COMP2610 / COMP6261 - Information Theory

Lecture 4: Bayesian Inference

Australian National University

July 31, 2018

Last time

- Examples of joint, marginal and conditional distributions

- Wh

- What, if anything, does $p(X = x|Y =$
  $p(Y = y|X = x)$?

Review Exercise

Suppose we have binary random variables X, Y such that

$$p(X = 1) = 0.6$$
$$p(Y = 1|X = 0) = 0.7$$
$$p(Y = 1|X = 1) = 0.8$$

Then,

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$$= \frac{p(Y = 1|X = 1)p(X = 1)}{p(Y = 1|X = 1)p(X = 1) + p(Y = 1|X = 0)p(X = 0)}$$

$$= \frac{(0.8)(0.6)}{(0.8)(0.6) + (0.7)(0.4)}$$

$$\approx 0.63$$

This time

- More examples on Bayes' theorem:
  - Eating hamburgers
  -
  - The Monty Hall problem
- Are there notions of probability beyond frequ

Outline

1. Bayes' Rule: Examples
   - Eating Hamburgers
   - Detecting Terrorists
   - Th
2. Mom
3. The meaning of Probability
4. Wrapping Up

Bayesian Inference:

Example 1 (Barber, BRML, 2011)

- 90% of people with McD syndrome are frequent hamburger eaters

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- Proportion of hamburger eaters is about 50

What is the probability that a hamburger eater will have McD syndrome?

Bayesian Inference:

Example 1: Formalization

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We need to compute $p(McD = 1|H = 1)$, r *eater having McD syndrome.*

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[4]

We need to compute $p(McD = 1|H = 1)$, *r*
eater having McD syndrome.

Any ballpark estimates of this probability?

Bayesian Inference:

Example 1: Solution

$$1)$$

$$= \quad . \quad \times$$

1)

$= \; . \; \times$

Repeat the above computation if the proportion of h rather small. (say in France) 0.001.

Example 2: Detecting Terrorists:

From understandinguncertainty.org

- Scanner detects true terrorists with 95% accuracy

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- The shifty looking man sitting next to you tests p

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- There is 1 terrorist on your plane with 100 passe

- The shifty looking man sitting next to you tests p

What are the chances of this man being a terrorist?

Example 2: Detecting Terrorists:

Simple Solution Using “Natural Frequencies” (David Spiegelhalter)

Figure reproduced from `understandin`

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- Relation to disease example
- Consequences when catching criminals

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Formalization with Actual Probabilities

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$p(T = 1) = 0.01$

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Let $T \in \{0, 1\}$ denote the variable regarding whether the person is a terrorist a

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$$p(T = 1) = 0.01$$

We want to compute $p(T = 1|S = 1)$, *the terrorist given that he has tested positive.*

Example 2: Detecting Terrorists:
Solution with Bayes' Rule

$p(T =$ ———— $0)p(T = 0)$

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$(0.95)(0.01) + (0.05)$

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≈ 0.16

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≈ 0.16

The probability of the man being a terrorist is $\overline{6}$

Example 2: Detecting Terrorists:

Posterior Versus Prior Belief

While the man has a low probability of being a terrorist, our belief has increased compared to our prior:

$$\underline{p(T = 1 \; S = 1)} \qquad \underline{0.16}$$

i.e. our beli

Since terrorists are so rare, a factor of 16 does not resul
(absolute) probability of belief

(Aside: They are indeed very rare. For an intruiging (and surprising) example of the implications of inability to take account of actual base rates (in the example above we made the numbers up), and the effect on people's subsequent decisions, see Gerd Gigerenzer, Dread Risk, September 11, and Fatal Traffic Accidents, *Psychological Science* 15(4), 286–287, (2004); Gerd Gigerenzer, Out of the Frying Pan into the Fire: Behavioural Reactions to Terrorist Attacks, *Risk Analysis* 26(2), 347–351 (2006). His calculation (which of course is based on some assumptions) is that in the year following 9/11, 6 times the number of people who were killed as passengers *additionally* died on roads (that is the increase in road deaths due to people chosing to drive instead of flying)! He calls the reaction to very low probability events with a bad outcome "dread risk".)

Example 3: The Monty Hall Problem

Problem Statement

- Three boxes, one with a prize and the other two are empty

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Should you switch to the other box? Would that increase your chances of winning the prize?

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Formalization

Let $C \in \{r, g, b\}$ denote the box that contains the prize where r, g, b refer to the identity of each box

WLOG assume the following:

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We want to compute $p(C = r|H = b)$ and $p(C = g|H = b)$ to decide if we should switch from our initial choice.

Example 3: The Monty Hall Problem:

Solution

We have that:

$$p(H = b) = \sum_{c \in \{r, g, b\}} p(H = b | C = c) p(C = c)$$

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Similarly, $p(C = g | H = b) = 2/3$.

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Similarly, $p(C = g|H = b) = 2/3$.

You should switch from your initial choice to the other box in order to increase your chances of winning the prize!

Example 3: The Monty Hall Problem:

Illustration of the Solution

Illustration of the solution when you have initially selected box r.

Example 3: The Monty Hall Problem:

Another Perspective

Switching is bad if, and only if, we initially picked the prize box (because if not, the oth

We picke he host's action

Hence, with probability 2/3, switching will rev

Example 3: The Monty Hall Problem:

Variants to Ponder

Would switching be rational if:

- The host only revealed a box when he knew we picked the right one?

- The
  one?

- The host is himself unaware of the prize box, an
  random, which by chance does not have the prize?

The Expected Value of a Function of Two Discrete Random Variables

(Assuming you have met Expectation $E[X]$ and Variance $\mathrm{Var}(X)$ before ...)

The expected value of a function $g(X, Y)$ of two discrete random variables is defined

(1)

In particular, the expected value of X is giv

$$E[X] = \sum_x \sum_y xp(X \qquad (2)$$

It should be noted that if we have already calculated the marginal distribution of X, then it is simpler to calculate $E[X]$ using this.

Covariance and the Correlation Coefficient

The covariance between X and Y, $\mathrm{Cov}(X, Y)$ is given by

$$\mathrm{Cov}(X, Y) = E(XY) \quad E(X)E(Y) \tag{3}$$

Note that b ... $\mathrm{ar}(X)$

The coeffi

$$\rho(X, Y) = \frac{\mathrm{Cov}(}{\sqrt{\mathrm{Var}(}} \tag{4}$$

Always in $[-1, 1]$.

Example

Discrete random variables X and Y have the following joint distribution:

	$Y = -1$	$Y = 0$	$Y = 1$
$X = 0$	0	$\frac{1}{3}$	0
	1		1

Calculat

1. $p(X$
2. marginal distributions of X and Y

3. expected values and variances of X
4. coefficient of correlation between X and Y

Are X and Y independent?

Example

To calculated the probability of such an event, note that we sum over all the cells w

$$+ p(X = 1, Y = \quad -$$

Example

Recall that

$$p(X = x) = \sum_{y} p(X = x, Y = y).$$

Hence,

$$\frac{1}{3}$$

$$p(X = 1) = \sum_{y=-1}^{1} p(X = 1, Y =$$

Note that after obtaining $p(X = 0)$, we could c (=) using the fact that

$$p(X = 1) = 1 - p(X = 0), \tag{5}$$

since X only takes the values 0 and 1.

Example

Similarly,

$$p(Y=0) = \sum_{x=0}^{1} p(X=x, Y=0) = - + 0 = -$$

$$p(Y=1) = 1 - p(Y=0) = -$$

Example

We then calculate the expected values and variances of X and Y from these marginal distributions.

$$E(Y) = \sum_{y=-1}^{1} y\,p(Y=y) = -1 \times$$

Example

To calculate the variances of X and Y, $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$, we use the formula

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2.$$

$$E(X^2 \quad \overset{1}{} \quad {}^2 \quad {}^2 \; \frac{1}{\;} \quad {}^2 \; \frac{2}{\;} \quad \frac{2}{\;}$$

$$E(Y^2) = \sum_{y=-1}^{1} y^2 p(Y = y) = (-1)^2 \quad - \quad - \quad - \quad \frac{2}{3}.$$

Thus we get

$$\mathrm{Var}(X) = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$$

$$\mathrm{Var}(Y) = \frac{2}{3} - (0)^2 = \frac{2}{3}$$

Example

To calculate the correlation coefficient, we first calculate the covariance between X and Y. We have

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y).$$

where

$E(XY$

$$= 0(-1)0 + 0(0)\frac{\;}{3} + 0(1)0 + \quad - \quad \frac{1}{\;} = 0$$

Thus we get

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0 - \frac{\;}{3} \times 0 = 0.$$

From the definition of the correlation coefficient,

$$\rho(X, Y) = 0.$$

Example - is X and Y independent

We have t

1. Bayes' Rule: Examples
   - Eating Hamburgers
   - Detecting Terrorists
   - The Monty Hall Problem

2. Mom

3. The meaning of Probability

4. Wrapping Up

The meaning of Probability

Frequentist : Frequencies of random repeatable experiments

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Cox Axio

Given $B(x)$, $B(\bar{x})$, $B(x,y)$, $B(x|y)$, $B(y)$:

1. Degrees of belief can be ordered
2. $B(x) = f[B(\bar{x})]$
3. $B(x,y) = g[B(x|y), B(y)]$

The meaning of Probability

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1. Degrees of belief can be ordered
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3. $B(x, y) = g[B(x|y), B(y)]$

If a set of Beliefs satisfy these axioms they can be mapped onto probabilities satisfying the rules of probability.

Image from http://xkcd.com/1132/

Image from `http://normaldeviate.wordpress.com/2012/11/09/anti-xkcd/`

In practice one needs to make use of both interpretations. Wise to be open to both. This is a huge topic which we can not get into further here. Note that Mackay was firmly in the Bayesian camp…

Summary

- Examples of application of Bayes' rule
  - Formalization
- Intu
- Frequentist v Bayesian probabilities
- Cox axioms

Next time

- Working through some useful probability distributions

- More on Bayesian inference