Advanced Network Technologies

Queueing Theory

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Outline

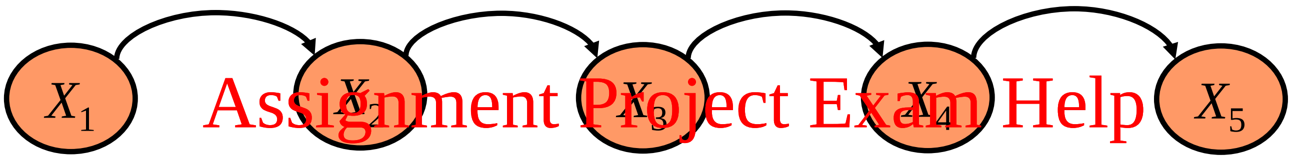
- Markov Chain
- Queueing System and Little's Theorem
- M/M/1 Queue foundations
- M/M/1 Queue

- Markov Chain

Markov Chain

- A stochastic process
  - $X_1, X_2, X_3, X_4 \ldots$
  - $\{X_n, n = 1, 2, \ldots\}$
  - X_n takes on a fini of possible values.
  - $X_n \in \{1, 2, \ldots, S\}$
  - i: ith state
  - Markov Property: The state of the system at time $n+1$ depends only on the state of the system at time n

$$\boxed{\begin{aligned}&\Pr\left[X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_2 = x_2, X_1 = x_1\right] = \\ &\Pr\left[X_{n+1} = x_{n+1} \mid X_n = x_n\right]\end{aligned}}$$



- Stationary Assumption: Transition probab ependent of time (n)

$$\boxed{\Pr\left[X_{n+1} = b \mid X_n = a\right] = p_{ab}}$$

An example

Weather:

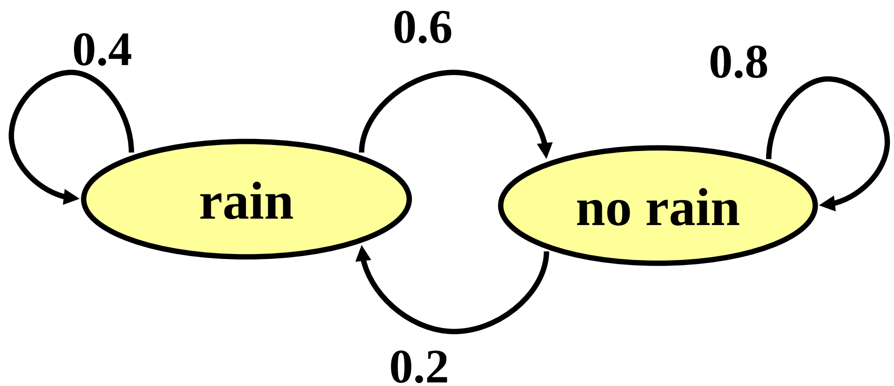
- raining today → 40% rain tomorrow

  → 60% no rain tomorrow

- not raining today → rrow

  → 80% rrow

Stochastic FSM:



An example

Weather:

- **raining today** → **40% rain tomorrow**
  → **60% no rain tomorrow**

- **not raining today** → **rrow**
  → **80% rrow**

Matrix:

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$$

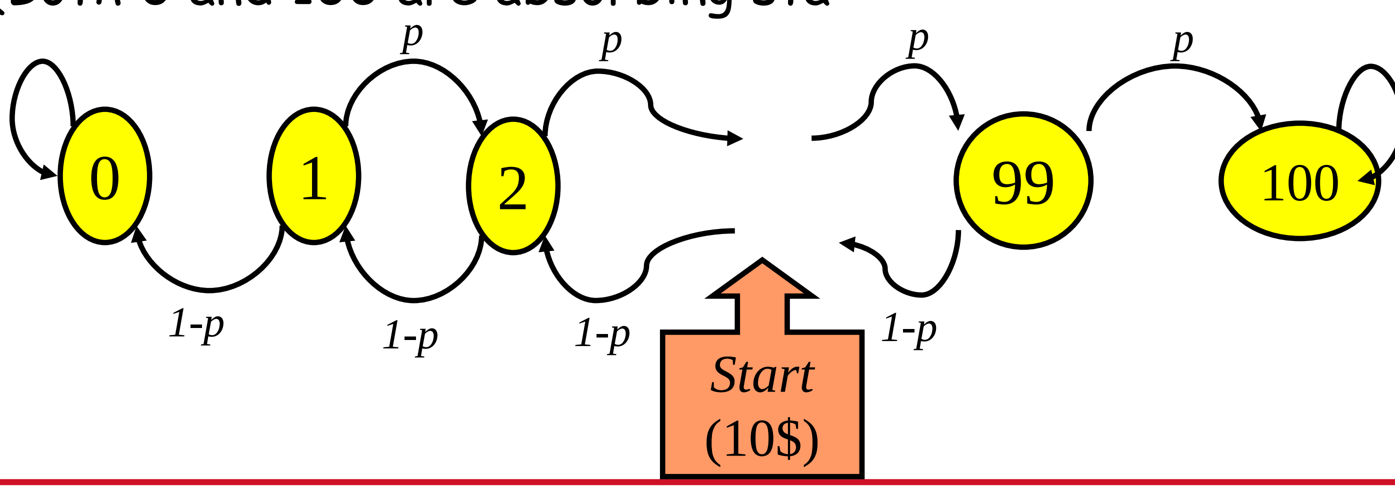
- Stochastic matrix:
  Rows sum up to 1

Transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1S} \\ p_{21} & p_{22} & \cdots & p_{2S} \\ \vdots & \vdots & & \vdots \\ p_{S1} & p_{S2} & \cdots & p_{SS} \end{pmatrix}$$

Gambler’s Example

- Gambler starts with $10
- At each play we have one of the following:
  - Gambler wins $1 with probability p
  - Gambler looses
- Game ends when gambler goes br s a fortune of $100
  (Both 0 and 100 are absorbing sta



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Gambler's Example

- **transient state**

if, given that we start in state *i*, there is a non-zero probability that we will never return to *i*

- **recurrent state**

Non-transient

- **absorbing** state

impossible to leave this state.

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Coke vs. Pepsi Example

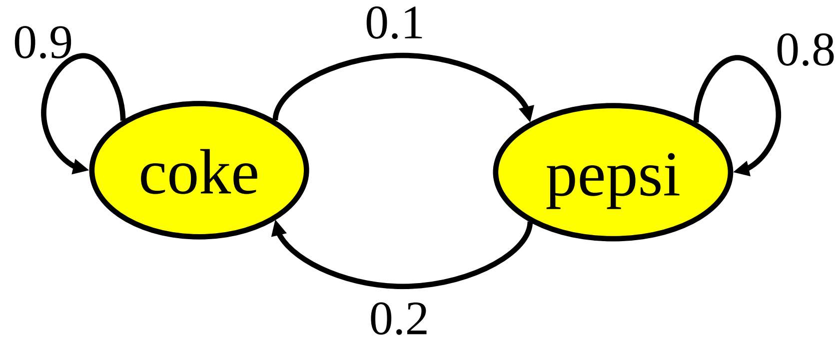
- Given that a person's last cola purchase was Coke, there is a 90% chance that his next cola purchase will also be Coke.

- If a person's last cola purchase was Pepsi, there is an 80% chance t … ase will also be Pepsi.

… n diagram:

transition matrix:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$



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Coke vs. Pepsi Example

Given that a person is currently a Pepsi purchaser, what is the probability that he will purchase Coke two purchases from now?

Pr[Pepsi→?→Coke

Pr[Pepsi→Coke→C

oke] =

0.2 * 0.9 + 0.

= 0.34

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

Pepsi → ?   ? → Coke

Coke vs. Pepsi Example

Given that a person is currently a Coke purchaser, what is the probability that he will purchase Pepsi **three** purchases from now?

$$P^3 = \begin{bmatrix}0.9 & 0.1\\ 0.2 & 0.8\end{bmatrix}\begin{bmatrix}[illegible] & [illegible]\\ 0.34 & 0.[illegible]\end{bmatrix} \begin{bmatrix}.781 & 0.219\\ .438 & 0.562\end{bmatrix}$$

Coke vs. Pepsi Example

- Assume each person makes one cola purchase per week
- Suppose 60% of all people now drink Coke, and 40% drink Pepsi
- What fraction of people will be drinking Coke three weeks from now?

$$P = \begin{bmatrix} 0. & \\ 0.2 & 0.8 \end{bmatrix} \qquad \begin{bmatrix} & 0.219 \\ & 0.562 \end{bmatrix}$$

$\Pr[X_3=\text{Coke}] = 0.6 * 0.781 + 0.4 * 0.438 = 0.6438$

Q_i - the distribution in week i

$Q_0=(0.6,0.4)$ - initial distribution

$Q_3= Q_0 * P^3 =(0.6438,0.3562)$

Coke vs. Pepsi Example

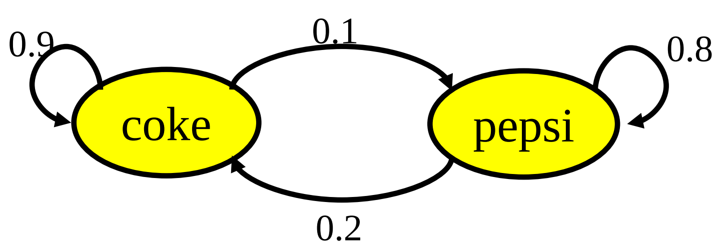
Simulation:

2/3

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

stationar

$\Pr[X_i = \text{Coke}]$



week - i

Steady State and Stationary distribution

$$\lim_{n\to\infty} P(X_n = i) = \pi_i$$

$$n \to \infty$$

$$\boldsymbol{\pi} = \boldsymbol{\pi} \cdot P$$

Steady State and Stationary distribution

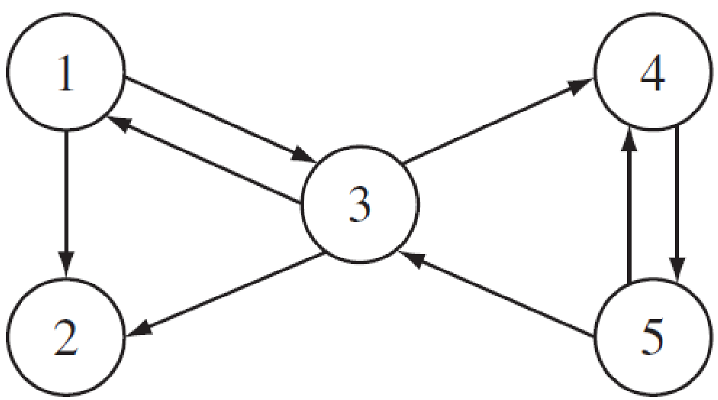
$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0 & \end{bmatrix} \qquad \boldsymbol{\pi} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.6761 & 0.3239 \\ 0.6478 & 0.3522 \end{bmatrix} \qquad \begin{bmatrix} 0.6667 & 0.3333 \\ 0.6667 & 0.3333 \end{bmatrix}$$

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

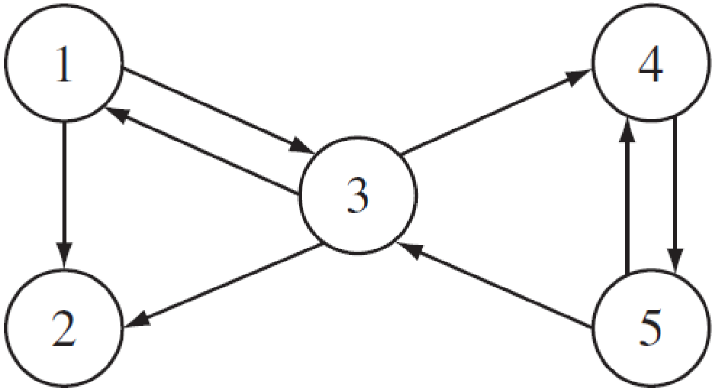
Steady State and Stationary distribution

PageRank: A Web surfer browses pages in a five-page Web universe shown in figure. The surfer selects the next page to view by selecting with equal probability from the pages pointed to by the current page. If a page has no outgoing link (e.g., page 2), the the pages in the universe with eq obability that the surfer views page i.



Steady State and Stationary distribution

Transition matrix P



Steady State and Stationary distribution

Stationary Distribution:
Solve the following equations:

$$\sum_{i=1} \pi_i$$

$$\boldsymbol{\pi} = (0.12195,\ 0.18293,\ 0.25610,\ 0.12195,\ 0.317072)$$

Search engineer. page rank: 5, 3, 2, 1, 4

› Queueing System and Little’s Theorem

- Customers = Data packets
- Service Time = Packet Transmission Ti f packet length and transmission speed)
- Queueing delay = time spent in buffer before transmission
- Average number of customers in systems
  – Typical number of customers either waiting in queue or undergoing service
- Average delay per customer
  – Typical time a customer spends waiting in queue + service time

Queueing System

$=$ N *Number*

$=$ T *Time*

- W: average waiting time in queue
- X: average service time
- T: average time spent in system ($T = W + X$)
- N_Q = average number of customers in queue
- ρ = utilization = average number of customers in service
- N = average number of customer in system ($N = N_Q + \rho$)
- Want to show later: $N = \lambda T$ (Little's theorem)
- λ Average arrival rate

Little’s Theorem

$\alpha(t)$ = Number of customers who arrived in the interval $[0, t]$

$\beta(t)$ = Number of customers who departed in the interval $[0, t]$

$N(t)$ = Number of customers in the system at time t, $N(t) = \alpha(t) - \beta(t)$

T_i = Time spent in the system by the i-th arriving customer

Average # of customers until t

$$N_t = \frac{1}{t}\int_0^t N(\tau)d\tau$$

Average # of customers in long-term

$$N = \lim_{t\to\infty}\frac{1}{t}\int_0^t N(\tau)d\tau$$

Average # arrival rate until t

$$\lambda_t = \frac{\alpha(t)}{t}$$

Average # arrival rate in long-term

$$\lambda = \lim_{t \to \infty} \frac{\alpha(t)}{t}$$

Average customer delay till t

$$T_t = \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

Average customer delay in long-term

$$T = \lim_{t \to \infty} \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

t'

Shaded area when the queue is empty: two ways to compute

$$\int_0^t N(\tau)d\tau = \sum_{i=1}^{\alpha(t)} T_i$$

t'

Shaded area when the queue is empty: two ways to compute

$$\frac{1}{t}\int_0^t N(\tau)d\tau = \frac{1}{t}\sum_{i=1}^{\alpha(t)} T_i$$

t'

Shaded area when the queue is empty: two ways to compute

N_t $\quad \lambda_t \quad T_t$

$$\frac{1}{t}\int_0^t N(\tau)d\tau = \frac{\alpha(t)}{t} \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

t'

Shaded area when the queue is empty: two ways to compute

$$N_t = \lambda_t T_t$$

$$N = \lambda T$$

Little’s Theorem

Note that the above Little’s Theorem is valid for any service disciplines (e.g., first-in-first-out, last-in-first-out), interarriva d service time distributions.

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Little's Theorem

λ R

$=$ N *Number*

$=$ T *Time*

λ *Rate*

Number

- $N = \lambda T$
- $N_Q = \lambda W$
- ρ = proportion of time that the server is busy = λX
- $T = W + X$
- $N = N_Q + \rho$

- M/M/1 Queue foundations

Exponential Distribution

- Exponential Distribution
- • The cumulative distribution function $F(x)$ and probability
- density function $f(x)$ are:
- $F(x) = 1 - e^{-\lambda x} f(x) = \lambda e^{-\lambda x} \quad \geq \quad \lambda >$

The mean is equal to its standard deviation: $E[X] = \sigma_X = 1/\lambda$

- $P(X > s + t \mid X > t) = P(X > s)$ for all $s, t \geq 0$
- The only continuous distribution with this property
- Practice Q2 in Tutorial Week 4

Other Properties of Exponential Distribution

- Let $X_1, \ldots, X_n$ be i.i.d. exponential r.v.s with mean $1/\lambda$,
- then $X_1+X_2+\ldots+X_n$ (Practice Q2 in Tutorial Week 4)

- gamma distribution with parameters n and λ.
- Suppose X_1 and X_2 are independent exponential r.v.s with means
- $1/\lambda_1$ and $1/\lambda_2$, respectively, then

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Counting Process

› A stochastic process $\{N(t), t \geq 0\}$ is a counting process if $N(t)$ represents the total # of events that have occurred up to time t.

› 1. $N(t) \geq 0$ and $N(t)$ is integer valued.

› 2. If $s < t$, then $N(s) \leq N(t)$

› 3. For $s < t$, $N(t) - N(s)$ in (s, t).

› • Examples:

› – # of people who have entered a particular

› – # of packets sent by a mobile phone

› • A counting process is said to be independent increment if # of events which occur in disjoint time intervals are independent.

› • A counting process is said to be stationary increment if the distribution of # of events which occur in any interval of time depends only on the length of the time interval.

Poisson Process

- The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda > 0$, if
- 1. $N(0) = 0$
- 2. The process has independent increments (i.e., # of events which occur in disjoint time intervals ar

- – for $0 < t_1 < t_2 < t_3 < t_4$,

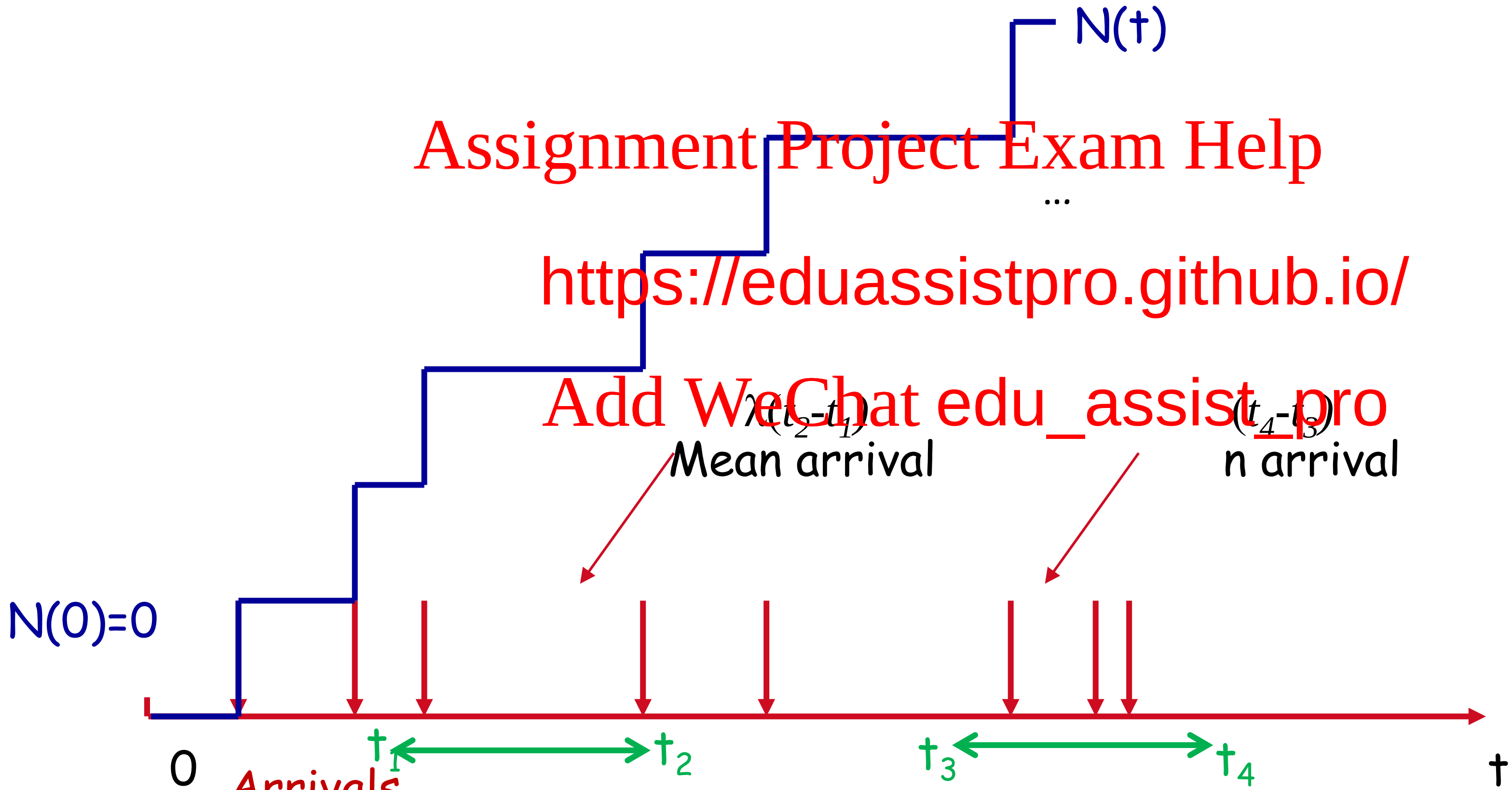
- – $P\{\, N(t_4) - N(t_3) = n \mid N(t_2) - N(t_1) = j \,\} = \ldots = n \,\}$

- 3. Number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$

$$E(N(t+s) - N(s)) = \lambda t$$

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Poisson Process



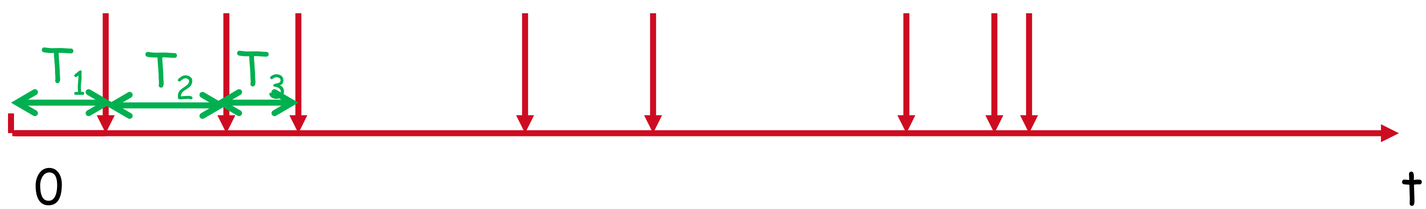
Arrivals in $[t_1, t_2]$ is independent of arrivals in $[t_3, t_4]$

Poisson Process: Inter arrival time distribution

Exponential distribution with parameter λ
(mean 1/ λ)

$P(T_2 >$ $\quad\quad$ $^{-\lambda t}$

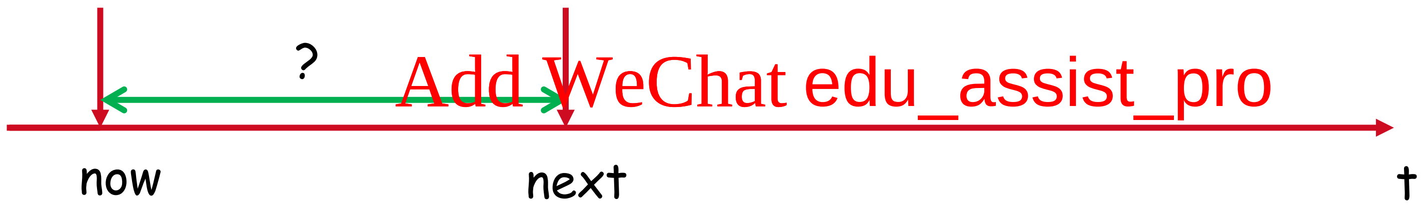
$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$



Poisson Process: Inter arrival time distribution

Given that an event arrives now, what is the distribution of T, where T is the time duration between now and next arrival event?

Exponentially distributed with parameter λ



Poisson Process: Inter arrival time distribution

Given that an packet event arrives at t_0 time ago, what is the distribution of T, where T is the time duration between now and next arrival event?

Exponentially distributed with parameter λ

Reason: Memoryless!

Number of arrivals in a short period of time

Number of arrival events in a very short period

$P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$; and

$P\{N(t+h) - N(t) \geq$

$o()$. Small o notati d to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

Summery

Poisson process:

Independent increments

# of arrivals: Poisson distributed

# of arrivals in a s rival, probability λh

Inter-arrival time istribution

› M/M/1 Queue

Queues: Kendall's notation

- Notations Used in Queueing Systems
- *X*/*Y*/*Z*
- – *X* refers to the distribution of the interarrival times
- – *Y* refers to the distribution of service times
- – *Z* refers to the number
- Common distributions:
- – *M* = Memoryless = exponential distribution
- – *D* = Deterministic arrivals or fixed-length se
- – *G* = General distribution of interarrival times or service times
- *M*/*M*/*1* refers to a single-server queuing model with exponential interarrival times (i.e., Poisson arrivals) and exponential service times.
- In all cases, successive interarrival times and service times are assumed to be statistically independent of each other.

M/M/1 Queue

- Arrival:
- Poisson arrival with rate λ
- Service:

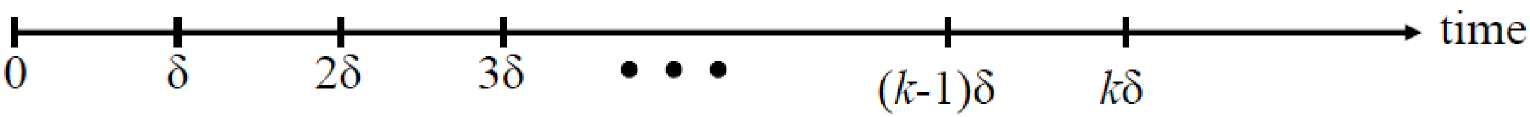
- Service time: ex

on with mean $1/\mu$

- μ: service rate,

- $\lambda < \mu$: Incoming rate < outgoing rate

Markov Chain Formulation



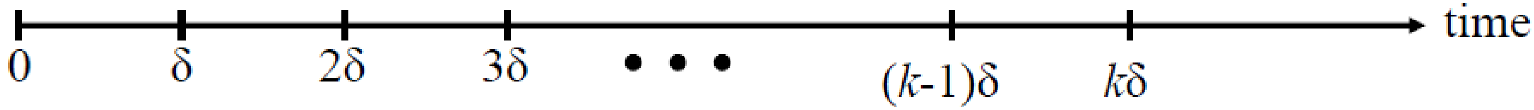
δ: a small value

N_k = Number of customers in t time $k\delta$

N_0 N_1 N_2... is a Markov Chain!

Q: How to compute the transition probability?

Markov Chain Formulation

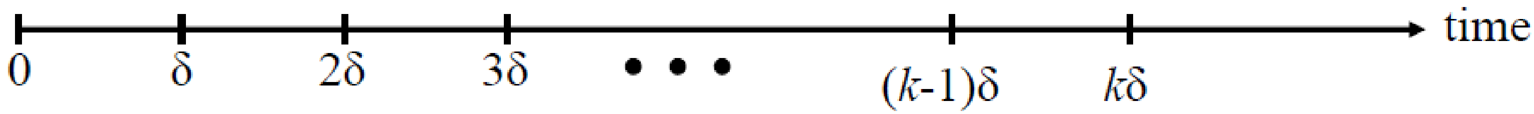


$P(0 \quad \text{cust}$ $\delta + o(\delta)$

$P(1 \quad \text{customer} \quad \text{arrive}$ (δ)

$$P(\geq 2 \quad \text{customer} \quad \text{arrives}) = o(\delta)$$

Markov Chain Formulation



$P(0$ custome (δ)

$$P(1 \text{ customer leaves}) = \begin{cases} & i \geq 1 \\ 0 & i = 0 \end{cases}$$

$$P(\geq 2 \text{ customer leaves}) = o(\delta)$$

No one in the system

Markov Chain Formulation

Aim to compute $P_{ij} = P\{N_{k+1} = j \mid N_k = i\}$

For example, $P\{N_{k+1} = i \mid N_k = i\}$, i>0

$P($0 custome ... departs)

$+P($1 customer arrives) ... mer departs)

$+P($other)

$\text{Result}: 1-\lambda\delta-\mu\delta+o(\delta)$

$$[1-\lambda\delta+o(\delta)][1-\mu\delta+o(\delta)] = 1-\lambda\delta-\mu\delta+o(\delta)$$

$$[\lambda\delta+o(\delta)][\mu\delta+o(\delta)] = o(\delta)$$

$$o(\delta)o(\delta) = o(\delta)$$

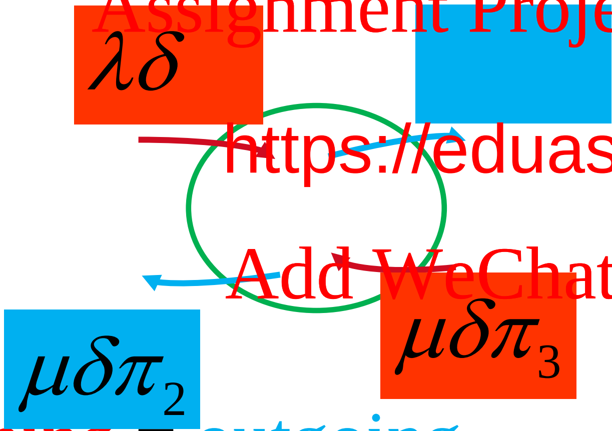
Result:

π_i Stationary distribution of state *i*

The probability that there are *i* units in the system

Stationary Distribution Derivation

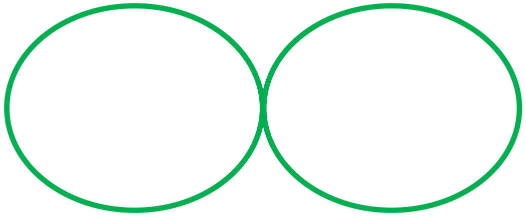
How to derive π_i balance equation satisfied



Incoming = outgoing

$$\lambda\delta\pi_2 + \mu\delta\pi_2 = \lambda\delta\pi_1 + \mu\delta\pi_3$$

How to derive π_i



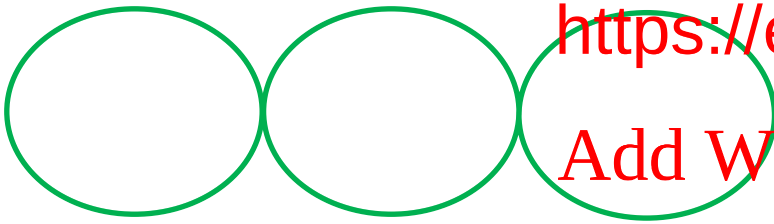
balance equation is performed at each state

$$\lambda\delta\pi_0 = \mu\delta\pi_1$$

$$\lambda\delta\pi_1 + \mu\delta\pi_1 = \lambda\delta\pi_0 + \mu\delta\pi_2 \Rightarrow \lambda\delta\pi_1 = \mu\delta\pi_2$$

Stationary Distribution Derivation

How to derive π_i



balance equation is performed at each state

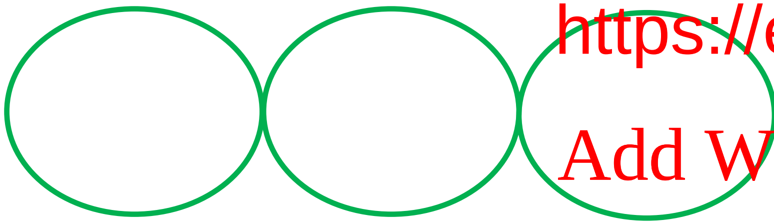
$$\lambda\delta\pi_0 = \mu\delta\pi_1$$

$$\lambda\delta\pi_1 = \mu\delta\pi_2$$

$$\lambda\delta\pi_2 + \mu\delta\pi_2 = \lambda\delta\pi_1 + \mu\delta\pi_3 \Rightarrow \lambda\delta\pi_2 = \mu\delta\pi_3$$

Stationary Distribution Derivation

How to derive π_i



balance equation is performed at each state

$$\lambda\delta\pi_0 = \mu\delta\pi_1$$

$$\lambda\delta\pi_1 = \mu\delta\pi_2$$

$$\lambda\delta\pi_2 = \mu\delta\pi_3$$

$$\lambda\delta\pi_i = \mu\delta\pi_{i+1}$$

For any i

How to derive π_i

balance equation is performed at each state

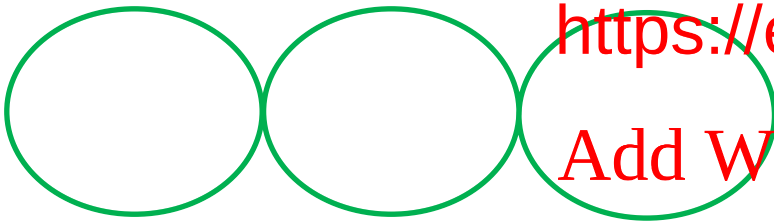
$$\pi_1 = \frac{\lambda}{\mu}\pi_0 \qquad \pi_2 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0 \qquad \cdots \qquad \pi_i = \left(\frac{\lambda}{\mu}\right)^i \pi_0$$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

Sum of geometric sequence

Stationary Distribution Derivation

How to derive π_i



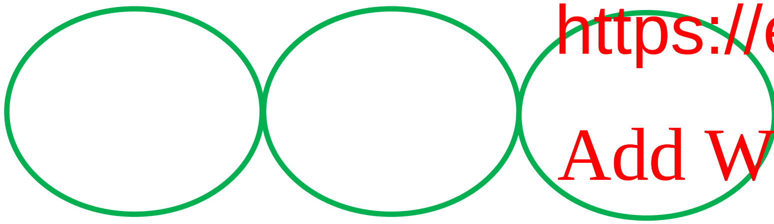
balance equation is performed at each state

$\pi_1 = \rho\pi_0$ $\quad \pi_2 = (\rho)^2 \pi_0$ $\quad \cdots \quad$ $\pi_i = (\rho)^i \pi_0$ $\quad \rho = \frac{\lambda}{\mu} < 1$

$$\sum_{i=0}^{\infty} \pi_i = 1$$

Sum of geometric sequence

How to derive π_i



balance equation is performed at each state

$$\lim_{N\to\infty}\frac{\pi_0(1-\rho^N)}{1-\rho}=\frac{\pi_0}{1-\rho} \quad =1$$

$$\pi_0 = 1 - \rho$$

$$\pi_i = (1-\rho)\rho^i$$

Sum of geometric sequence

Average number of users in the system

$$E(N) = \sum_{n=0}^{\infty} n(1-\rho)\rho^{n}$$

$$= \rho(1-\rho)\sum_{n=0}^{\infty}$$

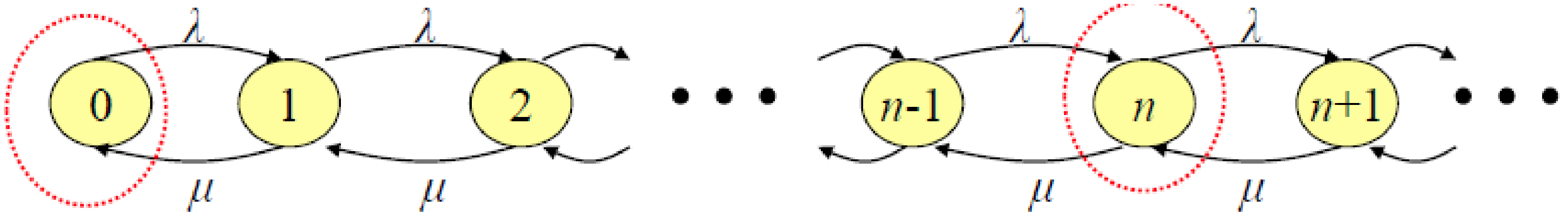
$$= \rho(1-\rho)\frac{\partial\left[\sum_{n=0}^{\infty}\rho^{n}\right]}{\partial\rho}$$

$$= \rho(1-\rho)\frac{\partial\left[\frac{\rho}{1-\rho}\right]}{\partial\rho} = \frac{\rho}{1-\rho}$$

Average waiting time

Average waiting time Little’s Theorem

$$E(T) \quad \frac{\qquad}{\lambda}$$


λ
λ
λ
λ
0
1
2
n-1
n
n+1
μ
μ
μ
μ

Stationary Distribution Derivation

balance equation is perform tate

$$\lambda\pi_0 = \mu\pi_1$$

$$\lambda\pi_1 + \mu\pi_1 = \lambda\pi_0 + \mu\pi_2$$

$$\cdots$$

balance equation is perform tate

Following the same step, derive the same result

Stationary Distribution Derivation

Queueing delay goes to infinity when arrival rate approaches service rate!

M/M/m Queue

M/M/m Queue

- Arrival:
- Poisson arrival with rate λ
- Service:

- Service time for ntial distribution with mean $1/\mu$
- service rate is i μ, if there are $i<m$ users in the system
- service rate is mμ, if there are $i>=m$ users in the system

$$\lambda\pi_{i-1} = i\mu\pi_i \qquad i$$

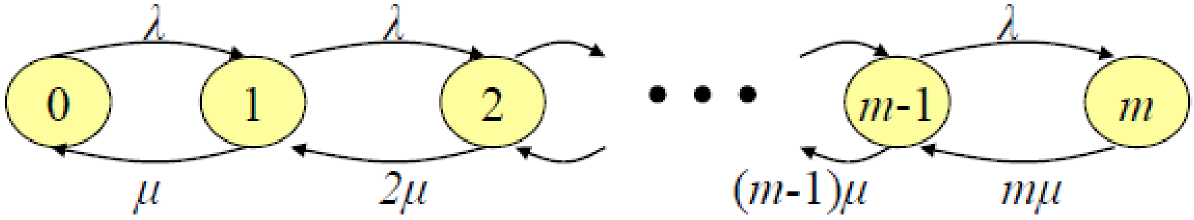
$$\lambda\pi_{i-1} = m\mu\pi_i \qquad i > m$$

$$\pi_n = \begin{cases} \pi_0 \dfrac{(m\rho)^n}{n!} & n \le m \\ \pi_0 \dfrac{m^m \rho^n}{m!} & n > m \end{cases} \qquad \rho = \frac{\lambda}{m\mu} < 1$$

Then, π_0 can be solved

M/M/∞ Queue

M/M/m/m Queue



M/M/m/n Queue

Arrivals will dropped if there are n users in the system.

Buffer size is n-m

How do you derive its stationary distribution?

Practice

- Analyze M/M/ ∞, M/M/m/n queues
  - Draw the state transition diagrams
  - Derive their stationary distributions
  - For M/M/m/n queue, c ncoming user is dropped. Calculate the probabili all users are served in the servers or there are no users at all.)