School of Computer Science and En
University of New South Wales

3. RECURRENCES - part A

Asymptotic notation

- **"Big Oh" notation:** $f(n) = O(g(n))$ is an abbreviation for:

  "*There exist positive constants* c *and* n_0 *such that* $0 \leq f(n) \quad c\, g(n)$ *for all* $n \quad n_0$".

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- Clearly, multiplying constants c of interest will be larger than 1, thus "enlarging" $g(n)$.

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- **"Omega" notation:** $f(n) = \Omega(g(n))$ is an abbreviation for:

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- **"Theta" notation:** $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$; thus, $f(n)$ and $g(n)$ have the same asymptotic growth rate.

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2. **t** ——
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4. Merge-Sort$(A, q+1, r)$
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- Since Merge(A, p, q, r) runs in linear time, the runtime $T(n)$ of Merge-Sort(A, p, r) satisfies

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

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but it can be shown that ignoring the integer parts and additive constants is OK when it comes to obtaining the asymptotics.

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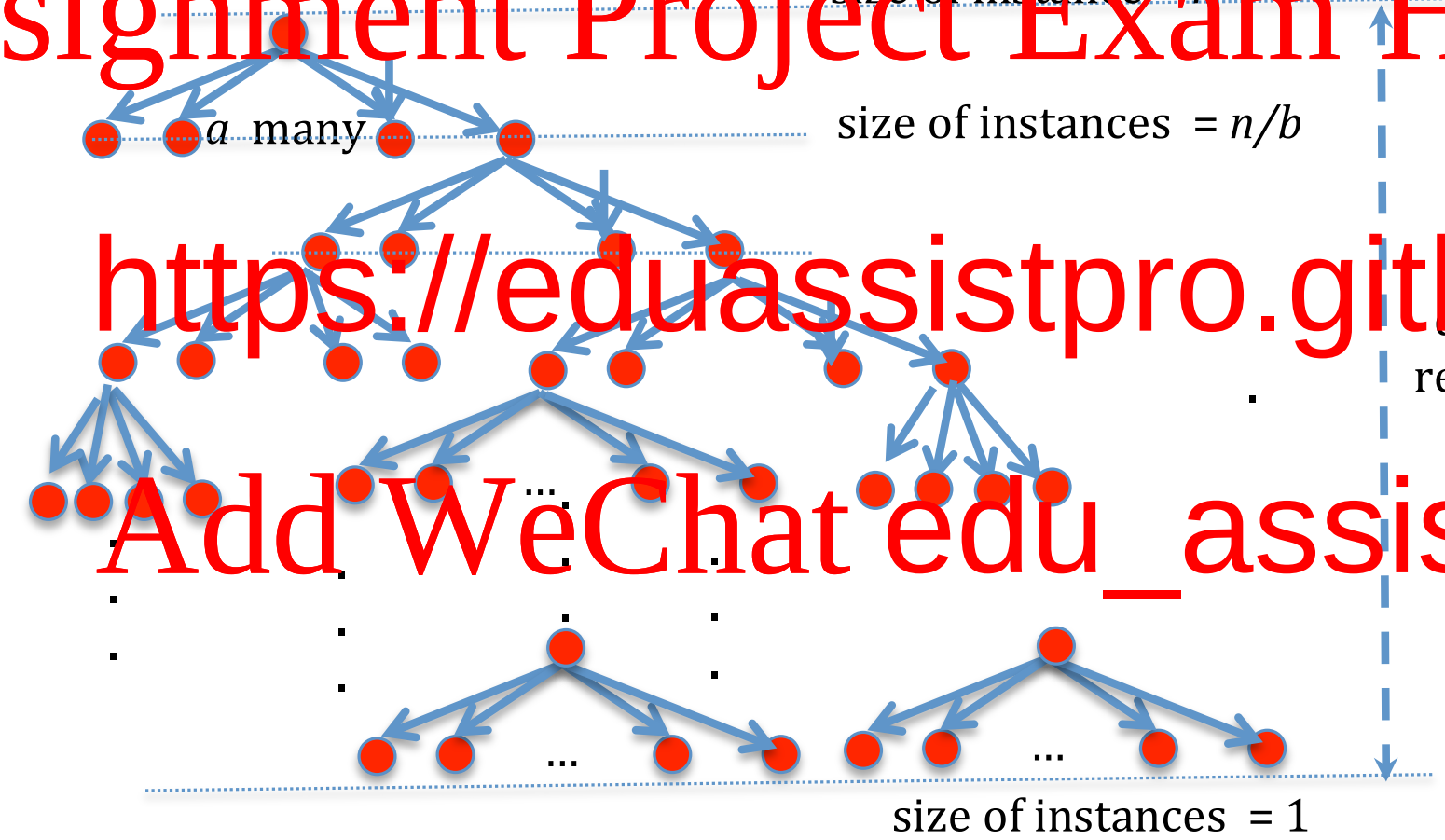

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  1. the order of growth (e.g. n^2, or $n \log n$, ...)
  2. the (approximate) **sizes of the constants** involved (more about that later)

- This is what the **Master Theorem** (see next) gives us (if applicable).

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(But often the proof of the Master Theorem can be tweaked to obtain the asymptotic of the solution $T(n)$ in such a case when the Master Theorem does not apply; an example is $T(n) = 2T(n/2) + n \log n$).

Master Theorem - a remark

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and also

$$\log_2 n = \Theta(\log_b$$

- So whenever we have $f = \Theta(g(n) \log n)$ we do not have to specify what base the log is - all bases produce equivalent asymptotic estimates (but we do have to specify b in expressions such as $n^{\log_b a}$).

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$$T(n) = \Theta(n^{\log_2 2} \log n) = \Theta(n \log n).$$

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    Hint: Use *de L'Hôpital's Rule* to show that $\log n/n^{\varepsilon} \to 0$.

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  - Thus, $f(n) = n \log_2 n = \Omega(n)$.
  - However, $f(n) = n \log_2 n \neq \Omega(n^{1+\varepsilon})$ for $\varepsilon > 0$.
  - This is because for every $\varepsilon > 0$, and e small, $\log_2 n < c \cdot n^{\varepsilon}$ for all sufficiently large n.
  - **Homework:** Prove this.
    Hint: Use *de L'Hôpital's Rule* to show that $\log n/n^{\varepsilon} \to 0$.
  - Thus, in this case the Master Theorem does **not** apply!

Master Theorem - Proof:

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$$= a^2\,\underbrace{T\left(\frac{n}{b^2}\right)}_{(3L)} + a\,f\left(\frac{n}{b}\right) + f(n) = a^2\,\underbrace{\left(a\,T\left(\frac{n}{b^3}\right) + f\left(\frac{n}{b^2}\right)\right)}_{(3R)} + a\,f\left(\frac{n}{b}\right) + f(n)$$

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$$= a^3\,\underbrace{T\left(\frac{n}{b^3}\right)}_{} + a^2 f\left(\tfrac{n}{b^2}\right) + a\,f\left(\tfrac{n}{b}\right) + f(n) = \ldots$$

Master Theorem Proof:

Continuing in this way $\log_b n - 1$ many times we get ...

$$T(n) = a^3\, T\left(\frac{n}{b^3}\right) + a^2 f\left(\frac{n}{b^2}\right) + a\, f\left(\frac{n}{b}\right) + f(n) =$$

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$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \tag{4}$$

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Note that so far we did not use any assumptions on $f(n)$...

$$T(n) \approx n^{\log_b a} T(1) + \underbrace{\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)}$$

Case 1: $f(n) = O(n^{\log_b a - \varepsilon})$

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$$= O\left(\frac{n^{\log_b a - \varepsilon}}{a^{i(\log_b a - \varepsilon)}}\right)$$

$$= O\left(n^{\log_b a - \varepsilon} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a - \varepsilon}}\right)^i\right)$$

$$T(n) \approx n^{\log_b a} T(1) + \underbrace{\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right)}$$

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$$= O\left(n^{\log_b a - \varepsilon} \frac{(b^{\varepsilon})^{\lfloor \log_b n \rfloor} - 1}{b^{\varepsilon} - 1}\right); \quad \text{we are using } \sum_{i=0}^{m-1} q^i = \frac{q^m - 1}{q - 1}$$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

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$$= O\left(\frac{n^{\log_b a}}{n^\varepsilon} \frac{n^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

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Case 1 - continued:

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$$= O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\log_b n} - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(\frac{n^{\log_b a - \varepsilon} (n^\varepsilon - 1)}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a}\right)$$

Master Theorem Proof:

Case 1 - continued:

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$$= O\left(\frac{n^{\log_b a - \varepsilon} n^\varepsilon - 1}{b^\varepsilon - 1}\right)$$

$$= O\left(n^{\log_b a}\right)$$

Since we had: $T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \frac{}{b^i}$

Master Theorem Proof:

Case 1 - continued:

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) = O\left(n^{\log_b a - \varepsilon} \frac{(b^\varepsilon)^{\lfloor \log_b n \rfloor} - 1}{b^\varepsilon - 1}\right)$$

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Since we had: $T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1}$

$$T(n) \approx n^{\log_b a} T(1) + O\left(n^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a}\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

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$$= \Theta\left(\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a}\right)$$

$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i\right)$$

Master Theorem Proof:

Case 2: $f(m) = \Theta(m^{\log_b a})$

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$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1\right)$$

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Case 2: $f(m) = \Theta(m^{\log_b a})$

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$$= \Theta\left(n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1\right)$$

$$= \Theta\left(n^{\log_b a} \lfloor \log_b n \rfloor\right)$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i}^{\lfloor \log_b n \rfloor - 1} \frac{n}{} \log_b a \log_b a$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \frac{n}{b^i} a^i = n \log_b a \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \frac{a^i}{b^i} = n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1^i$$

because log

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1^i$$

Master Theorem Proof:

Case 2 (continued):

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \frac{n}{b^i} a^i = n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} \left(\frac{a}{b^{\log_b a}}\right)^i = n^{\log_b a} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} 1^i$$

because log

$$= n^{\log_b a} \left(\lfloor \log_b n \rfloor - 1 + 1\right) = n^{\log_b a} \lfloor \log_b n \rfloor$$

we get:

$$T(n) \approx n^{\log_b a} T(1) + \Theta\left(n^{\log_b a} \log_2 n\right)$$

$$= \Theta\left(n^{\log_b a} \log_2 n\right)$$

Master Theorem Proof:

Case 3: $f(m) = \Omega(m^{\log_b a+\varepsilon})$ and $a\,f(n/b) \le c\,f(n)$ for some $0 < c < 1$.

We get by substitution: $f(n/b) \le \frac{c}{a} f(n)$

$f(n/b^2) \le \frac{c}{a} f(n/b)$

$f(n/b^3) \quad \frac{c}{-} f(n/b^2)$

Master Theorem Proof:

Case 3: $f(m) = \Omega(m^{\log_b a+\varepsilon})$ and $a\,f(n/b) \le c\,f(n)$ for some $0 < c < 1$.

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By chainin

$f(n/b^2) \le \frac{c}{a} f(n/b) \le \frac{c}{a} \cdot \frac{c}{a} f($ —

$f(n/b^3) \le \frac{c}{a} \underbrace{f(n/b^2)} \le \frac{c}{a} \cdot \underbrace{\frac{c^2}{a^2}} \quad \overline{a^3}$

$\ldots$

$f(n/b^i) \le \frac{c}{a} \underbrace{f(n/b^{i-1})} \le \frac{c}{a} \cdot \underbrace{\frac{c^{i-1}}{a^{i-1}} f(n)} = \frac{c^i}{a^i} f(n)$

Master Theorem Proof:

Case 3 (continued):

We got $f(n/b^i) \leq \frac{c^i}{a^i} f(n)$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$a^i f\left(\frac{n}{b^i}\right) \leq a^i \frac{c^i}{a^i} f(n) < f(n) \qquad c^i = \frac{f(n)}{c}$$

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < \sum_{i=0}^{\infty} c^i = \frac{f(n)}{c}$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$a^i f\left(\frac{n}{b^i}\right) \leq a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} c^i = \frac{f(n)}{c} \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} c^i < \frac{f(n)}{c} \sum_{i=0}^{\infty} c^i$$

Since we ha

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we ha

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

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Since we ha

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{b-1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ we get:

$$T(n) < n^{\log_b a} T(1) + O(f$$

but we also have

$$T(n) = aT(n/b) + f(n) > f(n)$$

Master Theorem Proof:

Case 3 (continued):

We got $$f(n/b^i) \leq \frac{c^i}{a^i} f(n)$$

Thus,

$$\sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i f\left(\frac{n}{b^i}\right) \leq \sum_{i=0}^{\lfloor \log_b n \rfloor - 1} a^i \frac{c^i}{a^i} f(n) < f(n) \sum_{i=0}^{\infty} c^i = \frac{f(n)}{1-c}$$

Since we have

$$T(n) \approx n^{\log_b a} T(1) + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

and since $f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ we get:

$$T(n) < n^{\log_b a} T(1) + O\left(f(n)\right)$$

but we also have

$$T(n) = aT(n/b) + f(n) > f(n)$$

thus,

$$T(n) = \Theta\left(f(n)\right)$$

Master Theorem Proof: Homework

Exercise 1: Show that condition

$$f(n) = \Omega(n^{\log_b a + \varepsilon})$$

follows from the condition

Example: Let us estimate the asymptotic growth rate of

$$T(n) = 2T(n/2) +$$

Note: we have seen that the Master Theorem does **NOT** apply, but the technique used in its proof still works! So let us just unwind the recurrence and sum up the logarithmic overheads.

$$T(n) = 2T\left(\frac{n}{2}\right) + n\log n$$

$$= 2\left(2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\log\frac{n}{2}\right) + n\log n$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + n\log\tfrac{n}{2} + n\log n$$

$$= 2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\log\frac{n}{2^2}\right) + n\log\tfrac{n}{2} + n\log n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + \ldots$$

$$\ldots$$

$$= 2^{\log n} T\left(\tfrac{n}{2^{\log n}}\right) + n\log\tfrac{n}{2^{\log n-1}} + \ldots + n\log\tfrac{n}{2^2} + n\log\tfrac{n}{2} + n\log n$$

$$= nT(1) + n(\log n \times \log n - \log 2^{\log n-1} - \ldots - 1$$

$$= nT(1) + n((\log n)^2 - (\log n - 1) - \ldots - 3 - 2$$

$$= nT(1) + n((\log n)^2 - \log n(\log n - 1)/2$$

$$= nT(1) + n((\log n)^2/2 + \log n/2)$$

$$= \Theta\left(n(\log n)^2\right).$$

PUZZLE!

Five pirates have to split 100 bars of gold. They all line up and proceed as follows:

1. The first pirate in line gets to propose a way to split up the gold (for example: everyone gets 20 bars)
2. The pirates, including the one who proposed, vote on whether to accept the proposal. If the proposal is rejected, the pirate who made the proposal is killed.
3. The next pirate in line then makes his proposal, and the 4 pirates vote again. If the vote is tied (2 vs 2) then the proposing pirate is still killed. Only majority can accept a prop

pirat

-
- given maximal possible amount of gold, he wants to s
  just for fun;
- each pirate knows his exact position in line;
- all of the pirates are excellent puzzle solvers.

Question : What proposal should the first pirate make?

Hint: assume first that there are only two pirates, and see what happens. Then assume that there are three pirates and that they have figured out what happens if there were only two pirates and try to see what they would do. Further, assume that there are four pirates and that they have figured out what happens if there were only three pirates, try to see what they would do. Finally assume there are five pirates and that they have figured out what happens if there were only four pirates.