COMP2610/6261 - Information Theory

Lecture 21: Hamming Codes & Coding Review

Australian National University

16 October, 2018

Today's plan

Noisy Channel Coding Theorem proves there exists codes with rate $R < C$ with arbitrarily low probability of error.

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What abo

We will foc

- repetition codes
- Hamming codes

We will sketch what can be said about the rate and reliability of the latter

1. Repetition Codes

2. The (7, 4) Hamming code
   - Co
   - De
   - Sy
   - M
   - Error Probabilities

3. Coding: Review

Repetition Codes

Simplest channel code: add *redundancy* by repeating every bit of the message (say) 3 times:

0	0 0 0
1	1 1 1

This *repetition code* is called R_3.

Repetition Codes for the BSC

Example

On a binary symmetric channel with flip probability f, receiver sees

$$\mathbf{r} = \mathbf{t} + \boldsymbol{\eta}$$

where $\boldsymbol{\eta}$ is a *noise* vector

- $p(\boldsymbol{\eta}$

Repetition Codes for the BSC

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- $p(\boldsymbol{\eta}$

Example

s	0	0	1	0	1	1	0
t	0 0 0	0 0 0	1 1 1	0 0 0	[illegible]	[illegible]	[illegible]
$\boldsymbol{\eta}$	0 0 0	0 0 1	0 0 0	0 0 0	1 0 1	0 0 0	0 0 0
r	0 0 0	0 0 1	1 1 1	0 0 0	0 1 0	1 1 1	0 0 0

Note that elements of $\boldsymbol{\eta}$ are not replicated like those of $\boldsymbol{t}$

- noise acts independently on every bit

Beyond Repetition Codes

Goal: Communication with small probability of error and high rate:

- Repetition codes introduce redundancy on a per-bit basis
- Can we improve on this?

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Idea: Introduce redundancy to blocks of data instead

Block Co

A block co s
into a length-N sequence of transmitted bits **t**.

- Introduce redundancy: $N > K$
- Focus on *Linear codes*

We will introduce a simple type of block code called
the (7,4) Hamming code

An Example

The (7, 4) Hamming Code

Consider $K = 4$, and a source message $\mathbf{s} = 1\ 0\ 0\ 0$

The repetition code R_2 produces

The (7,4) Hamming code produces

$$\mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0$$

- Redundancy, but not repetition
- How are these magic bits computed?

1 Repetition Codes

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The (7,4) Hamming code

Coding

- Consider $K = 4$, $N = 7$ and $\mathbf{s} = 1\,0\,0\,0$

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The (7,4) Hamming code
Coding

Copy the source bits into the the first 4 target bits:

The (7,4) Hamming code

Coding

Set *parity-check* bits so that the number of ones within each circle is even:

So we have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1$

The (7,4) Hamming code

Coding

Algebraically, we have set:

$$t_6 = s_2 \oplus s_3 \oplus s$$

$$t_7 = s_1 \oplus s_3 \oplus s$$

where we use modulo-2 arithmetic

In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where $\mathbf{s} = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix}^T$

G is called the *Generator matrix* of the code.

The Hamming code is linear!

The (7,4) Hamming code:

Codewords

Each (unique) sequence that can be transmitted is called a *codeword*.

Codewo

```
1010    1010010
1110       ?
```


For the (7,4) Hamming code we have a total of 16 code

The (7,4) Hamming code

Codewords

Write

$$\mathbf{G}^T = \begin{bmatrix} \mathbf{G}_{1\cdot} & \mathbf{G}_{2\cdot} & \mathbf{G}_{3\cdot} & \mathbf{G}_{4\cdot} \end{bmatrix}$$

where ea

Then, the t

$$\begin{aligned} \mathbf{t} &= \mathbf{G}^T \mathbf{s} \\ &= \begin{bmatrix} \mathbf{G}_{1\cdot} & \mathbf{G}_{2\cdot} & \mathbf{G}_{3\cdot} \end{bmatrix} \\ &= s_1 \mathbf{G}_{1\cdot} + \ldots + s_4 \mathbf{G}_{4\cdot} \end{aligned}$$

The (7,4) Hamming code:

Codewords

There are 2^7 possible transmitted bit strings

- There are $2^7 - 2^4$ other bit strings that immediately imply corruption
- If we s

Any two codewords differ in at least three bits

- Each original bit belongs to at least two circles

- Useful in constructing reliable decoders

The (7,4) Hamming code:

Decoding

We can encode a length-4 sequence **s** into a length-7 sequence **t** using 3 parity check bits

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How should we decode **r**?

- We could do this exhaustively using the 16 codewords
- Assuming BSC, uniform $p(\mathbf{s})$: Get the most probable explanation
- Find **s** such that $\|\mathbf{t}(\mathbf{s}) \ominus \mathbf{r}\|_1$ is minimum

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We can get the most probable source vector in an more *efficient* way.

The (7,4) Hamming code:

Decoding Example 1

We have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1 \xrightarrow{\text{noise}} \mathbf{r} = 1\ 1\ 0\ 0\ 1\ 0\ 1$:

(1) Detect circles with wrong (odd) parity

- What bit is responsible for this?

The (7,4) Hamming code:

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(2) Detect culprit bit and flip it

- The decoded sequence is $\hat{\mathbf{s}} = 1\ 0\ 0\ 0$

The (7,4) Hamming code:

Decoding Example 2

We have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1 \xrightarrow{\text{noise}} \mathbf{r} = 1\ 0\ 0\ 0\ 0\ 0\ 1$:

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The (7,4) Hamming code:

Optimal Decoding Algorithm: Syndrome Decoding

Given $\mathbf{r} = r_1, \ldots, r_7$, assume BSC with small noise level f:

1. Define the syndrome as the length-3 vector $\mathbf{z}$ that describes the pattern of violations of the parity bits r_5, r_6, r_7.
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$\mathbf{z}$	0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
Flip bit	none	r_7	r_6	r_4	r_5	r_1	r_2	r_3

The (7,4) Hamming code:

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Flip bit	none	r_7	r_6	r_4	r_5	r_1	r_2	r_3

The optimal decoding algorithm unflips at most one bit

The (7,4) Hamming code:

Syndrome Decoding: Matrix Form

Recall that we just need to compare the expected parity bits with the actual ones we received:

$$z_1 = r_1 \oplus r_2 \oplus r_3 \ominus r_5$$

The (7,4) Hamming code:

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but in modulo-2 arithmetic $-1 \equiv 1$ so we can r e have:

$$\mathbf{z} = \mathbf{Hr} \text{ with } \mathbf{H} = \begin{bmatrix} \mathbf{P} & \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 1\,0\,1\,1\,0\,0\,1 \end{bmatrix}$$

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Homework: What is the syndrome for a codeword?

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3. Coding: Review

The (7,4) Hamming code:

Optimal Decoding Algorithm: Syndrome Decoding

When the noise level f on the BSC is small, it may be reasonable that we see only a si

The syndr

this case

- c.f. Noise flipping one bit in the repetition code

But what happens if the noise flips more than one bit?

The (7,4) Hamming code:

Decoding Example 4: Flipping 2 Bits

We have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1 \xrightarrow{\text{noise}} \mathbf{r} = 1\ 0\ 1\ 0\ 1\ 0\ 0$:

(1) Detect circles with wrong (odd) parity

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The (7,4) Hamming code:

Decoding Example 4: Flipping 2 Bits

We have $\mathbf{s} = 1\ 0\ 0\ 0 \xrightarrow{\text{encoder}} \mathbf{t} = 1\ 0\ 0\ 0\ 1\ 0\ 1 \xrightarrow{\text{noise}} \mathbf{r} = 1\ 0\ 1\ 0\ 1\ 0\ 0$:

(2) Detect culprit bit and flip it

- The decoded sequence is $\hat{\mathbf{s}} = 1\ 1\ 1\ 0$
  - We have made 3 errors but only 2 involve the actual message

The (7,4) Hamming code:

Decoding Exercises

[Mackay, Ex 1.5]: Decode the following sequences using the syndrome decoding for the (7,4) Hamming code:

(a) $\mathbf{r} =$

(b) $\mathbf{r} =$

(c) $\mathbf{r} = 0100111 \rightarrow \hat{\mathbf{s}} = ??$

(d) $\mathbf{r} = 1111111 \rightarrow \hat{\mathbf{s}} = ??$

Work out the answers on your own.

The (7,4) Hamming code: Solution

For each exercise we simply compute the *syndrome* and use the optimal decoding algorithm (Table above) to determine which bit we should unflip.

(a) $\mathbf{r} = 1101011 \rightarrow$: $z_1 = r_1 \oplus r_2 \oplus r_3 \oplus r_5 = 0$ $z_2 = r_2 \oplus r_3 \oplus r_4 \oplus r_6 = 1$ ... r_4, $\hat{\mathbf{s}} =$

(b) $\mathbf{r} = 0110110 \rightarrow \mathbf{z} = 111$, we unflip r_3

(c) $\mathbf{r} = 0100111 \rightarrow \mathbf{z} = 001$, we unflip r_7

(d) $\mathbf{r} = 1111111 \rightarrow \mathbf{z} = 000$, we don't unflip any bit, $\hat{\mathbf{s}} = 1111$

The (7,4) Hamming code:

Zero-Syndrome Noise Vectors

[Mackay
(so that the
of these vectors are there?

Solution

By definition we have that the all-zero syndrome implies that the corresponding noise components should cancel out. For example for the first component we have:

$z_1 = r_1 \oplus r_2 \oplus r_3 \oplus r_5 = t_1 \oplus t_2 \oplus t_3 \oplus t_5 \oplus \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_5$. But $t_i = s_i$ for $i = 1, \ldots, 4$ and $t_5 = s_1 \oplus s_2 \oplus s_3$. Therefore $z_1 = 2s_1$. Thus, we have:

$$z_2 = \eta_2 \oplus \eta_3 \oplus \eta_4$$

$$z_3 = \eta_1 \oplus \eta_3 \oplus \eta_4$$

which is equivalent to:

$$\eta_5 = \eta_1 \oplus \eta_2 \oplus \eta_3$$

$$\eta_6 = \eta_2 \oplus \eta_3 \oplus \eta_4$$

$$\eta_7 = \eta_1 \oplus \eta_3 \oplus \eta_4$$

Solution (cont.)

As η_5 is determined by η_1, η_2, η_3 we have $2^3 = 8$ possibilities here.

Now, for fixed η_1, η_2 (and η_3) in the previous step we only have two possibilities for η_4, which determines η_6.

We have now that all the variables are set and η_7 is fully determined by their valu

Thus, we h

syndrom

The trivial noise vectors that yield this syndrome are: $\boldsymbol{\eta} = 0000000$ and $\boldsymbol{\eta} = 1111111$.

However, we can follow the above procedure and s

variables.

This is equivalent to having arbitrary settings for η_1, η_2, η_3 and η_4 which gives us 16 possible noise vectors which exactly correspond to the 16 codewords of the (7,4) Hamming code.

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Assignment Project Exam Help

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   - Co
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3. Coding: Review

The (7,4) Hamming code:

Error Probabilities

- Decoding Error : Occurs if at least one of the decoded bits $\hat{s}_i$ does not match the corresponding source bit s_i for $i = 1, \ldots 4$

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Rate : $R = \frac{K}{N} = \frac{4}{7}$

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Error Probabilities

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p(Bit Err $\frac{1}{K} \sum_{k=1}$

Rate : $R = \frac{K}{N} = \frac{4}{7}$

What is the probability of block error for the (7,4) Hamming code with $f = 0.1$?

The (7,4) Hamming code:

Leading-Term Error Probabilities

Block Error: This occurs when 2 or more bits in the block of 7 are flipped.

We can ap

$$_B = \quad m \quad (\quad)$$

$$m=2$$

$$\approx \binom{7}{2} f^2 = 21 f$$

The (7,4) Hamming code:

Leading-Term Error Probabilities

Bit Error: Given that a block error occurs, the noise must corrupt 2 or more bits

The most p

errors in th

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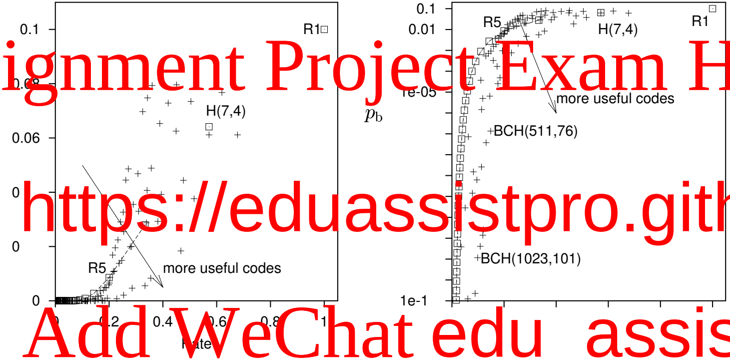
- $p_b \approx \frac{3}{7} p_B \approx 9f^2$

What Can Be Achieved with Hamming Codes?



- H(7,4) improves p_b at a moderate rate $R = 4/7$
- BCH are a generalization of Hamming codes.

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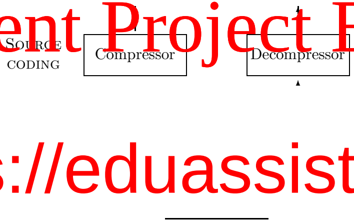
3 Coding: Review

Coding: Review

The Big Picture



Source Coding for Compression

- Shrink sequences
- Identify and remove redundancy
- Size limited by entropy
- Source Coding Theorems (Block & Variable Length)

Cha

-
- Add known form of redundancy
- Rate limited by capacity
- Noisy-Channel Coding Theorem