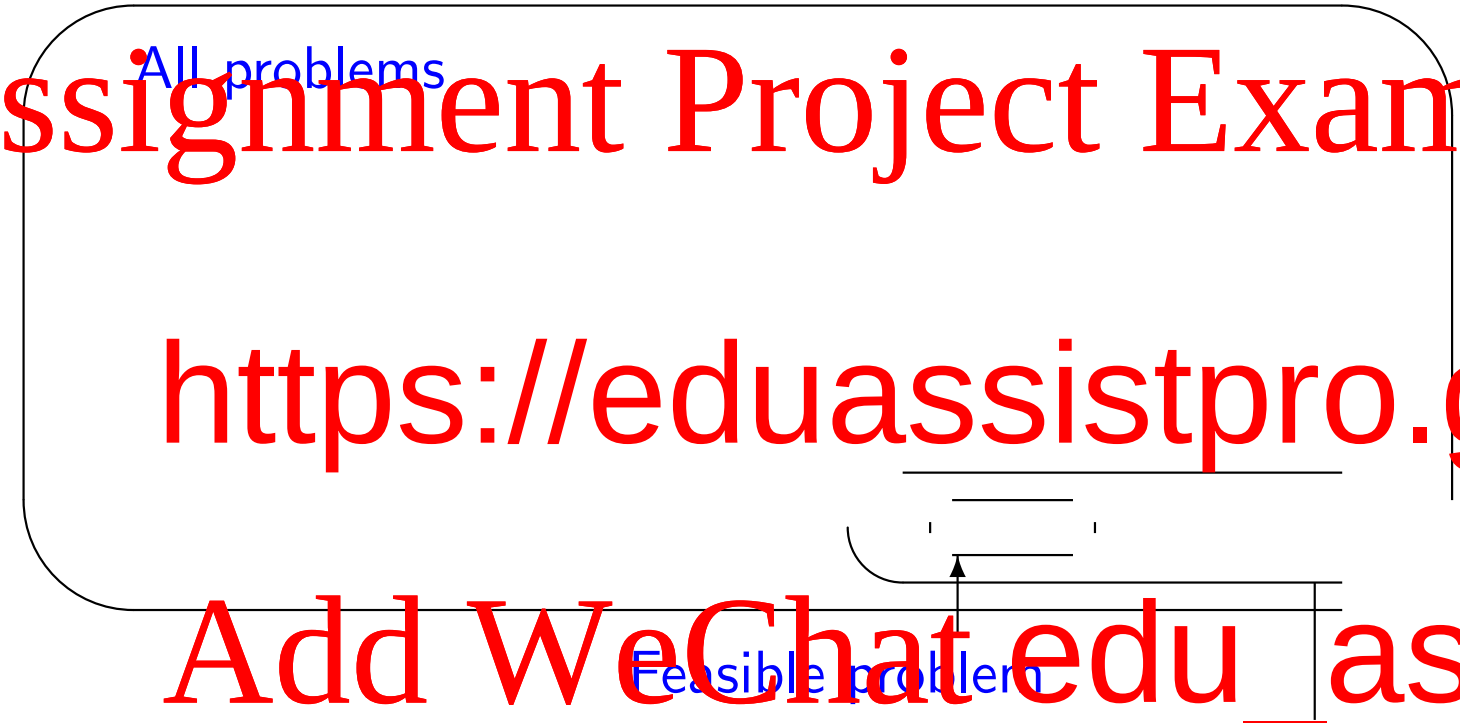
Turing Machines: Limits of Decidability

COMP1600 / COMP6260

Semester 2, 202

Efficient Computation



- Each of the inner sets is a tiny proportion of the set that contains it.

Time Complexity

Q. It's good about solving problems in general. How about efficiency?

Inverting booleans. Easy: constant time.

Multiplication of integers. Easy: polynomial time

- takes time that is proportional to the square of the total number of digi
- if we d
- if *n* *of* *the o*

Matrix Multiplication. Polynomial of order

Theorem Proving. Hard, sometimes of order 2

Feasible Problems.

- can be solved in *polynomial time*, i.e. in time of the order n^k, for some k
- n is the length of (the representation of) the input.

P vs. NP: Signpost

Polynomial Time. The complexity class P

- problems (languages) that can be decided in the order of n^k steps
- usually considered feasible

Non-deterministic polynomial time. The complexity class NP

- pro e
- alte

Example

- give
- can guess solution (assignment)
- alternatively, can verify correctness of assign

As a slogan. Coming up with a solution within a time bound intuitively harder than checking someone else's solution in that time.

Big Open Problem. Is $P = NP$?

- most important open problem in our discipline
- $1,000,000$ prize by the Clay maths foundation

More Detail

Computational Problem.

- given by a *Language* L, say
- Question: for a string w, is w L?

Solution

- alw
- and accepts w if and only if $w \in L$.

Time Complexity

- Given w, can count the number of steps of
- this defines a function $f(w)$ *dependent on the input*

Time Complexity – Abstraction

Problem. Number of steps function usually *very complicated*

- for example, $n^{17} + 23n^2 - 5$
- and hard to find in the first place

Solution

- foc
- as we

Landau Symbols. for f and g functions on na

- $f \in \mathcal{O}(g)$ if $\exists c.\exists n_0.\forall n \geq n_0(f(n) \leq c$
- "for large n, g is an upper bound to f

Idea. Abstract details away by just focussing on upper bounds

- e.g. $n^{17} + 23n^2 - 5 \in \mathcal{O}(n^{17})$

Landau Symbols: Examples

Examples.

- Polynomials: leading exponent dominates
  - e.g. $x^n +$ lower powers of x $\quad (x^n)$
- Exp
- e.g. 2

Important Special Cases.

- *linear*. f is linear if $f \in \mathcal{O}(n)$
- *polynomial*. f is polynomial if $f \in \mathcal{O}($
- *exponential*. f is exponential if $f \in \mathcal{O}(2^n)$

Important Special Cases, Graphically

(Image copyright Lauren Kroner)

Application to Computational Problems

Definition.

A computational problem (given by a language L) is *in* $\mathcal{O}(f)$ if there is a Turing machine M that

- always terminates, and accepts precisely all strings in L
- on every input string of length n, terminates in $g(n)$ steps and $g \in$

Example

- Given: regular language L
- Question: what's the complexity of deciding wh

More Detail.

- need to construct a Turing machine that decides whether $w \in L$ or not.
- how many steps (depending on length of input string) does M take?

Application to Computational Problems

Definition.

A computational problem (given by a language L) is *in* $\mathcal{O}(f)$ if there is a Turing machine M that

- always terminates, and accepts precisely all strings in L
- on every input string of length n, terminates in $g(n)$ steps and $g \in$

Example

- Given: regular language L
- Question: what's the complexity of deciding wh

More Detail.

- need to construct a Turing machine that decides whether $w \in L$ or not.
- how many steps (depending on length of input string) does M take?

A. This is *linear*. Think of finite automata.

Example: Graph Connectedness

Reminder. A *graph* is a pair $G = (V, E)$ where

- V is the set of *vertices* of the graph
- E is the set of *edges*, a collection of two-element subsets of V

Example.

Formally. $G = (V, E)$ with

- $V = \{0, 1, 2, 3, 4\}$
- E consisting of $\{0, 2\}$, $\{0, 1\}$, $\{0, 3\}$, $\{1, 2\}$, $\{1, 3\}$, and $\{3, 4\}$.
- Note: Edges are *not directional*.

Connected and non-connected Graphs

Definition. A graph is *connected* if there is a path between any two nodes.

Connected Graph Example.

Non-Connected Graph Example.

The Connected Graph Problem

Problem. Given a graph $G = (V, E)$

- is G connected?
- what is the complexity of deciding whether G is connected?

Algorithm.

- Pic
- do a b
- if the t
  the graph, we know that G is connecte

By Church-Turing Thesis. The problem "Is computable.

Q. How many steps does an algorithm take to figure this out?

- number of steps refers to "on a Turing machine"
- but input to TMs are strings, not graphs ...

Coding of Graphs as Strings

Recall. We have coded TM transition tables as strings.

Coding of graphs:

- vertices: numbered 0 to n, can be coded by n in binary
- single edge: pair (n, k), can be coded by $0\underline{1}\ldots\underline{0}\#1\underline{1}\ldots\underline{1}$.
- set of

Complete Graph

$10\ldots1\#\#1\ldots1\#1\ldots0\#\#$

- green number of vertices
- blue set of edges
- black separators

Question, reloaded.

Computational Problem. Given a graph $G = (V, E)$, how many steps does a TM take to determine whether the *encoding* of G is a connected graph?

- the e
- nu

Difficulty. Exact answers require *way too mu*

- Landau symbols $\mathcal{O}(\cdot)$ allow us to be "genero
- required answer of the form "in $\mathcal{O}(f)$

Complexity Analysis

Algorithm in Turing machine form.

- pick vertex 0 initially.
- designate vertex 0 as "to explore" (e.g by writing its binary encoding to th
- for e
  -
  - if a connected vertex is neither explored nor marked "to explore", mark it as "to explore" (e.g. by writing its binary encod the input, with a special separator)
  -
  - mark the vertex v as "explored"
- check that the number of vertices found is equal to the number of vertices in the graph.

Worst Case Analysis

Worst Case for a given graph G with n vertices

- When exploring a vertex, need to check n^2 edges
- For every edge checked, one more vertex to explore
- this needs to be done for every vertex

High Leve

- nee

Overhea

- checking whether two vertices match:
  - at most n (in fact $\log n$) bitwise comp
  - and going back and forth over the tape, at most
- adding another vertex to the list: *pol*
  - at most n bits to add
  - and going back and forth over the tape, at most $n^2 \cdot n$ steps

Summary. Polynomial Complexity

- polynomially many "high level" steps
- each of which takes polynomial time

Other Problems: Propositional Satisfiability

Given. A propositional formula, constructed from $\wedge$, $\vee$, $\rightarrow$, $\neg$, T, F and variables.

Question. Is there a truth value assignment to the propositional variables such that t

Naive Alg

- loo

Questions

1. How many truth assignments do we need to check?
2. How do we measure the size of the input?
3. What is the worst case complexity of this algorithm?

Complexity Class: Polynomial Time

Definition. The class **P** of *polynomial time* decision problems consists of all problems that can be answered in time *polynomial* in the input

- of order $\mathcal{O}(n), \mathcal{O}(n^2), \mathcal{O}(n^3), \ldots$

Example

- che
- check whether a list is sorted
- check whether a propositional formula is true for

Last Problem. Have *two* inputs

- need only *one line* of the truth table
- according to the valuation given

Other Problems: Propositional Satisfiability

Given. A propositional formula, constructed from $\wedge$, $\vee$, $\rightarrow$, $\neg$, T, F and variables.

Coding:

- have new tape symbols for $\wedge$, $\rightarrow$, etc.
- ass

Worst Ca

- vari $p_3 \wedge \ldots)$
- *exponentially* many valuations

This Algorithm

- *at least* exponential, e.g. $\mathcal{O}(2^n)$
- and in fact exponential

Q. Can we do better?

Other Problems: Propositional Satisfiability

Given. A propositional formula, constructed from $\wedge$, $\vee$, $\rightarrow$, $\neg$, T, F and variables.

Coding:

- have new tape symbols for $\wedge$, $\rightarrow$, etc.
- ass

Worst Ca

- vari $p_3 \wedge \ldots)$
- *exponentially* many valuations

This Algorithm

- *at least* exponential, e.g. $\mathcal{O}(2^n)$
- and in fact exponential

Q. Can we do better?

A. Probably not . . . this is the $1,000,000 Clayton math problem

Propositional Satisfiability

Verifying whether a formula evaluates to T for an assignment

- takes polynomial time

Determining whether a satisfying assignment exists

- tak
- *but*

Observation. The (coding of) a valuation is *polynomially large*

- in fact, shorter than the formula, a sequence of 0s an

Non-Deterministic Machines (informally)

- like non-deterministic finite automata: more than one transition possible
- propositional satisfiability: *guess* a valuation, then check

Complexity Class: Nondeterministic Polynomial Time

Definition. The class **NP** of *non-deterministic polynomial time* decision problems consists of all decision problems L that can be solved by a *non-deterministic* Turing machine in polynomial time

- we don't define this type of machine formally
- idea
- acc

Alternati L there exists a *certificate* c such that

- c is of polynomial length (in the length of
- determining whether c is a certificate for

Example. Propositional Satisfiability

- certificates are valuations
- checking the formula under a valuation is polynomial

More Problems

The Independent Set Problem Assume you want to throw a party. But you know that some of your friends don't get along. You only want to invite people that *do* get along.

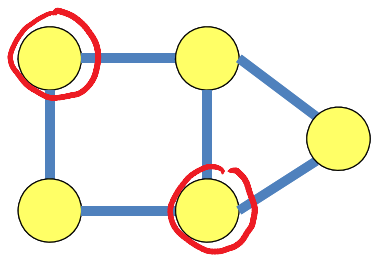
As a Graph:

- vert
- dra

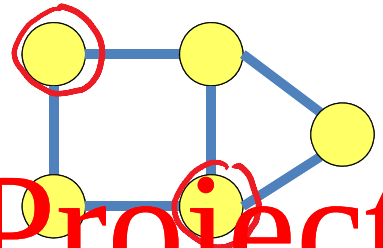
Problem I of $\geq k$ vertices so that

- no two elements of I are connected with a
- i.e. everybody in I gets along

Example of an independent set of size 2



Independent Set: Naive Algorithm



- loo
- and c

Alternative Formulation using guessing:

- *guess* a subset of vertices of size $\geq k$
- *check* whether it is an independent set

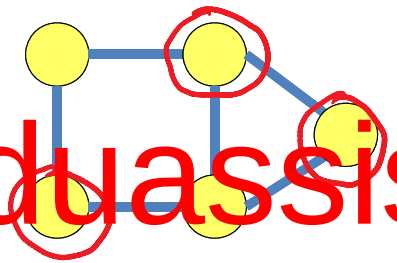
Complexity. Independent Set is in **NP**

- represent subsets as bit-vectors (certificates)
- checking is polynomial

Vertex Cover

Vertex Cover. Given a graph $G = (V, E)$, a *vertex cover* is a set C of vertices such that every edge in G has at least one vertex in C.

Example.



Vertex Cover Problem. Given a graph G, what is the smallest vertex cover of size $\leq k$?

Naive Algorithm.

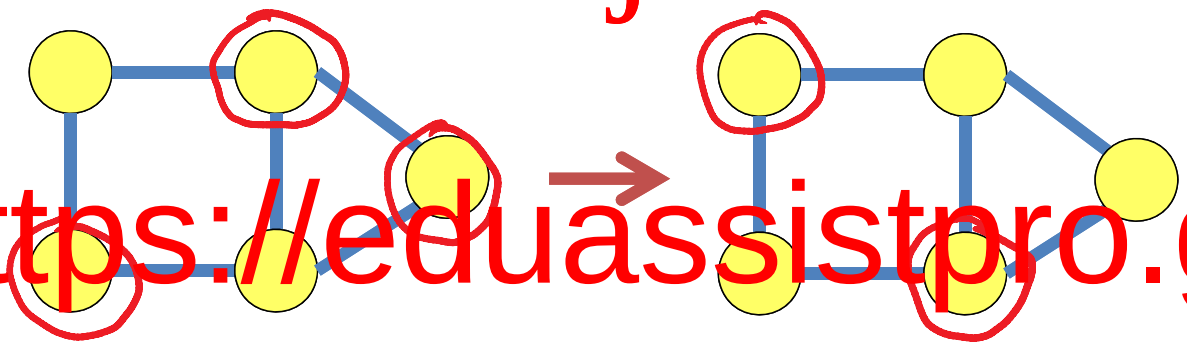
- search through all subsets of size $\leq k$
- check whether it's a vertex cover

From Independent Set to Vertex Cover

Reductions. Use solutions of one problem to solve another problem

Observation. Let G be a graph with n vertices and $k \geq 0$.

- G has a v. c. of size $\leq k$ iff G has an i. s. of size $n - k$



Reduction. A *polynomial reduction* from decision problem A to decision problem B is a function f that transforms instances such that

- $w \in A \iff f(w) \in B$ and f is computable in polynomial time

Example. Vertex cover to independent set

$$(G, k) \mapsto (G, n - k)$$

where n is the number of vertices of G

Reductions and Difficulty

Recall. A reduction from decision problem A to B is a polytime function f such that

Informal

- Giv
- Compute $f(w)$, and decide whether f

Q. If A is reducible to B, which of A and

NP-Completeness

Q. What is the "hardest" or most difficult **NP**-problem?

A. It's a problem that all other **NP**-problems can be reduced to

- a solution would yield solutions to *all* **NP**-problems
- recall that B is more difficult than A if A can be reduced to B

NP-Har

- A de an be reduced to it.
- A decision problem is NP-complete if it is NP-har

Hard Theorem. (Stephen Cook 1974) The propositi problem is NP-complete

- have seen that satisfiability is in **NP**
- hard part: reduce *all* **NP**-problems to satisfiability.

The P vs NP Problem

Big Open Question. is **P** = **NP** or not?

- Given that propositional satisfiability is NP-complete
- "all

Ramifica

- could break public-key cryptography
- this includes https-protocoll
- could solve optimisation problems efficiently
- lots of AI (learning) problems have *fast* solutions

Summary.

Undecidable Problems.

- Problems for which we cannot find an algorithmic answer
- most famous: halting problem – determine whether a computation terminates

Efficientl

- usu

More difficult Problems. Polynomial time with gues

- complexity class **NP**, not considered feasi
- **NP**-complete problems, like propositional s

Open Problem. Is $\mathbf{P} = \mathbf{NP}$ or not?

- neither have proof nor counter-example
- most important open problem in the discipline

Weighted Graphs

Definition. A *weighted graph* is an undirected graph where every edge is (additionally) labelled with a non-negative integer.

Formally: A *weighted graph* is a triple $G = (V, E, l)$ where

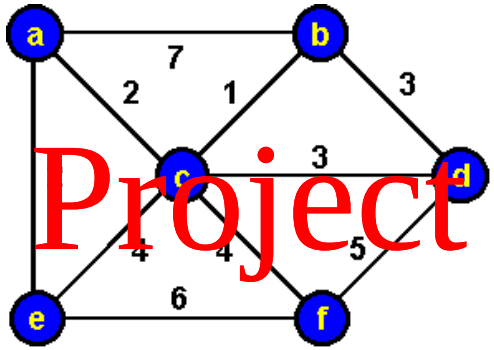
- (V, E) is an undirected graph (with vertices V and edges E)
- $l : E$ is a labelling function that assigns a *weight* (or cost) to each

Example

Intuition.

- the vertices can be locations
- the edges indicate whether we can travel between two locations
- the labels indicate the cost of travelling between two locations

Travelling Between Two Nodes



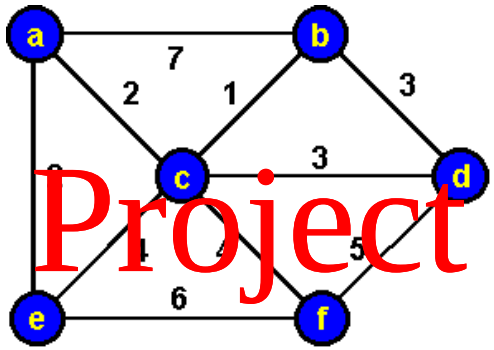
Q. Given t
from v_1 2

In More detail.

- as a computation problem: given vertices t
  path that connects v_1 and v_2
- as a decision problem: given graph G and $k \geq 0$, is there a path that connects vertices v_1 and v_2 of total cost $\leq k$?

Q. Easy or hard? Solvable in polynomial time?

Dijkstra's Algorithm



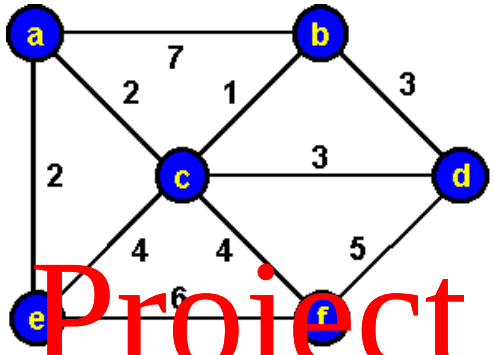
Main Idea

- to fin
  intermediate nodes
- more general problem: shortest path between x v_2
- a bit like breadth-first search

Data Structures. Given a start vertex s

- `cheapest[v]`: For every vertex v, the "price" of the "cheapest" path from s to v
- `explored[v]`: a boolean marker whether `v` has been explored

Algorithm Detail for source vertex s



Initialisation.

- set
- set

Iteration

1. Select a vertex in v that is *not explored* and *minimal*:
   - `explored[v]` $=$ `false`
   - `cheapest[v]` $\leq$ `cheapest[w]` for all
2. for each vertex w that is adjacent to
   - update, if the path $s \rightsquigarrow v \rightarrow w$ is cheaper, that is
   - if `cheapest[v] + cost(v, w)` $\leq$ `cheapest[w]`
   - then `cheapest[w] = cheapest[v] + cost(v, w)`
3. mark `v` as explored: `explored[v] = true`
   - once a node `v` is marked explored, `cheapest[v]` is the price of the cheapest path from source to `v`

Dijkstra's Algorithm: Correctness

Invariant. if E is the set of explored nodes, then

- for $e \in E$, `cheapest[e]` is the cost of cheapest path from source to `e`
- for $u \in V \setminus E$, `cheapest[u]` is the cost of the cheapest path to u that only visits explored nodes.

True after Initialisation.

- trivi

Invariant

- pick
- cheapest path from source to u *cann*
- after (possible) update: `cheapest` is st explored vertices

At End of Iteration.

- `cheapest[v]` is cost of cheapest path from source to `v`

Recognise the While-Rule in Hoare Logic?

- *could* formalise this in Hoare logic
- here: Hoare-Logic as informal guidance principle

Dijkstra's Algorithm: Complexity

Worst Case Focus.

- need to explore all nodes

In Every Iteration

- pos

Overall C

- run t
  - find minimal unexplored vertex – n
  - update `cheapest[v]` – n steps
- so overall $\mathcal{O}(n^2)$ steps

Low-Level Operations are "harmless" (polynomial)

- comparing two n-bit binary numbers
- marking / checking explored status

Shortest Paths

Decision Problem. Given G, v_1, v_2 and $k \geq 0$ is there a path of cost $\leq k$ from v_1 to v_2?

- run $t[v_2] \leq k$

Comput h from v_1 to v_2

- Dijkstra's algorithm only gives cost
- paths needs to be re-constructed

- idea: remember *penultimate* nodes

Computing Shortest Paths

Initialisation.

- set `cheapest[s] = 0`, `cheapest[v] = undef` for `v` $\neq$ `s`
- set `explored[v] = false` all $v \in V$
- set `penultimate[v] = undef`, all $v \in V$

Iteration

1. Select a
   - `explored[v] = true`
   - `cheapest[v]` $\leq$ `cheapest[w]` for all `w` with `explored[w]` $=$ `false`
2. for each vertex w that is adjacent to
   - update, if the path $s \rightsquigarrow v \rightarrow w$ is ch
   - if `cheapest[v] + cost(v, w)` $\leq$
   - then `cheapest[w] = cheapest[v] + cost(v, w)`
   - and put `penultimate[w] = v`
3. mark `v` as explored: `explored[v] = true`
   - once a node `v` is marked explored, `cheapest[v]` is the price of the cheapest path from source to `v`

Path Reconstruction

Idea.

- `penultimate[v]` is the *penultimate* node of a cheapest path from `source` to `v`

Reconstruction of a path from `source` to `v`

- initialisation: the *last* node is `v`
- iter
  -
- termination: if the constructed path is of the form $\texttt{source}, \ldots, \texttt{v}_\texttt{n}$

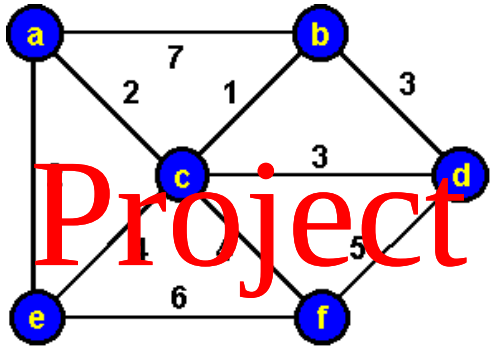
Complexity

- reconstruction phase: at most n additi
- iteration phase: adds constant overhead
- so overall, still polynomial

How About Coding Graphs as Strings?

- our analysis in terms of *number of vertices*
- bounded above by length of encoding

Travelling Salesman Problem



Given. A we

- vert
- edges represent travel time

Q. Find a path $v_1 \to v_2 \to \cdots \to v_n \to v_1$

- that begins and ends in the same city
- that visits each city *exactly once*
- for which the overall travel time is minimal.

Q. Easy or hard? Solvable in Polytime?

Travelling Salesman: Naive Approach

Naive Algorithm Given n cities, and their distances $\texttt{dist}(c, d)$

- initialise: construct set S consisting of all possible sequences of nodes
  - ▶
  - ▶
- com
- for each $s = (c_1, \ldots, c_n) \in S$, compute the total distance $\sum_i \texttt{dist}(c_i, c_j)$
- report the smallest such distance

Q. What is the complexity of this algorithm?

Travelling Salesman: Naive Approach

Counting Permutations of n vertices

- n possibilities for 1st city, $n-1$ for 2nd, $n-2$ for third ...
- overall: $n!$ different paths to check

Estimate

- sim

size	tim
20	7
40	$2.6 \cdot 10$
80	$2.3 \cdot 10$

Q. What is the unit of time in the right-hand column?

(we have seen this before)

Travelling Salesman as Decision Problem

Travelling Salesman as Decision Problem. Given weighted graph G and $k \geq 0$, is there a path that

- visits each vertex in G exactly once
- is of total cost $\leq k$

Guessing

- Cer
- of po
  per vertex)

Certificate Checking. As for Hamiltonian paths, plus

- check that the total cost of the path is $\leq$
- $n - 1$ additions, one comparison – *polynomial*
- so checking is polynomial, overall

Corollary. Travelling Salesman *as a decision problem* is in NP.

The Knapsack Problem

Given: A knapsack with maximal capacity C and items with weights $w_1, \ldots, w_n$ and values $v_1, \ldots v_n$

What's th

- sum
- sum

Assumptions.

- all weights are strictly positive, i.e. w_i
- there is an unlimited supply of each item

Knapsack: Main Idea

Construct a Table

0	1	2	...	C
$m(0)$	$m(1)$	$m(2)$	...	$m(C)$

- $m(w)$ = value of the "best" knapsack with weight w
- $m(0$
- $m($

Solution

Correctness Argument.

- the "best" knapsack with weight w [illegible] i
- removing this item, we get the best knapsack with [illegible] w_i (otherwise, can have better knapsack with weight w)

Solution.

- iteratively compute $m(0), m(1), \ldots, m(C)$
- store already computed values in a table (don't recompute)

Knapsack: Complexity Analysis

Complexity Analysis given capacity C and n items

- nee
- for e ute ma
- Overall complexity: $n \cdot C$

Q. Does that mean that knapsack is *linear*?

Knapsack: Encoding

Recall. Complexity depends on *encoding* of input data

- usual encoding for numbers: as binary strings

For knapsack:

- Capacity: C as a binary integer ($\log C$ bits)
- nu
- wei
- valu

Q. How large is C as a function of the *len*
knapsack problem *in the worst case*?

Knapsack: Encoding

Recall. Complexity depends on *encoding* of input data

- usual encoding for numbers: as binary strings

For knapsack:

- Capacity: C as a binary integer ($\log C$ bits)
- nu
- wei
- valu

Q. How large is C as a function of the *len*
knapsack problem *in the worst case*?

A. In the worst case:

- one item, value 1, weight 1: 3 bits (plus separators)
- encoding of C has $\log C$ many bits, so $C = 2^{\log C}$ is *exponential*

Overall Complexity. *Exponential* in the size of the problem

- if numbers are coded in binary
- only polynomial if C is coded in unary – also called *pseudo polytime*

Summary

Bad News.

- there are *lots* of problems that are undecidable about TMs
- Rice's theorem gives plenty – any property of languages

But ...

- spe
- ofte

Example. Provability in First-Order Logic

- not decidable, but there are plenty of implement
- implementations may not terminate ...
- goal: try and solve *as many problems as possible*
- problem of interest: usually human-generated

Course Summary

Alignment

- you can already do programming (functional / imperative)

Questions.

- Wh
- Wh

Answers.

- use *logic* to formally describe systems

- functional programs: induction proofs
- imperative programs: Hoare logics
- computation in general: Turing machines