ANLY-601 Pattern Recognition

Homework 1
Due Tuesday, January 29, 2018

Use only your course notes — no internet or texts.

1. **Moments of Gaussian Densities (10 points)**

   Consider the one-dimensional Gaussian pdf

   $$p(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\ \exp - \left(\frac{(x-m)^2}{2\,\sigma^2} \right) \quad .$$

   Use the fact that

   $$\int_{-\infty}^{\infty} \exp -(\alpha\, u^2)\ \ du \ = \ \sqrt{\frac{\pi}{\alpha}}$$

   and the identity

   $$\int u^2 \exp -(\alpha\, u^2)\ \ du \ = \ -\frac{d}{d\alpha} \int \exp -(\alpha\, u^2)\ \ du$$

   to show that the *even* central moments of the Gaussian density are

   ven .

   Use symmetry argu) to show that the *odd* central moments are all zero

   $$E[(x-m)^n] = 0 \quad \text{for}$$

2. **Conditional and Unconditional Variance (10 points)**

   In class we showed the relationship between conditional means and unconditional means. Specifically for random variables $x \in R^N$ and $y \in R^M$, the conditional mean of x is

   $$E[x|y] \ = \ \int x\, p(x|y)\ d^N x$$

   and the *unconditional mean* is

   $$\begin{aligned} E[x] \ &= \ \int x\, p(x)\ d^N x \ = \ \int x \left(\int p(x|y)\, p(y)\ d^M y \right)\ d^N x \\ &= \ \int \left(\int x\, p(x|y)\, d^N x \right) p(y)\, d^M y \ = \ E_y[\, E_x[x|y]\,] \ . \end{aligned}$$

   The relationship between the conditional variance and the unconditional variance is a bit more interesting. For simplicity, take $x \in R$ and $y \in R$ (scalar random variables). The conditional variance is

   $$\mathrm{var}(x|y) \ = \ \int (x - E[x|y])^2\, p(x|y)\, dx \quad . \qquad (1)$$

(Note that like the mean, the conditional variance is a function of x_2.) Show that the unconditional variance is related to the condition variance by

$$\mathrm{var}(x) \;=\; \int (x - E[x])^2\, p(x)\, dx \;=\; E_y[\, \mathrm{var}_x(x|y)\,] \;+\; \mathrm{var}_y(\, E[x|y]\,). \tag{2}$$

Your derivation must show explicitly what $\mathrm{var}_y(\, E[x|y]\,)$ means in terms of integral averages over quantities.

(Hint: Rewrite

$$\begin{aligned}(x - E[x])^2 \;&=\; (x - E[x] + E[x|y] - E[x|y])^2 \;=\; (x - E[x|y] + E[x|y] - E[x])^2 \\ &=\; (x - E[x|y])^2 + (E[x|y] - E[x])^2 \\ &\quad\; + 2\,(x - E[x|y])(E[x|y] - E[x]) \quad .\end{aligned}$$

)

3. **A Maximum likelihood estimation (5 points)**

This problem has an interesting practical origin, that I'll explain after you hand your solution back.

I have a bag filled with m plastic balls, numbered consecutively $1, 2, \ldots, m$. I don't tell you what the value of m is; I want you to make a (statistically informed) guess.

So I give you one piece o

(i.e. with probabili

Let's compute the m. Mathematically, this is the value of m that maximizes p likelihood function — since there is one ball with each number 1 , any number on a ball in the range $1 - m$ can be observed with equal pro

$$p(1|m) = p(2|m) = \cdots = p(m|m) = 1/m \quad .$$

Note also that it's not possible to observe a number on a ball greater than (the unknown) m

$$p(n|m) = 0 \text{ for } n > m \quad .$$

These two pieces of information fix the likelihood function $p(x|m)$. Given this information, what is the value of m that maximizes the likelihood of the data $p(19|m)$?