CIS 471/571(Fall 2020): Introduction to Artificial Intelligence

Lecture 17 rkov Model

Thanh H. Nguyen

Source: http://ai.berkeley.edu/home.html

Reminder

- Homework 4: Bayes Nets
  - Deadline: Nov 24^{th}, 2020

Hidden Markov Model

Reasoning over Time or Space

- Often, we want to reason about a sequence of observations
  - Speech recogniti
  - Robot localizatio
  - User attention
  - Medical monitoring

- Need to introduce time (or space) into our models

Markov Models

- Value of X at a given time is called the state

$$X_1 \to X_2 \to X_3 \to X_4 \dashrightarrow$$

$$P(X_1)$$

- Parameters: called transition probabilities or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action

Conditional Independence

- Basic condce:
  - Past and future indepen the present

  - Each time step only dep e previous
  - This is called the (first order) Markov property

- Note that the chain is just a (growable) BN
  - We can always use generic BN reasoning on it if we truncate the chain at a fixed length

Example Markov Chain: Weather

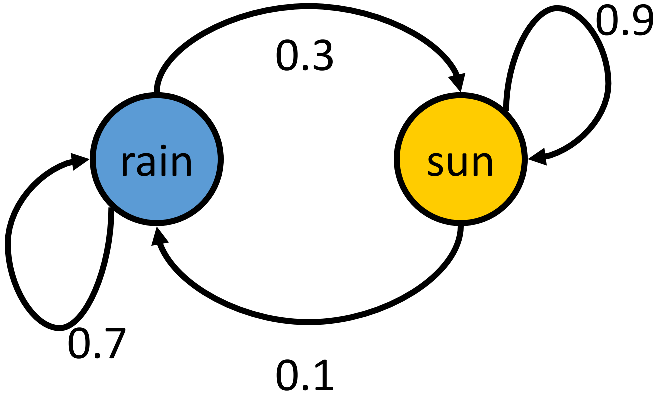
- States: $X = \{\text{rain}, \text{sun}\}$

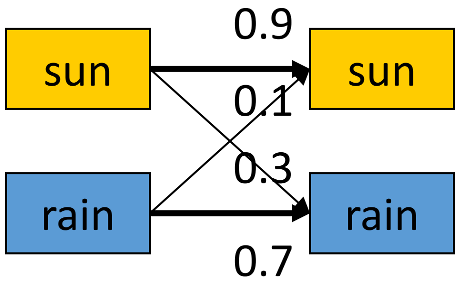
- Initial distribution: 1

- CPT $P(X_t \mid X_{t-1})$:

Two new presenting the same CPT

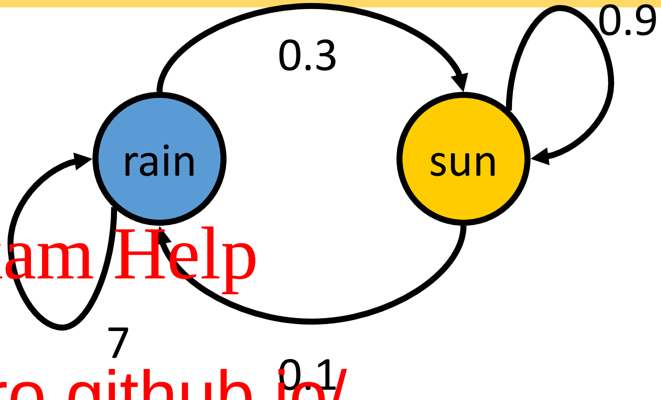
X_{t-1}	X_t	$P(X_t \mid X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7





Example Markov Chain: Weather

- Initial distribution: 1.0 sun



- What is the probability distrib ter one step?

$$P(X_2 = \mathsf{sun}) = P(X_2 = \mathsf{sun} | X_1 = \mathsf{sun}) P(X_1 = \mathsf{sun}) + P(X_2 = \mathsf{sun} | X_1 = \mathsf{rain}) P(X_1 = \mathsf{rain})$$

$$0.9 \cdot 1.0 + 0.3 \cdot 0.0 = 0.9$$

Mini-Forward Algorithm

- Question: What's P(X) on some day t?

$$X_1 \to X_2 \to X_3 \to X_4 \to \cdots$$

$$P(x_1) = \text{known}$$

$$P(x_t) = \sum_{x_{t-1}} P(x_{t-1}, x_t)$$

$$= \sum_{x_{t-1}} P(x_t \mid x_{t-1}) P(x_{t-1})$$

Forward simulation

Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{pmatrix} 1.0 \\ 0.0 \end{pmatrix} \quad \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix} \quad \begin{pmatrix} 0.84 \\ 0.16 \end{pmatrix} \quad \begin{pmatrix} 0.804 \\ 0.196 \end{pmatrix} \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

$P(X_1)$ $P(X_2)$ $P(X_3)$ $P(X_4)$ $P(X_\infty)$

- From initial observat

$$\begin{pmatrix} 0.0 \\ 1.0 \end{pmatrix} \quad \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \quad \begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

$P(X_1)$ $P(X_2)$ $P(X_3)$ $P(X_4)$ $P(X_\infty)$

- From yet another initial distribution $P(X_1)$:

$$\begin{pmatrix} p \\ 1-p \end{pmatrix} \quad \ldots \quad \Rightarrow \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$$

$P(X_1)$ $P(X_\infty)$

Stationary Distributions

- For most chains:
  - Influence of the initial distribution gets less and less over time.
  - The distribution we end u independent of the initial distribution

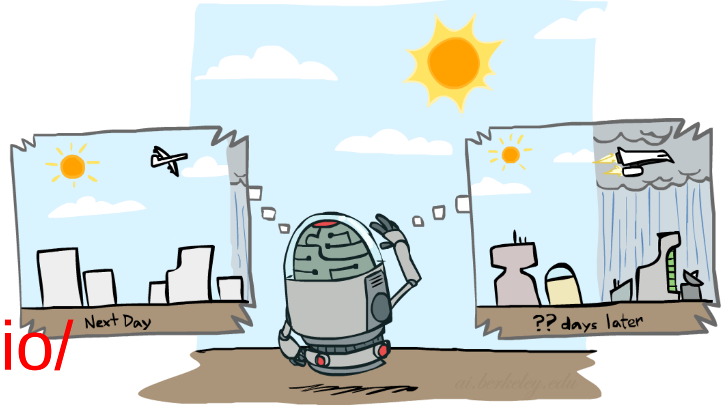
- Stationary distribution:
  - The distribution we end up with is called the stationary distribution P_∞ hain
    fies

$$= P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$

Example: Stationary Distributions

- Question: What's P(X) at time t = infinity?

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_4 \dashrightarrow$$



$$P_\infty(sun) = P(sun|sun)P_\infty(sun)$$
$$P_\infty(rain) = P(rain|sun)P_\infty(sun$$

$$P_\infty(sun) = 0.9P_\infty(sun) + 0.3P_\infty(rain)$$
$$P_\infty(rain) = 0.1P_\infty(sun) + 0.7P_\infty(rain)$$

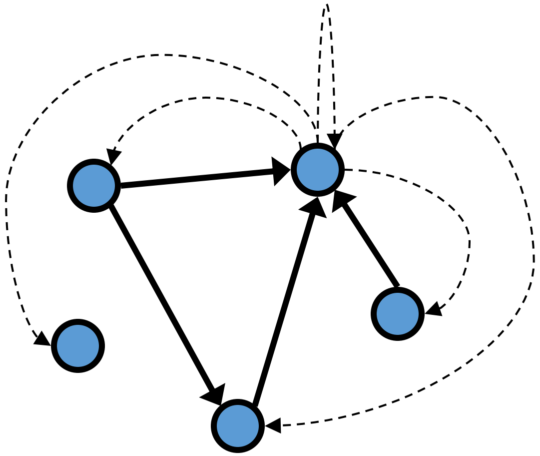
$$P_\infty(sun) = 3P_\infty(rain)$$
$$P_\infty(rain) = 1/3P_\infty(sun)$$

Also: $P_\infty(sun) + P_\infty(rain) = 1$ $\Rightarrow$ $P_\infty(sun) = 3/4$, $P_\infty(rain) = 1/4$

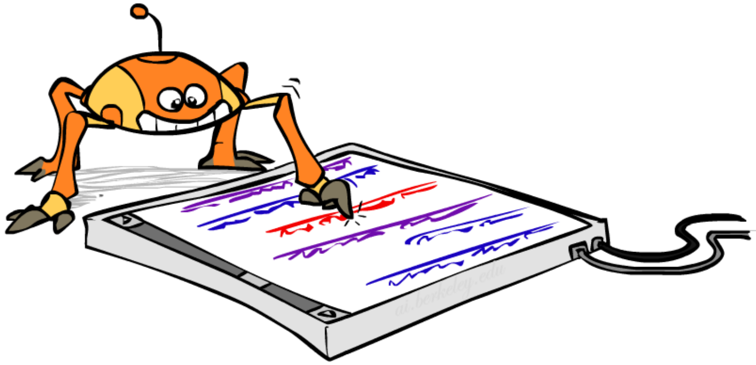
X_{t-1}	X_t	$P(X_t\|X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob. c, uniform jump to a r l shown)
    - With prob. 1-c, follow a random o



- Stationary distribution
  - Will spend more time on highly reachable pages
  - E.g. many ways to get to the Acrobat Reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors



Application of Stationary Distributions: Gibbs Sampling*

- Each joint instantiation over all hidden and query variables is a state: $\{X_1, \ldots, X_n\} = H \cup Q$

- Transitions:
  - With probability 1/n resampl

$$P(X_j \mid x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, e_1, \ldots, e_m)$$

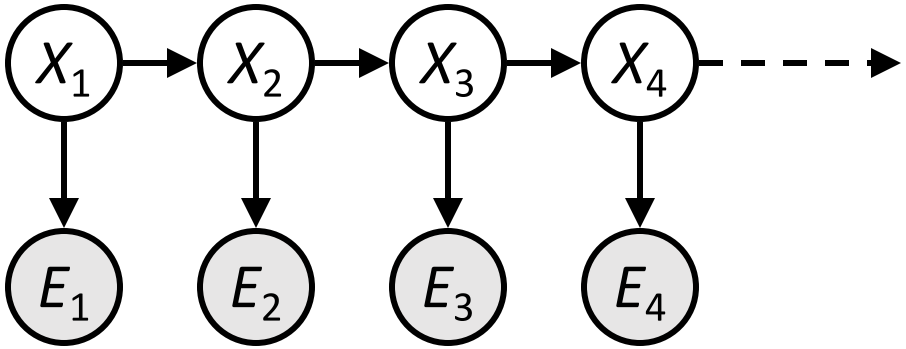
- Stationary distribution:
  - Conditional distribution $P(X_1, X_2, \ldots, X_n \mid e_1, \ldots, e_m)$
  - Means that when running Gibbs sampling long enough we get a sample from the desired distribution
  - Requires some proof to show this is true!

Hidden Markov Models

Hidden Markov Models

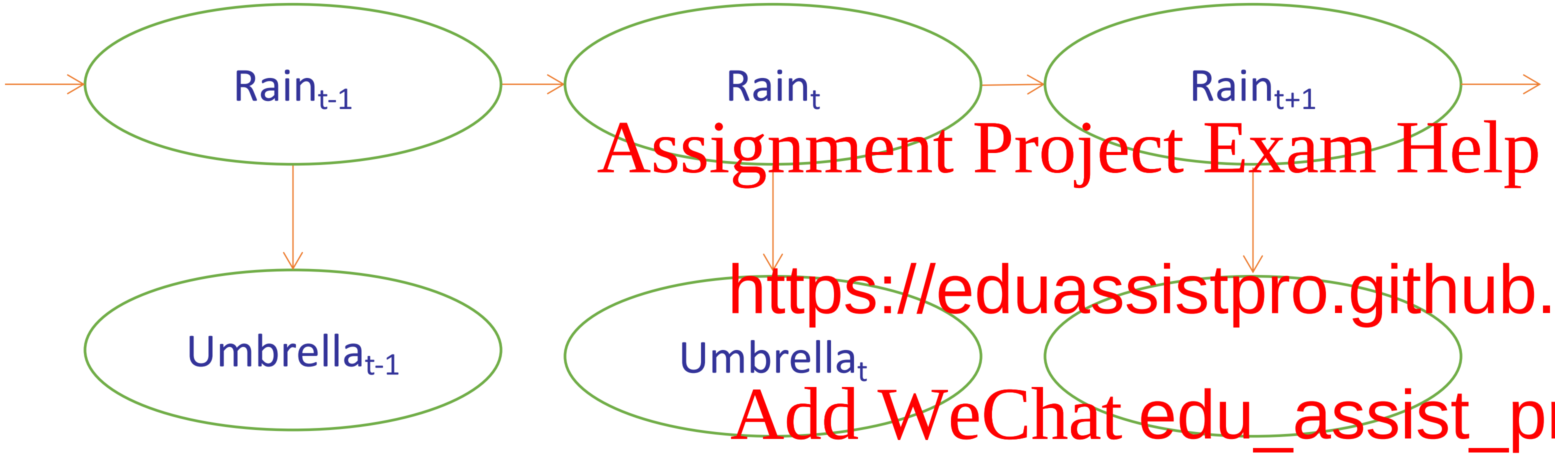
- Markov chains not so useful for most agents
  - Need observations to update your beliefs

- Hidden Markov models (H
  - Underlying Markov chain ov
  - You observe outputs (effects)



Example: Weather HMM

$P(X_t \mid X_{t-1})$



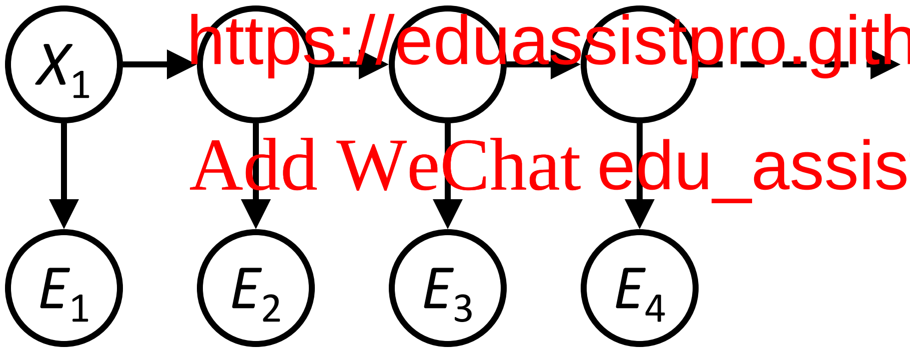
- An HMM is defined by:
  - Initial distribution: $P(X_1)$
  - Transitions: $P(X_t \mid X_{t-1})$
  - Emissions: $P(E_t \mid X_t)$

R_{t-1}	R_t	$P(R_t \mid R_{t-1})$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process: future depends on past via the present
  - Current observation independent of all else given current state



- Quiz: does this mean that evidence variables are guaranteed to be independent?
  - [No, they tend to correlated by the hidden state]

Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translatio
  - Observations are words (tens of thou
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

Filtering / Monitoring

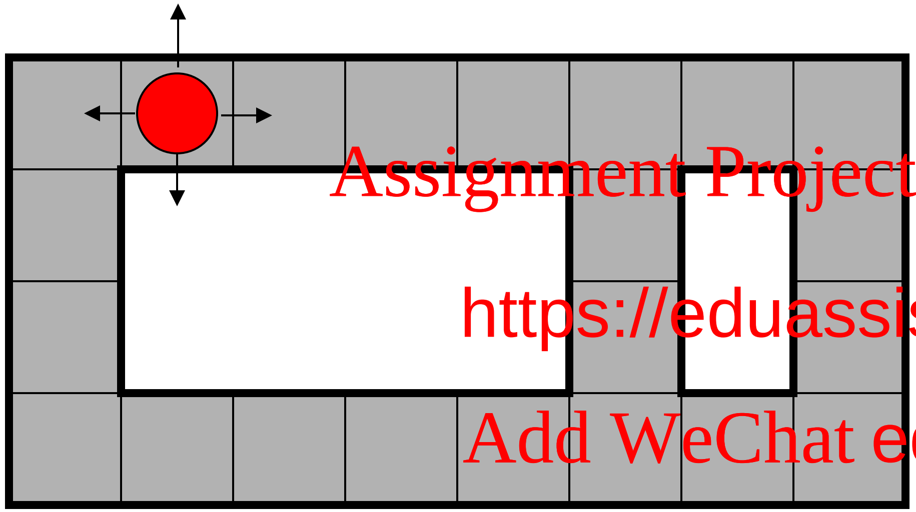
- Filtering, or monitoring, is the task of tracking the distribution $B_t(X) = P_t(X_t \mid e_1, \ldots, e_t)$ (the belief state) over time

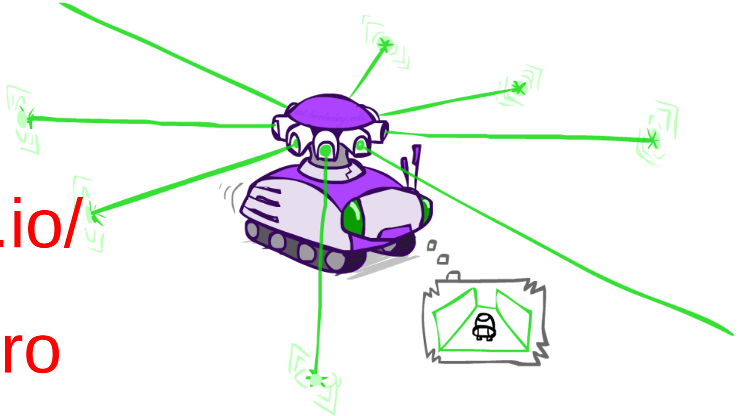
- We start with $B_1(X)$ i ally uniform

- As time passes, or we get observati date $B(X)$

Example: Robot Localization

Example from
Michael Pfeiffer





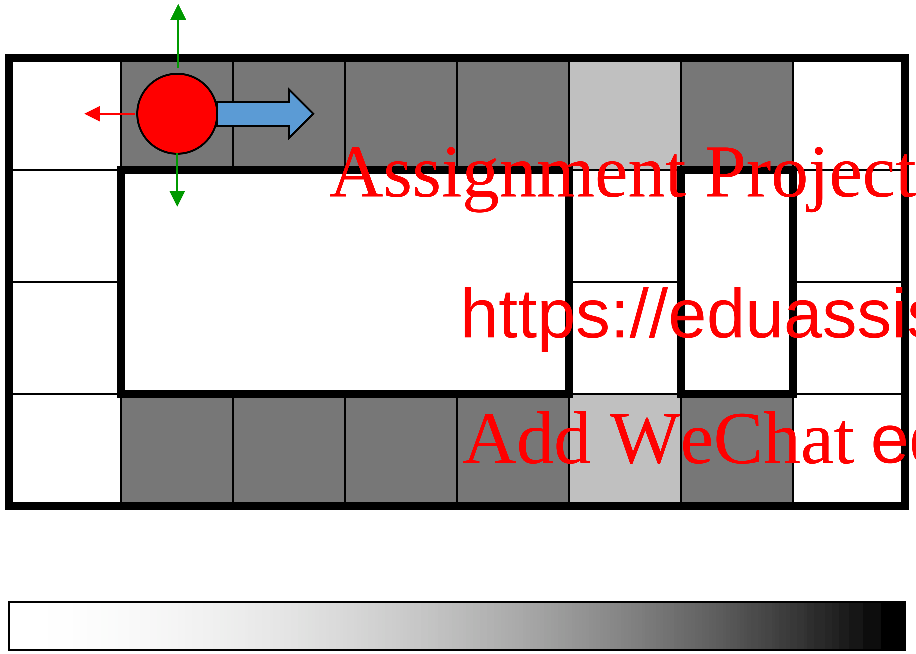
Prob 0 1

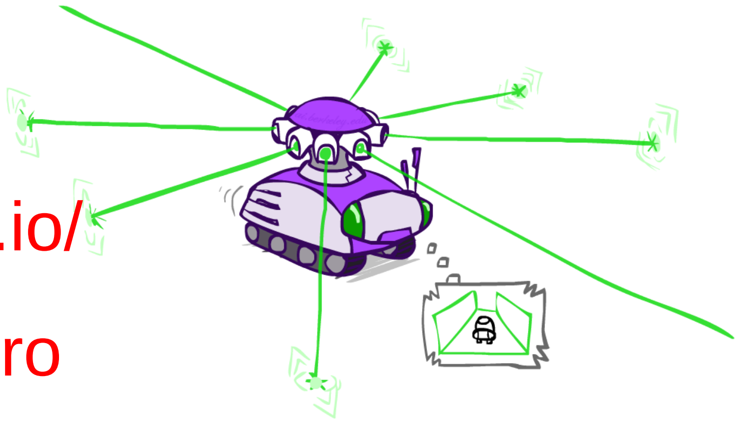
t=0

Sensor model: can read in which directions there is a wall, never more than 1 mistake

Motion model: may not execute action with small prob.

Example: Robot Localization



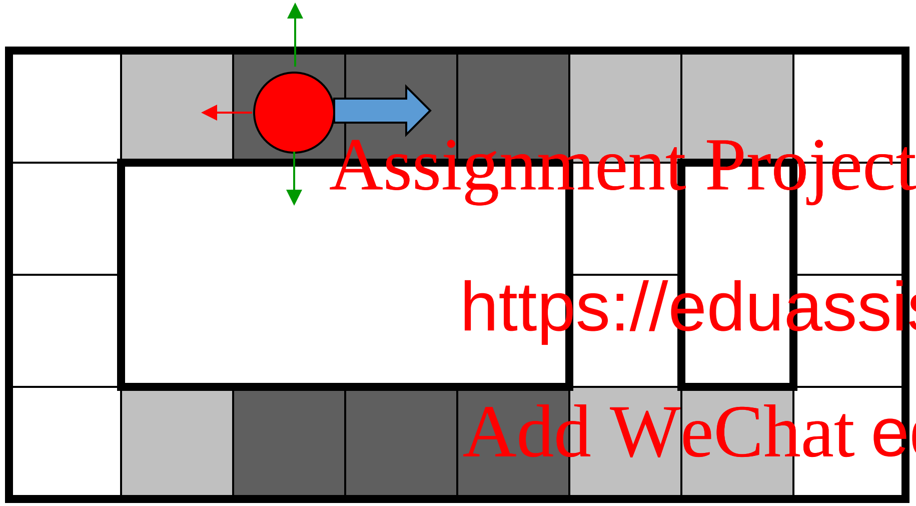


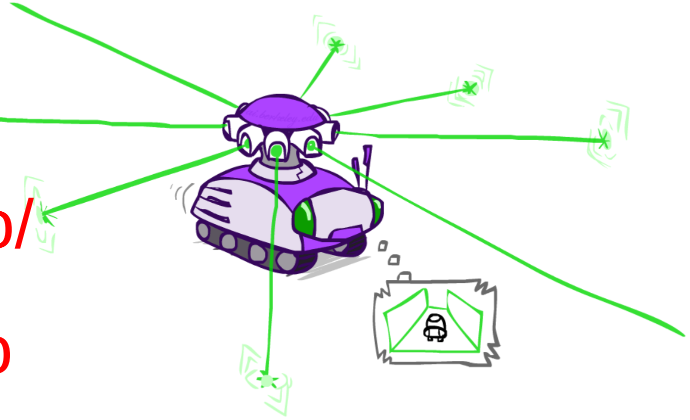
Prob 0 1

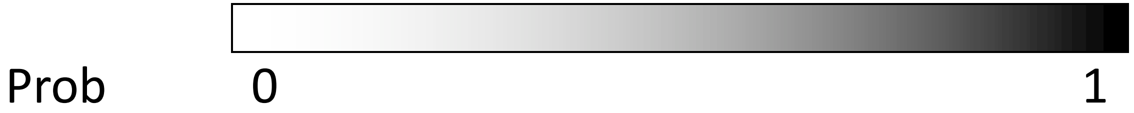
t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake

Example: Robot Localization

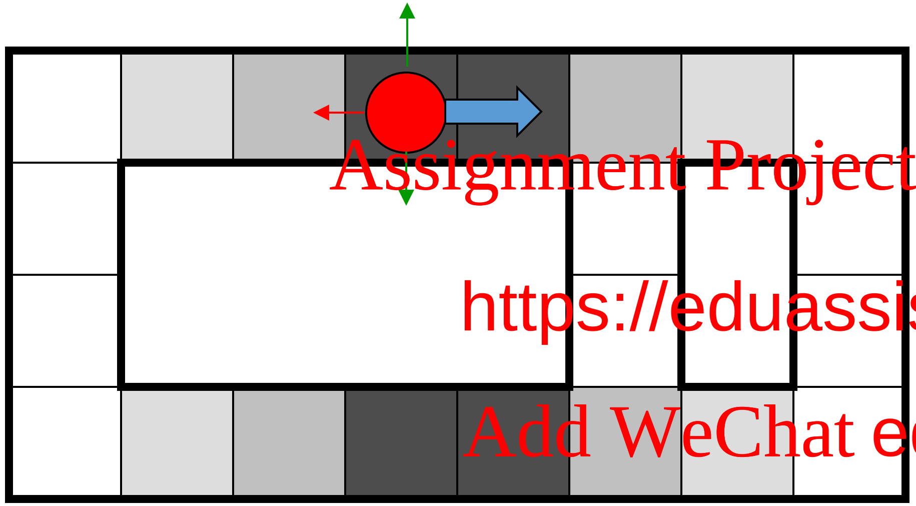


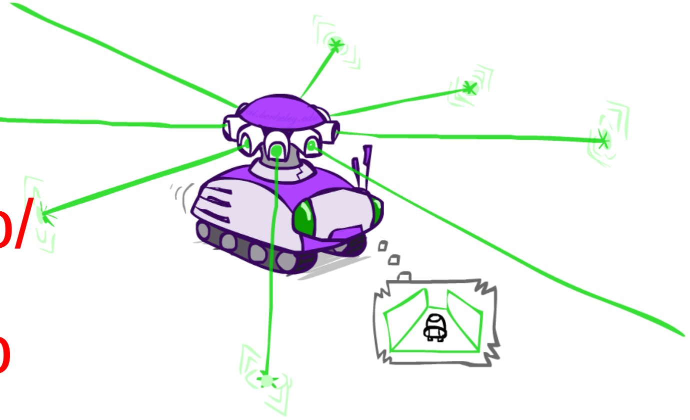




t=2

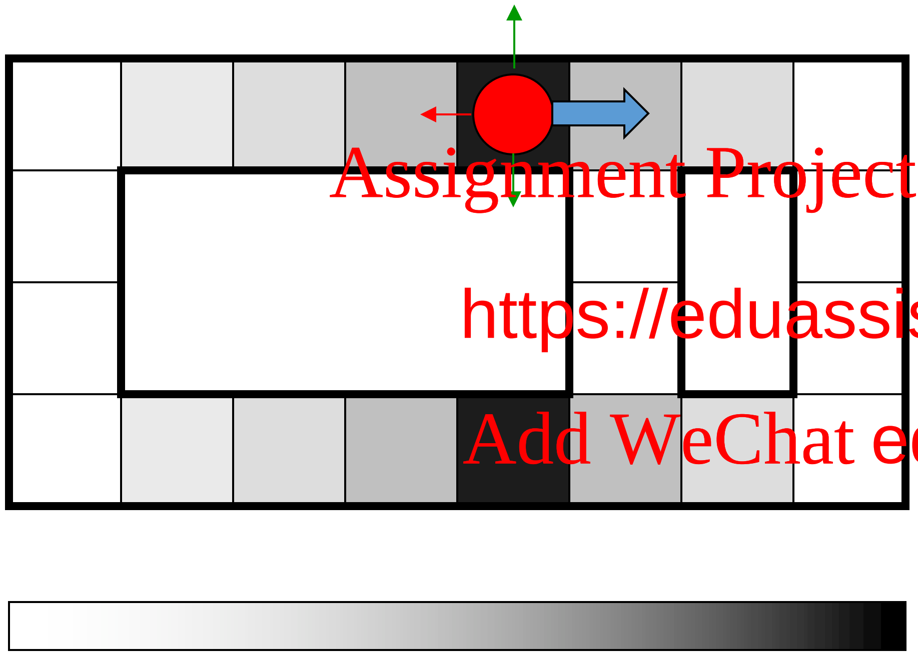
Example: Robot Localization

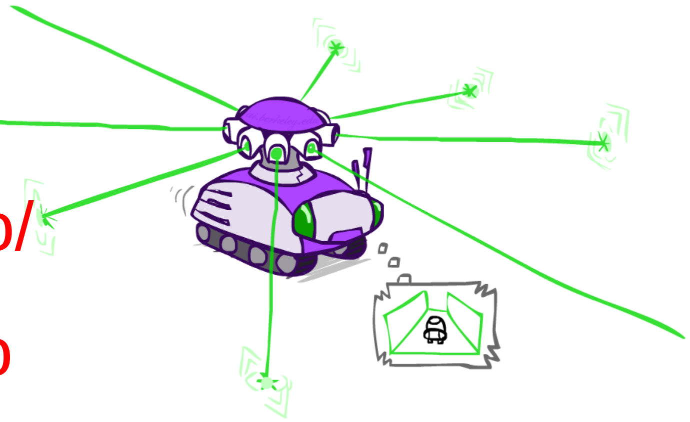




t=3

Example: Robot Localization

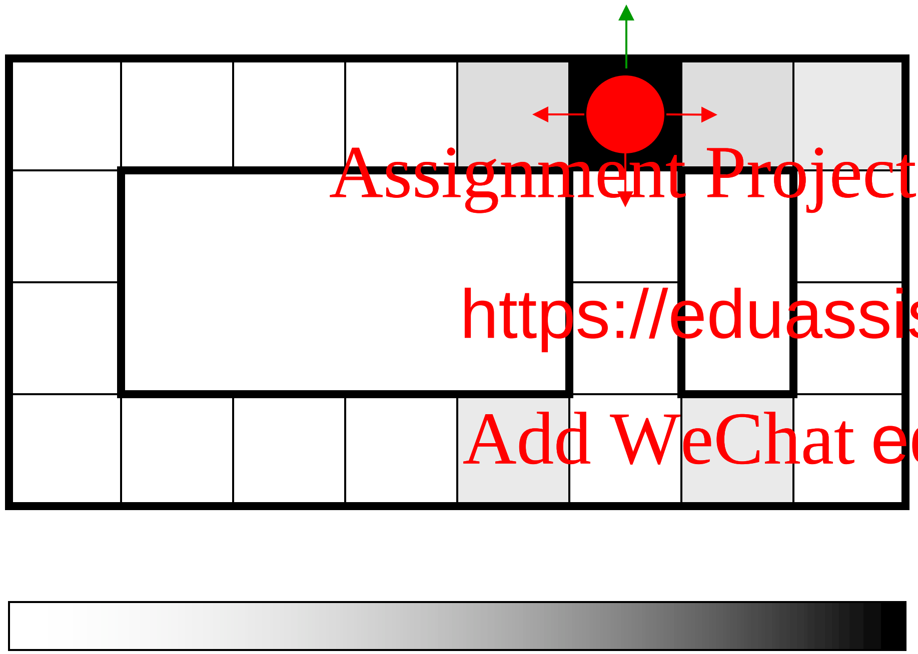


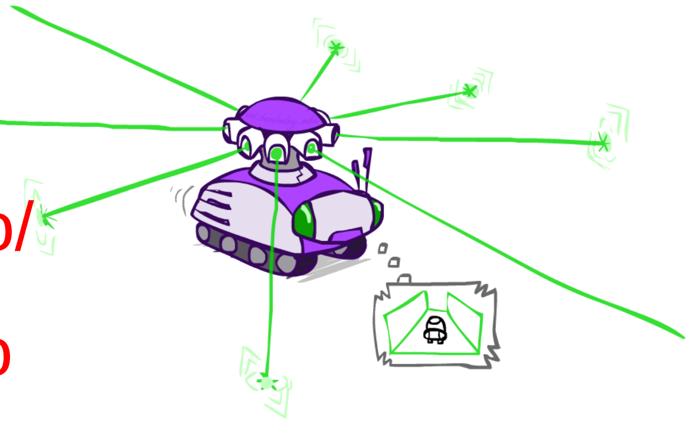


Prob 0 1

t=4

Example: Robot Localization





Prob 0 1

t=5

The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t | e_{1:t})$$

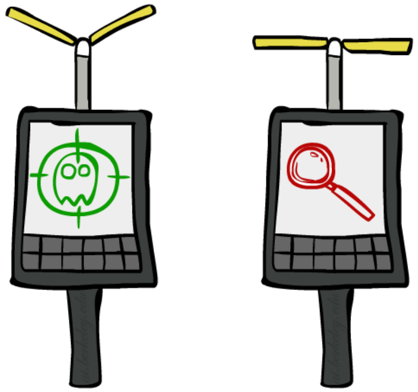
- Induction: assuming $B(X_t) = P(X_t | e_{1:t})$

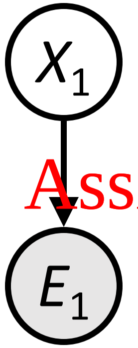
$$P\left(X_{t+1} | e_{1:(t+1)}\right) \leftarrow P(X_t \quad) \quad P(X_t | e_{1:t})$$

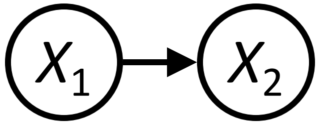
Observation update

Passage of time update

Inference: Base Cases







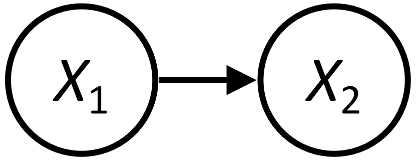
$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$

$$P(X_2)$$

$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$

Passage of Time



- Assume we have current belief P(X | evidence to date)

$$B(X_t) = P(X_t|e_{1:t})$$

- Then, after one time step passes:

$$P(X_{t+1}|e_{1:t}) = \sum_{x_t} P(X_{t+1}$$

$$= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(x_t|e_{1:}$$

$$= \sum_{x_t} P(X_{t+1}|x_t) P(x_t|e_{1:t})$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X'|x_t) B(x_t)$$

- Basic idea: beliefs get "pushed" through the transitions
  - With the "B" notation, we have to be careful about what time step t the belief is about, and what evidence it includes

Observation

- Assume we have current belief P(X | previous evidence):

$$B'(X_{t+1}) = P(X_{t+1}|e_{1:t})$$

- Then, after evidence comes in:

$$P(X_{t+1}|e_{1:t+1}) = P(X_{t+1}, e_{t+1}|e_{1:t})$$

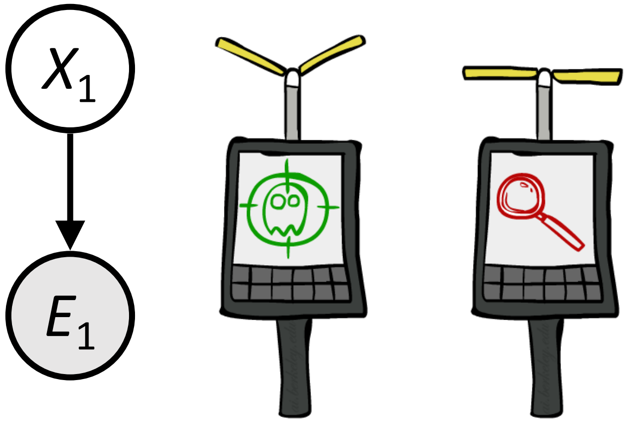
$$\propto_{X_{t+1}} P(X_{t+1}, e_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|e_{1:t}, X_{t+1})P(X_{t+1}|e_{1:t})$$

$$= P(e_{t+1}|X_{t+1})P(X_{t+1}|e_{1:t})$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1}|X_{t+1})B'(X_{t+1})$$



- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

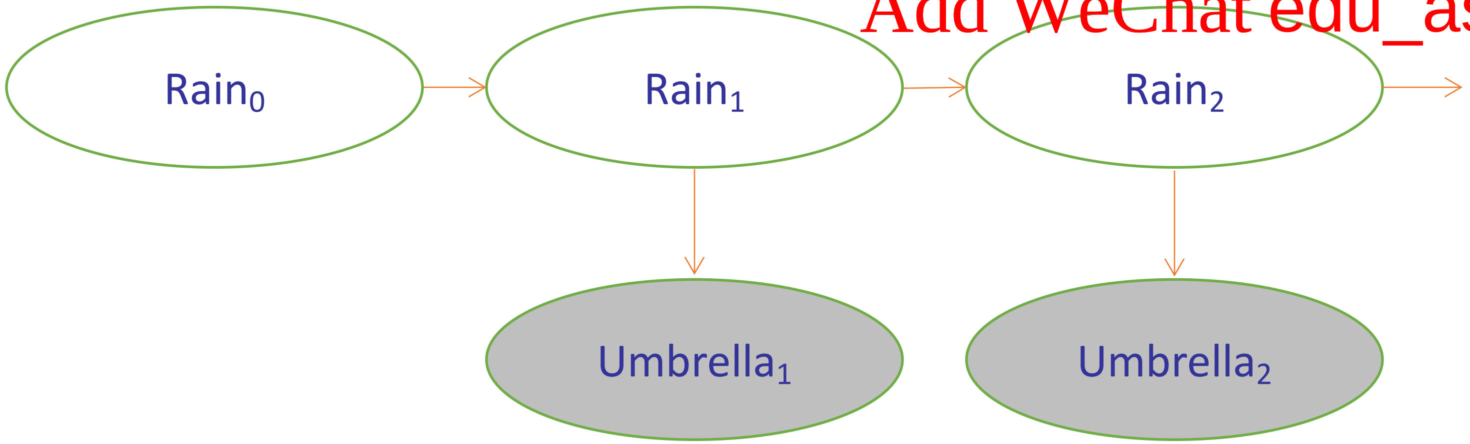
Example: Weather HMM

B(+r) = 0.5
B(-r) = 0.5

B'(+r) = 0.5
B'(-r) = 0.5

B(+r) = 0.818
B(-r) = 0.182

B'(+r) = 0.627
B'(-r) = 0.373

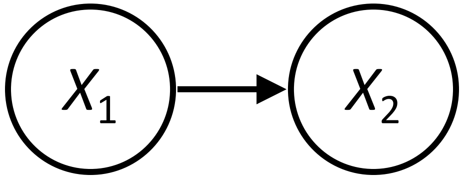


R_t	R_{t+1}	$P(R_{t+1} \mid R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

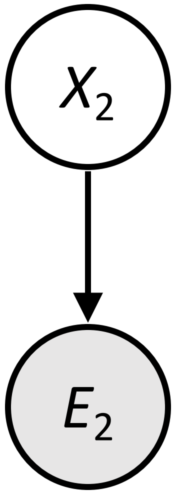
R_t	U_t	$P(U_t \mid R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

Online Belief Updates

- Every time step, we start with current P(X | evidence)
- We update for time:



- We update for evidence:



$$P(x_t|e_{1:t}) \propto_X P(x_t|e_{1:t-1}) \cdot P(e_t|x_t)$$

Next Time: Particle Filtering and Applications of HMMs