

16–18 September 2020

## The exercises

51. Let  $A$ ,  $B$ , and  $C$  be sets. Show:

- (a)  $A \not\subseteq B \Leftrightarrow A \setminus B \neq \emptyset$ .
- (b)  $A \cap B = A \setminus (A \setminus B)$ .

Hint: Use the formal (logical) definitions of the concepts involved.

52. Recall that the *symmetric difference* of two sets  $A$  and  $B$  is  $A \oplus B = (A \setminus B) \cup (B \setminus A)$ . For each of the following set equations, determine whether it is true for all sets  $A$  and  $B$ . However, do not simply replace  $A$  and  $B$  with simple sets to test the equation.

- (a)  $A \oplus B = A$
- (b)  $A \oplus B = A \cap B$
- (c)  $A \oplus B = A \cup B$
- (d)  $A \oplus B = A \cap B$
- (e)  $A \oplus B = A^c$

53. Consider this statement:  $(A \cap B) \subseteq C \Leftrightarrow (A \subseteq C) \wedge (B \subseteq C)$ . If the statement is true, provide a simple proof. If it is false, provide a counterexample.

54. Show that a relation  $R$  on  $A$  is transitive iff  $R \circ R \subseteq R$ . Note that  $R \circ R = R$  is a transitive relation  $R$  for which  $R \circ R = R$  fails to hold.

55. Relations are sets. To say that  $R(x, y) \wedge S(x, y)$  holds is the same as saying that  $(x, y)$  is in the relation  $R$  and also in the relation  $S$ , that is,  $(x, y) \in R \cap S$ .

Suppose  $R$  and  $S$  are reflexive relations on a set  $A$ . Then  $\Delta_A \subseteq R$  and  $\Delta_A \subseteq S$ , so  $\Delta_A \subseteq R \cap S$ . That is,  $R \cap S$  is also reflexive. We say that intersection *preserves* reflexivity. It is easy to see that union also preserves reflexivity. Similarly, if  $R$  is reflexive then so is  $R^{-1}$ , but the complement  $A^2 \setminus R$  is clearly not. The following table lists these results. Complete the table, indicating which operations on relations preserve symmetry and transitivity.

Property	Reflexivity	Symmetry	Transitivity
preserved under $\cap$ ?	yes		
preserved under $\cup$ ?	yes		
preserved under inverse?	yes		
preserved under complement?	no		

56. Continuing from the previous question, now assume that  $R$  and  $S$  are equivalence relations. From your table's first two rows, determine whether  $R \cap S$  necessarily is an equivalence relation, and whether  $R \cup S$  is.

57. Suppose we know about functions  $f : A \rightarrow B$  and  $g : B \rightarrow A$  that  $f(g(y)) = y$  for all  $y \in B$ . What, if anything, can be deduced about  $f$  and/or  $g$  being injective and/or surjective?

58. Suppose  $h : X \rightarrow X$  satisfies  $h \circ h \circ h = 1_X$ . Show that  $h$  is a bijection. Also give a simple example of a set  $X$  and a function  $h : X \rightarrow X$  such that  $h \circ h \circ h = 1_X$ , but  $h$  is not the identity function (hint: think paper-scissors-rock).

59. (Drill.) The *Cartesian product* of two sets  $A$  and  $B$  is defined  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$ . That is, a pair whose first component comes from  $A$  and whose second component comes from  $B$  is an element of  $A \times B$  (and no other pairs are). Recall that  $\cap$  and  $\cup$  are absorptive, commutative and associative. Does  $\times$  have any of those properties?
60. (Drill.) Consider this conjecture: If a binary relation  $R$  on some set  $A$  is both symmetric and anti-symmetric then  $R$  is reflexive. Prove or disprove the conjecture.
61. (Drill.) Suppose  $A$  is a set of cardinality 42, that is,  $A$  has 42 elements. What, if anything, can we say about  $B$ 's cardinality if we know that some function  $f : A \rightarrow B$  is injective? What, if anything, can we say about  $B$ 's cardinality if we know that some function  $f : A \rightarrow B$  is surjective?
62. (Optional.) Let  $\leq$  be a partial order on a set  $X$ . We say that a function  $h : X \rightarrow X$  is:
- *idempotent* iff  $\forall x$
  - *monotone* iff  $\forall x, y$ ,  $x \leq y \implies h(x) \leq h(y)$
  - *increasing* iff  $\forall x, y$ ,  $x < y \implies h(x) < h(y)$

Note that an idempotent function does all of its work “in one go”; repeated application will not change its result. A monotone function is one that respects order: if its input grows, its output must grow too (or stay the same).

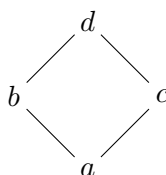
A function which is idempotent and monotone is a *closure operator*. If it is also increasing, we call it an *upper* closure operator. Closure operators are important and appear in many different contexts.

Then the functions *refl*, *trans*, *sym*, *trans-sym*, *trans-sym-refl*, *trans-sym-refl-idemp*, *trans-sym-refl-idemp-increasing*, and *trans-sym-refl-idemp-increasing-monotone* are all closure operators. We will meet an “ $\epsilon$ -closure” function in the next section. We will also see how to convert a non-deterministic automaton into an equivalent deterministic automaton—yet another use of closure operators.

Consider  $D = \{a, b, c, d\}$  and the partial order:

$$x \leq y \text{ iff } x = a \vee (x = b \wedge y = d) \vee (x = c \wedge y = d)$$

Below is the so-called Hasse diagram for  $D$ . A Hasse diagram provides a helpful way of depicting a partially ordered set. The nodes are the elements of the set, and the order is given by the edges:  $x \leq y$  if and only if there is a path from  $x$  to  $y$  travelling upwards only, along edges (and the path can have length 0).



Define eight functions  $f_1, \dots, f_8 : D \rightarrow D$ , exhibiting all possible combinations of the three properties. That is, find some

- $f_1$  which is idempotent, monotone, and increasing;
- $f_2$  which is idempotent and monotone, but not increasing;
- $f_3$  which is idempotent and increasing, but not monotone;
- $f_4$  which is monotone and increasing, but not idempotent;
- $f_5$  which is idempotent, but neither monotone nor increasing;
- $f_6$  which is monotone, but neither idempotent nor increasing;
- $f_7$  which is increasing, but neither idempotent nor monotone;
- $f_8$  which is neither idempotent, monotone, nor increasing.