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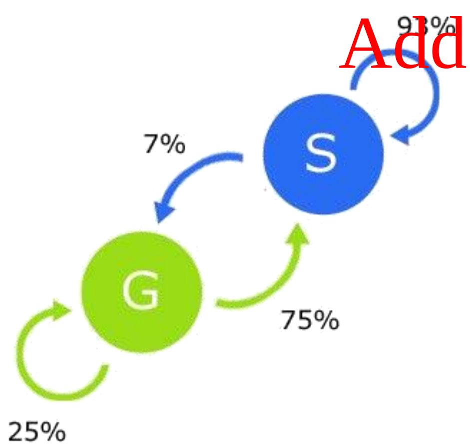
CIS 418

Markov chains: Background

- Markov chain is a generalization o arge Numbers.
- Law of Large Numbers: assumes that every random outcome that we generate is independent of the previous outcomes.
- Markov chain: is called a memoryless process. The idea is that it has "short-term memory" – the future depends only on the current state of the process. This is the key assumption underlying Markov chains.

- Example: Class all



The robabilities are:

P(S)=91.5%

P(G)=8.5%

Markov chains: applications

- Examples of Markov chains applic
  - Describing communication systems
  - Queueing systems
  - Signal processing
  - Manufacturing
  - Music: for exa /watch?v=qOZ2Q-Ls48U
  - Finance. Today of bonds' default.

Videos that explain what Markov chains are:

- https://www.youtube.com/watch?v=63HHmjlh794
- https://www.youtube.com/watch?v=EqUfuT3CC8s
- https://www.youtube.com/watch?v=Ws63I3F7Moc

Debt Rating by Major Agencies

Moody's	Standard & Poor's	Interpretation
Aaa	AAA	Highest quality
Aa	AA	High quality
A	A	Strong payment capacity
Baa	BBB	yment capacity
Ba	BB	tion, ongoing inty
B	B	High -risk obligations
Caa/Ca/C	CCC/CC/C	Current vulnerability to default
C	D	In Default

Future bond payments may be uncertain

This is an example with a 7-year bond, i 00$ face value, 6% annual coupon. Table illustrates some scenarios of possible cash flows .

Question: What is the probability for different scenarios?

Price a Bond based on *expected* return

(as opposed to [illegible] n)

Need to consider how to model bond default:

- Probability of default at a given time
- Percentage of principal that will be recovered in case of default

Expected payoff from a one-year bond

Symbol	Definition
F	Face value
q	Annual **promised** coupon rate (%)
π_D	Probability of default
λ	Fraction o collect in case of default

$$\text{Expected Payo} \quad \overbrace{\quad \cdot \quad}^{default} + \overbrace{1-\pi_D)\cdot\left(F+qF\right)}^{no\_default\_at\_maturity}$$

Expected payments from a two-year bond

Symbol	Definition
F	Face value
q	Annual **promised** coupon rate (%)
π_D^1	Probability of default in year 1
π_D^2	Probability of default in year 2, **given** that there was no default i
λ	Fraction o collect in case of default

$$\text{Exp. Payoff Year } 1 = \overbrace{\pi_D^1 \cdot \lambda F}^{default\_year\_1} + \overbrace{\left(1-\pi_D^1\right) \cdot qF}^{no\_default\_year\_1}$$

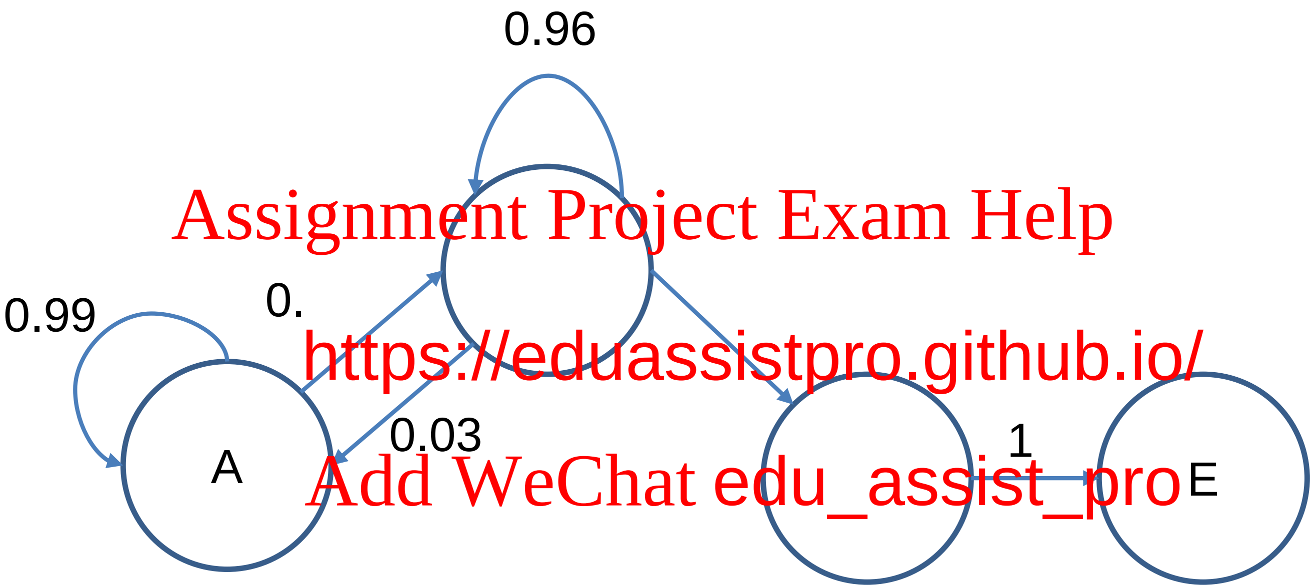
$$\text{Exp. Payoff Year } 2 = \overbrace{\pi_D^1 \cdot 0}^{already\_in\_default} + \overbrace{\left(1-\pi_D^1\right) \cdot \left(\underbrace{\pi_D^2 \cdot \lambda F}_{default\_year\_2} + \underbrace{\left(1-\pi_D^2\right) \cdot (1+q) F}_{no\_default\_year\_2} \right)}^{no\_default\_year\_1}$$

A simplified model of upgrades and downgrades depends he *latest* rating

		bility of transition during *one period* to rating:			
Rating	Interpretation	A	B	D	E
A	Highest quality, pays as promised	99%	1%	0	0
B	Next highest, pays as promised	3%	96%	1%	0
D	First time def percenta		0	0	100%
E	Defaulted in a previous period, pa nothing		0	0	100%

- The lower the ranking, the higher the probability of going into default.
- Possible to get payment only upon initial default, hence two default states (*D* and *E*) are needed for the model.

Graphic representation



In mathematical notation the same information is ca a matrix

Transition Matrix

Rating	A	B	D	E
A	99%	1%		
B	3%	96%		
D	0	0	0	100%
E	0	0	0	100%

$$\begin{bmatrix} .99 & .01 & 0 & 0 \\ & & & \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Transition Matrix

More generally, this matrix r sition probabilities

$$\begin{bmatrix} \pi_{AA} & \pi_{AB} & \pi_{AD} & \pi_{AE} \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & \pi_{BE} \\ \pi_{DA} & \pi_{DB} & \pi_{DD} & \pi_{DE} \\ \pi_{EA} & \pi_{EB} & \pi_{ED} & \pi_{EE} \end{bmatrix}$$

π_{XY} Represents the probability of transitioning from state (rating) X to state (rating) Y in one period.

How to read the transition matrix?

$$\Pi = \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \pi_{AA} & \pi_{AB} & \pi_{AD} & \pi_{AE} \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & \pi_{BE} \\ \pi_{DA} & \pi_{DB} & \pi_{DD} & \pi_{DE} \\ & & & \pi \end{bmatrix}$$

- Say we start with an lity of not going into default in year 1?
  - Given the values we assigned to our things can happen to an A-rated bond over one period: it ating, or get downgraded to B. The probability of going into default, π_{AD} , is zero.

So if the bond matures in one period, what payout should we expect?

$$\underbrace{\left(\pi_{AA} + \pi_{AB}\right)\cdot\left(1+q\right)F}_{not\_in\_default\_and\_at\_maturity} + \underbrace{\pi_{AD}\cdot\lambda F}_{just\_defaulted} + \underset{already\_in\_default}{\pi_{AE}\cdot 0}$$

Second period transition probability

In this model, what will happen to t ted bond in the second period? Given data in our model, it can end up either rated A or B or it can go into default for the first time:

$$\begin{bmatrix} \pi_{AA} & \pi_{AB} & 0 & 0 \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\pi_{AA}^{2} = \pi_{AA} \cdot \pi_{AA} + \pi_{AB} \cdot \pi_{BA}$$

$$\pi_{AB}^{2} = \pi_{AA} \cdot \pi_{AB} + \pi_{AB} \cdot \pi_{BB}$$

$$\pi_{AB} \cdot \pi_{BD}$$

$$\pi_{AE}^{2} = 0$$

So if the bond matures in the second perio t should we expect in the second period?

$$\underbrace{\left(\pi_{AA}^{2} + \pi_{AB}^{2}\right) \cdot (1+q)F}_{not\_in\_default\_and\_at\_maturity} + \underbrace{\pi_{AD}^{2} \cdot \lambda F}_{just\_defaulted} + \underbrace{\pi_{AE}^{2} \cdot 0}_{already\_in\_default}$$

Expected payout in period *n*

If we start out with an A-rated bond, we c expected payout in period *n*.
If *n* is prior to maturity:

$$\underbrace{\left(\pi_{AA}^{n}+\pi_{AB}^{n}\right)\cdot qF}_{not\_in\_default\_and\_before\_maturity} + \underbrace{\pi_{AD}^{n}\cdot\lambda F}_{just\_defaulted} + \underbrace{\pi_{AE}^{n}\cdot 0}_{already\_in\_default}$$

If *n* is at maturity, face the coupon

$$\underbrace{\left(\pi_{AA}^{n}+\pi_{AB}^{n}\right)\cdot(1+q)F}_{matured\_and\_not\_in\_default} + \underbrace{\pi_{AD}^{n}\cdot\lambda F}_{just\_defaulted} + \underbrace{\pi_{AE}^{n}\cdot 0}_{already\_in\_default}$$

After maturity we do not expect any cash flows…

How to calculate the n-period transition probabilities?

We can think of transitions as two-step e consider the rating in period n-1, and then the next one-period transition:

$$\pi_{AA}^{n} = \pi_{AA}^{n-1} \cdot \pi_{AA} + \pi_{AB}^{n-1} \cdot \pi_{BA}$$

$$\pi_{AB}^{n} = \pi_{AA}^{n-1} \cdot \pi_{AB} + \pi_{AB}^{n-1} \cdot \pi_{BB}$$

$$\pi_{AD}^{n} = \pi_{AB}^{n-1} \cdot \pi_{BD}$$

$$\pi_{AE}^{n} = \pi_{AD}^{n-1} \cdot 1 + \pi_{AE}^{n-1} \cdot 1$$

We use matrices multiplication:

$$\begin{bmatrix} \pi_{AA}^{n} & \pi_{AB}^{n} & \pi_{AD}^{n} & \pi_{AE}^{n} \end{bmatrix} = \begin{bmatrix} \pi_{AA}^{n-1} & \pi_{AB}^{n-1} & \pi_{AD}^{n-1} & \pi_{AE}^{n-1} \end{bmatrix} \begin{bmatrix} \pi_{AA} & \pi_{AB} & 0 & 0 \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternative calculation

Another way to find the probabilitie ings in period *n* for a bond initially rated *A* is to multiply the initial probabilities by the transition matrix *n* times

$$\begin{bmatrix} \pi_{AA}^{n} & \pi_{AB}^{n} & \pi_{AD}^{n} & \pi_{AE}^{n} \end{bmatrix} \qquad \begin{bmatrix} \pi_{AA} & \pi_{AB} & 0 & 0 \\ & & & 0 \\ & & & \\ & 0 & 0 & 1 \end{bmatrix}^{n}$$

Calculate the cash flow of period n

If we know the probabilities of diff [illegible] or a bond in period n, we can calculate the cash flow during that period.

If n is prior to maturity, then the cash flow is

$$\begin{bmatrix}\pi_{AA}^{n} & \pi_{AB}^{n} & \pi_{AD}^{n} & \pi_{AE}^{n}\end{bmatrix} \times \begin{bmatrix} qF \\ \text{[illegible]} \\ \text{[illegible]} \\ 0 \end{bmatrix}$$

If n is at maturity, then the cash flow is

$$\begin{bmatrix}\pi_{AA}^{n} & \pi_{AB}^{n} & \pi_{AD}^{n} & \pi_{AE}^{n}\end{bmatrix} \times \begin{bmatrix} (1+q)F \\ (1+q)F \\ \lambda F \\ 0 \end{bmatrix}$$

In class exercise

- A 10-year bond issued today *at p* face value) with an A rating is assumed to bear a coupon rate of 7 percent. The transition matrix is given in the file *Bond Default Example.xlxs*. What should be its actual coupon rate?

- If another 10-year rating and with the recovery percenta coupon rate so that it is also sold *at par*?

Hints

- Think about creating a schedule of rresponding to bond payoffs every year.
- Start out testing your spreadsheet with the A-rated bond
- Use Excel NPV function to calculate the PV of the bond.
- What is the NPV discount rate implied by the fact that the A-rated bond is sold at par?
- Use Excel's GoalS riate coupon rate for the B-rated bond.