COMP2610 / COMP6261 - Information Theory

Lecture 10: Typicality and Asymptotic Equipartition Property

Australian National University

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Last time

Markov'

Chebys

Law of large numbers

Law of Large Numbers

Theorem

Let $X_1, \ldots, X_n$ be a sequence of iid random variables, with

$$\mathbb{E}[X_i] = \mu$$

and $\mathbb{V}[X_i$

Then, for a

$$\lim_{n\to\infty} p(|\bar{X}_n - \mu| <$$

This is also called $\bar{X}_n \to \mu$ **in probability**.

Definition: For random variables $v_1, v_2, \ldots$, we say $v_n \to v$ **in probability** if for all $\beta > 0$ $\lim_{n\to\infty} P(|v_n - v| > \beta) = 0$.

β is fixed (not shrinking like $\frac{1}{n}$). Not max/min. Reduction in variability.

This time

- Ensembles and sequences

- Typ

- Asymptotic Equipartition Property (AEP)

1 Ensembles and sequences
- Counting Types of Sequences

2 Typi

3 Asymptotic Equipartition Property (AEP)

4 Wrapping Up

Ensembles

Ensem

An **ense** … taking

values i … $p_2, \ldots, p_I\}$.

We will call $\mathcal{A}_X$ the alphabet of the ensemble

Ensembles

Example: Bent Coin

Let X be an ensemble with outcomes h for

r *tails*

- X t
- The proba

$\mathcal{P}_X = \{p_{\mathtt{h}}$

Extended Ensembles

We can also consider blocks of outcomes, which will be useful to describe sequences:

Example (Coin Flips):

hhhh … blocks)

… blocks)

$\rightarrow$ hhhh thht hthh (3 4 outcome blocks)

Extended Ensembles

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$\rightarrow$ `hhhh thht hthh` (3 4 outcome blocks)

Extended Ensemble

Let X be a single ensemble. The **extended ensemble** of blocks of size N is denoted X^N. Outcomes from X^N are denoted $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. The **probability** of $\mathbf{x}$ is defined to be $P(\mathbf{x}) = P(x_1)P(x_2) \ldots P(x_N)$.

Extended Ensembles

Example: Bent Coin

Let X be an ensemble with outcomes $\mathcal{A}_X = \{\mathtt{h}, \mathtt{t}\}$ with $p_\mathtt{h} = 0.9$ and $p_\mathtt{t} = 0.1$

$\mathtt{t}\}$

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$\mathtt{t}\}$

$P(\mathtt{hhhh}) = (0.9)^4 \approx 0.6561$

$P(\mathtt{tttt}) = (0.1)^4 = 0.0001$

$P(\mathtt{hthh}) = 0.9 \cdot 0.1 \cdot 0.9 \cdot 0.9 = (0.9)\ (0.1) \approx 0.0729$

$P(\mathtt{htht}) = 0.9 \cdot 0.1 \cdot 0.9 \cdot 0.1 = (0.9)^2(0.1)^2 \approx 0.0081.$

Extended Ensembles

Example: Bent Coin

Entropy of extended ensembles

We can view X^4 as comprising 4 independent random variables, based on the ense

Entropy i

Thus,

$$H(X^4) = 4H(X) = 4.(-0.9\log_2 0.9 \quad \ldots \text{s.}$$

More generally,

$$H(X^N) = NH(X).$$

Counting Types of Sequences

Criteria for dividing 2^N sequences into **types**

In the bent coin example,

$$= P$$

$$= P$$

$$= P$$

The order of outcomes in the sequence is irrelevant

Counting Types of Sequences

Let X be an ensemble with alphabet $\mathcal{A}_X = \{a_1, \ldots, a_I\}$.

Let $p(X = a_i) = p_i$.

For a sequ

let n_i = # of t

Given the n_i's, we can compute the probability of s

$$
\begin{aligned}
P(\mathbf{x}) &= P(x_1) \cdot P(x_2) \ldots \\
&= P(a_1)^{n_1} \cdot P(a_2)^{n_2} \cdot \ldots \cdot (\quad_I)^{\ I} \\
&= p_1^{n_1} \cdot p_2^{n_2} \ldots p_I^{n_I}
\end{aligned}
$$

Sufficient statistics: $\{n_1, n_2, \ldots, n_I\}$. Use it as a criteria of partitioning.

Each unique choice of $(n_1, n_2, \ldots, n_l)$ gives a different type of sequence

- 4 heads, (3 heads, 1 tail), (2 heads, 2 tails), ...
- Sequences in each type are equiprobable

For a give

symbol c

# of sequences with n_i copies of

$$\frac{\quad !}{\qquad}$$

$$\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\cdots$$

$$= \frac{N!}{n_1!(N-n_1)!} \cdot \frac{(N-n_1)!}{n_2!(N-n_1-n_2)!} \cdot \frac{(N-n_1-n_2)!}{n_3!(N-n_1-n_2-n_3)!} \cdots$$

Probability of types

Let $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ with $P(\mathtt{a}) = 0.2$, $P(\mathtt{b}) = 0.3$, $P(\mathtt{c}) = 0.5$.

Probability of types

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There are $\frac{6!}{2!1!3!} = 60$ such sequences.

Counting Types of Sequences

Example

Probability of types

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Each seq ability

$(0.2)^2(0$

There are $\frac{6!}{2!1!3!} = 60$ such sequences.

The probability **x** is of type $(2, 1, 3)$ is $(0.0$

Study probabilities at the level of types (most likely, average/typical)

Extended Ensembles

Example

With $p_{\mathtt{h}} = 0.75$, what are the probabilities for X^N?

$N = 2$

x	
`hh`	
`ht`	
`th`	0.1875
`tt`	0.0625

With $p_{\mathtt{h}} = 0.75$, what are the probabilities for X^N?

$N = 2$

x	
hh	[illegible]
ht	[illegible]
th	0.1875
tt	0.0625

$N = 3$

hth	0.1406
thh	0.1406
htt	0.0469
tht	0.0469
tth	0.0469
ttt	0.0156

With $p_h = 0.75$, what are the probabilities for X^N?

$N = 2$

$\mathbf{x}$	[illegible]
hh	[illegible]
ht	[illegible]
th	0.1875
tt	0.0625

$N = 3$

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
hth	0.1406
thh	0.1406
htt	0.0469
tht	0.0469
tth	0.0469
ttt	0.0156

$N = 4$

[illegible]	[illegible]	[illegible]	$P(\mathbf{x})$
[illegible]	[illegible]	[illegible]	0.0352
[illegible]	[illegible]	[illegible]	0.0352
h	[illegible]	[illegible]	.0352
h	[illegible]	[illegible]	.0117
t	[illegible]	[illegible]	.0117
h	[illegible]	[illegible]	.0117
htth	0.0352	ttth	0.0117
hhtt	0.0352	tttt	0.0039

Observations

As N increases, there is an increasing spread of probabilities

The most li

Howeve h t

Not surprising because $3 = N \cdot p_{\mathrm{h}}$, pretty mu

Symbol Frequency in Long Sequences

To judge if a sequence is typical/average, a natural question to ask is:

How often does each symbol appear in a sequence $\mathbf{x}$ from X^N?

Intuitively, in a sequence of length N, let a_i appear for n_i times.

Then **in expectation**

Note p_i

$$P(\mathbf{x}) = P(a_1)^{n_1} P(a_2)^{n_2} \ldots P(a_I$$

Symbol Frequency in Long Sequences

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So the *information content* $-\log_2 P(\mathbf{x})$ of th y

$$-p_1 N \log_2 p_1 - \dots - p_I N \log_2 p_I = -N \sum_{i=1}^{I} p_i \log_2 p_i = NH(X)$$

Typical Sets

We want to consider elements $\mathbf{x}$ that have $-\log_2 P(\mathbf{x})$ "close" to $NH(X)$

Typical Set

For "close

$$\beta \qquad 2$$

$$= \left\{ \mathbf{x} : \left| -\frac{1}{N} \log_2 P(\mathbf{x} \right. \right.$$

Union of types

Typical Sets

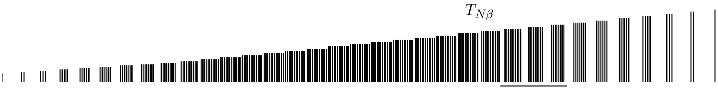
We want to consider elements $\mathbf{x}$ that have $-\log_2 P(\mathbf{x})$ "close" to $NH(X)$

Typical Set

For "closeness" $\beta > 0$ the typical set $T_{N\beta}$ for X^N is

def

Union of types



What when $\beta = 0$ (and replace $<$ by $\leq$)?

Criterion based on information content. Other criterion (KL divergence)?

Typical Sets

The name "typical" is used since $\mathbf{x} \in T_{N\beta}$ will have roughly $p_1 N$ occurrences of symbol a_1, $p_2 N$ of a_2, ..., $p_K N$ of a_K.

$\mathbf{x}$	$\log_2(P(\mathbf{x}))$
...1.....................1.....1....1.1.......1.......1.........1...................1......11...	−50.1
.....................1.....1.....1.......1...1.........1....................................1....	−37.3
.......1....1..1...1....11..1.1.........11.......................1...1.1..1...1................1.	65.9
[illegible]	56.4
[illegible]	53.2
[illegible]	43.7
[illegible]	46.8
.....1..1..1...............111...................1..............1........1.1...1...1.............1	−56.4
........1.........1....1.....1.........1...1................	
.....1.......................1.............1....1..1.1.1..1....	
1.........................1.........1..1..........................1...1...	
..........11.1...........1..............1.......1......	
.1.........1...1.1...........1.......11..........1.1...1.......	
.....1...1..1.....1..11.1.1.1...1...................1...........1...........1..1............	−59.5
...........11.1......1....1..1...........................1.......1.............1......1........	−46.8
....	−15.2
1111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111	−332.1

Randomly drawn sequences for $P(1) = 0.1$. Note: $H(X) \approx 0.47$

Typical sequences are nearly equiprobable: Every $\mathbf{x} \in T_{N\beta}$ has

Variatio

Number of sequences in the typical set: For any

$$|T_{N\beta}| \leq 2^{N(H(X$$

Typical Sets

Proof of Cardinality Bound

For every $\mathbf{x} \in T_{N\beta}$,

$$p(\mathbf{x}) \geq 2^{-N(H(X)-\beta)}.$$

Thus,

$$\sum_{\mathbf{x}\in T_{N\beta}}$$

$$\geq \sum_{\mathbf{x}\in T_{N\beta}} 2^{-N(H(\cdots}$$

$$= 2^{-N(H(X)-\beta)} \cdot |T_{N\beta}|.$$

Thus

$$|T_{N\beta}| \leq 2^{N(H(X)+\beta)}$$

Typical Sets

Most Likely Sequence

The most likely sequence may not belong to the typical set

e.g. with

whereas $H(X) = 0.8113$

The most likely single sequence $\rightarrow$ $\mathtt{hhhh}$

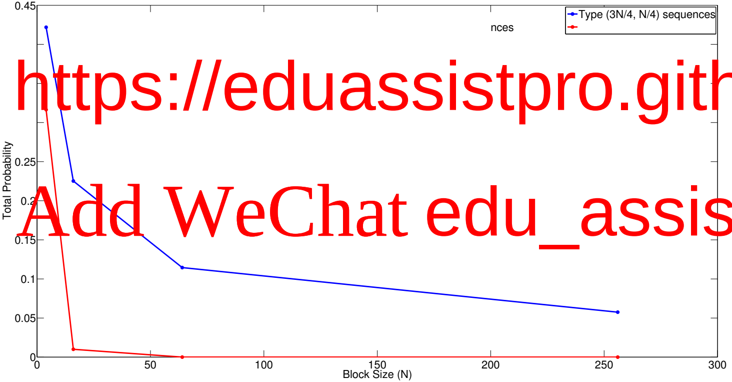
The most likely single sequence type $\rightarrow$ $\{\mathtt{hhht}, \mathtt{hthh}, \ldots\}$

Typical Sets

Most Likely Sequence

Probability of most likely sequence decays like $(p_{\mathrm{h}})^N$ ($p_{\mathrm{h}} = 0.75$)

Sequences with $N \cdot p_{\mathrm{h}}$ heads contain much more total probability mass



Blue curve corresponds to typical set with $\beta = 0$. What if $\beta > 0$?

$\underbrace{\text{Asymptotic}}_{\text{Eventually}}$ $\underbrace{\text{Equipartition}}_{\text{Equally Divided}}$ Property

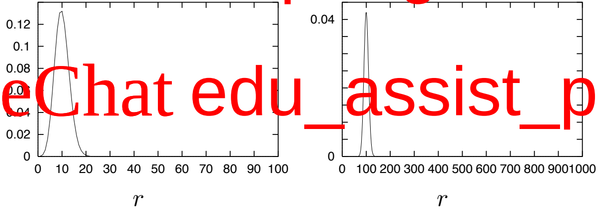
Informally

Asymptotic Equipartition Property (Informal)

As $N \to \infty$, $\log_2 P(x_1, \ldots, x_N)$ is close to $-NH(X)$ with high probability.

For la … $T_{N\beta}$)

$$n(r)P(\mathbf{x}) = \binom{N}{r} p_1^r (1 - p_1)^{N-r}$$



Probability sequence $\mathbf{x}$ has r heads for $N = 100$ (left) and $N = 1000$ (right). Here $P(X = \text{head}) = 0.1$.

Asymptotic Equipartition Property

Formally

Asymptotic Equipartition Property

If $x_1, x_2, \ldots$ are i.i.d. with distribution P then, in probability

$$\frac{1}{}$$

In precise l

$$(\forall \beta > 0) \lim_{N \to \infty} p \left(\left| -\frac{1}{N} \log_2 P(x_1, \ldots, \right. \right. \quad 1.$$

Exactly the probability of $\mathbf{x} \in T_{N\beta}$.

Asymptotic Equipartition Property

Formally

Asymptotic Equipartition Property

If $x_1, x_2, \ldots$ are i.i.d. with distribution P then, **in probability**:

$$\frac{1}{}$$

In precise

$$(\forall \beta > 0) \lim_{N \to \infty} p \quad \left| -\frac{1}{N} \log_2 P(x_1, \ldots, \right. \qquad 1.$$

Exactly the probability of $\mathbf{x} \in T_{N\beta}$.

Recall definition: for random variables $v_1, v_2, \ldots$, we say $v_N \to v$ **in probability** if for all $\beta > 0$ $\lim_{N \to \infty} P(|v_N - v| > \beta) = 0$

Here v_N corresponds to $-\frac{1}{N} \log_2 P(x_1, \ldots, x_N)$.

Why is it surprising/significant?

For an ens

i.e. the typical set is a small fraction of all possible sequences

AEP says that for N sufficiently large, we are vi
a sequence from this small set

Significance in information theory

Asymptotic Equipartition Property

Proof

Since $x_1, \ldots, x_N$ are independent,

$$-\frac{1}{N}\log p(x_1, \ldots, x_N) = -\frac{1}{n}\log\prod_{n=1}^{N} p(x_n)$$

Let $Y = -\log p(X)$ and $y_n = -\log p(x_n)$. T

$$\mathbb{E}[Y] = H(X$$

But then by the law of large numbers,

$$(\forall\beta > 0)\ \lim_{N\to\infty} p\left(\left|\frac{1}{N}\sum_{n=1}^{N} y_n - H(X)\right| > \beta\right) = 0.$$

1. Ensembles and sequences
   - Counting Types of Sequences

2. Typi

3. Asymptotic Equipartition Property (AEP)

4. Wrapping Up

Summary & Conclusions

- Ensembles and sequences
- Typ

- Asymptotic Equipartition Property (AEP)

Next: Source Coding.