COMP2610 / 6261 - Information Theory

Lecture 14: Source Coding Theorem for Symbol Codes

Australian National University

17 September, 2018

Last time

Variable-length codes

Uniquely

- Pre

Kraft's inequality:

Lengths $\{\ell_i\}_{i=1}^{I}$ can form a prefix code

How to **generate** prefix codes?

Prefix Codes (Recap)

A simple property of codes **guarantees** unique decodeability

Prefix property

A codeword $\mathbf{c}$ 0, 1 $^{+}$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0,$ **t**.

Can you cr

- **Example**: 01101 has prefixes 0, 01, 011, 0110.

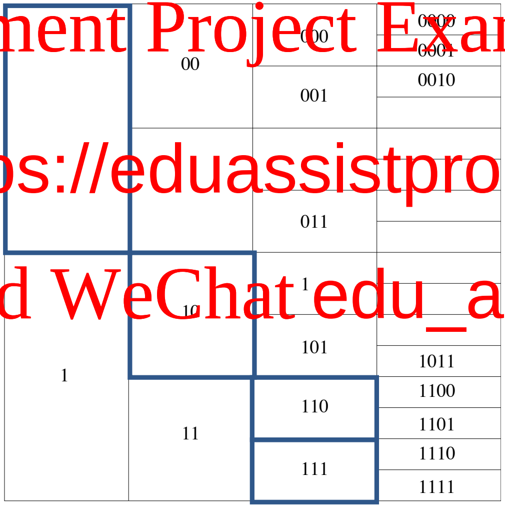
Prefix Codes

A code $C = \{\mathbf{c}_1, \ldots, \mathbf{c}_I\}$ is a **prefix code** C there is no prefix of $\mathbf{c}_i$ in C.

In a stream, no confusing one codeword with another

Prefix Codes as Trees (Recap)

$$C_2 = \{0, 10, 110, 111\}$$



This time

Bound on expected length for a prefix code

Shannon codes

Huffman coding

1. Expected Code Length
   - Minimising Expected Code Length
   - Shannon Coding

2. The S

3. Huffman Coding
   - Algorithm and Examples
   - Advantages and Disadvantages

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Expected Code Length

With uniform codes, the length of a message of N outcomes is trivial to compute

With variable-length codes, the length of a message of N outcomes will depend on the outcomes we observe

Outcom

- On a

Expected Code Length

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Expected Code Length

The **expected length** for a code C for ense . . . , $a_I\}$ and $\mathcal{P}_X = \{p_1, \ldots, p_I\}$ is

$$L(C, X) = \mathbb{E}[\ell(x)] = \sum_{x \in \mathcal{A}_X} p(x)\,\ell(x) = \sum_{i=1}^{I} p_i\,\ell_i$$

Expected Code Length: Examples

Example: X has $\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\}$

1. The code $C_1 = \{0001, 0010, 0100, 1000\}$ has

$$L(C_1, X) = \sum^{4} p_i \, \ell_i = 4$$

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$$L(C_1, X) = \sum_{i=1}^{4} p_i \, \ell_i = 4$$

2. The

$$L(C_2, X) = \sum_{i=1}^{4} p_i \, \ell_i = \frac{1}{2} \times 1 + \frac{1}{4} \ldots .25$$

Code Lengths and Probabilities

The *Kraft inequality* says that $\{\ell_1, \ldots, \ell_I\}$ are prefix code lengths **iff**

$$\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$$

If it were tru

$$i=1$$

then we could interpret

$$\mathbf{q} = (2^{-\ell_1}, \ldots,$$

as a probability vector over I outcomes

General lengths ℓ?

Code Lengths and Probabilities

Probabilities from Code Lengths

Given code lengths $\ell = \{\ell_1, \ldots, \ell_I\}$ such that $\sum_{i=1}^{I} 2^{-\ell_i} \leq 1$, we define $\mathbf{q} = \{q_1, \ldots, q_I\}$, the **probabilities for** ℓ, by

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2. Lengths $\{2, 2, 3\}$ give $z = \frac{5}{8}$ so $q_1 = \frac{2}{5}$, $q_2 = \frac{2}{5}$, and $q_3 = \frac{1}{5}$

Minimising Expected Code Length

The probability view of lengths will be useful in answering:

Goal of compression

Given an ensemble X with probabilities $\mathcal{P}_X = \mathbf{p} = \{p_1, \ldots, p_I\}$ how can we minimise the expected code length?

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Limits of c

Given an ensemble X with probabilities $\mathbf{p}$, a codeword length probabilities $\mathbf{q}$ and normali

$$L(C, X) = H(X) + D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) + \log_2 \frac{1}{z}$$

$$\geq H(X),$$

with equality only when $\ell_i = \log_2 \frac{1}{p_i}$.

Minimising Expected Code Length

Suppose we use code C with lengths $\ell = \{\ell_1, \ldots, \ell_I\}$ and corresponding probabilities $\mathbf{q} = \{q_1, \ldots, q_I\}$ with $q_i = \frac{1}{z} 2^{-\ell_i}$. Then,

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$$= H(X) + D_{\mathrm{KL}}(\mathbf{p} \| \mathbf{q}) + \log_2(1/z) \cdot 1$$

Minimising Expected Code Length

So if $\mathbf{q} = \{q_1, \ldots, q_I\}$ are the probabilities for the code lengths of C then under ensemble X with probabilities $\mathbf{p} = \{p_1, \ldots, p_I\}$

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(Gibb's inequality)

For $\mathbf{q} = \mathbf{p}$, we have $z \stackrel{\text{def}}{=} \sum_i q_i = \sum_i p_i = 1$ and so $\log_2 \frac{1}{z} = 0$

Entropy as a Lower Bound on Expected Length

We have shown that for a code C with lengths corresponding to $\mathbf{q}$,

$$L(C, X) \quad H(X)$$

with equa

Once again, the entropy determines a lower bound
compression is possible

- $L(C, X)$ refers to *average* compressio
- Individual message length could be bigger th

Shannon Codes

If we pick lengths $\ell_i = \log_2 \frac{1}{p_i}$, we get optimal expected code lengths

But $\log_2 \frac{1}{p_i}$ is not always an integer—a problem for code lengths!

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Shanno

Given an e ... s

$\ell = \{\ell_1, \ldots$

$$\ell_i = \log_2 \frac{}{p_i} \geq \quad -$$

A code C is called a **Shannon code** if it has c

Here $\lceil x \rceil$ is "smallest integer not smaller than x". e.g., $\lceil 2.1 \rceil = 3$, $\lceil 5 \rceil = 5$.

This gives us code lengths that are "closest" to $\log_2 \frac{1}{p_i}$

Shannon Codes: Examples

Examples:

1. If $\mathcal{P}_X = \ \frac{1}{\ }, \frac{1}{\ }, \frac{1}{\ }\ $ then $\ell = \ 1, 2, 2\ $ so $C = \ 0, 10, 11\ $ is a Shannon cod

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$C = \{00, 10, 11\}$ (or $C = \{01, 10, 11$

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Entropy also gives a guideline upper bound of comp

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Since $\lceil x \rceil$ is the *smallest* integer bigger than or equal to x it must be the case that $x \leq \lceil x \rceil < x + 1$.

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Therefore, if we create a Shannon code C for $\mathbf{p} = \{p_1, \ldots, p_n\}$ with $\ell_i = \left\lceil \log_2 \frac{1}{p_i} \right\rceil < \log_2 \frac{1}{p_i} + 1$ it will satisfy

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Furthermore, since $\ell_i \geq -\log_2 p_i$ we have 2 [illegible]
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2. If $\mathcal{P}_X = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ then $\ell = \{2, 2, 2\}$ and $H(X) = \log_2 3 \approx 1.58 \leq L(C, X) = 2 \leq 2.58 \approx H(X) + 1$

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   - Minimising Expected Code Length
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The previous arguments have established:

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In particular, **Shannon codes** C — those wit $-\rceil$ —

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This is good, but is it

Shannon codes are suboptimal

Example: Consider $p_1 = 0.0001$ and $p_2 = 0.9999$. (Note $H(X) \approx 0.0013$)

- The Shannon code C has lengths $\ell_1 = \lceil \log_2 10000 \rceil = 14$ and $\ell_2 = \left\lceil \log_2 \frac{10000}{9999} \right\rceil = 1$
- The ... $99 = 1.0013$
- But c...

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Shannon codes do not necessarily have ... th

This is perhaps disappointing, as these codes were constructed very naturally from the theorem

- Fortunately, there is another simple code that **is** provably optimal

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Constructing a Huffman Code

Huffman Coding is a procedure for making provably optimal prefix codes

It assigns the longest codewords to least probable symbols

Basic alg

- Tak

- Prepend bits 0 and 1 to current codewords of sy

- Combine these two symbols into a single "met

- Repeat

Huffman Coding: Example 1

Start with $\mathcal{A} = \{a, b, c\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

Step 1

c 0.25

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

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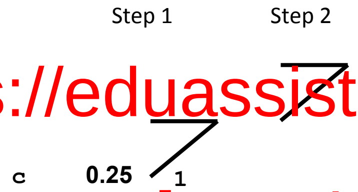
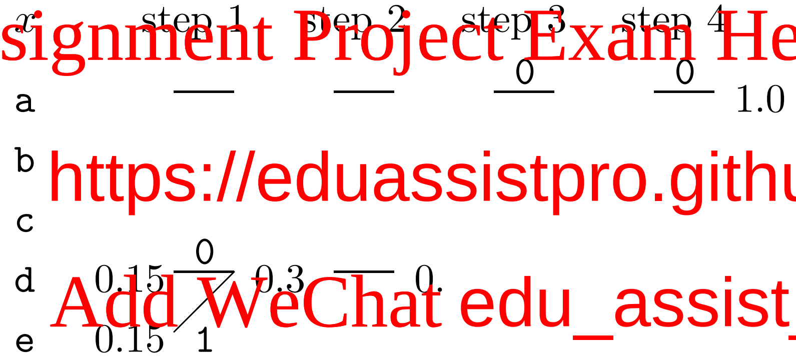


Now we read off the labelling implied by path from the la
each of the original symbols: $C = \{0, 10, 11\}$

Huffman Coding: Example 2

$\mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P}_X = \{0.25, 0.25, 0.2, 0.15, 0.15\}$



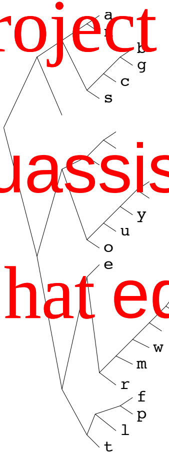
From Example 5.15 of MacKay

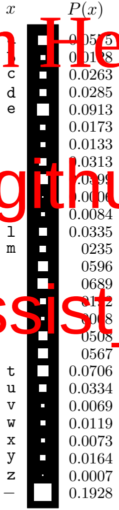
$C = \{00, 10, 11, 010, 011\}$

Huffman Coding: Example 3

English letters – Monogram statistics

a_i	p_i	$\log_2 \frac{1}{p_i}$	l_i	$c(a_i)$
a	0.0575	4.1	4	0000
b	0.0128	6.3	6	001000
c	0.0263	5.2	5	00101
d	0.0285	5.1	5	10000
e				
f				
g				
h				
i				
j				
k				
l	0.0335	4.9	5	11101
m	0.0235	5.4	6	110101
n	0.0596	4.1	4	0001
o	0.0689	3.9	4	1011
p	0.0192	5.7	6	111001
q	0.0008	10.3	9	110100001
r	0.0508	4.3	5	11011
s	0.0567	4.1	4	0011
t	0.0706	3.8	4	1111
u	0.0334	4.9	5	10101
v	0.0069	7.2	8	11010001
w	0.0119	6.4	7	1101001
x	0.0073	7.1	7	1010001
y	0.0164	5.9	6	101001
z	0.0007	10.4	10	1101000001
–	0.1928	2.4	2	01





Huffman Coding: Formally

HUFFMAN($\mathcal{A}, \mathcal{P}$):

1. If $|\mathcal{A}| = 2$ return $C = \{0, 1\}$; else
2. Let $a, a' \in \mathcal{A}$ be *least probable* symbols
3. Let $' =$ a, a' aa'
4. Let
5. Co
6. Defi
   - $( ) = ( )$
   - $c(a') = c'(aa')1$
   - $c(x) = c'(x)$ for $x \in \mathcal{A}'$
7. Return C

Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

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  - b and c are least probable with $p_a = p_b = \frac{1}{4}$

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  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$

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  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  -
    -
    -
  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = 10$
    - $c(\mathtt{c}) = c'(\mathtt{bc})1 = 11$
    - $c(\mathtt{a}) = c'(\mathtt{a}) = 0$

Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_\mathtt{b} = p_\mathtt{c} = \frac{1}{4}$

  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$

  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = 10$
    - $c(\mathtt{c}) = c'(\mathtt{bc})1 = 11$
    - $c(\mathtt{a}) = c'(\mathtt{a}) = 0$
  - Return $C = \{0, 10, 11\}$

Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN$(\mathcal{A}, \mathcal{P})$:
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_{\mathtt{a}} = p_{\mathtt{b}} = \frac{1}{4}$
  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  -
    -
    -
  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = 10$
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Huffman Coding: Example 1

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ and $\mathcal{P} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathtt{b}$ and $\mathtt{c}$ are least probable with $p_\mathtt{a} = p_\mathtt{b} = \frac{1}{4}$
  - $\mathcal{A}' = \{\mathtt{a}, \mathbf{bc}\}$ and $\mathcal{P}' = \{\frac{1}{2}, \mathbf{\frac{1}{2}}\}$
  -
  - Define
    - $c(\mathtt{b}) = c'(\mathtt{bc})0 = 10$
    - $c(\mathtt{c}) = c'(\mathtt{bc})1 = 11$
    - $c(\mathtt{a}) = c'(\mathtt{a}) = 0$
  - Return $C = \{0, 10, 11\}$

The constructed code has $L(C, X) = \frac{1}{2} \times 1 + \frac{1}{4} \times (2 + 2) = 1.5$.
The entropy is $H(X) = 1.5$.

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN(′, ′):

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($'$, $'$):
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Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($'$, $'$):
    -
    -
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}, \texttt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\texttt{a}, \texttt{b}, \texttt{c}, \textbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($\mathcal{A}'$, $\mathcal{P}'$):
    -
    -
      - Return $c'''(\texttt{ade}) = 0, c'''(\texttt{bc}) = 1$
    - Return $c''(\texttt{a}) = \mathbf{0}0, c''(\texttt{bc}) = 1, c$

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN(′, ′):
    -
    -
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{00}$, $c''(\mathtt{bc}) = 1$, c
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{10}$, $c'(\mathtt{c}$

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($'$, $'$):
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      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, c
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{1}0$, $c'(\mathtt{c}$
- Return $c(\mathtt{a}) = 00$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 11$, $c(\mathtt{d}) = \mathbf{01}0$, $c(\mathtt{e}) = \mathbf{01}1$

Huffman Coding: Example 2

Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($'$, $'$):
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  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = \mathbf{1}0$, $c'(\mathtt{c}$
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Start with $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$ and $\mathcal{P} = \{0.25, 0.25, 0.2, 0.15, 0.15\}$

- HUFFMAN($\mathcal{A}, \mathcal{P}$):
  - $\mathcal{A}' = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathbf{de}\}$ and $\mathcal{P}' = \{0.25, 0.25, 0.2, \mathbf{0.3}\}$
  - Call HUFFMAN($'$, $'$):
    -
    -
      - Return $c'''(\mathtt{ade}) = 0$, $c'''(\mathtt{bc}) = 1$
    - Return $c''(\mathtt{a}) = \mathbf{0}0$, $c''(\mathtt{bc}) = 1$, c
  - Return $c'(\mathtt{a}) = 00$, $c'(\mathtt{b}) = 10$, $c'(\mathtt{c}$
- Return $c(\mathtt{a}) = 00$, $c(\mathtt{b}) = 10$, $c(\mathtt{c}) = 11$, $c(\mathtt{d}) = \mathbf{01}0$, $c(\mathtt{e}) = \mathbf{01}1$

The constructed code is $C = \{00, 10, 11, 010, 011\}$.
It has $L(C, X) = 2 \times (0.25 + 0.25 + 0.2) + 3 \times (0.15 + 0.15) = 2.3$.
Note that $H(X) \approx 2.29$.

Huffman Coding in Python

See full example code with examples at:

https://gist.github.com/mreid/fdf6353ec39d050e972b

```python
def huffman(p):
    '''Return a Huffman code for an ensemble with distribution p'''
    assert(sum(p.values()) == 1.0) # Ensure probabilities sum to 1

    # Base cas
    if(le
        r

    # Create a ne
    p_prime = p.copy()
    a1, a2 = lowest_prob_pair(p)
    p1, p2 = p_prime.pop(a1), p_prime.pop(a2)
    p_prime[a1 + a2] = p1 + p2

    # Recurse and construct code on new distribution
    c = huffman(p_prime)
    ca1a2 = c.pop(a1 + a2)
    c[a1], c[a2] = ca1a2 + '0', ca1a2 + '1'

    return c
```

Advantages of Huffman coding

- Produces prefix codes automatically (by design)

- Algorithm is simple and efficient
- Huf

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If C_{Huff} is a Huffman code, then for any other uniquely decodable code C',

$$L(C_{\text{Huff}}, X) \leq$$

It follows that

$$H(X) \leq L(C_{\text{Huff}}, X) < H(X) + 1$$

Disadvantages of Huffman coding

- Assumes a fixed distribution of symbols
- The
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Disadvantages of Huffman coding

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    Huffman codes are the best possibl

– but symbol coding is not always the best t

Next Time: Stream Codes!

Summary

Key Concepts:

1. The expected code length $L(C, X) = \sum_i p_i \ell_i$
2. Probabilities and codelengths are interchangeable

   $q_i = 2^{-\ell_i} \iff \ell_i = \log_2 \frac{1}{q_i}$

3. Rel … py

   $H($ …

4. The … (Shannon) code C for ensemble X with $\ell_i =$ log – so that

   $$H(X) \leq L(C; X)$$

5. Huffman codes are optimal symbol codes

Reading:

- §5.3-5.7 of MacKay
- §5.3-5.4, §5.6 & §5.8 of Cover & Thomas