School of Computing and Information Systems
COMP30026 Models of Computation Problem Set 12

18–22 October 2021

Content: Undecidability, reductions, well-founded relations, termination

P12.1 Show that

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a } TM \text{ and } L(M) = \emptyset\}$$

is undecidable. Hint: We know that A_{TM} is undecidable. Show that if E_{TM} was decidable, then A_{TM} would be decidable.

P12.2 Show that

$$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are } TM\text{s and } L(M_1) = L(M_2)\}$$

is undecidable. Hint: Use the fact that E_{TM} is undecidable.

P12.3 Recall that a binary relation $\prec$ over set S is a well-founded relation iff there is no infinite sequence $s_0, s_1, s_2, s_3, \ldots$ such that $s_i \succ s_{i+1}$ for all $i \in \mathbb{N}$. That is, each sequence of elements from S, when listed in decreasing order, is finite. For each of the following, say whether it is well-founded.

(a) The usual "smaller than" relation, $<$, on the natural numbers $\mathbb{N}$.

(b) The usual "smal $\mathbb{Q}$

(c) The relation "is a

(d) The (strict) lexicographic ordering of pairs of natural t is, the relation $\prec$ defined by $(m, m') \prec (n, n')$ iff $m < n \vee (m$

For the last question, it may help to draw the Hasse diagram fo ially ordered set $\mathbb{N} \times \mathbb{N}$, ordered by $\preceq$, the reflexive closure of $\prec$.

P12.4 (Optional.) Consider the function $c : \mathbb{N} \hookrightarrow \mathbb{N}$, defined recursively like so:

$$c(n) = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ c(n/2) & \text{if } n \text{ is even and } n > 1 \\ c(3n+1) & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

Write a Haskell function `hailstone :: Integer -> Int` which calculates the number of recursive calls made when computing $c(n)$. For example, `hailstone 5` should evaluate to 5, and `hailstone 27` should evaluate to 111.

c is known to terminate for all natural numbers up to 10^{20}. It is conjectured to terminate for all $n \in \mathbb{N}$, but whether this is is actually the case is an open problem. There are examples where similar conjectures have been refuted. One famous example has to do with prime factorisations. Say that $n > 1$ is *peven* if its prime factorisation has an even number of factors; otherwise n is *podd*. So $28 = 2 \cdot 2 \cdot 7$ is podd, and $40 = 2 \cdot 2 \cdot 2 \cdot 5$ is peven. Pólya conjectured that, for any k, the set $\{2, 3, 4, \ldots, k\}$ never has a majority of peven elements. However, that turned out to be false, the smallest counter-example being $k = 906150257$.

P12.5 (Optional.) Consider the alphabet $\Sigma = \{0, 1\}$. The set Σ^* consists of all the *finite* bit strings, and the set, while infinite, turns out to be countable. (At first this may seem obvious, since we can use the function $binary : \mathbb{N} \to \Sigma^*$ defined by

$$binary(n) = \text{ the binary representation of } n$$

as enumerator; however, that is not a surjective function, because the legitimate use of leading zeros means there is no unique binary representation of n. For example, both 101 and 00101 denote 5. Instead the idea is to list all binary strings of length 0, then those of length 1, then those of length 2, and so on: $\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, \ldots$.)

Now consider instead the set $\mathcal{B}$ of *infinite* bit strings. Show that $\mathcal{B}$ is much larger than Σ^*. More specifically, use diagonalisation to show that $\mathcal{B}$ is not countable.