School of Computer Science and En
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10. LINEAR PROGRAMMING

Problem:

- You are given a list of food sources $f_1, f_2, \ldots, f_n$;
- for each source f_i you are given:
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  - ent f_i.

- Your task: to find a combination of quantities of foo
  - the total number of calories in all of the chosen f recommended daily value of 2000 calories;
  - the total intake of each vitamin V_j daily intake of w_j milligrams for all $1 \leq j \leq 13$;
  - the price of all food per day is as low as possible.

- To obtain the corresponding constraints let us assume that we take x_i grams of each food source f_i for $1 \leq i \leq n$. Then:
  - the total number of calories must satisfy

$$\sum_{i=1}^{n} x_i c_i = 2000;$$

  -

  - an implicit assumption is that all the quanti
    non-negative numbers,

$$x_i \geq 0,$$

- Our goal is to minimise the objective function which is the total cost

$$y = \sum_{i=1}^{n} x_i p_i.$$

- Note that all constraints and the objective function, are **linear**.

Problem:

- Assume now that you are politician and you want to make certain promises to the electorate which will ensure that your party will win in the forthcoming elections.

- You can promise that you will build
  - a certain number of bridges, each 3 billion a piece;
  -
  -
- You

  -
    suburban votes and 9% of rural votes;
  - each rural airport you promise brings you no suburban votes and 15% of rural votes;

  - each olympic swimming pool promised bri
    3% of suburban votes and no rural votes.
- In order to win, you have to get at least 51% of each of the city, suburban and rural votes.
- You wish to win the election by cleverly making a promise that **appears** that it will blow as small hole in the budget as possible, i.e., that the total cost of your promises is as low as possible.

Linear Programming problems - Example 2

- We can let the number of bridges to be built be x_b, number of airports x_a and the number of swimming pools x_p.
- We now see that the problem amounts to minimising the objective $y = 3x_b + 2x_a + x_p$, while making sure that the following constraints are satisfied:

$0.05x_b \quad + 0.12x_p \quad 0.51$ (securing majority of city votes)

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- However, there is a very significant difference wit
  - you can eat 1.56 grams of chocolate, but
  - you cannot promise to build 1.56 bridges, 2. swimming pools!

- The second example is an example of an **Integer Linear Programming problem**, which requires all the solutions to be integers.
- Such problems are MUCH harder to solve than the "plain" Linear Programming problems whose solutions can be real numbers.

- In the **standard form** the *objective* to be maximised is given by

$$\sum_{j=1}^{n} c_j x_j$$

- the

$$j=1$$

$$x_j \geq 0; \qquad (2)$$

- Let the boldface $\mathbf{x}$ represent a (column) vector, $\mathbf{x} = \langle x_1 \dots x_n \rangle^{\intercal}$.
- To get a more compact representation of linear programs we introduce a partial ordering on vectors $\mathbf{x} \in \mathbf{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_j \leq y_j$ for all $1 \leq j \leq n$.

- Letting $\mathbf{c} = \langle c_1 \ldots c_n\rangle^\intercal \in \mathbb{R}^n$ and $\mathbf{b} = \langle b_1 \ldots b_m\rangle^\intercal \in \mathbb{R}^m$, and letting A be the matrix $A = (a_{ij})$ of size $m \times n$, we get that the above problem can be formulated simply as:

  -
  -

  and

  $\mathbf{x}$

- Thus, to specify a Linear Programming optimis
  have to provide a triplet $(A, \mathbf{b}, \mathbf{c})$;
- This is the usual form which is accepted by most standard LP solvers.

- The value of the objective for any value of the variables which makes the constraints satisfied is called a *feasible solution* of the LP problem.
- Equality constraints of the form $\sum_{i=1}^{n} a_{ij}x_i = b_j$ can be replaced by two inequalities: $\sum_{i=1}^{n} a_{ij}x_i \geq b_j$ and $\sum_{i=1}^{n} a_{ij}x_i \leq b_j$; thus, we can assume that all constraints are inequalities.

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  vari
- This poses no problem, because each occurrence o
  variable x_j can be replaced by the expression $\quad {}^{*}_{j}$ are
  new variables satisfying the constraints

- If $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector, we let $|\mathbf{x}|$ $\quad |x_1|, \ldots, |x_n|$
  problems are naturally translated into constraints of the form $|A\mathbf{x}| \leq \mathbf{b}$. This also poses no problem because we can replace such constraints with two linear constraints: $A\mathbf{x} \leq \mathbf{b}$ and $-A\mathbf{x} \leq \mathbf{b}$ because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

Linear Programming - Standard Form

- Standard Form: maximize $\mathbf{c}^\intercal \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
- Any vector $\mathbf{x}$ which satisfies the two constraints is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^\intercal \mathbf{x}$ might be.
- As an example, let us consider the following optimisation problem:

maximize $\quad z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \quad (3)$

subj

$\leq 30 \quad (4)$

$\leq 24 \quad (5)$

- How large can the value of the objective $+ 2x_3$ be, without violating the constraints?
- If we add inequalities (4) and (5), we get

$$3x_1 + 3x_2 + 8x_3 \leq 54 \tag{8}$$

- Since all variables are constrained to be non-negative, we are assured that

$$3x_1 + x_2 + 2x_3 \leq 3x_1 + 3x_2 + 8x_3 \leq 54$$

$$
\begin{aligned}
\text{maximize:} \quad & z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 && (3)\\
\text{with constraints:} \quad & x_1 + x_2 + 3x_3 \leq 30 && (4)\\
& 2x_1 + 2x_2 + 5x_3 \leq 24 && (5)\\
& 4x_1 + x_2 + 2x_3 \leq 36 && (6)
\end{aligned}
$$

- Thu
  $z(x_1$

- Can
  $y_1, y_2, y_3 \geq 0$ to be used to for a linear combination o

$$y_1(x_1 + x_2 + 3$$

$$y_2(2x_1 + 2x_2 + 5$$

$$y_3(4x_1 + x_2 + 2x_3) \leq 36y_3$$

- Then, summing up all these inequalities and factoring, we get

$$x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3) \leq 30y_1 + 24y_2 + 36y_3$$

$$\text{maximize:} \quad z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \tag{3}$$

$$\text{with constraints:} \quad x_1 + x_2 + 3x_3 \leq 30 \tag{4}$$

$$2x_1 + 2x_2 + 5x_3 \leq 24 \tag{5}$$

$$4x_1 + x_2 + 2x_3 \leq 36 \tag{6}$$

$$x_1, x_2, x_3 \geq 0 \tag{7}$$

- So w

$x_1(y_1 + 2y_2$ ... $_1 + 24y_2 + 36y_3$

(9)

- If we c ... y_1, y_2 and y_3 so that:

$$y_1 + 2y_2 + 4$$

$$y_1 + 2y_2 +$$

$$3y_1 + 5y_2 + 2$$

then

$$3x_3 + x_2 + 2x_3 \leq x_1(y_1 + 2y_2 + 4y_3) + x_2(y_1 + 2y_2 + y_3) + x_3(3y_1 + 5y_2 + 2y_3)$$

Combining this with (9) we get:

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$$

- Consequently, in order to find as tight upper bound for our objective $z(x_1, x_2, x_3)$ of the problem P:

maximize: $$z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \quad (3)$$

with constraints: $$x_1 + x_2 + 3x_3 \leq 30 \quad (4)$$

$$2x_1 + 2x_2 + 5x_3 \leq 24 \quad (5)$$

we h

minimise: $$z^*(y_1, y_2, y_3) = 30 \quad (10)$$

with constraints:

$$y_1 \quad y_2 \quad y_3 \geq$$

$$y_1, y_2, y_3 \geq 0 \quad (14)$$

then $z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3)$ will be a tight upper bound for $z(x_1, x_2, x_3)$

- The new problem P^* is called the *dual problem* for the problem P.

- Let us now repeat the whole procedure with P^* in place of P, i.e., let us find the dual program $(P^*)^*$ of P^*.
- We are now looking for $z_1, z_2, z_3 \geq 0$ to multiply inequalities (11)-(13) and obtain

$$z_1(y_1 + 2y_2 + 4y_3) \geq 3z_1$$

$$z_2(y_1 + 2y_2 + y_3) \geq z_2$$

- Su

$$y_1(z_1 + z \quad \ldots \quad + z_2 + 2z_3 \tag{15}$$

- If we choose multipliers z_1, z_2, z_3 so that

$$z_1 + z_2 + 3 \tag{16}$$

$$2z_1 + 2z_2 + 5 \tag{17}$$

$$4z_1 + z_2 + 2z_3 \leq \tag{18}$$

we will have:

$$y_1(z_1 + z_2 + 3z_3) + y_2(2z_1 + 2z_2 + 5z_3) + y_3(4z_1 + z_1 + 2z_3) \leq 30y_1 + 24y_2 + 36y_3$$

- Combining this with (15) we get

$$3z_1 + z_2 + 2z_3 \leq 30y_1 + 24y_2 + 36y_3$$

- Consequently, finding the dual program $(P^*)^*$ of P^* amounts to maximising the objective $3z_1 + z_2 + 2z_3$ subject to the constraints

$$
\begin{aligned}
z_1 + z_2 + 3z_3 &\leq 30\\
2z_1 + 2z_2 + 5z_3 &\leq 24
\end{aligned}
$$

- But n xactly our starting program P. Thus, the dual program $(P^*)^*$ for program P^* is just P itself, i.e., $(P^*)^* = P$.

- So, at the first sight, looking for the multipliers p much, because it only reduced a maximisation pr hard minimisation problem.
- It is now useful to remember how we proved that the Ford - Fulkerson Max Flow algorithm in fact produces a **maximal flow**, by showing that it terminates only when we reach the capacity of a **minimal cut**.

- The original, *primal* Linear Program P and its *dual* Linear Program can be easily described in the most general case:

$$P: \quad \text{maximize} \qquad z(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j,$$

$$\text{subject to the constraints} \qquad \sum^{n} a_{ij} x_j \quad b_i; \quad 1 \quad i \le m$$

$$i \quad i$$

subject to the constraints

$$y_1, \ldots, y_m \ge 0,$$

or, in matrix form,

$$P: \quad \text{maximize} \quad z(\mathbf{x}) = \mathbf{c}^{\intercal}\mathbf{x}, \quad \text{subject to the constraints} \quad A\mathbf{x} \le \mathbf{b} \quad \text{and} \quad \mathbf{x} \ge 0;$$

$$P^*: \quad \text{minimize} \quad z^*(\mathbf{y}) = \mathbf{b}^{\intercal}\mathbf{y}, \quad \text{subject to the constraints} \quad A^{\intercal}\mathbf{y} \ge \mathbf{c} \quad \text{and} \quad \mathbf{y} \ge 0.$$

Weak Duality Theorem

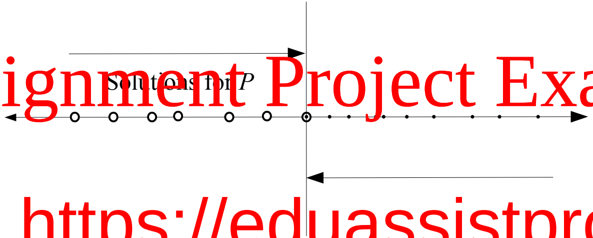
- Recall that any vector $\mathbf{x}$ which satisfies the two constraints, $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^{\mathsf{T}}\mathbf{x}$ might be.
- **Theorem** If $x = \langle x_1 \ldots x_n \rangle$ is any basic feasible solution for P and $y = \langle y_1 \ldots y_m \rangle$ is any basic feasible solution for P^*, then:

$$\sum^{n} \qquad \sum^{n}$$

Pro ... respectively, we ca ... P^* ... P to obtain

$$z(x) = \sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \ldots \sum_{i} \ldots = z^*(y)$$

- Thus, the value of (the objective of P^* for) any feasible solution of P^* is an upper bound for the set of all values of (the objective of P for) all feasible solutions of P, and
- every feasible solution of P is a lower bound for the set of feasible solutions for P^*.



- Thus, if we find a feasible solution for P which is equal to a feasible solution to P^*, such solution must be the maxi objective of P and the minimal feasible value P^*.
- If we use a search procedure to find an optimal solutio now when to stop: when such a value is also a feasible soluti .
- This is why the most commonly used LP solving method, the SIMPLEX method, produces optimal solution for P, because it stops at a value of the primal objective which is also a value of the dual objective.
- See the Lecture Notes for the details and an example of how the SIMPLEX algorithm runs.

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