# COMP2610 / COMP6261 Information Theory

## Lecture 2: First Steps and Basic Probability



Australian National University

July 24, 2018

# Outline

# A General Communication System



Source : The information source that generat communicated

Encoder : Operates on the message to produce transmission

Channel : The medium used to transmit the signal

Decoder : Reconstructs the message from the signal

Destination : The entity that receives the message

# Communication over Noisy Channels

A channel is some medium for transmitting messages



A noisy ch
sender's

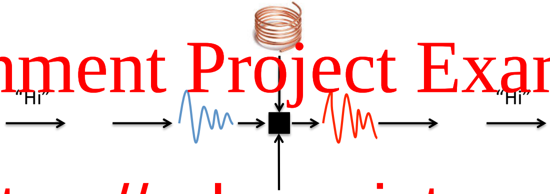


**The Prob**

"The fundamental problem of communication is th
point either exactly or approximately a message se
(Claude Shannon, 1948)

# Example: Telephone Network










Source : Aditya

Encoder : Telephone handset

Channel : Analogue telephone line

Decoder : Telephone handset

Destination : Mark

# Examples of Noisy Communication Channels

Other examples of noisy channels:

- A radio communication link



- VDSL NBN connection
- The
  inte



- Reproducing cells



- A magnetic hard disk drive
  - Channel does not need to involve physical movement

What would the other components be for each of these channels?

1 A General Communication System



2 The Role of Uncertainty

3 Basic



4 Relating Joint, Marginal and Conditional Prob



5 Wrapping Up

# Uncertainty in Communication



We can not avoid uncertainty when

1. Dealing with noise (imperfections) in the channel



2. “Compressing” the messages (compare a h
   of a manuscript with the typed text that capture
   transcript of a spoken utterance)

## Channel Noise



A noisy ch

Thus, rec

How to model, quantify, and mitigate this uncertain

# Message Compression – I

## Cover and Thomas, Example 1.1.2



- Suppose we'd like to relay the outcome of an 8 horse race to a friend
  - We wish to convey one of `A, B, ..., H`
- Sup
  sch



$$
\begin{aligned}
\texttt{B} &\rightarrow \texttt{0} \\
\texttt{C} &\rightarrow \texttt{0} \\
&\vdots \\
\texttt{H} &\rightarrow \texttt{111}
\end{aligned}
$$

# Message Compression – II

Cover and Thomas, Example 1.1.2

- Now say the probabilities of the horses winning are $(1/2, 1/4, 1/8, 1/16, 1/64, 1/64, 1/64, 1/64)$
- Encoding messages based on their probability of the being chosen will g



$$\mathtt{C} \rightarrow \mathtt{110}$$

$$\mathtt{D} \rightarrow \mathtt{111}$$

$$\mathtt{E} \rightarrow \mathtt{111}$$

$$\mathtt{F} \rightarrow \mathtt{111100}$$

$$\mathtt{G} \rightarrow \mathtt{111101}$$

$$\mathtt{H} \rightarrow \mathtt{111111}$$

## What is "Information"?



For noise correction and message compression, we will need to quantify the information contained in a message

Roughly



- Un



- The receiver's uncertainty is reduced on seei

# The Case for Probability

We run into the notion of uncertainty when trying to pin down:



1. How to deal with channel noise
2. Ho



3. Wh

To make progress, we need to formalise uncertain



We will do this using probability theory

We now commence our *review* of probability; this will be hard going if you have not met it before!

# Probability: Example

Quantification and Manipulation of Uncertainty (Bishop, PRML, 2006)

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1. Pick a box at random

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## Quantification and Manipulation of Uncertainty (Bishop, PRML, 2006)

1. Pick a box at random
2. From the selected box, pick a fruit (apple or orange) at random
3. Ob

- Identity of the box is a random variable $\in \{ \ , \ \}$
- Identity of the fruit is a random variable $F \in \{a, o\}$

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- Identity of the fruit is a random variable $F \in \{a, o\}$

Probability of an event: Proportion of times it happens out of a large number of trials

# Probability: Example



Suppose we repeat the selection process many tim picking up the blue box 60% of the time and the red box 40% of the time

- $p(B = r) = 0.4$, $p(B = b) = 0.6$

# Probability: Basic Properties

By definition, $0 \leq p(B = b) \leq 1$ and $0 \leq p(B = r) \leq 1$



Outcomes are mutually exclusive:



Outcomes are jointly exhaustive:



$$
\begin{aligned}
p(B = r \text{ OR } B = b) &= p(B = r) + p(B \quad b) \\
&= p(B = r) + p(B = b) \\
&= 1
\end{aligned}
$$

- What is the probability of picking the red box, and an apple within that box?



- Wh

- Given that we selected a red box, what is the pro an apple?



We can answer these and more complex questions by using the *rules of probability*.

What is the probability of selecting the red box and selecting an apple?

# Joint Probability

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**Joint Probability of a Set of Events**

The proportion of times these events happened together out of the total number o

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If we repea of the trials we sa



$$
\begin{aligned}
p(B = r \texttt{ AND } F = a) &= p \\
&= \frac{}{1} \\
&= 0.1
\end{aligned}
$$

## Marginal Probability

What is the probability of an apple being picked, regardless of the box we selected?

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Remem



Say that in 45 of the trials, we selected a blue box and an a



So, irrespective of $B$, an apple was selected 45 + 1

$$p(F = a) = \frac{55}{100} = \frac{11}{20}$$

# Conditional Probability

What is the probability of an apple being picked up, given that a red box was selected?

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We selected a red box and an apple 10 out of 100 times



We selected a red box (regardless of the fruit) 40 out of 1

$$p(F = a \,\texttt{GIVEN}\, B = r) = p(F = a | B = r) = \frac{10}{40} = \frac{1}{4}$$

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Can we write this in terms of the joint and marginal probabilities?

# Joint, Marginal and Conditional Probabilities:

## A More General Formulation (1)

Consider the more general case of two random variables:

$$X \in \{x_1, x_2, \ldots, x_M\} \text{ and } Y \in \{y_1, y_2, \ldots, y_L\}$$



$N$ : Total number of trials

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$r_j$ : $\sharp(Y = y_j) = \sum_i n_{ij}$ = $\sharp$ of times that $y_j$ happens

## Joint, Marginal and Conditional Probabilities:

### A More General Formulation (2)





By definiti



$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

$$p(X = x_i) = \frac{c_i}{N}$$

$$p(Y = y_j) = \frac{r_j}{N} \text{ (Marginal)}$$

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i} \text{ (Conditional)}$$

# Joint, Marginal and Conditional Probabilities:
## A More General Formulation (1)



Bins and fr



| | | |
|---|---|---|
| | | |
| | | |
| **Red** | 30 | |

Verify the computations from previous section wit

Observe 

$$p(X = x_i) = \frac{\sum_j n_{ij}}{N}$$



$$= \sum_j p(X$$

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i} = \frac{n_{ij}}{N} \Big/ \frac{c_i}{N}$$

$$= p(X = x_i, Y = y_j)/p(X = x_i)$$

# The Rules of Probability

Sum Rule / Marginalization :

$$p(X = x_i) = \sum_j p(X = x_i, Y = y_j)$$

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Therefore:

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j) = \sum_j P(X = x_i | Y = y_j) P(Y = y_j)$$

## An Illustration of a Distribution over Two Variables

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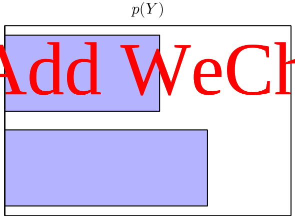


marginal

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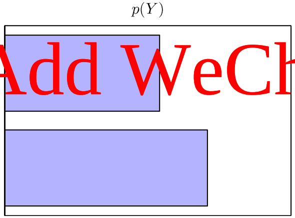


marginal

conditional

# Joint, Marginal and Conditional Probabilities:

## An even More General Formulation

Given $D$ random variables $X_1, \ldots, X_D$:

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## An even More General Formulation



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Chain Ru

$p(X_1, X_2) = p(X_1)p(X_2|X_1)$ What are we

# Joint, Marginal and Conditional Probabilities:

## An even More General Formulation



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Chain Ru

$$p(X_1, X_2) = p(X_1)p(X_2|X_1)$$

What are we

$$p(X_1, X_2, X_3) = p(X_1, X_2)p(X_3|X_1, X_2)$$

$, X_2)$

# Joint, Marginal and Conditional Probabilities:

## An even More General Formulation



Given $D$ random variables $X_1, \ldots, X_D$:



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$$p(X_1, X_2) = p(X_1)p(X_2|X_1)$$

What are we



$$p(X_1, X_2, X_3) = p(X_1, X_2)p(X_3|X_1, X_2) \quad \ldots, X_2)$$

$$p(X_1, \ldots, X_D) = p(X_1)p(X_2|X_1)p(X_3|X_2, X_1)\ldots p(X_D|X_1, \ldots, X_{D-1})$$

## Summary

- General architecture for communication systems



- Wh



- Probability theory: joint, marginal and condit



- Reading: Mackay § 2.1 and § 2.2; Bishop § 1.2

# Exercise

## Coming Back to our Original Example

Given: $p(B = r) = 0.4$, $p(B = b) = 0.6$

Assume the fruit are selected uniformly from each box



- Write down all conditional probabilities $P(F|B)$
- Evaluate the overall probabilities $P(F)$

# Next time

- More on joint, marginal and conditional distributions
- Wh
- What, if anything, does $p(X = x | Y =$ $p(Y = y | X = x)$?

## Next time



- Sign-up for tutorials open at 9am wednesday 25 July. Will offer **three** tuto



- Class rep.