Natural Deduction

COMP1600 / COMP6260

Semester 2, 202

Criticism of Equational Proofs

The good.

Completeness tells us that if an equation is true, we can prove it.

The bad.

So

$$x \vee x \quad \vee \quad \wedge \quad \vee \quad \wedge \quad \vee \neg \quad x \quad F = x$$

The ugly.

Equational reasoning is not *natural*, i.e. it doesn't mirror the *meaning* of $\wedge, \vee$ and $\neg$.

New Connective. Implication, written $\rightarrow$

In English. $x \rightarrow y$ means "if x is true, then so is y".

Truth Ta

- the p
- x

x	y	x
F	F	T
F	T	T
T	F	F
T	T	T

Interlude: Logic to English

Exercise. Use the predicates I – I'm going surfing, Y – you're going surfing, and W – there'll be a big wave that kills us all, to translate the following statements to English:

1. $I \wedge$
2. $(I$

Possible Answer.

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(Both formulae have the same truth table!)

Propositional Formulae

Definition. Given a set V of variables, *propositional formulae* are constructed as follows:

- T (true) and F (false) and all variables $x \in V$ are boolean formulae
- if ϕ and ψ are boolean formulae, then so are $\phi \wedge \psi$ and $\phi \vee \psi$ and $\phi \quad \psi$
- if ϕ

Precede an $\vee$ binds more strongly than $\to$:

$$\neg x \to y \vee z \text{ reads as } ($$

Boolean Formulae vs Propositional Formulae

- propositional formulae are boolean formulae with addition of $\to$
- $\to$ is expressible using boolean formulae: $x \to y = \neg x \vee y$
- but included as implication is used *very* frequently

Contradictions and Contingencies

Types of Propositional Formulae. A propositional formula is

- *valid*, if it evaluates to T in all situations / under all truth value assignments.
- a *c*
- a *c*
  eval

Example.

- 'John had toast for breakfast' is a contingency.
- 'John had toast for breakfast' $\wedge \neg$ 'Jo
  contradiction.
- $p \to (\neg q \vee p) \to (p \wedge q) \vee r$ – can be complicated

Example proof using truth tables

Statement to be proved:

$$\phi \equiv (p \land (q \lor r)) \rightarrow ((p \land q) \lor r))$$

For all 8 $(= 2^3)$ possibilities of p, q, r, calculate truth value of the statement

	q	r					ϕ
		T					T
T	T	F	T	T			
T	F	T	T	T			
T	F	F	F	F			
F	T	T	T	F			
F	T	F	T	F	F	F	T
F	F	T	T	F	F	T	T
F	F	F	F	F	F	F	T

Always exponential in size!

Natural Deduction

Truth Tables: Can be exponential

Equatio

Natural

- formal system that imitates human reasoning
- explains one connective at a time: intro and elim rul
- used to prove *validity* of formulae
- also used in all formal theorem provers

Informal Proof

Goal. Show that $\phi \equiv (p \wedge (q \vee r)) \rightarrow (q \rightarrow s) \vee p$ is valid.

Informal Proof.

1. First, assume that $p \wedge (q \vee r)$ (is true) and show that $(q \rightarrow s) \vee p$.
2. under this assumption, we have that p (is true).
3. still und
4. That is

Formal Natural Deduction Proof.

1	$p \wedge (q \vee r)$	Assumption
2	p	$\wedge$-E, 1
3	$(q \rightarrow s) \vee p$	$\vee$-I, 2
4	$(p \wedge (q \vee r)) \rightarrow ((q \rightarrow s) \vee p)$	$\rightarrow$-I, 1–3

Conjunction rules

And Introduction ($\wedge$-I)

$$\frac{p \qquad q}{p \wedge q}$$

- as p

And Elim

$$\frac{p \wedge q}{p}$$

- as $p \wedge q$ is true, we have that p is true.
- as $p \wedge q$ is true, we have that q is true.

Example

Example. Commutativity of conjunction (derived rule)

$$\frac{p \wedge q}{q \wedge p}$$

- ass is true).

Inform

Natural Deduction Proof.

1. Assume that $p \wedge q$.
2. because of $p \wedge q$, we have p.
3. because of $p \wedge q$, we have q.
4. therefore, we also have $q \wedge p$.

3	q	$\wedge$-E, 1
4	$q \wedge p$	$\wedge$-I, 2, 3

Implication rules

Implication Introduction ($\rightarrow$-I)

$$\begin{array}{c} [\ p\ ] \\ \vdots \\ q \\ \hline \end{array}$$

- if q … *ithout* the assumption p.

- […] means that the assumption p is d

Implication Elimination ($\rightarrow$-E)

$$\frac{p \qquad p \rightarrow q}{q}$$

- if both p and $p \rightarrow q$ hold (are true), then so does q.

Example - transitivity of implication (derived rule)

We prove $\dfrac{p \to q \qquad q \to r}{p \to r}$

Informal Proof

1. fix the assumption $p \to q$.
2. fix the ass
3. additio
4. because p and $p \to q$, we have q.
5. because q and $q \to r$, we have r.
6. hence $p \to r$ holds *without* assuming p

Natural Deduction Proof

1	$p \quad q$	
		-E, 1, 3
		-E, 2, 4
6	$p \to r$	$\to$-I, 3–5

- lines 1 and 2 are assumptions, can be used anywhere
- line 3 is an assumption we make, can be used *only* in scope (l 3–5).

Aside: Justification of Proof Steps

Silly Proof. (we prove what we already know!)

1	$p \to q$	
2		p

Line Number Notation

- $\to$-E,1,2 means that rule $\to$-E proves lin
- $\to$-I,2-3 means rule $\to$-I proves line 4 from *the fact that* we could *assume* line 2 and (using that assumption) *prove* line 3.
- In $\to$-I, 2–3 is the *entire* scope of the assumption p.

Rules involving assumptions

1	$p \to q$	
2	$q \to r$	
3		p

- statements inside the scope of an assumption de assumption.
- we only know that they are true if the assumption is true!
- we have assumed p and "proved" $q \wedge r$, but $q \wedge r$ depends on p.
- Indentation and vertical lines indicate scoping
- Similar to programming: p is a "local variable".

Useless assumptions

You can assume anything, but it might not be useful.

1	$p \wedge q$

1	$p \wedge$ You are a giraffe	
2	You are a giraffe	, 1
3	$p \wedge$ You are a giraffe $\rightarrow$ You are a giraffe	$\rightarrow$-I, 1–2

Disjunction rules

Or Introduction (∨-I)

$$\frac{p}{p \vee q} \qquad \frac{p}{q \vee p}$$

- if p

Or Elimin

$$\frac{p \vee q \qquad \begin{matrix} \vdots \\ r \end{matrix} \qquad r}{r}$$

- assuming that we have a proof of $p \vee q$ and
- for the case that p holds, we have a proof of r
- for the case that q holds we have a proof of r
- then we have a proof of r *just* from $p \vee q$.

∨-E template

1. know that $p \lor q$
2. in case p is true

   . . .

a. we know

b. and in case that q is true

   . . .

c. we also know that r

d. so we know r as long as $p \lor q$!

1	$p \lor q$	
2	p	
$\vdots$	$\vdots$	
$\vdots$		
c		
	r	∨-E, 1, 2–a, b–c

Example: commutativity of disjunction (derived rule)

$$\frac{p \vee q}{q \vee p}$$

Informal

1. fix the ass
2. first assume that p is true.
3. then also have that $q \vee p$.
4. now assume that q is true.
5. then also have that $q \vee p$.
6. hence $q \vee p$, without assuming either p or q.

2		
3		
4	q	
5	$q \vee p$	$\vee$-I, 4
6	$q \vee p$	$\vee$-E, 1, 2–3, 4–5

Negation and Truth Rules

not introduction (¬-I)

$$\begin{array}{c} [p] \\ \vdots \\ F \\ \hline \end{array}$$

not elimination (¬-E)

$$\frac{p \qquad \neg p}{F}$$

- if ass ... ust hold

Proof by Contradiction (PC)

$$\begin{array}{c} [\neg p] \\ \vdots \\ F \\ \hline p \end{array}$$

Truth

$$\frac{}{T}$$

- to prove p, assume $\neg p$ and derive a contradiction.
- truth, i.e. T, can always be established without assumptions.

Example: double negation introduction (derived rule)

$$\frac{p}{\neg\neg p} \qquad \frac{\text{It is raining}}{\text{It is not the case that it is not raining}}$$

Informal

1. Fix the as
2. additionally assume that $\neg p$.
3. then F as p and $\neg p$ (under assn p)
4. hence $\neg p$ is contradictory, so $\neg\neg p$.

		, 2
4	$\neg\neg p$	$\neg$I, 2–3

Example: contradiction elimination (derived rule)

"Anything follows form a contradiction"

$$\frac{F}{q}$$

3	F	
4	q	

- R stands for "repeat".
- F holds and continues to hold within the scope of the assumption $\neg q$.
- assuming $\neg q$ a "technical trick".

Example: double negation elimination (derived rule)

$$\frac{\neg\neg p}{p}$$

1	$\neg\neg p$	
2	$\quad \neg p$	
3	$\quad F$	
4	p	PC, 2–3

Equivalence

$p \leftrightarrow q$ means p is true if and only if q is true

We can make the definition

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

which wo

introduc

$$\frac{\qquad}{p \leftrightarrow q}$$

elimination rules:

$$\frac{p \leftrightarrow q}{p \rightarrow q} \qquad \frac{p \leftrightarrow q}{q \rightarrow p}$$

Which rule to use next?

- Guided by the "form" of your goal, and what you already have proved
- "form" — ie, look at the connective: , , ,
- alw

- to p

$$\frac{p}{p \vee q} \qquad \frac{\quad}{\quad}$$

p may not be necessarily true, q may n

To prove $p \vee q$, sometimes you need to do this:

1. Using PC, assume $\neg(p \vee q)$ (hoping to prove some contradiction)
2. When is
3. From
4. Havin n

Tutorial Exercise. $\dfrac{\neg p \to q}{p \vee q}$

Not-or elimination (derived rule)

$$\frac{\neg(p \vee q)}{\neg p}$$

2	p	
3	$p \vee q$	
4	F	
5	$\neg p$	$\neg$I, 2–4

Proving a contrapositive rule

In the same way, whenever you can prove any $\dfrac{p}{q}$

then you can prove $\dfrac{\neg q}{}$

2	p	
3	q	your pr
4	F	$\neg$E, 1, 3
5	$\neg p$	$\neg$I, 2–4

Law of the excluded middle (derived)

$$p \vee \neg p$$

"Everything must either be or not be." — Russell

2	$\neg p$	$\neg$
3	$\neg\neg p$	$\neg$
4	F	$\neg$E, 2, 3
5	$p \vee \neg p$	PC, 1–4

Summary: Major Proof Techniques

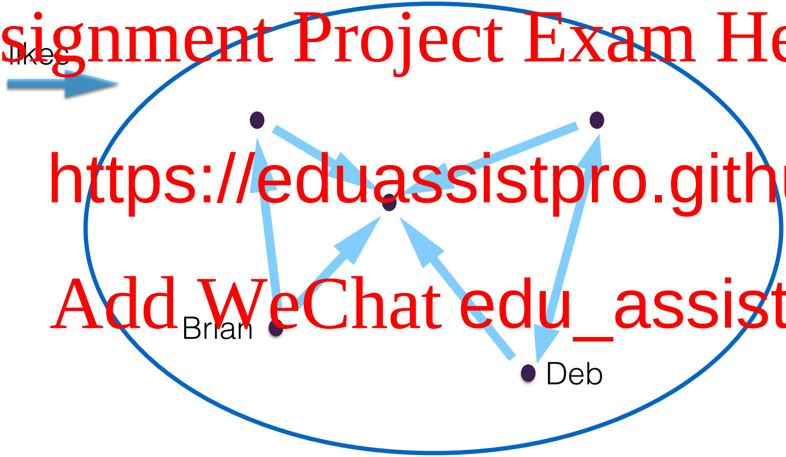
Three major styles of proof in logic and mathematics

- **Mo**
- **Alg**
- **De**

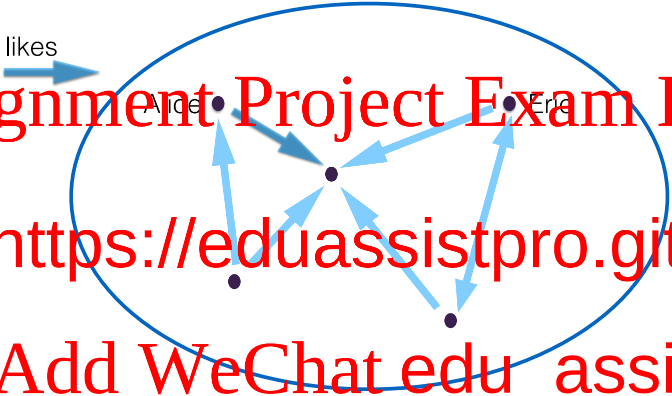
Q. Why bother? Why not write a program that does tru

- propositional logic is *decidable*: can write
- other logics are *not*: first order logic (next)

What can we say about the following situation?

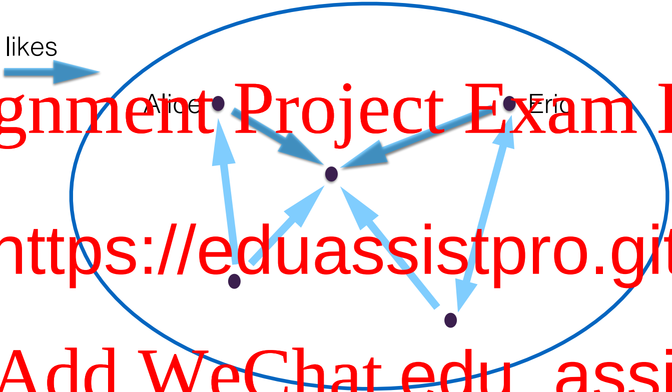


Some (English) Sentences



- Brian likes Alice.
- Eric and Deb like one another.
- Nobody likes Brian.
- Everybody likes Charlie.
- Two people like each other.
- Someone is liked by everyone.

Key Ingredients



Set

- $U = \{A, B, C, D, E\}$
- thought of as "individuals" (can be things)

Relation

- $R = \{(A, C), (B, A), (B, C), (D, C), (D, E), (E, C), (E, D)\}$
- set of ordered pairs (directional)

Limits of Propositional Logic

Propositional Logic.

- atomic propositions, no “inner structure”
- could be ‘Brian likes Alice’ and ‘someone likes someone (else)’

How abo

- Alic

- Everybody likes someone who doesn’t like anyone
- ...

Propositional Logic is not enough!

- What is the limit of what we can say?
- What are the relationships between all these propositions?

Limits of Propositional Logic

Propositional logic talks (only) about *statements*, or *facts*

- e.g. 'I am going surfing'
- can be true or false.

Propositi

- **Obj**
- **Rel**

First Order Logic

- Predicated *depending* on variables (e.g. lik
- Combined using universal ($\forall$) and existential ($\exists$) quantifiers
- In the example: everyone ($\forall$) and someone ($\exists$)
- more complex concepts by nesting (everybody loves someone who . . .)

Quantifiers, Informally

Universal Quantification. $\forall x$: "All x . . .

- (logic) $\forall x.\text{likes}(x, \text{Charlie})$
- (dir
- (na

Existential Quantification. $\exists x$: "There is an

- (logic) $\exists x.\text{likes}(x, \text{Alice})$
- (direct translation) There is an x s.t.
- (natural English) Alice is liked by someone.

More Complex Sentences.

Nobody likes Brian.

$$\forall x(\neg \text{likes}(x, \text{Brian}))$$

Two people like each other.

Someone is liked by everyone.

$$\exists x\, \forall y\, (\text{likes}(y$$

Q. What does the following translate to?

$$\forall x\, \exists y\, (\text{likes}(y, x))$$

First Order Logic: Vocabulary

Vocabulary. relation symbols

- have an *arity*: number of "things" that are related.
- Examples: is_elephant(x) (unary), likes(x, y) (binary)

Informa

- is_el
- likes: the set of all pairs (x, y) s.t. x

Informal interpretation: Syntax

- is_elephant: a proposition depending on an argument
- likes: a proposition depending on two arguments

First Order Logic: Official Syntax

Vocabulary. A *vocabulary* for first order logic is a set R (of relation symbols) where each relation symbol has an *arity* (number of arguments).

Syntax of First Order Logic. Let R be a vocabulary and V a set of variables. The *formulae* of first-order logic (over R and V) are constructed as follows:

1. If r ...) is a for...
2. If ϕ and ψ are formulae, then so are ... $\neg\phi$.
3. If ϕ is a formula, and $x \in V$, then $\exists x$ ...

Dot Notation saves outermost parentheses:

$$\forall x.\text{very complex formula} \equiv \forall x(\text{very complex formula})$$

$$\exists x.\text{very complex formula} \equiv \exists x(\text{very complex formula})$$

Happy and Unhappy Dragons

Vocabulary.

unary predicates

$D(x)$ – x is a dragon, $H(x)$ – x is happy, $F(x)$ – x can fly

binary predicate

English.

1. All dragons can fly unless they are unhappy.
2. At least one dragon can fly despite being unhappy.
3. A dragon is happy if all its children can fly.

1. $\forall x.$
2. $\exists x.$
3. $\forall x. D(x) \to (\forall y. C(y, x) \to F(y)) \to H(x)$

Common Patterns

Existential Quantifier. often goes with $\wedge$

- There is a dragon that . . .
- The
- $\exists x.$

Universal Quantifier. often goes with $\rightarrow$

- All dragons . . .
- For all x, if x is a dragon, then . . .
- $\forall x.\text{dragon}(x) \rightarrow \ldots$

Situations for First Order Logic

Recall. Formulae of *propositional logic* depend on variables

- a situation tells us whether these variables are true or false
- given a situation, can evaluate a formula to true or false

Q. Wha

or false?

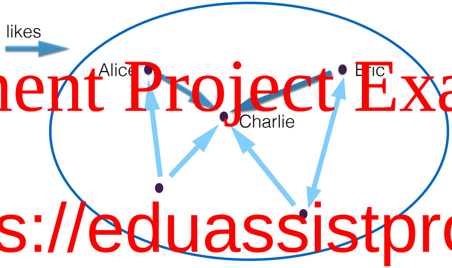
- Exa

Situations θ for first-order logic are given by:

- a domain of discourse U – simply a *s*
- for every $r \in R$ n-ary, an n-ary *relatio*
- for all variables x, an element $\theta(x)$ of U.

Notation: (U, θ)

Example: The Taxonomy of "likes"



Situation. $S = (U, \theta)$ where

- $U = \{A, B, C, D, E\}$
- $\theta(\text{likes}) = \{(A, C), (B, A), (B, C), (D, C), (D, E), (E, C), (E, D)\}$
- $\theta(x) = \theta(y) =$ Alice, $\theta(x) =$ Deb. (not required to be injective – many variables can point to same object)

Formal Semantics

Given. Vocabulary R, situation $S = (U, \theta)$

Truth of formula ϕ in situation (U, θ)

Base case. Truth of $r(x_1, \ldots, x_n)$

- if $r(x_1, \ldots x)$ is true in the situation
- tha i
- tha i

Propositi

- $\phi \wedge$
- $\phi \vee \psi$ is true $\iff$ ϕ is true or ψ is tr
- $\phi \rightarrow \psi$ is true if ϕ is not true or ψ is tr
- $\neg\phi$ is true $\iff$ ϕ is not true

Quantifier Cases.

- $\forall x.\phi$ is true if ϕ is true for *all* values of $\theta(x)$ (and everything else unchanged)
- $\exists x.\phi$ is true if ϕ is true for *some* value of $\theta(x)$ (and everything else unchanged)

Digression on Quantifiers

Universal Quantifier. To see that $\forall x.\phi$ is true in situation (U, θ)

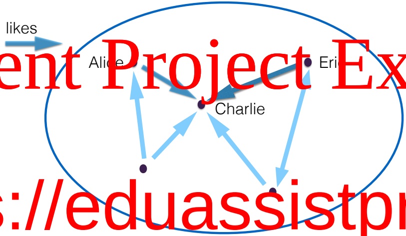
- need to take θ and vary the value of x
- show that ϕ is true in situation $(U, \theta\ )$
- beh

Existential Quantifier. To see that $\exists x.\phi$):

- need to take θ and vary the value of
- need to exhibit *one* varied θ' such that $, \theta')$
- behaves line an "infinite or" over all elements of the domain

Back to the Example

Variables $V = \{a, b, c, \ldots\}$ with $\theta(a) =$ Alice, $\theta(b) =$ Brian etc.



Q. Does everyone like Charlie, i.e. is $\forall x($li
above?

- vary the value of x for θ, i.e. consider
- show likes(x, c) is true for all such θ'
- show likes(Alice, Charlie) and likes(Brian, Charlie) and ...
- but it is *false* that likes(Charlie, Charlie)!
- so $\forall x.$likes(x, c) is not true in the situation above as Charlie doesn't like her/himself!

Special Case: Quantifiers over the Empty Set

Given.

- Situation $S = (U, \theta)$ for first order logic
- unary relation unicorn with $\theta(\text{unicorn}) = \emptyset$

Existent

- “Th s_trees(x)
- alw
- irrespective of value of x, unicorn(x) is a

Universal Quantifier.

- “All unicorns climb trees”: $\forall x.\text{unicorn}(x) \to \text{climbs\_trees}(x)$
- always *true*: $\forall x.\text{unicorn}(x) \to \text{climbs\_trees}(x)$ is true in S.
- irrespective of value of x, unicorn(x) is *always* false.

Propositional vs First Order Logic

Propositional Logic.

- Given propositional logic formula ϕ, can decide validity by constructing all truth tables.

First Order Logic.

- To s con

- But t

Decidability.

- Propositional logic is *decidable*: can writ validity.
- Naive checking for validity *doesn't* work for first order logic – but maybe there's a better way?
- Can *formally prove* that there *cannot* be a program that checks first-order validity!

Interlude: The Drinker's Paradox

Drinker's Paradox (Smullyan 1978, in "What is the name of this book?")

> *"In every non-empty pub there is someone so that if (s)he is drinking, s*

In Logic

$$\phi \equiv \exists x(D(x) \to \forall$$

Q. Is ϕ valid in all situations (U, θ) where U is not empty?

Laws for Quantifiers: Negating "there exists"

$$\neg(\exists x.\ P(x)) \leftrightarrow (\forall x.\ \neg P(x))$$

Examples.

- "No elephant is unhappy. "
- $\neg\exists x$
- $\forall x.$

- $\forall x.\neg$elephant$(x) \vee$ happy(x)
- "Everything is either happy or not an elephant"

De Morgan Laws. Consider domain $U =$ 0 1

- $\neg\exists x.P(x)$ intuitively equivalent to $\neg(P(a_0) \vee P(a_1) \vee \dots)$
- $\neg(P(a_0) \vee P(a_1) \vee \dots)$ equiv to $\neg P(a_0) \wedge \neg P(a_1) \wedge \dots$ by De Morgan
- $\neg P(a_0) \wedge \neg P(a_1) \wedge \dots$ intuitively equivalent to $\forall x.\neg P(x)$

Negating ∃, Formally

Theorem. Let ϕ be a first-order formula, $S = (\mathcal{I}, \theta)$ a situation. Then $\neg\exists x.\phi$ is true in S if and only if iff $\forall x.\neg\phi$ is true in S.

Proof (Sk … $\forall x.\neg\phi$ is true in S … .

We need to s … S'.

Suppose for a contradiction that ϕ is true in … except possibly on the value of x, this means that … iction.

The reverse direction is analogous.

Negating "for all"

$$\neg(\forall x.\ P(x)) \leftrightarrow (\exists x.\ \neg P(x))$$

Exampl

- "No
- $\neg\forall x$

- $\exists x.\neg(\text{elephant}(x) \rightarrow \text{happy}(x))$
- $\exists x.\text{elephant}(x) \wedge \neg\text{happy}(x)$
- There exists an unhappy elephant.

Theorem. Let ϕ be a first-order formula, S a situation for first order logic. Then $\neg\forall x.\phi$ is true in S iff $\exists x.\neg\phi$ is true in S.

Mixed negated quantifiers

Here are four different expressions of the fact that there is no upper bound to the natural numbers.

We can shif

$$\forall m. \quad \neg\forall n. \quad m$$

$$\forall m. \quad \exists n. \quad \neg m$$

$$\forall m. \quad \exists n. \quad m$$