COMP2610 / COMP6261 - Information Theory

Lecture 9: Probabilistic Inequalities

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Australian National University

20 August, 2018

Last time

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Mutual information chain rule

Jensen'

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"Informa

Data processing inequality

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Review: Data-Processing Inequality

Theorem

if $X \to Y \to Z$ then $I(X;Y) \geq I(X;Z)$

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- X is t … s the pro …

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- No “clever” manipulation of the data can impro … can be made from the data

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- No processing of Y, deterministic or random, can increase the information that Y contains about X

This time

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- Markov's inequality
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- Law of large numbers

Outline

1. Properties of expectation and variance
2. Mark
3. Cheb
4. Law of large numbers
5. Wrapping Up

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1 Properties of expectation and variance

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2 Markov's Inequality

3 Chebyshev's Inequality

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4 Law of large numbers

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5 Wrapping Up

Expectation and Variance

Let X be a random variable over $\mathcal{X}$, with probability distribution p

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Expected value:

Varianc

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$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}$$

$$= \mathbb{E}[X^2] - ($$

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Standard deviation is $\sqrt{\mathbb{V}[X]}$

Properties of expectation

A key property of expectations is linearity:

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$$\mathbb{E}\left[\sum^{n} X_i\right] = \sum^{n} \mathbb{E}\left[X_i\right]$$

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$x_1 \in \mathcal{X}_1$ $x_n \in \mathcal{X}_n$ $i=1$

This holds even if the variables are dependent!

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We have for any $a \in \mathbb{R}$,

$$\mathbb{E}[aX] = a \cdot \mathbb{E}[X].$$

Properties of variance

We have linearity of variance for independent random variables:

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$$\mathbb{V}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{V}[X_i].$$

Does not h

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(prove this: expand the definition of variance and rely upo $X_i)\mathbb{E}(X_j)$ when $X_i \perp X_j$)

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We have for any $a \in \mathbb{R}$,

$$\mathbb{V}[aX] = a^2 \cdot \mathbb{V}[X].$$

1. Properties of expectation and variance

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2. Markov's inequality

3. Cheb

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4. Law of large numbers

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5. Wrapping Up

Markov's Inequality

Motivation

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1000 school students sit an examination

The busy p

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The principal wants to estimate the maximum poss
who scored more than 80

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- A question about the *minimum* number of s

Markov's Inequality

Motivation

Call x the number of students who score > 80

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Call S is the total score of students who score ≤ 80

We know:

$$40 \cdot 1 \text{[illegible]} \} > 80x$$

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Exam scores are nonnegative, so certainly

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Thus, $80x < 40 \cdot 1000$, or

$$x < 500.$$

Can we formalise this more generally?

Markov's Inequality

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Theorem

Let X be a nonnegative random variable. Then, for any $\lambda > 0$,

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Bounds probability of observing a large outcome

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Vacuous if $\lambda < \mathbb{E}[X]$

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Corollary

Let X be

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Observations of nonnegative random variable u
than expected value

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Vacuous if $\lambda < 1$

Markov's Inequality

Proof

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$$\mathbb{E}[X] = \quad x \cdot p(x)$$

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$$\geq \sum_{x \geq \lambda} x \cdot p(x)$$ nonn

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$$\geq \sum_{x \geq \lambda} \lambda \cdot p(x)$$

$$= \lambda \cdot p(X \geq \lambda).$$

Markov's Inequality

Illustration from

https://justindomke.wordpress.com/2008/06/19/markovs-inequality/

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Chebyshev's Inequality

Motivation

Markov's inequality only uses the mean of the distribution

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What about the spread of the distribution (variance)?

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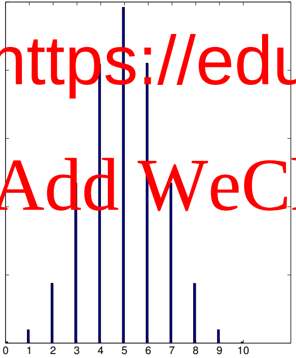

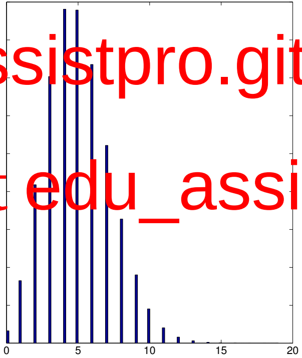

Chebyshev's Inequality

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Theorem

Let X be a random variable with $\mathbb{E}[X] < \infty$. Then, for any $\lambda > 0$,

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Bounds the probability of observing an "unexpect

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Does not require non negativity

Two-sided bound

Chebyshev's Inequality

Alternate Statement

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Corollar

Let X be 0.

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$$(| \quad - \quad [\quad]| \geq \lambda \cdot \quad \overline{[\quad]}) \quad -.$$

Observatio s are unlikely to occur several standa the mean

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Chebyshev's Inequality

Proof

Define

$$Y = (X - \mathbb{E}[X])^2.$$

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Then, by Markov's inequality, for any $\nu > 0$,

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But,

$$\mathbb{E}[Y] = \mathbb{V}[X$$

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Also,

$$Y \geq \nu \iff |X - \mathbb{E}$$

Thus, setting $\lambda = \sqrt{\nu}$,

$$p(|X - \mathbb{E}[X]| \geq \lambda) \leq \frac{\mathbb{V}[X]}{\lambda^2}.$$

Chebyshev's Inequality

Illustration

For a binomial X with N trials and success probability θ, we have e.g.

$$p\left(|X - N\theta| \geq \sqrt{2N\theta(1-\theta)}\right) \leq \frac{1}{2}.$$

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Chebyshev's Inequality

Example

Suppose we have a coin with bias θ, i.e. $p(X = 1) = \theta$

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Say we flip the coin n times, and observe $x_1, \ldots, x_n \in \{0, 1\}$

We use th

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Estimate how large n should be such that

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$$p(|\hat{\theta}_n - \theta| \geq 0.05)$$

1% probability of a 5% error

(Aside: the need for two parameters here is generic: "Probabably Approximately Correct")

Observe that

$$\mathbb{E}[\hat{\theta}_n] = \frac{\sum_{i=1}^{n} \mathbb{E}[x_i]}{n} = \theta$$

$$\mathbb{V}[\hat{\theta}_n] = \frac{\sum_{i=1}^{n} \mathbb{V}[x_i]}{n^2} = \frac{\theta(1-\theta)}{n}$$

Thus, app[illegible]

$$p(|\hat{\theta}_n - \theta| \geq 0.05) \leq [illegible]$$

We are guaranteed this is less than 0.01 if

$$n \geq \frac{\theta(1-\theta)}{(0.05)^2(0.01)}.$$

When $\theta = 0.5$, $n \geq 10,000$ (!)

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Independent and Identically Distributed

Let $X_1, \ldots, X_n$ be random variables such that:

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- Each X_i is independent of X_j
- The

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Then, we say that $X_1, \ldots, X_n$ are independ
(or iid)

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Example: For n independent flips of an unbiase $_1$ $_n$ re iid from Bernoulli$\left(\frac{1}{2}\right)$

Law of Large Numbers

Theorem

Let $X_1, \ldots, X_n$ be a sequence of iid random variables, with

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$$\mathbb{E}[X_i] = \mu$$

and $\mathbb{V}[X_i$ [illegible]

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$${}_n = \frac{[illegible]}{n}.$$

Then, for any $\epsilon > 0$,

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$$\lim_{n \to \infty} p(|\bar{X}_n - \mu| > \epsilon) = 0.$$

Given enough trials, the empirical "success frequency" will be close to the expected value

Since the X_i's are identically distributed,

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$$\mathbb{E}[\bar{X}_n] = \mu.$$

Since the

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$$= \frac{\mathbb{V}[X_1 + \dots}{n^2}$$

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$$= \frac{n\sigma^2}{n^2}$$

$$= \frac{\sigma^2}{n}.$$

Law of Large Numbers

Proof

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Applying Chebyshev's inequality to $\bar{X}_n$,

$$\frac{\mathbb{V}[\bar{X}_n]}{}$$

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For any fixed $\epsilon > 0$, as $n \to \infty$, the right hand s

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Thus,

$$p(|\bar{X}_n - \mu| < \epsilon) \to 1.$$

Law of Large Numbers

Illustration

$N = 1000$ trials with Bernoulli random variable with parameter $\frac{1}{2}$

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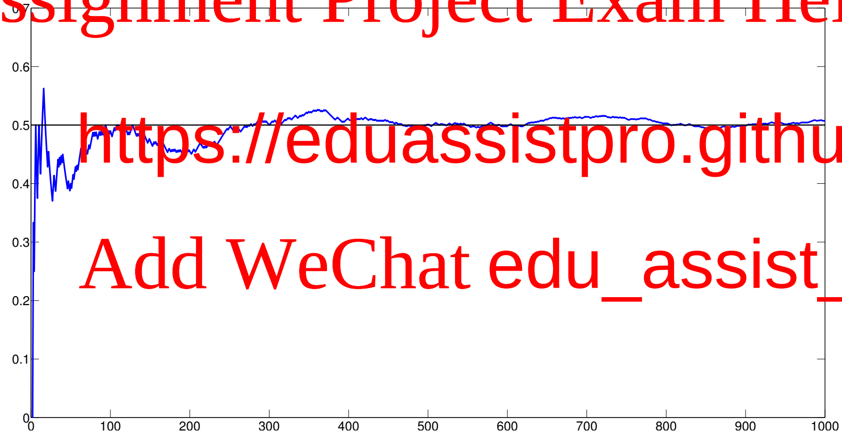

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Law of Large Numbers

Illustration

$N = 50000$ trials with Bernoulli random variable with parameter $\frac{1}{2}$

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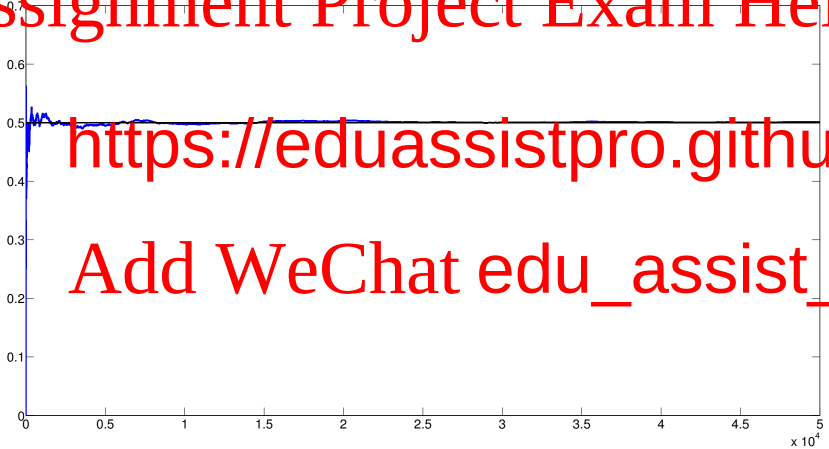

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Summary & Conclusions

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- Markov's inequality

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- Ch

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- Law of large numbers

Next time

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- Ensembles and sequences

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- Typ

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- Approximation Equipartition (AEP)