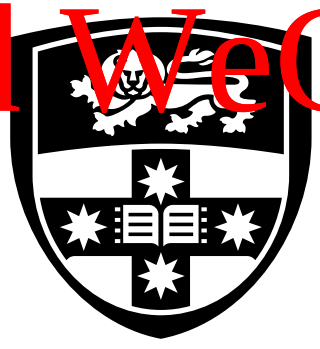
COMP2022: Formal Languages and Logic

2018, Semester 2, Week 2



THE UNIVERSITY OF SYDNEY

OUTLINE

- **Revision : Lambda Calculus**
-
- Encodings
-
- Functional Programming: LISP

Operations

- Application

Operations

- Application

- Abstraction
  - $\lambda x.M$
    - Variable x is abstracted in expressio

Rewriting

- $M[x := N]$

d with

- e.g.
  - $(xyz\lambda x.(zxz))[x := A] =$
  - $(xyz\lambda x.(zxz))[y := B] =$
  - $(xyz\lambda x.(zxz))[z := C] =$

Rewriting

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  - $(xyz\lambda x.(zxz))[z := C] = (xyC\lambda x.(CxC))$

α-reduction

- Rename a λ to remove a name conflict
-
-
- y must be a *new* variable
  - You must not choose a symbol that is alrea

β-reduction

- Solve an abstraction

- Note: the free occurrences of x i
  occurrences which bound to the $\lambda x.$ $(\lambda x.M)$

OUTLINE

- Revision - Lambda Calculus
- https://eduassistpro.github.i
- Encodings

- Functional Programming

Two arguments

- Suppose we have a function $j(x, y)$ which requires two arguments.

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- F is a function which takes one input, and ret F_x, which will take the *next* inpu

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- F is a function which takes one input, and ret F_x, which will take the *next* inpu

- The output of the second function will be $f(x, y)$.

Example

Normal arithmetic: $f(x, y) = (x + y)/2$

Lambda calculus: $(\lambda x.(\lambda y.(x + y)/2))$

$$(\lambda x.(\lambda y.(x + y)/2)) \quad 5 \quad 7$$

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$$= (\lambda y.(5 + y)/2) \quad 7$$

Example

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$$
\begin{aligned}
&(\lambda x.(\lambda y.(x + y)/2))\ \ 5\ \ 7 \\
&= (\lambda y.(5 + y)/2)\ 7 \\
&= (5 + 7)/2
\end{aligned}
$$

Example

Normal arithmetic: $f(x, y) = (x + y)/2$

Lambda calculus: $(\lambda x.(\lambda y.(x + y)/2))$

$$
\begin{aligned}
& (\lambda x.(\lambda y.(x + y)/2)) \;\; 5 \;\; 7 \\
&= (\lambda y.(5 + y)/2) \; 7 \\
&= (5 + 7)/2 \\
&= 6
\end{aligned}
$$

Currying

- A m-ary parameter function can be represented in the lambda calculus through *Currying*

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- A n-ary parameter function can be represented in the lambda calculus through *Currying*
-
- function, which returns an $(n-2$

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-
- function, which returns an $(n-2$
-
- e.g. $(\lambda x.(\lambda y.(\lambda z.f(x,y,z)))) \cdot 1 = (\lambda y.(\lambda z.f(1,y,z)))$

Evaluation

Recall the example from earlier:

$$(\lambda x.(\lambda y.(x + y)/2))\ \ 5\ \ \ 7$$

The function is *partially evaluated* at ea

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Evaluation

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$$(\lambda x.(\lambda y.(x + y)/2))\ \ 5\ \ \ 7$$

The function is *partially evaluated* at ea

- The first function returns $(\lambda y.(5 +$
- 7 is then applied to the new function
- $(5 + 7)/2$ is evaluated and returned

Notation

- Too many parentheses. Let's make it simpler:

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- $ABCDEF \equiv (((((A$

Notation

- Too many parentheses! Let's make it simpler:
-
- $ABCDEF \equiv (((((A$

- i.e. the leftmost application happens *first*

Notation

- for function abstraction we use association to the *right*

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$$1 \quad 2 \quad 3 \quad k$$

Notation

- for function abstraction we use association to the *right*

$$\underset{1}{} \quad \underset{2}{} \quad \underset{3}{} \quad \underset{k}{}$$

- This means the leftmost λ will first be applied to the function

Notation

- Abstraction is right associative
- Application is left associative
-

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$$= \quad \lambda yz.((z-4) \quad y) \quad 2\ 3$$

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$$= \lambda yz.((z-4) \quad y) \quad 2\ 3$$

$$\equiv (\lambda z.((z-4)$$

Notation

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- Application is left associative
-

$$
\begin{aligned}
&= \lambda yz.((z-4) \quad y) \quad 2\ 3 \\
&\equiv (\lambda z.((z-4) \\
&= (3-4) \times 2 \\
&= -2
\end{aligned}
$$

Notation

- Abstraction is right associative
- Application is left associative
- If we wrote it out in full...

$$\lambda xyz.((z \quad x) \quad y) \quad 4\ 2\ 3$$

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- If we wrote it out in full...

$$\lambda xyz.((z \quad x) \quad y) \quad 4\ 2\ 3$$

$$= \quad \lambda x.\ \lambda y.(\lambda z.((z - x) \quad y)) \quad 4 \quad 2 \quad 3$$

Notation

- Abstraction is right associative
- Application is left associative
- If we wrote it out in full...

$$\lambda xyz.((z \quad x) \quad y) \quad 4\ 2\ 3$$

$$= \quad \lambda x.\ \lambda y.(\lambda z.((z - x) \quad y)) \quad 4 \quad 2 \quad 3$$

$$= (\lambda y.(\lambda z.((z - 4) \times y))) \cdot 2 \cdot 3$$

$$= \big(\lambda z.((z - 4) \times 2)\big) \cdot 3$$

$$= (3 - 4) \times 2$$

$$= -2$$

Notation

- Question:
  1. Is $\lambda x.xy = (\lambda x.(xy))$, or

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- Answer: (1), it's $(\lambda x.(xy))$

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- Use parentheses to limit the scope of the λ if needed

Currying

- Suppose we wanted to abstract a function with k arguments:

Currying

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$$(\lambda x_1 x_2 x_3 ... x_k . N)$$

-

$$(\lambda x_1 x_2 x_3 ... x_k . N)\, v_1\, v_2\, v_3 ... v_k$$

Currying

- Suppose we wanted to abstract a function with k arguments:

$$(\lambda x_1 x_2 x_3 ... x_k . N)$$

- $$(\lambda x_1 x_2 x_3 ... x_k . N) v_1 v_2 v_3 ... v_k$$

- Each β-reduction partially evaluates t
  - v_1 replaces x_1. The resulting function takes $k-1$ arguments:

$$(\lambda x_2 x_3 ... x_k . N[x_1 : v_1]) v_2 v_3 ... v_k$$

  - ... then v_2 would replace x_2, etc.

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- **Encodings**

- Functional Programming: LISP

Untyped Lambda Calculus

- Lambda calculus does not have primitives
  - No numbers
  - No arithmetic operators

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Untyped Lambda Calculus

- Lambda calculus does not have primitives
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- However, I'm claiming that it is computati
  a Turing Machine!

- So, how can we represent data types?
  - They must be expressed as functions, known as *encodings*

Encodings: truth

- Boolean constants:
  - TRUE := $\lambda xy.x$

Encodings: truth

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- Now we can do conditional logic:
  - IFELSE := $\lambda fxy.fxy$ has semanti
    - `if <cond> then <x> else <y>`
    - If `<cond>` is true, return result of `<x>` `<y>`

Encodings: truth

$$\mathit{IFELSE}\ \mathit{TRUE}\ A\ B$$

Encodings: truth

$$\begin{aligned} & \mathit{IFELSE}\ \mathit{TRUE}\ A\ B \\ &= (\lambda fxy.fxy)\ (\lambda xy.x)\ A\ B && \text{(macro substitution)} \end{aligned}$$

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$$\begin{aligned}
&IFELSE\ TRUE\ A\ B \\
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&= (\lambda ab.(\lambda xy.x)ab)\ A\ B && \beta \text{ duction)}
\end{aligned}$$

Encodings: truth

$$
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&= (\lambda b.(\lambda xy.x)Ab)\ B && \text{ion)} \\
&= (\lambda xy.x)AB) && \text{ion)} \\
&= (\lambda y.A)B) && \text{ion)}
\end{aligned}
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&= (\lambda b.(\lambda xy.x)Ab)\ B && \text{ion)} \\
&= (\lambda xy.x)AB) && \text{ion)} \\
&= (\lambda y.A)B) && \text{ion)} \\
&= A && (\beta\text{-reduction})
\end{aligned}
$$

Encodings: truth

$TRUE\ FALSE\ A\ B$

Encodings: truth

$$IFELSE\ FALSE\ A\ B$$

$$= (\lambda fxy.fxy)\ (\lambda xy.y)\ A\ B \qquad \text{(macro substitution)}$$

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& = (\lambda b.(\lambda xy.y)Ab)\ B && \text{ion)} \\
& = (\lambda xy.y)AB) && \text{ion)} \\
& = (\lambda y.y)B) && \text{ion)}
\end{aligned}
$$

Encodings: truth

$$
\begin{aligned}
& IFELSE\ FALSE\ A\ B && \\
& = (\lambda fxy.fxy)\ (\lambda xy.y)\ A\ B && \text{(macro substitution)} \\
& = (\lambda ab.(\lambda xy.y)ab)\ A\ B && \beta\text{-reduction)} \\
& = (\lambda b.(\lambda xy.y)Ab)\ B && \text{ion)} \\
& = (\lambda xy.y)AB) && \text{ion)} \\
& = (\lambda y.y)B) && \text{ion)} \\
& = B && (\beta\text{-reduction)}
\end{aligned}
$$

Encodings: truth

- Boolean constants:
  - TRUE := $\lambda xy.x$
-
- Boolean operators:
  - NOT := $\lambda fxy.fyx$
  - OR := $\lambda xy.xxy$
  - AND := $\lambda xy.xyx$

Encodings: NOT

- NOT $:= \lambda fxy.fyx$
  - https://eduassistpro.github.i
    - x, y would be those arguments

Encodings: NOT

- NOT $:= \lambda fxy.fyx$
  - x, y would be those arguments

- i.e. NOT outputs f, except its argume around!

Encodings: truth

NOT TRUE

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NOT TRUE

Encodings: truth

NOT TRUE

tution)

duction)

Encodings: truth

NOT TRUE

tution)

duction)

duction)

Encodings: truth

$NOT\ TRUE$

tution)

duction)

duction)

$= \lambda xy.(\lambda ab.a)yx$

ion)

Encodings: truth

$$NOT\ TRUE$$

tution)

duction)

duction)

$$= \lambda xy.(\lambda ab.a)yx$$

ion)

$$= \lambda xy.(\lambda b.y)x$$

ion)

Encodings: truth

$$NOT\ TRUE$$

tution)

duction)

duction)

$$= \lambda xy.(\lambda ab.a)yx$$

ion)

$$= \lambda xy.(\lambda b.y)x$$

ion)

$$= \lambda xy.y$$

ion)

Encodings: truth

$$
\begin{aligned}
& NOT\ TRUE && \\
& && \text{tution)} \\
& && \text{duction)} \\
& && \text{duction)} \\
&= \lambda xy.(\lambda ab.a)yx && \text{ion)} \\
&= \lambda xy.(\lambda b.y)x && \text{ion)} \\
&= \lambda xy.y && \text{ion)} \\
&= FALSE && \text{(macro substitution)}
\end{aligned}
$$

Encodings: numbers

- The natural numbers can be thought of as a sequence, starting from 0, and successively increasing by one.

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- Inductive clause: for any element x, $x + 1$ is an element of the natural numbe

Encodings: numbers

- The natural numbers can be thought of as a sequence, starting from 0, and successively increasing by one.

- Inductive clause: for any element x, $x + 1$ is an element of the natural numbe
- Extremal clause: nothing is in the set of natural numbers unless it is obtained by the inductive clause and basis clause

Church numerals

- Natural numbers in lambda calculus have two constructors:

Church numerals

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- SUCCESSOR $:= \lambda xyz.\, y(xyz)$
  - Returns the next number in the sequence

Church numerals

- Natural numbers in lambda calculus have two constructors:
  - SUCCESSOR $:= \lambda xyz.\, y(xyz)$
    - Returns the next number in the sequence
- We're now ready to start constructing the natural numbers!

Church numerals

$$ONE$$

Church numerals

$$ONE$$

Church numerals

$$ONE$$

(macro)

$$(\alpha)$$

Church numerals

$$ONE$$

$$= \lambda yz.y((\lambda ab.b)$$ (macro) (α) (β)

Church numerals

$$ONE$$

$$= \lambda yz.y((\lambda ab.b)\ \ (\text{macro})$$

$$(\alpha)$$

$$= \lambda yz.y((\lambda ab.b)\ \ (\beta)$$

$$= \lambda yz.y(\lambda b.b)z \quad (\beta)$$

Church numerals

$$ONE$$

$$\begin{aligned}
&= \lambda yz.y((\lambda ab.b)\ \ldots) && (\beta)\\
&= \lambda yz.y(\lambda b.b)z && (\beta)\\
&= \lambda yz.yz && (\beta)
\end{aligned}$$

Church numerals

$$TWO$$

(macro)

Church numerals

$$TWO$$

(macro)

$$(\alpha)$$

Church numerals

$$TWO$$

$$= \lambda yz.y((\lambda ab.ab \qquad (\alpha)$$

$$(\beta)$$

Church numerals

$$TWO$$

$$= \lambda yz.y((\lambda ab.ab \quad (\alpha)$$

$$= \lambda yz.y((\lambda b.yb) \quad (\beta)$$

$$= \lambda yz.y(yy) \quad (\beta)$$

Church numerals

TWO

$$\begin{aligned}
&= \lambda yz.y((\lambda ab.ab && (\alpha)\\
&= \lambda yz.y((\lambda b.yb) && (\beta)\\
&= \lambda yz.y(yz) && (\beta)
\end{aligned}$$

Church numerals

$$THREE = SUCCESSOR\ TWO$$

Church numerals

$$THREE$$
$$= SUCCESSOR\ TWO$$

$$FOUR$$
$$= SUCCESSOR\ T$$
$$= ...$$
$$= \lambda yz.y(y(y(yz)))$$

Arithmetic?

- We have numbers. Do they work?

Arithmetic?

- We have numbers. Do they work?

-
  - ADD $:= \lambda xypq.xp(ypq)$
  - MULT $:= \lambda xyz.x(yz)$
  - EXP $:= \lambda xy.yx$

Addition example

ADD TWO THREE

Addition example

$$ADD\ TWO\ THREE$$

$$= \lambda xypq.xp(ypq)\quad(\lambda yz.y(yz))\ (\lambda yz.y(y(yz)))$$

Addition example

$$ADD\ TWO\ THREE$$

$$= \lambda xypq.xp(ypq)\ \ (\lambda yz.y(yz))\ (\lambda yz.y(y(yz)))$$

$$\stackrel{(\alpha)}{=}$$

Addition example

$$\begin{aligned} & ADD\ TWO\ THREE \\ = \ & \lambda xypq.xp(ypq)\ \ (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) \\ \xrightarrow{(\alpha)} \ & \\ \xrightarrow{(\beta)} \ & \end{aligned}$$

Addition example

$$\begin{aligned} ADD\ TWO\ THREE & \\ = \quad & \lambda xypq.xp(ypq) \quad (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha) \\ = \quad & \lambda ypq.(\lambda b.p(pb))(ypq) \quad (\lambda cd.c(c(cd))) \qquad\qquad (\beta) \\ = \quad & \lambda ypq.(\lambda b.p(pb))(ypq) \quad (\lambda cd.c(c(cd))) \qquad\qquad (\beta) \end{aligned}$$

Addition example

$$
\begin{aligned}
ADD\ TWO\ THREE &= \lambda xypq.xp(ypq)\ \ (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) && (\alpha)\\
&= \lambda ypq.(\lambda b.p(pb))(ypq)\ \ (\lambda cd.c(c(cd))) && (\beta)\\
&= (\lambda ypq.p(p(ypq)))\ (\lambda cd.c(&& (\beta)
\end{aligned}
$$

Addition example

$$
\begin{aligned}
ADD\ TWO\ THREE & \\
= \quad & \lambda xypq.xp(ypq) \quad (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) && (\alpha) \\
= \quad & \lambda ypq.(\lambda b.p(pb))(ypq) \quad (\lambda cd.c(c(cd))) && (\beta) \\
= \quad & (\lambda ypq.p(p(ypq)))\ (\lambda cd.c(&& (\beta) \\
= \quad & \lambda xq.p(p(\lambda cd.c(c(cd))xp && (\beta)
\end{aligned}
$$

Addition example

$$
\begin{aligned}
ADD\ TWO\ THREE \\
&= \lambda xypq.xp(ypq)\ \ (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) && (\alpha) \\
&= \lambda ypq.(\lambda b.p(pb))(ypq)\ \ (\lambda cd.c(c(cd))) && (\beta) \\
&= (\lambda ypq.p(p(ypq)))\ (\lambda cd.c(&& (\beta) \\
&= \lambda xq.p(p(\lambda cd.c(c(cd))x)) && (\beta) \\
&= \lambda pq.p(p((\lambda d.p(p(pd)))q) && (\beta)
\end{aligned}
$$

Addition example

$$
\begin{aligned}
ADD\ TWO\ THREE &= \lambda xypq.xp(ypq)\ \ (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) && \\
&= \lambda ypq.(\lambda b.p(pb))(ypq)\ \ (\lambda cd.c(c(cd))) && (\alpha) \\
&= \lambda ypq.p(p(ypq))\ (\lambda cd.c(c(cd))) && (\beta) \\
&= \lambda pq.p(p((\lambda cd.c(c(cd)))p)q) && (\beta) \\
&= \lambda pq.p(p((\lambda d.p(p(pd)))q) && (\beta) \\
&= \lambda pq.p(p(p(p(pq)))) && (\beta)
\end{aligned}
$$

Addition example

$$
\begin{aligned}
ADD\ TWO\ THREE & \\
&= \lambda xypq.xp(ypq)\ \ (\lambda yz.y(yz))\ (\lambda yz.y(y(yz))) && (\alpha)\\
&= \lambda ypq.(\lambda b.p(pb))(ypq)\ \ (\lambda cd.c(c(cd))) && (\beta)\\
&= (\lambda ypq.p(p(ypq)))\ (\lambda cd.c(&& (\beta)\\
&= \lambda pq.p(p((\lambda cd.c(c(cd)))p && (\beta)\\
&= \lambda pq.p(p((\lambda d.p(p(pd)))q) && (\beta)\\
&= \lambda pq.p(p(p(p(pq)))) && (\beta)\\
&= FIVE
\end{aligned}
$$

Multiplication example

Multiplication example

Multiplication example

Just kidding

Multiplication example

MULT TWO THREE

Multiplication example

$$MULT\ TWO\ THREE$$
$$= (\lambda xyz.x(yz))\ TWO\ THREE$$

Multiplication example

$$MULT\ TWO\ THREE$$
$$= (\lambda xyz.x(yz))\ TWO\ THREE$$

Multiplication example

$$MULT\ TWO\ THREE$$
$$= (\lambda xyz.x(yz))\ TWO\ THREE$$

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Multiplication example

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&= \lambda zx.z(z(z(z(z(zx))))) \\
&= SIX
\end{aligned}
$$

Recursion

In imperative languages, we can easily write recursive code:

d

... by referencing the method itself by name!

So far, we haven't directly seen iteration or recursion in the lambda calculus.

Recursion

In the last tutorial you tried to reduce:

$$(\lambda x.xx)(\lambda x.xx)$$

Recursion

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This is

Combinator:

$$Y \equiv \lambda f.(\lambda x.f\,(x\,x))$$

Recursion

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This is
Combinator:

$$\mathsf{Y} \equiv \lambda f.(\lambda x.f\,(x\,x))$$

Next week, we'll use this to compute recursive functions in the lambda calculus.

OUTLINE

- Revision - Lambda Calculus
- https://eduassistpro.github.i
- Encodings

- **Functional Programming: LISP**

LISP

- LISP is the second oldest programming language in common
-
-
- Is a functional programming language
- Is a practical implementation of the Lambd
- Has many dialects (e.g. Clojure, Common L Scheme, etc.)

LISP = LISt Processing

- LISP has atoms
  - Numbers, e.g. 10
  - Identifiers, e.g. Foo
-
  - can contain atoms
  - can contain nothing (empty)
-
- *very* small syntax:

```
<object> ::= <atoms> | <list>
<list> ::= "(" { <object> } ")"
```

List Examples in LISP


```
(1 2 3)
()
(+ 1 2)
(* (+ 1 2) (
(sq 1 2)
(setq a 100)
(defun sq (n) (* n n))
(let ((a 6)) a)
(if t 5 6)
(cons 5 6)
(cons (cons 6 7))
```

Concepts of LISP

-
-
- Even *LISP* is written as a list.
- No other data structures

Evaluation

- Prefix notation of function calls as lists
-
-

```
(+ 4 2)
(+ 3 (- 3 2))
(sq (* 4 2))
```

Numerical Functions

- Numerical operations:
  - Addition: `(+ 1 2)`
  - Subtraction: `(- 1 2)`

- Base Exponent: `(expt x y)`
- Trigonometric Functions: `(sin x)`

- Absolute Value: `(abs x)`
- Modulo: `(mod x y)`
- Rounding: `(round x)`

Interaction

- Interaction with lisp is done in a *read-eval-print loop*
-

- Example:


```
>> (+ 1 2)
3
```

Variables

- Variables can be defined by:

  (setq <var> <value>)

-
-
- value of the variable
- Example:

```
>> (setq a (+ 5 3))
8
>> a
8
```

Quote

- If lists should not be evaluated, use function quote

- There is a short-hand form, using a single quo


```
>> (setq a '(+ 1 2))
(+ 1 2)
```

Condition Function

- Definition: `(if <cond> <true-value> <false-value>)`
- Boolean values in LISP are given by two symbols


```
>> ( if (= 10 10) 1 2)
1
>> ( if () 1 2)
2
```

Predicates

- Type checking predicates
  - `(atom x)` checks whether x is not a list
  - `(integerp x)` checks whether x is an integer
  - `(numberp x)` checks whether x is a number
- Equality
  - `(equal x y)` checks structural eq
  - `(eql x y)` checks atom equality
  - `(eq x y)` checks identity
  - `(= x y)` checks numerical equality
- Logical operators
  - `(or x y)` logical OR
  - `(and x y)` logical AND

Functions

- Function declaration:

```
(defun <name> (<arg1> ... <argn>) body)
```

- Translates to:

```
(define FACTORIAL
  (lambda (x)
    (* x (factorial (- x 1)))))
```

```
>> (factorial 4)
24
```

- Next week we'll do this in lambda calculus directly - without the impurity of defining variables

Bindings

- Definition:

```
(let ((<name1> <value1>) ... (<namen> <valuen>))
```



```
(+ (* a b) c))
17
>> a
Error: variable A is unbound
```

Bindings (2)

- Let allows local bindings of variables
- Bindings might be nested – innermost variable is taken


```
((a 5))
 a
))
5
```

List Construction

- Construction with cons: (cons <element> <list>)
  - Cons returns a new list with <element> as first element,
  - https://eduassistpro.github.i

```
(1)
>> (cons 'a '(b c))
(a b c)
>> (list 1 2 3)
(1 2 3)
```

List Access

- Access first element: `(first <list>)`
-

```
>> (rest '(a b c))
(b c)
```

λ in LISP

λ in Haskell

λ in Python

```
>>> (lambda x: x + 1) 4
5
>>> (lambda x: lambda y: lambda z: (x + x) * z)(1)
1
>
>
<f
>>> f(1)
<function <lambda>.<locals><lambda> at 0x02
>>> f(1)(3)
<function <lambda>.<locals>.<lambda>.<locals>.<lam
>>> f(1)(3)(5)
10
```

λ in Python


```
>>> NOT = lambda f: lambda x: lambda y: f(y)(x)
>>
>>
>>
>>
>>
'a'
>>> IF(NOT(NOT(TRUE)))("a")("b")
'b'
```

Review

- Lambda Calculus revision
  - Application, Abstraction
  - Rewriting
- Encodings
  - Boolean logic
  - Church numerals, arithmetic
- Functional programming
  - Introduction to LISP
  - Brief look λ in other languages