

COMP2610/COMP6261 - Information Theory

Tutorial 5: Probabilistic inequalities and Mutual Information

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1. Consider a discrete variable X taking on values from the set \mathcal{X} . Let p_i be the probability of each state, with $i = 1, \dots, |\mathcal{X}|$. Denote the vector of probabilities by \mathbf{p} . We saw in lectures that the entropy of X satisfies:

$$H(X) \leq \log |\mathcal{X}|,$$

with equality if and only if $p_i = \frac{1}{|\mathcal{X}|}$ for all i , i.e. \mathbf{p} is uniform. Prove the above statement using Gibbs' inequality, which says

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for any probability distributions \mathbf{p}, \mathbf{q} over outcomes, with equality if and only if $\mathbf{p} = \mathbf{q}$.

2. Let X be a discrete r.v. with probability distribution \mathbf{p} . Let g be a function from \mathcal{X} to \mathcal{Y} . Then the entropy of $g(X)$ is less than or equal to the entropy of X by justifying that

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$$\begin{aligned} H(X, g(X)) &\stackrel{(a)}{=} H(X) \\ &\stackrel{(b)}{=} H(X|g(X)) + H(g(X)) \\ &\stackrel{(c)}{=} H(g(X)) + H(X|g(X)) \\ &\stackrel{(d)}{\geq} H(g(X)). \end{aligned}$$

Thus $H(g(X)) \leq H(X)$.

3. Random variables X, Y, Z are said to form a Markov chain in that order (denoted by $X \rightarrow Y \rightarrow Z$) if their joint probability distribution can be written as:

$$p(X, Y, Z) = p(X) \cdot p(Y|X) \cdot p(Z|Y)$$

- (a) Suppose (X, Y, Z) forms a Markov chain. Is it possible for $I(X; Y) = I(X; Z)$? If yes, give an example of X, Y, Z where this happens. If no, explain why not.
- (b) Suppose (X, Y, Z) does *not* form a Markov chain. Is it possible for $I(X; Y) \geq I(X; Z)$? If yes, give an example of X, Y, Z where this happens. If no, explain why not.
4. If $X \rightarrow Y \rightarrow Z$, then show that
- (a) $I(X; Z) \leq I(X; Y)$
- (b) $I(X; Y|Z) \leq I(X; Y)$

5. A coin is known to land heads with probability $\frac{1}{5}$. The coin is flipped N times for some even integer N .
- (a) Using Markov's inequality, provide a bound on the probability of observing $\frac{N}{2}$ or more heads.
 - (b) Using Chebyshev's inequality, provide a bound on the probability of observing $\frac{N}{2}$ or more heads. Express your answer in terms of N .
 - (c) For $N \in \{2, 4, \dots, 20\}$, in a single plot, show the bounds from part (a) and (b), as well as the *exact* probability of observing $\frac{N}{2}$ or more heads.

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