Deterministic Finite Automata

COMP1600 / COMP6260

Semester 2, 202

The Story So Far . . .

Logic.

- language and proofs to speak about systems *precisely*
- useful to *express properties* and *do proofs*

Functio

- esta
- mai

Imperative Programs.

- again: focus on *properties* of programs
- main tool: Hoare Logic

Q. Is there a *general* notion of computation? That encompasses both?

First Shot: Your Laptop

Abstract Characteristics.

- can do computation
- has memory – a finite amount
- has (lots of) internal states

From Laptops to Formal Models

Concrete (your laptop)

- realistic (it exists!)
- com
- hard t

Abstract (mathematical model)

- exists only as a model
-
-

Q. What is a "*good*" simple model of computati

- should match what really exists (possibly by a lon
- should be *conceptually simple*

First Answer: Finite State Automata

Basic Components.

- internal states – finitely many
- stat
- sim

Data.

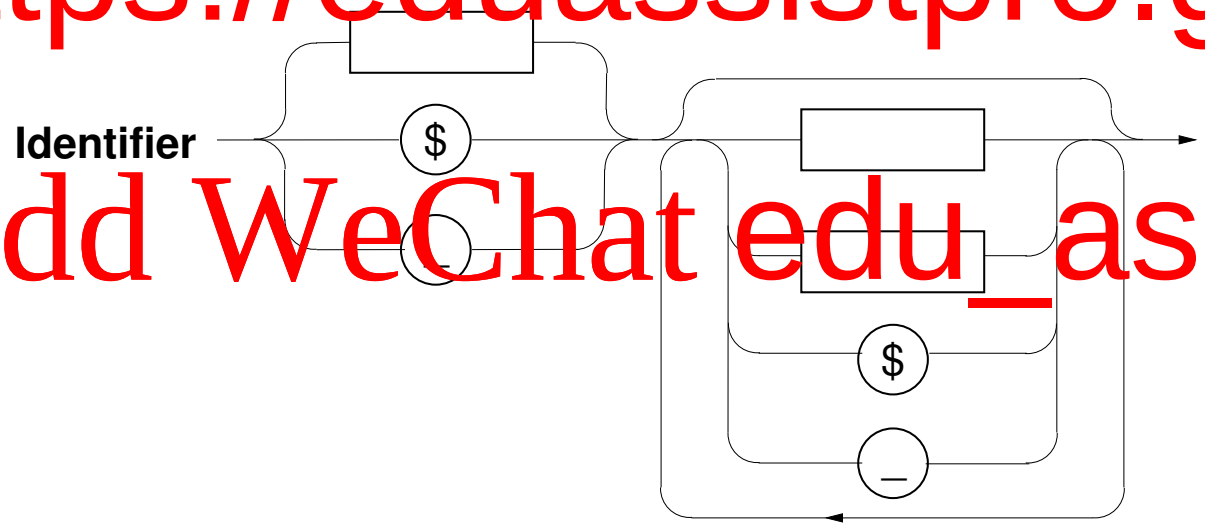
- basic input: strings (what you type in, text/xml fil
- characters: drawn from *finite* set (alph

Example: Java Identifiers

From Oracle's Java Language Specification.

> *An identifier is a sequence of one or more characters. The first character must be a valid first character (letter, $, _) in an identifier of the Java programming language, hereafter in this chapter called simply "Java". Each subsequent character in the sequence must be a valid nonfirst char*

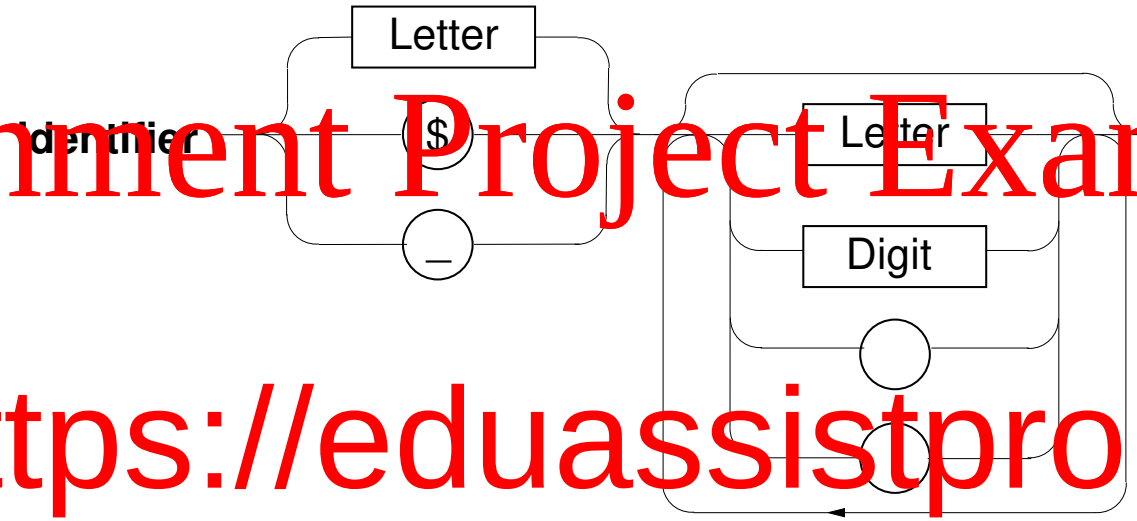
Graphic



Q. Can you "see" a machine that *recognises* Java identifiers?

Java Identifiers

Example: Main Components



Data.

- drawn form a finite alphabet (unicode or ASCII)

Control.

- "yes" if I can get from the left to the right, "no" otherwise
- have *states* after taking a transition (implicit in diagram)

Computational Problem with yes/no answer:

- it a given sequence of characters a valid Java identifier?

Preview.

Next two weeks. Finite Automata

- start with simplest model: finite automata
- relate to regular languages, non-determinism
- conclusion: finite automata "too simple"

The wee

- like fi
- usef
- still "too simple" for general computation

Then. Turing machines

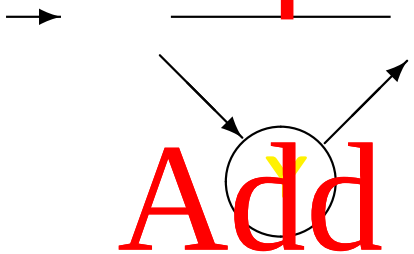
- *The* most widely accepted model of computa
- *infinite* memory
- idea: buy another hard disk whenever your computation runs out of memory
- *limits* of what can be computed

Finite State Automata: First Example

The simplest useful abstraction of a **"computing machine"** consists of:

- A fixed, finite set of **states**
- A **transition relation** over the states

Exampl



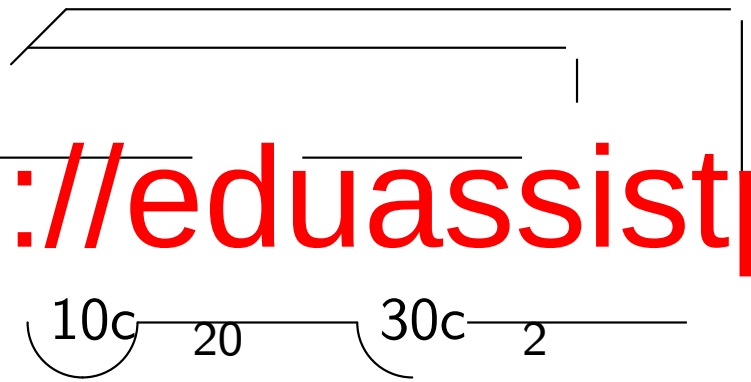
Y names stat
R names stat

System designs are often in terms of state machines.

Second Example: Vending Machine

Operation

- accept 10c and 20c coins
- delivers if it has received at least 40c and selection is made



Note.

- transitions are *labelled*
- new ingredient: *final states* (doubly circled)

Computation. Sequences of actions (labels) from initial to final state.

Language Examples

Main Idea.

- input: a string over a fixed character set
- operation: transitions labelled with characters
- output: yes if in final state after reading the input

More Ge

- Set
- Pro
- Task: decide computationally which strings are "good"

Example Languages.

1. A finite set.

$$\{a, aa, ab, aaa, aa$$

2. Palindromes consisting of bits (0,1):

$$\{0, 1, 00, 11, 010, 101, 000, 111, 0110, ...\}$$

Languages in this sense are called *formal* languages.

Terminology

Alphabet.

A finite set (of symbols). Usually denoted by Σ.

Strings over an alphabet Σ

finite sequence of characters (elements of Σ), can be the empty sequence

Langua

are just s

Sentences of the language

just another name for the elements (strings) of the la

Notation:

- Σ^* is the set of all strings over Σ.
- Therefore, every language with alphabet Σ is some **subset** of Σ^*.

Automata

First Model of Computation. Deterministic Finite Automata

- solve computational problem: given string s, is s accepted?

Basic Ingredients. (see e.g. traffic light and vending machine example)

- The at an aut
- a DF
- One of the states is the **initial** state —
- At least one of the states is a **final** st
- A **transition function** *(next state functi*

$$State \times Token \rightarrow State$$

Recurring Theme

Diagrammatic Notation.

- useful for humans
  - e.g. the transition diagram of the vending machine

Mathe

- usef
- useful for computer implementation

Glue between Diagrams and Maths

- both notions convey *precisely* the same information
- crucial: being able to switch back and forth!

Formal Definition of DFA

A Deterministic Finite State Automaton (DFA) consists of five parts:

$$A = (\Sigma, S, s_0, F, N)$$

- an in
- a set o
- an **"initial"** state $s_0 \in S$ (we start here
- a set of **"final"** states $F \subseteq S$ (we hop
- a **transition function** $N : S \times \Sigma \rightarrow$

Aside. Having a transition *function* is what makes the automaton deterministic.

Finite State Automata as String Acceptors

Idea. A finite state automaton

- works on *strings* over an alphabet Σ
- determines which strings in Σ are "good" (accepted) and which strings are "bad" (rejected)

Accepta

A *accepts*

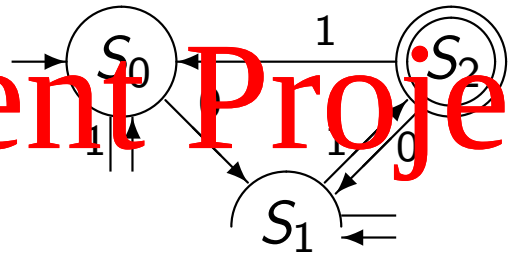
$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}}$$

where s_0 is the starting state, $s_n \in F$ is an ac $\xrightarrow{a}$ t if $\delta(s, a) = t$.

Informally. Run the automaton from the starting state, move states according to the individual letters of the word, and accept if you end up in a final state.

Example 1

As a diagram.



In Mathe

- Alp
- States - $\{S_0, S_1, S_2\}$
- Initial state - S_0
- Final states - $\{S_2\}$
- Transition function (as a table) -

S_2	S_1	S_0

Q1. Which strings are accepted by this automaton?
Q2. What changes if we re-name the states?

Example 1, ctd

Recall. $N : S \times \Sigma \to S$ is the transition function.

	0	1
S_0	S_1	S_0
S_1	S_1	S_2
S_2	S_1	S_0

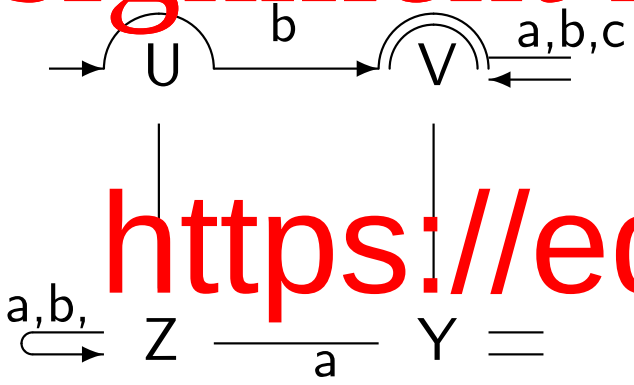
Single S

- $N($ S_0 reading letter 0.
- Here: $N(S_0, 0) = S_1$.

Multiple Steps of the automaton

- $N(N(S_0, 0),\ 1)$ is the state of the automation when starting in S_0 and reading first 0, then 1.
- Here: $N(N(S_0, 0),\ 1) = S_2$.

Example 2



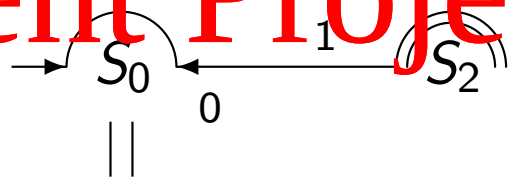
	a	b	c

(the table carries the same information as the diagram)

Q. What is the language of this automaton?

Eventual State Function

Revisit example 1:



- Input 0101 takes the DFA from S_0 to
  Input 1011 takes the DFA from S_1 to
- A complete list of such possibilities is a function fro
  and a string to an 'eventual state.'

This is the idea of **Eventual State Function.**

Eventual State Function — Definition

Definition. Let A be a DFA with states S, alphabet Σ, and transition function N.

The *eventual state function* for A is of type

$$N^* : S \times \Sigma^* \to S$$

and is defin

(N1)

Or in Haskell, where strings are lists of elements of type

% n :: State -> Sigma -> State -- given

```
nstar :: State -> [Sigma] -> State
nstar s [] = s
nstar s (a:as) = nstar (n s a) as
```

Informally. $N^*(s, w)$ is the state A reached by starting in state s and reading string w.

An Important (but Unsurprising) Theorem about N^*

Theorem. For all states $s \in S$ and for all strings $\alpha, \beta \in \Sigma^*$:

$$N^*(s, \alpha\beta) = N^*(N^*(s, \alpha), \beta)$$

Proof b

Base case:

$$\text{LHS} = N^*(s, \epsilon\beta) =$$

$$\text{RHS} = N^*(N^*(s, \epsilon), \beta)$$

$$= N^*(s, \beta) =$$

y (NI))

Proof ctd: Step case:

Step Case. Show that $N^*(s,(x\alpha)\beta) = N^*(N^*(s,x\alpha),\beta)$

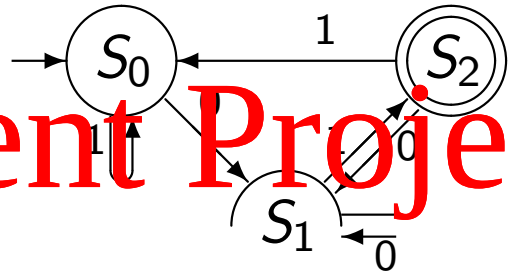
$$\begin{aligned}\text{LHS} &= N^*(s,(x\alpha)\beta)\\ &= N^*(s,x(\alpha\beta))\end{aligned}$$

$$\begin{aligned}\text{RHS} &= N^*(N^*(s,x\\ &= N^*(N^*(s,\\ &= \end{aligned}$$

Corollary — when β is a single token

$$N^*(s,\alpha y) \;=\; N(N^*(s,\alpha),y)$$

Example



$$
\begin{aligned}
&= N^*(S_2, 011) \\
&= N^*(S \\
&= N^*(S \\
&= N^*(S_0, \epsilon) \\
&= S_0
\end{aligned}
$$

Language of an Automaton, Revisited

Acceptance, with eventual states. Let $A = (\Sigma, S, s_0, F, N)$ be an DFA and w b

Then w

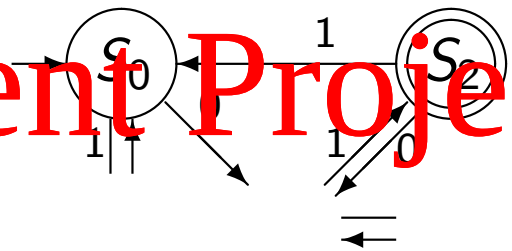
0

Q1. How does this compare with the earlier notion of a

Q2. How can we prove that both are equivalent?

Example 1 again

A_1:

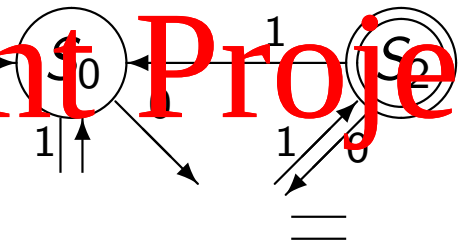


Q. Whic

- e.g. 0011101 takes the machine from state S_1, S_1, S_2, S_0, S_0, S_1 to S_2 (a final state).
- $N^*(S_0, 0011101) = N^*(S_1, 011101) = \ldots N^*(S_1, 1) = S_2$
- others: 01, 001, 101, 0001, 0101, 00101101 ...

Example 1 (ctd.)

A_1:



Accepte

01, 001, 101, 0001, 0101, 00101101 . . .

Strings that are not accepted.

ϵ, 0, 1, 00, 10, 11, 100 . . .

Q. What do the accepted strings have in common? How do we justify this?

Proving an Acceptance Predicate — in General

Our Claim. The automaton A accepts precisely the strings that are elements

(P is som

Proof Ob

1. Show that any string satisfying P is ac
2. Show any string accepted by A satisfi

Proving an Acceptance Predicate for A_1

Proof obligation 1.

If a string ends in 01, then it is accepted by A_1. That is:

Proof obl

If a string is accepted by A_1, then it ends in 01. Th

For all $w \in \Sigma^*$, if $M^*(s_0, w) \in F$ t

Part 1: $\forall \alpha \in \Sigma^*,\ N^*(S_0, \alpha 01) \in F$

Lemma:

$$\forall s \in S,\ N^*(s, 01) \equiv S_2$$

Proof by cases:

$$N^*(S_2, 01) = N^*(S$$

So, by the "append" theorem above,

$$N^*(S_0, \alpha 01) = N^*(N^*(S_0, \alpha), 01) = S_2 \square$$

Part 2: $N^*(S_0, w) = S_2 \implies \exists \alpha.\ w = \alpha 01$

Proof. Suppose $N^*(S_0, \alpha xy) = S_2$.

By corollary to append-theorem (case of single token):

By the defi $\qquad\qquad 0 \qquad\qquad 1$.

Similarly,

$$N(N^*(S_0, \alpha y), x)$$

and x is 0, again by the definition of N.

□

Another Example

Answer for *SOB*

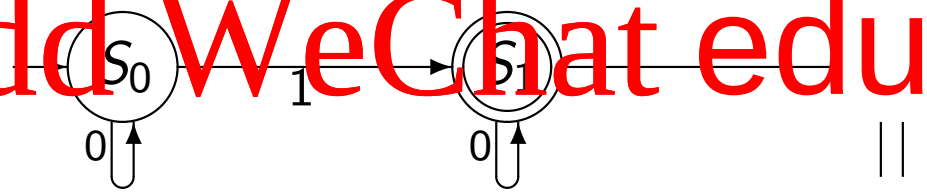
SOB accepts the language of bitstrings containing **exactly one 1-bit.**

Proof obligations:

- Sho
  by
- Sho
  1-bit.

y one

SOB:



Mapping to Mathematics

Expressed mathematically, the main conclusion is:

$$n \quad m$$

The two sets

1. If w
2. If $N^*(S_0, w) = S_1$ then $w = 0^n 1 0^m$.

For this DFA the phrase "w is accepted by expression $N^*(S_0, w) = S_1$.

Proving these subgoals

The first subgoal follows immediately from the following two lemmas, which are easily proved by induction:

Lemma 1: $\forall n \geq 0.\ N^*(S_1, 0^n) = S_0$

Lemma 2: $\forall n \geq 0.\ N^*(S_1, 0^n) = S_1$

Therefor

$$
\begin{aligned}
&= N^*(S_0, 10^m) && \text{(by Lemma 1)} \\
&= N^*(N^*(S_0, 1), 0^m) && \text{(by def.} \\
&= N^*(S_1, 0^m) && \text{(by def.} \\
&= S_1 && \text{(by Lemma 2)}
\end{aligned}
$$

The second subgoal, stated more formally as

$$\forall w:\ N^*(S_0, w) = S_1 \implies \exists n, m \geq 0.\ w = 0^n 10^m$$

can be proved in a similar fashion to Example 1 on earlier slides.

Limitations of FSAs

Q. Is an FSA a "good" model of computation?

- Suppose we have a program P that always terminates
- and outputs "yes" or "no" for every input string
- Is the P says "ye

Technic

A very important example: $L = \{ a^n b$

- $L = \{\epsilon, ab, aabb, aaabbb, a^4b^4, a^5b^5$
- **Claim.** There is no FSA that recognises this lan

(because an FSA's memory is limited.)

Q. Given the claim above, are FSA's *realistic* models of computation?

Proof of Claim

Proof by contradiction.

Suppose A is an FSA that accepts L. That is $L = L(A)$.

Then each

But A only has finitely many states, so some state mu

There are distinct i and j such that $N^*(S_0$

- that is, the automaton *cannot* tell a^i and a^j apart.

Proof by contradiction (ctd)

Since $a^i b^i$ is accepted, we know

$$N^*(S_0, a^i b^i) \in F$$

By the a

Now, since $N^*(S_0, a^i) = N^*(S_0, a^j)$

$$N^*(N^*(S_0, a^i), b^i) = N$$

So $a^j b^i$ is accepted by A but $a^j b^i$ is not in L, contradicting the initial assumption.

Pigeon-Hole Principle

The proof used the **pigeon-hole principle**:

No function from one set to a smaller finite set can be one-to-one.

(Finiteness is not really necessary — no function from one set t
smaller *cardinality* can be one-to-one.)

"You cannot fit $n + 1$ pigeons into n holes"

Equivalence of Automata

Two automata are said to be **equivalent** if they accept the same language.

Example

Q. Can FSAs be simplified? is there an equivalent FSA with *fewer states?*

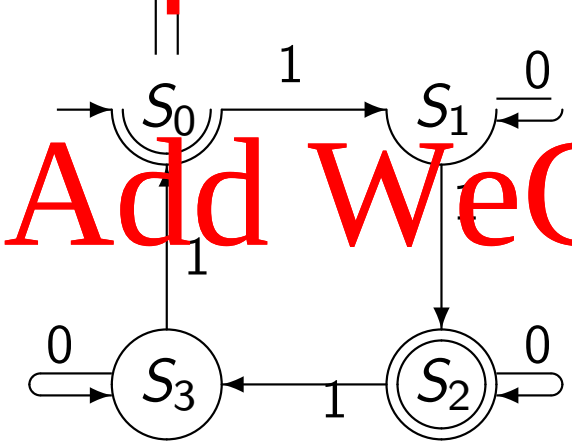
Equivalence of States

Two states S_j and S_k of an FSA are equivalent if, for all input strings w

$N^*(S_j, w) \in F$ if and only if $N^*(S_k, w) \in F$

Exampl S_3.

A_4:



Elimination of Equivalent States

Assumptions

- $A = (\Sigma, S, S_0, F, N)$ is an FSA
- S_k
- $S_k \neq S_0$

Elimination of S_k from A: new automaton $A' = (\Sigma, S', S_0, F', N')$

- S' is S without S_k
- F' is F without S_k
- $N'(s, w) = (\text{if } N(s, w) = S_k \text{ then } S_j \text{ else } N(s, w))$

Example

Since $S_2 \equiv S_0$ in A_4, let's eliminate S_2

- New set of states is $\{S_0, S_1, S_3\}$
- Ne
- Ne