Algorithms:

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2. DIVIDE-AND-CONQUER

A Puzzle

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- We split the array into two, sort the two parts recursively and then merge the two sorted arrays.
- We now look at a closely related but more interesting problem of counting inversions in an array.

Counting the number of inversions

- Assume that you have m users ranking the same set of n movies. You want to determine for any two users A and B how similar their tastes are (for example, in order to make a recommender system).

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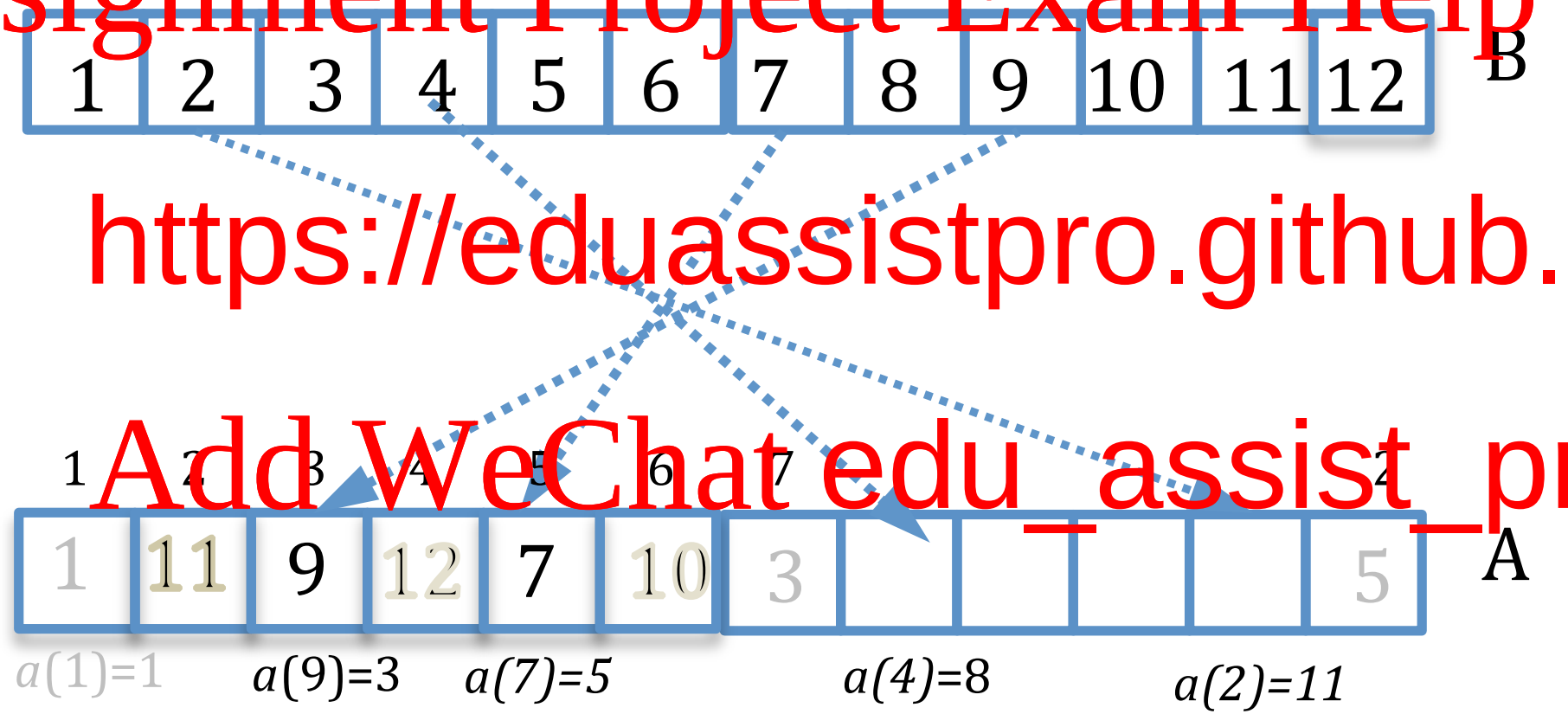
- How to measure the similarity between A and B?
- Lets enumerate the movies on the ranking list of A by assigning to the top choice of user A index 1, to the second choice index 2 and so on.

Counting the number of inversions

- Assume that you have m users ranking the same set of n movies. You want to determine for any two users A and B how similar their tastes are (for example, in order to make a recommender system).
- How to compare A and B?
- Lets enumerate the movies on the ranking list of A by assigning to the top choice of user A index 1, next choice index 2 and so on.
- For the i^{th} movie on $B's$ list we can now look at the position (i.e., index) of that movie on $A's$ list, denoted by $a(i)$.

Counting the number of inversions



Counting the number of inversions

- A good measure of how different these two users are, is the total number of *inversions*, i.e., total number of pairs of movies i, j such that movie i precedes movie j on $B's$ list but movie j is higher up on $A's$ list than the movie i.

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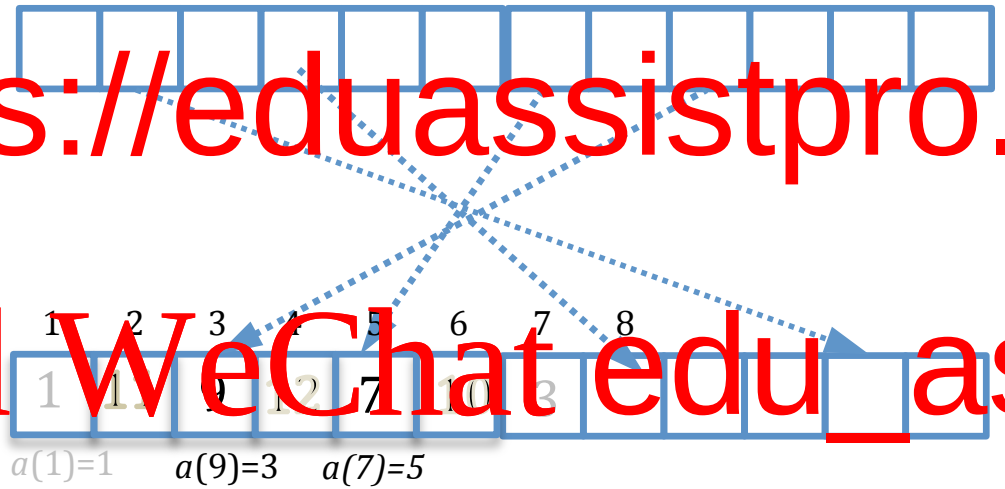

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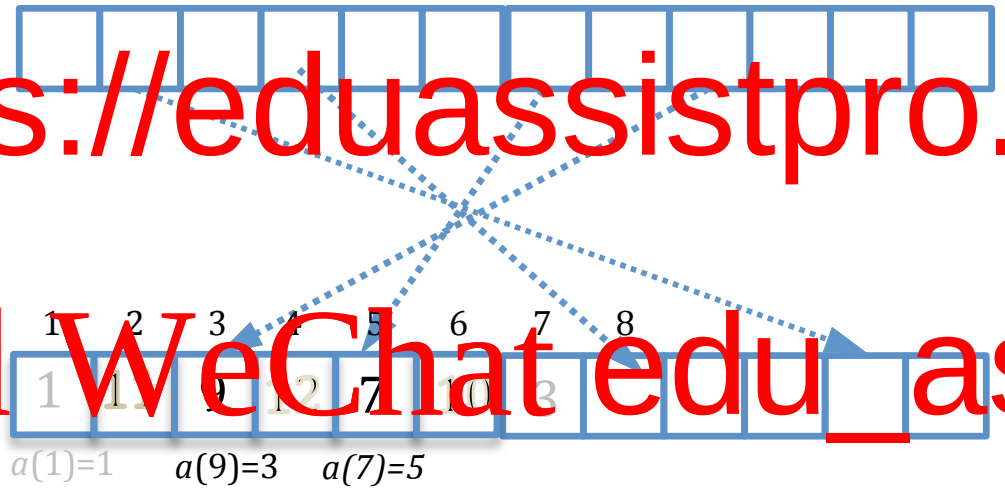

- For example 1 and 2 do not form an inversion because $a(1) < a(2)$ ($a(1) = 1$ and $a(2) = 11$ because $a(1)$ is on the first and $a(2)$ is on the 11^{th} place in A);

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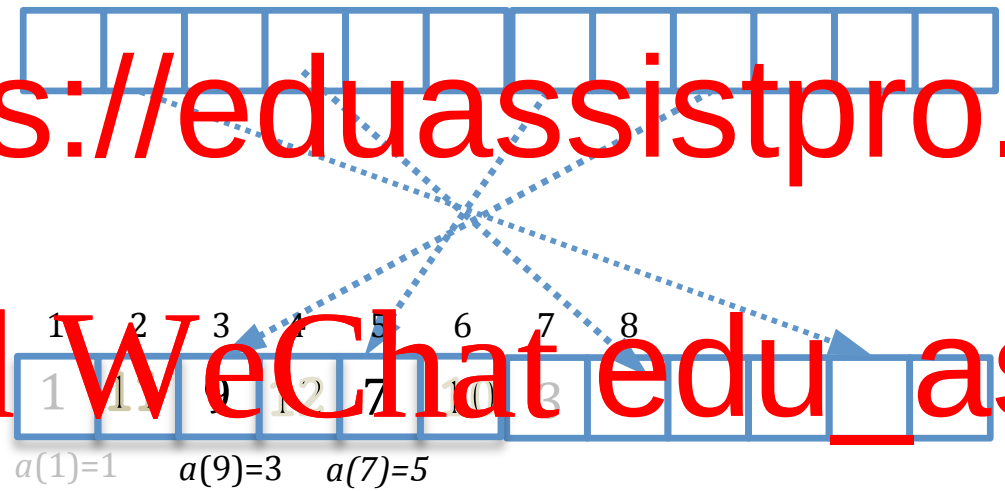

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- However, for example 4 and 7 do form an inversion because $a(7) < a(4)$ ($a(7) = 5$ because seven is on the fifth place in A and $a(4) = 8$)

Counting the number of inversions

- An easy way to count the total number of inversions between two lists is by looking at all pairs $i < j$ of movies on one list and determining if they are inverted in the second list, but this would produce a quadratic time algorithm, $T(n) = \Theta(n^2)$.

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- Clearly, since the total number of pairs is quad cannot afford to inspect all possible pairs.
- The main idea is to tweak the MERGE-SORT algorithm, by extending it to recursively both sort an array A **and** determine the number of inversions in A.

Counting the number of inversions

- We split the array A into two (approximately) equal parts $A_{top} = A[1 \ldots \lfloor n/2 \rfloor]$ and $A_{bottom} = A[\lfloor n/2 \rfloor + 1 \ldots n]$.

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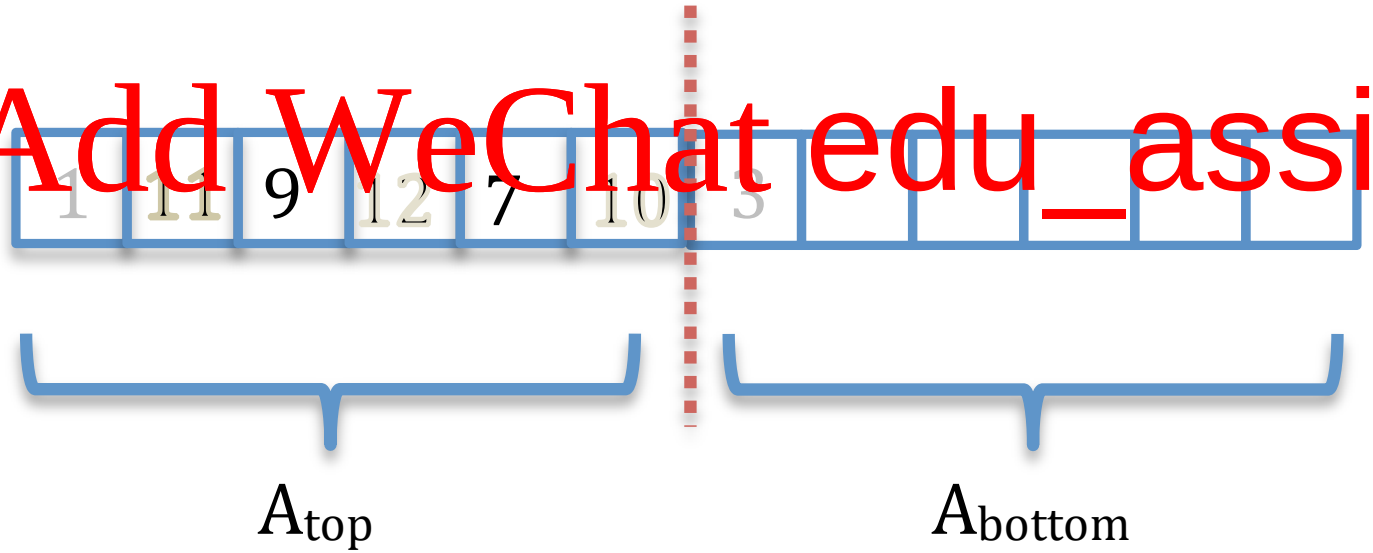

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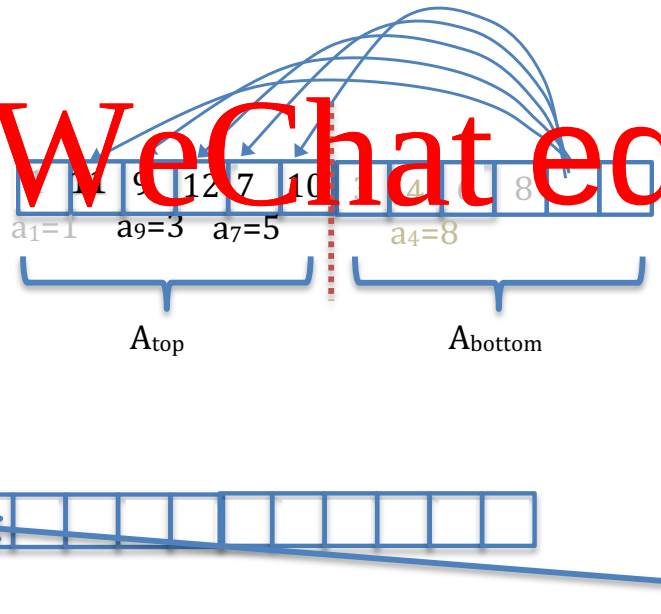

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- **Next:** we study applications of divide and conquer to arithmetic of very large integers.

Basics revisited: how do we add two numbers?

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- "Simple" problems can actually turn out to be difficult!

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What we mean is that the product AB can be calculated recursively by the following program:

```
1:  function MULT(A, B)
2:      if |A| = |B| = 1 then return AB
3:      else
4:          A_1 ← MoreSignificantPart(A);
5:          A_0 ← LessSignificantPart(A);
6:
7:
8:
9:                                  0    1
10:         Z ← MULT(A_1, B_0);
11:         W ← MULT(A_1, B_1);
12:         return W 2^n + (Y + Z) 2^{n/2}
13:     end if
14: end function
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In 1952, one of the most famous mathematicians of th try, Andrey Kolmogorov, conjectured that you cann $\Omega(n^2)$ elementary operations. In 1960, Karatsuba, then a 23-year-old student, found an algorithm (later it was called "divide and conquer") that multiplies two n-digit numbers in $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.58\ldots})$ elementary steps, thus disproving the conjecture!! Kolmogorov was shocked!

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Take again our two input numbers A and B, and split them into two halves:

$$A = \overline{a_{n-1} \ldots a_{n/2}}\ \overline{a_{n/2-1} \ldots a_0}$$

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$$A = \overline{a_{n-1}\ldots a_{n/2}}\;\overline{a_{n/2-1}\ldots a_0} = 10^{n/2}\overline{a_{n-1}\ldots a_{n/2}} + \overline{a_{n/2-1}\ldots a_0}$$

$$B = \overline{b_{n-1}\ldots b_{n/2}}\;\overline{b_{n/2-1}\ldots b_0} = 10^{n/2}\overline{b_{n-1}\ldots b_{n/2}} + \overline{b_{n/2-1}\ldots b_0}$$

- AB can now be calculated as follows:

The Karatsuba trick

How did Karatsuba do it??

Take again our two input numbers A and B, and split them into two halves:

$$A = A_1 2^{\frac{n}{2}} + A_0$$

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$$AB = A_1 B_1 2^n + (A_1 B_0 + A_0 B_1) 2^{\frac{n}{2}}$$

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$$= A_1 B_1 2^n + ((A_1 + A_0)(B_1 + B_0) - A_1 B_1 - A_0 B_0) 2^{\frac{n}{2}} + A_0 B_0$$

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- So we have saved one multiplication at each recursion round!

- Thus, the algorithm will look like this:

```
1:  function MULT(A, B)
2:      if |A| = |B| = 1 then return AB
3:      else
4:          A_1 ← MoreSignificantPart(A);
5:          A_0 ← LessSignificantPart(A);
6:
7:
8:
9:
10:         X ← MULT(A_0, B_0);
11:         W ← MULT(A_1, B_1);
12:         Y ← MULT(U, V);
13:         return W 2^n + (Y − X − W)
14:     end if
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- How fast is this algorithm?

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$$T(n) = 3T\left(\tfrac{n}{2}\right) + cn = 3\left(3T\left(\tfrac{n}{2^2}\right) + c\tfrac{n}{2}\right) + cn = 3^2\, T\left(\frac{n}{2^2}\right) + c\tfrac{3n}{2} + cn$$

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$$\approx 3^{\log_2 n} T(1) + cn\,\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1} = 3^{\log_2 n} T(1) + 2cn\left(\left(\tfrac{3}{2}\right)^{\log_2 n} - 1\right)$$

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So we got

$$T(n) \simeq 3^{\log_2 n} T(1) + 2cn \left(\left(\frac{3}{2} \right)^{\log_2 n} - 1 \right)$$

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$$= O(n^{\log_2 3}) = O(n^{1.58\ldots}) \ll n^2$$

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Please review the basic properties of logarithms and the asymptotic notation from the review material (the first item at the class webpage under "class resources".)

A Karatsuba style trick also works for matrices: Strassen's algorithm for faster matrix multiplication

- If we want to multiply two $n \times n$ matrices P and Q, the product will be a matrix R also of size $n \times n$. To obtain each of n^2 entries in R we do n multiplications, so matrix product by brute force is $\Theta(n^3)$.

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- We s $n/2$:

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad Q = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

- Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \tag{4}$$

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- Prima facie, there are 8 matrix multiplications, each running in time $T\left(\frac{n}{2}\right)$ and 4 matrix additions, each running in time
calculation would result in time complexity given

$$T(n) = 8T\left(\frac{n}{2}\right)$$

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- The first case of the Master Theorem gives $T(n) = \Theta(n^3)$, so nothing gained.

Strassen's algorithm for faster matrix multiplication

- However, we can instead evaluate:

$$A = a\,(f - h); \qquad B = (a + b)\,h; \qquad C = (c + d)\,e \qquad D = d\,(g - e);$$
$$E = (a + d)(e + h); \qquad F = (b - d)(g + h); \qquad H = (a - c)(e + f).$$

- We now obtain

$$E + D - B = (a\,e + d\,e + a\,h + d\,h) + (d\,g - b\,h - d\,h)$$
$$= a\,e + d\,e + d\,g - b\,h = t;$$

$$C + D = (c\,e + d\,e) + (d\,g - d\,e) = c\,e + d\,g = t;$$
$$E + A - C - H = (a\,e + d\,e + a\,h + d\,h) + (a\,f - c\,f + a\,f - c\,f)$$
$$= c\,f + d\,h = u.$$

- We have obtained all 4 components of C u
  and 18 matrix additions/subtractions.
- Thus, the run time of such recursive algorithm satisfies $T(n) = 7T(n/2) + O(n^2)$ and the Master Theorem yields $T(n) = \Theta(n^{\log_2 7}) = O(n^{2.808})$.
- In practice, this algorithm beats the ordinary matrix multiplication for $n > 32$.

Next time:

1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??

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1. Can we multiply large integers faster than $O\left(n^{\log_2 3}\right)$??
2. Can we avoid messy computations like:

$$T(n) = 3T\left(\frac{n}{2}\right) + cn = 3\left(3T\left(\frac{n}{2^2}\right) + c\frac{n}{2}\right) + cn = 3^2 T\left(\frac{n}{2^2}\right) + c\frac{3n}{2} + cn$$

$$= 3^2\left(3T\left(\frac{n}{2^3}\right) + c\frac{n}{2^2}\right) + c\frac{3n}{2} + cn = 3^3 T\left(\frac{n}{2^3}\right) + c\frac{3^2 n}{2^3} + c\frac{3n}{2^2} + cn$$

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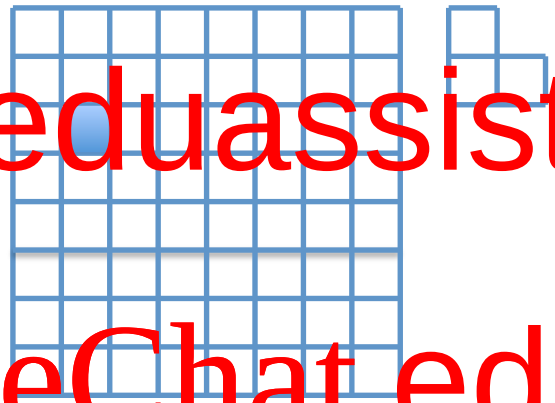
$$\approx 3^{\log_2 n} T(1) + cn\frac{\left(\frac{3}{2}\right)^{\log_2 n} - 1}{\frac{3}{2} - 1}$$

$$= 3^{\log_2 n} T(1) + 2cn\left(\left(\frac{3}{2}\right)^{\log_2 n} - 1\right)$$

PUZZLE!

You are given a $2^n \times 2^n$ board with one of its cells missing (i.e., the board has a hole); the position of the missing cell can be arbitrary. You are also given a supply of "dominoes" each containing 3 such squares; see the figure



Your task is to design an algorithm which covers the entire board with such "dominoes" except for the hole.

Hint: Do a divide-and-conquer recursion!

That's All, Folks!!