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5. THE FAST FOURIER TRANSFORM
(not examinable material)

Our strategy to multiply polynomials fast:

- Given two polynomials of degree at most n,

$$P_A(x) = A_n x^n + \ldots + A_0; \qquad P_B(x) = B_n x^n + \ldots + B_0$$

1. convert them into value representation at $2n+1$ distinct points $x_0, x_1, \ldots, x_{2n}$:

$$\ldots (x_{2n}))\}$$

$$\ldots (x_{2n}))\}$$

2. multiply them point by point using 2

$$\Big\{(x_0, \underbrace{P_A(x_0)P_B(x_0)}_{P_C(x_0)}), (x_1, \underbrace{P_A(x_1)P_B(x_1)}_{P_C(x_1)}), \ldots, (x_{2n}, \underbrace{P_A(x_{2n})P_B(x_{2n})}_{P_C(x_{2n})})\Big\}$$

3. Convert such value representation of $P_C(x)$ to its coefficient form

$$P_C(x) = C_{2n}x^{2n} + C_{2n-1}x^{2n-1} + \ldots + C_1 x + C_0;$$

Our strategy to multiply polynomials fast:

- So, we need $2n+1$ values of $P_A(x_i)$ and $P_B(x_i)$, $0 \leq i \leq 2n$.

- If we use $2n+1$ integers which are the smallest by their absolute value, i.e.,

$$n, \quad (n \quad 1), \dots, \quad 1, 0, 1, \dots, n \quad 1, n$$

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$P_A($

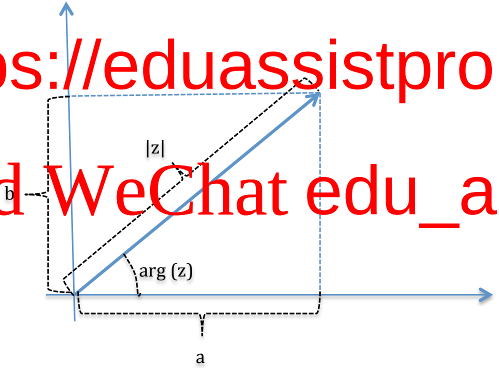
- We saw that the trouble is that, as the degree (x) and $P_B(x)$ increases, the value of n^n increases v
of the computational complexity of the algorithm for
which we used in the generalised Karatsuba algorith

- **Key Question:** What values should we take for $x_0, \dots, x_{2n}$ to avoid "explosion" of size when we evaluate x_i^n while computing $P_A(x_i) = A_0 + A_1 x + \dots + A_n x_i^n$?

Complex numbers $z = a + ib$ can be represented using their *modulus* $|z| = \sqrt{a^2 + b^2}$ and their *argument*, $\arg(z)$, which is an angle taking values in $(-\pi, \pi]$ and satisfying:

$$z = |z|e^{i \arg(z)} = |z|(\cos \arg(z) + i \sin \arg(z)),$$

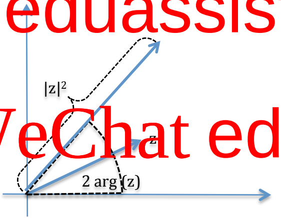
see figure below



Recall that

$$z^n = \left(|z|\, e^{i \arg(z)}\right)^n = |z|^n e^{i\, n \arg(z)} = |z|^n \left(\cos(n \arg(z)) + i \sin(n \arg(z))\right),$$

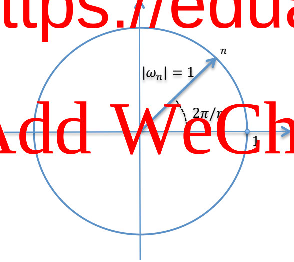
see the figure.

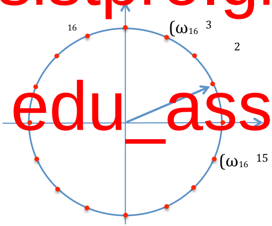


Complex roots of unity

- *Roots of unity of order* n are complex numbers which satisfy $z^n = 1$.
- If $z^n = |z|^n(\cos(n \arg(z)) + i \sin(n \arg(z))) = 1$ then

- $|z| = 1$ and $n \arg(z)$ is an integer multiple of 2π;
- Thus, $n \arg(z) = 2\pi\, k$, i.e., $\arg(z) = \frac{2\pi\, k}{n}$
- We d er n.





Roots of unity of order 16

- A root of unity ω of order n is *primitive* if all other roots of unity of the same order can be obtained as its powers ω^k.

Complex roots of unity

- For $\omega_n = e^{i\,2\pi/n}$ and for all k such that $0 \leq k \leq n-1$,

$$(\omega_n{}^k)^n = (\omega_n)^{nk} = (\omega_n{}^n)^k = 1^k = 1.$$

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  prim

- Sinc n
  distinct roots of unity of order n (i.e., solutions to the equation $x^n - 1 = 0$) we conclude that every root of unity of order
- For the product of any two roots of unity
  have $\omega_n^k \omega_n^m = \omega_n^{k+m}$

- If $k + m \geq n$ then $k + m = n + l$ for $l = ($
  $\omega_n^k \omega_n^m = \omega_n^{k+m} = \omega_n^{n+l} = \omega_n^n \omega_n^l = 1 \cdot \omega_n^l = \omega_n^l$ where $0 \leq l < n$.
- Thus, the product of any two roots of unity of the same order is just another root of unity of the same order.

Complex roots of unity

- So in the set of all roots of unity of order n, i.e., $\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ we can multiply any two elements or raise an element to any power without going out of this set.

- Note that this is not true for addition, i.e., the sum of two roots of unity is NOT another root of unity!

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  Proof:

$$\omega_{kn}^{km} = (\omega_{kn})^{km} = (e^{\frac{2\pi}{kn}})^{km} = e^{\frac{2\pi}{k}}$$

- Thus, in particular, $(\omega_{2n}^k)^2 = \omega_{2n}^{2k} = (\omega_{2n}^2)^k = \omega_n^k$.

- So the squares of the roots of unity of order $2n$ are just the roots of unity of order n.

- Let $A = \langle A_0, A_1, \dots, A_{n-1} \rangle$ be a sequence of n real or complex numbers.

- We can form the corresponding polynomial $P_A(x) = \sum_{j=0}^{n-1} A_j x^j$.

- We can evaluate it at all complex roots of unity of order n, i.e., we compute $P_A($

- The s … lled **the Dis** … $A = \langle A_0, A_1, \dots, A_{n-1} \rangle$.

- The value $P_A(\omega_n^k)$ is usually denoted by $\widehat{A}$ … $\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2), \dots, P_A(\omega_n^{n-1}) \rangle$ is u … $\widehat{A} = \langle \widehat{A}_0, \widehat{A}_1, \dots, \widehat{A}_{n-1} \rangle$.

- The DFT $\widehat{A}$ of a sequence A can be computed VERY FAST using a divide-and-conquer algorithm called the **Fast Fourier Transform**.

- If we multiply a polynomial

$$P_A(x) = A_0 + \ldots + A_{n-1}x^{n-1}$$

of degree $n-1$ with a polynomial

of de

$$C(x) = P_A(x)P_B(x) = C_0 + \ldots + C \quad x^{m+n-2}$$

of degree $n-1+m-1 = m+n-2$ wit

- To uniquely determine such a polynomial 2 we need $m+n-1$ many values.

- Thus, we will evaluate both $P_A(x)$ and $P_B(x)$ at all the roots of unity of order $n+m-1$ (instead of at $-(n-1), \ldots, -1, 0, 1, \ldots, m-1$ as we would in Karatsuba's method!)

New way for fast multiplication of polynomials

- Note that we defined the DFT of a sequence of length n as the values of the corresponding polynomial of degree $n-1$ at the n roots of unity of order n, i.e., ω_n^k $(0 \le k \le n-1)$.
- So the DFT of a sequence A is another sequence $\hat{A}$ of exactly the same length; we do

  $n -$

- For t ... $P_B(x)$, we pad ... 0 1 ... $n-1$ ... ake it of

  length $m+n-1$, and similarly we pad B

  $(B_0, B_1, \ldots, B_{m-1}, \underbrace{0, \ldots, 0}_{n-1})$ to also obtain a seque ... -1.

- Note that this does not change the associated polynomials because the added higher powers have the corresponding coefficients equal to zero.

New way for fast multiplication of polynomials

- We can now compute the DFTs of the two (0 padded) sequences:

$$DFT(\langle A_0, A_1, \ldots, A_{n-1}, \underbrace{0, \ldots, 0}_{m-1}\rangle) = \langle \widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_{n+m-2}\rangle$$

and

$$\ldots_{-2}\rangle$$

- For each k we multiply the corresponding values $A_k = P_A(\omega_{n+m-1})$ and $\widehat{B}_k = P_B(\omega^k_{n+m-1})$, thus obtaining

$$\widehat{C}_k = \widehat{A}_k \widehat{B}_k = P_A(\omega^k_{n+m-1})P_B$$

- We then use the inverse transformation for DFT, called IDFT, to recover the coefficients $\langle C_0, C_1, \ldots, C_{n+m-1}\rangle$ of the product polynomial $P_C(x)$ from the sequence $\langle \widehat{C}_0, \widehat{C}_1, \ldots, \widehat{C}_{n+m-1}\rangle$ of its values $C_k = P_C(\omega^k_{n+m-1})$ at the roots of unity of order $n + m - 1$.

$$P_A(x) = A_0 + \ldots + A_{n-1}x^{n-1} + 0 \cdot x^n + \ldots + 0 \cdot x^{n+m-2};$$

$$P_B(x) = B_0 + \ldots + B_{m-1}x^{m-1} + 0 \cdot x^m + \ldots + 0 \cdot x^{n+m-2}$$

$$\Downarrow \text{DFT} \qquad \Downarrow \text{DFT}$$

$$\{P_A(1), \ldots, P_A(\omega_{n+m-1}^{n+m-2})\}$$

$$\Big\{\underbrace{P_A(1)P_B(1)}_{P_C(1)}, \underbrace{P_A(\omega_{n+m-1})P_B(\omega_{n+m-1})}_{P_C(\omega_{n+m-1})}, \ldots, P$$

$$\Downarrow \text{IDFT}$$

$$P_C(x) = P_A(x) \cdot P_B(x) = \sum_{j=0}^{n+m-2} C_j x^j = \sum_{j=0}^{n+m-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)}_{C_j} x^j$$

- Crucial fact: the values $P_A(\omega_n^k)$ for all k such that $0 \leq k < n$ can be computed in $\mathbf{O}(\mathbf{n} \log \mathbf{n})$ time!

- Note that a direct evaluation of a polynomial of degree $n - 1$ at n roots of unity of order n would take n^2 many multiplications, even if we p
mul
$0 \leq$

  - We can assume that n is a power of 2 - oth $P_A(x)$ with zero coefficients until its number of coe to the nearest power of 2.

  - <u>Exercise:</u> Show that for every n which is not a power of two the smallest power of 2 larger or equal to n is smaller than $2n$.

  - *Hint:* consider n in binary. How many bits does the nearest power of two have?

- **Problem:** Given a sequence $A = \langle A_0, A_1, \ldots, A_n \rangle$ compute its DFT.
- This amounts to finding values of $P_A(x)$ for all $x = \omega_n^k$, $0 \le k \le n-1$.
- **The main idea of the FFT algorithm:** divide-and-conquer by splitting the polynomial $P_A(x)$ into the even powers and the odd powers:

$$P_A(x) = (A_0 + A_2x^2 + A_4x^4 + \ldots + A_{n-2}x^{n-2}) + (A_1x + A_3x^3 + \ldots + A_{n-1}x^{n-1})$$

$$\ldots (x^2)^{n/2-1})$$

- Let us define $A^{[0]} = \langle A_0, A_2, A_4, \ldots A_{n-2} \rangle$ an ... ; then

$$P_{A^{[0]}}(y) = A_0 + A_2y + A_4y \ldots$$

$$P_{A^{[1]}}(y) = A_1 + A_3y + A_5y \ldots n-1$$

$$P_A(x) = P_{A^{[0]}}(x^2) + x\,P_{A^{[1]}}(x^2)$$

- Note that the number of coefficients of the polynomials $P_{A^{[0]}}(y)$ and $P_{A^{[1]}}(y)$ is $n/2$ each, while the number of coefficients of the polynomial $P_A(x)$ is n.

The Fast Fourier Transform (FFT)

- **Problem of size n:**
  Evaluate a polynomial with n coefficients at n many roots of unity.

- **Problem of size $n/2$:**
  Evaluate a polynomial with $n/2$ coefficients at $n/2$ many roots of unity.

- We reduced evaluation of our polynomial $P(x)$ with n coefficients at inputs $x =$ … $P_{A[0]}(y)$ and $P_{A[1]}$ … uts x.

- How … $y = x^2$ ranges through $\{\omega_{n/2}^0, \omega_{n/2}^1, \omega_{n/2}^2, \ldots, \omega_{n/2}^{n-1}\}$, and **there are only $n/2$ distinct such values.**

- Once we get these $n/2$ values of $P_{A[0]}(x^2)$ and … multiplications with numbers ω_n^k to obtain the val

$$P_A(\omega_n^k) = P_{A[0]}((\omega_n^k)^2) + \omega_n^k \cdot P_{A[1]}((\omega_n^k)^2)$$
$$= P_{A[0]}(\omega_{n/2}^k) + \omega_n^k \cdot P_{A[1]}(\omega_{n/2}^k).$$

- Thus, we have reduced a problem of size n to two such problems of size $n/2$, plus a linear overhead.

- Note that by the Cancelation Lemma $\omega_n^{n/2} = \omega_{2n/2}^{n/2} = \omega_2 = -1$.
- Thus,

$$\omega_n^{k+n/2} = \omega_n^{n/2}\omega_n^k = \omega_2\omega_n^k = -\omega_n^k;$$

- We can now simplify evaluation of

for n

let $k = n/2 + m$ where $0 \leq m < n/2$; then

$$\begin{aligned} P_A(\omega_n^{n/2+m}) &= P_{A[0]}(\omega_{n/2}^{n/2+m}) + \omega_n^{n/2+m} P_{A[1]}(\omega_{n/2}^{n/2+m}) \\ &= P_{A[0]}(\omega_{n/2}^{n/2}\omega_{n/2}^m) + \omega_n^{n/2}\omega_n^m P_{A[1]}(\omega_{n/2}^{n/2}\omega_{n/2}^m) \\ &= P_{A[0]}(\omega_{n/2}^m) - \omega_n^m P_{A[1]}(\omega_{n/2}^m) \end{aligned}$$

- Compare this with $\quad P_A(\omega_n^m) = P_{A[0]}(\omega_{n/2}^m) + \omega_n^m P_{A[1]}(\omega_{n/2}^m) \quad$ for $0 \leq m < n/2$.

- So we can replace evaluations of

$$P_A(\omega_n^k) = P_{A^{[0]}}(\omega_{n/2}^k) + \omega_n^k P_{A^{[1]}}(\omega_{n/2}^k)$$

for k

with

and ju −

$$P_A(\omega_n^k) = P_{A^{[0]}}(\omega_{n/}^k$$

$$P_A(\omega_n^{n/2+k}) = P_{A^{[0]}}(\omega_{n/}^k$$

- We can now write a pseudo-code for our FFT algorithm:

function FFT(A)
  $n \leftarrow \text{length}[A]$
  if $n = 1$ **then return** A
  else
    $A^{[0]} \leftarrow (A_0, A_2, \ldots A_{n-2})$;
    $A^{[1]} \leftarrow (A_1, A_3, \ldots A_{n-1})$;

    % a variable to ho
    for $k = 0$ to $k = n/2 - 1$ **do**:   % $P(\omega^k) = P_{[0]}(\omega^k) + \omega^k \underbrace{P_{[1]}(\omega^k_{n/2})}_{y_k^{[1]}}$
      $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
      $y_{n/2+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$;   % $\underbrace{P_A(\omega)}_{y_{n/2+k}} \ldots \underbrace{\ldots}_{y_k^{[0]}} \ldots \underbrace{P_{[1]}(\omega^k_{n/2})}_{y_k^{[1]}}$
      $\omega \leftarrow \omega \cdot \omega_n$;
    end for
    return y
  end if
end function

- We have recursively reduced evaluation of a polynomial $P_A(x)$ with n coefficients at n roots of unity of order n to evaluations of two polynomials $P_{A^{[0]}}(y)$ and $P_{A^{[1]}}(y)$, each with $n/2$ coefficients, at $n/2$ many roots of unity of order $n/2$.

- Once we get these $n/2$ values of $P_{A^{[0]}}(y)$ and $P_{A^{[1]}}(y)$ we need $n/2$ additional multiplications to obtain the values of

$$k \qquad k \qquad k \qquad k \tag{1}$$

and

$$\underbrace{P_A(\omega_n^{n/2+k})}_{y_{n/2+k}} = \underbrace{A^{[0]}(\omega_{n/}^{k}\ )}_{y_k^{[0]}} \quad \omega^k A^{[1]}(\omega^k \quad) \tag{2}$$

for all $0 \leq k < n/2$.

- Thus, we have reduced a problem of size n $n/2$, plus a linear overhead.
- Consequently, our algorithm's run time satisfies the recurrence

$$T(n) = 2\,T\,(n/2) + c\,n$$

- The Master Theorem gives $T(n) = \Theta(n \log n)$.

- Evaluation of a polynomial $P_A(x) = A_0 + A_1 x + \ldots + A_{n-1}x^{n-1}$ at roots of unity ω_n^k of order n can be represented in the matrix form as follows:

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \ldots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} = \begin{pmatrix} \widehat{A}_0 \\ \widehat{A}_1 \\ \widehat{A}_2 \\ \vdots \\ \widehat{A}_{n-1} \end{pmatrix}$$

- The FFT is just a method of replacing this matrix-vector multiplication taking n^2 many multiplications with an $n \log n$ procedure.
- From $P_A(1) = P_A(\omega_n^0), P_A(\omega_n), P_A(\omega_n^2), \ldots, P$

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \ldots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} \widehat{A}_0 \\ \widehat{A}_1 \\ \widehat{A}_2 \\ \vdots \\ \widehat{A}_{n-1} \end{pmatrix} \quad (3)$$

- To obtain the inverse of the above matrix, all we have to do is just change the signs of the exponents and divide everything by n:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ & & & & \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1}$$

$$\frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & & \\ 1 & \omega_n^{-1} & & & \\ 1 & \omega_n^{-2} & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

To see this, note that if we compute the product

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

the (i,j) entry in the product is the product of the i^{th} row and j^{th} column:

$$\begin{pmatrix} 1 & \omega_n^i & \omega_n^{2\cdot i} & \dots & \omega_n^{i(n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-j} \\ \omega_n^{-2j} \\ \vdots \\ \omega_n^{-(n-1)j} \end{pmatrix} = \sum_{k=0}^{n-1} \omega_n^{ik}\omega_n^{-jk} = \sum_{k=0}^{n-1} \omega_n^{(i-j)k}$$

We now have two possibilities:

1. $i = j$: then
$$\sum_{k=0}^{n-1} \omega_n^{(i-j)k} = \sum_{k=0}^{n-1} \omega_n^{0} = \sum_{k=0}^{n-1} 1 = n;$$

2. $i \neq j$: then $\sum_{k=0}^{n-1} \omega_n^{(i-j)k}$ represents a sum of a geometric progression with the ratio ω_n^{i-j} and thus

[illegible] $= 0$

So,

$$\begin{pmatrix} 1 & \omega_n^{i} & \omega_n^{2\cdot i} & \dots & \omega_n^{i\cdot(n-1)} \end{pmatrix} \begin{pmatrix} 1 \\ \omega_n^{-j} \\ \omega_n^{-2j} \\ \vdots \\ \omega_n^{-(n-1)j} \end{pmatrix} = \sum_{k=0} \omega_n^{(i-j)k} = \begin{cases} n & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{4}$$

So we get:

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

$$= \begin{pmatrix} n & 0 & 0 & \dots & 0 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n \end{pmatrix} = n \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ 1 & \omega_n^{-2} & \omega_n^{-2\cdot 2} & \dots & \omega_n^{-2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{-(n-1)} & \omega_n^{-2(n-1)} & \dots & \omega_n^{-(n-1)(n-1)} \end{pmatrix}$$

- We now have

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^{2\cdot 2} & \dots & \omega_n^{2\cdot(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{pmatrix}^{-1} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n^{-1} & \omega_n^{-2} & \dots & \omega_n^{-(n-1)} \\ \end{pmatrix} \begin{pmatrix} P_A(1) \\ P_A(\omega_n) \\ P_A(\omega_n^2) \\ \vdots \\ P_A(\omega_n^{n-1}) \end{pmatrix}$$

- This means that to covert from the values

$$\langle P_A(1), P_A(\omega_n), P_A(\omega_n^2$$

which we denoted by $\langle \widehat{A}_0, \widehat{A}_1, \widehat{A}_2, \dots, \widehat{A}$ rm

$$P_A(x) = A_0 + A_1 x + A_2 x^2 + A_{n-1}x^{n-1}$$

we can use **the same** FFT algorithm with the only change that:

1. the root of unity ω_n is replaced by $\overline{\omega_n} = e^{-i\frac{2\pi}{n}}$,
2. the resulting output values are divided by n.

function IFFT$^*(\widehat{A})$
    $n \leftarrow \text{length}(\widehat{A})$
    if $n = 1$ **then return** $\widehat{A}$
    else
        $\widehat{A}^{[0]} \leftarrow (\widehat{A}_0, \widehat{A}_2, \dots \widehat{A}_{n-2})$
        $A^{[1]} \leftarrow (A_1, A_3, \dots A_{n-1})$;
        $y^{[0]}$ $IFFT^*(A^{[0]})$;

            $y_k \leftarrow y_k^{[0]} + \omega \cdot y_k^{[1]}$;
            $y_{n/2+k} \leftarrow y_k^{[0]} - \omega \cdot y_k^{[1]}$
            $\omega \leftarrow \omega \cdot \omega_n$;
        end for
        return y;
    end if
end function

function IFFT$(\widehat{A})$ $\Leftarrow$ different from FFT
    return $IFFT^*(\widehat{A})/\text{length}(\widehat{A})$
end function

Important note:

Computer science books take the forward DFT operation to be the evaluation of the corresponding polynomial at all roots of unity $\omega_n^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ and the InverseDFT to be the evaluation of the polynomial at the complex conjugates of the roots of unity, i.e., at $\omega_n^{-k} = \cos\frac{2\pi k}{n} - i\sin\frac{2\pi k}{n}$.

However, Electrical engineering books do it just opposite, the direct DFT evaluates the polynomial at ω_n^{-k} and the InverseDFT at ω_n^{k}!

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We did here only multiplication of polynomials, and did not apply it t
large integers. This is possible to do but one has to be careful because roo
represented by floating point numbers so you have to show that if yo
sufficient precision you can round off the results and obtain correct i
this is tricky.

Earlier results along this line produced algorithms for multiplication of large integers which operate in time $n\log n\log(\log n)$ but very recently David Harvey of the School of Mathematics at UNSW came up with an algorithm to multiply large integers which runs in time $n\log n$.

$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1}x^{n-1}$ $\qquad$ $P_B(x) = B_0 + B_1 x + \ldots + B_{n-1}x^{n-1}$

$\Downarrow$ DFT $O(n \log n)$ $\qquad$ $\Downarrow$ DFT $O(n \log n)$

$\{P_A(1), P_A(\omega_{2n-1}), P_A(\omega_{2n-1}^2), \ldots, P_A(\omega_{2n-1}^{2n-2})\}$; $\{P_B(1), P_B(\omega_{2n-1}), P_B(\omega_{2n-1}^2), \ldots, P_B(\omega_{2n-1}^{2n-2})\}$

$\{P$

$\Downarrow$ IDFT $O(n \log$

$$P_C(x) = \sum_{j=0}^{2n-2} \underbrace{\left(\sum_{i=0}^{j} A_i B_{j-i} \right)}_{C_j} x^j = \sum_{j=0}^{2n-2} C_j x^j = P_A(x) \cdot P_B(x)$$

Thus, the product $P_C(x) = P_A(x)\,P_B(x)$ of two polynomials $P_A(x)$ and $P_B(x)$ can be computed in time $O(n \log n)$.

$A = \langle A_0, A_1, \ldots, A_{n-1} \rangle$ $\qquad$ $B = \langle B_0, B_1, \ldots, B_{n-1} \rangle$

$\Downarrow \quad O(n)$ $\qquad$ $\Downarrow \quad O(n)$

$P_A(x) = A_0 + A_1 x + \ldots + A_{n-1} x^{n-1}$ $\qquad$ $P_B(x) = B_0 + B_1 x + \ldots + B_{n-1} x^{n-1}$

$\Downarrow$ DFT $\quad O(n \log n)$ $\qquad$ $\Downarrow$ DFT $\quad O(n \log n)$

$$\{P_A(1), P_A(\omega_{2n-1}), P_A(\omega_{2n-1}^2), \ldots, P_A(\omega_{2n-1}^{2n-2})\} \; ; \; \{P_B(1), P_B(\omega_{2n-1}), P_B(\omega_{2n-1}^2), \ldots, P_B(\omega_{2n-1}^{2n-2})\}$$

$\{P$

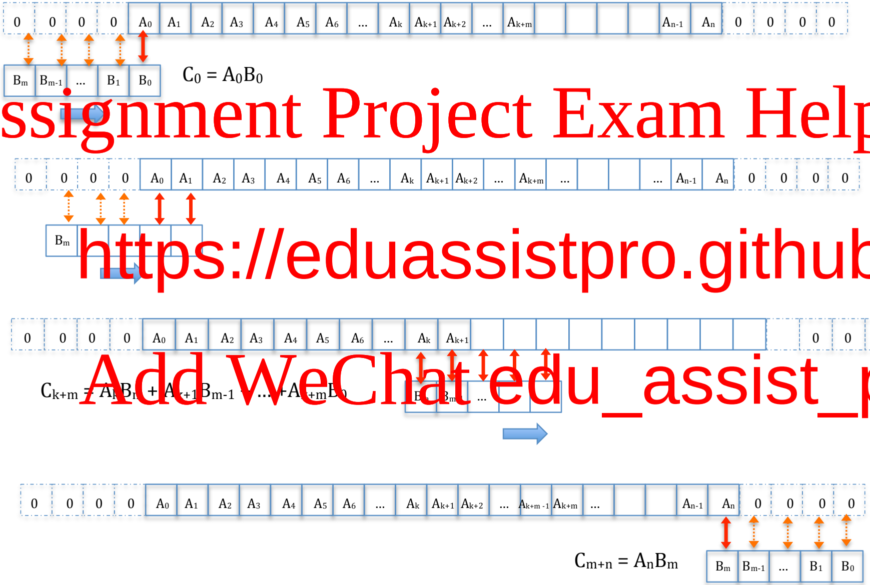
$$P_C(x) = \sum_{j=0}^{2n-2} \Big(\underbrace{\sum_{i=0}^{j} A_i B_{j-i}}_{C_j} \Big) x^j$$

$$\Downarrow$$

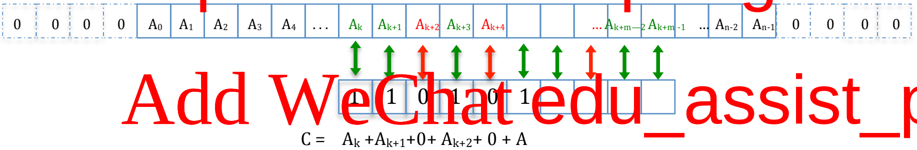
$$C = \left\langle \sum_{i=0}^{j} A_i B_{j-i} \right\rangle_{j=0}^{j=2n-2}$$

Convolution $C = A * B$ of sequences A and B is computed in time $O(n \log n)$.

Visualizing Convolution $C = A * B$



- Assume you are given a map of a straight sea shore of length $100n$ meters as a sequence on $100n$ numbers such that A_i is the number of fish between i^{th} meter of the shore and $(i+1)^{th}$ meter, $0 \leq i \leq 100n - 1$. You also have a net of length n meters but unfortunately it has holes in it. Such a net is described as a sequence N of n ones and zeros, where 0's denote where the holes are. If you throw such a net starting at meter k and ending at meter $k + n$, then you will catch only the fish in one met
see t



Find the spot where you should place the left end of your net in order to catch the largest possible number of fish using an algorithm which runs in time $O(n \log n)$.
Hint: Let N' be the net sequence N in the reverse order; Compute $A * B'$ and look for the peak of that sequence.

PUZZLE!!

- On a circular highway there are n petrol stations, unevenly spaced, each con
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  of the
  and take the fuel from that station, you can continu
  round trip around the highway, never emptying
  reaching the next station to refuel.