COMP2610/6261 - Information Theory

Lecture 18: Channel Capacity

Australian National University

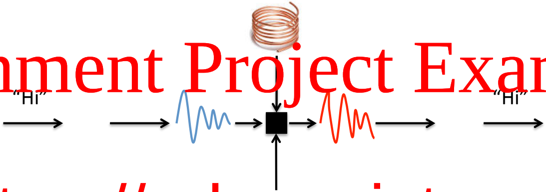
8 October, 2018

1. Channel Capacity

2. Com

3. Summary

Channels: Recap



Source	: Aditya
Encoder	: Telephone handset
Channel	: Analogue telephone line
Decoder	: Telephone handset
Destination	: Mark

Channels: Recap

A **discrete channel** Q consists of:

- an *input alphabet* $\mathcal{X} = \{a_1, \ldots, a_I\}$

- an
- *tran*

The channel Q can be expressed as a matrix

$$Q_{j,i} = P(y = b_j|$$

This represents the probability of observing b_j given that we transmit a_i

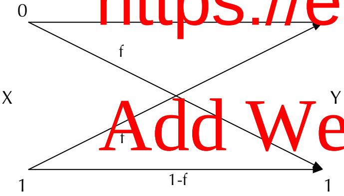
The Binary Noiseless Channel

One of the simplest channels is the **Binary Noiseless Channel**. The received symbol is always equal to the transmitted symbol – there is no probability of error, hence *noiseless*.

$= \{0, 1\}$

The Binary Symmetric Channel

Each symbol sent across a **binary symmetric channel** has a chance of being "flipped" to its counterpart ($0 \to 1$; $1 \to 0$)

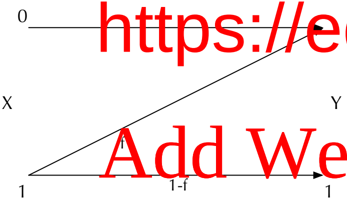


$= \{0, 1\}$;
flip$) = f$

The Z Channel

Symbols may be corrupted over the channel asymmetrically.

$= \{0, 1\};$



1 f

Communicating over Noisy Channels

Suppose we know we have to communicate over some channel Q and we want build an *encoder/decoder* pair to reliably send a message $\mathbf{s}$ over Q.

$$\boxed{\mathbf{s}_{in}} \longrightarrow \boxed{} \longrightarrow \boxed{} \longrightarrow \boxed{\mathbf{s}_{out}}$$

Reliability is measured via **probability of err** f
incorrectly decoding $\mathbf{s}_{out}$ given $\mathbf{s}_{in}$ as input:

$$P(\mathbf{s}_{out} \neq \mathbf{s}_{in}) = \sum_{\mathbf{s}} P(\mathbf{s}_{out} \neq \mathbf{s}$$

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Mutual Information for a Channel

A key quantity when using a channel is the mutual information between its inputs X and outputs Y:

This mea *ty* about what was transmitted

This requires we specify some particular p

- A channel is only specified by its transition matr

For noisel

If $\mathbf{p}_X = ($

Mutual Information for a Channel: Example

For binary symmetric channel with $f = 0.15$ and $\mathbf{p}_X = (0.9, 0.1)$ we have

$$p(Y = 1) = p(Y = 1 \mid X = 1) \cdot p(X = 1) + p(Y = 1 \mid X = 0) \cdot p(X = 0)$$

and so $H(Y) = 0.76$

Further, $H(Y \mid X = 0) = H(Y \mid X = 1) =$

So, $I(X; Y) = 0.15$ bits

Mutual Information for a Channel: Example

For Z chan 42,

$H(Y|X)$

So, intuitively, the reliability is “noiseless $>$ Z $>$ symmetric”

Channel Capacity

The mutual information measure for a channel depends on the choice of input distribution $\mathbf{p}_X$. If $H(X)$ is small then $I(X;Y) \leq H(X)$ is small.

The *largest possible* reduction in uncertainty achievable across a channel is its **ca**

Channe

The capa een its input and output for any choice of input ensemble. T

$$C = \max_{\mathbf{p}_X} I(X$$

Later, we will see that the capacity determines the rate at which we can communicate across a channel with arbitrarily small error.

Assignment Project Exam Help

1 Channel Capacity

2 Com https://eduassistpro.github.i

3 Summary Add WeChat edu_assist_pr

Computing Capacities

Definition of capacity for a channel Q with inputs $\mathcal{A}_X$ and ouputs $\mathcal{A}_Y$:

$$C = \max_{p_X} I(X; Y)$$

How do we actually calculate this quantity?

1. Co
2. Det
3. Use t

Binary Symmetric Channel:
We first consider the *binary symmetric chann* 0, 1}
and flip probability f. It has transition matrix

$$Q = \begin{bmatrix} 1-f & f \\ f & 1-f \end{bmatrix}$$

Computing Capacities

Binary Symmetric Channel - Step 1

The mutual information can be expressed as $I(X;Y) = H(Y) - H(Y|X)$. We therefore need to compute two terms: $H(Y)$ and $H(Y|X)$ so we need the distributions $P(y)$ and $P(y|x)$.

Comput

- $P(y$ … $p_0 + f \cdot p_1$
- $P(y$ … $(1-f) \cdot p_1$

In general, $\mathbf{q} := \mathbf{p}_Y = Q\mathbf{p}_X$, so above calculat

$$\mathbf{q} = \mathbf{p}_Y = \begin{bmatrix} 1-f & f \\ f & 1 \end{bmatrix}$$

Using $H_2(q) = -q \log_2 q - (1-q)\log_2(1-q)$ and letting $q = q_1 = P(y=1)$ we see the entropy

$$H(Y) = H_2(q_1) = H_2(f \cdot p_0 + (1-f) \cdot p_1)$$

Computing Capacities

Binary Symmetric Channel - Step 1

Computing $H(Y|X)$:

Since $P(y|x)$ is described by the matrix Q, we have

$H(Y|x=0) = H_2(P(y=1|x=0)) = H_2(Q_{1,0}) = H_2(f)$

and similarly,

$H(Y|x$

So,

$$H(Y|X) = \quad_{x} \quad H(Y|x)P(x) = \quad_{x} \quad H_2(f)P(x) = H\ (f) \quad P(x) = H_2(f)$$

Computing $I(X;Y)$:

Putting it all together gives

$$\boxed{I(X;Y) = H(Y) - H(Y|X) = H_2(f \cdot p_0 + (1-f) \cdot p_1) - H_2(f)}$$

Computing Capacities

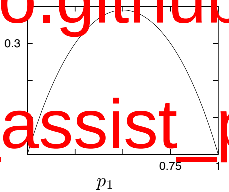
Binary Symmetric Channel - Steps 2 and 3

Binary Symmetric Channel (BSC) with flip probability $f \in [0, 1]$:

$$I(X; Y) = H_2(f p_0 + (1 - f) p_1) - H_2(f)$$

Example

- BSC (...
  $I(X;$ ...
- BSC ($f = 0.15$) and $\mathbf{p}_X = (0.5, 0.5)$:
  $I(X; Y) = H_2(0.5) - H_2(0.15) \approx 0.39$
- BSC ($f = 0.15$) and $\mathbf{p}_X = (0.9, 0.1)$:
  $I(X; Y) = H_2(0.22) - H_2(0.15) \approx 0.1$



$I(X; Y), f = 0.15$

Maximise $I(X; Y)$: Since $I(X; Y)$ is symmetric in p_1 it is maximised when $p_0 = p_1 = 0.5$ in which case $C = 0.39$ for BSC with $f = 0.15$.

Channel Capacity: Example

For a binary symmetric channel, we could also argue

$$I(X;Y) = H(Y) - H(Y \mid X)$$

$$x$$

$$= H(Y) - H(f)$$

$$\leq 1 - H(f),$$

where equality of the last line holds for **uniform** $\mathbf{p}_X$

Symmetric Channels

The BSC was easy to work with due to considerable symmetry

Symme

A channe metric if $\mathcal{A}_Y$
can be part Q'
containing only rows for outputs Y' has:

- Columns that are all permutations of each oth
- Rows that are all permutations of each other

Symmetric Channels: Examples

$\mathcal{A}_X = \mathcal{A}_Y = \{0, 1\}$ $\mathcal{A}_X = \{0, 1\}, \mathcal{A}_Y = \{0, ?, 1\}$ $\mathcal{A}_X = \mathcal{A}_Y = \{0, 1\}$

$$Q = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \qquad Q = \begin{bmatrix} 0.7 & 0.1 \\ 0.2 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} \qquad Q = \begin{bmatrix} 0.9 & 0 \\ 0.1 & 1 \end{bmatrix}$$

Sym **mmetric**

Subse

If one of our partitions has just one row, then every ele
equal for the columns to be permutations of each oth

Simplest case: all rows and columns are permutations of each other

- But this is not a requirement

Channel Capacity for Symmetric Channels

For symmetric channels, the optimal distribution for the capacity has a simple for

Theore

If Q is sy

over $\mathcal{X}$.

Exercise 10.10 in MacKay

Computing Capacities in General

What can we do if the channel is not symmetric?

- We can still calculate $I(X;Y)$ for a general input distribution $\mathbf{p}_X$
- Finding the maximising $\mathbf{p}_X$ is more challenging

What to do

- $I(X$

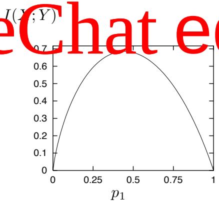
- For *binary* inputs, just look for stationary poi 2) i.e., where $\frac{d}{dp} I(X;Y) = 0$ for $\mathbf{p}_X = (1-p$

- In general, need to consider distributions that place 0 probability on one of the inputs

Computing Capacities in General

Example (Z Channel with $P(y=0|x=1)=f$):

$$H(Y) = H_2(P(y=1)) = H_2(0p_0 + (1-f)p_1)$$
$$= H_2((1-f)p_1)$$
$$H(Y|X) = p_0 H_2(P(y=1|x=0)) + p_1 H_2(P(y=0|x=1))$$



$I(X;Y)$ for Z channel with $f = 0.15$

Computing Capacities in General

Example (Z Channel):

Showed earlier that $I(X;Y) = H_2((1-f)p) - pH_2(f)$ so solve

$$\frac{d}{dp} I(X;Y) = 0 \iff (1-f)\log_2\left(\frac{1-(1-f)p}{(1-f)p}\right) - H_2(f) = 0$$

$$\frac{}{}$$

$$\iff \quad = \frac{}{1 + 2^{H_2(f)}}$$

For $f = 0.15$, we get $p = \frac{1/0.85}{1+2^{0.61/0.85}} \approx 0.$

$C = H_2(0.38) - 0.44 H_2(0.15) \approx 0.685$

Homework: Show that $\frac{d}{dp} H_2(p) = \log_2 \frac{1-p}{p}$

Why Do We Care?

We have a template for computing channel capacity for generic channels

But what does this tell us?

- How, if at all, does it relate to the error probability when decoding?
- Wh
  can

We will se
and the best achievable rate of transmission

- Rates above the capacity cannot be achieve
  arbitrarily small error probabilities

Summary and Conclusions

Mutual information between input and output should be large

- Depends on input distribution

Capacity is the maximal possible mutual information

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