Differential Kinematics

King's College London

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We have talked about:

- Forward Kinematics—figuring out the end point of a robot
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- Mostly we have talked about this in the conte
  what positions to end up in, but not about how t
  positions.

- So now we'll talk about how to get there.

- One way to program robots to move is through trial and error.
- We have talked about this kind of motion for mobile robots
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- So you can think of:
- Forward Kinematics as figuring out how a r motors work in a given way

- Inverse Kinematics as figuring out how to m get the robot to do what we want.

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Here is a mathematical way to represent the problem.

Given:

- n joints:
-

  $\overrightarrow{t} = (t_1 \dots t_k)^T$ end effector target p
- Each position is a coordinate in space, e.g. z).

- $e_i = t_i - s_i$ desired change in position of
- The Inverse Kinematics problem:
  Find values for all θ_j's such that $t_i = s_i(\theta)\ \forall\ i$
- There may be multiple solutions (if there is a solution at all).

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- There are many ways to solve this problem.
- We will look at one incremental way.
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- There are multiple incremental approac
- Gradient Descent — start with an initial gue incrementally change values to reduce err

- Jacobian — we'll look at this one here

- The Jacobian approach is based on a matrix of partial derivatives, where each partial indicates the influence on the end effector position (p_i) by the joint angle (θ_j).
- The Jacobian matrix J is defined as:

$$J_{ij} = \frac{\partial p_i}{\partial \theta_j}$$

- We assume that J is an $m \times n$ m

$$J(\theta) = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_0} & \frac{\partial p_x}{\partial \theta_1} & \cdots & \frac{\partial p_x}{\partial \theta_n} \\ \frac{\partial p_y}{\partial \theta_0} & \frac{\partial p_y}{\partial \theta_1} & \cdots & \frac{\partial p_y}{\partial \theta_n} \\ \frac{\partial \omega}{\partial \theta_0} & \frac{\partial \omega}{\partial \theta_1} & \cdots & \frac{\partial \omega}{\partial \theta_n} \end{bmatrix}$$

($m = 3$ in the case of a 2-DOF arm)

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- A derivative is a “measure of the rate of change of one quantity with respect to another.” [MW, p47]

- The slope of a line is a derivative: change in x with respect to
- th
- Definition:

  The derivative $y = f(x)$ at x_0 is w

$$f'(x_0) = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- The partial derivatives "of a function of several variables are its ordinary derivatives with respect to each variable separately." [MW, p686]

- When we take the partial derivative of a function with two

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$$f(x) = x^2 + \quad ^2$$

the partial derivative of $f(x)$ with

$$\frac{\partial f(x)}{\partial x} = 2x + y$$

- We pretend that y is a constant and compute the derivative of the function $f(x)$ with respect to x.

- Below (a) is an illustration of the function $z = f(x, y) = x^2 + xy + y^2$.
- The red line highlighted in (a), shows the value of the function at the point $y = 1$.

- The red line is shown within the xz plane in (b), illustrating

$- \qquad + y.$

(a) (b)

Differential Kinematics and the Jacobian Method for computing Inverse Kinematics

- Differential Kinematics gives the relationship between the joint velocities (how the joints change) and the corresponding end effector and its angular velocity

- ν_e is the angular velocity vector of the end effector
- $\dot{p}_e$ is the position velocity (change in positi effector

- ω_e is the angular velocity (change in rotati effector
- J is a square matrix representing the Jacobian—the partial derivatives of joint positions with respect to the angular velocity of the joints
- $\dot{q}$ is a vector of joint velocities

- For the contribution to the linear velocity (translation), we have:

$$\dot{p}_e = \sum_{i=1}^{n} JP_i \dot{q}_i$$

- JP is

- Each term represents the contribution of the velocity of a single joint (i) to the end effector, as if all the o were stationary.

$$\dot{p}_e = [JP_1 JP_2 \ldots JP_n] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_2 \\ \dot{q}_n \end{bmatrix}$$

- For the contribution to the angular velocity (rotation), we have:

$$\dot{\omega}_e = \sum_{i=1}^{n} J\theta_i \dot{q}_i$$

- $J\theta$ is ... $\dot{q}$.

- Each term represents the contribution of th single joint angle (i) to the end effector, joints were stationary.

- Thus:

$$\dot{\omega}_e = [J\theta_1 J\theta_2 \dots J\theta_n] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_2 \\ \dot{q}_n \end{bmatrix}$$

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- We put these together in the Jacobian matrix:

position components

JP

-

$$\nu_e = \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = \begin{bmatrix} JP_1 & JP \\ J\theta_1 & J\theta_2 \end{bmatrix} \quad n \quad \begin{bmatrix} \dot{q}_n \end{bmatrix}$$

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- The cross product between two vectors, vector that is perpendicular to both of them, o n.

- The cross product can be calculated as:

$$A \times B = |A|\ |B|\ sin(\theta)\ n$$

where: A and B are our vectors,
θ is the angle between them,
and n is the unit vector perpendicular to both A and B

- The cross product, when A and B intersect at the origin $(0, 0)$ is computed as follows:

$$C = A \quad B = \quad A_x \quad A \qquad B_x \quad B$$

$$-A_y \quad A_x \quad 0 \quad B_z$$

$$= \begin{bmatrix} (0 * B_x) + (-A_z * \\ (A_z * B_x) + (0 * B \\ (-A_y * B_x) + (A_x \end{bmatrix}$$

$$= \begin{bmatrix} -A_z * B_y + A_y * B_z \\ A_z * B_x - A_x * B_z \\ -A_y * B_x + A_x * B_y \end{bmatrix}$$

- For example, for a 3-link arm, J can be written as:

$$J = \begin{bmatrix} JP \\ JR \end{bmatrix} = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} & \frac{\partial p_x}{\partial \theta_3} \\ \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} & \frac{\partial p_y}{\partial \theta_3} \end{bmatrix}$$

- (p_x, p_y) represent the position of the end e
- ω represents the rotation of the end effecto
- θ_i represents the angle at each of the 3 joints
- and

$$JP = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} & \frac{\partial p_x}{\partial \theta_3} \\ \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} & \frac{\partial p_y}{\partial \theta_3} \end{bmatrix} \quad JR = \begin{bmatrix} \frac{\partial \omega}{\partial \theta_1} & \frac{\partial \omega}{\partial \theta_2} & \frac{\partial \omega}{\partial \theta_3} \end{bmatrix}$$

$$P0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad P1 = \begin{bmatrix} L_1 \; cos(\omega_1) \\ L_1 \; sin(\omega_1) \\ 0 \end{bmatrix}$$

$$P2 = \begin{bmatrix} L_1 \; cos(\omega_1) + L_2 \; cos(\omega_1 + \omega_2) \\ L_1 \; sin(\omega_1) + L_2 \; sin(\omega_1 + \omega_2) \\ 0 \end{bmatrix}$$

$$P3 = \begin{bmatrix} L_1 \ cos(\omega_1) + L_2 \ cos(\omega_1 + \omega_2) + L_3 \ cos(\omega_1 + \omega_2 + \omega_3) \\ L_1 \ sin(\omega_1) + L_2 \ sin(\omega_1 + \omega_2) + L_3 \ sin(\omega_1 + \omega_2 + \omega_3) \\ 0 \end{bmatrix}$$

- if we want to compute the position of the end effector, $P3$, we can use the Jacobian matrix to do that.

- Given:

[illegible] $_2 + \omega_3)$

[illegible] are:

$$\frac{\partial P3_x}{\partial \omega_1} = -L_1\ sin(\omega_1) - L_2\ sin(\omega \text{[illegible]} + \omega_3)$$

$$\frac{\partial P3_x}{\partial \omega_2} = -L_2\ sin(\omega_1 + \omega_2) - L_3 \text{[illegible]}$$

$$\frac{\partial P3_x}{\partial \omega_3} = -L_3\ sin(\omega_1 + \omega_2 + \omega_3)$$

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- and for $P3_y$:

$$P3_y = L_1 \; sin(\omega_1) + L_2 \; sin(\omega_1 + \omega_2) + L_3 \; sin(\omega_1 + \omega_2 + \omega_3)$$

are:

$$\frac{\partial P3_y}{\partial \omega_1} \quad \ldots \; 1 \quad 1 \quad 2 \quad 1 \quad 2 \quad 3 \quad \ldots \; 1 + \omega_2 + \omega_3)$$

$$\frac{\partial P3_y}{\partial \omega_2} = L_2 \; cos(\omega_1 + \omega_2) + L_3 \; c\ldots$$

$$\frac{\partial P3_y}{\partial \omega_3} = L_3 \; cos(\omega_1 + \omega_2 + \omega_3)$$

- and for $\omega_{total} = \omega_1 + \omega_2 + \omega_3$:

$$\frac{\partial \omega_{total}}{\partial \omega_1} = 1 \qquad \frac{\partial \omega_{total}}{\partial \omega_2} = 1 \qquad \frac{\partial \omega_{total}}{\partial \omega_3} = 1$$

- We can put these all together into the Jacobian matrix as follows

$$
\begin{matrix}
\frac{\partial P3_x}{\partial \omega_1} & \frac{\partial P3_x}{\partial \omega_2} & \frac{\partial P3_x}{\partial \omega_3} \\
\frac{}{} & \frac{}{} & \frac{}{} \\
\frac{}{\partial \omega_1} & \frac{}{\partial \omega_2} & \frac{}{\partial \omega_3}
\end{matrix}
$$

- Substitute for each of the partial derivative values from the previous slides

- Then we can use this matrix to determine ho moves when each of the joint angles change—Differential Kinematics !!

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- Remember where we started today:

  Differential Kinematics gives the relationship between the joint velocities (how the joints change) and the corresponding end effector and its angular velocity

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

where:

- ν_e is the angular velocity vector of the end eff
- $\dot{p}_e$ is the position velocity (change in positi effector
- ω_e is the angular velocity (change in rotation) of the end effector
- $\dot{q}$ is a vector of joint velocities

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- What if we know ν_e and we want to know $\dot{q}$?
- i.e., we want to determine what joint velocities are required to obtain a desired position and orientation of the end effector?

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- So if we know J:

  then we first compute its inverse:

  and then we multiply the inverse by the angular velocity vector of the end effector (ν_e)

$\rightarrow$ and that will tell us how much each joint angle changes

Jacobian: Example Code

```
float[] solveJacobian( px, py, pw, w1, w2, w3, ṗx, ṗy, ṗw ) {
  float new_angles[] = { 0.0, 0.0, 0.0 };
  float νe[] = { ṗx − px, ṗy − py, ṗw − pw }; // velocity of he
  float J[][] = new float[3][3]; // Jacobian
  J[0][0] = −L1 ∗ sin(w1) − L2 ∗ sin(w1 + w2) − L3 ∗ sin(w1 + w2 + w3);
  J[0
  J[0
  J[1                                                              w3);
  J[1
  J[1
  J[2][0] = 1;
  J[2][1] = 1;
  J[2][2] = 1;
  float Jinv[][] = inv3( J ); // compute
  float q̇[] = multiply3x1( Jinv, νe );
  new_angles[0] = w1 + q̇[0];
  new_angles[1] = w2 + q̇[1];
  new_angles[2] = w3 + q̇[2];
  return( new_angles );
} // end of solveJacobian()
```


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- That's all!
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- Coursework 2 is due in 2 weeks (27th Novemb
- Look at starter and helper code uploaded on K

- sum rule

$$\frac{d}{}(u+v) = \frac{du}{} + \frac{dv}{}$$

-

- constant multiple rule:

$$\frac{d}{dx} A * u =$$

- power rule:

$$\frac{d}{dx}(x^n) = n * x^{(n-1)}$$

- power of a function rule:

- quotient rule:

- reciprocal rule:

$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$$

-

  if y is a function of u and u is a function of x

- quadratic function rule:

$$\frac{d(ax^2 + bx + c)}{}$$

-

$$\frac{d(bx + c)}{dx}$$

$$\frac{d(c)}{dx} =$$

- The change in value of $sin(x)$ wit

$$\frac{d\ sin(x)}{dx} =$$

- The change in value of $cos(x)$ with respect to x is $-sin(x)$:

$$\frac{d\ cos(x)}{dx} = -sin(x)$$

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