COMP2610/6261 - Information Theory

Lecture 20: Joint-Typicality and the Noisy-Channel Coding Theorem

Australian National University

October 24th, 2018

Channel Capacity: Recap

The *largest possible* reduction in uncertainty achievable across a channel is its **capacity**

Channe

The capa een its input and output for any choice of input ensemble. That is,

$$C = \max_{\mathbf{p}_X} I(X$$

Block Codes: Recap

(N, K) Block Code

Given a channel Q with inputs and outputs , an integer $N > 0$, and $K > 0$, an ds

where each $\mathbf{x}^{(s)} \in \mathcal{X}^N$ consists of N symb

Rate of a block code is $\frac{K}{N} = \frac{\log_2 S}{N}$

Reliability: Recap

$$\mathbf{s}_{in} \xrightarrow{\text{Encoder}} \mathbf{x} \xrightarrow{Q} \mathbf{y} \xrightarrow{\text{Decoder}} \mathbf{s}_{out}$$

Probabi

Given a ch

$$p_B = P(\mathbf{s}_{out} \neq \mathbf{s}_{in}) = \sum_{\mathbf{s}_{in}} P(\quad / $$

and its **maximum probability of (block) error**

$$p_{BM} = \max_{\mathbf{s}_{in}} P(\mathbf{s}_{out} \neq \mathbf{s}_{in} | \mathbf{s}_{in})$$

The Noisy-Channel Coding Theorem: Recap

Informal Statement

Recall that a rate R is achievable if there is a block code with this rate and arbitrarily small error probability

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Noisy-C

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Ideally, we would like to know:

- Can we go above C if we allow some fixed probability of error?
- Is there a maximal rate for a fixed probability of error?

1. Noisy-Channel Coding Theorem

2. Joint Typicality

3. Proof

4. Good Codes vs. Practical Codes

5. Linear Codes

The Noisy-Channel Coding Theorem

Formal Statement

Recall: a rate is achievable if for any tolerance $\epsilon > 0$, an (N, K) code with rate $K/N \geq R$ exists with max. block error $p_{BM} < \epsilon$

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Note that as $p_b \to \frac{1}{2}$, $R(p_b) \to +\infty$, while as $p_b \to \{0, 1\}$, $R(p_b) \to C$, so we cannot achieve rate greater than C with probability of bit error arbitrarily small

Implications of NCCT

Suppose we know a channel has capacity 0.6 bits

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We **cannot** achieve a rate of 0.8 with arbitrarily small error

We **can** achieve a rate of 0.8 with probability of bit error 5%, since $\frac{0.6}{1-H_2(0.05}$

Joint Typicality

Recall that a random variable $\mathbf{z}$ from Z^N is typical for an ensemble Z whenever its average symbol information is within β of the entropy $H(Z)$

$$\left| -\frac{1}{N} \log_2 P(\mathbf{z}) - H(Z) \right| < \beta$$

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Joint Ty

A pair of sequences $\mathbf{x} \in \mathcal{A}_X^N$ and $\mathbf{y} \in \mathcal{A}_Y^N$, e **ntly typical** (to tolerance β) for distribution $P(x$

1. $\mathbf{x}$ is typical of $P(\mathbf{x})$ … above]
2. $\mathbf{y}$ is typical of $P(\mathbf{y})$ … $=$ above]
3. $(\mathbf{x}, \mathbf{y})$ is typical of $P(\mathbf{x}, \mathbf{y})$ [$\mathbf{z} = (\mathbf{x}, \mathbf{y})$ above]

The **jointly typical set** of all such pairs is denoted $J_{N\beta}$.

Joint Typicality

Example

Example ($\mathbf{p}_X = (0.9, 0.1)$ and BSC with $f = 0.2$):

$\mathbf{x}$ 1111111111000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

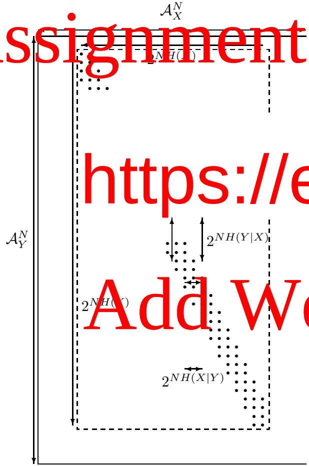
$\mathbf{y}$ 0011111111000000000000000000000000000000000000000000000000000000000000000000000000111111111111111111

Here:

- x has 10 1's (c.f. $p(X = 1) = 0.1$)

- y has 26 1's (c.f. $p(Y = 1) = (0.8)(0.$

- x, y differ in 20 bits (c.f. $p(X \neq Y) = 0.2$)
  - This is essential in addition to the above two facts

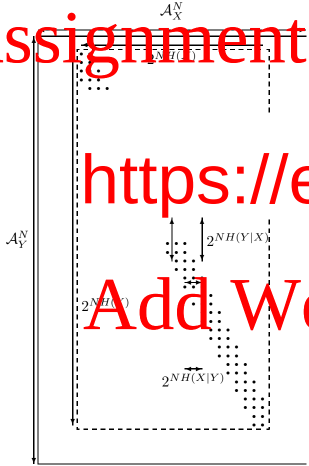


There are approximately:

- $2^{NH(X)}$ typical $\mathbf{x} \in \mathcal{A}_X^N$

Joint Typicality

Counts



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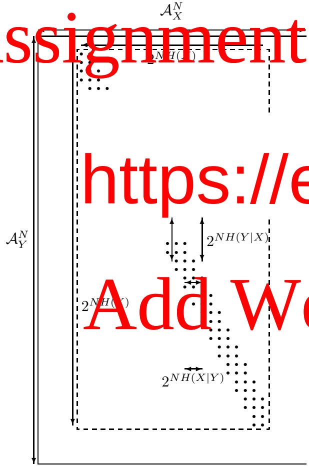

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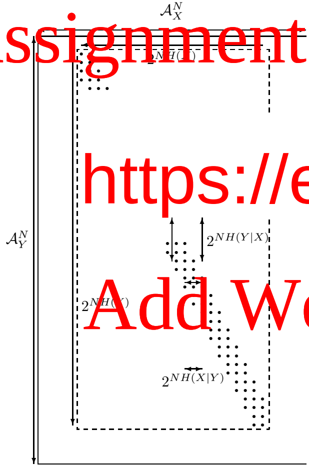

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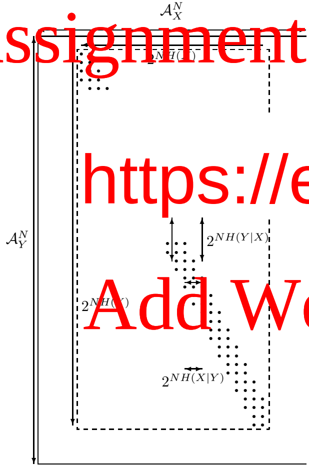

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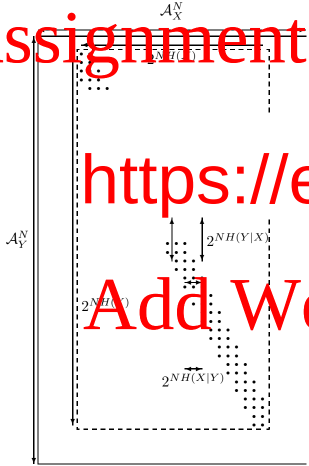

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Here we used

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

Joint Typicality Theorem

Let $\mathbf{x}, \mathbf{y}$ be drawn from $(XY)^N$ with $P(\mathbf{x}, \mathbf{y}) = \prod_n P(x_n, y_n)$.

Joint Typicality Theorem

For all tolerances $\beta > 0$

1. Almost every pair is eventually jointly typical

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3. For $\mathbf{x}'$ and $\mathbf{y}'$ drawn independently from th
   marginals of $P(\mathbf{x}, \mathbf{y})$,

$$P((\mathbf{x}', \mathbf{y}') \in J_{N\beta}) \leq 2^{-N(I(X;Y)-3\beta)}$$

The Noisy-Channel Coding Theorem

Let Q be a channel with inputs $\mathcal{A}_X$ and outputs $\mathcal{A}_Y$.

Let $C = \max_{p_X} I(X;Y)$ be the capacity of Q and $H_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$.

The Noisy-Channel Coding Theorem

1. Any rate $R < C$ is *achievable* for Q (i. … 0, an (N, K) code with rate $K/N \geq R$ exist … $_M < \epsilon$)
2. If probability of bit error $p_b := p_B/K$ i … (N, K) codes with rates
$$\frac{K}{N} \leq R(p_b) = \frac{C}{1 - H_2(p_b)}$$
3. For any p_b, rates greater than $R(p_b)$ are not achievable.

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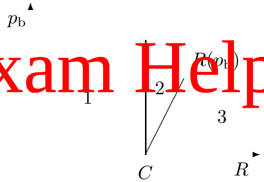

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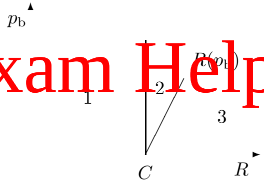

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The proof of the NCCT is based on the following observations:

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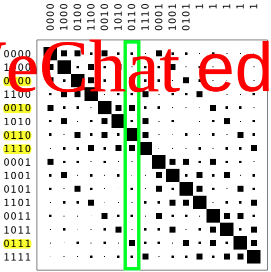

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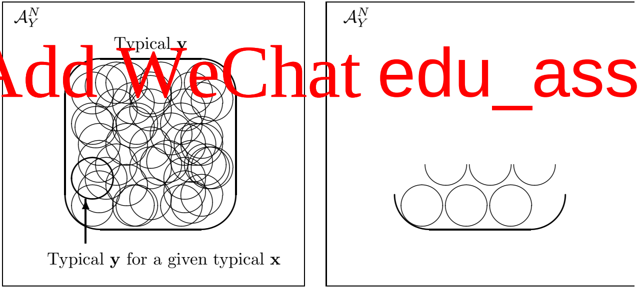

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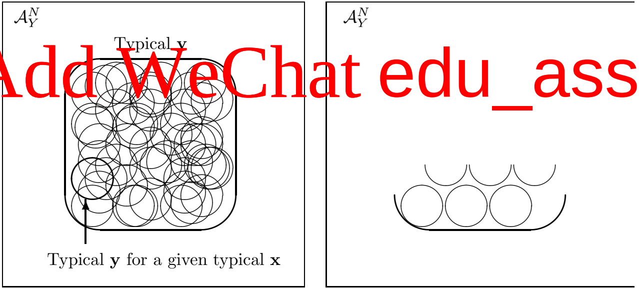

Proof Sketch of NCCT Part 1

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- define a family of random codes, which rely on joint typicality, and which achieve the given rate
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- “expurgate” the above code so that it has low m
  error

This will establish that the final code achieves low maximal probability of error, while achieving the given rate!

Random Coding and Typical Set Decoding

Make **random code** $\mathcal{C}$ with rate R':

- Fix $\mathbf{p}_X$ and choose $S = 2^{NR'}$ codewords, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(S)}$ each with $P(\mathbf{x}) = \prod_n P(x_n)$

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- If the
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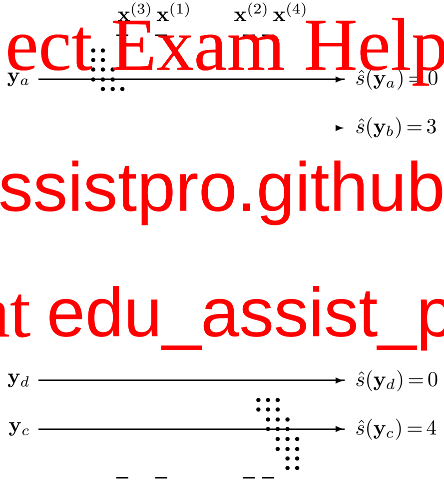
- Otherwise, fail ($\hat{s} = 0$)



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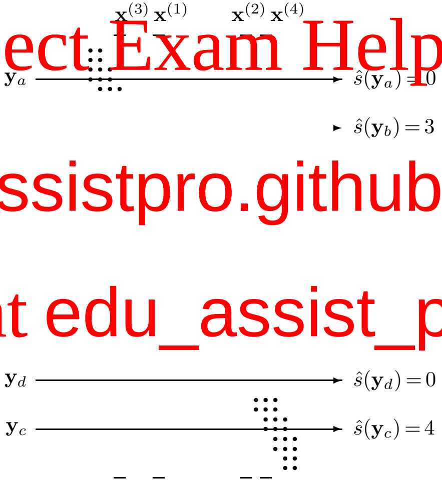
Errors:

- $p_B(\mathcal{C}) = P(\hat{s} \neq s|\mathcal{C})$
- $\langle p_B \rangle = \sum_{\mathcal{C}} P(\hat{s} \neq s|\mathcal{C}) P(\mathcal{C})$
- $p_{BM}(\mathcal{C}) = \max_s P(\hat{s} \neq s|s, \mathcal{C})$
  (Aim: $\exists \mathcal{C}$ s.t. $p_{BM}(\mathcal{C})$ small)



Average Error Over All Codes

Let's consider the average error over random codes:

$$\langle p_B \rangle = \quad P(\hat{s} \neq s|\mathcal{C})P(\mathcal{C})$$

A bound p es $z \in \mathcal{Z}$
with prob such that
$f(z^*)$ is smaller than the bound.

[1]If $\langle f \rangle < \delta$ but $f(z) \geq \delta$ for all z, $\langle f \rangle = \sum_z f(z)P(z) \geq \sum_z \delta P(z) = \delta$!!

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So $\langle p_B \rangle < \delta \implies p_B(\mathcal{C}^*)$

Analogy: Suppose the average height of class is not more than 160 cm. Then one of you *must* be shorter than 160 cm.

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Proof Sketch of NCCT Part 1

Want to prove

> Any rate $R < C$ is *achievable* for Q (i.e., an (N, K) code with rate $N/K \geq R$ exists with max. block error $p_{BM} < \epsilon$ for any tolerance ϵ)

Let us thus

Choose s

1. Part one of the Joint Typicality Theorem says we can find an $N(\delta)$ such that the probability $(\mathbf{x}, \mathbf{y})$ are not jo δ.

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Let us thus bound p_B for our random code

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1. Par $N(\delta)$
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2. Thus, the average probability of error satisfie

$$\langle p_B \rangle = \sum_{\text{atypical } (\mathbf{x},\mathbf{y})} P(\hat{s} \neq s|\cdot) \qquad \sum_{\text{typical } (\mathbf{x},\mathbf{y})} / \quad |\cdot$$

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3. Increasing N will make $\langle p_B \rangle < 2\delta$ if $R' < I(X;Y) - 3\beta$

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3. Increasing N will make $\langle p_B \rangle < 2\delta$ if $R' < I(X; Y) - 3\beta$
4. Choosing maximal $P(x)$ makes required condition $R' < C - 3\beta$

Code Expurgation

The last main "trick" is to show that if there is an (N, K) code with rate R' and $p_B(\mathcal{C}) < \delta$ we can construct a new (N, K') code $\mathcal{C}'$ with rate $R' - \frac{1}{N}$ and maximum probability of error $p_{BM}(\mathcal{C}') < 2\delta$.

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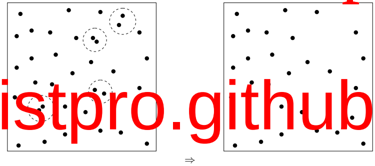
We create $\mathcal{C}'$ by **expurgating** (throwing out) half the codewords from , specifically the half with probabilit



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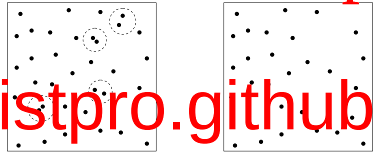

$\Rightarrow$

Proof:

- Code $\mathcal{C}'$ has $2^{NR'}/2 = 2^{NR'-1}$ messa $' - \frac{1}{N}$
- Suppose $p_{BM}(\mathcal{C}') = \max_s P(\hat{s} \neq s|s,$ that was thrown out must have conditional probability $P(\hat{s} \neq s|s, \mathcal{C}) \geq 2\delta$
- But then

$$p_B(\mathcal{C}) = \sum_s P(\hat{s} \neq s|s, \mathcal{C})P(s) \geq \frac{1}{2}\sum_{s \notin \mathcal{C}'} 2\delta + \frac{1}{2}\sum_{s \in \mathcal{C}'} P(\hat{s} \neq s|s, \mathcal{C}) \geq \delta$$

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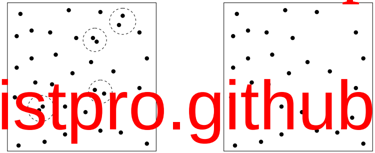

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Wrapping It All Up

From the previous slide, $\langle p_B \rangle < 2\delta \implies$ some $\mathcal{C}'$ such that $p_{BM}(\mathcal{C}') < 4\delta$ with rate $\frac{1}{}$

Setting

NCCT Part 1: Comments

NCCT shows the existence of good codes; actually constructing practical codes is another matter

In principl

- Ho
  the t

Over the past few decades, some codes (e.g. Turb
shown to achieve rate close to the Shannon capacit

- Beyond the scope of this course!

NCCT Converse: Comments

One can in fact make a stronger statement about

$$p_{B,\text{avg}} = \frac{1}{2^K} \quad P(\mathbf{s}_{\text{out}} \neq \mathbf{s}_{\text{in}} \mid \mathbf{s}_{\text{in}}),$$

the proba

We have:

$$p_{B,\text{avg}} \geq 1 - O(e$$

Thus, if $R > C$, the probability of block error shoo ases!

- We have a “phase transition” around C between perfectly reliable and perfectly unreliable communication!

Theory and Practice

The difference between theory and practice is that, in theory, there is no difference between theory and practice but, in practice, there is.

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Theory v

- The
  cha
- However, the theorem is non-constructive: ... ow to create *practical* codes for a given noisy cha

- The construction of practical codes that achieve rates up to the capacity for general channels is ongoing research

Types of Codes

When we talk about types of codes we will be referring to schemes for creating (N, K) codes for any size N. MacKay makes the following distinctions:

- **Bad**: Cannot achieve arbitrarily small error, or only achieve it if the rate

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  make a code with $K/N = C$ and $p_{BM} < \epsilon$)
- **Practical**: Can be coded and decoded in time that is polynomial in the block length N.

Random Codes

During the discussion of the Noisy-Channel Coding Theorem we saw how to construct very good **random codes** via typical set decoding

Properties:

- Ver ... rror
- Co
- Not Practical:
  - The 2^K codewords have no structure an
  - Typical set decoding is expensive

Linear Codes

(N, K) Block Code

An (N, K) **block code** is a list of $S = 2^K$ codewords $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(S)}\}$, each of length N. A message $s \in \{1, 2, \ldots, 2^K\}$ is encoded as $\mathbf{x}^{(s)}$.

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Example: Suppose $(N, K) = (7, 4)$. To send $s = 3$, first create $\mathbf{s} = 0011$ and send $\mathbf{t} = \mathbf{G}^\top \mathbf{s} = \mathbf{G}^\top(\mathbf{e}_0 + \mathbf{e}_1) = \mathbf{G}^\top \mathbf{e}_0 + \mathbf{G}^\top \mathbf{e}_1 = \mathbf{t}_0 + \mathbf{t}_1$ where $\mathbf{e}_0 = 0001$ and $\mathbf{e}_1 = 0010$.

Types of Linear Code

Many commonly used codes are linear:

- Repetition Codes: e.g., 0 $\rightarrow$ 000 ; 1 $\rightarrow$ 111
- Convolution Codes: Linear coding plus bit shifts
- Concatenation Codes: Two or more levels of error correction
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An NCCT
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Practical linear codes:

- Use very large block sizes N
- Based on semi-random code constructions
- Apply probabilistic decoding techniques
- Used in wireless and satellite communication

Linear Codes: Examples

(7,4) Hamming Code

(6,3) Repetition Code

1 0 0 0

1 0 0

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

For $\mathbf{s} = 0011$,

$$\mathbf{G}^\top \mathbf{s} (\bmod 2) = [0\ 0\ 1\ 1\ 1\ 0\ 0]^\top$$

For

$$\mathbf{G}^\top \mathbf{s} (\bmod 2) = [0\ 1\ 0\ 0\ 1\ 0]^\top$$

Decoding

We can construct codes with a relatively simple encoding but how do we decode them? That is, given the input distribution and channel model Q how do we find the posterior distribution over $\mathbf{x}$ given we received $\mathbf{y}$?

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Simple? J

$$\overline{\quad}$$
$$\mathbf{x}' \in \mathcal{C} \; P(\mathbf{y}\ \mathbf{x})P(\mathbf{x})$$

But:

- the number of codes $\mathbf{x} \in \mathcal{C}$ is 2^K so, n
- linear codes provide structure that the above method doesn't exploit

Summary and Reading

Main Points:

- Joint Typicality and the Joint Typicality Theorem

- The (Longer) Noisy Channel Coding Theorem
- Pro

  - Average Error Over Random Codes

  - Code Expurgation

Reading:

- MacKay §9.7, §10.1-§10.5