

16–18 September 2020

The exercises

51. Let A , B , and C be sets. Show:

- (a) $A \not\subseteq B \Leftrightarrow A \setminus B \neq \emptyset$.
- (b) $A \cap B = A \setminus (A \setminus B)$.

Hint: Use the formal (logical) definitions of the concepts involved.

52. Recall that the *symmetric difference* of two sets A and B is $A \oplus B = (A \setminus B) \cup (B \setminus A)$. For each of the following set equations, determine whether it is true for all sets A and B . However, do not simply replace A and B with simple sets to test the equation.

- (a) $A \oplus B = A$
- (b) $A \oplus B = A \cap B$
- (c) $A \oplus B = A \cup B$
- (d) $A \oplus B = A \cap B$
- (e) $A \oplus B = A^c$

53. Consider this statement: $(A \cap B) \subseteq C \Leftrightarrow (A \subseteq C) \wedge (B \subseteq C)$. If the statement is true, provide a simple proof. If it is false, provide a counterexample.

54. Show that a relation R on A is transitive iff $R \circ R \subseteq R$. Note that $R \circ R = R$ is a transitive relation R for which $R \circ R \not\subseteq R$ fails to hold.

55. Relations are sets. To say that $R(x, y) \wedge S(x, y)$ holds is the same as saying that (x, y) is in the relation R and also in the relation S , that is, $(x, y) \in R \cap S$.

Suppose R and S are reflexive relations on a set A . Then $\Delta_A \subseteq R$ and $\Delta_A \subseteq S$, so $\Delta_A \subseteq R \cap S$. That is, $R \cap S$ is also reflexive. We say that intersection *preserves* reflexivity. It is easy to see that union also preserves reflexivity. Similarly, if R is reflexive then so is R^{-1} , but the complement $A^2 \setminus R$ is clearly not. The following table lists these results. Complete the table, indicating which operations on relations preserve symmetry and transitivity.

Property	Reflexivity	Symmetry	Transitivity
preserved under \cap ?	yes		
preserved under \cup ?	yes		
preserved under inverse?	yes		
preserved under complement?	no		

56. Continuing from the previous question, now assume that R and S are equivalence relations. From your table's first two rows, determine whether $R \cap S$ necessarily is an equivalence relation, and whether $R \cup S$ is.

57. Suppose we know about functions $f : A \rightarrow B$ and $g : B \rightarrow A$ that $f(g(y)) = y$ for all $y \in B$. What, if anything, can be deduced about f and/or g being injective and/or surjective?

58. Suppose $h : X \rightarrow X$ satisfies $h \circ h \circ h = 1_X$. Show that h is a bijection. Also give a simple example of a set X and a function $h : X \rightarrow X$ such that $h \circ h \circ h = 1_X$, but h is not the identity function (hint: think paper-scissors-rock).

59. (Drill.) The *Cartesian product* of two sets A and B is defined $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. That is, a pair whose first component comes from A and whose second component comes from B is an element of $A \times B$ (and no other pairs are). Recall that \cap and \cup are absorptive, commutative and associative. Does \times have any of those properties?
60. (Drill.) Consider this conjecture: If a binary relation R on some set A is both symmetric and anti-symmetric then R is reflexive. Prove or disprove the conjecture.
61. (Drill.) Suppose A is a set of cardinality 42, that is, A has 42 elements. What, if anything, can we say about B 's cardinality if we know that some function $f : A \rightarrow B$ is injective? What, if anything, can we say about B 's cardinality if we know that some function $f : A \rightarrow B$ is surjective?
62. (Optional.) Let \leq be a partial order on a set X . We say that a function $h : X \rightarrow X$ is:
- *idempotent* iff $\forall x$
 - *monotone* iff $\forall x, y$, $x \leq y \implies h(x) \leq h(y)$
 - *increasing* iff $\forall x, y$, $x < y \implies h(x) < h(y)$

Note that an idempotent function does all of its work “in one go”; repeated application will not change its result. A monotone function is one that respects order: if its input grows, its output must grow too (or stay the same).

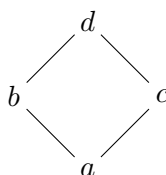
A function which is idempotent and monotone is a *closure operator*. If it is also increasing, we call it an *upper* closure operator. Closure operators are important and appear in many different contexts.

Then the functions *refl*, *trans*, *transitive closure*, *symmetric*, and *transitive symmetric closure* are all closure operators. We will meet an “ ϵ -closure” function in the next section. We will also see how to convert a non-deterministic automaton into an equivalent deterministic automaton—yet another use of closure operators.

Consider $D = \{a, b, c, d\}$ and the partial order:

$$x \leq y \text{ iff } x = a \vee$$

Below is the so-called Hasse diagram for D . A Hasse diagram provides a helpful way of depicting a partially ordered set. The nodes are the elements of the set, and the order is given by the edges: $x \leq y$ if and only if there is a path from x to y travelling upwards only, along edges (and the path can have length 0).



Define eight functions $f_1, \dots, f_8 : D \rightarrow D$, exhibiting all possible combinations of the three properties. That is, find some

- f_1 which is idempotent, monotone, and increasing;
- f_2 which is idempotent and monotone, but not increasing;
- f_3 which is idempotent and increasing, but not monotone;
- f_4 which is monotone and increasing, but not idempotent;
- f_5 which is idempotent, but neither monotone nor increasing;
- f_6 which is monotone, but neither idempotent nor increasing;
- f_7 which is increasing, but neither idempotent nor monotone;
- f_8 which is neither idempotent, monotone, nor increasing.