COMP30026 Models of Computation

Predicate Logic: Syntax

Lecture Week 3 Part 2 (Zo

Semester 2, 2021

This Lecture is Being Recorded

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From Propositional to Predicate Logic

Propositional logic is useful for many purposes, but there is much in our everyday vocabulary (and in the mathematician's arguments) that it cannot express.

Proposi … econd half of the …

By 1879 Gottlob Frege had designed his "Begriffschrift", which was really the language of first-order predicate logic, although it looked very different, with statements being written as tree structures.

Why Predicate Logic?

Unlike propositional logic, predicate logic allows us to

- finitely express statements that deal with infinite collections of obj
- ex e pro

To enable this, predicate logic uses vari range over collections of individuals, such as integ ople, or whatever, as well as quantifiers.

Propositional letters become generalised to predicates, that is, functions that map tuples of individuals to **f** or **t**.

No emus fly: $\forall x\ (Emu(x) \Rightarrow \neg Flies(x))$

There are $\wedge\ Swan(x))$

or: $\wedge\ Swan(y))$

If all push the cart, the donkey will be happy: $\forall x\ (P(x)) \Rightarrow H$

If somebody pushes, the donkey will be happy: $)) \Rightarrow H$

or (strangely): $) \Rightarrow H)$

Expressing Relations

Tom found Rover and returned him to Anne:

$$Found(tom, rover) \land Gave(tom, rover, anne)$$

Tom fou

$$\ldots e))$$

Jill inhabits the house that Jack built:

$$\exists x\,(House(x) \land Inhabits(jill, x) \ldots$$

Mothers' mothers are grandmothers:

$$\forall x, y, z\ ((Mother(x, y) \land Mother(y, z)) \Rightarrow Grandmother(x, z))$$

Existential and Universal Quantification

Tom found an amount of money and gave it to Red Cross:

Existential quantification, $\exists$, is generalised

Universal quantification, $\forall$, is generalised

Sidebar: Other Variable Binders

$\sum_{i=1}^{100} i^2$ is another example of variable binding.

Similarly with the lambda term $\lambda x.\ x^2 + 1$, or in Haskell notation

A lambda t a
name. This is surprisingly useful.

Mathematicians sometimes talk about 'the fun
really that notation is ambiguous, because it does n o
distinguish, say, $\lambda x.\ x^2 + 1$ and $\lambda xy.\ x^2 + 1$.

Mechanical Inference with Predicate Logic

Consider this argument:

- Ev
- All l
- Col

- Any tadpole eaten by a deep water fish is misera
- Therefore some tadpole is miserable

Later we shall see how to automate reasoning about such arguments.

Our Vocabulary

The alphabet of a first-order language:

- variables (x, y, z, u, v, w, ...)
- function symbols (f, g, h, ..., $+$, $\cdot$, ...)
- co
- pre
- co
- quantifiers
- parentheses
- (sometimes: **f**, **t**)

Each function symbol comes with an arity: a number that says how many arguments the function takes. Each predicate symbol similarly comes with an arity.

Terminology

A term is a variable or a constant or a construction

$$f(t_1, \ldots, t_n)$$

where n $_i$ is a term.

(Or we ma

An atomic formula (or atom) is a constructio re

$n \geq 0$ and P is a predicate symbol of arity .

Term	$\longleftrightarrow$	Individua
Atom	$\longleftrightarrow$	Assertion (false or true)

A literal is an atomic formula or its negation.

Case Matters

Note carefully the convention we adopt:
A predicate starts with an upper case letter; nothing else does.

So *fath*

(Most lik

On the other hand, *Father*(*ron*) is a for

(Most likely we intend this to mean: “Ron is a father”)

Well-formed formulas (wffs) are generated by the grammar

$$
\begin{array}{rcl}
\mathit{wff} & \to & \mathit{atom} \\
 & \mid & \neg\ \mathit{wff} \\
 & \mid & \mathit{wff} \\
 & \mid & \mathit{wff} \\
 & \mid & \forall \mathit{var} \\
 & \mid & \exists \mathit{var} \\
 & \mid & (\ \mathit{wff}\ )
\end{array}
$$

Assume precedence rules like for propositional logic.

Bound and Free Variables

A variable which is in the scope of a quantifier (binding that variable) is bound. If it is not bound then it is free.

A variable may occur both free and bound in a formula—witness y in

A formul

It is possible for scopes to have “holes”:

$$\forall x \exists y\ (x < y \wedge \exists x \quad <$$

The last occurrence of x is bound by the closest quantifier, so the scope of $\forall x$ is not all of $\exists y(\ldots)$.

Bound Variable Renaming and Capture

The bound variable of a quantified formula is just a placeholder—its exact name is inessential.

$\exists x \forall y$ (

If a variab f
that expr f that
variable are replaced by another one.

However, to avoid variable capture, we ca
bound by $\forall y$ to a variable in an enclosing scope:

$\exists x \forall y \ (x \leq y)$ is very different to $\exists x \forall x \ (x \leq x)$.

From English to Predicate Logic

Introduce symbols for predicates.

Sentence: "He is a man"   Predicate: "is a man"   Symbol: $M(\ )$

"x is a ma

Kim is a ma

Sentence: "Alice is taller than Kim" $e, kim)$

Quantifier examples:

"Every human is mortal"	$\forall x\ (Human(x) \Rightarrow Mortal(x))$
"Some cat is mortal"	$\exists x\ (Cat(x) \land Mortal(x))$

Note: Very common to use $\Rightarrow$ with $\forall$ and $\land$ with $\exists$.

Let $L(x, y)$ stand for "x loves y".

Let $I(x, y)$ stand for "x is y".

$L(bob$	Bob loves Eva
$\forall x\ L($	Everyone lov
$\forall x\ (\neg$	
$\exists x\ (\neg I(x, bob) \land L(x, bob))$	Someone oth
$\forall x\ (\exists y\ L(x, y))$	Everybody loves somebody
$\exists y\ (\forall x\ L(x, y))$	Someone is loved by everybody
$\exists x\ (\forall y\ L(x, y))$	Someone loves everybody

Quiz: Translate This

Translate the following statement to predicate logic:

Every Melburnian barracks for a footy team.

Use these predicates:

Quiz: Translate This

Translate the following statement to predicate logic:

Every Melburnian barracks for a footy team.

Use these predicates:

$$\forall x\ (M(x) \Rightarrow \exists y\ (T(y$$

or, equivalently:

$$\forall x\ \exists y\ (M(x) \Rightarrow (T(y) \land B(x, y)))$$

Word Order

Consider word order with care:

- “T
- “T):
  $\neg\exists y\ P(y)$
- “All S are not P” vs “not all S ar
  $\forall x\ (S(x) \Rightarrow \neg P(x))$ vs $\neg\forall x\ (S(x)$

Quantifier Order

The order of different quantifiers is important.

$\forall x \exists y$ is n

The form ter says there's an individual y that satisfies $P($

But $\forall x \forall y$ is the same as $\forall y \forall x$ and $\exists x$.

Quantified Formulas as a Two-Person Game

The truth or falsehood of a quantified formula can be expressed as a question of winning strategies for a two-person game. Say I make a claim (the quantified statement) and you try to disprove it. You get to supply values for the universally quantified variables.

- If I cla
  an), but I get
  to k

- If I claim $\exists y \forall x \; P(x, y)$, then you can ch
  to provide the y, and then you just have to fi oes not satisfy $P(x, y)$, knowing the y

- If I claim $\exists x \exists y \; P(x, y)$, then I have to find both x and y, so it doesn't matter what order they appear.
- If I claim $\forall y \forall x \; P(x, y)$, then you get to pick both x and y, so again their order does not matter.

Implicit Quantifiers

Often quantifiers are implicit in English. Look for nouns (especially plural) without determiners (words to indicate which members of a group are intended).

“Huma

$\forall x\ (Ma$

“A woman is stronger than a man” would usually me

$\forall x \forall y\ ((Woman(x) \land Man(y)) \Rightarrow Stron$

“If a girl owns a poodle, she spoils it”:

$$\forall x \forall y\ ((Girl(x) \land Poodle(y) \land Owns(x, y)) \Rightarrow Spoils(x, y))$$

Reading Materials

Remember, if you feel like reading more about these topics, check out the resources that are linked under “Subject Overview” (also accessible from “Modules” → “Reading Resources”).

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e

use of a styl

(not covered by us, and not examinable).

A rather different introduction to predicate logic is i n's Chapter 9.

The book by Jenkyns also looks good.

Next Week Tune In for More Predicate Logic

We next c

detail.

Plus, we g ate

logic.