the current estimate of the shortest path to v
- The algorithm proceeds by gradually reducing these estimates

Bellman-Ford

Relaxing edge (u, v) checks if $s \rightsquigarrow u \rightarrow v$ reduces `distance[v]`

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Relax (Input: weighted edge (u, v))

- If $distance[v]$ is greater than $distance[u] + w(u, v)$ then:
  - $distance[v]$ is $distance[u] + w(u, v)$
  - Parent of v is u

Bellman-Ford

Bellman-Ford (Input: weighted graph $G = (V, E)$ and vertex s)

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- Set $distance[v] = \infty$ for all vertices
- Set $distance[s] = 0$
- Rep
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- For each edge $(u, v) \in E$
  - If $distance[v]$ is greater than *dist*
    - Return *FALSE*
- Return *TRUE*

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Question

Why does the loop run $|V| - 1$ times?

Bellman-Ford

Bellman-Ford (Input: weighted graph G and vertex s)

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- All edges are relaxed $|V| - 1$ times so all paths are tried
- The algorithm returns $FALSE$ if a negative weight cycle occurs

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- In iteration i all edges in paths containing i edges have been relaxed
- The most edges in any (simple) path is $|V| - 1$

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Bellman-Ford

Definition (Negative Weight Cycle)

A path $C = \langle v_1, v_2, \ldots, v_n \rangle$ in a directed graph is a negative weight cycle iff C is a cycle and $\sum_{i=1}^{n-1} w(v_i, v_{i+1}) < 0$.

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If a directed graph G contains a negative weight cycle $\langle v_1, v_2, \ldots, v_n \rangle$ then:

- The shortest paths to all vertices reachable from $v_1, \ldots, v_n$ are undefined
- In this case Bellman-Ford will return *FALSE*

Time

Question

What is the time complexity of Bellman–Ford?

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Bellman

- Set
- Set

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- Repeat $|V| - 1$ times:
  - For each edge $e \in E$
    - Relax e

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- For each edge $(u, v) \in E$
  - If *distance*$[v]$ is greater than *distance*$[u] + w(u, v)$
    - Return *FALSE*
- Return *TRUE*

Dijkstra's Algorithm

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If G has non-negative edges only then we can use Dijkstra's Algorithm

- Bellman-Ford relaxes every edge of every path
- The r
- Dijk

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Dijkstra's Algorithm

Basic idea:

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- Relax edges from one vertex
- Will
- Rep

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Dijkstra's Algorithm

Dijkstra's algorithm maintains a set of vertices whose *distance*[*v*] is correct

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Dijkstra (Input: weighted graph $G = (V, E)$, vertex s)

```
distance[v] = infinity for all vertices
di
S = {}
wh
  u is vertex in V - S with least distance[u]
  for v in G.adj[u]
    relax (u,v)
  S = S + {u}
```

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- The next vertex added to S is the one with the least *distance*[*u*]
- This value is now assumed to be minimal. Is this correct?

Correctness

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In the following, the function p represents the (actual) length of the shortest path from the source to a given vertex

- If the
- ∞

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Theorem (Correctness of Dijkstra)

At the start of the while loop of Dijkstra's algorithm, run o directed graph $G = (V, E)$ with non-negati vertex $s \in V$: if $distance[v] = p(v)$ for all ve $distance[u] = p(u)$ for u, the next vertex added to S.

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Proof

First we prove two useful properties

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Lemma ([illegible]

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Let $G = (V, E)$ be a weighted, directed graph with weight function w, and source vertex s. If (u, v) is an edge in [illegible] *(u, v).*

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Proof.

If there is no path $s \rightsquigarrow u$, then $p(u) = \infty$, so $p(v) \leq p(u)$ and the lemma holds. If there is a path $s \rightsquigarrow u$, then $s \rightsquigarrow u \rightarrow v$ is a path to v. The length of one such path to v is $p(u) + w(u, v)$. The *shortest* path to v cannot be *longer* than this, so the lemma also holds in this case. □

Proof

First we prove two useful properties

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https://eduassistpro.github.i

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Proof

First we prove two useful properties

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Lemma (

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Proof

This lemma shows that $distance[u]$ is always an upper bound for $p(u)$

Lemma (Upper Bound Lemma)

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Let $G = (V, E)$ be a weighted, directed graph with weight function w, and sourc [illegible] *ce$[v]$, for all $v \in V$ w* [illegible] *), for all $u \in V$, aft* [illegible]

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Proof.

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Firstly, consider a sequence of 0 relaxed edges.

- $distance[u] = \infty$, for $u \neq s$
- $distance[s] = 0$

If s is part of a negative weight cycle, then $p(s) = -\infty$, otherwise $p(s) = 0$. So, $distance[u] \geq p(u)$ for all $u \in V$ in this case. □

Proof

Proof (continued).

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Now consider the relaxation of edge (x, y) within some sequence of relaxations.

- Ass $x, y)$

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When $(x$ *distance*

- $distance[y] = distance[x] + w(x, y)$, so
- $distance[y] \geq p(x) + w(x, y)$, by the assu

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- $distance[y] \geq p(y)$, by the Triangle Lemma

So after relaxing (x, y), $distance[u] \geq p(u)$ still holds for all vertices in G, and by the principle of induction $distance[u] \geq p(u)$ is always true for any sequence of edge relaxations. $\square$

Proof

Theorem (Correctness of Dijkstra)

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At the start of the while loop of Dijkstra's algorithm, run on weighted, directed graph $G = (V, E)$ with non-negative weight function w, and vertex s

distance

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Proof.

If there is no path $s \rightsquigarrow u$ then $p(u) = \infty$. Sin

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- $distance[u] \geq p(u)$, by the Upper Bound Lem
- $distance[u] = \infty$, so
- $distance[u] = p(u)$.

and the theorem is true. □

Proof

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Proof (co

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If there is a path $s \rightsquigarrow u$, then consider the shortest such path. Let this path be $s \rightsquigarrow^{p} x \to y \rightsquigarrow^{q} u$, where y is the first vertex o . First, it is shown that $distance[y] = p(y)$, as follow shortest path from s to y. (Or there would be a sho) Then,

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- $distance[x] = p(x)$
- $distance[y] = distance[x] + w(x, y) = p(x) + w(x, y)$

since x is in S and (x, y) was relaxed when x was added to S. □

Proof

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Proof (co

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And, since $s \rightsquigarrow^{p} x \rightarrow y$ is a shortest path from s to y, then:

- $p(y) = p(x) + w(x, y) = distance[y]$

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Next we show that $distance[u] = distance$

observations that

- (1) $distance[u] \leq distance[y]$, since u is added next to S
- (2) $p(y) \leq p(u)$, since all edges are non-negative.

$\square$

Proof

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Proof (co

So, beginning with Observation (1):

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- $distance[u] \leq distance[y]$, and therefore
- $distance[u] \leq p(y)$, and
- $distance[u] \leq p(u)$, by Observation (2).

But $distance[u] \geq p(u)$ by the Upper Bound Lemma, so $distance[u] = p(u)$ and the theorem is true. □

Dijkstra's Algorithm

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Discussion

What is the time complexity of Dijkstra's algorithm?

Dijkstra (I

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```
di
di
S = {}
while V - S != {}
  u is vertex in V - S with least distance[u]
  for v in G.adj[u]
    relax (u,v)
  S = S + {u}
```

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Dijkstra's Algorithm

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Dijkstra (I

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```
di
di
S = {}
while V - S != {}
  u is vertex in V - S with least distance[u]
  for v in G.adj[u]
    relax (u,v)    affects ordering of vertices
  S = S + {u}
```

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Performance

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The running time of Dijkstra's algorithm depends on the way in which the ordering of the vertices is managed

- Imp
- The
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- Each edge is relaxed once, giving an aggregate of E

With a binary-heap-based priority queue adding, re
(changing key) all run in $O(\log_2 V)$ time.

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- Overall running time is then $O(E \log_2 V)$