Grammars and Pushdown Automata

COMP1600 / COMP6260

Semester 2, 202

Formal Languages – A Reminder of Terminology

- The **alphabet** or **vocabulary** of a formal language is a set of **tokens** (or **letters**). It is usually denoted Σ.
- A **string** over Σ is a *sequence* of tokens, or the null-string ϵ.
  -
  -
- A l
  - For example, the set of all strings Σ^*
  - or the set of all strings of even length, th$\}$

Notation:

- Σ^* is the set of all strings over Σ.
- Therefore, every language with alphabet Σ is some *subset* of Σ^*.

Specifying Languages

Languages can be given ...

- as a finite enumeration, e.g. $L = \{\epsilon, a, ab, abb\}$
- as a set, by giving a predicate, e.g. $L = \{w \in \Sigma^* \mid P(w)\}$ for some alphabet Σ
- alge ... r
- by a
- *by a gr*

Grammar

- a concept that has been invented in linguistics to d languages
- describes how strings are *constructed* rather than how membership can be *checked* (e.g. by an automaton)
- *the* main tool to describe syntax.

Grammars in general

Formal Definition. A *grammar* is a quadruple $\langle V_t, V_n, S, P \rangle$ where

- V_t is a finite set of *terminal symbols* (the alphabet)
- V_n is a finite set of **non-terminal symbols** disjoint from V_t
  (No
- S is *mbol*
- P is

$$\alpha \to$$

where

- $\alpha \in V^* V_n V^*$ (i.e. at least one non-terminal in α)
- $\beta \in V^*$ (i.e. β is *any* list of symbols)

Example

The grammar

$$G = \langle \{a, b\},\ \{S, A\},\ S,\ \{S \to aAb,\ aA \to aaAb,\ A \to \epsilon\} \rangle$$

has the following components:

- *Terminals:* a, b
- *N*
- *St*
- *Productions:*

Notation.

- Often, we just list the productions P, a
  inferred (S is the standard notation for the sta
- The notation $\alpha \to \beta_1 \mid \cdots \mid \beta_n$ abbreviates the *set* of productions

$$\alpha \to \beta_1, \quad \alpha \to \beta_2, \quad \ldots, \quad \alpha \to \beta_n$$

(like for inductive data types)

Derivations

Intuition.

- A production $\alpha \to \beta$ tells you what you can “make” if you have α: you can turn it into β.
- The production $\alpha \to \beta$ allows us to re-write any string $\gamma\alpha\rho$ to $\gamma\beta\rho$.
- Notation: $\gamma\alpha\rho \quad \gamma\beta\rho$

Derivati

- α
- so

Language of a grammar.

- informally: all strings of terminal symbols that ca the start symbol S
- formally: $L(G) = \{w \in V_t^* \mid S \Rightarrow^* w\}$

Sentential Forms of a grammar.

- informally: all strings (may contain non-terminals) that can be generated from S
- formally: $S(G) = \{w \in V^* \mid S \Rightarrow^* w\}$.

Example

Productions of the grammar G.

$$S \to aAb, \qquad aA \to aaAb, \qquad A \to \epsilon.$$

Example Derivation.

- last s

Language of grammar G.

$$L(G) = \{a^n b^n \mid n \in$$

Alternative Grammar for the *same* language

$$S \to aSb, \qquad S \to ab.$$

(Grammars and languages are not in 1-1 correspondence)

The Chomsky Hierarchy

By Noam Chomsky (a linguist!), according to the form of productions:

Unrestricted: (type 0) no constraints.

Context

Context-free: (type 2) the left of each production must be a *single non-terminal*.

Regular: (type 3) As for type 2, and the right of each
also constrained (details to come).

(There are *lots* of intermediate types, too.)

Classification of Languages

Definition. A language is *type n* if it can be generated by a type n grammar.

Immediate Fact.

- Eve

Establis

- give a grammar of type n that generates the language
- usually the easier task

Disproving that a language is of type n

- must show that *no* type n-grammar generates the language
- usually a *difficult* problem

Example — language $\{a^n b^n \mid n \in \mathbb{N},\ n \geq 1\}$

Different grammars for this language

- *Unrestricted (type 0):*

$$S \rightarrow aAb$$
$$aA \qquad aaAb$$

- *Con*

$$S \rightarrow a$$

$$S \rightarrow a$$

Recall. We know from last week that there is no DFA accepting L

- We will see that this means that there's no regular grammar
- so the language is context-free, but not regular.

Regular (Type 3) Grammars

Definition. A grammar is *regular* if all its productions are either *right-linear*, i.e. of the form

$$A \to aB \quad \text{or} \quad A \to a \quad \text{or} \quad A \to \epsilon$$

or *left-li*

- right and left linear grammars are equivalent: th
  languages

- we focus on *right linear* (for no deep reaso
- i.e. one symbol is *generated* at a time (cf. DFA/NFA!)
- termination with terminal symbols or ϵ

Next Goal. Regular Grammars generate precisely all regular languages.

Regular Languages - Many Views

Theorem. Let L be a language. Then the following are equivalent:

- L is the language generated by a *right-linear grammar*;
- L is the language generated by a *left-linear grammar*;
- L is the language accepted by some *DFA*;
- L is t
- L is t

So far.

- have seen that NFAs and DFAs generate the sam
construction)
- have hinted at regular expressions and NFAs generate the same languages

Goal. Show that NFAs and right-linear grammars generate the same languages.

From NFAs to Right-linear Grammars

Given. Take an NFA $A = (\Sigma, S, s_0, F, R)$.

- alphabet, state set, initial state, final states, transition relation

Construction of a right-linear grammar

- *terminal symbols* are elements of the alphabet Σ;
- *non*
- *star*

- *productions* are constructed as follows:

Each *transition*

$T \longrightarrow U$

Each *final state* gives *production*

$T \in F$ $\qquad$ $T \rightarrow \epsilon$

(Formally, a transition $T \xrightarrow{a} U$ means $(T, a, U) \in R$.)

Observation. The grammar so generated is right-linear, and hence regular.

NFAs to Right-linear Grammars - Example

Given. A non-deterministic automaton

Equival

S

$$S \to aS_1$$

$$S_1 \to bS_1$$

$$S_1 \to bS_2$$

Exercise. Convince yourself that the NFA accepts precisely the words that the grammar generates.

From Right-linear Grammars to NFAs

Given. Right-linear grammar (V_t, V_n, S, P)

- terminals, non-terminals, start symbol, productions

Construction of an equivalent NFA has

- *alphabet* is the terminal symbols V_t;
- *stat* … e S_f (for

- *star*
- *final states* are S_f and *all* non-termin … production $T \rightarrow \epsilon$;

- *transition relation* is constructed as follow

Each *production*	gives *transition*
$T \rightarrow aU$	$T \xrightarrow{a} U$
Each *transition*	gives *transition*
$T \rightarrow a$	$T \xrightarrow{a} S_f$

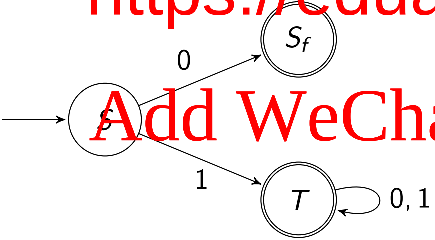
Right-linear Grammars to NFAs - Example

Given. Grammar G with the productions

$$S \to 0 \qquad S \to 1T$$
$$T \to \varepsilon \qquad T \to 0T \qquad T \to 1T$$

(generates binary strings without leading zeros)

Equival



Exercise. Convince yourself that the NFA accepts precisely the words that the grammar generates.

Context-Free (Type 2) Grammars (CFGs)

Recall. A grammar is type-2 or *context free* if all productions have the form

$$A \to \omega$$

where A V_n is a non-terminal, and ω V^* is an (arbitrary) string.

- left si
- righ
- *inde*

In Contrast. Context-Sensitive grammars m

$$\alpha A \beta \to \alpha \omega$$

- may only replace A by ω if A appears in context α_β

Example

Goal. Design a CFG for the language

$$L = \{a^m b^n c^{m-n} \mid m \geq n \geq 0\}$$

Strategy. Every word $\omega \in L$ can be split

and hence

- con
- generate $a^k \dots c^k$, i.e. same number of leadin c^s
- fill $\dots$ in the middle by $a^n b^n$, i.e. same nu
- use different non-terminals for both phases of th

Resulting Grammar. (productions only)

$$\begin{aligned} S &\rightarrow aSc \mid T \\ T &\rightarrow aTb \mid \epsilon \end{aligned}$$

Example ctd.

Grammar

$$S \to aSc \mid T$$

Exampl

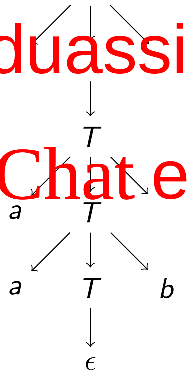
$$\begin{aligned} S &\Rightarrow aSc \\ &\Rightarrow aTc \\ &\Rightarrow aaTbc \\ &\Rightarrow aaaTbbc \\ &\Rightarrow aaabbc \end{aligned}$$

Parse Trees

Idea. Represent derivation as *tree* rather than as list of rule applications

- describes where and how productions have been applied
- generated word can be collected at the leaves

Example for the grammar that we have just constructed



A fun exam

http://

Parse Trees Carry Semantics

Take the code

```
if e1 then if e2 then s1 else s2
```


where **e1**, **e2** are boolean expressions and s1, s2 are subprograms.

Two Rea


```
if e1 t
```

and


```
if e1 then ( if e2 then s1 ) else s2
```

Goal. *unambiguous* interpretation of the code leading to *determined* and *clear* program execution.

Ambiguity

Recall that we can present CFG derivations as *parse trees*.

Until now t

A context

derived by one parse tree.

G is **ambiguous** iff there exists any word

than one parse trees.

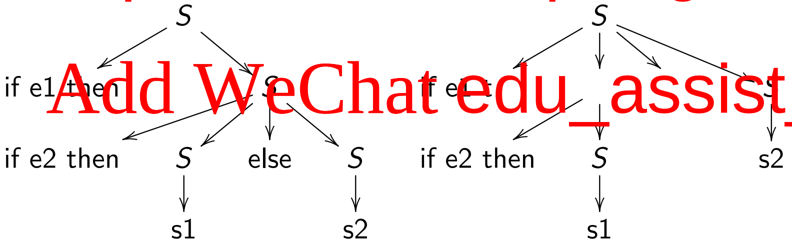
Example: If-Then and If-Then-Else

Consider the CFG

$$S \rightarrow \text{if } \textbf{bexp} \text{ then } S \mid \text{if } \textbf{bexp} \text{ then } S \text{ else } S \mid \textbf{prog}$$

where **bexp** and **prog** stand for boolean expressions and (if-statement free) prog

The string if e

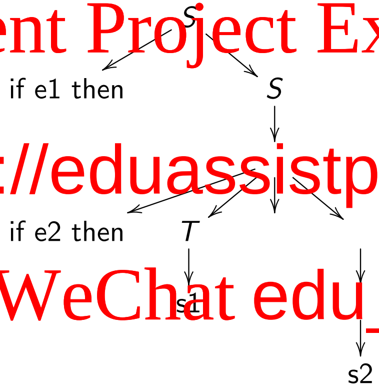


Example: If-Then and If-Then-Else

That grammar was **ambiguous**. But here's a grammar accepting the *exact sam*

There is now **only one** parse for if e1 then if e2 then s1
This is given on the next slide.

Example: If-Then and If-Then-Else



You **cannot** parse this string as if e1 then (if e2 then s1) else s2.

Reflecting on This Example

Observation.

- more than one grammar for a language
- some are ambiguous, others are not
- ambiguity is a property of *grammars*

Gramm

- am

- replace ambiguous grammar with unambiguous one

Choices for converting ambiguous grammars to u

- *decide* on just *one* parse tree
- e.g. `if e1 then ( if e2 then s1 ) else s2` vs `if e1 then ( if e2 then s1 else s2 )`
- in example: we have *chosen* `if e1 then ( if e2 then s1 else s2 )`

What Ambiguity Isn't

Question. Is the grammar with the following production ambiguous?

$$T \quad \to \quad \text{if } \mathbf{bexp} \text{ then } T \text{ else } S$$

Reasoning.

- Sup
- we c

A. This is *not* ambiguity.

- both options give rise to the *same* pa
- indeed, for context-free languages it ion is applied first.
- thinking about parse trees, both expansions happen in parallel.

Main Message. Parse trees provide a better representation of syntax than derivations.

Inherently Ambiguous Languages

Q. Can we always remove ambiguity?

Example: Language $L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$

Q. Why is t

A. Note t

- idea

- $S \to T \mid W$ where T is "left" and W is "right"

Complete Grammar:

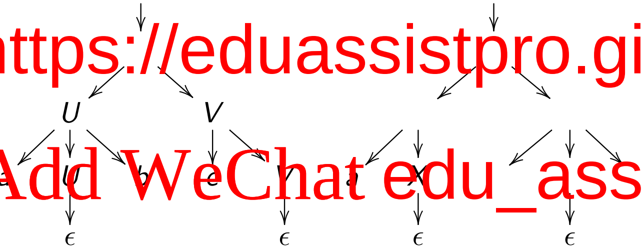
$$S \to T \mid W$$

$$\begin{aligned} T &\to UV \\ U &\to aUb \mid \epsilon \\ V &\to cV \mid \epsilon \end{aligned} \qquad \begin{aligned} W &\to XY \\ X &\to aX \mid \epsilon \\ Y &\to bYc \mid \epsilon \end{aligned}$$

Inherently Ambiguous Languages

Problem. Both left part $a^i b^i c^k$ and right part $a^i b^j c^j$ has non-empty intersection: $a^i b^i c^i$

Ambiguity where a, b and c are equi-numerous:



Fact. There is *no* unambiguous grammar for this language (we don't prove this)

The Bad News

Q. Can we *compute* an unambiguous grammar whenever one exists?

Q. Can we even *determine* whether an unambiguous grammar exists?

A. If we int
program

- There is *no* program that solves this proble
- input: CFG G, output: ambiguous or not. Thi
  undecidable

(More undecidable problems next week!)

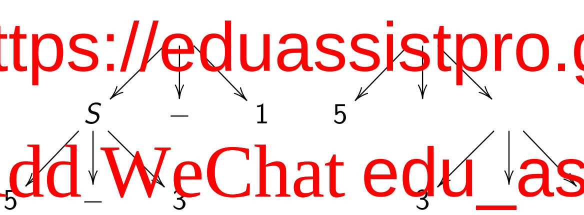
Example: Subtraction

Example.

$$S \quad\rightarrow\quad S - S \;\mid \mathrm{int}$$

- int stands for integers
- the intended meaning of $-$ is subtraction

Ambigu



Evaluation.

- left parse tree evaluates to 1
- right parse tree evaluates to 3
- so ambiguity matters!

Technique 1: Associativity

Idea for ambiguity induced by binary operator (think: $-$)

- prescribe "implicit parentheses", e.g. $a - b - c \equiv (a - b) - c$
- make operator associate to the left or the right

Left Associativity.

Result.

- $5 - \quad - \quad -$
- this is *left associativity*

Right Associativity.

$$S \rightarrow \textbf{int} - S \mid \textbf{int}$$

Idea. Break the symmetry

- one side of operator forced to lower level
- here: force right hand side of i to lower level

Example: Multiplication and Addition

Example. Grammar for addition and multiplication

$$S \to S * S \mid S + S \mid \mathbf{int}$$

Ambigu

- $1 + 2$ … nt results
- also

Take 1. The trick we have just seen

- strictly evaluate from left to right
- but this gives $1 + 2 * 3 \equiv (1 + 2) * 3$, *not* intended!

Goal. Want $*$ to have *higher precedence* than $+$

Technique 2: Precedence

Example Grammar giving $*$ higher precedence:

$$S \to S + T \mid T$$

$$T \to T * \textbf{int} \mid \textbf{int}$$

Given e.

- *forc*

- so $+$ w

Example. Derivation of $1 + 2 * 3$

- suppose we start with $S \Rightarrow T \Rightarrow T *$
- stuck, as cannot generate $1 + 2$ from

Idea. Forcing operation with *higher* priority to *lower* level

- three levels: S, (highest), T (middle) and integers
- lowest-priority operation generated by highest-level nonterminal

Example: Basic Arithmetic

Repeated use of $+$ and $*$:

$$S \to S + T \mid S - T \mid T$$

Main Differences.

- have *parentheses* to break operator priorit
- parentheses at *lowest* level, so *highest*
- lower-priority operator can be inside parentheses
- expressions of arbitrary complexity (no nesting in previous examples)

Example: Balanced Brackets

$S \quad \epsilon \quad (S) \quad SS$

Ambigu

- ass
  ... tw $\Rightarrow$
- indeed, *any* expression has *infinitely ma*

Reason. More than one way to derive ϵ.

Technique 3: Controlling ϵ

Alternative Grammar with only *one* way to derive ϵ:

$$S \to \epsilon \mid T$$

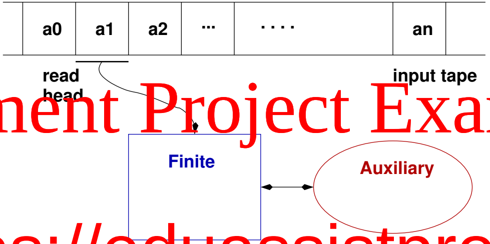
- ϵ can only be derived from S
- all other derivations go through T
- here: combined with multiple level technique
- ambiguity with ϵ can be easy to miss!

So Far.

- reg

Q. What automata correspond to *context fr*

General Structure of Automata



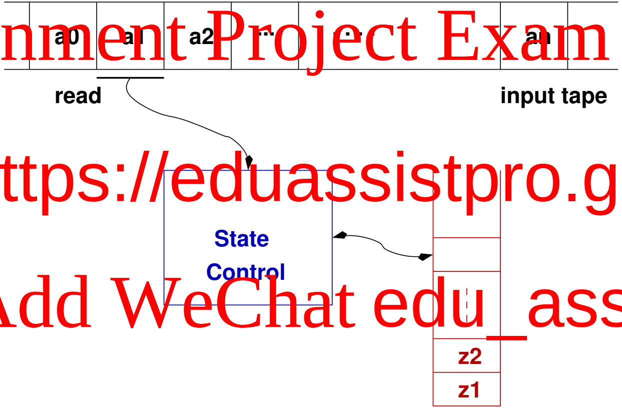
- *inp*
- *finite state control* is just like for DFAs / NFA
- symbols are processed and head advances
- new aspect: *auxiliary memory*

Auxiliary Memory classifies languages and grammars

- no auxiliary memory: NFAs / DFAs: regular languages
- *stack*: *push-down automata* : context free languages
- *unbounded tape*: Turing machines: all languages

39 / 59

Push-down Automata — PDAs



PDAs ctd.

Actions of a push-down automaton

- change of internal state
- pushing or popping the stack
- advance to next input symbol

Action de

- curr
- inp
- sym

Acceptance. The machine accepts if

- input string is fully read
- machine is in accepting state
- stack is *empty*

Variations.

- acceptance with empty stack: input fully read, stack empty
- acceptance with final state: input fully read, machine in final state

41 / 59

Example

Language (that cannot be recognised by a DFA)

$$L = \{a^n b^n \mid n \geq 1\}$$

- *can*
- *can*
- *can*

PDA design. (ad hoc, but showcases the idea)

- *phase 1:* (state S_1) *push* a's from the inpu
- *phase 2:* (state S_2) *pop* a's from the stack, if there is a b on input
- *finalise:* if the stack is empty and the input is exhausted in the final state (S_3), accept the string.

Deterministic PDA – Definition

Definition. A *deterministic PDA* has the form $(S, s_0, F, \ \Sigma, \ \Gamma, Z, \ \delta)$, where

- S is the set of *states*, $s_0 \in S$ is the *initial state* and $F \subseteq S$ are the *final states*;
- Σ is th
- Γ is th ... *ymbol*;
- δ is a (

$$\delta \ : \ S \times (\Sigma \cup \{\epsilon\})$$

$\delta \ : \ $ (state, input token or ϵ, top-of-st ... ck string)

Additional Requirement to ensure determinism:

- if $\delta(s, \epsilon, \gamma)$ is defined, then $\delta(s, a, \gamma)$ is undefined for all $a \in \Sigma$
- ensures that automaton has *at most* one execution

Notation

Given. Deterministic PDA with transition function

$$\delta : S \times (\Sigma \cup \{\epsilon\}) \times \Gamma \nrightarrow S \times \Gamma^*$$

$$\delta : (\text{state, input token or } \epsilon, \text{top-of-stack}) \nrightarrow (\text{new state, stack string})$$

Notatio

- writ
- σ is a *string* that replaces top stack symbol
- *final states* are usually underlined ($\underline{s}$)

Rationale.

- *replacing* top stack symbol gives just *one* operation for push and pop
- pop: $\delta(s, a, \gamma) = s'/\epsilon$
- push: $\delta(s, a, \gamma) = s'/w\gamma$

Two types of PDA transition

Input consuming transitions

- δ contains $(s_1, x, \gamma) \mapsto s_2/\sigma$
- aut
- sym

Non-consuming transitions

- δ contains $(s_1, \epsilon, \gamma) \mapsto s_2/\sigma$
- independent of input symbol
- can happen *any time* and does not consume input symbol

45 / 59

Example ctd.

Language $L = \{a^n b^n \mid n \geq 1\}$

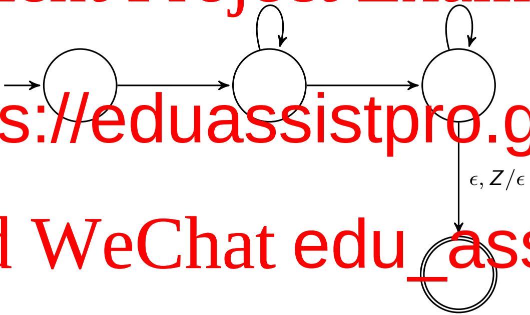
Push-down automaton

- starts with Z (initial stack symbol) on stack
- fina
- tran

$$
\begin{array}{llll}
\delta(S_0, a, Z) & \mapsto & S_1/aZ & \text{push first } a \\
\delta(S_1, a, a) & \mapsto & S_1/aa & \text{push f} \\
\delta(S_1, b, a) & \mapsto & S_2/\epsilon & \text{start} \\
\delta(S_2, b, a) & \mapsto & S_2\ /\ \epsilon & \text{pop fu} \\
\delta(S_2, \epsilon, Z) & \mapsto & \underline{S_3}/\epsilon & \text{accept}
\end{array}
$$

(δ is partial, i.e. undefined for many arguments)

Example ctd. – Diagrammatically



Example ctd. — PDA Trace

PDA configurations

- triples: *(state, remaining input, stack)*
- top of stack on the *left* (by convention)

Example Execution.

a)

a's)

$\Rightarrow (S_1, bbb, a$ (push f a's)

$\Rightarrow (S_2, bb, aa$ (start poppi a's)

$\Rightarrow (S_2, b, aZ)$ (pop further a's)

$\Rightarrow (S_2, \epsilon, Z)$ (pop further a's)

$\Rightarrow (\underline{S_3}, \epsilon, \epsilon)$ (accept)

Accepting execution. Ends in final state, input exhausted, empty stack.

Example ctd. — Rejection

PDA execution.

$$(S_0, aaba, Z) \Rightarrow (S_1, aba, aZ)$$

Non-accepting execution

- No transition possible, stuck without reaching
- rejection happens when transition function is undefined for current configuration (state, input, top of stack)

Example: Palindromes with ‘Centre Mark’

Example Language.

$$L = \{wcw^R \mid w \in \{a, b\}^* \wedge w^R \text{ is } w \text{ reversed}\}$$

Determ

- Pus
- Wh
- Now try to match the tokens we are reading with th the stack, popping as we go;

- If the top of the stack is the empty stack symbol the final state via an ϵ-transition. Hopefully our input has been used up too!

Exercise. Define this formally!

Non-Deterministic PDAs

Deterministic PDAs

- transitions are a partial function

$$\delta : S \times (\Sigma \cup \{\epsilon\}) \times \Gamma \nrightarrow S \times \Gamma^*$$

$$\delta : (\text{state}, \text{input token or } \epsilon, \text{top-of-stack}) \nrightarrow (\text{new state}, \text{stack string})$$

- side c

Non-De

- transitions given by *relation*

$$\delta \subseteq S \times (\Sigma \cup \{\epsilon\})$$

- no side condition (at all).

Main differences

- for deterministic PDA: *at most* one transition possible
- for non-deterministic PDA: *zero or more* transitions possible

Non-Deterministic PDAs ctd.

Finite Automata

- non-determinism is *convenient*
- but doesn't give extra power (subset construction)
- can convert every NFA to an equivalent DFA

Push-do

- non
- can A
- there are context free languages that can
  non-deterministic PDA
- intuition: non-determinism allows "guessin

Grammar / Automata correspondence

- non-deterministic PDAs are more important
- they correspond to context-free languages

Example: Even-Length Palindromes

Palindromes of even length, *without* centre-marks

$$L = \{ww^R \mid w \in \{a,b\}^* \wedge w^R \text{ is } w \text{ reversed}\}$$

- this is a context-free language
- *cannot* be recognised by deterministic PDA
- intu
  wor

Non-de

$$\delta(s, \epsilon, \gamma) \quad =$$

- $x \in \{a, b, Z\}$, s is the 'push' state and

Intuition

- "guess" (non-deterministically) whether we need to enter "match-and-pop"-state
- automaton gets stuck if guess is not correct (no harm done)
- automaton accepts if guess is correct

Grammars and PDAs

Theorem. Context-free languages and *non-deterministic* PDAs are equivalent

- for e
- if L

Proof. We only do one direction: construct PDA fro

- this is the "interesting" direction for parser gene
- other direction quite complex.

From CFG to PDA

Given. Context-Free Grammar $G = (V_t, V_n, S, P)$

Construct non-deterministic PDA $A = (Q, Q_0, F, \Sigma, \Gamma, Z, \delta)$

States. Q_0 (initial state), Q_1 (working state) and $\underline{Q_2}$ (final state).

Alphabe

Stack Al

Initialisation.

- push start symbol S onto stack, enter wor
- $\delta(Q_0, \epsilon, Z) \mapsto Q_1/SZ$

Termination.

- if the stack is empty (i.e. just contains Z), terminate
- $\delta(Q_1, \epsilon, Z) \mapsto \underline{Q_2}/\epsilon$

From CFGs to PDAs: working state

Idea.

- build the derivation on the stack by expanding non-terminals according to productions productions
- if a terminal appears that matches the input, pop it
- terminate, if the entire input has been consumed

Expand

- non [illegible] productions
- $\delta(Q_1, \epsilon, A) \mapsto Q_1/\alpha$ for all productions

Pop Terminals.

- terminals on the stack are popped if they match the input
- $\delta(Q_1, t, t) \mapsto Q_1/\epsilon$ for all terminals t

Result of Construction. *Non-deterministic* PDA

- may have more than one production for a non-terminal

Example — Derive a PDA for a CFG

Arithmetic Expressions as a grammar:

$$
\begin{aligned}
S &\rightarrow S + T \mid T \\
T &\rightarrow T * U \mid U
\end{aligned}
$$

1. *Initialis*

$$\delta(Q_0, \epsilon, Z) \mapsto$$

2. *Expand non-terminals:*

$$
\begin{aligned}
\delta(Q_1, \epsilon, S) &\mapsto Q_1/S + T & \delta(Q_1, \epsilon, T) &\mapsto Q_1/U \\
\delta(Q_1, \epsilon, S) &\mapsto Q_1/T & \delta(Q_1, \epsilon, U) &\mapsto Q_1/(S) \\
\delta(Q_1, \epsilon, T) &\mapsto Q_1/T * U & \delta(Q_1, \epsilon, U) &\mapsto Q_1/\textbf{int}
\end{aligned}
$$

CFG to PDA ctd.

3. *Match and pop terminals:*

$$\delta(Q_1, +, +) \mapsto Q_1/\epsilon$$

$$\delta(Q_1,),))$$

4. *Terminate:*

$$\delta(Q_1, \epsilon, Z) \mapsto \underline{Q_2}/\epsilon$$

Example Trace

$$
\begin{aligned}
(q_0,\ \textbf{int} * \textbf{int},\ Z) &\Rightarrow (Q_1,\ \textbf{int} * \textbf{int},\ SZ) \\
&\Rightarrow (Q_1,\ \textbf{int} * \textbf{int},\ TZ) \\
& (Q_1,\ \textbf{int}\ \ \textbf{int},\ T\ \ UZ) \\
&\Rightarrow (Q_1,\ \textbf{int},\ UZ) \\
&\Rightarrow (Q_1, \\
&\Rightarrow (Q_1,\ \epsilon, \\
&\Rightarrow (Q_2,\ \epsilon,\ \epsilon \\
&\Rightarrow \mathit{accept}
\end{aligned}
$$