

LECTURE 12

- **Readings:** Section 4.3; parts of Section 4.5 (mean and variance only; no transforms)

Lecture outline

- Conditional expectation
 - Law of iterated expectations
 - Law of total variance
- Sum of a random number of independent r.v.'s
 - mean, variance

Conditional expectations

- Given the value y of a r.v. Y :

$$\mathbf{E}[X | Y = y] = \sum_x x p_{X|Y}(x | y)$$

(integral in continuous case)

- Stick example: stick of length ℓ
break at uniformly chosen point Y
break again at uniformly chosen point X
- $\mathbf{E}[X | Y = y] = \frac{y}{2}$ (number)

$$\mathbf{E}[X | Y] = \frac{Y}{2} \text{ (r.v.)}$$

- **Law of iterated expectations:**

$$\mathbf{E}[\mathbf{E}[X | Y]] = \sum_y \mathbf{E}[X | Y = y] p_Y(y) = \mathbf{E}[X]$$

- In stick example
 $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X | Y]] = \mathbf{E}[Y/2] = \ell/4$

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$\text{var}(X | Y)$ and its expectation

- $\text{var}(X | Y = y) = \mathbf{E}[(X - \mathbf{E}[X | Y = y])^2 | Y = y]$
- $\text{var}(X | Y)$: a r.v. with value $\text{var}(X | Y = y)$ when $Y = y$
- **Law of total variance:**
 $\text{var}(X) = \mathbf{E}[\text{var}(X | Y)] + \text{var}(\mathbf{E}[X | Y])$

Proof:

- Recall: $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$
- $\text{var}(X | Y) = \mathbf{E}[X^2 | Y] - (\mathbf{E}[X | Y])^2$
- $\mathbf{E}[\text{var}(X | Y)] = \mathbf{E}[X^2] - \mathbf{E}[(\mathbf{E}[X | Y])^2]$
- $\text{var}(\mathbf{E}[X | Y]) = \mathbf{E}[(\mathbf{E}[X | Y])^2] - (\mathbf{E}[X])^2$

Sum of right-hand sides of (c), (d):
 $\mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \text{var}(X)$

variances

Two sections:

$y = 1$ (10 students); $y = 2$ (20 students)

$$y = 1: \frac{1}{10} \sum_{i=1}^{10} x_i = 90 \quad y = 2: \frac{1}{20} \sum_{i=11}^{30} x_i = 60$$

$$\mathbf{E}[X] = \frac{1}{30} \sum_{i=1}^{30} x_i = \frac{90 \cdot 10 + 60 \cdot 20}{30} = 70$$

$$\mathbf{E}[X | Y = 1] = 90, \quad \mathbf{E}[X | Y = 2] = 60$$

$$\mathbf{E}[X | Y] = \begin{cases} 90, & \text{w.p. } 1/3 \\ 60, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbf{E}[\mathbf{E}[X | Y]] = \frac{1}{3} \cdot 90 + \frac{2}{3} \cdot 60 = 70 = \mathbf{E}[X]$$

$$\begin{aligned} \text{var}(\mathbf{E}[X | Y]) &= \frac{1}{3}(90 - 70)^2 + \frac{2}{3}(60 - 70)^2 \\ &= \frac{600}{3} = 200 \end{aligned}$$

Section means and variances (ctd.)

$$\frac{1}{10} \sum_{i=1}^{10} (x_i - 90)^2 = 10 \quad \frac{1}{20} \sum_{i=11}^{30} (x_i - 60)^2 = 20$$

$$\text{var}(X | Y = 1) = 10 \quad \text{var}(X | Y = 2) = 20$$

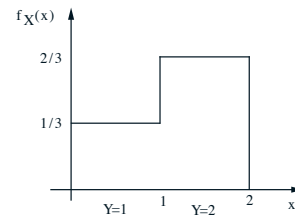
$$\text{var}(X | Y) = \begin{cases} 10, & \text{w.p. } 1/3 \\ 20, & \text{w.p. } 2/3 \end{cases}$$

$$\mathbb{E}[\text{var}(X | Y)] = \frac{1}{3} \cdot 10 + \frac{2}{3} \cdot 20 = \frac{50}{3}$$

$$\begin{aligned} \text{var}(X) &= \mathbb{E}[\text{var}(X | Y)] + \text{var}(\mathbb{E}[X | Y]) \\ &= \frac{50}{3} + 200 \\ &= (\text{average variability within sections}) \\ &\quad + (\text{variability between sections}) \end{aligned}$$

Example

$$\text{var}(X) = \mathbb{E}[\text{var}(X | Y)] + \text{var}(\mathbb{E}[X | Y])$$



$$\mathbb{E}[X | Y = 1] = \quad \mathbb{E}[X | Y = 2] =$$

$$\text{var}(X | Y = 1) = \quad \text{var}(X | Y = 2) =$$

$$\mathbb{E}[X] =$$

$$\text{var}(\mathbb{E}[X | Y]) =$$

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Sum of a random number of independent r.v.'s

- N : number of stores visited (N is a nonnegative integer r.v.)
- X_i : money spent in store i
 - X_i assumed i.i.d.
 - independent of N
- Let $Y = X_1 + \dots + X_N$

$$\begin{aligned} \mathbb{E}[Y | N = n] &= \mathbb{E}[X_1 + X_2 + \dots + X_n | N = n] \\ &= \mathbb{E}[X_1 + X_2 + \dots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n] \\ &= n \mathbb{E}[X] \end{aligned}$$
- $\mathbb{E}[Y | N] = N \mathbb{E}[X]$

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y | N]] \\ &= \mathbb{E}[N \mathbb{E}[X]] \\ &= \mathbb{E}[N] \mathbb{E}[X] \end{aligned}$$

Sum of a random number of independent r.v.'s

- $\text{var}(Y) = \mathbb{E}[\text{var}(Y | N)] + \text{var}(\mathbb{E}[Y | N])$
 - $\mathbb{E}[Y | N] = N \mathbb{E}[X]$

$$\text{var}(\mathbb{E}[Y | N]) = (\mathbb{E}[X])^2 \text{var}(N)$$
 - $\text{var}(Y | N = n) = n \text{var}(X)$

$$\begin{aligned} \text{var}(Y | N) &= N \text{var}(X) \\ \mathbb{E}[\text{var}(Y | N)] &= \mathbb{E}[N] \text{var}(X) \end{aligned}$$
- $$\begin{aligned} \text{var}(Y) &= \mathbb{E}[\text{var}(Y | N)] + \text{var}(\mathbb{E}[Y | N]) \\ &= \mathbb{E}[N] \text{var}(X) + (\mathbb{E}[X])^2 \text{var}(N) \end{aligned}$$

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