School of Computer Science and En
University of New South W

10. LINEAR PROGRAMMING

Linear Programming problems - Example 1

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  - the price of all food per day is as low as possible.

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- Note that all constraints and the objective function, are **linear**.

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- In order to win, you have to get at least 51% of each of the city, suburban and rural votes.

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- In order to win, you have to get at least 51% of each of the city, suburban and rural votes.
- You wish to win the election by cleverly making a promise that **appears** that it will blow as small hole in the budget as possible, i.e., that the total cost of your promises is as low as possible.

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$$0.05x_b \qquad + 0.12x_p \quad 0.51 \quad \text{(securing majority of city votes)}$$

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- The second example is an example of an **Integer Linear Programming problem**, which requires all the solutions to be integers.
- Such problems are MUCH harder to solve than the "plain" Linear Programming problems whose solutions can be real numbers.

Linear Programming problems

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- Let the boldface $\mathbf{x}$ represent a (column) vector, $\mathbf{x} = \langle x_1 \dots x_n \rangle^\mathsf{T}$.
- To get a more compact representation of linear programs we introduce a partial ordering on vectors $\mathbf{x} \in \mathbf{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_j \leq y_j$ for all $1 \leq j \leq n$.

- Letting $\mathbf{c} = \langle c_1 \ldots c_n \rangle^{\mathsf{T}} \in \mathbf{R}^n$ and $\mathbf{b} = \langle b_1 \ldots b_m \rangle^{\mathsf{T}} \in \mathbf{R}^m$, and letting A be the matrix $A = (a_{ij})$ of size $m \times n$, we get that the above problem can be formulated simply as:

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- Thus, to specify a Linear Programming optimisation we have to provide a triplet $(A, \mathbf{b}, \mathbf{c})$;
- This is the usual form which is accepted by most standard LP solvers.

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- Equality constraints of the form $\sum_{i=1}^{n} a_{ij}x_i = b_j$ can be replaced by two inequalities: $\sum_{i=1}^{n} a_{ij}x_i \geq b_j$ and $\sum_{i=1}^{n} a_{ij}x_i \leq b_j$; thus, we can assume that all constraints are inequalities.

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- If $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector, we let $|\mathbf{x}|$ $|x_1|, \ldots, |x_n|$ problems are naturally translated into constraints of the form $|A\mathbf{x}| \leq \mathbf{b}$. This also poses no problem because we can replace such constraints with two linear constraints: $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq \mathbf{b}$ because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

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- As an example, let us consider the following optimisation problem:

$$\text{maximize} \quad z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \tag{3}$$

subj

$$\leq 30 \tag{4}$$

$$\leq 24 \tag{5}$$

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- How large can the value of the objective $+ 2x_3$ be, without violating the constraints?

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- How large can the value of the objective be, without violating the constraints?
- If we add inequalities (4) and (5), we get

$$3x_1 + 3x_2 + 8x_3 \leq 54 \quad (8)$$

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- Since all variables are constrained to be non-negative, we are assured that

$$3x_1 + x_2 + 2x_3 \leq 3x_1 + 3x_2 + 8x_3 \leq 54$$

Linear Programming - Standard Form

$$\begin{aligned}
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& && 2x_1 + 2x_2 + 5x_3 \leq 24 && (5)\\
& && 4x_1 + x_2 + 2x_3 \leq 36 && (6)
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- Then, summing up all these inequalities and factoring, we get

$$x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3) \leq 30y_1 + 24y_2 + 36y_3$$

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$$\begin{aligned}
y_1 + 2y_2 + 4 \\
y_1 + 2y_2 + \\
3y_1 + 5y_2 + 2
\end{aligned}$$

y_1, y_2

then

$$3x_3 + x_2 + 2x_3 \leq x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3)$$

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with constraints:

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y_1, y_2

$$y_1 + 2y_2 + 4$$
$$y_1 + 2y_2 +$$
$$3y_1 + 5y_2 + 2$$

then

$$3x_3 + x_2 + 2x_3 \leq x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3)$$

Combining this with (9) we get:

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$$

Linear Programming - Standard Form

- Consequently, in order to find as tight upper bound for our objective $z(x_1, x_2, x_3)$ of the problem P:

$$\begin{aligned} &\text{maximize:} && z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 && (3)\\ &\text{with constraints:} && x_1 + x_2 + 3x_3 \leq 30 && (4)\\ & && 2x_1 + 2x_2 + 5x_3 \leq 24 && (5) \end{aligned}$$

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then $z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$ will be a tight upper bound for $z(x_1, x_2, x_3)$

- The new problem P^* is called the *dual problem* for the problem P.

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- So, at the first sight, looking for the multipliers much, because it only reduced a maximisation pr hard minimisation problem.
- It is now useful to remember how we proved that the Ford - Fulkerson Max Flow algorithm in fact produces a **maximal flow**, by showing that it terminates only when we reach the capacity of a **minimal cut**.

Linear Programming - primal/dual problem forms

- The original, *primal* Linear Program P and its *dual* Linear Program can be easily described in the most general case:

$$P: \text{ maximize } z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to the constraints } \sum_{j=1}^{n} a_{ij} x_j \leq b_i; \quad 1 \leq i \leq m$$

$$\text{subject to the constraints } \sum_{i=1}^{m} a_{ij} y_i \geq c_j; \quad 1 \leq j \leq n$$

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or, in matrix form,

$$P: \text{ maximize } z(\mathbf{x}) = \mathbf{c}^{\mathsf{T}}\mathbf{x}, \text{ subject to the constraints } A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq 0;$$

$$P^*: \text{ minimize } z^*(\mathbf{y}) = \mathbf{b}^{\mathsf{T}}\mathbf{y}, \text{ subject to the constraints } A^{\mathsf{T}}\mathbf{y} \geq \mathbf{c} \text{ and } \mathbf{y} \geq 0.$$

Weak Duality Theorem

- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^\mathsf{T}\mathbf{x}$ might be.

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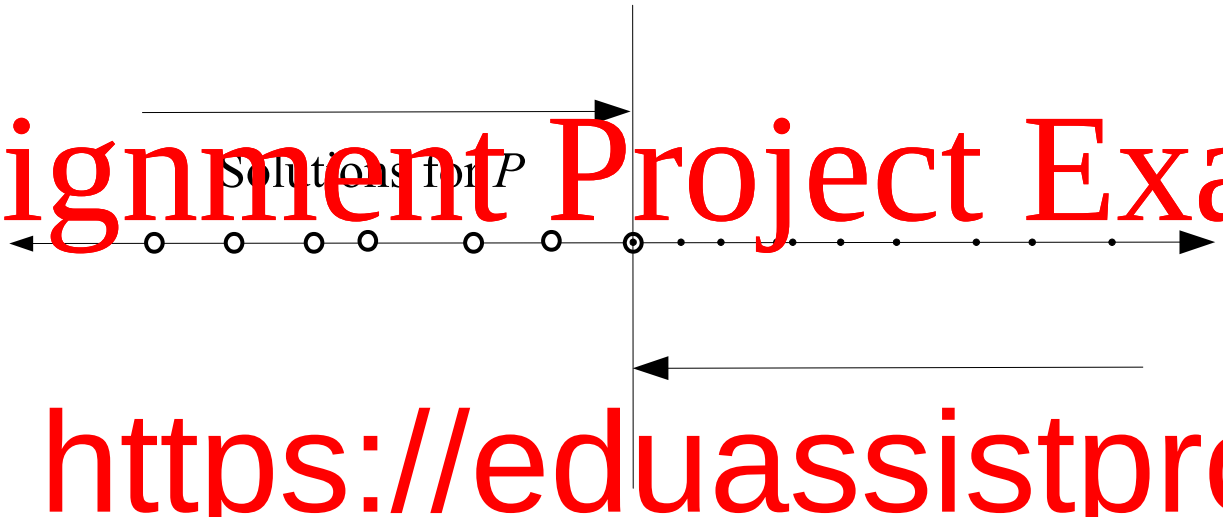
Proof Assume P and P^* are in standard form, respectively; we can always convert P^* and P to obtain

$$z(x) = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i} \equiv z^*(y)$$

- Thus, the value of (the objective of P^* for) any feasible solution of P^* is an upper bound for the set of all values of (the objective of P for) all feasible solutions of P, and
- every feasible solution of P is a lower bound for the set of feasible solutions for P^*.

Weak Duality Theorem



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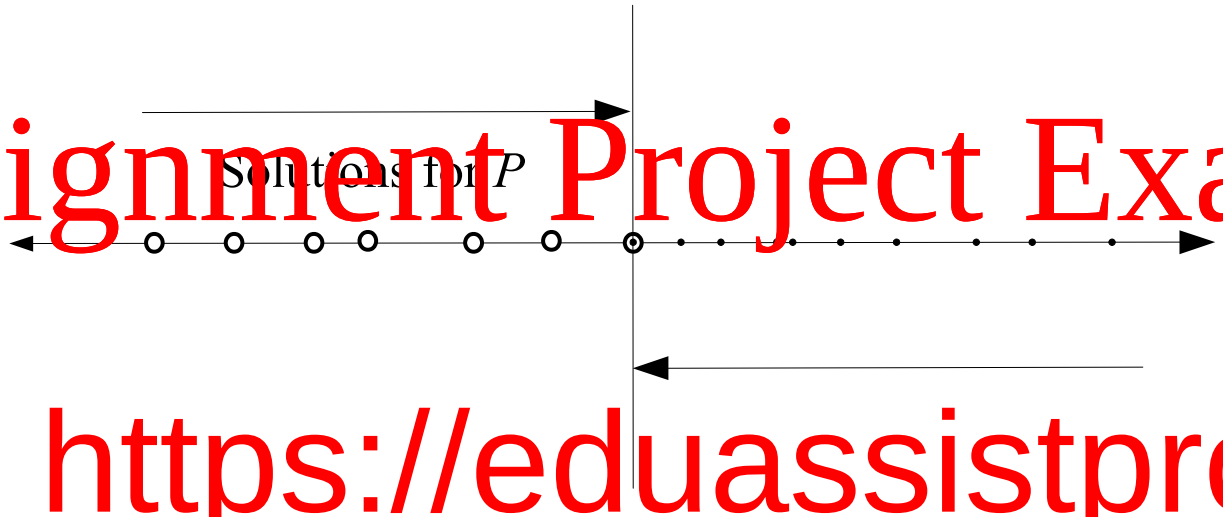

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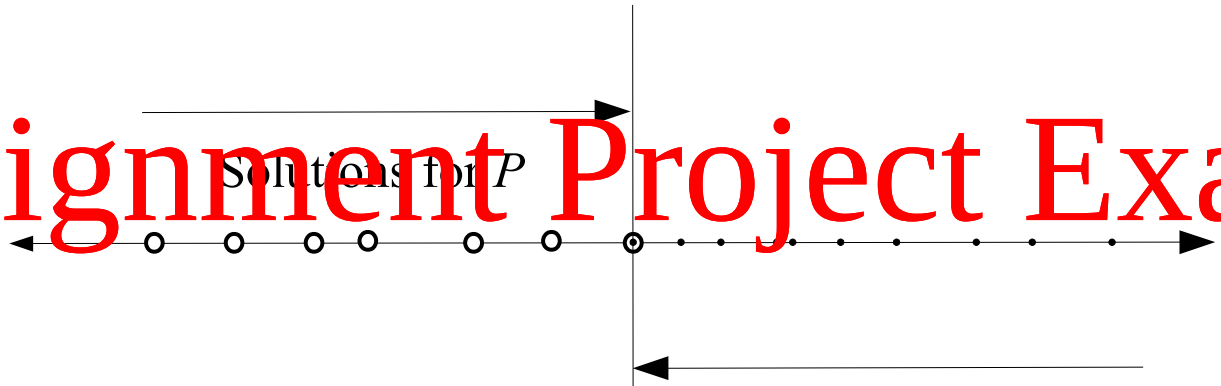

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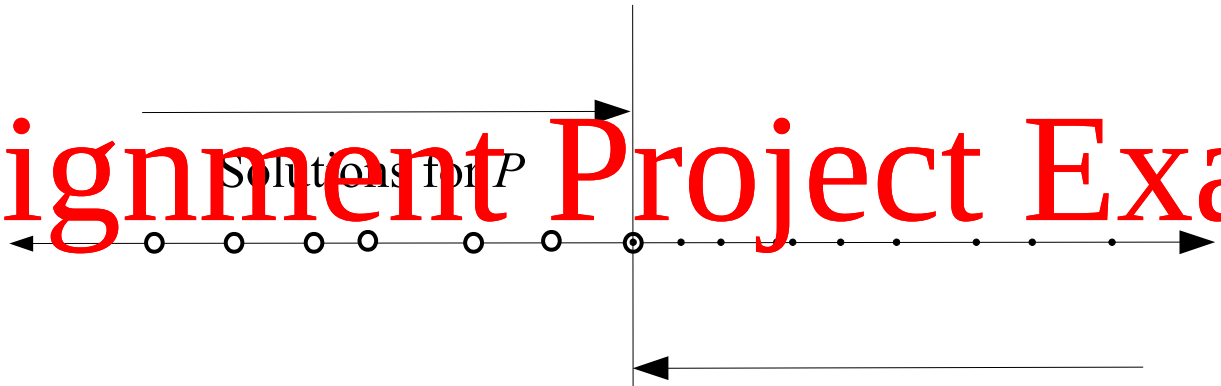

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- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for P, because it stops at a value of the primal objective which is also a value of the dual objective.
- See the Lecture Notes for the details and an example of how the SIMPLEX algorithm runs.

PUZZLE!!

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