COMP2610/COMP6261 - Information Theory

Tutorial 6: Source Coding

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Week 8 (25th – 29th Sep), Semester 2, 2017

1. *Probabilistic inequalities*

   Suppose a coin is tossed n times. The coin is known to land "heads" with probability p. The number of observed "heads" is recorded as a random variable X.

   (a) What is the *exact* probability of X being $n - 1$ or more?

   (b) Using Markov's inequality, compute a *bound* on the same probability as the previous part.

   (c) Suppose $n = 2$. For what values of p will the bound from Markov's inequality be within 1% of the exact probability?

2. *AEP and source coding* (cf. Cover & Thomas, Problem 3.7)

   A sequence of bits its g $p_0 = 0.995$ and $p_1 = 0.005$. Sequences are code word. Those blocks with more tha

   (a) What is the minimum required length of the assigned codeword

   (b) Calculate the probability of observing a 100-bit block that has no a

   (c) (*Harder*) Use Chebyshev's inequality to bound the probabil hich no codeword has been assigned. Compare the bound to the proba

3. *Typical Sets and Smallest δ-Sufficient Subsets* (cf. Cover & Thomas, Problem 3.13)

   Let X^N be an extended ensemble for X with $\mathcal{A}_X = \{0, 1\}$ and $\mathcal{P}_X = \{0.4, 0.6\}$.

   (a) Calculate the entropy $H(X)$.

   (b) Let $N = 25$ and $\beta = 0.1$.

      i. Which sequences in X^N fall in the typical set $T_{N\beta}$? (You may find it helpful to refer to Table 1 below.)

      ii. Compute $P(\mathbf{x} \in T_{N\beta})$, the probability of a sequence from X^N falling in the typical set.

      iii. With reference to Table 1 below, how many elements are there in $T_{N\beta}$?

      iv. How many elements are in the smallest δ-sufficient subset S_δ for $\delta = 0.9$?

      v. What is the essential bit content $H_\delta(X^N)$ for $\delta = 0.9$?

4. *Source Coding Theorem*

Recall that the source coding theorem (for uniform codes) says that for any ensemble X:

$$(\forall \epsilon > 0)\,(\forall \delta \in (0,1))\,(\exists N_0)\,(\forall N > N_0)\,\left|\frac{1}{N}H_\delta(X^N) - H(X)\right| \leq \epsilon.$$

(a) Near, an enthusiastic software developer, has just learned about the source coding theorem. He exclaims: *"The theorem allows us to pick any $\epsilon > 0$. So, if I pick $\epsilon = H(X) - \epsilon'$, I get that for sufficiently large N,*

$$\frac{1}{N}H_\delta(X^N) \geq \epsilon'.$$

This means that by making ϵ' tiny, I can get away with using virtually zero bits per outcome. Great!".

Is Near's reasoning correct? Explain why or why not.

(b) Mello, a skeptical econometrician, has also just learned about the source coding theorem. He complains: *"The thereom is not really relevant to me. I am interested in coding blocks of outcomes where each outcome is dependent on the previous outcome, rather than them all being independent of each other. The source coding theorem is not useful in this case."*

Is Mello's reasoning correct? Explain why or why not.

5. *Prefix Codes*

Consider the codes $C_1 = \{0, 01, 1101, 10101\}$, $C_2 = \{00, 01, 100, 101\}$, and $C_3 = \{0, 1, 00, 11\}$

(a) Are C_1, C_2, and C_3 prefix codes? Are they uniquely decodable?

(b) Construct new prefix codes C_1', C_2', and C_3' that have the same lengths as C_1, C_2, and C_3, respectively. If this is not possible, explain why.

(c) Is it possible to con $4, 4, 4\}$? If so, construct one.

(d) Is it possible to con $4, 4, 4\}$? If so, construct one.

k	$\binom{N}{k}$	$\binom{N}{k} p_1^k p_0^{N-k}$	$-\frac{1}{N}\log_2 p(\mathbf{x})$
0	1	0.000000	1.321928
1	25	0.000000	1.298530
2	300	0.000000	1.275131
3	2300	0.000001	1.251733
4	12650	0.000007	1.228334
5	53130	0.000045	1.204936
6	177100	0.000227	1.181537
7	480700	0.000925	1.158139
8	1081575	0.003121	1.134740
9	2042975	0.008843	1.111342
10	3268760	0.021222	1.087943
11	4457400	0.043410	1.064545
12	5200300	0.075967	1.041146
13	5200300	0.113950	1.017748
18	480700	0.079986	0.900755
19	177100	0.044203	0.877357
20	53130	0.019891	0.853958
21	12650	0.007104	0.830560
22	2300	0.001937	0.807161
23	300	0.000379	0.783763
24	25	0.000047	0.760364
25	1	0.000003	0.736966

Table 1: Table for Question 3. Column 1 shows k, the number of 1s in a block of length $N = 25$. Column 2 shows the number of such blocks. Column 3 shows the probability $p(\mathbf{x}) = \binom{N}{k} p_1^k p_0^{N-k}$ of drawing such a block $\mathbf{x}$. Column 4 shows the Shannon information per symbol in $\mathbf{x}$.