Assignment Project Exam Help

Review of the Final

Reminders/Comments

This is a final exam that means

- You should work on your own
- We will answer clarification questions but not debug your code
- No extensions

but it is o

- https://eduassistpro.github.i
- We are happy to explain what error message should isolate the line of code that generates t

On the final

- Structured like a homework; 3 questions, 6 p
- Sub-parts are structured as (i): do the coding, (ii) and (iii): plot or comment on results.
- Happy to tell you what code should do, written responses are up to you.

Question 1: Control Functionals

Reminder: control variates

- Monte Carlo estimate if $X_i \sim f(\cdot)$

$$\bar{g}$$

- $\frac{1}{N}\sum_i g(X_i) - \alpha$

has smaller variance if $\alpha = \text{cov}(g(X), h(X))/\text{var}(h(X))$.

- α? Estimate from values of $g(X_i)$ and $h(X_i)$.

Example on Final

Part a: $X \sim N(0,1)$, $g(x) = \sin(x + 0.1)$

Use $h(x) = -e^{x} + e^{1/2}$ because $E[-e^{X}] = -e^{1/2}$.

Comparisons

Parts b/c with antithetic sampling

Note: for $X \sim N(0,1)$ $(X, -X)$ are antithetic.

We want to look at

1. $\bar{X}$
2. $\bar{X}$
3. $\bar{X}$
4. Both: $\frac{1}{N}\sum(g(X_i) + g(-X_i))/2$ $\bar{X}$ $/2$

For both, calculate alpha from values of $(h(X_i) + h(-X_i))/2$

Question about relative improvement are for you to think about.
Also bonus on using $g(x) = \sin(x)$ or $h(x) = x$.

Control Functionals

Part d taken from Oates, Girolami, Chopin, 2016

- Idea, if we can make $h_N(x)$ closer to $g(x)$ as $N \to \infty$ we will do better faster.
- Use $X_1, \ldots, X_{N/2}$ to get a better $h_N(x)$, $X_{N/2+1}, \ldots, X_N$ for
- https://eduassistpro.github.i

$$h_N(x) = \sum_{j=1}^{N/2} d_j$$

d_j make $h_N(x)$ approximate $g(x)$ but we know

$$\int k_j(x) f(x) dx = 0$$

Control Functional Details

Need $Ek_j(X) = 0$: modify a kernel function

$$k_j(x) = \phi'(x - X_j; s) + \phi(x - X_j; s)f'(x)/f(x)$$

where $\phi(z; s)$ is normal density with standard deviation s

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Why?

$$\int k_j(x)f(x)dx = \int (\phi'(x - X_j; s)f(x \qquad x$$

$$= \int \frac{d}{dx}(\phi(x - X_j; s)f(x))dx$$

$$= \phi(\infty - X_j; s)f(\infty) - \phi(-\infty - X_j; s)f(-\infty) = 0$$

Control Functional Implementation

1. Calculate matrix to store for $1, \ldots, N/2$

$$K_{ij} = k_j(X_i) = \phi'(X_i - X_j; s) + \phi(X_i - X_j; s) f'(X_i)/f(X_i)$$

remember $\phi(z; s)$ is $N(0, s)$, $f(x)$ is $N(0, 1)$.

2.

3. Plug in $X_{N/2+1}, \ldots, X_N$ to

$$\hat{m}_N(x) = \sum_{l=N/2+1}^{N} \frac{}{}$$

(note that the X_j used to define k_j stay the same.

k_j: sort of kernel/sort of basis. No smoothing penalty because $g(X_i)$ has no error.

Example

Using $g(x) = \sin(x)$

$N = 5$ $N = 50$

Convergence Rates

Parts e/f: simulation based on control functionals Example is based on $g(x) = \sin(x)$, control variate $h(x) = x$ (more effective than parts a-c). Notes changes with N.

End of Part f:

$$\sqrt{MSE} \approx CN^{\alpha} \rightarrow \log(\sqrt{MSE}) \approx \alpha \log(N) + log(C)$$

looks like a linear model.

Question 2: Kernel Density Estimates

Kernel Density Estimation, given $X_1, \ldots, X_n$:

$$\hat{f}_s(x) = \frac{1}{n} \sum_{i=1}^{n} \phi(x - X_i; s)$$

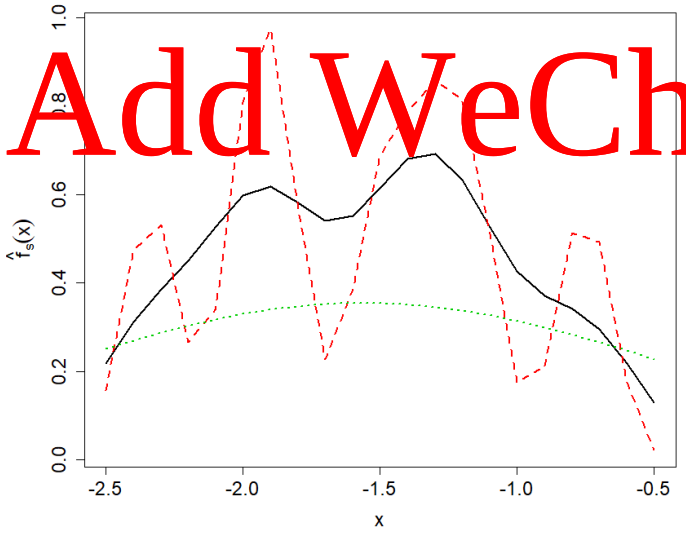
$(\phi(z$

Cross Validation

Part b to choose s, we want to make the density on a *new* point as high as possible.

- $\hat{f}_s^{-i}(X_i)$: density with bandwidth s evaluated at X_i, estimated *without* using X_i.
- nsity

Possible Densities



CV Scores

CV can be calculated efficiently

Simulation

Part c Form of density is like

$$Z|k \sim N(X_k, s), k \sim \text{Unif}(1, \ldots, n)$$

1. Choose an observation at random
2.

Can a

Hellinger Distance

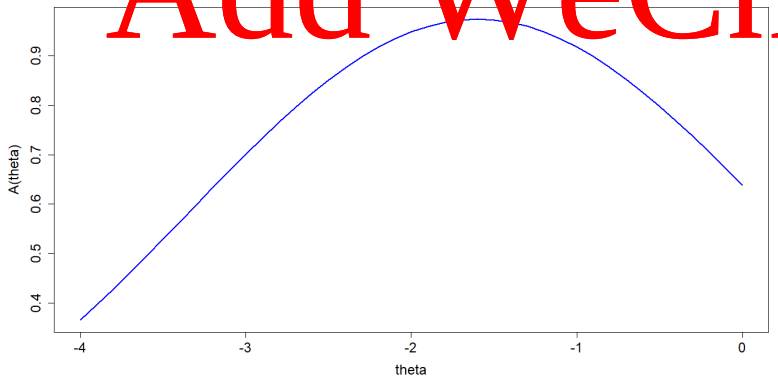
Part d Measure of distance between densities

$$\int (\sqrt{g(x)} - \sqrt{f(x)})^2 dx \equiv 2 - 2\int \sqrt{g(x)}\sqrt{f(x)}dx = 2 - 2A(g, f)$$

or use *affinity*

$$A(g, f) = \int \sqrt{g(x)}\sqrt{f(x)}dx$$

Statistical use: find θ to maximize affinity $A(g, f(x;\theta))$



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and $f(x;\theta)$ is $N(\theta, 1)$:

$$A(\theta) = \frac{1}{N}\sum \sqrt{\frac{\phi(Z_j - \theta, 1)}{\hat{f}(Z_j)}}$$

Optimization and Robustness

To find $\max_\theta A(\theta)$ use `optimize` function between -10 and 10.

Part e Reason for Hellinger is *robustness* – adding a large value to data doesn't change estimate much.

- Add O to data (8 points instead of 7)
- Don't change s with value of O (yes this is cheating).
- Do simulate new Monte Carlo points with each value.

Hellinger's Posterior

Hooker and Vidyashankar, 2014 suggested finding a posterior based on

$$P(\theta) = e^{2nA(\theta)}\pi(\theta)$$

Here we let $\pi(\theta)$ be $N(0, 10)$ and sample from $P(\theta)$ by MCMC

For this question:

- Keep same MC samples with $A(\theta)$ for each θ.
- Experiment to get acceptance to around 30%; decide on thinning from visual inspection (don't work too hard).

Stochastic Objective Functions

Monte Carlo integration $\Rightarrow$ evaluation of $A(\theta)$ random.

Using same samples makes problem smooth, still random.

New Sample Each θ | 10 samples same θ

In MCMC, stochastic posteriors are ok, but decrease acceptance rate.

Mixed Effects Logistic Models

Example data

- 12 Subjects, each measured 7 times
- Record mental health + time from treatment

Logistic model (can use `plogis`)

$$P(Y_{ij} = 1|Z_j) = \frac{e^{\beta 0+\beta_1 X_{ij}+Z_j}}{1 + e^{\beta 0+\beta_1 X_{ij}+Z_j}}$$

X_{ij} = time of visit, Z_j = effect of subject j.

Generative Model

Don't get to see subject effects Z_j: model for a new data set is

1. Generate new $Z_1, \ldots, Z_{12}$ from $N(0, \sigma^2)$
2. Then generate Y_{ij} from $P(Y_{ij} = 1|Z_j)$

Since we don't see Z_j probability of observed data is

Whe

$$P(Y_1, \ldots, Y_{7j}|z) = \prod_{i=1}^{7} P(Y_{ij}|z)$$

$$= \prod_{i=1}^{7} P(Y_{ij} = 1|z)^{Y_{ij}}(1 \qquad _{ij}$$

$$= \prod_{i=1}^{7} [Y_{ij} P(Y_{ij} = 1|z) + (1 - Y_{ij})(1 - P(Y_{ij} = 1|z))]$$

Use most convenient form, or `dbinom`.

Gauss Hermite Approximation

Part a Package `ecoreg` function `GaussHermite(21)` produces z_q, w_q in columns `Points` and `Weights` so that

$$\int F(Y_{i1},\ldots,Y_{ij}|z)\phi(z;\sigma)dz \approx \sum w_q F(Y_{i1},\ldots,Y_{ij}|\sigma z_q)$$

One s

log Likelihood

Parameter vector $\theta = (\beta_0, \beta_1, \sigma)$ has *negative* log likelihood

$$l(\theta) \equiv -\sum_j \log P(Y_{1j}, \ldots, Y_{7j}|\theta)$$

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nota

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- Obtain $P(Y_{ij}|z_q, X_{ij}, \theta)$ from `plog`).
- Evaluate $P(Y_{1j}, \ldots, Y_{7j}|z_q, X_{\cdot j}, \theta)$ m.
- Obtain

$$P(Y_{1j}, \ldots, Y_{7j}|X_{1j}, \ldots, X_{7j}, \theta) \approx \sum w_q P(Y_{1j}, \ldots, Y_{7j}|z_q, X_{\cdot j}, \theta)$$

- negative log likelihood is minus sum of logs.

Maximizing and Alternatives

Maximum likelihood estimator minimizes negative log likelihood.

Use

```
optim(par=theta,fn=logistic.nll,data=toenail)
```

to get

Part

Replace Gauss-Hermite approximation with

$$\int P(Y_{i1},\ldots,Y_{ij}|z)\phi(z,\sigma)dz \approx \frac{}{k}$$

with $Z_1,\ldots,Z_N \sim N(0,\sigma^2)$.

Possible to code so minimal changes from Part a.

Part c: MCMC

- Fix $\sigma = 1.2$ (MCMC techniques for variances are more fiddly).
- Set a prior $\beta_0 \sim N(0, 10)$, $\beta_1 \sim N(0, 10)$.
- Easiest to work with log posterior = log likelihood + log prior (remember `log=TRUE` in density functions)

Run

-

Obtain mean values and quantiles from chain.

Part d: Extended MCMC

What if we also include $Z_1, \ldots, Z_{12}$ as values to be sampled?

Whole likelihood is

$$P(Y, Z, \theta) = \left(\prod_{ij} P(Y_{ij}|X_{ij}, Z_j, \theta) \right) P(Z_j|\sigma) \pi(\theta)$$

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For M

1. $\beta_0 \sim N(\beta_0, 0.25)$, $\beta_1 \sim N(\beta_1, 0.05)$
2. $Z_j \sim N(0, 0.5)$

Z_1 chain gives $P(Z_1|Y_{11}, \ldots, Y_{71})$ (fe

Part e: SMC

Alternative random number generation from $f(z)$:

1. Generate $Z_1, \ldots, Z_N \sim g(z)$
2. Resample (with replacement) using probabilities $W_i = f(Z_i)/g(Z_i)$ to get Z_i^*

samp

Our e

$Z_i \sim N(0, \sigma^2)$

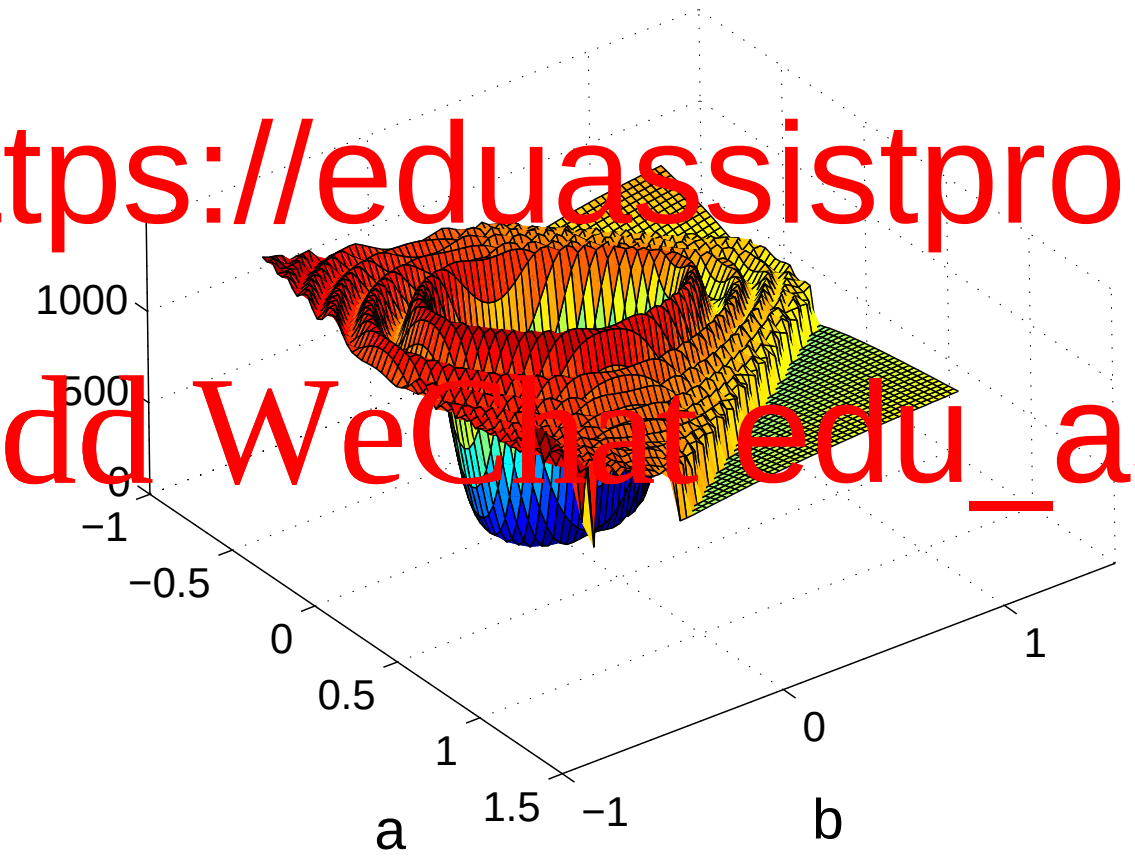
$W_i = P(Y_{\cdot 1} | Z_i, X_{\cdot 1}, \theta)$

$Y_{\cdot 1}, X_{\cdot 1}$ data from Subject 1, θ from Part a estimate.

Bonus

For a hard optimization problem:

1. Write squared error function
2. try function to avoid errors



Wrap Up

BTRY/STSCI 4520: Many Topics Covered Briefly

- Computational considerations
  - Programming strategies + computational complexity
  - Numerical stability
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  -
  -
  - Integration (numerical/Monte Carlo)
- Statistics
  - Simulation
  - Boostraps and permutation
  - Nonparametric smoothing
  - Maximum likelihood and LASSO penalties
  - MCMC

Many tools enable practical, modern statistics.