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DYNAMIC PROGRAMMING

Dynamic Programming

- **The main idea of Dynamic Programming:** build an optimal solution to the problem from optimal solutions for (carefully chosen) smaller size subproblems.

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Dynamic Programming: Activity Selection

- **Instance:** A list of activities a_i, $1 \leq i \leq n$ with starting times s_i and finishing times f_i. No two activities can take place simultaneously.
- **Task:** Find a subset of compatible activities of **maximal total duration**.
- Remember, we used the Greedy Method to solve a somewhat similar problem of finding a subset with the **largest possible number** of compatible acti lem.
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- For every $i \leq n$ we solve the following subproble
  Subproblem $P(i)$: find a subsequence σ_i $S_i = \langle a_1, a_2, \ldots, a_i \rangle$ such that:
  1. σ_i consists of non-overlapping activitie
  2. σ_i ends with activity a_i;
  3. σ_i is of maximal total duration among all subsequences of S_i which satisfy 1 and 2.
- Note: the role of Condition 2 is to simplify recursion.

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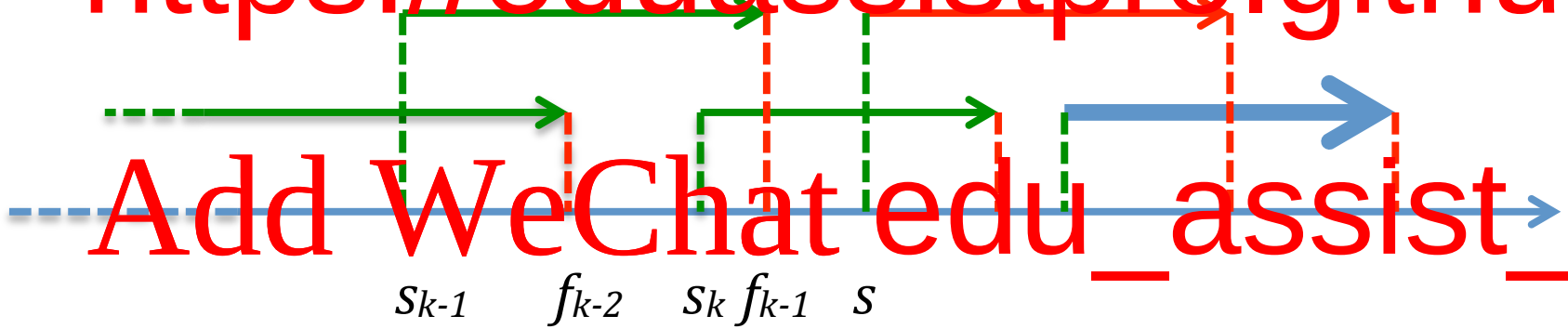
- Let $T(i)$ be the total duration of the optimal solution $S(i)$ of the subproblem $P(i)$.
- For $S(1)$ we choose a_1; thus $T(1) = f_1 - s_1$;
- Recursively assuming that we have solved subproblems $P(j)$ for all $j < i$ and stored them in a table, we let



- In the table, for every i, besides $T(i)$, we also store $\pi(i) = j$ for which the above max is achieved:

$$\pi(i) = \arg\max\{T(j) \ : \ j < i \ \ \& \ \ f_j \leq s_i\}$$

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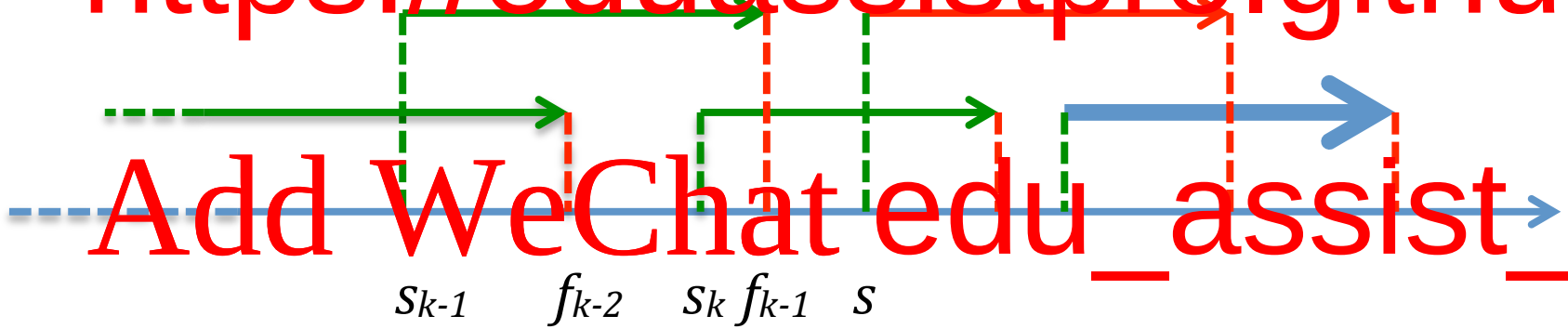

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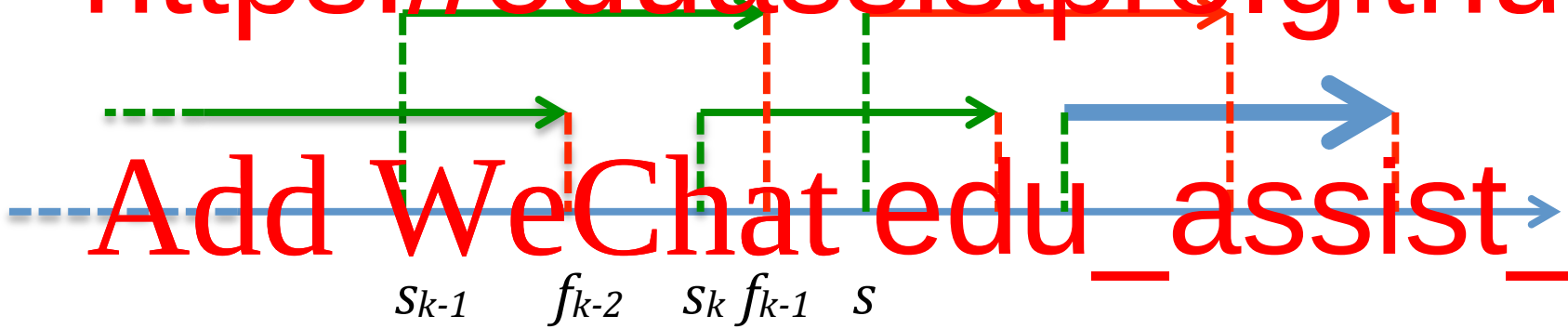

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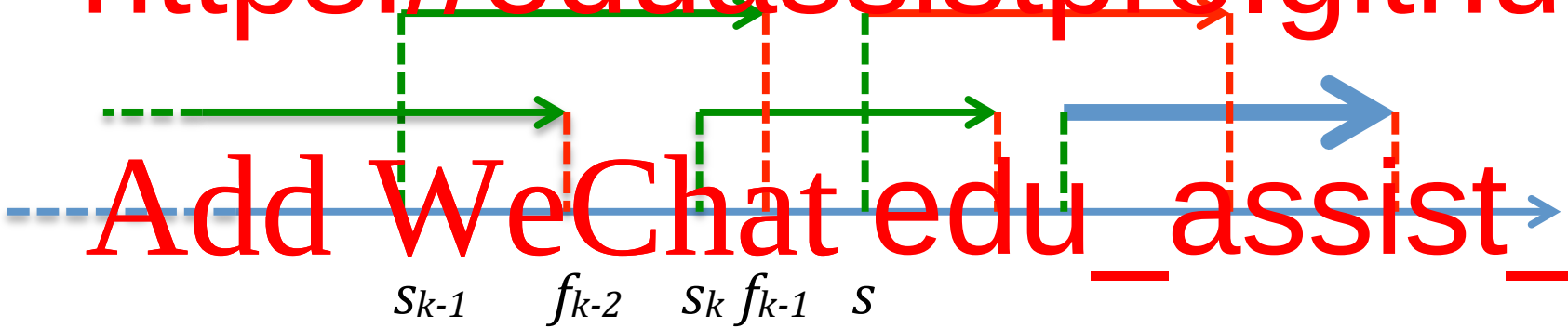

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- Why does such a recursion produce optimal solutions to subproblems $P(i)$?
- Let the optimal solution of subproblem $P(i)$ be the sequence $S = (a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ where $a_{k_1} = i$.
- We claim: the truncated subsequence $S' = (a_{k_1}, a_{k_2}, \ldots, a_{k_{m-1}})$ is an optimal solu
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- If there were a sequence S^* of a larger total dur sequence S' and also ending with activity by extending the sequence S^* with activity subproblem $P(i)$ with a longer total duration than t sequence S, contradicting the optimality of S.
- Thus, the optimal solution $S = (a_{k_1}, a_{k_2}, \ldots, a_{k_{m-1}}, a_{k_m})$ for problem $P(i)$ $(= P(a_{k_m}))$ is obtained from the optimal solution $S' = (a_{k_1}, a_{k_2}, \ldots, a_{k_{m-1}})$ for problem $P(a_{k_{m-1}})$ by extending it with a_{k_m}

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- Continuing with the solution of the problem, we now let

$$T_{max} = \max\{T(i) \ : \ i \leq n\};$$

$$\text{last} = \arg\max\{T(i) \ : \ i \leq n\}.$$

- Select (now recompute) the optimal sequence which solves our problem from the table of partial solutions, because in the i^{th} slot of the table, besides $T(i)$ we also store $\pi(i) = j$, $(j < i)$ such that the optimal solution of $P(i)$ extends the o

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- Why is such solution optimal, i.e., why looking for optimal solutions of $P(i)$ which must end with a_i did not cause us to miss th
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assume it ends with activity a_k; then it would have been obtained as the optimal solution of problem $P(k)$.

- Time complexity: having sorted the activities by their finishing times in time $O(n \log n)$, we need to solve n subproblems $P(i)$ for solutions ending in a_i; for each such interval a_i we have to find all preceding compatible intervals and their optimal solutions (to be looked up in a table). Thus, $T(n) = O(n^2)$.

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More Dynamic Programming Problems

- **Longest Increasing Subsequence:** Given a sequence of n real numbers $A[1..n]$, determine a subsequence (not necessarily contiguous) of maximum length in which the values in the subsequence are strictly increasing.

- Solution: For each $i \leq n$ we solve the following subproblems:
- *Sub* *max* *end* .i] of
- Recursion: Assume we have solved the subprobl nd that we have put in a table S the values hs ℓ_j of maximal increasing sequences which end w
- We now look for all $A[m]$ such that $m < i$ and such that $A[m] < A[i]$.
- Among those we pick m which produced the longest increasing subsequence ending with $A[m]$ and extend it with $A[i]$ to obtain the longest increasing subsequence which ends with $A[i]$:

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- **Longest Increasing Subsequence:** Given a sequence of n real numbers $A[1..n]$, determine a subsequence (not necessarily contiguous) of maximum length in which the values in the subsequence are strictly increasing.

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$$\ell_i = \max\{\ell_m \ : \ m < i \ \& \ A[m] < A[i]\} + 1$$

$$\pi(i) = \arg\max\{\ell_m \ : \ m < i \ \& \ A[m] < A[i]\}$$

- We store in the i^{th} slot of the table the length ℓ_i of the longest increasing subsequence ending with $A[i]$ and $\pi(i) = m$ such that the optimal solu

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$$\text{end}, \pi(\text{end}), \pi(\pi(\text{end})), \ldots$$

- Aga ... ective bec ... ve been constructed as the solution for $P(m)$.

- Time complexity: $O(n^2)$.

- Exercise: (somewhat tough, but very useful) Design an algorithm for solving this problem which runs in time $n \log n$.

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- **Making Change.** You are given n types of coin denominations of values $v(1) < v(2) < \ldots < v(n)$ (all integers). Assume $v(1) = 1$, so that you can always make change for any integer amount. Give an algorithm which makes change for any given integer amount C with as few coins as possible, assuming that you have an unlimited supply of coins of each den

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- We consider optimal solutions $opt(i - v(k))$ for every amount of the form $i - v(k)$, where k ranges from 1 to n. (Recall $v(1), \ldots, v(n)$ are all of the available denominations.)
- Among all of these optimal solutions (which we find in the table we are constructing recursively!) we pick one which uses the fewest number of coins, say t
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- However, we do not know which coins the optimal solution includes, so we try all the available coins and then pick m for which the optimal solution for amount $i - v(m)$ uses the fewest number of coins.
- It is enough to store in the i^{th} slot of the table such m and $opt(i)$ because this allo ored in the i^{th} slot, t stored in the slot
- $opt(C)$ is the solution we need.
- Time complexity of our algorithm is nC.
- **Note:** Our algorithm is **NOT** a polynomial gth of the input, because the length of a representation of C is only $\log C$, while the running time is nC.
- But this is the best what we can do...

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More Dynamic Programming Problems

Integer Knapsack Problem (Duplicate Items Allowed) You have n types of items; all items of kind i are identical and of weight w_i and value v_i. You also have a knapsack of capacity C. Choose a combination of available items which all fit in the knapsack and whose value is as large as possible. You can take any number of items of each kind.

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- We h $i \leq C$.
- Assume we have solved the problem for all knapsacks of capacities $j < i$.

- We now look at optimal solutions $opt(i - w_1), \ldots, opt(i - w_n)$ for all $1 \leq m \leq n$.
- Chose the one for which $opt(i - w_m) + v_m$ is the largest;
- Add to such optimal solution for the knapsack of size $i - w_m$ item m to obtain a packing of a knapsack of size i of the highest possible value.

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$$\pi(i) = \arg\max\{opt(i - w_m) + v_m : 1 \le m \le n\}.$$

- Afte $opt(C)$.

- Whi
backtracking: if $\pi(C) = k$ then the first object is alue v_k;
if $\pi(C - w_k) = m$ then the second object is

- Note that $\pi(i)$ might not be uniquely determined; i good solutions we pick arbitrarily among them.

- Again, our algorithm is **NOT** polynomial in the **length** of the input.

More Dynamic Programming Problems

- **Integer Knapsack Problem (Duplicate Items NOT Allowed)** You have n items (some of which can be identical); item I_i is of weight w_i and value v_i. You also have a knapsack of capacity C. Choose a combination of available items which all fit in the knapsack and whose value is as large as possible.

- This is an example of a "2D" recursion; we will be filling a table of size $n \times C$, row by row; subproblems $P(i, c)$ for all i n and c C will be of the form:

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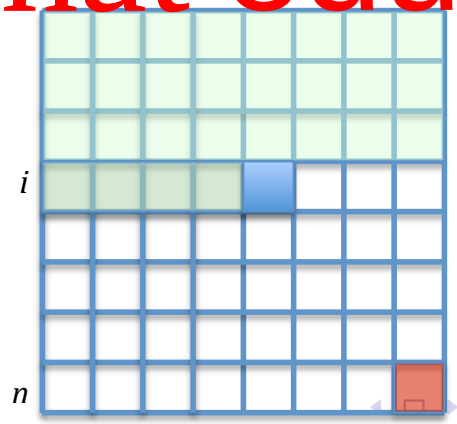

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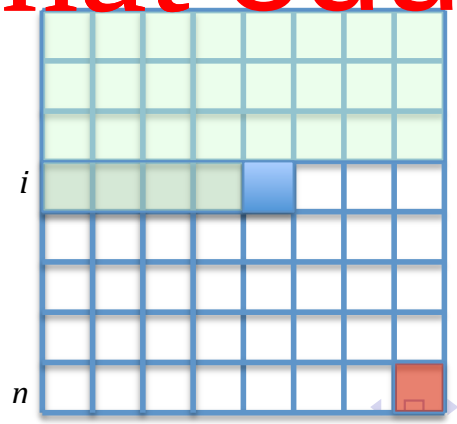

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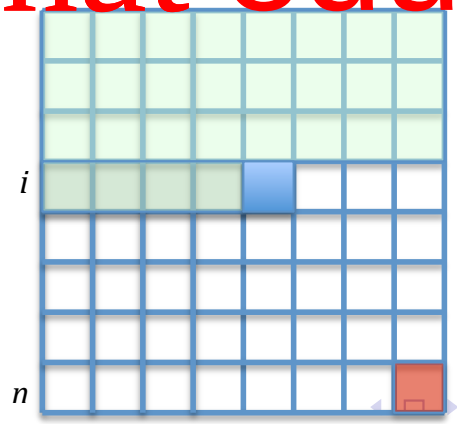

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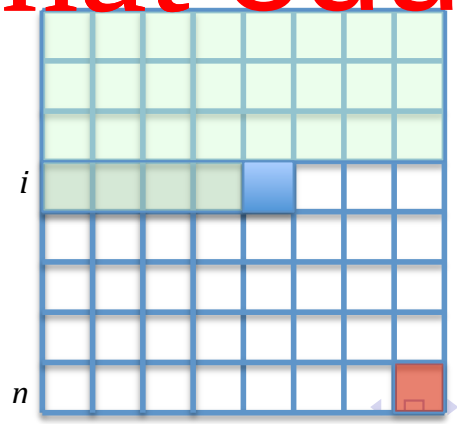

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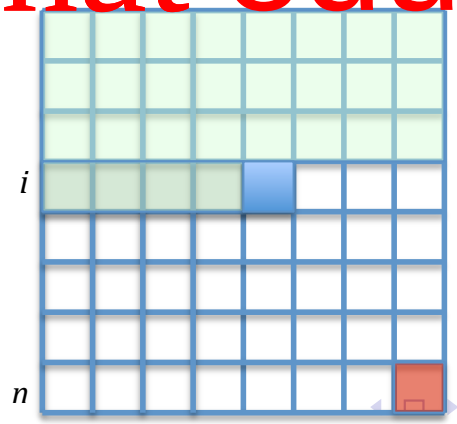

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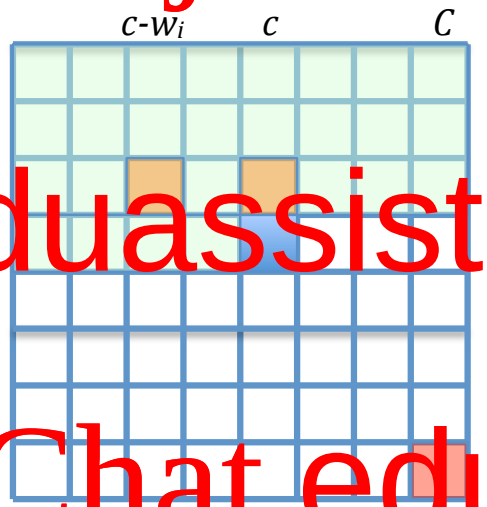

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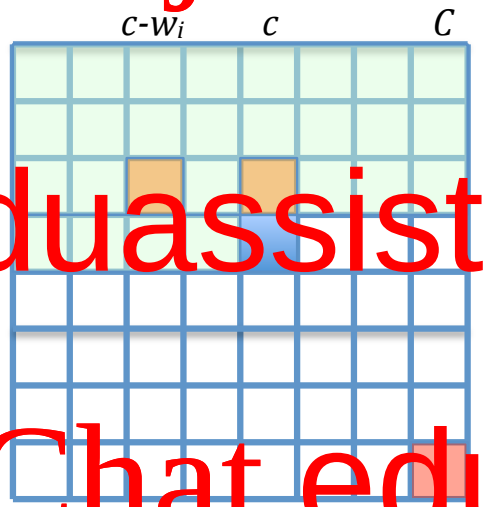

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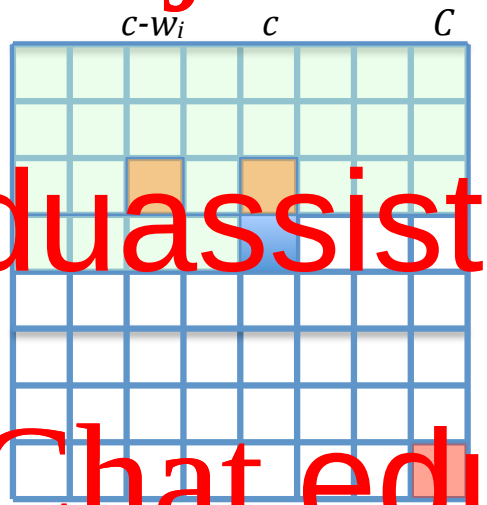

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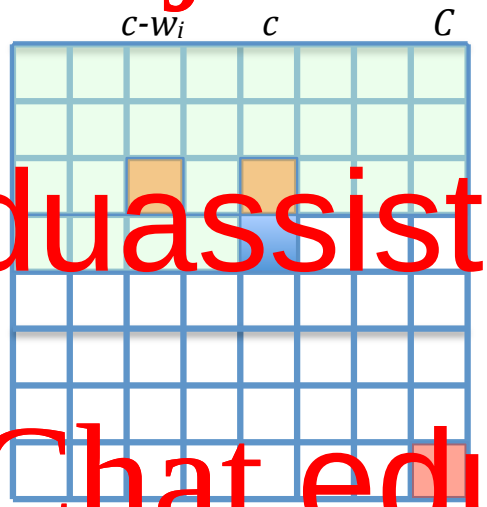

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- Solution: Let S be the total sum of all integers in the set, consider the Knapsack problem (with duplicate items not allowed) with the knapsack of size $S/2$ and with each integer x_i of both size and value equal to x_i.

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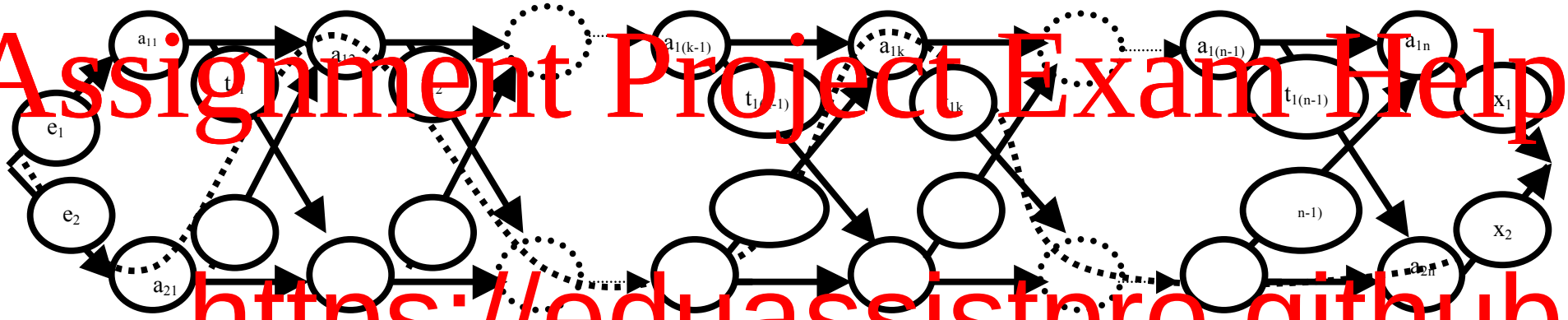
Dynamic Programming: Assembly line scheduling



Instance: Two assembly lines with workstations for

- On the first assembly line the k^{th} job takes $a_{1,k}$ units of time to complete; on the second assembly line the same job takes $a_{2,k}$ units of time.
- To move the product from station $k-1$ on the first assembly line to station k on the second line it takes $t_{1,k-1}$ units of time.
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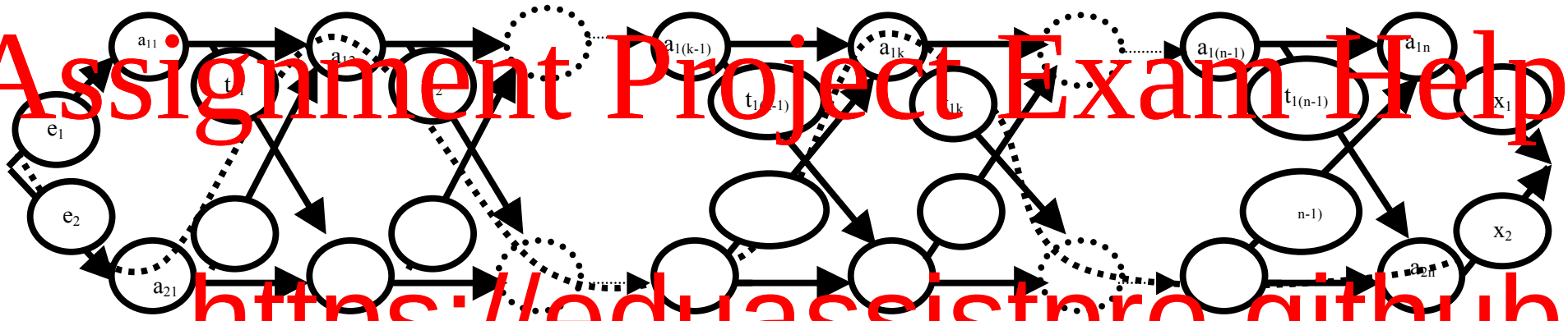

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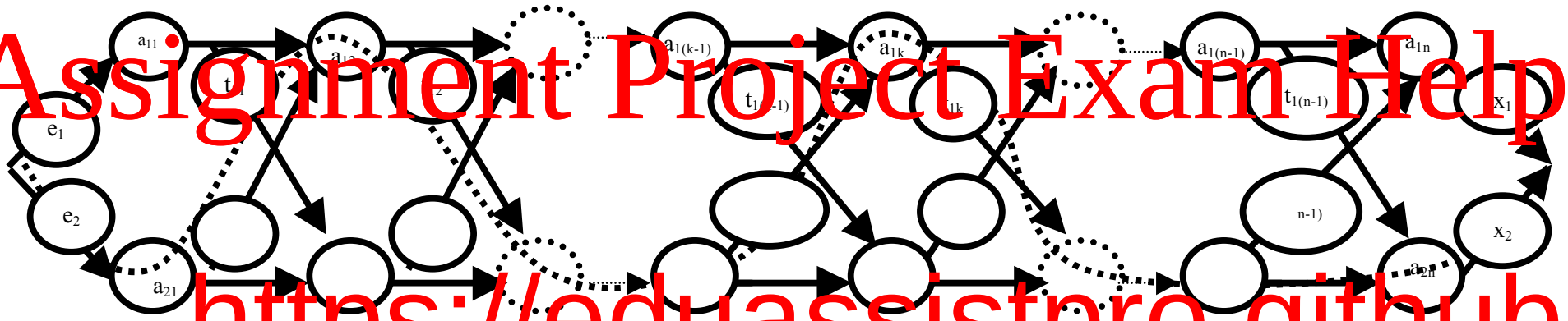

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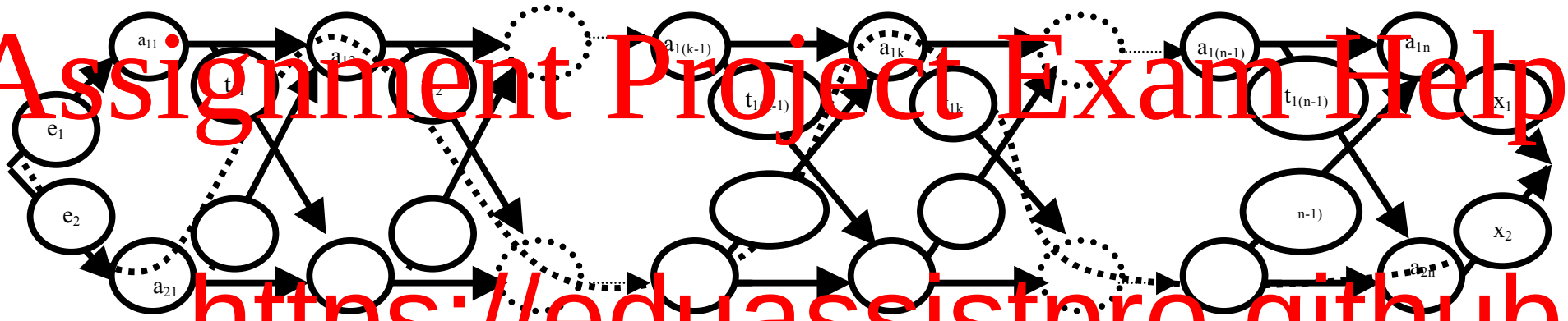

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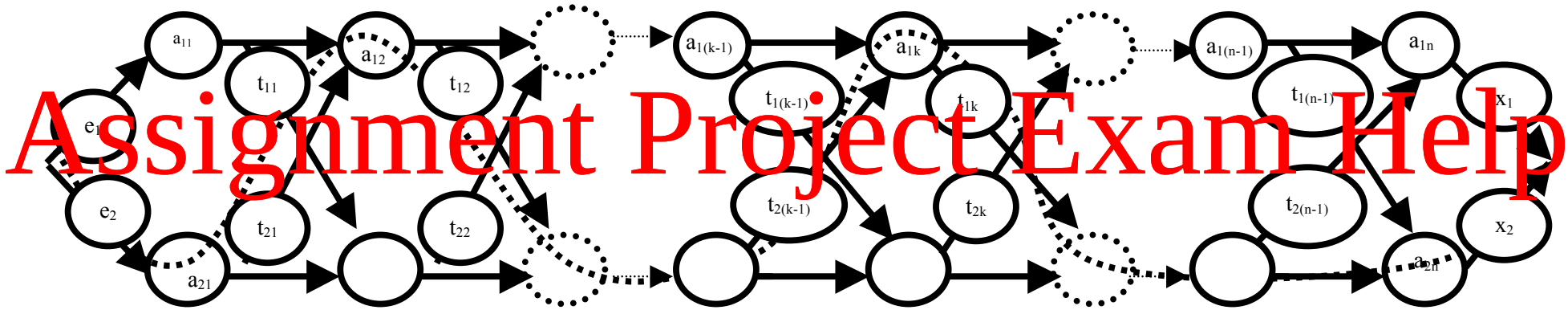

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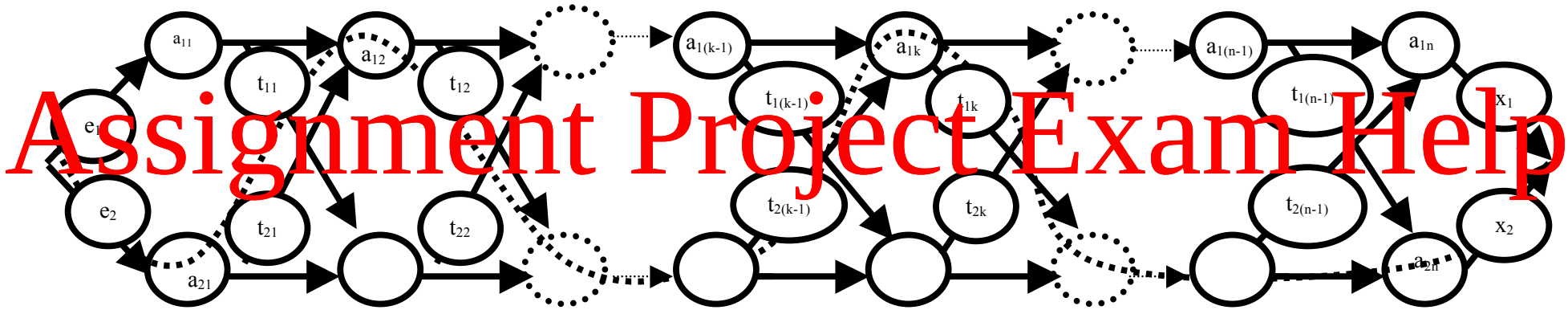

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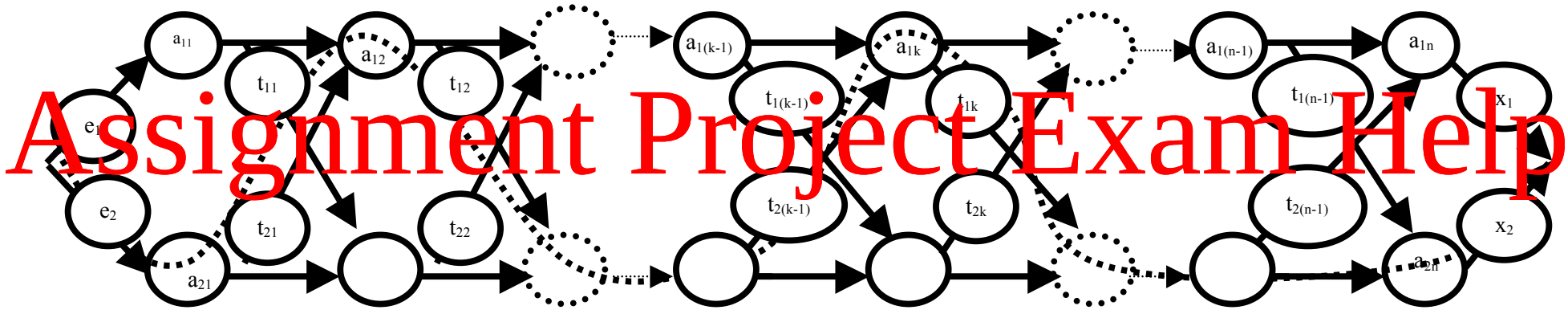

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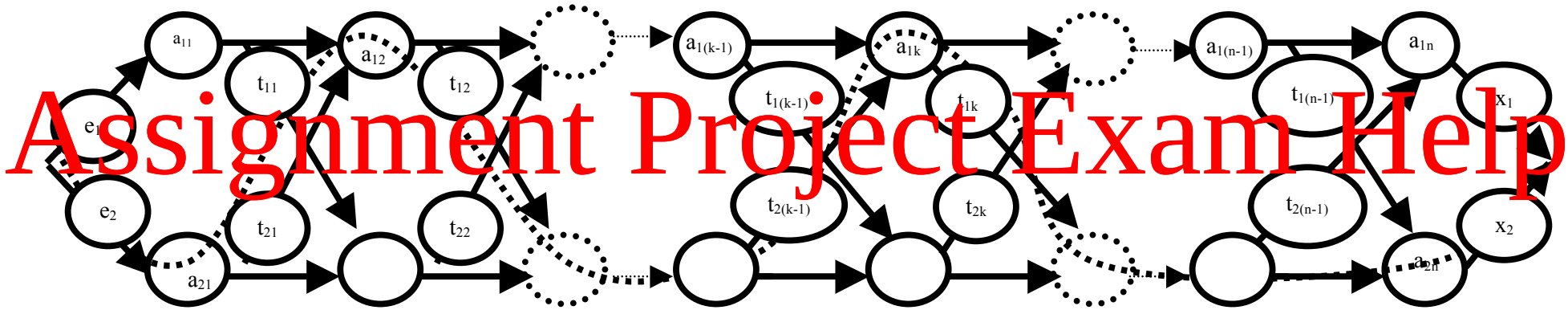

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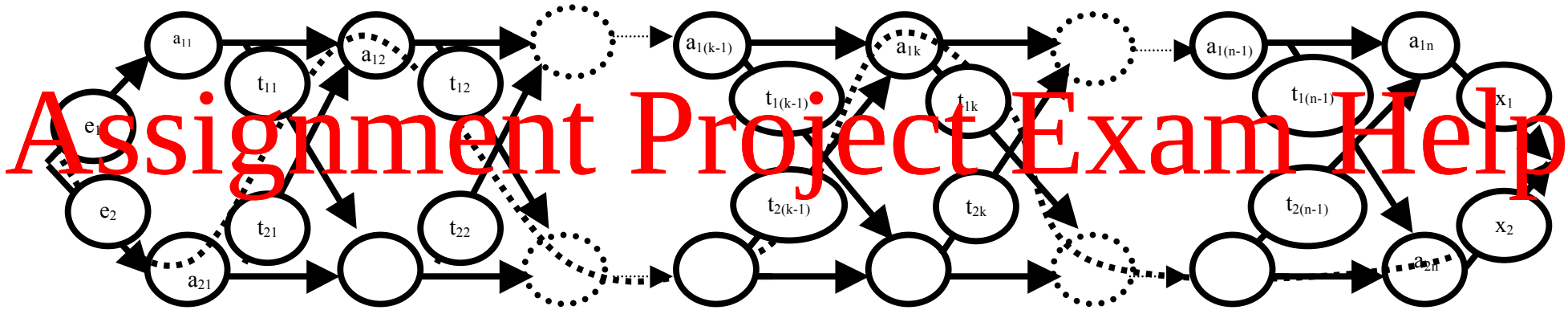

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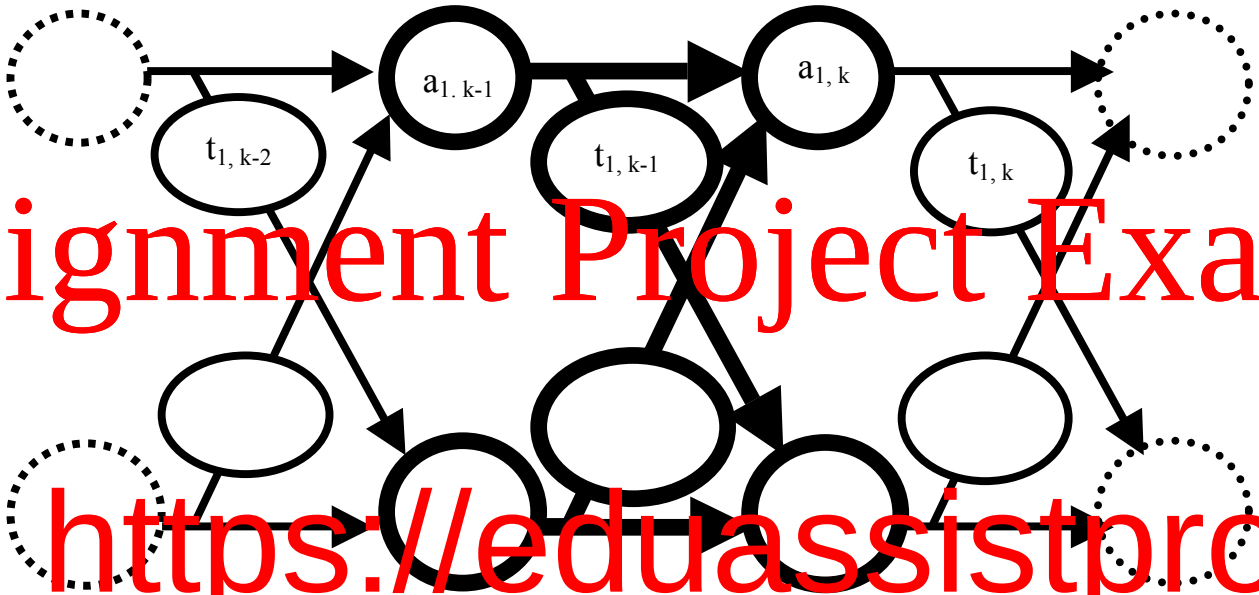

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- $P(1,k)$: find the minimal amount of time k jobs, such the k^{th} job is finished on the k^{th} workstation on the **first** assembly line;
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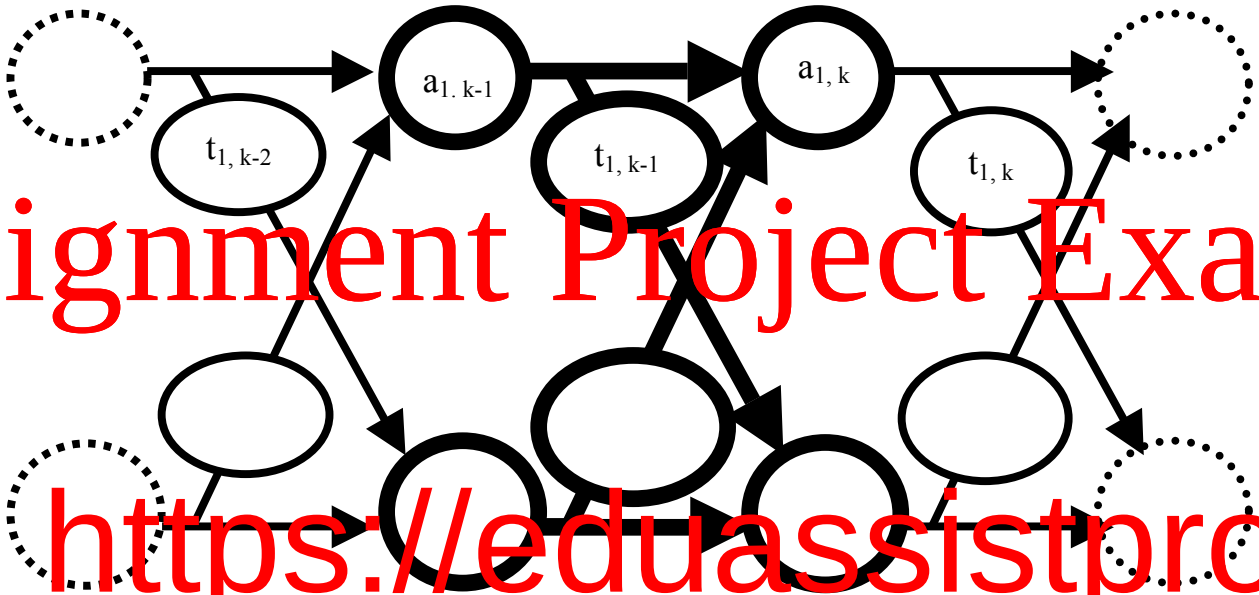
Dynamic Programming: Assembly line scheduling



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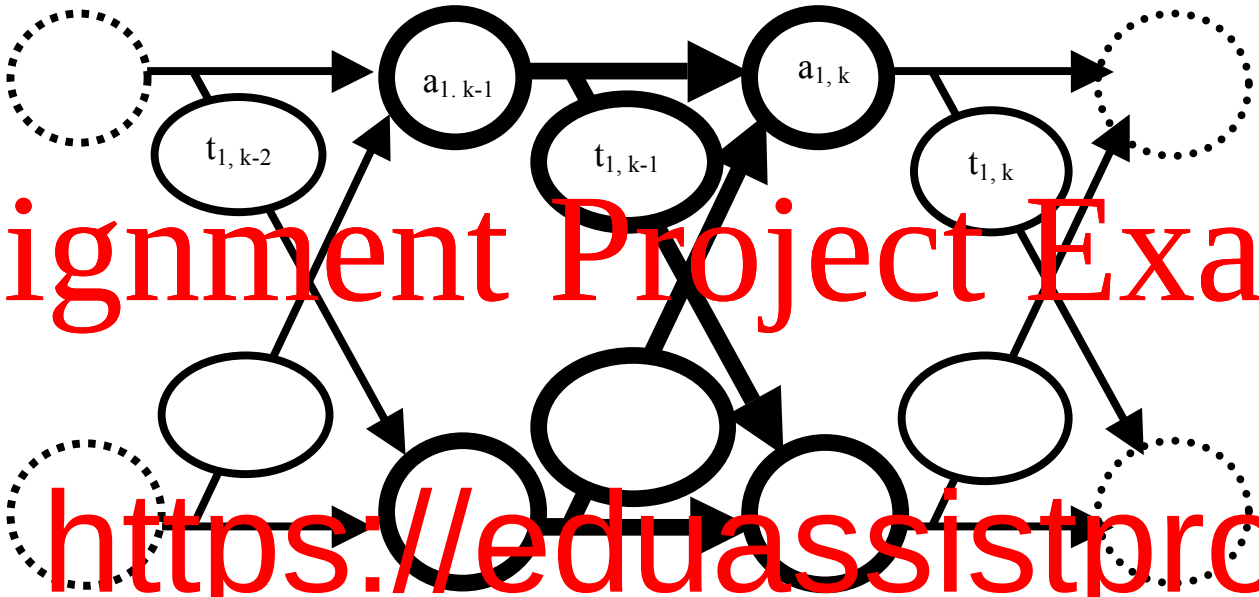

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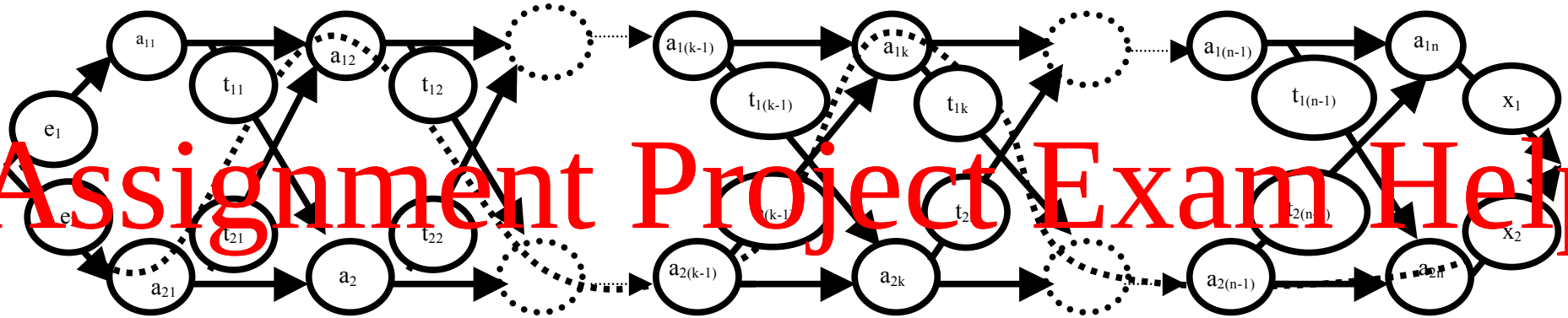

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- Finally, after obtaining $m(1,n)$ and $m(2,n)$ we choose

$$opt = \min\{m(1,n) + x_1, \quad m(2,n) + x_2\}.$$

- This problem is important because it has the same design logic as the Viterbi algorithm, an extremely important algorithm for many fields such as speech recognition, decoding convolutional codes in telecommunications etc, covered

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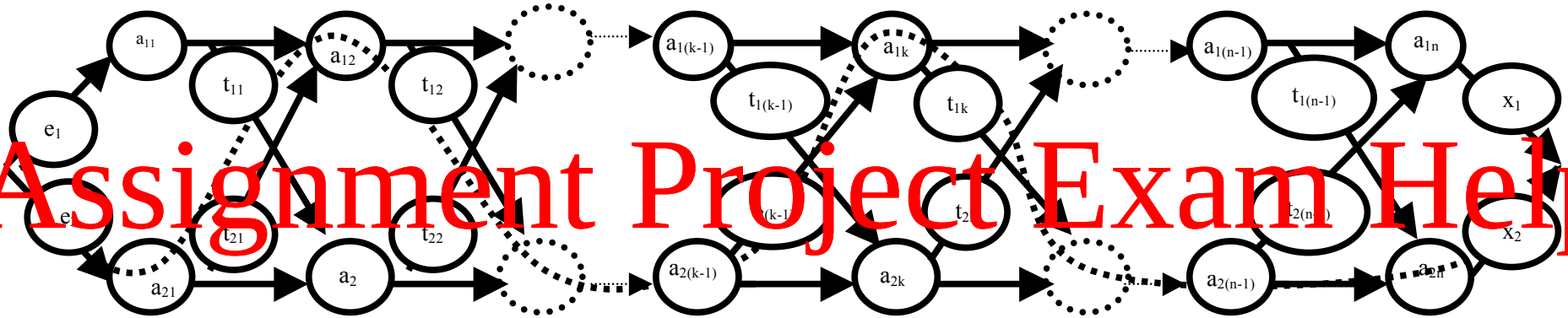

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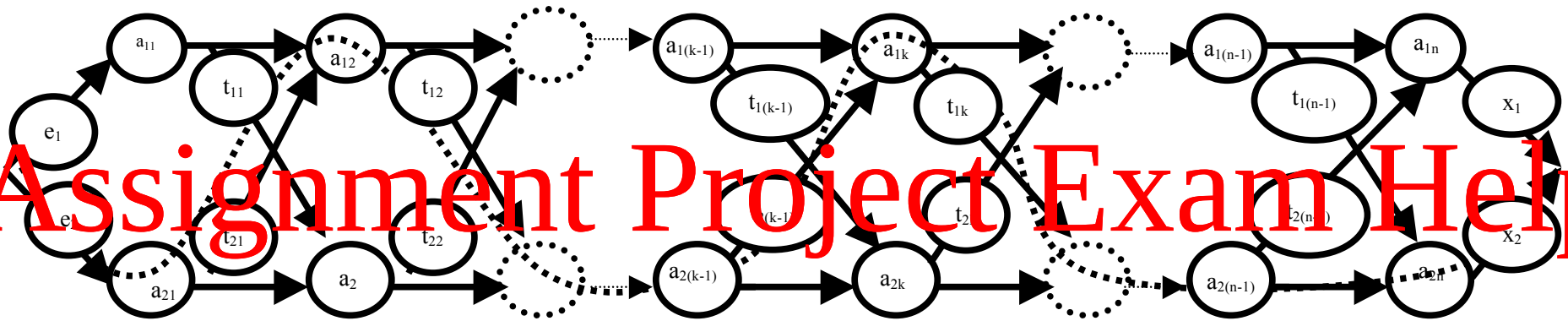

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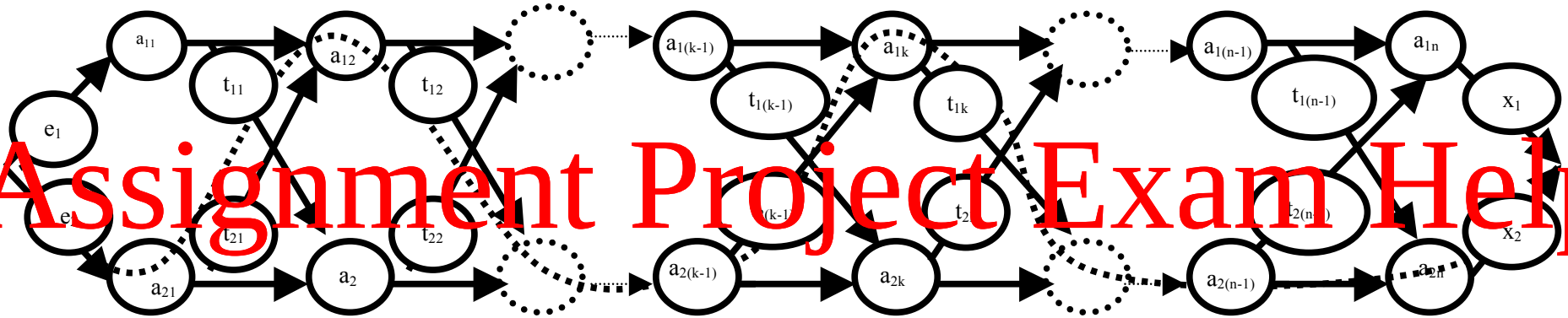

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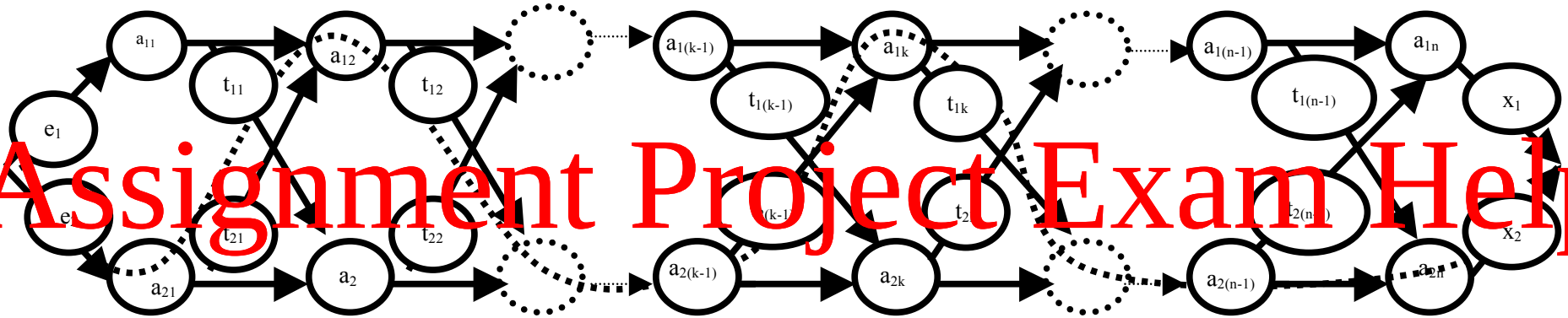

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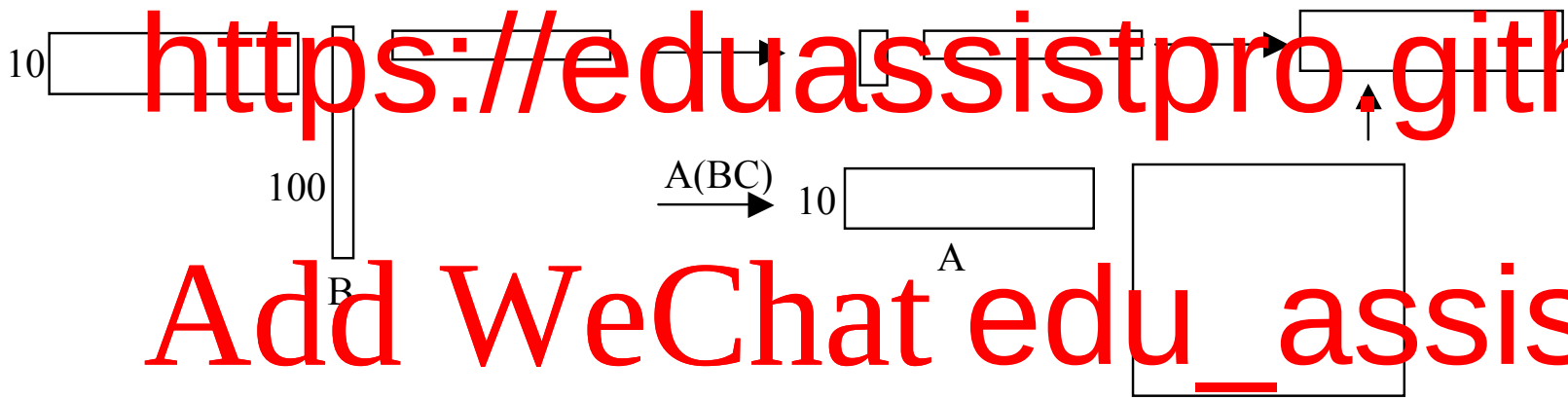
Dynamic Programming: Matrix chain multiplication

- For any three matrices of compatible sizes we have $A(BC) = (AB)C$.
- However, the number of real number multiplications needed to perform in order to obtain the matrix product can be very different:

$A = 10 \times 100, \quad B = 100 \times 5, \quad C = 5 \times 50$



- To evaluate $(AB)C$ we need
  $(10 \times 5) \times 100 + (10 \times 50) \times 5 = 5000 + 2500 = 7500$ multiplications;
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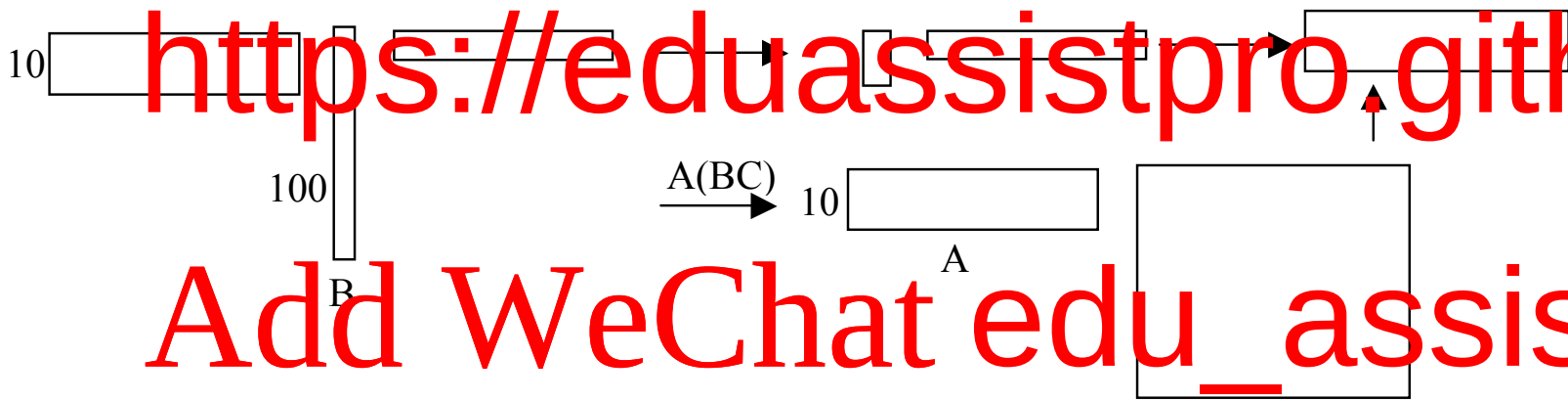

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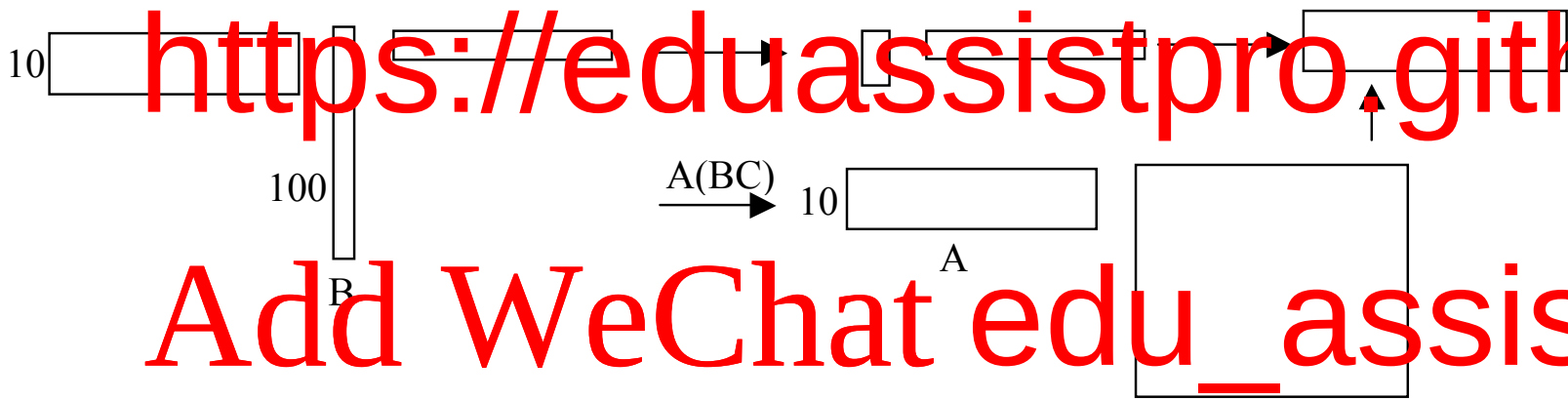

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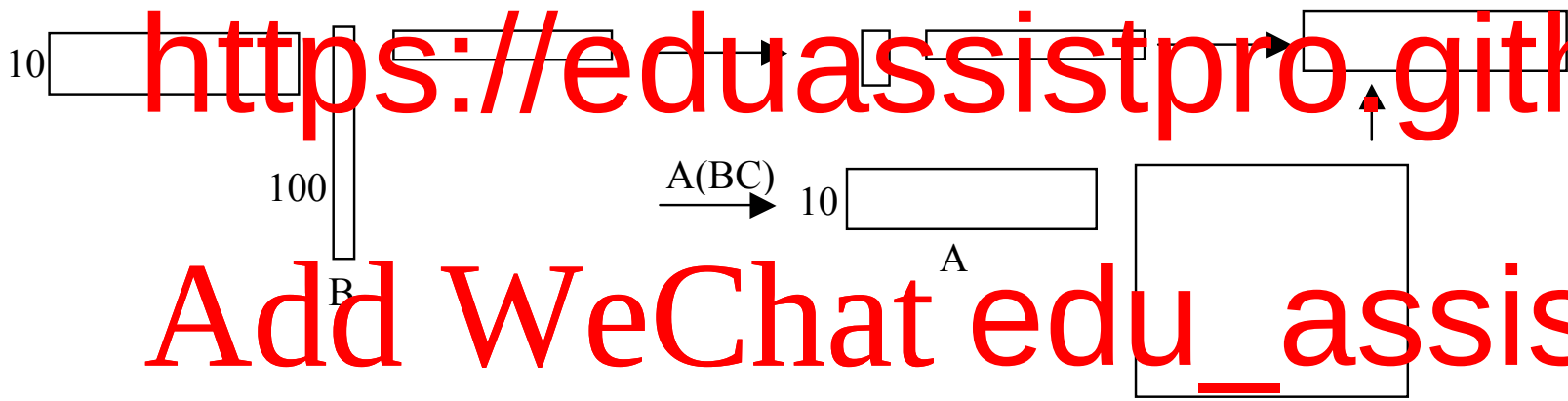

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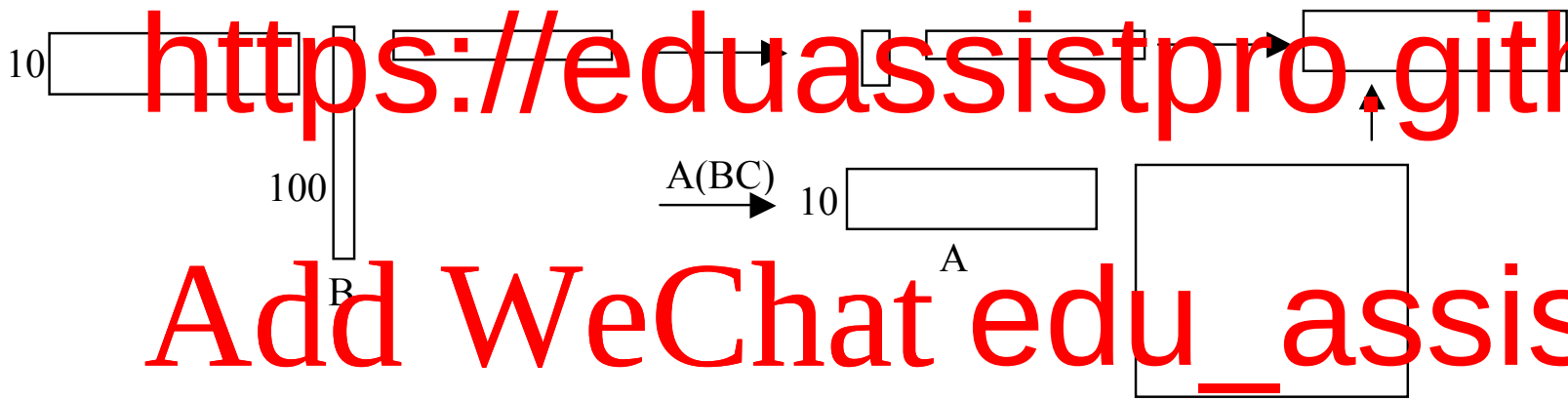

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- **Problem Instance:** A sequence of matrices $A_1 A_2 ... A_n$;
- **Task:** Find the way to multiply the matrices that minimises the total number of multiplications needed to find the product matrix.
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recursion (why?):

$$T(n) = \sum_{i=1}^{n-1} T(i$$

- One can show that the solution satisfies $T(n) = \Omega(2^n)$.
- Thus, we cannot do an exhaustive search for the optimal placement of the brackets.

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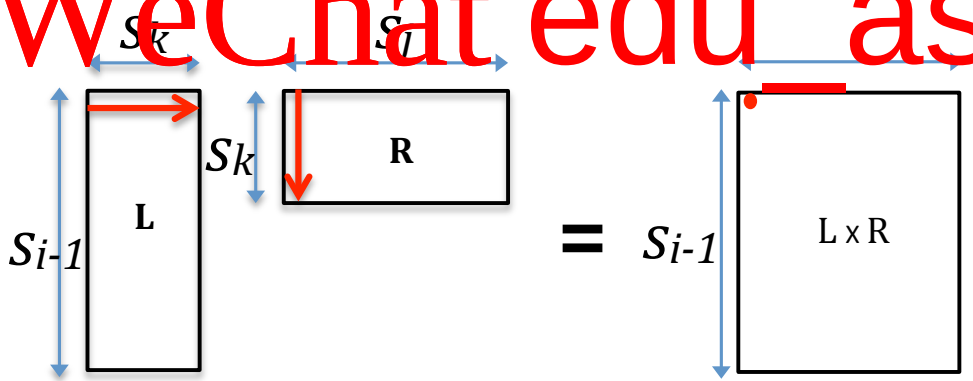

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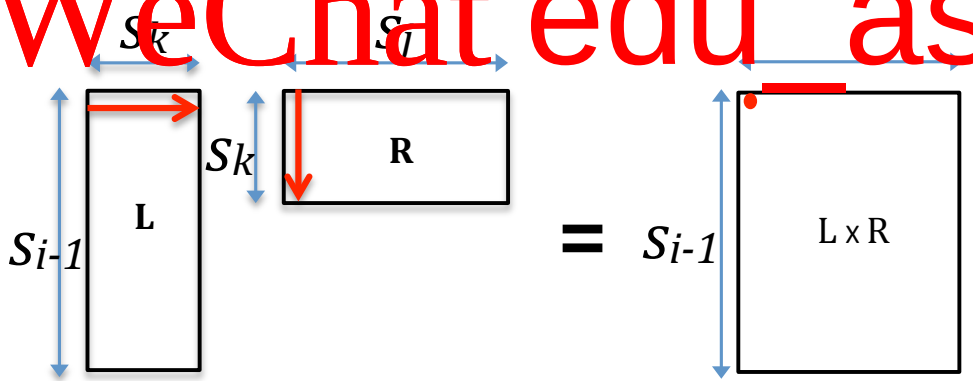

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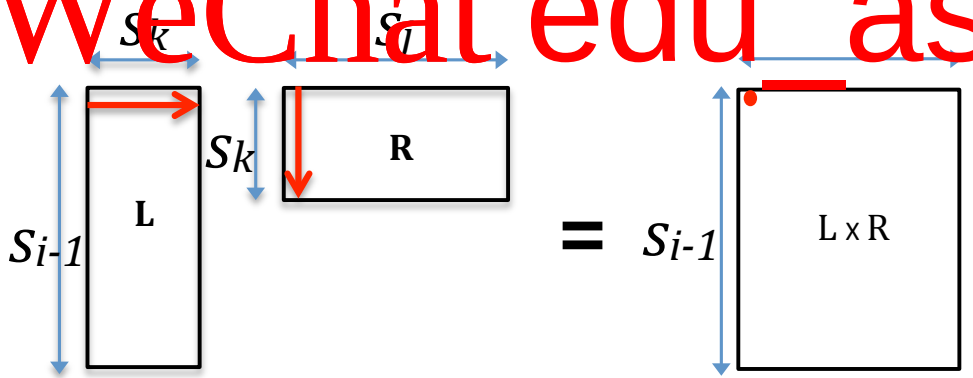

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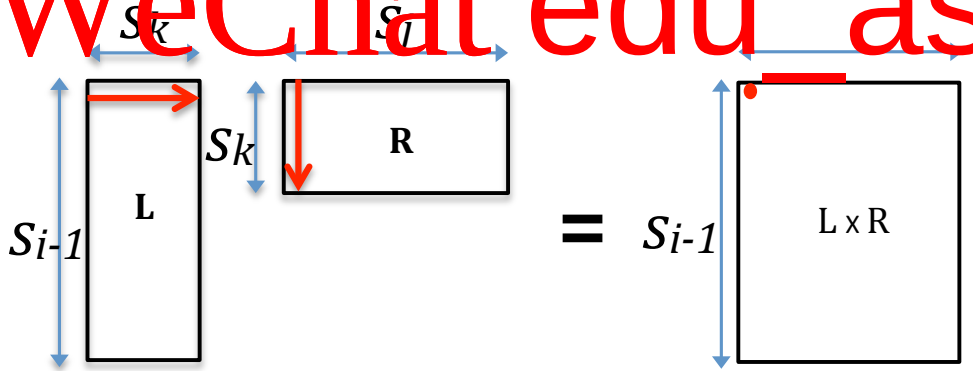

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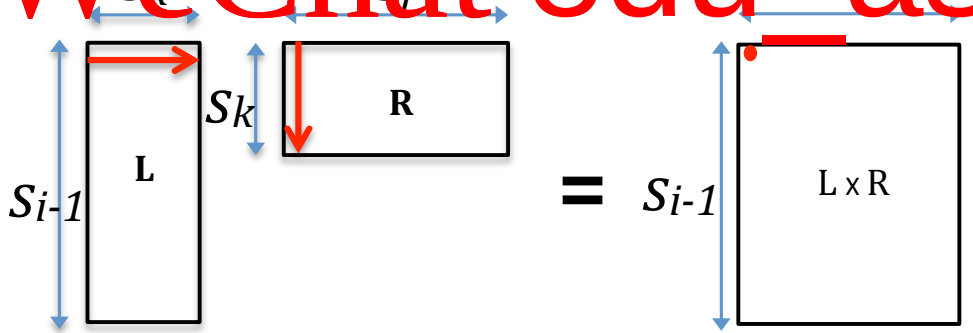

Total number of multiplications: $S_{i-1}\ S_j\ S_k$

Dynamic Programming: Matrix chain multiplication

- The recursion:

$$m(i,j) = \min\{m(i,k) + m(k+1,j) + s_{i-1}s_j s_k \; : \; i \leq k \leq j-1\}$$

- Not algo ther

- k for which the minimum in the recursive definiti ieved can b stored to achieve the optimal placement of br chain $A_i \ldots A_j$.

- Thus, in the m^{th} slot of the table we are constructing we store all pairs $(m(i,j),k)$ for which $j - i = m$.

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Dynamic Programming: Longest Common Subsequence

- Assume we want to compare how similar two sequences of symbols S and S^* are.
- Example: how similar are the genetic codes of two viruses.
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$$c[i,j] = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0; \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } a_i = b_j; \\ \max\{c[i-1, j], c[i, j-1]\} & \text{if } i, j > 0 \text{ and } a_i \neq b_j. \end{cases}$$

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Retrieving a longest common subsequence:

Dynamic Programming: Longest Common Subsequence

- What if we have to find a longest common subsequence of three sequences S_1, S_2, S_3?
- Can we do $\mathrm{LCS}(\mathrm{LCS}(S_1, S_2), S_3)$?
- Not necessarily! Consider

$$\mathrm{LCS}(\mathrm{LCS}(S_1, S_2), S_3) = \mathrm{LCS}(ABE$$
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$$\mathrm{LCS}(S_1, S_2, S_3) = ACEG$$

- So how would you design an algorithm which computes correctly $\mathrm{LCS}(S_1, S_2, S_3)$?

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- Recurrence:

$$d[i,j,l] = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \text{ or } l = 0; \\ d[i-1, j-1, l-1] + 1 & \text{if } i,j,l > 0 \text{ and } a_i = b_j = c_l; \\ \max\{d[i-1,j,l], d[i,j-1,l], d[i,j,l-1]\} & \text{otherwise.} \end{cases}$$

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- **Instance:** Two sequences $s = \langle a_1, a_2, \ldots a_n \rangle$ and $s^* = \langle b_1, b_2, \ldots, b_n \rangle$
- **Task:** Find a shortest common super-sequence S of s, s^*, i.e., the shortest possible sequence S such that both s and s^* are subsequences of S.
- **Sol** ... s and s^* and then ... orde...

$LCS(s, s$

shortest super-sequence $S = ax\mathbf{b}ya\mathbf{ca}z\mathbf{d}a$

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Dynamic Programming: Edit Distance

- **Edit Distance** Given two text strings A of length n and B of length m, you want to transform A into B. You are allowed to insert a character, delete a character and to replace a character with another one. An insertion costs c_I, a deletion costs c_D and a replacement costs c_R.
- Tas
- Not num called *the edit distance* between A and B.
- If the sequences are sequences of DNA bases and the cost probabilities of the corresponding mutations, then the probability that one sequence mutates into anot of DNA copying.
- Subproblems: Let $C(i, j)$ be the minimum cost of transforming the sequence $A[1..i]$ into the sequence $B[1..j]$ for all $i \leq n$ and all $j \leq m$.

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- **Subproblems** $P(i, j)$**:** Find the minimum cost $C(i, j)$ of transforming the sequence $A[1..i]$ into the sequence $B[1..j]$ for all $i \leq n$ and all $j \leq m$.
- **Recursion:** we again fill the table of solutions $C(i, j)$ for subproblems $P(i, j)$ row by row, where $C(i, j)$:
  - cost $c_D + C(i-1, j)$ corresponds to the option if we transform $A[1..i-1]$ into $B[1..j]$ and then delete $A[i]$;
  - cost $C(i, j-1) + c_I$ corresponds to the option if we transform $A[1..i]$ to $B[1..j-1]$ and then append $B[j]$ at the end;
  - the third option corresponds to first transforming $A[1..i-1]$ to $B[1..j-1]$ and
    1. if $A[i]$ is already equal to $B[j]$ do nothing, thus incurring a cost of only $C(i-1, j-1)$;
    2. if $A[i]$ is not equal to $B[j]$ replace $A[i]$ by $B[j]$ with a total cost of $C(i-1, j-1) + c_R$.

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Dynamic Programming: Maximizing an expression

- **Instance:** a sequence of numbers with operations $+, -, \times$ in between, for example

$$1 + 2 - 3 \times 6 - 1 - 2 \times 3 - 5 \times 7 + 2 - 8 \times 9$$

- **Task:** Place brackets in a way that the resulting expression has the largest possible value.

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Dynamic Programming: Floyd Warshall algorithm

- Let again $G = (V, E)$ be a directed weighted graph where $V = \{v_1, v_2, \ldots, v_n\}$ and where weights $\mathrm{w}(e(v_p, v_q))$ of edges $e(v_p, v_q)$ can be negative, but there are no negative weight cycles.
- We can use a somewhat similar idea to obtain the shortest paths from **every** vertex v_p to **every** vertex v_q (including back to v_p).
- Let $\mathrm{opt}[k, v_p, v_q]$ be the length of a shortest path from v_p to a vertex v_q that only uses intermediary vertices from the set $\{v_1, \ldots, v_k\}$.
- Then

$$\mathrm{opt}[k, v_p, v_q] = \min\{\mathrm{opt}[k-1, v_p, v_q], \mathrm{opt}[k-1, v_p, v_k] + \mathrm{opt}[k-1, v_k, v_q]\}$$

- Thus, we gradually **relax** the constraint that the intermediary vertices have to belong to $\{v_1, v_2, \ldots, v_k\}$.
- Algorithm runs in time $|V|^3$.

Dynamic Programming: Floyd Warshall algorithm

- Let again $G = (V, E)$ be a directed weighted graph where $V = \{v_1, v_2, \ldots, v_n\}$ and where weights $\mathrm{w}(e(v_p, v_q))$ of edges $e(v_p, v_q)$ can be negative, but there are no negative weight cycles.
- We can use a somewhat similar idea to obtain the shortest paths from **every** vertex v_p to **every** vertex v_q (including back to v_p).
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Another example of relaxation:

- Compute the number of partitions of a positive integer n. That is to say the number of distinct multi-sets of positive integers $\{n_1, \ldots, n_k\}$ which sum up to n, i.e., such that $n_1 + \ldots + n_k = n$.

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Hin } of j
i.e., dition,
hav s $j_q \leq i$.

We are looking for nump(n, n) but the recursion is based on relaxation of the allowed size i of the parts of j for all
definition of nump(i, j) distinguish the case
components are $\leq i-1$ and the case where at le
size i.

PUZZLE

You have
minute ea
at differe
thick ness
45 second interval?