School of Computing and Information Systems
COMP30026 Models of Computation Problem Set 7

6–10 September 2021

Content: sets, relations, properties of relations, functions

P7.1 Compute the transitive closure and symmetric-transitive closure of the following relations on $\mathbb{Z}$. For each closure, determine if it is reflexive.

(i) $\{(2,3),(5,4),(0,3),(2,1),(1,5)\}$

(ii) $\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid |x-y| \leq 2\}$

P7.2 Recall that the *symmetric difference* of sets A and B is the set $A \oplus B = (A \setminus B) \cup (B \setminus A)$. For each of the following set equations, give an equivalent equation that does not use $\oplus$. However, do not simply replace $\oplus$ by its definition; instead try to find the simplest equivalent equation.

(a) $A \oplus B = A$

(b) $A \oplus B = A \setminus B$

(c) $A \oplus B = A \cup B$

(d) $A \oplus B = A \cap B$

(e) $A \oplus B = A^c$

(f) $A \oplus B = \emptyset$

P7.3 There are multiple way

- A subset R
- A function $\chi_R : A \times B \to \{0,1\}$
- A function $\alpha_R : A \to \mathcal{P}(B)$
- A function $\beta_R : B \to \mathcal{P}(A)$

Explain how we determine whether $a \in A$ and $b \in B$ are related under each representation of R. Then show that we can convert between these representations, starting with a subset $R \subseteq A \times B$, and converting to and from each of the other three representations.

P7.4 Relations are sets. To say that $R(x,y) \wedge S(x,y)$ holds is the same as saying that (x,y) is in the relation R and also in the relation S, that is, $(x,y) \in R \cap S$.

Suppose R and S are reflexive relations on a set A. Then $\Delta_A \subseteq R$ and $\Delta_A \subseteq S$, so $\Delta_A \subseteq R \cap S$. That is, $R \cap S$ is also reflexive. We say that intersection *preserves* reflexivity. It is easy to see that union also preserves reflexivity. Similarly, if R is reflexive then so is R^{-1}, but the complement $A^2 \setminus R$ is clearly not. The following table lists these results. Complete the table, indicating which operations on relations preserve symmetry and transitivity.

Property	Reflexivity	Symmetry	Transitivity
preserved under $\cap$?	yes		
preserved under $\cup$?	yes		
preserved under inverse?	yes		
preserved under complement?	no		

P7.5 Continuing from the previous question, now assume that R and S are equivalence relations. From your table's first two rows, determine whether $R \cap S$ necessarily is an equivalence relation, and whether $R \cup S$ is.

P7.6 The *Cartesian product* of two sets A and B is defined $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. That is, a pair whose first component comes from A and whose second component comes from B is an element of $A \times B$ (and no other pairs are). Recall that $\cap$ and $\cup$ are absorptive, commutative and associative. Does $\times$ have any of those properties?

P7.7 Suppose A is a set of cardinality 42, that is, A has 42 elements. What, if anything, can we say about B's cardinality if we know that some function $f : A \to B$ is injective? What, if anything, can we say about B's cardinality if we know that some function $f : A \to B$ is surjective?

P7.8 Consider this conjecture: If a binary relation R on some set A is both symmetric and anti-symmetric then R is reflexive. Prove or disprove the conjecture.

P7.9 Consider this statement: For all sets S and T, $S \times T = T \times S$ iff $S = T$.

If the statement is true, prove it. Otherwise provide a counter-example.

P7.10 Consider this conjecture: If a binary relation R on some set A is reflexive then R is either symmetric or anti-symmetric, or both. Prove or disprove the conjecture.

P7.11 Let R be a relation o R.

P7.12 Let R be a transitiv of R that its symmetric reflexive closure $R \cup R^{-1} \cup \Delta_A$ is also tra

P7.13 Consider the relations on the natural numbers listed i tick off their properties. The *successor* relation is the relation given b $m\}$. The *coprime* relation C on $\mathbb{N}$ is given by $C(n, m)$ iff $GCD(n, m) = 1$, that is, the only common factor of n and m is 1.

	$<$	$\leq$	successor	divides	coprime
irreflexive					
reflexive					
asymmetric					
antisymmetric					
symmetric					
transitive					
linear					

P7.14 (Optional.) Let $\leq$ be a partial order on a set X. We say that a function $h : X \to X$ is:

- *idempotent* iff $\forall x \in X\ (h(h(x)) = h(x))$
- *monotone* iff $\forall x, y \in X\ (x \leq y \Rightarrow h(x) \leq h(y))$
- *increasing* iff $\forall x \in X\ (x \leq h(x))$

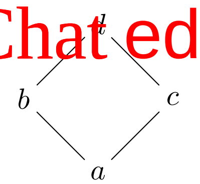
Note that an idempotent function does all of its work "in one go"; repeated application will not change its result. A monotone function is one that respects order; if its input grows, its output must grow too (or stay the same).

A function which is idempotent and monotone is a *closure operator*. If it is also increasing, we call it an *upper* closure operator. Closure operators are important and appear in many different contexts. We have met several—let $\mathcal{R}$ be the set of all binary relations. Then the functions *refl*, *symm*, and *trans*, in $\mathcal{R} \to \mathcal{R}$, producing a relation's reflexive, symmetric, and transitive closure, respectively, are all upper closure operators. Soon we will meet an "ϵ-closure" function that is part of the algorithm for turning a non-deterministic automaton into an equivalent deterministic automaton—yet another upper closure operator.

Consider $D = \{a, b, c, d\}$ and the partial order $\leq$ on D, defined by

$$x \leq y \text{ iff } x = a \lor x = y \lor y = d$$

Below is the so-called H ul way of depicting a partially d the order is given by the edges: x ing upwards only, along edges (and the path ca



Define eight functions $f_1, \ldots, f_8 : D \to D$, exhibiting all possible combinations of the three properties. That is, find some

(a) f_1 which is idempotent, monotone, and increasing;

(b) f_2 which is idempotent and monotone, but not increasing;

(c) f_3 which is idempotent and increasing, but not monotone;

(d) f_4 which is monotone and increasing, but not idempotent;

(e) f_5 which is idempotent, but neither monotone nor increasing;

(f) f_6 which is monotone, but neither idempotent nor increasing;

(g) f_7 which is increasing, but neither idempotent nor monotone;

(h) f_8 which is neither idempotent, monotone, nor increasing.