# COMP2610/6261 - Information Theory



## Lecture 19: Block Codes and the Coding Theorem



Australian National University



9 October, 2018

# Channel Capacity: Recap



The *largest possible* reduction in uncertainty achievable across a channel is its **capacity**

**Channe** [illegible]



The capa [illegible] een its input and output for any choice of input ensemble. That is,



$$C = \max_{\mathbf{p}_X} I(X \ldots$$ [illegible]

1 Block Codes

2 The N

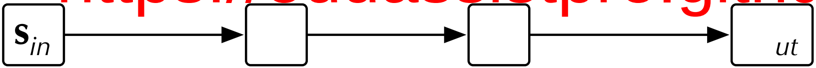
3 Extended Channels

# Communicating over Noisy Channels



Suppose we know we have to communicate over some channel $Q$ and we want buil **s** over $Q$.





# Block Codes

We now consider codes that make repeated use of a noisy channel to communicate a predefined set of messages $\mathcal{S} = \{1, 2, \ldots, S\}$

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Thus, we can imagine there being $S$ unique codewords $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(S)}\}$, where each codeword has block length $N$

# Block Codes: Example

Suppose $\mathcal{S} = \{1, 2, 3, 4\}$



| **Message ID** $s$ | **Message encoding** |
|---|---|
| 4 | |



Block size $N = 2$



Codewords $\mathbf{x}^{(1)} = 00, \mathbf{x}^{(2)} = 01$, and so on

# Block Codes: Formally

We formalise the preceding with the following notion:



**$(N, K)$ Block Code**

Given a channel $Q$ with inputs and outputs , an integer $N > 0$, and $K > 0$, an ds



where each $\mathbf{x}^{(s)} \in \mathcal{X}^N$ consists of $N$ symb



The code is parameterised by the length of the block, messages that are encoded

- We parametrise by $K = \log_2 S$ for mathematical convenience
- Doesn't have to be an integer

An $(N, K)$ block code makes $N$ uses of a channel to transmit one of $S$ possible

We can m



**Rate of an $(N, K)$ Block Code**

The **rate** of an $(N, K)$ block code is $\frac{\log_2 S}{N}$ —

## Block Codes: Examples



**Examples** (for Binary Symmetric Channel Q)

- A $(1, 1)$ block code: $\mathcal{C} = \{0, 1\}$ — Rate: 1



- A (



- A $(3, \log_2 3)$ block code: $\mathcal{C} = \{001, 0$ $\frac{}{3}$ 0.53

# Decoding Block Codes



An $(N, K)$ block code sends each message $s \in \mathcal{S} = \{1, 2, \ldots, 2^K\}$ over a channel $Q$ as $\mathbf{x}^s \in \mathcal{X}^N$

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Even if $\mathcal{X} = \mathcal{Y}$, the decoder must allow for any codewords $\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(2^K)}\}$

# Decoding Block Codes: Formally

**Block Decoder**

A **decoder** for a $(N, K)$ block code is a mapping that associates each $\mathbf{y} \in \mathcal{Y}^N$ with an $\hat{s} \in \{1, 2, \ldots, 2^K\}$.

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Ideally, w … o $s$



## Optimal Decoder

An **optimal decoder** for a code $\mathcal{S}$, channel … $\mathbf{y}$ to $\hat{s}$ such that $P(\hat{s}|\mathbf{y})$ is maximal.



That is, $\mathtt{dec}_{opt}(\mathbf{y}) = \arg\max_s P(s|\mathbf{y}) = \arg\max_s P(\mathbf{y}|s) \cdot P(s)$

# Decoding Block Codes: Examples



**Example** The $(2,1)$ block code $= \; 000, 111$ and majority vote decoder $d : \{0, 1$



$$d(111) = d(110) = d(10$$

1. Block Codes
2. The N
3. Extended Channels

# Rates and Reliability



Ideally, we would like to have high rate for our channel code

- Lo



But intuit

- If $N$ is small, we may be more easily "confuse



How to measure reliability?

# Reliability

Want an *encoder/decoder* pair to reliably send a messages over channel $Q$.

$$\mathbf{s}_{in} \xrightarrow{\text{Encoder}} \mathbf{x} \xrightarrow{Q} \mathbf{y} \xrightarrow{\text{Decoder}} \mathbf{s}_{out}$$

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### Probabi



Given a ch

$$p_B = P(\mathbf{s}_{out} \neq \mathbf{s}_{in}) = \sum_{\mathbf{s}_{in}} P( \quad / $$



and its **maximum probability of (block) error**

$$p_{BM} = \max_{\mathbf{s}_{in}} P(\mathbf{s}_{out} \neq \mathbf{s}_{in} | \mathbf{s}_{in})$$

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$$p_{BM} = \max_{\mathbf{s}_{in}} P(\mathbf{s}_{out} \neq \mathbf{s}_{in} | \mathbf{s}_{in})$$

As $P(\mathbf{s}_{out} \neq \mathbf{s}_{in} | \mathbf{s}_{in}) \leq p_{BM}$ for all $\mathbf{s}_{in}$ we get $p_B \leq \sum_{\mathbf{s}_{in}} p_{BM} P(\mathbf{s}_{in}) = p_{BM}$ and so if $p_{BM} \to 0$ then $p_B \to 0$.

# Reliability: Example

Suppose $\mathbf{s} \in \{\mathtt{a}, \mathtt{b}\}$ and we encode by $\mathtt{a} \rightarrow \mathtt{000}$ and $\mathtt{b} \rightarrow \mathtt{111}$.
To decode we count the number of 1s and 0s and set all bits to the majority count to determine $\mathbf{s}$.



$$\mathtt{000}, \mathtt{001}, \mathtt{010}, \mathtt{100} \rightarrow \mathtt{a} \quad \text{and} \quad \mathtt{111}, \mathtt{110}, \mathtt{101}, \mathtt{011} \rightarrow \mathtt{b}$$

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If the chan



$$\begin{aligned} {}_B &= \ ( \ _{in} \neq \ _{out}) \\ &= P(\mathbf{y} \in B|000)\, p_{\mathtt{a}} + P( \\ &= [f^3 + 3f^2(1-f)]\, p_{\mathtt{a}} + [ \\ &= f^3 + 3f^2(1-f). \end{aligned}$$



In fact,

$$p_{BM} = \max(P(\mathbf{y} \in B|000), P(\mathbf{y} \in A|111)) = f^3 + 3f^2(1-f).$$

## Achievable Rates

Ideally, we would like to consider rates of transmission for which we can guarantee small maximum probability of block error



Even more ideally, we would like rates for which we can guarantee arbitraril

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**Achievable Rate**

A rate $R$ over a channel $Q$ is said to be **ac** ere is a $(N, K)$ block code and decoder such that its ra its maximum probability of block error satisfies



$$p_{BM} = \max_{\mathbf{s}_{in}} P(\mathbf{s}_{out} \neq \mathbf{s}_{in} | \mathbf{s}_{in}) < \epsilon$$

# Achievable Rates



Achievable rates sound nice in theory, but surely they cannot exist?

- Surely we will have to drive $R$ 0 to get small error probability?

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Remark

**Noisy-Channel Coding Theorem (Brief)**

If $Q$ is a channel with capacity $C$ then the rat **only if** $R \leq C$, that is, the rate is no greater than the channe

## The Noisy-Channel Coding Theorem

### Example

**Example**:



- In last lecture: BSC $Q$ with $f = 0.15$ has capacity $C = 0.39$ bits.
- Sup .25
- The $(N, K)$ code with $K/N \geq 0.25$



Indeed, we showed the code $\mathcal{S} = \{000, 1$ r has probability of error $0.028 < 0.05$ for $Q$

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- For $N = 3$ there is a $(3, 1)$ code meeting the requirements.
- However, there is *no code* with same $\epsilon$ and rate $1/2 > 0.39 = C$.

## The Noisy Typewriter Channel

This channel simulates a noisy "typewriter". Inputs and outputs are 26 letters A through Z plus space. With probability $\frac{1}{3}$, each letter is either: unchanged; changed to the next letter; changed to the previous letter.

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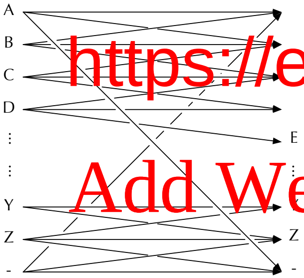


$$\text{Inputs} = \mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}, \_ \; ;$$



$$Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{1}{3} \\ \vdots & & & & & & \vdots \\ \frac{1}{3} & 0 & 0 & \cdots & & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

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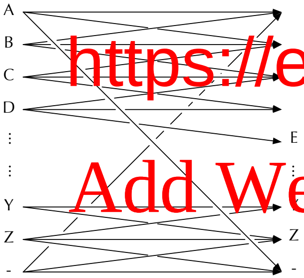


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The transition matrix for this channel has a diagonal structure: all of the probability mass is concentrated around the diagonal.

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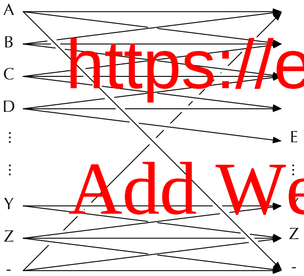


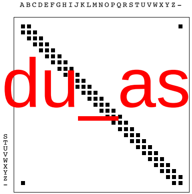




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# Noisy Channel Coding Theorem

## NCCT

Any rate $R < C$ is *achievable* for $Q$ (i.e., for any tolerance $\epsilon > 0$, an $(N, K)$ code with rate $K/N \geq R$ exists with max. block error $p_{BM} < \epsilon$)



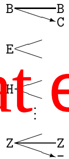
Conside

For nois



- Th
- For any $\epsilon > 0$ and $R < C$ we can choose $N = 1$ . . .
- . . . and code messages using $\mathcal{C} = \{\mathtt{B}, \mathtt{E}, \ldots, \mathtt{Z}\}$



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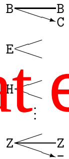


Since $|\mathcal{C}| = 9$ we have $K = \log_2 9$ so $K/N = \log_2 9 \geq R$ for any $R < C$, and $\mathcal{C}$ has zero error so $p_{BM} = 0 < \epsilon$

1. Block Codes

2. The N

3. Extended Channels

# Noisy Channel Coding Theorem: How Is This Possible?



The main "trick" to minimising $p_{BM}$ is to construct a $(N, K)$ block code with (almost)

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The main "trick" to minimising $p_{BM}$ is to construct a $(N, K)$ block code with (almost)

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This is possible because extended channels look li

# Extended Channels

When used $N$ times, a channel $Q$ from $\mathcal{X}$ to $\mathcal{Y}$ can be seen as an *extended channel* taking "symbols" from $\mathcal{X}^N$ to "symbols" in $\mathcal{Y}^N$.

**Extended Channel**



The $N^{th}$ **extended channel** of $Q$ from $\mathcal{X}$ to $\mathcal{Y}$ is a channel from $\mathcal{X}^N$ to $\mathcal{Y}^N$ with trans

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**Example**: $BSC$ $Q$ with $f = 0.1$ from $\mathcal{X} =$ $N = 2$ *extended channel* from $\mathcal{X}^2 = \{00, 01, 10,$ $11\}$ with



$$Q_2 = \begin{bmatrix} 0.81 & 0.09 & 0.09 & 0.01 \\ 0.09 & 0.81 & 0.01 & 0.09 \\ 0.09 & 0.01 & 0.81 & 0.09 \\ 0.01 & 0.09 & 0.09 & 0.81 \end{bmatrix}$$

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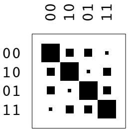

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As $N$ increases, any extended channel looks like the noisy typewriter!

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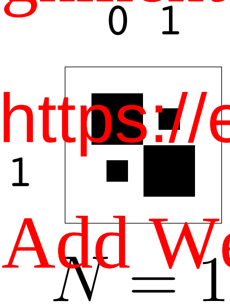


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Noisy Typewriter Channel

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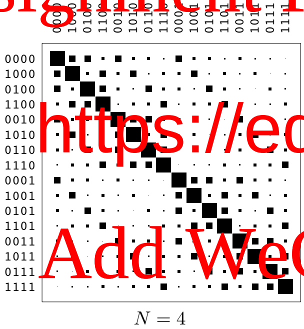


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Extended Z Channel


Noisy Typewriter Channel

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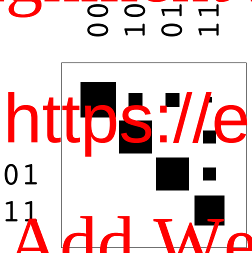


$N = 2$

Extended Z Channel


Noisy Typewriter Channel

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$N = 4$

Extended Z Channel


Noisy Typewriter Channel

Why does this happen?

Remem art looking

typical



For a given $\mathbf{x}$, the corresponding $p(\mathbf{y} \mid \mathbf{x})$ will also be concentrated on a few sequences



Formalising this will require a notion of joint typicalit

# Summary and Reading

**Main Points**

- The Noisy Typewriter



- Extended Channels
- Blo



- The

**Reading**

- MacKay §9.6



- Cover & Thomas §7.5

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**Next time**: Detail of the NCCT, joint typicality, and a sketch of the proof!