# COMP2610 / COMP6261 - Information Theory

## Lecture 3: Probability Theory and Bayes' Rule



Australian National University

July 30, 2018

# Last time



- A general communication system
- Wh



- Basics of probability theory



- Joint, marginal and conditional distributions

# Review Exercise

Suppose I go through the records for $N = 1000$ students, checking their admission status, $A = \{0, 1\}$, and whether they are "brilliant" or not, $B = \{0, 1\}$



(Aside: "Brilliance" is a dodgy concept, and does not predict scientific achievement as well as persistence and combinatorial ability; see e.g. Dean Simonton, *Scientific Genius: A Psychology of Science*, Cambridge University Press, 2009; this is just a toy example!)

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|---|---|---|
| $A = 0$ | 680 | 10 |
| $A = 1$ | 220 | 90 |



Then:

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- More on joint, marginal and conditional distributions



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- What, if anything, does $p(X = x|Y =$
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I am *not* providing a general answer; but keep it in mind!

# Outline

1. More on Joint, Marginal and Conditional Distributions
2. Stati
3. Bayes' Theorem
4. Wrapping up

1 More on Joint, Marginal and Conditional Distributions



2 Stati



3 Baye



4 Wrapping up

# Document Modelling Example

Suppose we have a large document of English text, represented as a sequence of characters:



$$X_1 X_2 X_3 \ldots X_N$$

- e.g. `hello␣how␣are␣you`



Treat eac ... variable



```
X = ‘h’, Y = ‘e’

X = ‘e’, Y = ‘l’

X = ‘l’, Y = ‘l’
```

$$\vdots$$

# Document Modelling: Marginal and Joint Distributions

| $i$ | $a_i$ | $p_i$ |
|---|---|---|
| 1 | a | 0.0575 |
| 2 | b | 0.0128 |
| 3 | c | 0.0263 |
| 4 | d | 0.0285 |
| 5 | e | [illegible] |
| 6 | f | [illegible] |
| [illegible] | g | 0.0133 |
| 8 | h | 0.0313 |
| 9 | i | 0.0599 |
| 10 | j | 0.0006 |
| 11 | k | 0.0084 |
| 26 | z | 0.0007 |
| 27 | – | 0.1928 |



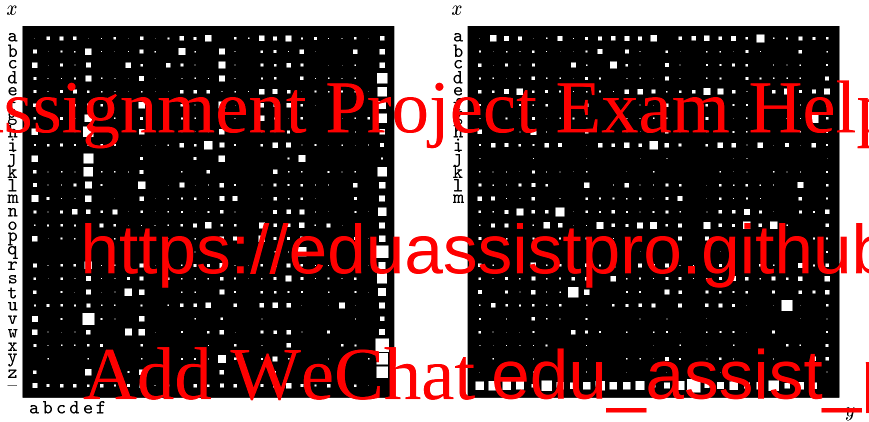


Unigram / Monogram

Bigram

Marginal and joint distributions for English alphabet, estimated from the "FAQ manual for Linux". Figure from Mackay (ITILA, 2003); areas of squares proportional to probability (the right way to do it!).

# Document Modelling: Conditional Distributions






Conditional distributions for English alphabet, estimated from the “FAQ manual for Linux”. Are these distributions “symmetric”? Figure from Mackay (ITILA, 2003)

$P(X = x|Y = y) = P(Y = y|X = x)$? $P(X = x|Y = y) = P(X = y|Y = x)$?.

## Recap: Sum and Product Rules



**Sum rule**:



**Product rule**:



$$p(X = x_i, Y = y_j) = p(Y = y$$

# Relating the Marginal, Conditional and Joint

Suppose we knew $p(X = x, Y = y)$ for all values of $x, y$. Could we compute all of $p(X = x|Y = y)$, $p(X = x)$ and $p(Y = y)$?

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These have the same marginals, but different joint distributions

# Joint as the "Master" Distribution



In genera
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The joint di
dependence

1 More on Joint, Marginal and Conditional Distributions



2 Stati



3 Baye



4 Wrapping up

## Statistical Independence

Suppose that both boxes (red and blue) contain the same proportion of apples and oranges



If fruit is sel

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If fruit is sel



*The probability of selecting an apple (or an orange) i box that is chosen*



We may study the properties of $F$ and $B$ separately: this often simplifies analysis

# Statistical Independence: Definition



**Definition: Independent Variables**

Two varia ⊥⊥ $Y$, if

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This definition generalises to more than two variabl

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Are the variables in the language example statistic

# A Note on Notation

When we write

$$p(X, Y) = p(X)p(Y)$$



we have n

This state



$$p(X = x, Y = y) = p(X$$

for every possible $x$ and $y$



This notation is sometimes called implied universality

## Conditional independence

We may also consider random variables that are conditionally independent given some other variable



**Definition: Conditionally Independent Variables**

Two varia enoted
$X \perp\!\!\!\perp Y|$



$$p(X, Y|Z) = p(X|Z)p(Y|Z)$$

Intuitively, $Z$ is a common cause for $X$ and



Example: $X$ = whether I have a cold
$Y$ = whether I have a headache
$Z$ = whether I have the flu

1 More on Joint, Marginal and Conditional Distributions



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# Revisiting the Product Rule

The product rule tells us:



$$p(X, Y) = p(Y|X)p(X)$$

This can e



probability:

$$p(Y|X) = \frac{p(X}{p(}$$



Can we use these to relate $p(X|Y)$ and $p(Y|X)$?

## Posterior Inference:
### Example 1 (Mackay, 2003)

- Dicksy Sick had a test for a rare disease
  - Only 1% people of Dicksy's background have the disease

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  - It correctly identifies a healthy individual 96
    $p(\text{identifies healthy} \mid \text{healthy}) = 96$

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- What is the probability of Dicksy having the disease?

# Posterior Inference:

## Example 1: Formalization



Let $D \in \{0, 1\}$ denote whether Dicksy has the disease, and $T \in \{0, 1\}$ the outcome

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$$p(T = 0|D = 1) = 0.05$$



We need to compute $p(D = 1|T = 1)$, *th*

*the disease given that the test has resulted positive.*

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$$\approx 0.19.$$

Despite testing positive and the high accuracy of the test, the probability of Dicksy having the disease is only 0.19!

# Why is the Probability So Low?

## A “Natural Frequency” Approach

In 100 people, only 1 is expected to have the disease ($p(D = 1) = 0.01$)



This sick person will most likely test positive ($p(T = 1\ D = 1) = 0.95$)

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So when the test is positive, the chance of being sick is

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(Aside: If you can correctly perform the calculation on the previous slide
most medical doctors! See Gird Gigerenzer and Adrian Edwards, Si
from innumeracy to insight, *British Medical Journal*, 327(7417), 741–744, 27 September 2003; Gird Gigerenzer, *Reckoning with risk: Learning to live with uncertainty*, Penguin, 2002.

Moral of the story — if you get sick, don't delegate conditional probability computations to your doctor!)

## Bayes' Theorem

We have implicitly used the following (at first glance remarkable) fact:

Bayes' Theorem:



$$
\begin{aligned}
p(Z|X) &= \frac{p(Z,X)}{p(X)} \\
&= \frac{p(X|Z)p(Z)}{p(X)} \\
&= \frac{p(X|Z)p(Z)}{\sum_{Z'} p(X|Z')p(Z')}
\end{aligned}
$$



If we can express what knowledge of $X$ (test) tells us about $Z$ (disease), then we can express what knowledge of $Z$ tells us about $X$

# The Bayesian Inference Framework

### Bayesian Inference

Bayesian inference provides a mathematical framework explaining how to change our (prior) beliefs in the light of new evidence.





$$\underbrace{\qquad}_{\text{posterior}} = \frac{\qquad}{\underbrace{p(X)}_{\text{evid}}}$$



**Prior**: Belief that someone is sick

**Likelihood**: Probability of testing positive given you are sick

**Posterior**: Probability of being sick given you test positive

## Posterior Inference:

### Example 2 (Bishop, 2006)

Recall our fruit-box example:

- The proportion of oranges and apples are given by

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- What is the probability that it came from the red box?

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## Example 2: Formalization



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$p(F = a|B = b) = 3/4$

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and therefore $p(B = b|F = o) = 1/3$.

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- In fact, the proportion of oranges is so much hig this is strong evidence that the orange came fro



  - *So after picking up the orange the red box is mu been selected than the blue one*

# Summary

- Recap on joint, marginal and conditional distributions



- Inte



- Stat



- Bayes rule: combination of prior, likelihood to

- Reading: Mackay § 2.1, § 2.2 and § 2.3

# Homework Exercise

Suppose we know that random variables $X, Y$ satisfy



$$p(X|Y) = p(Y|X)$$

What can y $Y$?



If $X$ and ?

Repeat the above questions for the statement



$$\frac{p(X|Y)}{p(Y|X)} = \frac{p}{p(Y)}$$

# Next time

- More examples on Bayes' theorem:



  - Eating hamburgers
  - 
  - The Monty Hall problem



  - Document modelling
- Are there notions of probability beyond frequency counting?