COMP proofs.

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Based on slides from (Langer,2012), (CRLS, 2009) & (Sora,2015)

Outline

- **Induction proofs**
  - Introduction
  - Definition
  - https://eduassistpro.github.io/
- **Loop invariants**
  - Definition
  - Example (Insertion sort)
  - Analogy with induction proofs
  - Example (Merge sort)

$$for\ any \quad n \geq 2, \quad n^2 \geq 2^n \quad ?$$

$f(x) = x^2$

$g(x) = 2^x$

$$\text{for any} \quad n \geq 5, \quad n^2 \leq 2^n \quad ?$$

$g(x) = 2^x$

$f(x) = x^2$

Motivation

How to prove these?

$$\text{for any } n \geq 1,\quad 1+2+3+4+\cdots+n = \frac{n\cdot(n+1)}{2}$$

$$\text{for any } n \quad n\cdot(n-1) = n^2$$

$$\text{for any } n \geq 5,\quad n^2 \leq 2^n$$

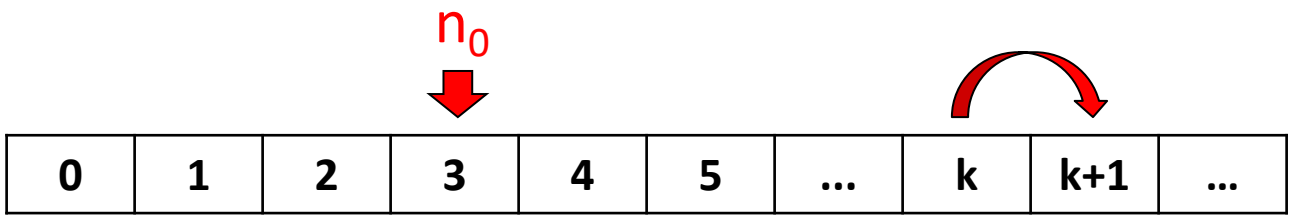
And in general, any statement of the form:
"*for all $n \geq n_0$, P(n)*" where *P(n) is some proposition.*

Mathematical induction

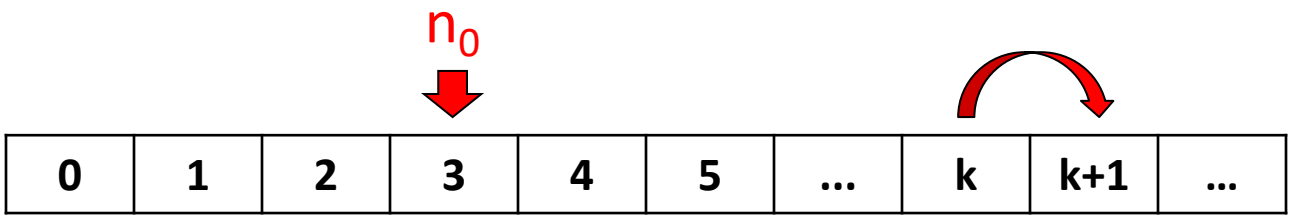
Many statement of the form "*for all* $n \geq n_0$, *P(n)*" can be proven with a logical argument call *mathematical induction.*

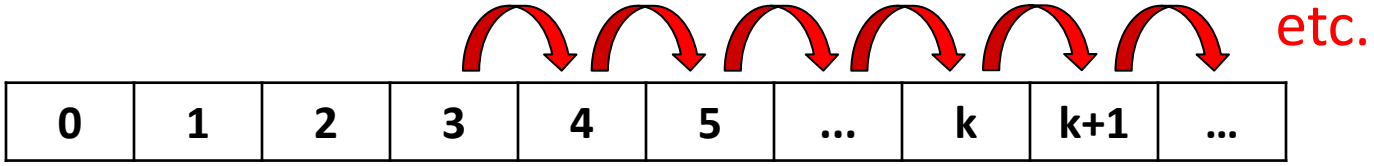
The proof has t

- **Base case**: P(n_0)
- **Induction step**: for any $n \geq n_0$, if *P(n)* then *P(n+1)*



Principle





Example 1

Claim: $for\ any\ n \geq 1, \quad 1+2+3+4+\cdots+n = \frac{n \cdot (n+1)}{2}$

Proof:

- Base case: $n = 1$

- Induction step:

$for\ any\ k \geq 1, \quad if \quad 1+2+3+4+\cdots+k = \frac{k \cdot (k+1)}{2}$

$then \quad 1+2+3+4+\cdots+k+(k+1) = \frac{(k+1) \cdot (k+2)}{2}$

Example 1

Assume $1+2+3+4+\cdots+k=\dfrac{k\cdot(k+1)}{2}$

then $1+2+$

$$=\frac{k\cdot(}{2}+(k+1)$$

Induction hypothesis

$$=\frac{k\cdot(k+1)+2\cdot(k+1)}{2}$$

$$=\frac{(k+2)\cdot(k+1)}{2}$$

Summary

Base case: P(1)

Induction step en P(k+1)

Thus for all n≥1, P(n) □

Example 2

Claim: $for\ any\ n \geq 1, \quad 1+3+5+7+\cdots+(2\cdot n-1) = n^2$

Proof:

- Base case: $n = 1$

- Induction step:

$for\ any\ k \geq 1, \quad if \quad 1+3+5+7+\cdots+(2\cdot k-1) = k^2$

$then \quad 1+3+5+7+\cdots+(2\cdot(k+1)-1) = (k+1)^2$

Example 2

Assume $1+3+5+7+\cdots+(2\cdot k-1)=k^2$

then $1+3+5+\cdots+(2\cdot k-1)+(2\cdot(k+1)-1)$

$=k^2+2\cdot(k+1)-1$

$=k^2+2\cdot k+1$

$=(k+1)^2$

Induction hypothesis

Example 3

$g(x) = 2^x$

$f(x) = 2x+1$

Example 3

Claim: $for\ any\ n \geq 3, \quad 2 \cdot n + 1 < 2^n$

Proof:

- Base case: $n = 3 \quad 2 \cdot 3 + 1 < 2^3$

- Induction step:

$$for\ any\ k \geq 3, \quad if \quad 2 \cdot k + 1 < 2^k$$

$$then \quad 2 \cdot (k+1) + 1 < 2^{k+1}$$

Example 3

Assume $2 \cdot k + 1 < 2^k$

then

tion

hesis

$$< 2^k + 2$$

$$\leq 2^k + 2^k \quad \textit{for } k \geq 1$$

$$= 2^{k+1}$$

Stronger than we need, but that works!

Example 4

$g(x) = 2^x$

$f(x) = x^2$

Example 4

Claim: $for\ any\ n \geq 5, \quad n^2 \leq 2^n$

Proof:

- Base case: $n = 5$

- Induction step:

$$for\ any\ k \geq 5, \quad if \quad k^2 \leq 2^n$$
$$then \quad (k+1)^2 \leq 2^{k+1}$$

Example 4

Assume $k^2 \leq 2^k$

k^{th}

$= k^2 + 2 \cdot k$

$\leq 2^k + 2 \cdot k + 1$

$\leq 2^k + 2^k$

$= 2^{k+1}$

uction hypothesis

From previous example

Example 5

Fibonacci sequence:

$Fib_0 = 0$ base case

$Fib_1 = 1$ base case

$Fib_n = Fib_{n-1} +$ ive case

Claim: For all $n \geq 0$, $Fib_n < 2^n$

Base case: $Fib_0 = 0 < 2^0 = 1$, $Fib_1 = 1 < 2^1 = 2$

Q: Why should we check both Fib_0 and Fib_1?

Induction step: for any $i \leq k$, if $Fib_i < 2^i$ then $Fib_{k+1} < 2^{k+1}$

Example 5

Assume that *for all i ≤ k*, $Fib_i < 2^i$ (Note variation of induction hypothesis)

Then $Fib_{k+1} = Fi$

ction hypothesis (x2)

<

$\leq \quad 2^k + 2^k \qquad\qquad 1$

$= \quad 2^{k+1}$

Proving the correctness of an algorithm

LOOP INVARIANTS

Algorithm specification

- An algorithm is described by:
  - Input data
  - Output data
  - ***Precondition*** input data suit
  - ***Postcondition***
- Example: Binary Search
  - Input data: `a:array of integer; x:integer;`
  - Output data: `index:integer;`
  - Precondition: `a` is sorted in ascending order
  - Postcondition: `index` of `x` if `x` is in `a`, and -1 otherwise.

Correctness of an algorithm

An algorithm is correct if:

- for any correct input data:
  - it stops and
  - it produ
- Correct input data: satisf dition
- Correct output data: sati ondition

Problem: Proving the correctness of an algorithm may be complicated when the latter is repetitive or contains loop instructions.

How to prove the correctness of an algorithm?

- Recursive a ction proofs

- Iterative algorithm (l ???

Loop invariant

A **loop invari** rty that hold

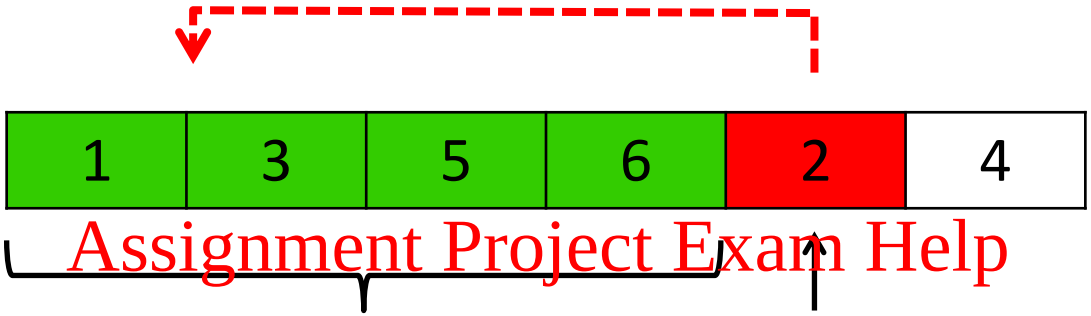
before and after each it f a loop.

Insertion sort

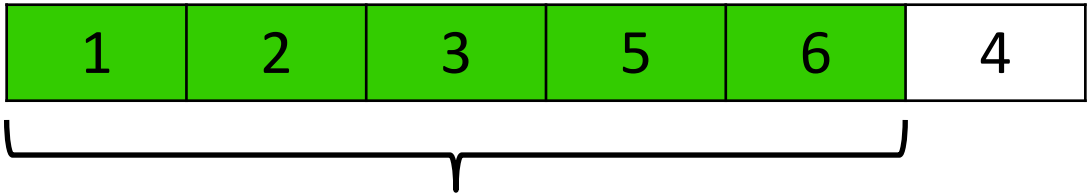
```
for i ← 1 to length(A) - 1
    j ← i
    while j > 0 and A[j-1] > A[j]
                                1]

    end while
end for
```


(Seen in previous lecture)

Insertion sort



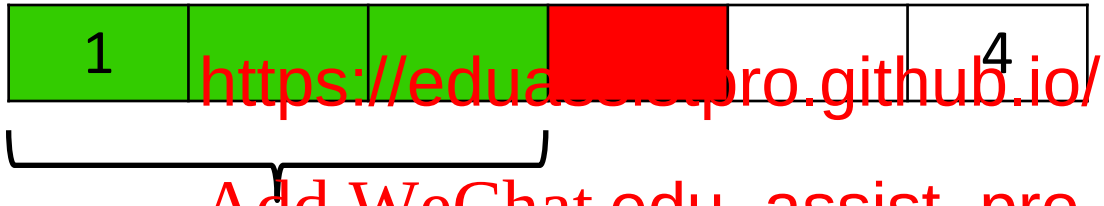
n elements: already sorted



n+1 elements sorted

Loop invariant

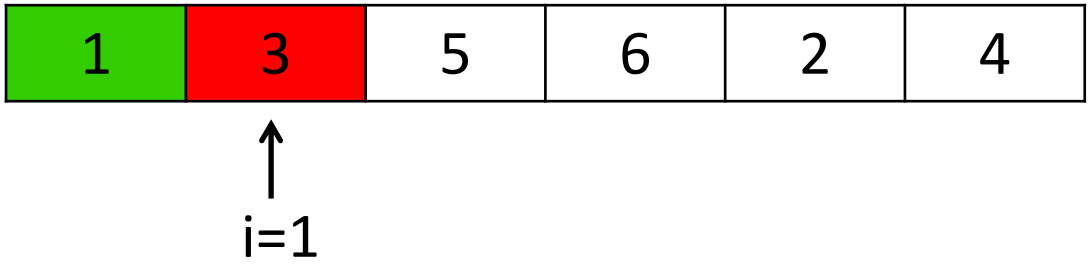
The array A[0…i-1] is fully sorted.



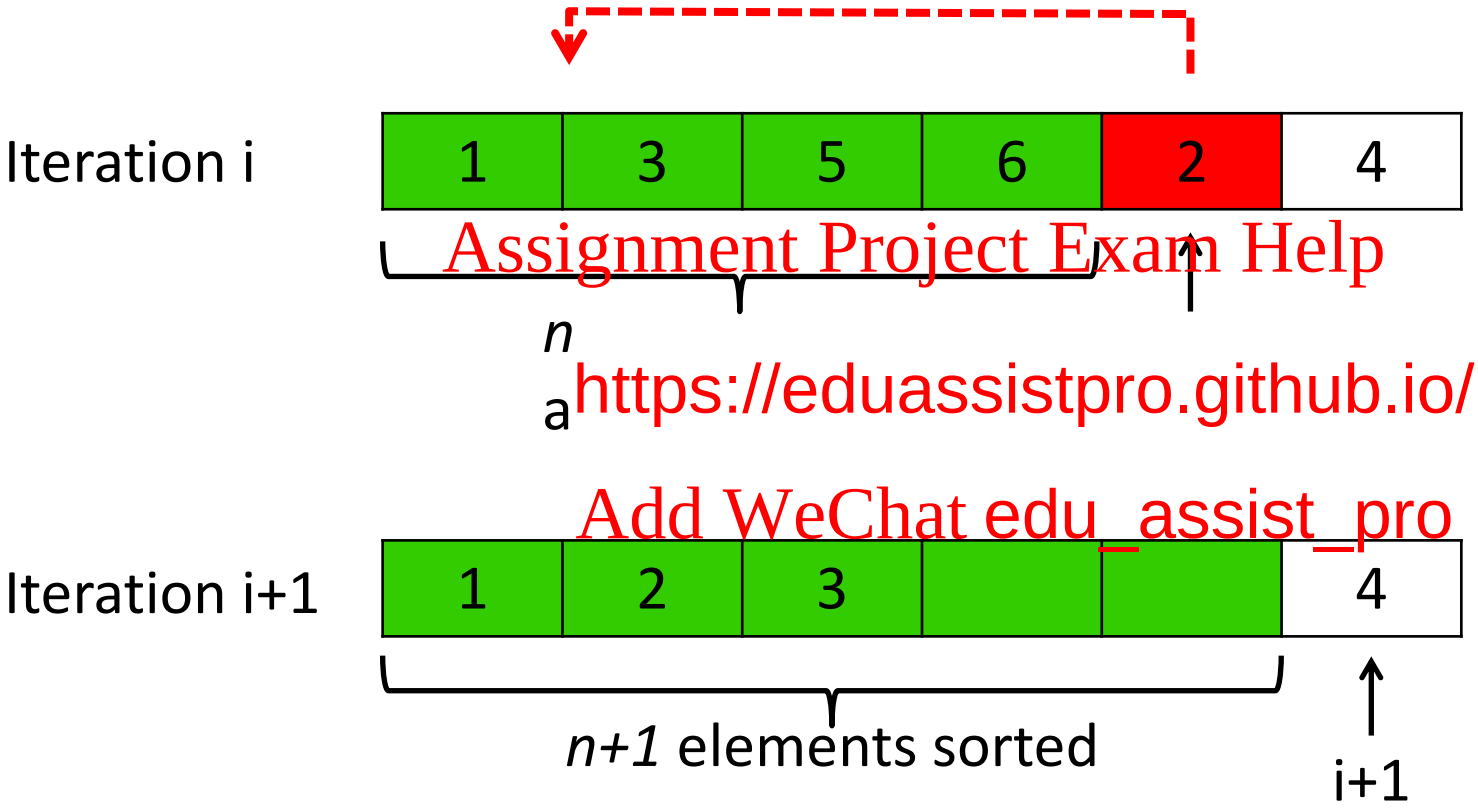
A[0…i-1]

Initialization

Just before the first iteration ($i = 1$), the sub-array $A[0 \ldots i-1]$ is the single element $A[0]$, which is the element original ... ially sorted.



Maintenance



Note: To be precise, we would need to state and prove a loop invariant for the ``inner'' **while** loop.

Termination

The outer **for** loop ends when $i \geq length(A)$ and increment by 1 at each iteration starting from 1.

Therefore, $i = length(A)$.

Plugging *length(* p invariant, the subarray *A*[0 … *length(A)-1*] c he elements originally in *A*[0 … *length(A)*-1 rted order.

A[0 … *length(A)*-1] contains *length(A) elements (i.e. all initial elements!) and no element is duplicated/deleted.*

In other words, **the entire array is sorted**.

Proof using loop invariants

We must show:

1. **Initialization:** It is true prior to the first iteration of the loop.
2. **Maintenanc** loop, it remains true befo **iteration of the** iteration.
3. **Termination:** When the lo **tes, the** invariant gives us a useful property that helps show that the algorithm is correct.

Analogy to induction proofs

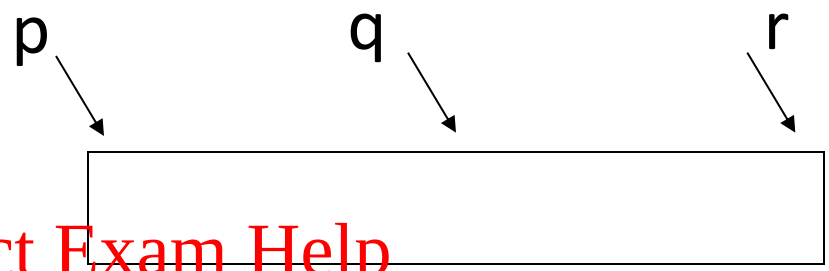
Using loop invariants is like mathematical induction.

- You prove a **base case** and an **inductive step**.
- Showing that th he first iteration is like the base cas
- Showing that the invariant hold ion to iteration is like the inductive step.
- The **termination** part differs from classical mathematical induction. Here, we stop the ``induction'' when the loop terminates instead of using it infinitely.

We can show the three parts in any order.

Merge Sort

```
MERGE-SORT(A,p,r)
    if p < r then
        q=(p+r)/2
        MERGE-SORT(A,p,q)
        MERGE-S
        MERGE(A
```



Precondition:

Array A has at least 1 element between indexes p and r (p≤r)

Postcondition:

The elements between indexes p and r are sorted

Merge Sort (Merge method)

- MERGE-SORT calls a function MERGE(A,p,q,r) to merge the sorted subarrays of A into a single sorted one
- The proof of ME done separately invariants

MERGE (A,p,q,r)

Precondition: A is an array and p, q, and r
ices into the
uch that p <= q
e
subarrays
and A[q +1..
orted

Postcondition: The subarray A[p..r] is sorted

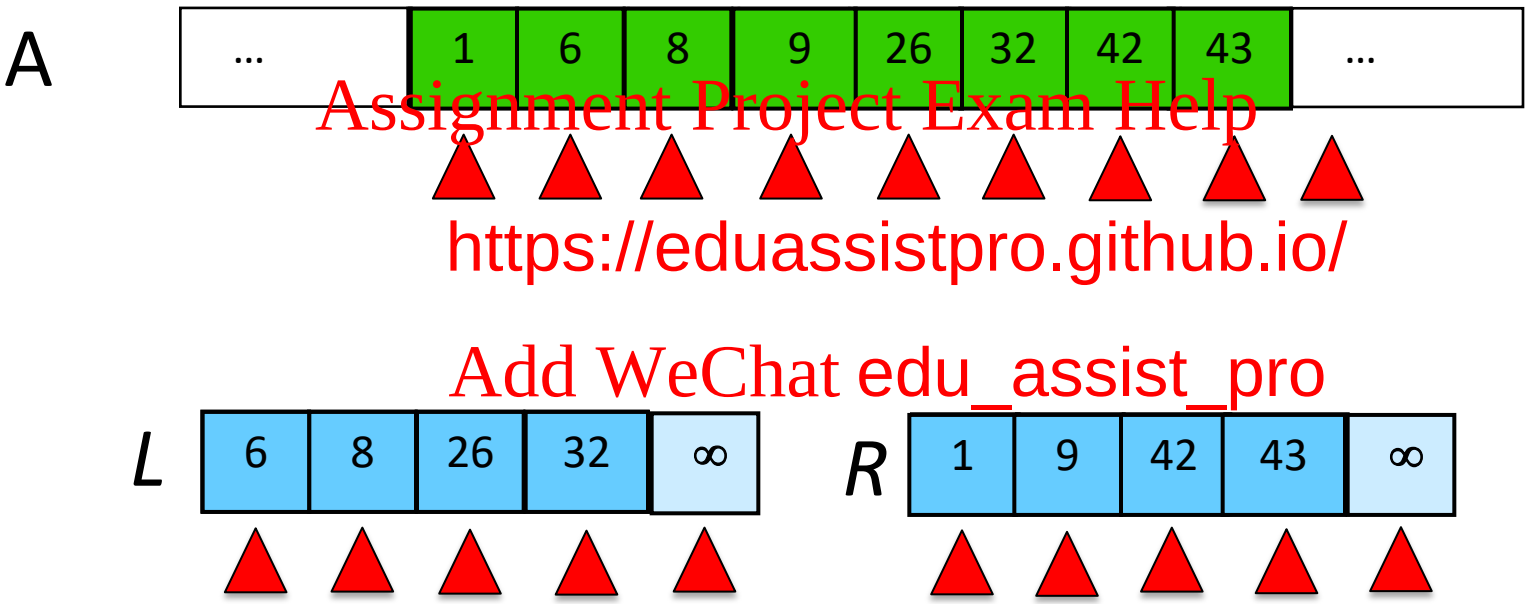
Procedure Merge

```
Merge(A, p, q, r)
1  n1 ← q − p + 1
2  n2 ← r − q
3       for i ← 1 to n1
4          do L[i] ← A[p + i − 1]
5       for j ← 1 to n2
6          do R[j] ← A[q + j]
7       L[n1+1] ← ∞
8       R[n2+1] ← ∞
9       i ← 1
10      j ← 1
11      for k ←p to r
12         do if L[i] ≤ R[j]
13            then A[k] ← L[i]
14                 i ← i + 1
15            else A[k] ← R[j]
16                 j ← j + 1
```

Input: Array containing sorted subarrays $A[p..q]$ and $A[q+1..r]$.

Output: Merged sorted y in $A[p..r]$.

Merge/combine – Example



Idea: The lists L and R are **already sorted**.

Correctness proof for **Merge**

- **Loop Invariant:** The array A[p,k] stored the (k-p+1) smallest elements of L and R sorted in increasing order.
- **Initialization**: k = p
  - A contains a single element (which is trivially "sorted")
  - A[p] is the smal
- **Maintenance:**
  - Assume that Merge satisfy the loo operty until k.
  - (k-p+1) smallest elements of L and sorted in A.
  - Next value to be inserted is the smallest one remaining in L and R.
  - This value is larger than those previously inserted in A.
  - Loop invariant property satisfied for k+1.
- **Termination Step:**
  - Merge terminates when k=r, thus when r-p+1 elements have been inserted in A ⇒ All elements are sorted.

Correctness proof for **Merge Sort**

- Recursive property: Elements in A[p,r] are be sorted.
- **Base Case**: p = r
  - A contains a single element (which is trivially "sorted")
- **Inductive Hypot**
  - Assume that M[illegible] k elements
- **Inductive Step:**
  - Show that MergeSort correctly sor[illegible] ments.
- **Termination Step:**
  - MergeSort terminate and all elements are sorted.

Note: Merge Sort is a recursive algorithm. We already proved that the Merge procedure is correct. Here, we complete the proof of correctness of the main method using induction.

Inductive step

- **Inductive Hypothesis:**
  - Assume MergeSort correctly sorts n=1, ... , **k** elements
- **Inductive Step:**
  - Show that M = **k + 1** elements.
- **Proof:**
  - First recursive call $n_1=q-p+1=$
    => subarray A[p .. q] is sort
  - Second recursive call $n_2=r-q=(k+1)/2 \leq k$
    => subarray A[q+1 .. r] is sorted
  - A, p q, r fulfill now the precondition of Merge
  - The post-condition of Merge guarantees that the array A[p .. r] is sorted => post-condition of MergeSort satisfied.

Termination Step

We have to find a quantity that decreases with every recursive call: the length of the subarray of A to be sorted MergeSort.

At each recursive call, the length of the subarray is strictly decreasing.

When MergeSort is called on an array of size ≤ 1 (i.e. the base case), the algorithm terminates without making additional recursive calls.

Calling MergeSort(A,0,n) returns a fully sorted array.