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11. INTRACTABILITY

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- We denote this by $A \in \mathbf{P}$.

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  - This can only change the constants involved in t $T(n) = O(n^k)$ but not the asymptotic bound.

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- In fact, every precise description without artificial redundancies will do.

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  2. $B$ can be computed in polynomial time (in the length of $x$).
- We call $y$ a *certificate* for $x$.
- For example, decision problem "integer $x$ is composite" is in $\mathbf{NP}$ because $A(x)$ is true just in case there exists integer $y$ such that $1 < y < x$ and $x$ is divisible by $y$" is true.

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  2. $y$ is at most $n^k$ for some $k$.
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- Clearly, the problem "$x$ is divisible by $y$" is decidable by an algorithm which runs in time polynomial in the length of $x$ only.
- In fact, "integer $x$ is not prime" is actually decidable in (deterministic) polynomial time, but this is a hard theorem to prove.

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- (Vertex Cover) Instance: a graph $G$ and an integer $k$. Problem: "There exists a subset $U$ consisting of at most $k$ vertices of $G$ (called a vertex cover of $G$) such that each edge has at least one end belonging to $U$."

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  Clearly, given an evaluation of the propositional variables one can determine in polynomial time if the formula is true for such an evaluation.

- If each clause $C_i$ involves exactly three variables we call such decision problem 3SAT.

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- However, so far, no one has been able to prove (or disprove) that this is indeed the case, despite a huge effort of very many very famous people!!
- Conjecture that $NP$ is a strictly larger class of decision problems than $P$ is known as "$P \neq NP$" hypothesis, and it is widely believed that it is one of the hardest open problems in the whole of Mathematics!!

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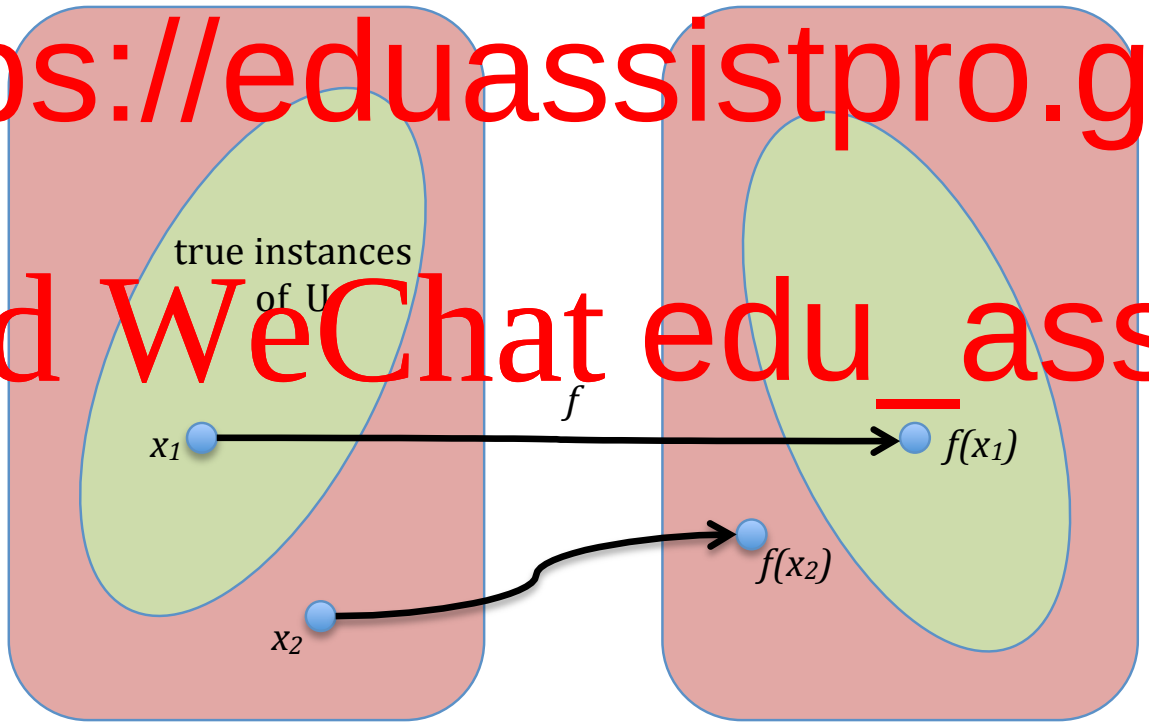

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with the variables $Q_1, Q_2, Q_3$:

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- Clearly, (??) can be obtained from (??) using a simple polynomial time algorithm.

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- **Theorem:** Every NP problem is polynomially reducible to the SAT problem.
- This means that for every NP decision problem $U(x)$ there exists a polynomial time computable function $f(x)$ such that:
  1. for every instance $x$ of $U$, $f(x)$ is a propositional formula $\Phi_x$;
  2.
- An N m is poly
- Thus, Cook's Theorem says that SAT is NP complete.
- NP complete problems are in a sense universal: if we had a solves one single NP complete problem $U$, t to solve every other NP problem:
- A solution of an instance $x$ of any other NP problem $V$ could simply be obtained by:
  1. computing in polynomial time the reduction $f(x)$ of $V$ to $U$,
  2. then running the algorithm that solves $U$ on instance $f(x)$.

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- Unfortunately, this cannot be farthest from the truth!
- A vast number of practically important decision problems are NP complete!

# NP complete problems are everywhere!

- **Traveling Salesman Problem**

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- Think of a mailman which has to deliver mail to several addresses and then return to the post office. Can he do it while traveling less than $L$ kilometres in total?

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- In graph theoretic terms: Is it possible to color the vertices of a graph $G$ with at most $K$ colors so that no edge has both vertices of the same color.

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- As we will see, many other practically important problems are NP complete.

# NP hard problems

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- Think of a mailman having to deliver mail to several addresses while having to travel as small total distance as possible.

# NP hard problems



- The Traveling Salesman Optimisation Problem is clearly NP hard:
- using a "black box" for solving it, we can solve the Traveling Salesman Decision problem:
- Giv ycle cont
- We d determining the length of the cycle of minimal possible length and comparing the length of such a cycle with $L$.
- Since all other NP problems are polynomial time redu Salesman Decision problem (which is NP complete) problem is solvable using a "black box" for the Traveling Salesman Optimisation Problem.

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  - prove that the problem is indeed NP hard, to justify not trying solving the problem exactly;
  - look for an approximation algorithm which provides a feasible sub-optimal solution that it is not too bad.

# Proving NP completeness

Warning: sometimes distinction between a problem in P and an NP complete problem can be subtle!

| in P | NP complete |
| --- | --- |
| • Given a graph $G$ and two vertices $s$ an of len | • Given a graph $G$ and two vertices ? |
| • G CNF form such that every clause has at most **two** propositional variables, does the formula have a satisfying assignment? | a in CNF form such that every clause has at m nal vari sati |
| • Given a graph $G$, does $G$ have a tour where every **edge** is traversed exactly once? (An *Euler tour*.) | • Given a graph $G$, does $G$ have a tour where every **vertex** is visited exactly once? (A *Hamiltonian* cycle.) |

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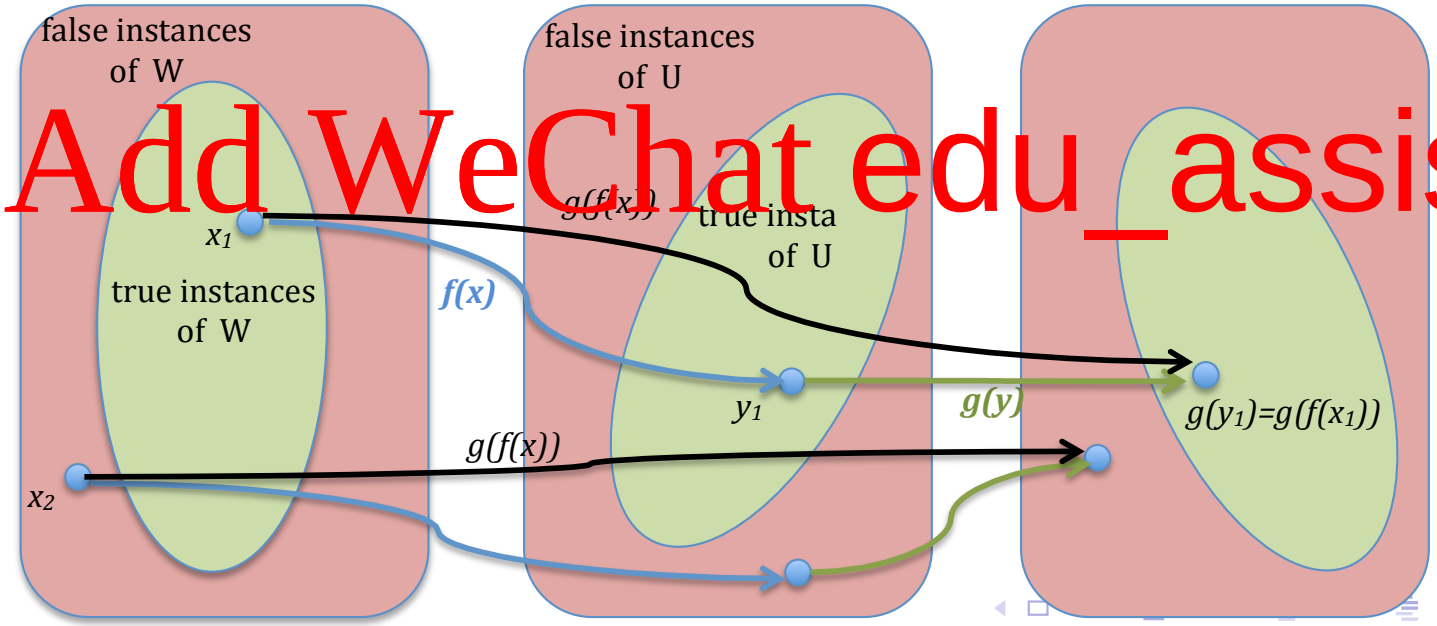

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  - In total, the computation of $g(f(x)$ terminates in at most $P(|x|) + Q(P(|x|))$ many steps, which is a polynomial bound in $|x|$.

# Reducing 3SAT to VC

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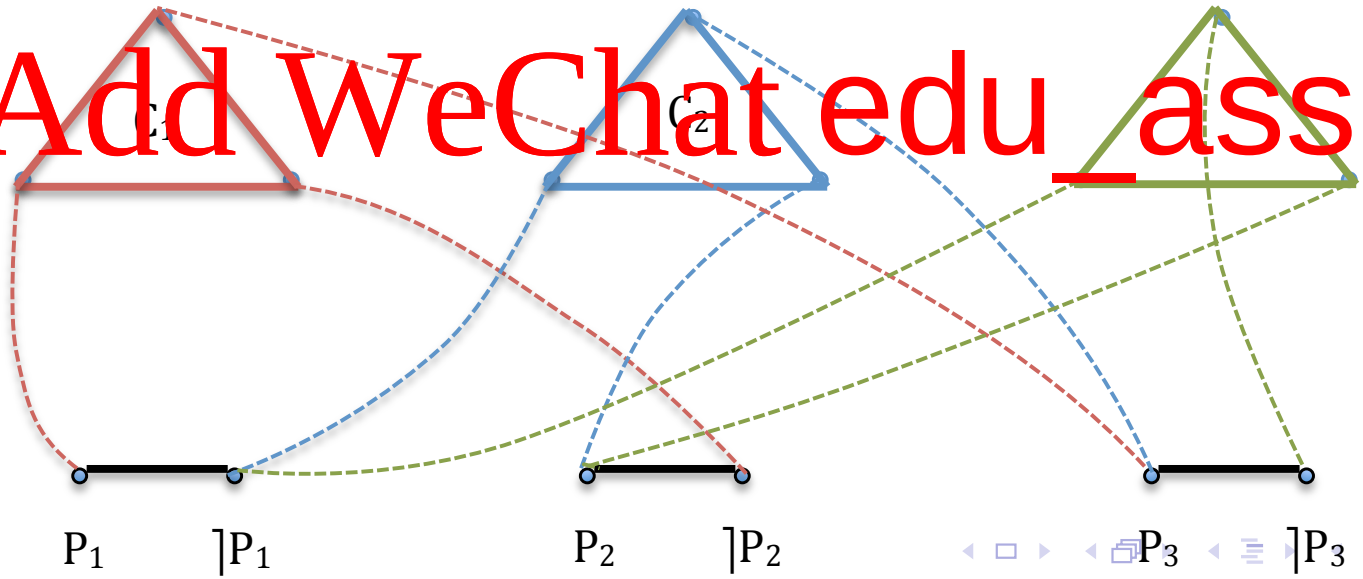

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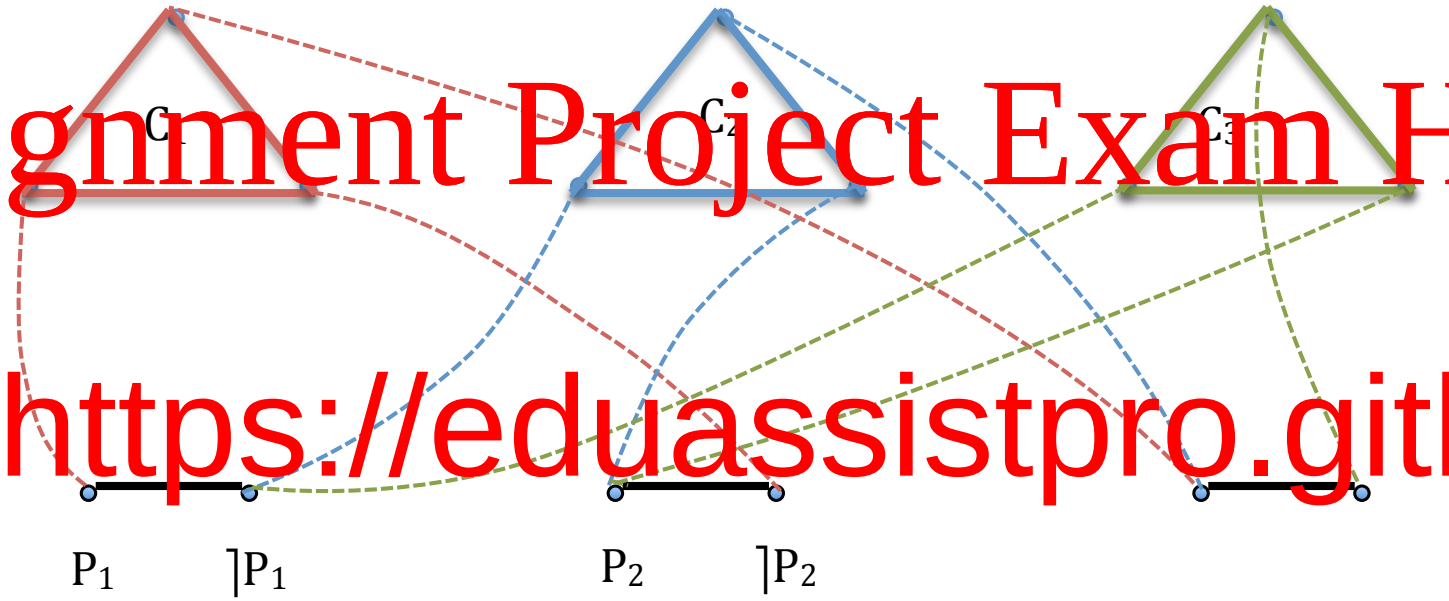

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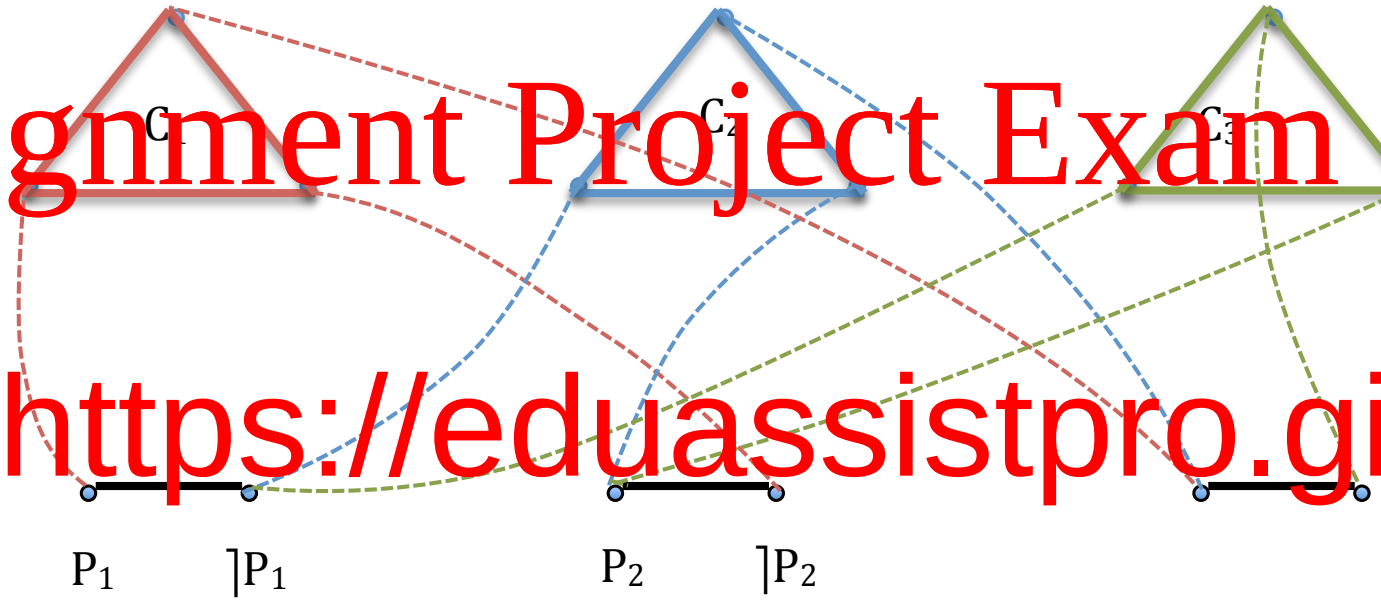




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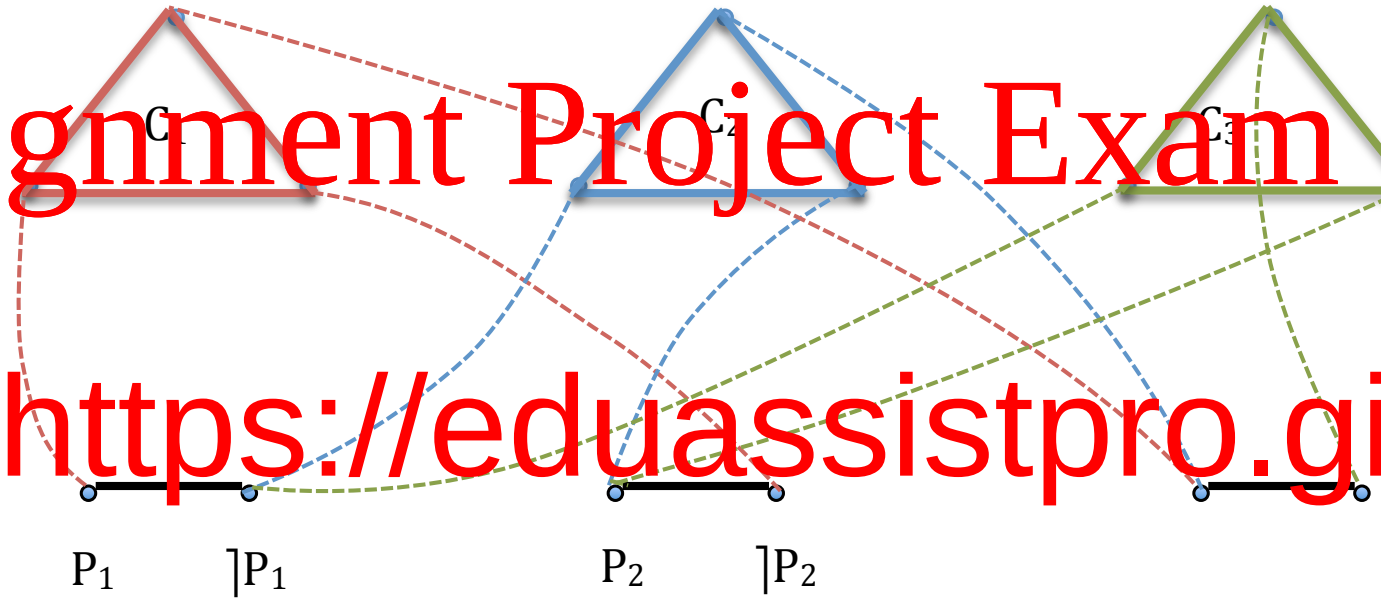




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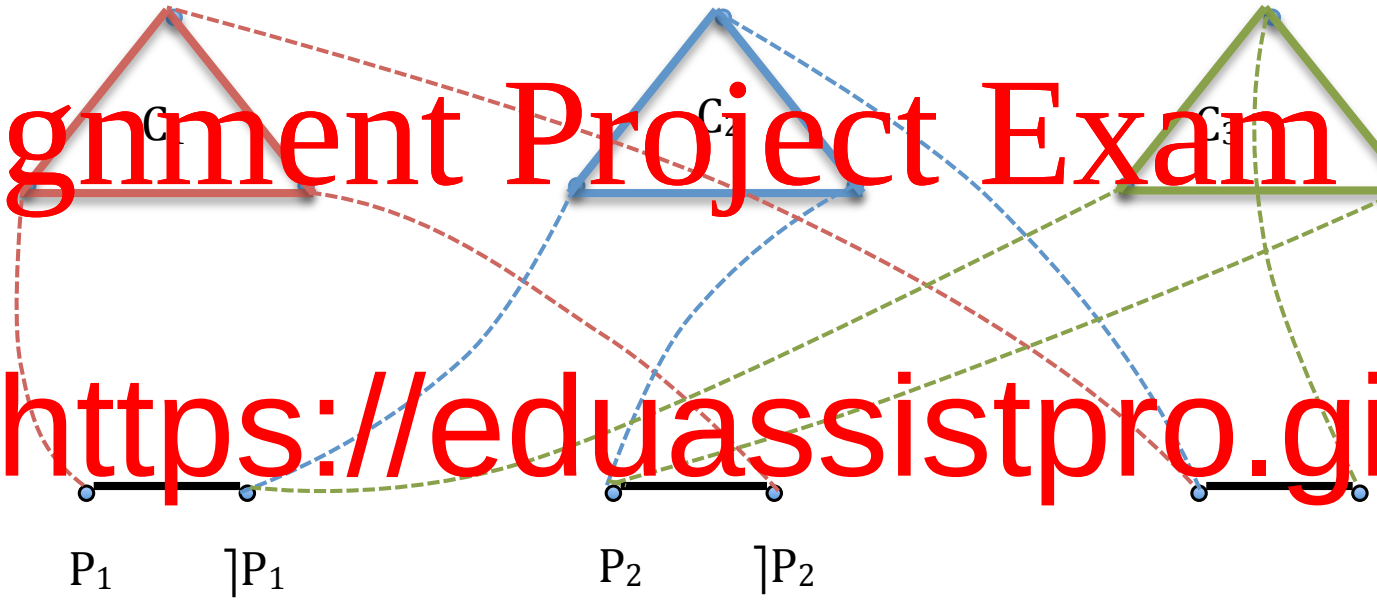




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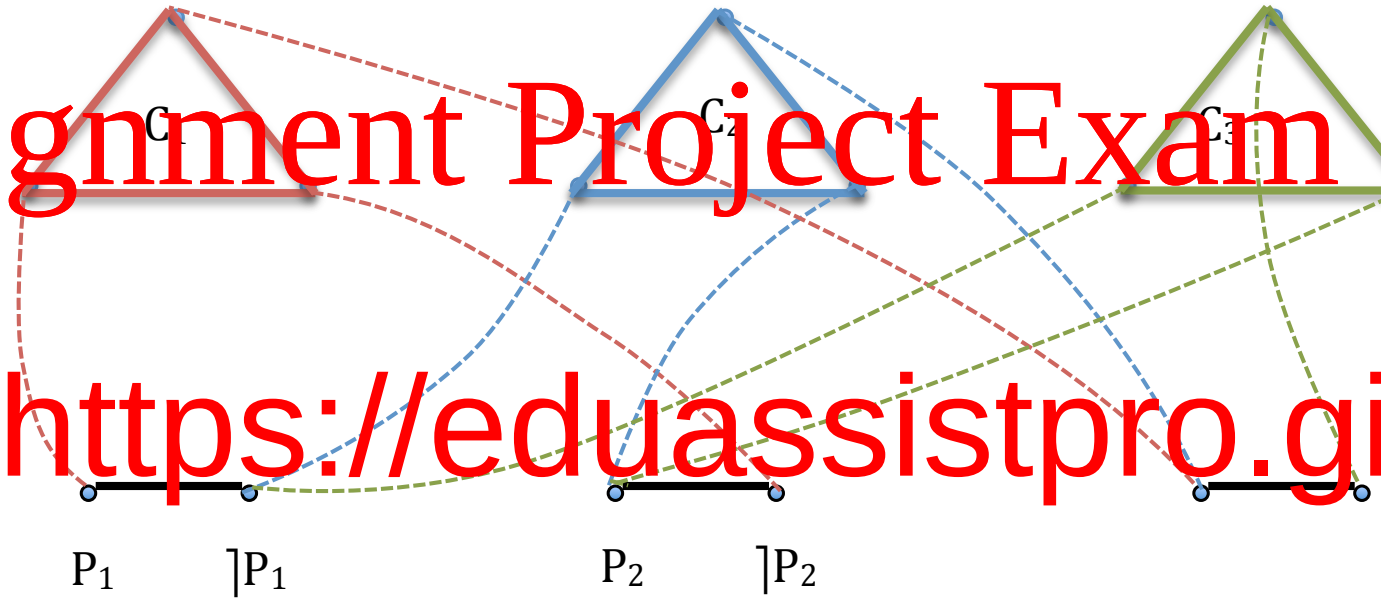




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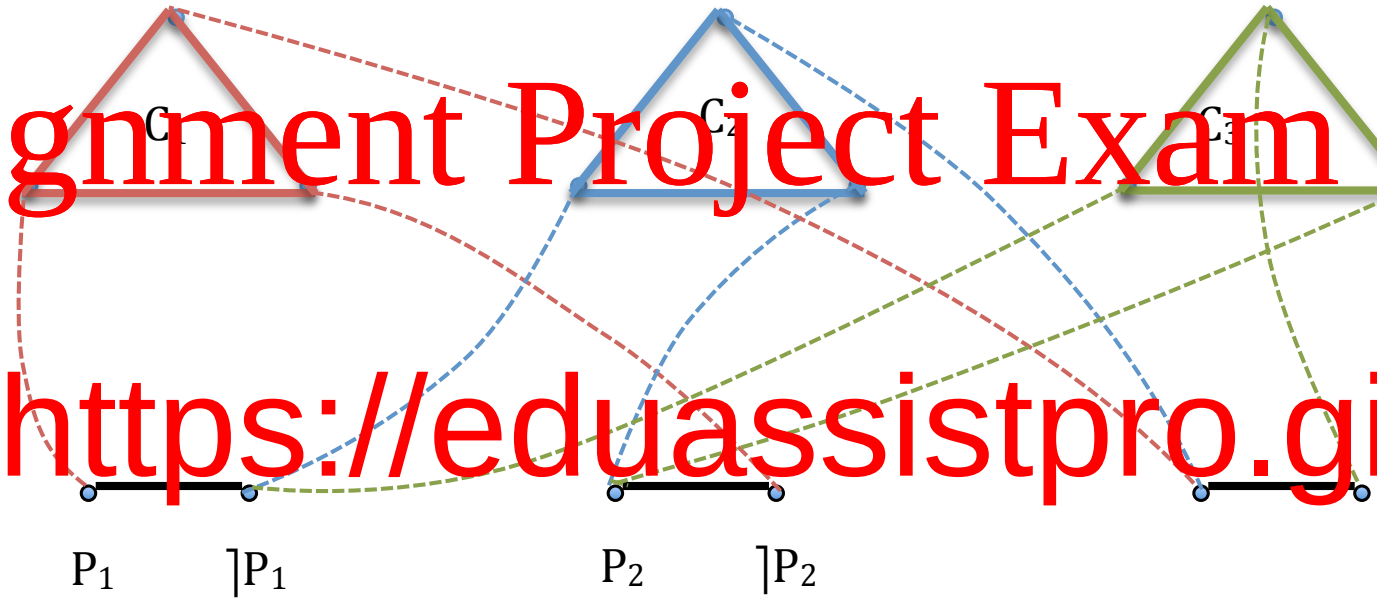




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- This is in total $2M + N$ points; thus each triangle must have *exactly* two vertices chosen and each segment must have *exactly* one of its ends chosen.

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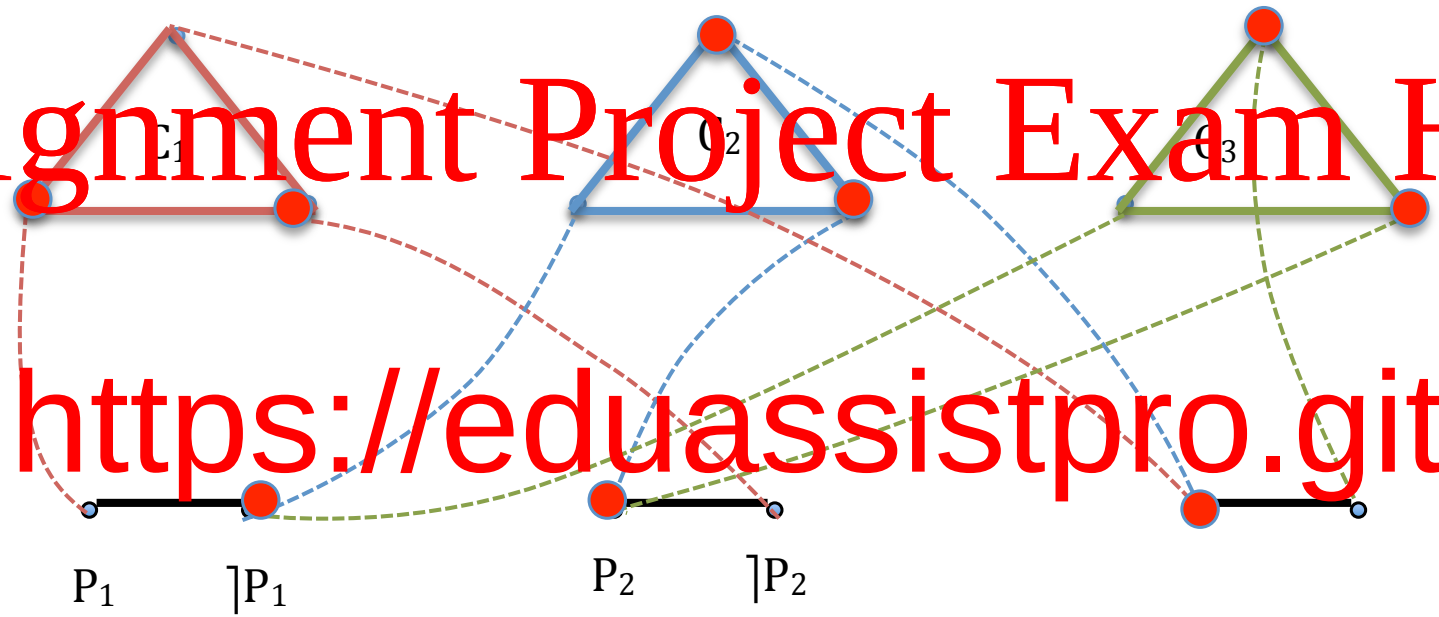




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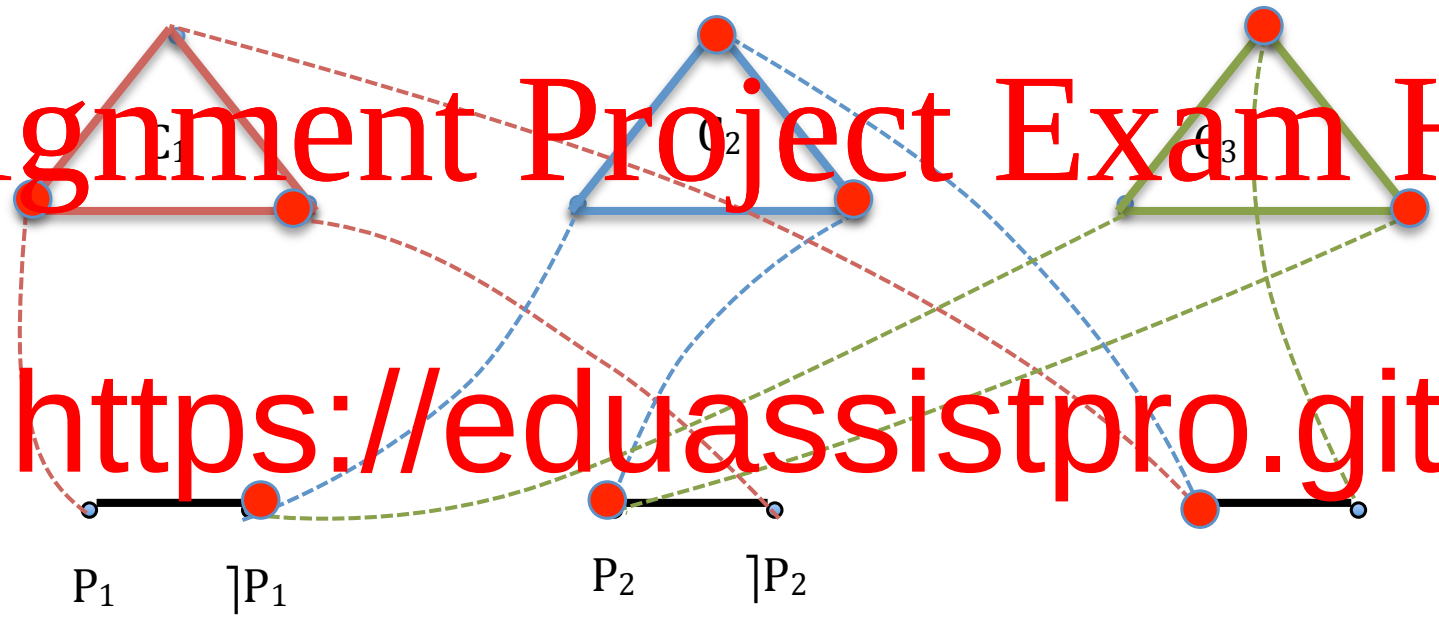




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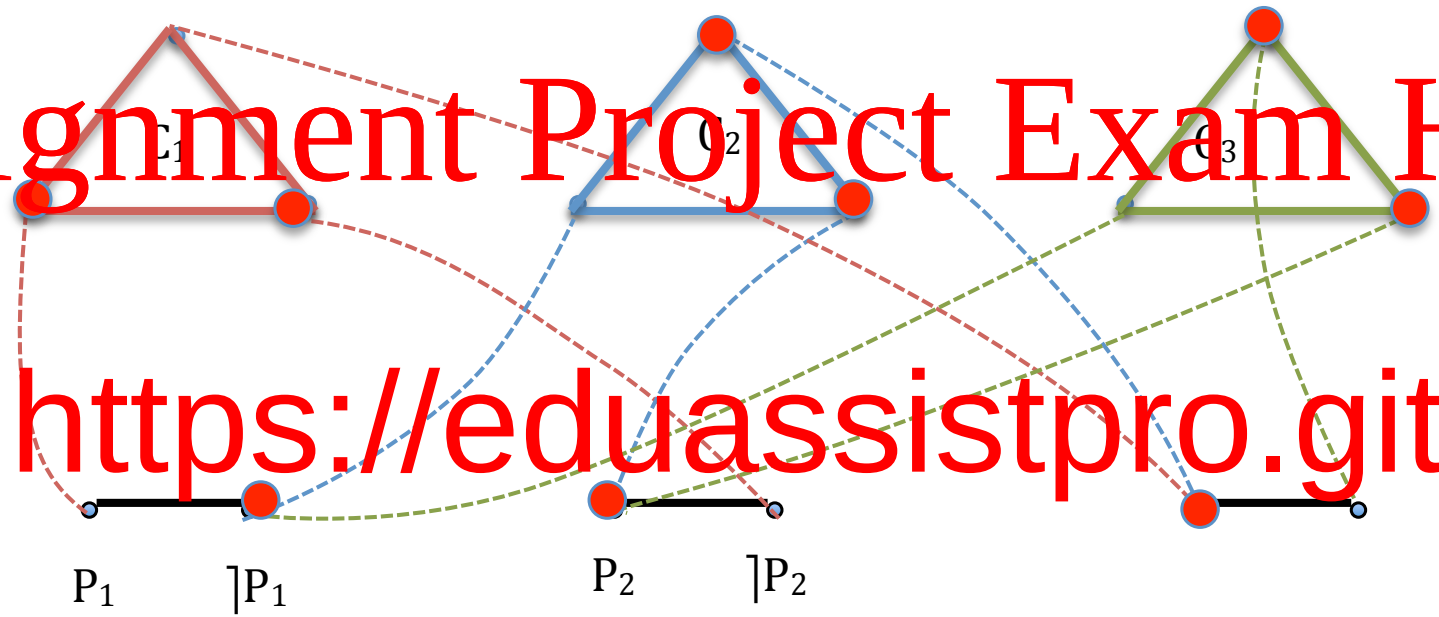




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- In a vertex cover of such a graph every uncovered vertex of each triangle must be connected to a covered end of a segment, which guarantees that the clause corresponding to each triangle is true.

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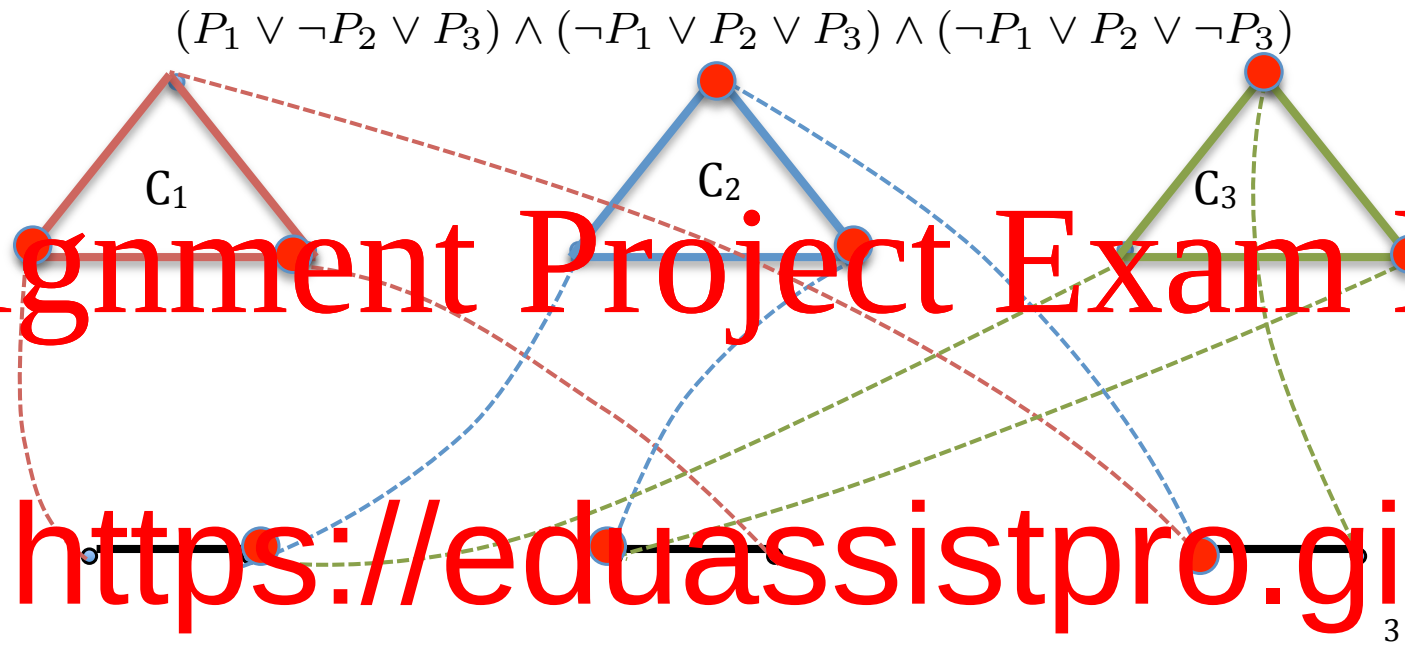




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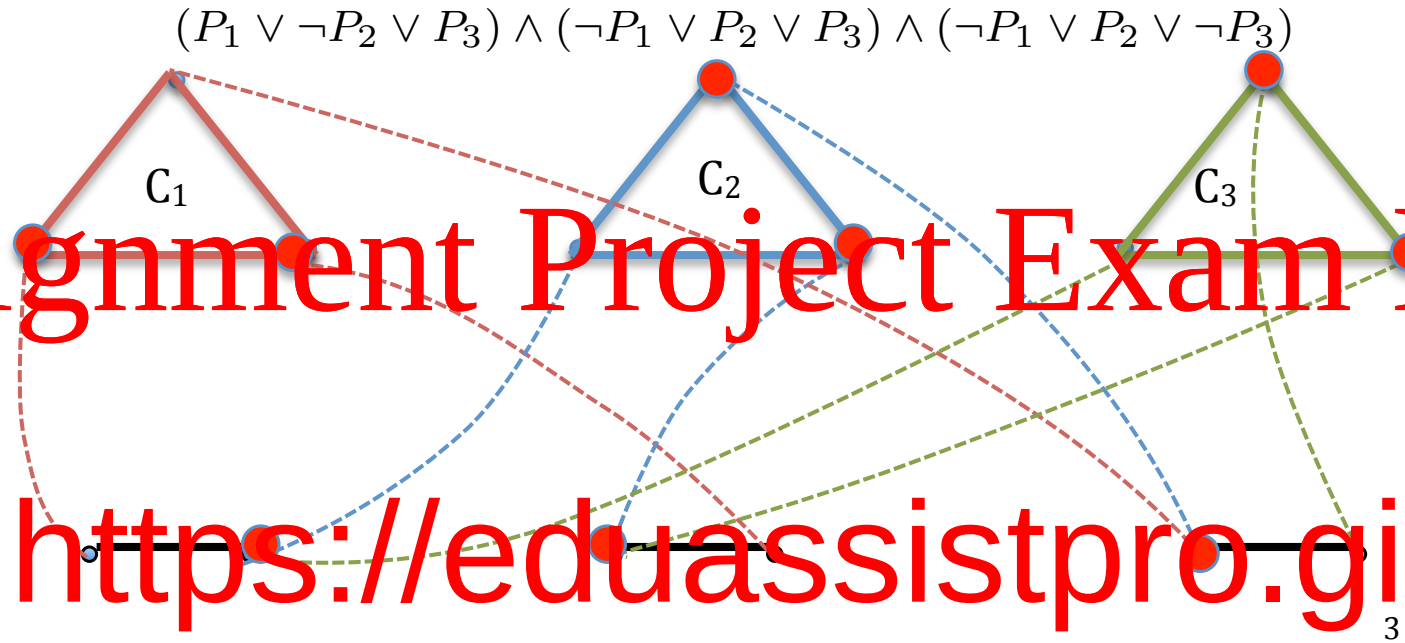




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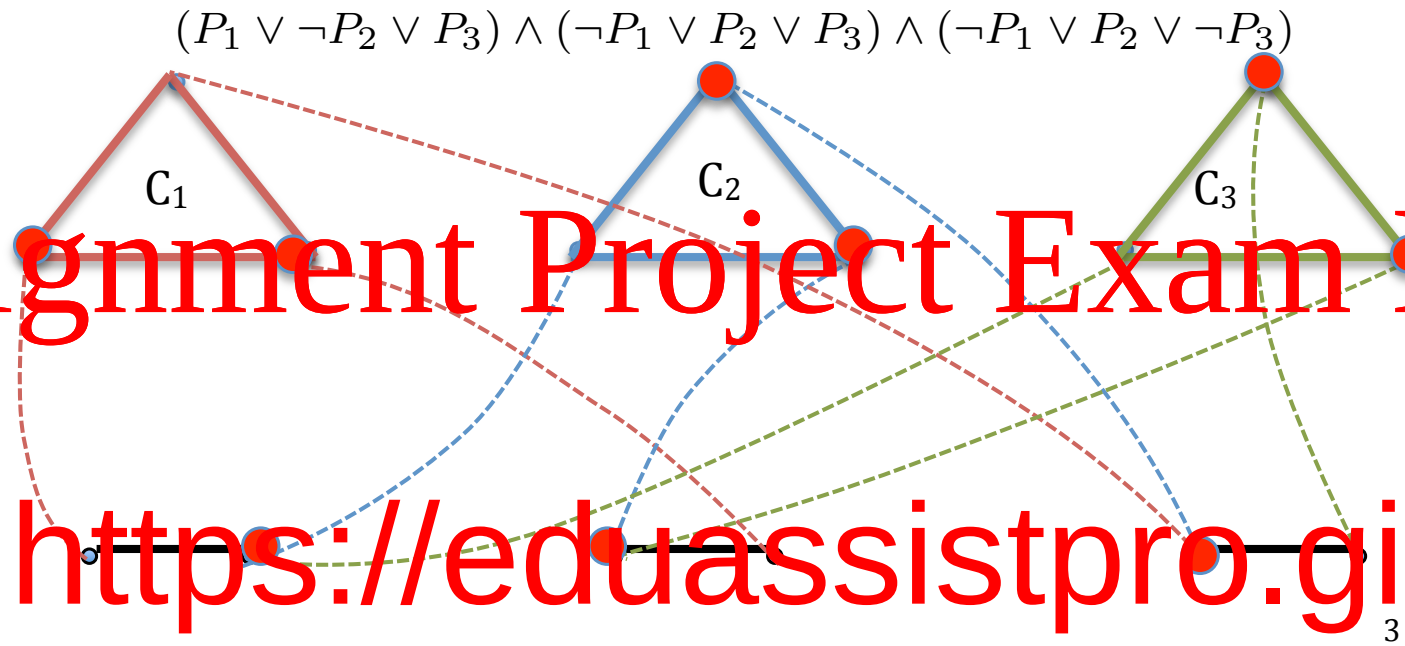




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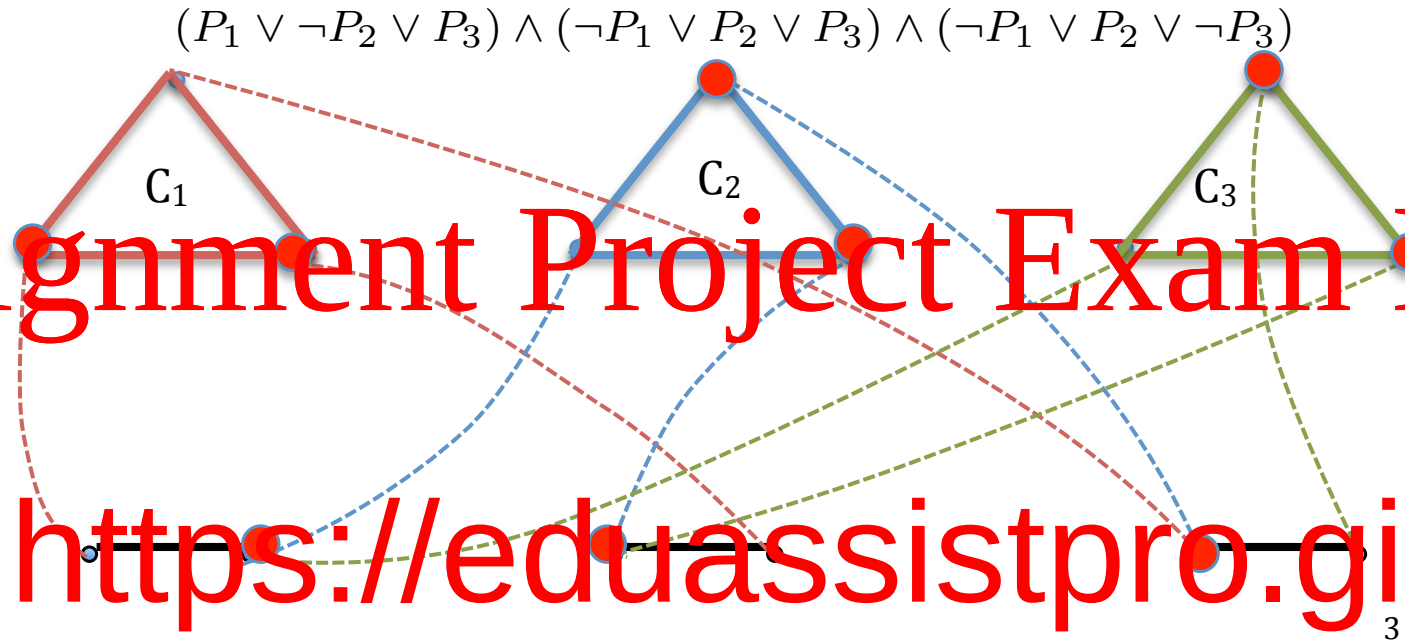




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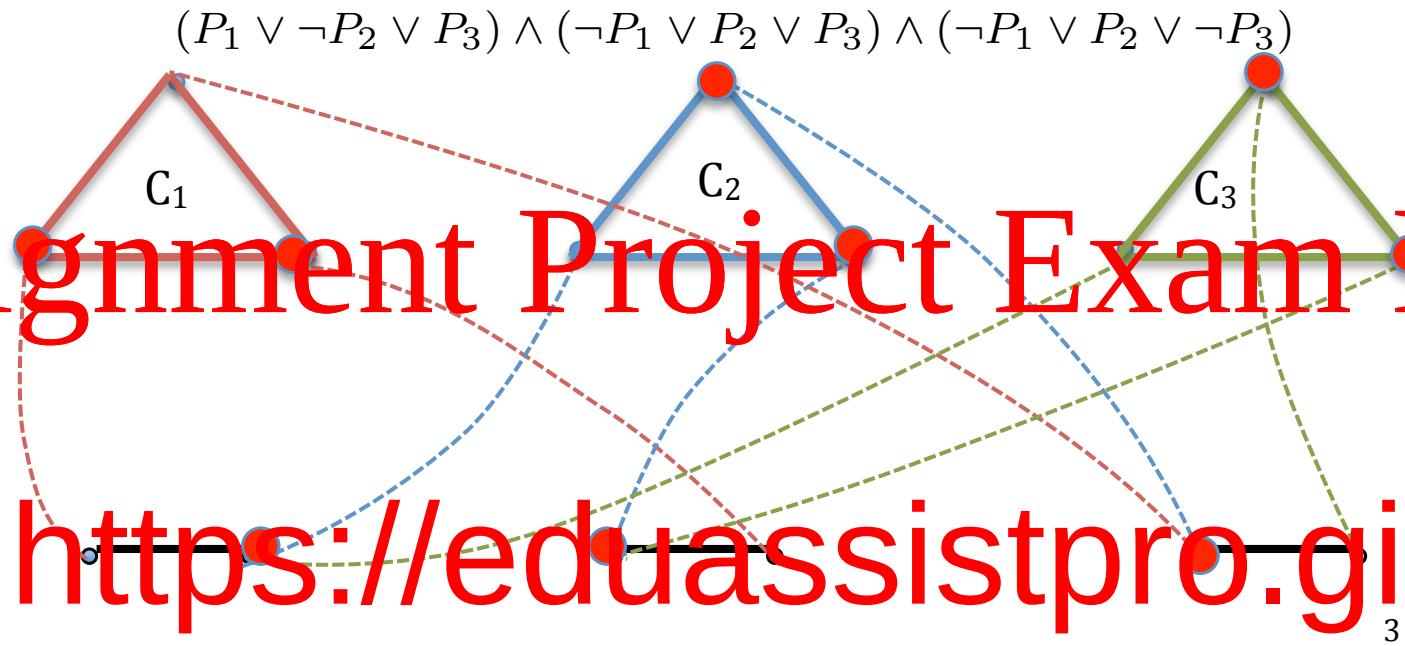




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- in this way we cover exactly $2M + N$ vertices of the graph and clearly every edge between a segment and a triangle has at least one end covered.

# Dealing with NP hard optimisation problems

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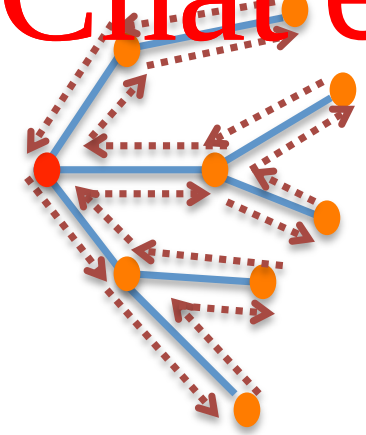

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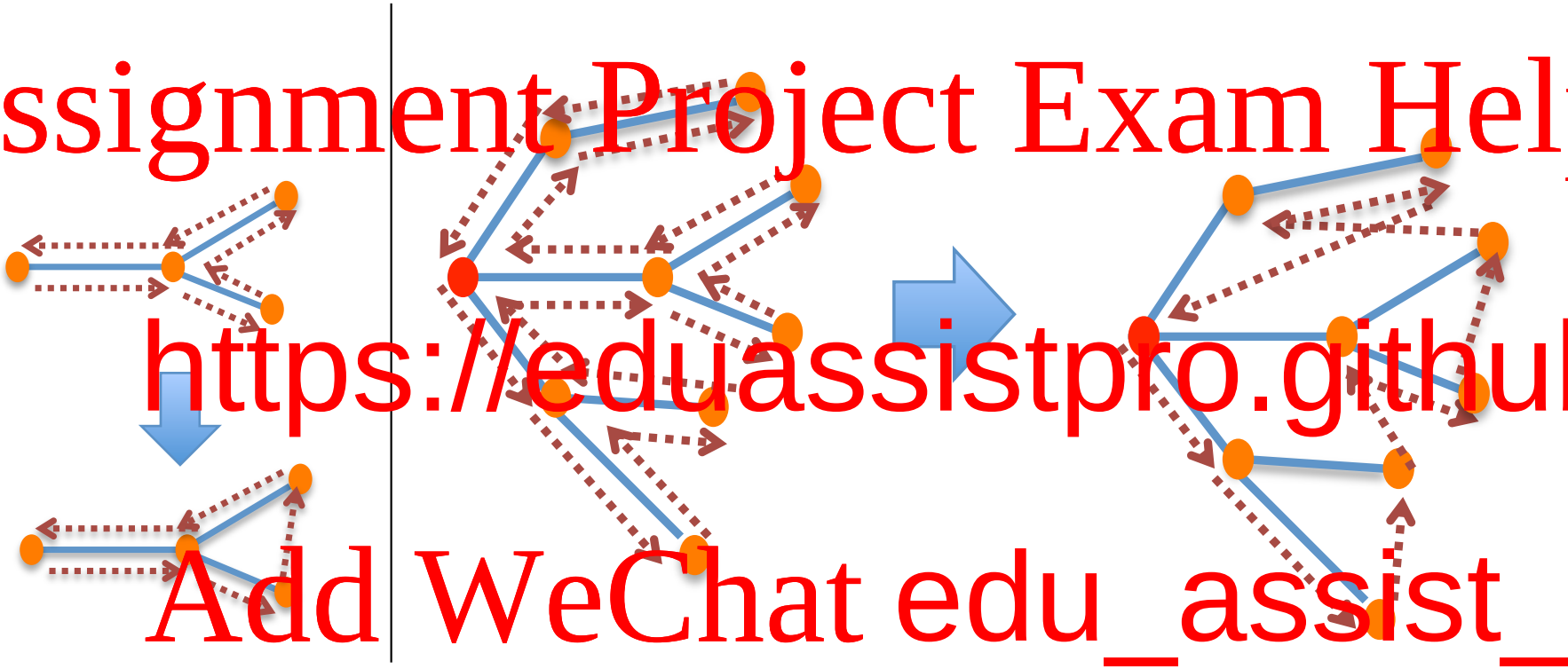

- We now take shortcuts to avoid visiting vertices more than once; because of the triangle inequality, this operation does not increase the length of the tour.

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- To prove this, we show that if there existed algorithm producing a tour which is at most $K$ times longer than the optimal tour, then we could obtain an algorithm which solves in polynomial time the Hamiltonian Cycle Problem, i.e., which for every graph $G$ determines if $G$ contains a cycle visiting all vertices exactly once, which is impossible because this problem is known to be NP complete.

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- This is impossible, because this would be a polynomial time decision procedure for determining in $G$ has a Hamiltonian cycle.