

### Solutions for Problem Set 3

P3.1 (a)

$$\begin{aligned} & \neg(A \wedge \neg(B \wedge C)) \\ & \neg A \vee (B \wedge C) && \text{(push negation in)} \\ & (\neg A \vee B) \wedge (\neg A \vee C) && \text{(distribute } \vee \text{ over } \wedge) \end{aligned}$$

The result is now in reduced CNF.

(b)

$$\begin{aligned} & A \vee (\neg B \wedge (C \vee (\neg D \wedge \neg A))) \\ & (A \vee \neg B) \wedge (A \vee C \vee (\neg D \wedge \neg A)) && \text{(distribute } \vee \text{ over } \wedge) \\ & (A \vee \neg B) \wedge (A \vee C \vee \neg D) \wedge (A \vee C \vee \neg A) && \text{(distribute } \vee \text{ over } \wedge) \end{aligned}$$

The result is in CNF but not RCNF. To get RCNF we need to eliminate the last clause which is a tautology, and we end up with  $(A \vee \neg B) \wedge (A \vee C \vee \neg D)$ .

(c)

$$\begin{aligned} & (A \vee B) \vee (C \wedge D) \\ & \neg(A \vee B) \vee (C \wedge D) && \text{(unfold } \Rightarrow) \\ & (\neg A \vee \neg B) \vee (C \wedge D) && \text{(de Morg)} \\ & (\neg A \vee \neg B) \vee C \vee D && \text{(distribute } \vee \text{ over } \wedge) \\ & (\neg A \vee \neg B) \vee C \vee D && \text{(distribute } \vee \text{ over } \vee) \end{aligned}$$

The result is in RCNF.

(d)

$$\begin{aligned} & A \wedge B \Rightarrow (A \Rightarrow B) \\ & A \wedge (\neg B \vee \neg A \vee B) && \text{(unfold both occurrences)} \\ & A && \text{(rightmost clause is tautological: remove it)} \end{aligned}$$

P3.2 Let us follow the method given in a lecture, except we do the double-negation elimination aggressively, as soon as opportunity arises:

$$\begin{aligned} & \neg((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)) \\ & \neg(\neg(B \vee \neg A) \vee \neg(B \vee A) \vee B) && \text{(unfold } \Rightarrow \text{ and eliminate double negation)} \\ & (B \vee \neg A) \wedge (B \vee A) \wedge \neg B && \text{(de Morgan for outermost neg; elim double neg)} \end{aligned}$$

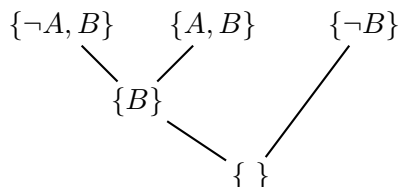
This is RCNF without further reductions.

We could also have used other transformations—sometimes this can shorten the process. For example, we could have rewritten the sub-expression  $\neg B \Rightarrow \neg A$  as  $A \Rightarrow B$  (the contraposition principle). You may want to check that this does not change the result.

The resulting formula, written as a set of sets of literals:

$$\{\{\neg A, B\}, \{A, B\}, \{\neg B\}\}$$

We can now construct the refutation:



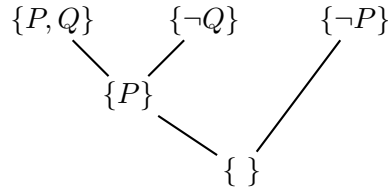
- P3.3 (a)  $(P \vee Q) \Rightarrow (Q \vee P)$ . First negate the formula (why?), to get  $\neg((P \vee Q) \Rightarrow (Q \vee P))$ . Then we can use the usual techniques to convert the negated proposition to RCNF. Here is a useful shortcut, combining  $\Rightarrow$ -elimination with one of de Morgan's laws:

$$\neg(A \Rightarrow B) \equiv A \wedge \neg B.$$

So:

$$\begin{aligned} & \neg((P \vee Q) \Rightarrow (Q \vee P)) \\ & (P \vee Q) \wedge \neg(Q \vee P) \quad (\text{shortcut}) \\ & (P \vee Q) \wedge \neg Q \wedge \neg P \quad (\text{de Morgan}) \end{aligned}$$

The result allows for a straight-forward refutation:



- (b)  $(\neg P \Rightarrow P) \Rightarrow P$ . Again, first negate the formula, to get  $\neg((\neg P \Rightarrow P) \Rightarrow P)$ . Then turn the result into RCNF:

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$$P \wedge \neg P \quad (\vee\text{-a})$$

The resolution proof is immediate; we will leave it out.

- (c)  $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$ . Again, negate the formula, to get  $\neg((P \Rightarrow Q) \Rightarrow P)$ . Then turn the result into RCNF:

$$\begin{aligned} & \neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P) \\ & ((P \Rightarrow Q) \Rightarrow P) \wedge \neg P \quad (\text{shortcut, outermost } \Rightarrow) \\ & ((\neg P \vee Q) \Rightarrow P) \wedge \neg P \quad (\text{unfold } \Rightarrow) \\ & (\neg(\neg P \vee Q) \vee P) \wedge \neg P \quad (\text{unfold } \Rightarrow) \\ & ((\neg\neg P \wedge \neg Q) \vee P) \wedge \neg P \quad (\text{de Morgan}) \\ & ((P \wedge \neg Q) \vee P) \wedge \neg P \quad (\text{double negation}) \\ & (P \vee P) \wedge (\neg Q \vee P) \wedge \neg P \quad (\text{distribution}) \\ & P \wedge (\neg Q \vee P) \wedge \neg P \quad (\text{absorption}) \end{aligned}$$

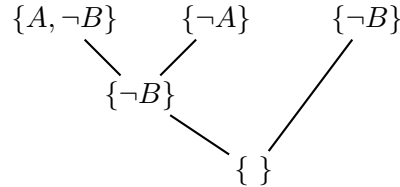
Again this gives an immediate refutation: just resolve  $\{P\}$  against  $\{\neg P\}$ .

- (d)  $P \Leftrightarrow ((P \Rightarrow Q) \Rightarrow P)$ . Negating the formula, we get  $P \oplus ((P \Rightarrow Q) \Rightarrow P)$ . Let us turn the resulting formula into RCNF:

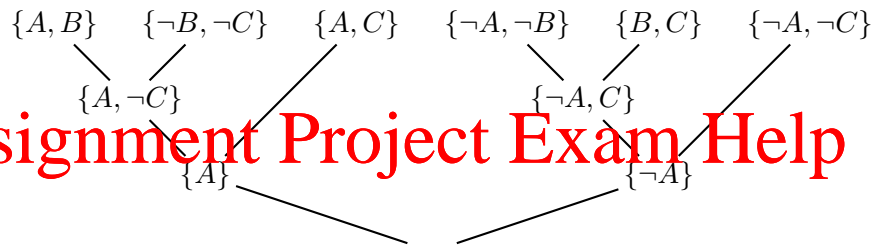
$$\begin{aligned} & P \oplus ((P \Rightarrow Q) \Rightarrow P) \\ & (P \vee ((P \Rightarrow Q) \Rightarrow P)) \wedge (\neg P \vee \neg((P \Rightarrow Q) \Rightarrow P)) \quad (\text{eliminate } \oplus) \\ & (P \vee ((P \Rightarrow Q) \Rightarrow P)) \wedge (\neg P \vee ((P \Rightarrow Q) \wedge \neg P)) \quad (\text{shortcut from above}) \\ & (P \vee (\neg(\neg P \vee Q) \vee P)) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) \quad (\Rightarrow\text{-elimination}) \\ & (P \vee (\neg\neg P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) \quad (\text{de Morgan}) \\ & (P \vee (P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) \quad (\text{double negation}) \\ & P \wedge (P \vee \neg Q) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) \quad (\vee\text{-absorption, distribution}) \\ & P \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge \neg P \quad (\vee\text{-absorption, distribution}) \end{aligned}$$

Once again, now just resolve  $\{P\}$  against  $\{\neg P\}$ .

- P3.4 (a)  $\{\{A, B\}, \{\neg A, \neg B\}, \{\neg A, B\}\}$  stands for the formula  $(A \vee B) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee B)$ . This is satisfiable by  $\{A \mapsto \mathbf{f}, B \mapsto \mathbf{t}\}$ .
- (b)  $\{\{A, \neg B\}, \{\neg A\}, \{B\}\}$  stands for  $(A \vee \neg B) \wedge \neg A \wedge B$ . A refutation is easy:



- (c)  $\{\{A\}, \emptyset\}$  stands for  $A \wedge \mathbf{f}$ , which is clearly not satisfiable.
- (d) We have  $\{\{A, B\}, \{\neg A, \neg B\}, \{B, C\}, \{\neg B, \neg C\}, \{A, C\}, \{\neg A, \neg C\}\}$ . This set is not satisfiable, as a proof by resolution shows:



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P3.5 Let us give names to the propositions:

- $C$ : Ann clears 2 meters
- $F$ : Ann gets the flu
- $K$ : The selectors are sympathetic
- $S$ : Ann is selected
- $T$ : Ann trains hard

The four assumptions then become:

- $C \Rightarrow S$
- $T \Rightarrow (F \Rightarrow K)$
- $(T \wedge \neg F) \Rightarrow C$
- $K \Rightarrow S$

It is easy to see that  $S$  is not a logical consequence of these, as we can give all five variables the value *false*, and all the assumptions will thereby be true.

To see that  $T \Rightarrow S$  is a logical consequence of the assumptions, we can negate it, obtaining  $T \wedge \neg S$ . Then, translating everything to clausal form, we can use resolution to derive an empty clause.

Alternatively, note that  $T \Rightarrow (F \Rightarrow K)$  is equivalent to  $(T \wedge F) \Rightarrow K$ . Since also  $K \Rightarrow S$ , we have  $(T \wedge F) \Rightarrow S$ . Similarly,  $(T \wedge \neg F) \Rightarrow C$  together with  $C \Rightarrow S$  gives us  $(T \wedge \neg F) \Rightarrow S$ .

But from  $(T \wedge F) \Rightarrow S$  and  $(T \wedge \neg F) \Rightarrow S$  we get  $T \Rightarrow S$ . (You may want to check that by massaging the conjunction of the two formulas.)

P3.6 Let us give names to the propositions:

- $A$ : The commissioner apologises
- $F$ : The commissioner can attend the function

- $R$ : The commissioner resigns

The four statements then become

- (a)  $F \Rightarrow (A \wedge R)$
- (b)  $(R \wedge A) \Rightarrow F$
- (c)  $R \Rightarrow F$
- (d)  $F \Rightarrow A$

The first translation may not be obvious. But to say “ $X$  does not happen unless  $Y$  happens” is the same as saying “it is not possible to have  $X$  happen and at the same time  $Y$  does not happen.” That is,  $\neg(X \wedge \neg Y)$ , which is equivalent to  $X \Rightarrow Y$ . Note that (a) entails (d) and (c) entails (b).

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