COMP30026 Models of Computation

Predicate Logic: Semantics

Lecture Week 4 Part 1 (Zo

Semester 2, 2021

This Lecture is Being Recorded

Reading Materials

Remember, if you need textbook support, check out the resources that are linked under "Subject Overview" (also accessible from "Modules" → "Reading Resources").

O'Donn
includin
use of a styl
(not covered by us, and not examinable).

A rather different introduction to predicate logic is i n's Chapter 9.

The book by Jenkyns also looks good.

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3. It is true if $D = \mathbb{R}$ and $<$ is the usual "smaller than".
4. It is true if $D = \{0\}$.

The Meaning of a Formula

In some cases, the meaning of a formula is independent of what its predicate (and function) names denote, and of what sort of things the varia

For exam what (it is valid).

Similarly, $\forall x\, P(x) \wedge (\neg P(a))$ is false no matt stand for (the formula is unsatisfiable).

Interpretations (or Structures)

An interpretation (or structure) consists of

1. A no
2. An a n-place fun
3. An assignment, to each n-ary functi place function $g : D^n \to D$.

4. An assignment to each constant a D.

Free Variables and Valuations

To give meaning to formulas that may have free variables, such as

we need t

- A valuation $\sigma : var \to D$ for free vari
- An interpretation as just discussed.

Connectives are always given their usual meaning.

We just said that a valuation is a function $\sigma : var \rightarrow D$.

But, given an interpretation $\mathcal{I}$ we get a valuation function from terms automatically, by natural extension:

where d is the element of D that $\mathcal{I}$ ass ^{n} D is the function that $\mathcal{I}$ assigns to g.

Example: Consider term $t = f(y, g(x,$ (with domain $\mathbb{Z}$) assign to a the value 3, to f the multiplication function, and to g addition. If $\sigma(x) = 9$ and $\sigma(y) = 5$ then $\sigma(t) = 60$.

Truth of a Formula

The truth of a closed formula should depend only on the given interpretation.

The only r riables (and her a formula

Notation:

$$\sigma_{x \mapsto d}(y) = \begin{cases} d & \\ \sigma(y) & \text{oth} \end{cases}$$

Read this as “the map σ, updated to map x to d.”

Making a Formula True

Given an interpretation $\mathcal{I}$ (with domain D), and a valuation σ,

- σ makes $P(t_1, \ldots, t_n)$ true iff $\mathbf{p}(\sigma(t_1), \ldots, \sigma(t_n)) = \mathbf{t}$, wh
- σ
- σ rue.

- σ makes $\forall x\ F$ true iff $\sigma_{x \mapsto d}$ make .

If we now define

$$\exists x\ F \equiv \neg \forall x\ \neg F$$

then the meaning of every other formula follows from this.

Quantifier Order

The order of different quantifiers is important.

$\forall x \exists y$ is n

The form ter says

there's an individual y that satisfies $P($

But $\forall x \forall y$ is the same as $\forall y \forall x$ and $\exists x$.

Quantified Formulas as a Two-Person Game

The truth or falsehood of a quantified formula can be expressed as a question of winning strategies for a two-person game. Say I make a claim (the quantified statement) and you try to disprove it. You get to supply values for the universally quantified variables.

- If I cla
  an), but I get
  to k

- If I claim $\exists y \forall x\ P(x, y)$, then you can ch
  to provide the y, and then you just have to fi oes
  not satisfy $P(x, y)$, knowing the y

- If I claim $\exists x \exists y\ P(x, y)$, then I have to find both x and y, so it doesn't matter what order they appear.
- If I claim $\forall y \forall x\ P(x, y)$, then you get to pick both x and y, so again their order does not matter.

Rules of Passage for the Quantifiers

We cannot in general "push quantifiers in".

For example, there is no immediate simplification of a formula of the form $\exists x\ (P(x) \land Q(x))$.

Howeve

$$\forall x\ (\neg F_1) \equiv \neg\exists x\ F$$
$$\exists x\ (F_1 \lor F_2) \equiv (\exists x\ F$$
$$\forall x\ (F_1 \land F_2) \equiv (\forall x\ F$$

It follows that

$$\exists x\ (F_1 \Rightarrow F_2) \equiv (\forall x\ F_1) \Rightarrow (\exists x\ F_2)$$

More Rules of Passage for Quantifiers

If G is a formula with no free occurrences of x, then we also get

$$\forall x\,(F \Rightarrow G) \;\equiv\; ($$

$$\forall x\,(G \Rightarrow F) \;\equiv\; G$$

no matter what F is. In particular F may have free occurrences of x.

Models and Validity of Formulas

A wff F is true in interpretation $\mathcal{I}$ iff every valuation makes F true (for $\mathcal{I}$). If not true then it is false in interpretation $\mathcal{I}$.

A model for F is an interpretation $\mathcal{I}$ such that F is true in $\mathcal{I}$.
We write

A wff F F.
In that case we write $\models F$.

F_2 is a logical consequence of F_1 iff $\mathcal{I}$
We write $F_1 \models F_2$.

F_1 and F_2 are logically equivalent iff $F_1 \models F_2$ and $F_2 \models F_1$.
We write $F_1 \equiv F_2$.

Summarising: Satisfiability and Validity

A closed, well-formed formula F is

- satisfiable iff $\mathcal{I} \models F$ for some interpretation $\mathcal{I}$;
- val
- un
- no

As in the propositional case, we have

- F is valid iff $\neg F$ is unsatisfiable;
- F is non-valid iff $\neg F$ is satisfiable.

Example of Non-Validity

Consider the formula

$$(\forall y \exists x \; P(x, y)) \Rightarrow (\exists x \forall y \; P(x, y))$$

It is not

For exam $\mathbb{Z}$, and the predicate P meaning "less than".

Or, let $D = \{0, 1\}$ and let P mean "equ

The formula is satisfiable, as it is true, for example, in the interpretation where $D = \{0, 1\}$ and P means "less than or equal".

Example of Validity

$F = (\exists y \forall x\ P(x,y)) \Rightarrow (\forall x \exists y\ P(x,y))$ is valid.

If we negate F (and rewrite it) we get

The right c D for which $\mathbf{p}(d_0, d)$ is false for all $d \in D$.

But the left conjunct requires that $\mathbf{p}(d_0$ d.

Since F's negation is unsatisfiable, F is valid.

Another Example of Validity

Consider

$$F = (\forall x\ P(x)) \Rightarrow P(t)$$

F is valid

To see this

$$\neg F = (\forall x\ P(x))$$

The term t denotes some element of the domai not be satisfied.

Puzzle for the Break

Deckard is a blade runner—his job is to identify replicants who look exactly like humans but who have actually been created in the laboratories of Tyrell Corp.

Deckar … cants are progra … ammed to always … t
Deckard deals with: some are always truthful, and the rest always lie.

One day, a suspect makes a simple statement that al … rd to conclude the suspect is a lying replicant.

What statement would do that?

After the break: Clausal form for first-order predicate logic.