Algorithms:

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2. DIVIDE-AND-CONQUER

A Puzzle

- **An old puzzle:** We are given 27 coins of the same denomination; we know that one of them is counterfeit and that it is lighter than the others. Find the counterfeit coin by weighing coins on a pan balance only three times.

- **Solution:**

- Thi

- We have already seen a prototypical "seriou
  using such a method: the MERGE-S

- We split the array into two, sort the two parts recursively and then merge the two sorted arrays.

- We now look at a closely related but more interesting problem of counting inversions in an array.

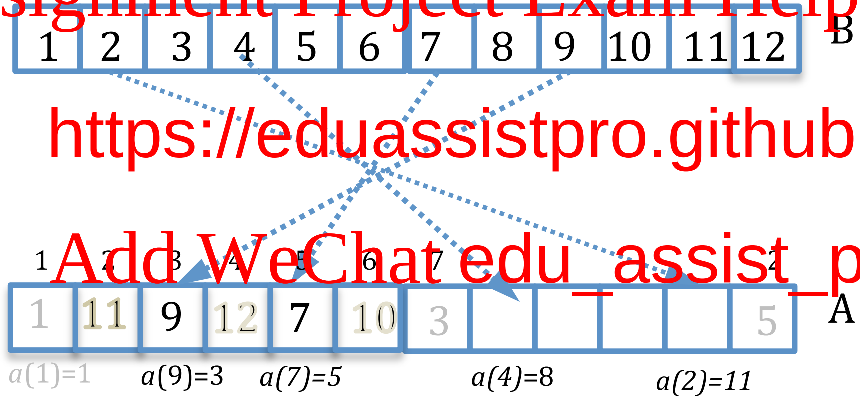
Counting the number of inversions

- Assume that you have m users ranking the same set of n movies. You want to determine for any two users A and B how similar their tastes are (for example, in order to make a recommender system).

- Ho [illegible] A and B?

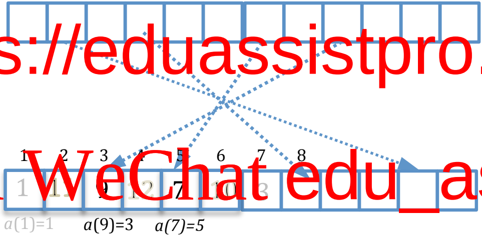
- Lets enumerate the movies on the ranking list o [illegible] assigning to the top choice of user [illegible] choice index 2 and so on.

- For the i^{th} movie on $B's$ list we can now look at the position (i.e., index) of that movie on $A's$ list, denoted by $a(i)$.


1 2 3 4 5 6 7 8 9 10 11 12
B
1 11 9 12 7 10 3 5
A
a(1)=1
a(9)=3
a(7)=5
a(4)=8
a(2)=11

- A good measure of how different these two users are, is the total number of *inversions*, i.e., total number of pairs of movies i, j such that movie i precedes movie j on $B's$ list but movie j is higher up on $A's$ list than the movie i.

- In other words, we count the number of pairs of movies i, j such that $i < j$ (movie i precedes movie j on $B's$ list) but $a(i) > a(j)$ (movie i is in the position $a(i)$ on $A's$ list which is after the position $a(j)$ of movie j on $A's$ list.



- For example 1 and 2 do not form an inversion because $a(1) < a(2)$ ($a(1) = 1$ and $a(2) = 11$ because $a(1)$ is on the first and $a(2)$ is on the 11^{th} place in A);

- However, for example 4 and 7 do form an inversion because $a(7) < a(4)$ ($a(7) = 5$ because seven is on the fifth place in A and $a(4) = 8$)

Counting the number of inversions

- An easy way to count the total number of inversions between two lists is by looking at all pairs $i < j$ of movies on one list and determining if they are inverted in the second list, but this would produce a quadratic time algorithm, $T(n) = \Theta(n^2)$.

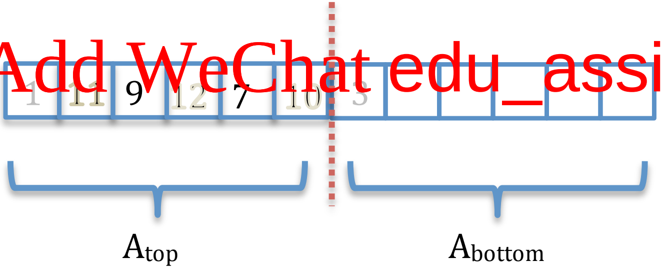
- We in ti ER strategy.

- Clearly, since the total number of pairs is quad cannot afford to inspect all possible pairs.

- The main idea is to tweak the MERGE-SORT algorithm, by extending it to recursively both sort an array A **and** determine the number of inversions in A.

- We split the array A into two (approximately) equal parts $A_{top} = A[1 \ldots \lfloor n/2 \rfloor]$ and $A_{bottom} = A[\lfloor n/2 \rfloor + 1 \ldots n]$.

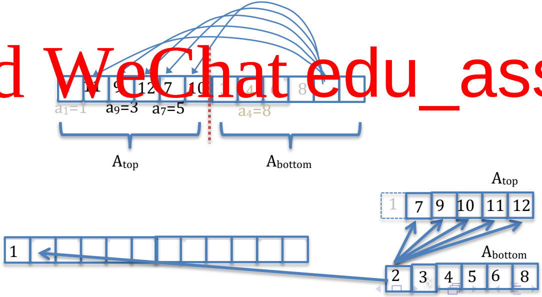
- Note that the total number of inversions in array A is equal to the sum of the number of inversions $I(A_{top})$ in A_{top} (such as 9 and 7) plu h as 4 and 2) p e two hal



- We now recursively sort arrays A_{top} and A_{bottom} also obtaining in the process the number of inversions $I(A_{top})$ in the sub-array A_{top} and the number of inversions $I(A_{bottom})$ in the sub-array A_{bottom}.

- We now merge the two sorted arrays A_{top} and A_{bottom} while counting the number of inversions $I(A_{top}, A_{bottom})$ which are across the two sub-arrays.

- Wh
elem
rem ing
in A $_{top}$ and
A_{bottom}.



- Whenever the next smallest element among all elements in both arrays is an element in A_{top}, such an element clearly is not involved in any inversions across the two arrays (such as 1, for example).

- Aft
  nu $_{bottom}$.

- The total number of inversions $I(A$ i ned as:

$$I(A) = I(A_{top}) + I(A_{botto}$$

- **Next:** we study applications of divide and conquer to arithmetic of very large integers.

Basics revisited: how do we add two numbers?

```
 C C C C C     carry
   X X X X X    first integer
+  X X X X X    second integer
  ---
 X X X X X X    result
```


- adding 3 bits can be done in constant time;
- the whole algorithm runs in linear time i.e.,

can we do it faster than in linear time?

- no, because we have to read every bit of the input
- no asymptotically faster algorithm

Basics revisited: how do we multiply two numbers?

```
      X X X X  <- first input integer
    * X X X X  <- second input integer
    -------
      X X X X  \
    X X X X     \ O(n^2) intermediate operations:
  X X X X       / O(n^2) el
X X X X        /  + O(n^2) elementa
-----
X X X X X X X X  <-  result of length 2n
```


- We assume that two X's can be multiplie X could be a bit or a digit in some other base).
- Thus the above procedure runs in time $O(n^2)$.
- Can we do it in **LINEAR** time, like addition?
- **No one knows!**
- "Simple" problems can actually turn out to be difficult!

Let us try a divide-and-conquer algorithm:
take our two input numbers A and B, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0 \qquad XX\ldots X\;XX\ldots X$$

- $A_0,\ B_0$ - the least significant bits; $A_1,\ B_1$ the most significant bits.
- AB can now be calculated as follows:

$$AB = A_1 B_1 2^n + (A_1 B_0 + \quad (1)$$

What we mean is that the product AB can be calculated recursively by the following program:

function $\textsc{Mult}(A, B)$
    if $|A| = |B| = 1$ **then return** AB
    else
        $A_1 \leftarrow$ MoreSignificantPart(A);
        $A_0 \leftarrow$ LessSignificantPart(A);
        0 1
        $Z \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_0)$;
        $W \leftarrow \textsc{Mult}(\mathrm{A}_1, \mathrm{B}_1)$;
        return $W\,2^n + (Y + Z)\,2^{n/2}$
    end if
end function

How many steps does this algorithm take?

Each multiplication of two n digit numbers is replaced by four multipli ... $_0B_0$,
plus we ha

$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \qquad (2)$$

Claim: if $T(n)$ satisfies

$$T(n) = 4T\left(\frac{n}{2}\right) + c\,n \qquad (3)$$

then

Proof: By "fast" induction. We assume it is true for $n/2$:

$$T\left(\frac{n}{2}\right) = \left(\frac{n}{2}\right)^2 (c \quad -$$

and prove that it is also true for n:

$$T(n) = 4\,T\left(\tfrac{n}{2}\right) + c\,n = 4\left(\left(\tfrac{n}{2}\right)^2 (c+1) - \tfrac{n}{2}\,c\right) + c\,n$$

$$= n^2(c+1) - 2c\,n + c\,n = n^2(c+1) - c\,n$$

Can we do multiplication faster than $O(n^2)$?

Thus, if $T(n)$ satisfies $\quad T(n) = 4\,T\left(\frac{n}{2}\right) + c\,n \quad$ then

$$T(n) = n^2(c+1) - c\,n = O(n^2)$$

i.e., we gained **nothing** with our divide-and-conquer!

Is there a sm $O(n^2)$
many ste

Remarkably, there is, but first some history:
In 1952, one of the most famous mathematicians of th tury,
Andrey Kolmogorov, conjectured that you cann

$\Omega(n^2)$ elementary operations. In 1960, Karatsuba, then a 23-year-old student, found an algorithm (later it was called "divide and conquer") that multiplies two n-digit numbers in $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.58\ldots})$ elementary steps, thus disproving the conjecture!! Kolmogorov was shocked!

How did Karatsuba do it??

Take again our two input numbers A and B, and split them into two halves:

- AB can now be calculated as follows:

$$AB = A_1B_12^n + (A_1B_0 + A_0B_1)2^{\frac{n}{2}}$$

$$= A_1B_12^n + ((A_1 + A_0)(B_1 + B_0) - A_1B_1 - A_0B_0)2^{\frac{n}{2}} + A_0B_0$$

- So we have saved one multiplication at each recursion round!

- Thus, the algorithm will look like this:

function $\textsc{Mult}(A, B)$
    if $|A| = |B| = 1$ **then return** AB
    else
        $A_1 \leftarrow \text{MoreSignificantPart}(A)$;
        $A_0 \leftarrow \text{LessSignificantPart}(A)$;
        $X \leftarrow \textsc{Mult}(\mathrm{A}_0, \mathrm{B}_0)$;
        $W \leftarrow \textsc{Mult}(A_1, B_1)$;
        $Y \leftarrow \textsc{Mult}(U, V)$;
        return $W\,2^n + (Y - X - W)$
    end if
end function

- How fast is this algorithm?

Clearly, the run time $T(n)$ satisfies the recurrence

$$T(n) = 3\,T\left(\frac{n}{2}\right) + c\,n$$

and this implies (by replacing n with $n/2$)

$$T\left(\frac{n}{2}\right) = 3\,T\left(\frac{n}{2^2}\right) + c\,\frac{n}{2}$$

and by replaci

So we get $T(n) = 3\,\underbrace{T\left(\frac{n}{2}\right)} + c\,n = 3\left(\underbrace{3T\left(\frac{n}{2^2}\right) + c\,\frac{n}{2}}\right)$

$$= 3^2\,\underbrace{T\left(\frac{n}{2^2}\right)} + c\frac{3n}{2} + c\,n = 3^2\left(\underbrace{3T\left(\frac{n}{2^3}\right) \quad \frac{}{2^2}}\right) \quad \frac{}{2}$$

$$= 3^3\,\underbrace{T\left(\frac{n}{2^3}\right)} + c\frac{3^2 n}{2^2} + c\frac{3n}{2} + c\,n = 3^3\left(\underbrace{3T\left(\frac{n}{2^4}\right) + c\frac{n}{2^3}}\right) + c\frac{3^2 n}{2^2} + c\frac{3n}{2} + c\,n = \ldots$$

The Karatsuba trick

$$T(n) = 3T\left(\frac{n}{2}\right) + c\,n = 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + c\,n = 3^2\,\underbrace{T\left(\frac{n}{2^2}\right)} + c\frac{3n}{2} + c\,n$$

$$= 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + c\frac{3n}{2} + c\,n = 3^3\,T\left(\frac{n}{2^3}\right) + c\frac{3^2 n}{2^2} + c\frac{3n}{2} + c\,n$$

$$= 3^3\,\underbrace{T\left(\frac{n}{2^3}\right)} + c\,n\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right)$$

$$= 3^3\underbrace{\left(3T\left(\frac{n}{2^4}\right) + c\frac{n}{2^3}\right)} + c\,n\left(\frac{3^2}{2^2} + \frac{3}{2} + 1\right)$$

$$= 3^4 T\left(\frac{n}{2^4}\right) + c\,n\left(\frac{3^3}{2^3} + \frac{3^2}{2^2} + \frac{3}{2} + 1\right)$$

$$\dots$$

$$= 3^{\lfloor \log_2 n \rfloor} T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + c\,n\left(\left(\frac{3}{2}\right)^{\lfloor \log_2 n \rfloor - 1} + \ldots + \frac{3^2}{2^2} + \frac{3}{2} + 1\right)$$

$$\approx 3^{\log_2 n} T(1) + c\,n\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} = 3^{\log_2 n} T(1) + 2c\,n\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

So we got

$$T(n) \approx 3^{\log_2 n} T(1) + 2c\,n\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

We now us [illegible]

$$T(n) \approx n \text{ [illegible] } \left(n^{\log_2 3 - 1} - 1\right)$$

$$= n^{\log_2 3} T(1) + 2c\,n^{\log_2 3} - 2c\,n$$

$$= O(n^{\log_2 3}) = O(n^{1.58\ldots}) \ll n^2$$

Please review the basic properties of logarithms and the asymptotic notation from the review material (the first item at the class webpage under "class resources".)

A Karatsuba style trick also works for matrices: Strassen's algorithm for faster matrix multiplication

- If we want to multiply two $n \times n$ matrices P and Q, the product will be a matrix R also of size $n \times n$. To obtain each of n^2 entries in R we do n multiplications, so matrix product by brute force is $\Theta(n^3)$.

- How

- We s … $n/2$:

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \qquad Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

- Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{4}$$

A Karatsuba style trick also works for matrices: Strassen's algorithm for faster matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \qquad (5)$$

- We o [illegible]

- Prima facie, there are 8 matrix multiplications, each r [illegible] $\left(\frac{n}{2}\right)$ and 4 matrix additions, each running in time [illegible] calculation would result in time complexity govern [illegible]

$$T(n) = 8T\left(\frac{n}{2}\right) + c\,n$$

- The first case of the Master Theorem gives $T(n) = \Theta(n^3)$, so nothing gained.

Strassen's algorithm for faster matrix multiplication

- However, we can instead evaluate:

$$A = a\,(f-h);\qquad B=(a+b)\,h;\qquad C=(c+d)\,e\qquad D=d\,(g-e);$$
$$E=(a+d)\,(e+h);\qquad F=(b-d)(g+h);\qquad H=(a-c)\,(e+f).$$

- We now obtain

$$E+D-B \quad \ldots \quad d\,g+b\,h-d\,h)$$

$$A \ldots$$
$$C+D=(c\,e+d\,e)+(d\,g-d\,e)=c\,e+d\,g=t;$$
$$E+A-C-H=(a\,e+d\,e+a\,h+d\,h)+(a\,f- \quad \ldots \quad +a\,f-c\,f)$$
$$=c\,f+d\,h=u.$$

- We have obtained all 4 components of C u and 18 matrix additions/subtractions.
- Thus, the run time of such recursive algorithm satisfies $T(n)=7T(n/2)+O(n^2)$ and the Master Theorem yields $T(n)=\Theta(n^{\log_2 7})=O(n^{2.808})$.
- In practice, this algorithm beats the ordinary matrix multiplication for $n>32$.

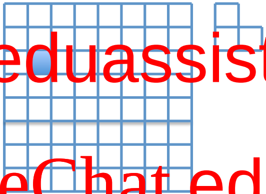
Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??
2. Can we avoid messy computations like:

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{2}\right) + cn = 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + cn = 3^2 T\left(\frac{n}{2^2}\right) + c\frac{3n}{2} + cn \\
&= 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + c\frac{3n}{2} + cn = 3^3 T\left(\frac{n}{2^3}\right) + c\frac{3^2 n}{2^2} + c\frac{3n}{2} + cn \\
&= 3^3\left(3T\left(\frac{n}{2^4}\right) + c\frac{n}{2^3}\right) + c\frac{3^2 n}{2^2} + c\frac{3n}{2} + cn \\
&= 3^4 T\left(\frac{n}{2^4}\right) + cn\left(\frac{3^3}{2^3} + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) \\
&\ldots \\
&= 3^{\lfloor \log_2 n \rfloor} T\left(\frac{n}{\lfloor 2^{\log_2 n} \rfloor}\right) + cn\left(\left(\frac{3}{2}\right)^{\lfloor \log_2 n \rfloor - 1} + \ldots + \frac{3^2}{2^2} + \frac{3}{2} + 1\right) \\
&\approx 3^{\log_2 n} T(1) + cn\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \\
&= 3^{\log_2 n} T(1) + 2cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)
\end{aligned}
$$

PUZZLE!

You are given a $2^n \times 2^n$ board with one of its cells missing (i.e., the board has a hole); the position of the missing cell can be arbitrary. You are also given a supply of “dominoes” each containing 3 such squares; see the figure.



Your task is to design an algorithm which covers the entire board with such “dominoes” except for the hole.

Hint: Do a divide-and-conquer recursion!

That's All, Folks!!