COMP30026 Models of Computation

Binary Relations and Functions

Lecture Week 6 Part

Semester 2, 2021

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Binary Relations

A binary relation is a set of pairs, or 2-tuples.

"Being unifiable", "$<$", "$\subseteq$", "divides" are all binary relations.

For small r

Paper	0	
Scissors	1	
Rock	0	

We can express membership of a relation in many ways:
$(x, y) \in Beats$, $Beats(x, y)$, or x $Beats$ y.

Domain and Range of a Relation

The domain of R is $dom(R) = \{x \mid \exists y\ R(x, y)\}$.

The range of R is $ran(R) = \{y \mid \exists x\ R(x, y)\}$.

We say th … nd $ran(R)$

A relatio

"Being unifiable" is a relation on *Term*.

"$<$" is a relation on integers.

"$\subseteq$" is a relation on $\mathcal{P}(A)$.

"Acted in" is a relation between actors and films.

Identity and Inverse

$A \times B$ is a relation—the full relation from A to B.

$\emptyset$ is a relation.

$\Delta_A = \{$ tion.

If R is a re $\in R\}$ is a relation from B to A, called the inverse

Clearly $(R^{-1})^{-1} = R$

Since relations are sets, all the set operations, such as $\cap$ and $\cup$, are applicable to relations.

Properties of Relations

Let A be a non-empty set and let R be a relation on A.

R is reflexive iff $R(x, x)$ for all x in A.

R is irre

R is sy

R is asymmetric iff $R(x, y) \Rightarrow \neg R(y, x)$ f

R is antisymmetric iff $R(x, y) \wedge R(y, x$ A.

R is transitive iff $R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ for all x, y, z in A.

Note that

1. The full relation is transitive.
2. Tr s, if R_1 an

Together, these two properties tell us that for any binary relation R, there is a unique smallest transitive relati R.

We call R^+ the transitive closure of R.

Similarly we have the (unique) reflexive closure and the (unique) symmetric closure of R.

Closures Quiz

What is th $\mid n \in \mathbb{N}\}$?

What is th $\mid n \in \mathbb{N}\}$?

What is th

Let R_1 and R_2 be relations on A. The composition $R_1 \circ R_2$ is the relation on A defined by

The n-f

$$\begin{aligned} R^1 &= R \\ R^{n+1} &= R^n \end{aligned}$$

If R is ^{2}?

Composition Quiz

If R is ^{2}?

What is

Composition Quiz

If R is ^{2}?

What is

If R is $<$ on $\mathbb{N}$, what is R^2?

Transitive Closure Again

The transitive closure of R can be defined in terms of union and composition:

$$R^+ = \bigcup_{n \geq 1} R^n$$

$(x, y), (y, z) \in R$, and hence $(x, y), (y, z) \in R^+$, but since R^+ is transitive, $(x, z) \in R^+$ (R^2 gives us all such p

$(x, z) \in R^2, (z, w) \in R$, and hence $(x,$ R^+ is transitive, $(x, w) \in R^+$ (R^3 gives us all such pairs)

The reflexive, transitive closure is

$$R^* = R^+ \cup \Delta_A$$

Equivalence Relations

A binary re ve is an

equivale

The ident A

A. The full relation A^2 is the largest equival .

Quiz: Equivalence Relations?

Which of these binary relations are equivalence relations?

- $\leq$
- $\equiv_m$

- “ar
- $\{(a, b) \mid |a - b| \leq 3\}$?
- “are compatriots” on the set of all people?

- “are logically equivalent” on the set of propos ulas?

R is a pre-order iff R is transitive and reflexive.

R is a s

R is a p

R is linear iff $R(x,y) \vee R(y,x) \vee x = y$

A linear partial order is also called a tot

In a total order, every two elements from A are comparable.

Quiz: Partial Orders?

Which of t

- Th
- The relation on $(\mathbb{N})$?
- The relation "divides" on $\mathbb{N}$?

A functio

$(x, y) \in$ here is

exactly o $f(x) = y$.

Domains and Co-Domains

We say that the function f is from X to Y, or

$$f : X \to Y$$

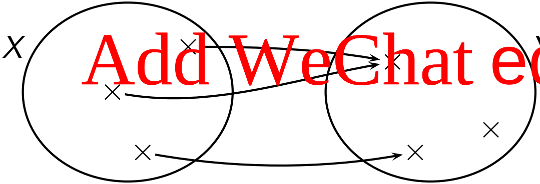
if *dom*(the
co-dom

Example: The range of the factorial function is $, \ldots\}$,
but we normally define it as having co-domain

The domain/co-domain specification is integr
definition, as we define functions $f : X \to Y$ and $f' : X' \to Y'$ to be equal iff $X = X'$, $Y = Y'$, and for all $x \in X, f(x) = f'(x)$.

A function $f : X \to Y$ is

- surjective (or onto) iff $ran(f)$ equals the co-domain of f.
- inj
- bij



$f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$ is neither surjective nor injective.

$g : \mathbb{Z} \to \mathbb{N}$ defined by $g(n) = |n|$ is surjective but not injective.

$s : \mathbb{N} \to$

$d : \mathbb{Z} \to \mathbb{N}$ defined by

$$d(n) = \begin{cases} 2n - 1 & \text{if} \\ -2n & \end{cases}$$

is bijective. It establishes a one-to-one mapping between $\mathbb{Z}$ and $\mathbb{N}$.

Function Composition

The composition of $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$ defined by

We assu ... hough the comp

Note the unfortunate inconsistency with the use of ... ing relations. For functions, $g \circ f$ is best read a

$\circ$ is associative, and for $f : X \to Y$, $f \circ 1_X = 1_Y \circ f = f$, where $1_X : X \to X$ is the identity function on X.