Structural Induction

COMP1600 / COMP6260

Semester 2, 202

Induction on Lists

Q. How do we *make* all finite lists?

A. All lists (over type A) can be obtained via the following:

- the empty list [] is a list (of elements of type A)
- give ... as with a is a list (

That is, list

Q. How do we prove a property $P(l)$ for *a*

A. We (only) need to prove it for list constructed as abov

- establish that the property holds for [], i.e. $P([])$
- if as is a list for which $P(as)$ holds, and a is arbitrary, show that $P(a : as)$ holds.

Making and Proving in Lockstep

Suppose we want to establish that $P(as)$ holds for all lists as.

Stage 0. $as = []$.

- need to establish $P([])$.

Stage 1.

- nee
- alre

$\dots$

Stage $n + 1$. $as = a : as'$ has length $n + 1$

- need to establish that $P(a : as')$
- already know that $P(as')$ and may use this knowledge

$\boxed{\text{May use the fact that } P(as) \text{ holds for lists constructed at previous stage}}$

List Induction, Informally

To prove that $\forall as. P(as)$ it suffices to show

Base Case. $P([])$, i.e. P holds for the empty list

Step Case. $\forall a. \forall as. P(as) \to P(a : as)$

- assuming that $P(as)$ holds for all lists as (considered at previous stag
- sho

Example.

$P([1, 3, 4, 7])$ follows from $P([$

$P([3, 4, 7])$ follows from $P([$

$P([4, 7])$ follows from $P([7])$ by step case

$P([7])$ follows from $P([])$ by step case

$P([])$ holds by base case.

Induction on Structure

List Induction as a proof rule:

Annotat

$$\frac{P([\,] :: [a]) \qquad \forall (x :: a).\ \forall (xs :: [}{\forall (xs :: [a]).\ P}$$

Standard functions

Recall the following (standard library) function definitions:


```
length []        = 0                      -- (L1)
le

ma    [          = []                     -- (M1)
map f (x:xs)     = f x : map f xs         -- (M2)

[]     ++ ys     = ys                     -- (A1)
(x:xs) ++ ys     = x : (xs ++ ys)         -- (A2)
```


We read (and use) each line of the definition as *equation*.

Example. Mapping over Lists Preserves Length

Show. $\forall$`xs.length (map f xs) = length xs`.

Need to establish both premises of induction rule:

- $P([])$
- $\forall x.$

Base Case: $P([\,])$


```
length (map f [])    = length []
```

Both sides are equal by `M1`: `map f [] = []`.

Step Case: $\forall x.\ \forall xs.\ P(xs) \to P(x : xs)$

Induction Hypothesis. Assume for an arbitrary list *as* that

```
length (map f as)        = length as      -- (IH)
```


Proof Goal. For arbitrary *a*, now prove that $P(a : as)$, i.e.

```
length (ma    (
```



```
length (map f (a
     = len
     = 1 + length (map f as)    -- by (L2)
     = 1 + length as            -- by (IH)
     = length (a:as)            -- by (L2)
```


Formally (using $\to I$ and $\forall I$)

- this gives $P(as) \to P(a : as)$
- as both *a* and *as* were arbitrary, have $\forall x. \forall xs. P(xs) \to P(x : xs)$

In terms of Natural Deduction

Fixing arbitrary a and as and assuming $P(as)$, we show $P(a : as)$. That is, we reason as follows:

6	$P(a : as)$	
7	$P(as) \to P(a : a$	
8	$\forall xs.\ P(xs) \to P(a :$	
9	$\forall x.\ \forall xs.\ P(xs) \to P(x : xs)$	$\forall$-I, 8

Concatenation

Show: `length (xs ++ ys) = length xs + length ys`

- statement contains *two* lists: *xs* and *ys*
- but induction principle only allows for *one*?

Formally. Show that

$$\forall \underbrace{\qquad\qquad}\;\underbrace{\qquad\qquad} .$$

Equivalent Alternative.

$$\forall ys. \underbrace{\forall xs. \texttt{length (xs ++ ys) = length xs + lengt}}_{P(ys)} .$$

As a slogan.

- list induction allows us to induct on one list *only*.
- the other list is treated as a constant.
- but on which list should we induct?

List Concatenation: Even more Options!

Show: `length (xs ++ ys) = length xs + length ys`

Option 1. Do induction on xs

$$\forall xs.\, \underbrace{\forall ys.\, \texttt{length (xs ++ ys) = length xs + length ys}}_{P(xs)}$$

Option 2.

$$\forall ys.\, \underbrace{\forall xs.\, \texttt{length (xs ++ ys) = length xs + length ys}}_{P(ys)}.$$

Option 3. Fix an arbitrary ys and show the below, the

$$\forall xs.\, \underbrace{\texttt{length (xs ++ ys) = length xs + length ys}}_{P(xs)}$$

Option 4. Fix an arbitrary xs and show the below, then use $\forall I$

$$\forall ys.\, \underbrace{\texttt{length (xs ++ ys) = length xs + length ys}}_{P(ys)}.$$

Choosing the most helpful formulation

Problem. For `length (xs ++ ys) = length xs + length ys`

- induct on *xs* (and treat *ys* as a constant), or
- induct on *ys* (and treat *xs* as a constant)?

Clue. Look at the definition of `xs ++ ys`:

```
[]      ++ ys   = ys                     -- (A1)
(x
```


- the list *xs* (i.e. the first argument of
- the second argument (i.e. `ys`) remains

Approach. Induction on *xs* and treat *ys* a

$$\forall xs.\underbrace{\forall ys.\texttt{length (xs ++ ys) = length xs + length ys}}_{P(xs)}.$$

The Base Case

Given.

```
length []       = 0                     -- (L1)
length (x:xs)   = 1 + length xs         -- (L2)

                = []                    -- (M1)

       ++ ys    = ys                    -- (A1)
(x:xs) ++ ys    = x : (xs ++ ys)        -- (A2)
```


Base Case. $P([\,])$ We want to prove

```
length ([] ++ ys) = length [] + length ys
```

```
length ([] ++ ys) = length ys -- by (A1)
                  = 0 + length ys
                  = length [] + length ys -- by (L1)
```

Concatenation preserves length: step case

Step Case. Show that $\forall x.\ \forall xs.\ P(xs) \rightarrow P(x : xs)$

Assume $P(as)$

$\forall ys.$ `length (as ++ ys) = length as + length ys -- (IH)`

Prove P

$\forall ys.$

For arbitrary ys we have:


```
length ((a:as) ++ ys)
    = length (a : (as ++ ys))     -- by (A2)
    = 1 + length (as ++ ys)       -- by (L2)
    = 1 + length as + length ys   -- by (IH)
    = length (a:as) + length ys   -- by (L2)
```

Theorem proved!

A few meta-points:

On the induction hypothesis:

- The *induction hypothesis* ties the recursive knot in the proof.
- If you haven't used it, the proof is likely wrong.
- It's i … *esis* act

On rules:

- Only use the rules that are given, that is
  - the function definitions
  - the induction hypothesis
  - basic arithmetic

Concatenation Distributes over Map

Show: `map f (xs ++ ys) = map f xs ++ map f ys`

Which list?

- as before `++` defined by recursion on `xs`
- treat `ys` as a constant.

Show.

$\forall$

So let $P(xs)$ be `map f (xs ++ ys) = map f xs ++ map f ys`

Base Case: $P([\,])$. Show for arbitrary ys

```
map f ([] ++ ys)  = map f [] ++ map f ys.

map f ([] ++ ys)  = map f ys                -- by (A1)
                  = [] ++ map f ys          -- by (A1)
                  = map f [] ++ map f ys    -- by (M1)
```

Concatenation Distributes over Map, Continued

Step Case: $\forall x.\ \forall xs.\ P(xs) \rightarrow P(x : xs)$

Assume $P(as)$

```
map f (as ++ ys)  = map f as ++ map f ys   -- (IH)
```


Prove P

```
map f ((a:as) ++ ys    m
```



```
map f ((a:as) ++ ys)
    = map f (a  : (as ++ ys))        -- by (A2)
    = f a : map f (as ++ ys)         -- by (M2)
    = f a : (map f as ++ map f ys)   -- by (IH)
    = (f a : map f as) ++ map f ys   -- by (A2)
    = map f (a:as) ++ map f ys       -- by (M2)
```

Theorem proved!

Observe a Trilogy

- **Inductive Definition** defines *all* lists


```
data [a] = [] | a : [a]
```

- **Rec**

```
f [] = ..
f (x:
```


- **Str** *all* lists

  Prove $P([\,])$

  Prove $\forall x.\ \forall xs.\ P(xs) \rightarrow P(x : xs)$ (pro)

- Each version has a base case and a step case.
- The form of the inductive type definition determines the form of recursive function definitions and the structural induction principle.

Induction on Finite Trees

Inductive Definition of finite trees (for an arbitrary type a)

1. `Nul` is of type `Tree a`
2. If `l` … `de l x r` is of ty…

No object is …

Tree Induction. To show that $P(t)$ for all …

- Show that $P(\texttt{Nul})$ holds
- Show that $P(\texttt{Node l x r})$ holds whenever both $P(\texttt{l})$ and $P(\texttt{r})$ are true.

Induction for Lists and Trees

Natural Numbers.

```
data Nat =
  O | S Nat
```

$$\frac{P(O) \quad \forall n.P(n) \rightarrow P(S\,n)}{\forall n.P(n)}$$

Lists.

```
data [a] =
  [] | a : [a]
```

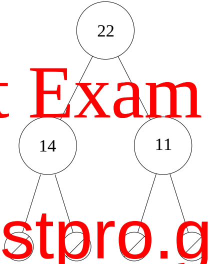
$$\frac{P([\,]) \quad \forall \mathtt{x}.\forall \mathtt{xs}.P(\mathtt{xs}) \rightarrow P(\mathtt{x : xs})}{\forall \mathtt{xs}.P(\mathtt{xs})}$$

Trees.


```
data Tree a =
  Nul
 | Node (Tree a) a (Tree a)
```

$$\frac{P(\mathtt{Nul}) \quad \forall \mathtt{l}.\forall \mathtt{x}.\forall \mathtt{r}.P(\mathtt{l}) \wedge P(\mathtt{r}) \rightarrow P(\mathtt{Node\,l\,x\,r})}{\forall t.P(\mathtt{t})}$$

Why does it Work?



Given.

- *Base Case:* P(`Nul`)
- *Step Case:*
  ∀`l`.∀

Show. P(`Node (Node Nul 14 Nul) 22 (Nod`

1. P(`Nul`) is given
2. P(`Node Nul 14 Nul`) follows from P(
3. P(`Node Nul 11 Nul`) follows from P(`Nul`) and P(`Nul`)
4. P(`Node (Node Nul 14 Nul) 22 (Node Nul 11 Nul)`)
   follows from P(`Node Nul 14 Nul`) and P(`Node Nul 11 Nul`)

Data Type.


```
data Tree a =
    Nul
  | Node (Tree a) a (Tree a)
```

Tree Ind

$$\frac{P(Nul) \qquad \forall t_1.\ \forall x.\ \forall t_2.\ P(t_1) \wedge}{\forall t P(t)}$$

with the following types:

- $x :: a$ is of type a
- $t_1 :: Tree\, a$ and $t_2 :: Tree\, a$ are of type $Tree\, a$.

Standard functions

```
mapT f Nul                  = Nul                    -- (M1)
mapT f (Node t1 x t2)
      = Node (mapT f t1) (f x) (mapT f t2)           -- (M2)

count Nul                   = 0                      -- (C1)
count (Node t1 x t2)
      = 1 + count t1 + count t2                      -- (C2)
```

Example. We use tree induction to show that

```
count (mapT f t) = count t
```

holds for all functions `f` and all trees `t`.

(Analogous to `length (map f xs) = length xs` for lists)

Show count (mapT f t) = count t

Base Cas

count (

This holds by (M1)

Step case

Show: $\forall$`l`.$\forall$`x`.$\forall$`r`.$P($`l`$) \land P($`r`$) \rightarrow P($`Node l x r`$)$

Induction Hypothesis for arbitrary u and u : $P(u\ )$ $P(u\ )$ written as

```
count (m                                   -- (
count (m                                   -- (
```


Proof Goal. For arbitrary a, show that P


```
count (mapT f (Node u1 a u2)) = count (Node u1 a u2)
```

Step case continued

Proof Goal. P(Node u1 a u2), i.e.


```
count (mapT f (Node u1 a u2)) = count (Node u1 a u2)
```

Our Reaso


```
  count (
= count (Node (mapT f u1) (f x) (mapT f u2)) -- by (M2)
= 1 + count (mapT f u1) + count (mapT f u2)  -- by (C2)
= 1 + count u1 + count u2                    -- by (IH1, IH2)
= count (Node u1 a u2)                       -- by (C2)
```


Theorem proved!

Observe the Trilogy Again

There are three related stories exemplified here, now for trees

- **Inductive Definition**

```
data Tree a = Nul | Node (Tree a) a (Tree a)
```

- **Rec**

```
f Nul = . .
f (No r
```


- **Structural Induction Principle**

  Prove $P(\texttt{Nul})$

  Prove $\forall \texttt{l}. \forall \texttt{x}. \forall \texttt{r}. P(\texttt{l}) \wedge P(\texttt{r}) \to P(\texttt{No}$

Similarities.

- One definition / proof obligation per Constructor
- Assuming that smaller cases are already defined / proved

Flashback: Accumulating Parameters

Two version of summing a list:


```
sum1 []           = 0                          -- (S1)
sum1 (x:xs)       = x + sum1 xs                -- (S2)

sum2 xs           = sum2' 0 xs                 -- (T1)
sum2' acc []      = acc                        -- (T2)
sum2' acc (x:xs) = sum2' (acc + x) xs          -- (T3)
```


Crucial Differences.

- one parameter in sum1, two in sum2
- *both* parameters change in the recursive call in sum2

Show: `sum1 xs = sum2 xs`

```
sum1 []              = 0                        -- (S1)
sum1 (x:xs)          = x + sum1 xs              -- (S2)
```



```
sum2 xs              = sum2’ 0 xs               -- (T1)
sum2’ a              = ac                       -- (T2)
sum2’ a
```


Base Case: $P([])$

```
sum2 []    = sum1 []
```



```
sum2 [] = sum2’ 0 []   -- by (T1)
        = 0            -- by (T2)
        = sum1 []      -- by (S1)
```

Step case

Step Case: $\forall x.\forall xs.P(xs) \rightarrow P(x : xs)$

Assume:

```
sum2 as = sum1 as                        -- (IH)
```


Prove:

```
sum2 (a
```



```
sum2 (a
          = sum2' (0 + a) as             -- by (T3)
```



```
sum1 (a:as) = a + sum1 as                -- by (S2)
            = a + sum2 as                -- by (IH)
            = a + sum2' 0 as             -- by (T1)
```

Problem.

- can't apply IH: as $0 \neq 0 + a$
- accumulating parameter in `sum2` has *changed*

Proving a Stronger Property

Solution. Prove a property that involved *both* arguments.

```
sum1 []              = 0                      -- (S1)
sum1 (x:xs)          = x + sum1 xs            -- (S2)
```



```
sum2 xs              = sum2' 0 xs             -- (T1)
sum2' a              = ac                     -- (T2)
sum2' a
```


Observa

```
sum2' acc xs  = acc + sum1 xs
```

Formally. We show that

$$\forall \texttt{xs}.\underbrace{\forall \texttt{acc}.\texttt{sum2' acc xs = acc + sum1 x}}_{P(\texttt{xs})}$$

Base Case: Show $P([])$, i.e. $\forall$acc.acc + sum1 [] = sum2' acc [].

```
acc + sum1 [] = acc + 0 = acc -- by (S1)
              = sum2' acc []  -- by (T2)
```

Step case

Step Case. $\forall \mathtt{x}.\forall \mathtt{xs}.P(\mathtt{xs}) \rightarrow P(\mathtt{x} : \mathtt{xs})$.

Induction Hypothesis

`∀acc.acc + sum1 as = sum2' acc as` (IH)

Show.

Our Reaso

```
acc + sum1 (a:as) = acc + a + sum1 as    -- by (S2)
                  = sum2' (acc + a) as   -- by (IH) (*)
                  = sum2' acc (a:as)     -- by (T3)
```


- Our induction hypothesis is ∀ `acc. ...`
- In `(*)` we instantiate ∀acc with it `acc + a`
- ∀ `acc` is *absolutely needed* in induction hypothesis

Proving the Original Property

We have. $\forall xs.\ P(\texttt{xs})$, that is:

```
∀ xs. ∀ acc. acc + sum1 xs = sum2' acc xs
```


Equival

```
∀
```


Instantiation `(acc = 0)`

```
∀ xs. 0 + sum1 xs = sum2' 0 xs          -- by
∀ xs. sum1 xs     = sum2' 0 xs          -- by arith
∀ xs. sum1 xs     = sum2 xs             -- by T1
```


That is, we have (finally) proved the original property.

When might a stronger property P be necessary ?

Alarm Bells.

```
sum2' acc (x:xs) = sum2' (acc + x) xs
```


Pattern.

- *bot*

Progra

- to evaluate `sum2'`, need evaluation steps w /

Proving Perspective.

- to prove facts about `sum2'`, need inducti / 0

Orthogonal Take.

- `sum2'` is *more capable* than `sum2` (works for *all* values of `acc`)
- when proving, need *stronger statement* that also works for all `acc`

Look at proving it for $xs = [2, 3, 5]$

Backwards Proof for a special case:


```
0 + sum1 [2,3,5] = sum2' 0 [2,3,5]  because
0 + 2 + sum1 [3
0 + 2 + 3 + sum1 [5
0 + 2 + 3 + 5 + sum1 [] = s                    ause
0 + 2 + 3 + 5 = (0+2+
```


Termination.

- the list gets shorter with every recursive call
- despite the accumulator getting larger!

Another example

```
flatten :: Tree a -> [a]
flatten Nul              = []                                  -- (F1)
flatten (Node l a r) = flatten l ++ [a] ++ flatten r  -- (F2)
```



```
flatten2 :: Tree a -> [a]
flatten                                                -- (G)

flatten
flatten2' Nul acc = acc                                -- (H1)
flatten2' (Node l a r) acc =
   flatten2' l (a:flatten2' r acc)                     -- (H2)
```


Show.

```
flatten2' t acc = flatten t ++ acc
```

for all t :: Tree a, and all acc :: [a].

Proof

Proof Goal.

$$\forall t.\underbrace{\forall acc.\texttt{flatten2' t acc = flatten t ++ acc}}_{P(t)}$$

Base Case `t = Nul`. Show that


```
flatt                                        -- by (H1)
     = [] ++ acc                             -- by (A1)
     = flatten Nul ++ acc                    -- by (F1)
```


Step Case: `t = Node t1 y t2`. Assume tha

```
flatten2' t1 acc = flatten t1 ++ acc          -- (IH1)
flatten2' t2 acc = flatten t2 ++ acc          -- (IH2)
```

Required to Show. For *all* `acc`,

```
flatten2' (Node t1 y t2) acc = flatten (Node t1 y t2) ++ acc
```

Proof (continued)

Proof (of Step Case): Let *a* be given (we will generalise *a* to ∀acc)


```
flatten2' (Node t1 y t2) a
  = flatten2' t1 (y : flatten2' t2 a)      -- by (H2)
  = flatten t1 ++ (y : flatten2' t2 a)     -- (IH1)(*)
  = flatt
  = flatt
  = (flat
  = flatten (Node t1 y t2) ++ a            -- by (F2)
```


Notes.

- in IH1, acc is instantiated with (y : flatten2' t2 a)
- in IH1, acc is instantiated with a

As a was arbitrary, this completes the proof.

General Principle

Inductive Definition

- `data Tree a = Nul | Node (Tree a) a (Tree a)`
- Con

Structu

- Pro

  Prove $\forall \mathtt{l}.\forall \mathtt{x}.\forall \mathtt{r}.P(\mathtt{l}) \wedge P(\mathtt{r}) \rightarrow P(\mathtt{No}$
- One proof obligation for each constructor
- All arguments universally quantified
- May assume property of *same type* arguments

General Principle: Example

Given. Inductive data type definition of type T

```
data T =
    C1 Int
  | C2 T T
  | C3 T Int T
```

```
Constructors:
    C1 :: Int -> T
  | C2 :: T -> T -> T
  | C3 :: T -> Int -> T -> T
```

Q. Wha

General Principle: Example

Given. Inductive data type definition of type T

```
data T =                      Constructors:
    C1 Int                      C1 :: Int -> T
  | C2 T T                    | C2 :: T -> T -> T
  | C3 T Int T                | C3 :: T -> Int -> T -> T
```


Q. Wha

A. To sho

- *three* things (*three* constructors)
- all arguments are *universally* quantifie
- P(t) may be assumed for arguments of type

More Concretely. To show $\forall t :: T, P(t)$, need to show

- $\forall \mathtt{n}. P(\mathtt{C1\,n})$
- $\forall \mathtt{t1}. \forall \mathtt{t2}. P(\mathtt{t1}) \wedge P(\mathtt{t2}) \rightarrow P(\mathtt{C2\,t1\,t2})$
- $\forall \mathtt{t1}. \forall \mathtt{n}. \forall \mathtt{t2}. P(\mathtt{t1}) \wedge P(\mathtt{t2}) \rightarrow P(\mathtt{C3\,t1\,n\,t2})$

Induction on Formulae

Boolean Formulae without negation as Inductive Data Type

```
data NFForm =
    TT
  | Var Int
  | Con
  | Dis
  | Imp
```


Induction Principle. $\forall \mathtt{f} : \mathtt{NFForm}.P(\mathtt{f})$ fol

- $P(\mathtt{TT})$
- $\forall \mathtt{n}.P(\mathtt{Var\,n})$
- $\forall \mathtt{f1}.\forall \mathtt{f2}.P(\mathtt{f1}) \wedge P(\mathtt{f2}) \rightarrow P(\mathtt{Conj\,f1\,f2})$
- $\forall \mathtt{f1}.\forall \mathtt{f2}.P(\mathtt{f1}) \wedge P(\mathtt{f2}) \rightarrow P(\mathtt{Disj\,f1\,f2})$
- $\forall \mathtt{f1}.\forall \mathtt{f2}.P(\mathtt{f1}) \wedge P(\mathtt{f2}) \rightarrow P(\mathtt{Impl\,f1\,f2})$

Recursive Definition

Given.

```
data NFForm =
    TT
  | Var Int
  | Conj NFForm NFForm
  | Dis
  | Imp
```


Evaluation of a (negation free) formula:

```
eval :: (Int -> Bool) -> NFForm -> Bool
eval theta TT = True
eval theta (Var n) = theta n
eval theta (Conj f1 f2) = (eval theta f1) && (eval theta f2)
eval theta (Disj f1 f2) = (eval theta f1) || (eval theta f2)
eval theta (Impl f1 f2) = (not (eval theta f1)) || (eval theta f2)
```

Example Proof

Theorem. If `f` is a negation free formula, then `f` evaluates to `True` under the valuation `theta _ = True`.

More precise formulation. Let `theta` be defined by `theta _ = True`. Then, for al .

Proof u

Base Case 1. Show that `eval theta TT = True`

Base Case 2. Show that `\n. eval theta (V`


```
eval theta (Var n) = theta n = True
```

(by definition of `eval` and definition of `theta`)

Proof of Theorem, Continued

Step Case 1. Assume that

- `eval theta f1 = True` (IH1) and
- `eval theta f2 = True` (IH2).

Show that

- `eva`

Proof (of Step Case 1).

```
  eval theta (Conj f1 f2)
= (eval theta f1) && (eval theta f2)  -- defn eval
= True            && True             -- IH1, IH2
= True                                -- defn &&
```

Wrapping Up

Step Case 2 and Step Case 3. In both cases, we may assume

- `eval theta f1 = True` (IH1) and
- `eval theta f2 = True` (IH2).

and need to s

- `eva`
- `eva`

The reasoning is almost identical to that of Step Case 1, and we use

```
True  || True  = True
False || True  = True
```

Summary. Having gone through all the (base and step) cases, the theorem is proved using induction for the data type `NFForm`.

Inductive Types: Degenerate Examples

Consider the following Haskell type definition:

```
data Roo a b =
  MkRoo a b
```


Q. Give

Inductive Types: Degenerate Examples

Consider the following Haskell type definition:

```
data Roo a b =
  MkRoo a b
```


Q. Give

A. It is the t

- To make an element of `Roo a b`, can us
  `b -> Roo a b`
- No other way to 'make' elements of

Let's give this type its usual name:

```
data Pair a b =
  MkPair a b
```

Recursion and Induction Principle

Data Type.


```
data Pair a b =
  MkPair a b
```

Pair Rec


```
f (MkPair x y) = ... x ... y ...
```

we may use *both* the values of x and y – sa

Pair Induction. To prove $\forall \mathtt{x}::\mathtt{Pair\ a\ b}.P(\mathtt{x})$

- show that $\forall \mathtt{x}.\forall \mathtt{y}.P(\mathtt{MkPair\ x y})$

just *one* constructor and *no* occurrences of arguments of pair type

Inductive Types: More Degenerate Examples

Consider the following Haskell type definition:

```
data Wombat a b =
  Left  a
| Right b
```


Q. Give

Inductive Types: More Degenerate Examples

Consider the following Haskell type definition:

```
data Wombat a b =
  Left  a
| Right b
```


Q. Give

A. It is the t

- use c
- use the constructor `Right:  b -> Wombat a b`
- No other way to "make" elements of

Let's give this type its usual name:

```
data CoPair a b =
  Left  a
| Right b
```

Recursion and Induction Principle

Data Type.

```
data CoPair a b =
  Left a
  | Right b
```


Copair R :


```
f (Right y) = ...  y ...  ...
```


we *have* to give equations for both cases: left and right

Copair Induction. To prove $\forall$z::CoPair a b.P(z)

- show that $\forall$x.P(Left x)
- show that $\forall$y.P(Right y)

here: *two* constructors and *no* occurrences of arguments of copair type

Limitations of Inductive Proof

Termination. Consider the following (legal) definition in Haskell


```
nt :: Int -> Int
nt x = nt x +1
```

Taking


```
0 = nt 0 - nt 0 = nt 0 + 1 - nt 0 = 1
```


I.e. a statement that is *patently* false.

Limitation 1. The proof principles outlined here only work if *all functions are terminating*.

Limitations, Continued

Finite Data Structures. Consider the following (legal) Haskell definition


```
blink :: [Bool]
blink = True:False:blink
```

and cons


```
length blink
```


Clearly, `length blink` is *undefined* and so n
statements.

Limitation 2. The proof principles outlined here only work for all *finite* elements of inductive types.

Addressing Termination

Q. How do we *prove* that a function terminates?

Example 1. The argument gets "smaller"

```
length [] = 0
len
```


Example 2. Only one argument gets "smaller"?


```
length' [] a = a
length' (x:xs) a = length' xs (a+1)
```

Q. What does "getting smaller" really mean?

Termination Measures

Given. The function `f` defined below as follows

```
f :: T1 -> T2
f x =  exp(x)
```


Q. Whe

A. Need

Informa

- in every recursive call, the measure `m` smaller

- termination, because natural numbers canno

Formally. A function `m:  T1 ->` $\mathbb{N}$ is a *termination measure* for `f` if

- for every defining equation `f x = exp`, and
- for every recursive call `f y` in `exp`

we have that `m y < m x`.

Example

List Reversal.


```
rev :: [a] -> [a]
rev [] = []
rev (x:xs) = (rev xs) + [x]
```

Termin


```
m :: [a] -> N
m xs = length xs
```


Recursive Calls only in the second line of function definition

- Show that `m xs < m (x:xs)`
- I.e. `length xs < length (x:xs)` – this is obvious.

Termination Measures: General Case

Consider a recursively defined function

```
f : T1 -> T2 -> ... -> Tn -> T0
```


that is defined using multiple equations of the form

```
f x1 .. xn = ex
```

taking n T

Definition. A *termination measure* for `f` i


```
m: T1 -> T2 -> ... -> Tn ->
```

such that

- for every defining equation `f x1 ... xn = exp`, and
- for every recursive call `f y1 ... yn` in `exp`

we have that `m y1 .. yn < m x1 ... xn`.

Termination Proofs

Theorem. Let `f: T1 -> ... -> Tn -> T` be a function with termination measure `m: T1 -> T2 -> ...-> Tn -> ℕ`. Then the evaluation of `f x1 .. xn` terminates for all `x1, ..., xn`.

Proof. We show the following statement by induction on $n \in \mathbb{N}$.

tes

Base Cas

Step Case. Assume that the statement is true for x1, ..., xn be given. Then the recursive call


```
f x1 .. xn = exp(x1, ..., xn)
```

only contains calls of the form `f y1 .. yn` for which `m y1 .. yn < m x1 ... xn` so that these calls terminate by induction hypothesis. Therefore `f x1 ... xn` terminates.

Example

```
rev_a :: [a] -> [a] -> [a]
rev_a [] ys = ys
rev_a (x:xs) = rev_a xs (x:ys)
```


Termin


```
m :: [a] -> [a] -> N
m  xs ys = length xs
```


Recursive Calls only in second line of function definition.

- Show that `m xs (x:ys) < m (x:xs) ys`.
- I.e. `length xs < length (x:xs)` – this is obvious.

Outlook: Induction Principles

More General Type Definitions

```
data Rose a =
  Rose a [Rose a]
```

```
data TTree a b =
  Wr b | Rd (a -> TTree a b)
```


Exampl


```
eat :: TT
eat  (Wr y) _ = y
eat  (Rd f) (x:xs) = eat (f x) xs
```


Induction Principles

- for `Rose`: may assume IH for all list elements
- for `TTree`: mayh assume IH for all values of `f`

Outlook: Termination Proofs

More Complex Function Definitions


```
ack :: Int -> Int -> Int
ack 0 y = y+1
ack x 0 = a
ack x y = a
```


Termination Measures

- `m x y = x` doesn't account for last line of fun
- difficulty: *nested* recursive calls

Digression. Both induction and termination proofs scratch the surface!

Outlook: Formal Proof in a Theorem Prover

The Coq Theorem Prover `https://coq.inria.fr`

- based on Theory *Coq*and's Calculus of Constructions
- *requires* that all functions terminate.

Exampl

- Nat

  Lem

```
      ((exists x, P x) -> Q) -> forall x,  P x -> Q.
```


- Inductive Proofs:

```
Lemma len_map {A B: Type} (f: A -> B): forall (l: list A),
  length l = length (map f l).
```


(and some other examples)