COMP2610/COMP6261
Tutorial 6 Sample Solutions

Tutorial 6: Source Coding

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1. (a) We know that X is a binomial with parameters n, p. Thus,

$$p(X \geq n - 1) = p(X = n - 1) + p(X = n)$$
$$= n \cdot p^{n-1} \cdot (1 - p) + p^n.$$

(b) By Markov's inequality,

$$p(X \geq n - 1) \leq \frac{\mathbb{E}[X]}{n-1} = \frac{n}{n-1} p$$

(c) When $n = 2$,

).

The bound from

The difference between the two is

$$2p - p^2$$

Thus, for the Markov bound to be within 0.01 of the e $\qquad\qquad p \leq 0.1$.

2. (a) We need to code all sequences of length 100 with three or less 1's. There is 1 sequence with zero 1's. There are 100 sequences with one 1 (one sequence for each possible position in which the 1 appears.) Similarly, there are $\binom{100}{2}$ and $\binom{100}{3}$ sequences with two and three 1's respectively. In total we have

$$1 + 100 + \binom{100}{2} + \binom{100}{3} = 166,751.$$

If we want a uniform code over this number of elements, we need $\lceil \log_2(166,751) \rceil = 18$

(b) We need to find the probability of observing more than 3 1's. Let K be the number of 1's observed. Then

$$P(K > 3) = 1 - P(K \leq 3) = 1 - P(K = 0) - P(K = 1) - P(K = 2) - P(K = 3)$$
$$= 1 - 0.995^{100} + 100 \times 0.995^{99} \times 0.005$$
$$+ \binom{100}{2} \times 0.995^{98} \times 0.005^2 + \binom{100}{3} \times 0.995^{97} \times 0.005^3$$
$$= 0.00167$$

(c) The number of 1's, K, is a binomial random variable with probability of success 0.995. It has expectation $\mathbb{E}[K] = 100 \times 0.005 = 0.5$ and variance $\mathbb{V}[K] = 100 \times 0.995 \times 0.005 = 0.4975$. By Chebyshev's inequality we have

$$P(K \geq 4) \leq P(|K - 0.5| \geq 3.5) = P(|K - \mathbb{E}[K]| \geq 3.5)$$
$$\leq \frac{\mathbb{V}[K]}{3.5^2} = \frac{0.4975}{3.5^2} = 0.0406$$

This is larger than the probability we found in part (b), it is off by a factor of around 20.

3. (a) Using the definition of entropy

$$H(X) = 0.4 \times \log \frac{1}{0.4} + 0.6 \times \log \frac{1}{0.6} = 0.971$$

(b) i. Recall that

$$T_{N\beta} := \{\mathbf{x} : |-\frac{1}{N} \log_2 P(\mathbf{x}) - H(X)| < \beta\}$$

so we are looking for k (the number of ones) such that

$$0.871 = H(X) - 0.1 < -\frac{1}{N} \log_2 P(k) < H(X) + 0.1 = 1.071.$$

Referring to the table we see this is the case for values of k ranging from 11 to 19. That is, the typical set consists of sequences with between 11 and 19 1s.

ii. Summing the corresponding entries in the table (rows 11-19 in the second column) of the table we see

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$$P(\mathbf{x} \in T_{N\beta}) = 0.043410 + 0.075967 + \ldots + 0.044203 = 0.936247$$

iii. The number o

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6,510.

iv. By definition, S_δ is the smallest set of sequences such tha $1 - \delta$. In particular, you always want to throw in the highest probability elements in t probability sequences are the ones with the most

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les from the third column of the table starting from the bottom (more $\geq 19) = 0.073564$ while $P(K \geq 18) = 0.15355$ so we need to add $\left\lceil \frac{(0.1 - 0.073564)}{0.079986} \times 480,700 \right\rceil = 158,876$ elements with 18 1s to the set of sequences with more than 18 1s. This gives us a total of

$$1 + 25 + 300 + 2,300 + 12,650 + 53,130 + 177,100 + 158,876 = 404,382$$

elements in S_δ.

v. The essential bit content is simply $\log_2 |S_\delta| = 18.625$.

4. (a) This is incorrect. Certainly it is true that $\frac{1}{N}H_\delta(X^N) \geq 0$, as the essential bit content is nonnegative. But this does not mean we can compress down to zero bits per outcome. If we were able to show that $\frac{1}{N}H_\delta(X^N) \leq \epsilon'$, for arbitrary ϵ', on the other hand, we would be able to make such a statement. Of course, we cannot show such a thing, because we know that the asymptotic fraction converges to the entropy $H(X)$.

   (b) This is correct. The source coding theorem assumes that we are coding blocks created from extended ensembles. By definition this involves performing independent trials. When there is dependence amongst outcomes in a block, the theorem no longer holds.

5. (a) The code C_1 is not prefix (0 is a prefix to 01) nor uniquely decodable (0110101 could decompose into 0 1101 01 or 01 10101). The code C_2 is prefix free and thus uniquely decodable. The code C_3 is not uniquely decodable (00 can be 00 or 0 0) and clearly not prefix free.

   (b) $C_1' = \{0, 10, 1110, 11110\}$, $C_2' = C_2$. For C_3' we need a code a with lengths 1,1,2, and 2. However, this cannot be done as the only codewords of length one are 0 and 1 and one of these will be a prefix to any other codeword.

   (c) No. The Kraft inequality tells us that for any prefix code

   $$\sum_i 2^{-l_i} \leq 1$$

   but for the given code lengths

   $$\sum_i 2^{-l_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 3 \times \frac{1}{16} = \frac{17}{16} > 1.$$

   (d) Yes, these lengths satisfy the Kraft inequality. For instance $C = \{0, 100, 101, 1100, 1101, 1110\}$.