COMP30026 Models of Computation

Sets

Lecture Week 6 Part

Semester 2, 2021

This Lecture is Being Recorded

Assignment 1

Assign er.
Solutio enges 5
and 6).

"Definition": (Georg Cantor) A set is a collection into a whole of definite, distinct objects of our intuition or of our thought. The objects are called the elements (members) of the set.

Notati A.

Examples: $42 \in \mathbb{N}$ and $\pi \notin \mathbb{Q}$.

Principle of Extensionality: For all sets

$$A = B \Leftrightarrow \forall x \; (x \in A \Leftrightarrow x \in B)$$

Set Notation

Small sets can be specified completely: $\{-2, -1, 0, 1, 2\}$, $\{\text{Huey}, \text{Dewey}, \text{Louie}\}$, $\{\}$. We often write the last one as $\emptyset$.

Note that, by the Principle of Extensionality, order and repetition are irreleva

For large sets, including infinite sets, we have

If P is a property of objects x then the

$$\{x \mid P(x)\}$$

denotes the set of things x that have the property P. Hence $a \in \{x \mid P(x)\}$ is equivalent to $P(a)$.

Haskell's list notation is clearly inspired by set notation:

Haskell	Set notation
	}
`[f n	n <- nats]`	
`[1,3..]`	

The dot-dot notation here assumes some systematic way of generating all elements (an enumeration).

Well-Foundedness

Call a set S well-founded if there is no infinite sequence $S = S_0 \ni S_1 \ni S_2 \ni \cdots$, and consider the set W of all well-founded sets.

If $W \in$ ... /

If $W \notin$ ...
$W = W_0$ ... $_1$ ... $_2$ ... $_1$ ... $_2$ ... $_3$ ... $_1$ s not well-founded, that is, $W_1 \notin W$. This contr ... $_1$.

Bertrand Russell's famous paradox similarly co ... perty $R = \{x \mid x \notin x\}$ which leads to an inconsistent set theory:

$$R \in R \Leftrightarrow R \notin R$$

Sets and Types

One way (a crude way) to curb set theory so as to obtain consistency is to impos

The purp ssible, by insisting t ”.

Russell's type concept is the root of type disciplines u any programming languages.

The Subset Relation

A is a subset of B iff $\forall x\ (x \in A \Rightarrow x \in B)$.

We write t

If $A \subseteq$ … B, and write this $A \subset B$.

Do not confuse $\subseteq$ with $\in$. We have $\{1$ … $1, 2\}$.

The Subset Relation Is a Partial Ordering

For all sets A, B, and C we have

- A (reflexivity)
- A (antisymmetry)
- A (transitivity)

These laws are easy to prove from the definition of

The three laws together state that $\subseteq$ is

The empty set satisfies $\emptyset \subseteq A$ for every set A.

A set with just a single element is a singleton.

For exam

The set

A set with two elements is a pair.

Ordinarily, and in programming languages, we re s a pair, but in set theory we would call that an ordered pair.

Algebra of Sets

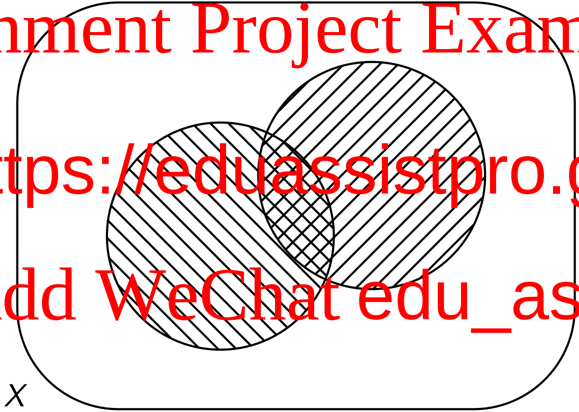
Let A and B be sets. Then

- $A \cap B = \{x \mid x \in A \wedge x \in B\}$ is the intersection of A and B;
- A
- $A \setminus$
- A e.

In the presence of a set X of which all sets ar
also define

- $A^c = X \setminus A$ is the complement of A.

Venn Diagrams



Some Laws

Absorption: $A \cap A = A$

$A \cup A = A$

Comm

Associativity: $A \cap (B \cap C) = (A \cap$

$A \cup (B \cup C) = (A \cup$

Distributivity: $A \cap (B \cup C) = (A \cap$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

More Laws

Double complement: $A = (A^c)^c$

De Morgan: $(A \cap B)^c = A^c \cup B^c$

$c \quad c \quad c$

Dualit

Identity: $A \cup \emptyset = A$ an

Dominance: $A \cap \emptyset = \emptyset$ and

Complementation: $A \cap A^c = \emptyset$ and $A \cup A^c = X$

Subset Equivalences

Subset characterisation: $A \subseteq B \equiv A = A \cap B \equiv B = A \cup B$

Contra

Subset Equivalences

Subset characterisation: $A \subseteq B \equiv A = A \cap B \equiv B = A \cup B$

Contra

All very similar to the equivalences we saw for propos logic—just substitute $\neg$ for complement, for $\subseteq$, $\bot$ for $\emptyset$, and $\top$ for X.

Powersets

The po f all subsets o

In particu

If X is finite, of cardinality n, then $\mathcal{P}(X$

Generalised Union and Intersection

Suppose we have a collection of sets A_i, one for each i in some (index) set I. For example, I may be $\{1..99\}$, or I may be infinite.

The uni

The intersection of the sets is

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \; (i \in I \Rightarrow x \in A_i)\}$$

Ordered Pairs

Can we capture the notion of ordered pairs (a, b) with set-theoretic notions? We want this to hold:

We can ac

$$(a, b) = \{\{a\}, \{$$

Hence we can freely use the notation (

meaning.

Cartesian Product and Tuples

The Cartesian product of A and B is defined

We defin

$$
\begin{aligned}
A^0 &= \{\emptyset \\
A^{n+1} &= A
\end{aligned}
$$

Of course we shall write (a, b, c) rather than $(a, (b, (c, \emptyset)))$.

Some Laws Involving Cartesian Product

$(A \times B$

$$
\begin{aligned}
(A \cap B) \times (C \cap D) &= (A \times C) \cap (\\
(A \cup B) \times (C \cup D) &= (A \times C) \cup (\quad \times D)
\end{aligned}
$$

Relations

An n-ary relation is a set of n-tuples.

That is, the relation is a subset of some Cartesian prod
$A_1 \times A_2 \times \cdots \times A_n$.

Or equivalently, we can think of a relation as a function from
$A_1 \times A_2 \times \cdots \times A_n$ to $\{0, 1\}$.