COMP90038

Algorit plexity

Lecture 19: Wars

(with thanks to Harald Sønde hael Kirley)

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Recap

- **Dynamic programming** is a bottom-up problem solving technique. The idea is to divide the problem into smaller, overlapping ones. The results are tabulated and used to find the complete solution.

  - Solutions often involves recursion.

- Dynamic programming i **atorial** **on** problems.

  - We are trying to find the **best** possible **co** bject to some **constraints**

- Two classic problems
  - Coin row problem
  - Knapsack problem

The coin row problem

- You are shown a group of coins of different denominations ordered in a row.

- **You can keep some of t** **ot pick two adjacent ones.**
  - Your objective is to **maxi** ant to take the largest amount of money.

- The solution can be expressed as the recurrence:

$$S(n) = \max\left(c_n + S\left(n-2\right), S\left(n-1\right)\right) \text{ for } n > 1$$

$$S(1) = c_1$$

$$S(0) = 0$$

The coin row problem

- Let’s quickly examine each step for [20 10 20 50 20 10 20]:

- S[0] = 0
- S[1] = 20
- S[2] = max(**S[1] = 20**, S[0] + 10 = 0 + 10) = 20
- S[3] = max(S[2] = 20, **S[1] + 20 = 20 + 20 = 40**) = 40
- S[4] = max(S[3] = 40, **S[2] + 50 = 20 + 50 = 70**) = 70
- S[5] = max(**S[4] = 70**, S[3] + 20 = 40 + 20 = 60) = 70
- S[6] = max(S[5] = 70, **S[4] + 10 = 70 + 10 = 80**) = 80
- S[7] = max(S[6] = 80, **S[5] + 20 = 70 + 20 = 90**) = 90

		1	2	3	4	5	6	7
	0	20	10	20	50	20	10	20
STEP 0	0							
	0	20						
	0	20	20					
		20	20	40				
ST		20	20	40	70			
ST		20	20	40	70	70		
STEP 6	0	20	20	40	70	70	80	
STEP 7	0	20	20	40	70	70	80	90

SOLUTION		1	1	1	1	1	1	1
				3	4	4	4	4
							6	7

The knapsack problem

- We also talked about the **knapsack problem**:

- Given a list of *n* items with:
  - Weights $\{w_1, w_2, \ldots, w_n\}$
  - Values $\{v_1, v_2, \ldots, v_n\}$

- and a knapsack (container) of capacity W

- Find the **combination** of items with the **highest value** that would **fit into the knapsack**

- All values are positive integers

The knapsack problem

- The critical step is to find a good answer to the question: **what is the smallest version of the problem that I could solve first?**
  - Imagine that I have a knapsack of capacity 1, and an item of weight 2. **Does it fit?**
  - What if the capacity was 2 and the weight 1. Does it fit? **Do I have capacity left?**

- Given that we have **two v** lation is formulated over **two parameters**:
  - the **sequence of items considered so far** {1, 2,
  - the **remaining capacity** $w \leq W$.
- Let $K(i,w)$ be the value of the best choice of items amongst the first i using knapsack capacity w.
  - Then we are after $K(n,W)$.

The knapsack problem

- By focusing on $K(i,w)$ we can express a recursive solution.

- Once a new item i arri it or not.

  - **Excluding *i*** means that at is, which items were selected before i arrived with the sam apacity.

  - **Including *i*** means that the solution also includes the subset of previous items **that will fit into a bag of capacity w-w_i** ≥ 0, i.e., $K(i\text{-}1,w\text{-}w_i) + v_i$.

The knapsack problem

- This was expressed as a recursive function, with a base **state**:

$$K(i, w) = 0 \text{ if } i = 0 \text{ or } w = 0$$

- And a general case:

- Our example was:
  - The knapsack capacity W = 8
  - The values are {42, 12, 40, 25}
  - The weights are {7, 3, 4, 5}

The knapsack problem

- Did you complete the table?

			j	0	1	2	3	4	5	6	7	8
v	w	i										
		0		0	0	0	0	0	0	0	0	0
42	7	1		0	0	0	0	0	0	0	42	42
				0	0	0	12	12	12	12	42	42
				0	0	0	12	40	40	40	52	52
				0	0	0	12	40	40	40	52	52

for $i \leftarrow 0$ to n **do**
    $K[i, 0] \leftarrow 0$
for $j \leftarrow 1$ to W **do**
    $K[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to W **do**
        if $j < w_i$ **then**
            $K[i, j] \leftarrow K[i-1, j]$
        else
            $K[i, j] \leftarrow max(K[i-1, j], K[i-1, j-w_i] + v_i)$
return $K[n, W]$

Solving the Knapsack Problem with Memoing

- To some extent the bottom-up (table-filling) solution is overkill:
  - It finds the solution to **tance**, most of which are unnecessary

- In this situation, a top-down approach, with **memoing**, is preferable.
  - There are many implementations of the memo table.
  - We will examine a simple array type implementation.

The knapsack problem

- Lets look at this algorithm, step-by-step

- The data is:
  - The knapsack capacity W = 8
  - The values are {42, 12, 40, 25}
  - The weights are {7, 3, 4, 5}

```
function MFKnap(i, j)
    if i < 1 or j < 1 then
        return 0
    if F(i, j) < 0 then
        if j < w(i) then
            value = MFKnap(i − 1, j)
        else
            value = max(MFKnap(i − 1, j), v(i) + MFKnap(i − 1, j − w(i)))
        F(i, j) = value
    return F(i, j)
```

- *F* is initialized to all -1, with the exceptions of *i*=0 and *j*=0, which are initialized to 0.

The knapsack problem

- We start with i=4 and j=8

			j	0	1	2	3	4	5	6	7	8
v	w	i										
		0		0	0	0	0	0	0	0	0	0
42	7	1		0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	**-1**

function MFKNAP(i, j)
    if $i < 1$ or $j < 1$ **then**
        return 0
    if $F(i, j) < 0$ **then**
        if $j < w(i)$ **then**
            $value = \text{MFKNAP}(i-1, j)$
        else
            $value = \max(\boxed{\text{MFKNAP}(i-1, j)}, v(i) \boxed{+ \text{MFKNAP}(i-1, j-w(i)))}$
        $F(i, j) = value$
    return $F(i, j)$

- i = 4
- j = 8
- K[4-1,8] = K[3,8]
- K[4-1,8-5] + 25 = K[3,3] + 25

The knapsack problem

- Next is i=3 and j=8

			j	0	1	2	3	4	5	6	7	8
v	w	i										
		0		0	0	0	0	0	0	0	0	0
42	7	1		0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	**-1**
				0	-1	-1	-1	-1	-1	-1	-1	-1

function MFKNAP(i, j)
  if $i < 1$ or $j < 1$ **then**
    return 0
  if $F(i, j) < 0$ **then**
    if $j < w(i)$ **then**
      $value = \text{MFKNAP}(i-1, j)$
    else
      $value = \max(\boxed{\text{MFKNAP}(i-1, j)}, \boxed{v(i) + \text{MFKNAP}(i-1, j-w(i))})$
    $F(i, j) = value$
  return $F(i, j)$

- i = 3
- j = 8
- K[3-1,8] = K[2,8]
- K[3-1,8-4] + 40 = K[2,4] + 40

The knapsack problem

- Next is *i*=2 and *j*=8

			j	0	1	2	3	4	5	6	7	8
v	*w*	*i*										
		0		0	0	0	0	0	0	0	0	0
42	7	1		0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	**-1**
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1

```
function MFKnap(i, j)
    if i < 1 or j < 1 then
        return 0
    if F(i, j) < 0 then
        if j < w(i) then
            value = MFKnap(i − 1, j)
        else
            value = max(MFKnap(i − 1, j), v(i) + MFKnap(i − 1, j − w(i)))
        F(i, j) = value
    return F(i, j)
```

- *i* = 2
- *j* = 8
- *K*[2-1,8] = K[1,8]
- *K*[2-1,8-3] + 12 = *K*[1,5] + 12

The knapsack problem

- Next is i=1 and j=8
- Here we reach the bottom of this recursion

			j	0	1	2	3	4	5	6	7	8
v	w	i										
		0		0	0	0	0	0	0	0	0	0
42	7	1		0	-1	-1	-1	-1	-1	-1	-1	42
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1

function $\text{MFKNAP}(i, j)$
  if $i < 1$ or $j < 1$ **then**
    return 0
  if $F(i, j) < 0$ **then**
    if $j < w(i)$ **then**
      $value = \text{MFKNAP}(i-1, j)$
    else
      $value = \max(\boxed{\text{MFKNAP}(i-1, j)}, \boxed{v(i) + \text{MFKNAP}(i-1, j-w(i))})$
    $F(i, j) = value$
  return $F(i, j)$

- i = 1
- j = 8
- K[1-1,8] = K[0,8] = 0
- K[1-1,8-7] + 42 = K[0,1] + 42 = 0 + 42 = 42

The knapsack problem

- Next is *i*=1 and *j*=5.
- As before, we also reach the bottom of this branch.

			j	0	1	2	3	4	5	6	7	8
v	*w*	*i*										
		0		0	0	0	0	0	0	0	0	0
42	7	1		0	-1	-1	-1	-1	0	-1	-1	42
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1
				0	-1	-1	-1	-1	-1	-1	-1	-1

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function MFKnap(i, j)
    if i < 1 or j < 1 then
        return 0
    if F(i, j) < 0 then
        if j < w(i) then
            value = MFKnap(i − 1, j)
        else
            value = max(MFKnap(i − 1, j), v(i) + MFKnap(i − 1, j − w(i)))
        F(i, j) = value
    return F(i, j)
```

- *i* = 1
- *j* = 5
- *K*[1-1,5] = K[0,5] = 0
- *j* - *w*[1] = 5-8 < 1 → return 0

The knapsack problem

- We can trace the complete algorithm, until we find our solution.

- The states visited (18) .

  - In the bottom-up approach we visited s (40).

- Given that there are a lot of places in the table never used, the algorithm is less space-efficient.
  - You may use a hash table to improve space efficiency.

i	j	value
0	8	0
0	1	0
1	8	42
0	5	0
1	5	0
2	8	42
0	4	0
1	4	0
0	1	0
1	1	0
2	4	12
3	8	52
0	3	0
1	3	0
1	0	0
2	3	12
3	3	12
4	8	52

A practice challenge

- Can you solve the problem in the figure?

  - W = 15
  - w = [1 1 2 4 12]
  - v = [1 2 2 10 4]

- Because it is a larger instance, **memoing** is preferable.
  - How many states do we need to evaluate?

- FYI the answer is $15 {1,2,3,4}

Dynamic Programming and Graphs

- We now apply dynami graph problems:

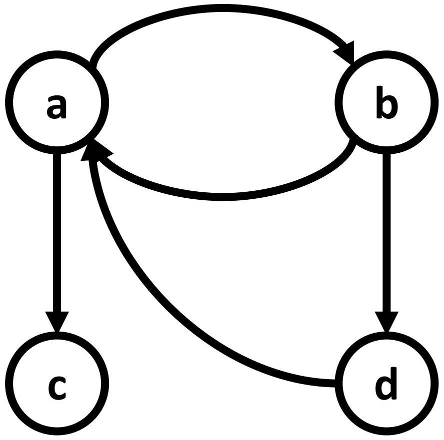
  - Computing the **transiti** raph; and
  - **Finding shortest distances** in weighte aphs.

Warshall’s algorithm

- Warshall's algorithm computes the **transitive closure** of a directed graph.
  - An edge (*a*,*d*) is in the transitive closure of graph *G* iff there is a path in *G* from *a* to *d*.

- Transitive closure is important in appli[illegible]ere we need to reach a “goal state” from some “initial state”.

- Warshall's algorithm was not originally thought of as an instance of dynamic programming, but it fits the bill



$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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    - An edge (*a*,*d*) is in the transitive closure of graph *G* iff there is a path in *G* from *a* to *d*.

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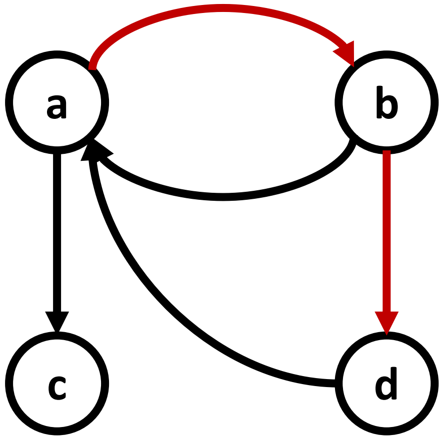


$$\begin{bmatrix} 0 & 1 & 1 & \boxed{1} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Warshall’s Algorithm

- Assume the nodes of graph *G* are numbered from 1 to *n*.

- **Is there a path** from n odes [1 … *k*] as “stepping stones”?

- Such path will exist if and only if we can:
  - step from *i* to *j* using only nodes [1 … *k*-1], or
  - step from *i* to *k* using only nodes [1 … *k*-1], and then step from *k* to *j* using only nodes [1 … *k*-1].

Warshall’s Algorithm

- If *G*’s adjacency matrix is *A* then we can express the recurrence relation as:

- This gives us a dynamic programmin :

function $\textsc{Warshall}(A[\cdot,\cdot], n)$
    $R[\cdot,\cdot,0] \leftarrow A$
    for $k \leftarrow 1$ to n **do**
        for $i \leftarrow 1$ to n **do**
            for $j \leftarrow 1$ to n **do**
                $R[i,j,k] \leftarrow R[i,j,k-1]$ **or** $(R[i,k,k-1]$ **and** $R[k,j,k-1])$
    return $R[\cdot,\cdot,n]$

Warshall’s Algorithm

- If we allow input *A* to be used for the output, we can simplify things.
  - If *R*[*i*,*k*,*k*-1] (that is, *A*[*i*,*k*]) is 0 then the assignment is doing nothing.
  - But if *A*[*i*,*k*] is 1 and if *A*[*k*,*j*] is also 1, then *A*[*i*,*j*] gets set to 1.

- But now we notice that *A*[*i*,*k*] does not depend on *j*, so testing it can be moved outside the innermost loop.

Warshall’s Algorithm

- This leads to a simpler version of the algorithm.

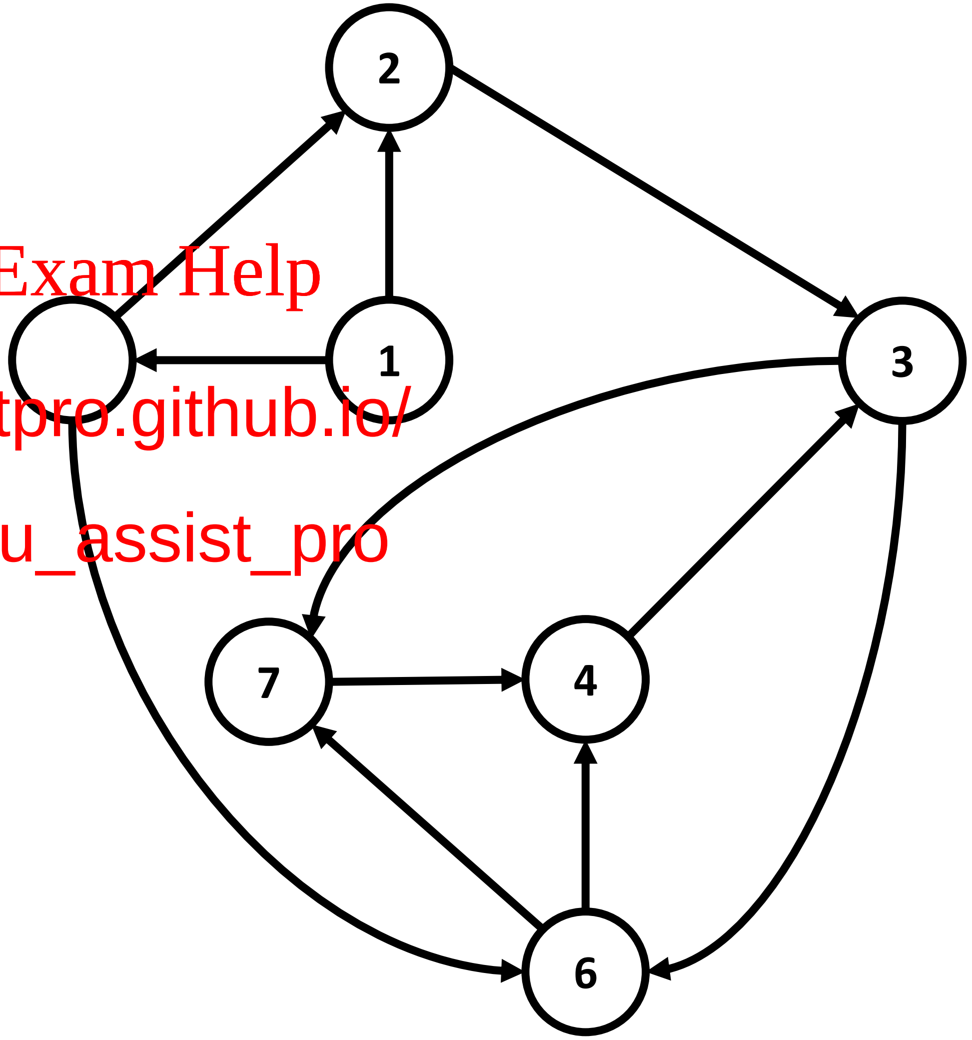
- If each row in the matrix is represented as a bit-string, then the innermost for loop (and *j*) can be gotten rid of – instead of iterating, just apply the “bitwise or” of row *k* to row *i*.

Warshall's Algorithm

- Let's examine this algorithm. Let our graph be.

- Then, the adjacency m

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



Warshall’s Algorithm

- For k=1, all the elements in the column are zero, so this **if** statement does nothing.

for $k \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to n **do**
    $\boxed{\textbf{if } A[i,k] \textbf{ then}}$
      for $j \leftarrow 1$ to n **do**
        if $A[k,j]$ **then**
          $A[i,j] \leftarrow 1$

Warshall's Algorithm

- For k=2, we have **A[1,2] = 1** and **A[5,2] = 1**, and **A[2,3]=1**

$$
\begin{array}{l}
\textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{if } A[i,k] \textbf{ then} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad \textbf{if } A[k,j] \textbf{ then} \\
\quad\quad\quad\quad\quad A[i,j] \leftarrow 1
\end{array}
$$

Warshall's Algorithm

- For k=2, we have **A[1,2] = 1** and **A[5,2] = 1**, and **A[2,3]=1**
  - Then, we can make **A[1,3] = 1** and **A[5,3] = 1**


```
for k ← 1 to n do
    for i ← 1 to n do
        if A[i, k] then
            for j ← 1 to n do
                if A[k, j] then
                    A[i, j] ← 1
```

Warshall's Algorithm

- For k=3, we have **A[1,3]**, **A[2,3]**, **A[4,3]**, **A[5,3]**, **A[3,6]** and **A[3,7]** equal to 1

$$
\begin{array}{l}
\textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{if } A[i,k] \textbf{ then} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad \textbf{if } A[k,j] \textbf{ then} \\
\quad\quad\quad\quad\quad A[i,j] \leftarrow 1
\end{array}
$$

Warshall's Algorithm

- For k=3, we have **A[1,3]**, **A[2,3]**, **A[4,3]**, **A[5,3]**, **A[3,6]** and **A[3,7]** equal to 1
  - Then, we can make **A[1,**
    A[4,6], **A[1,7]**, **A[2,7]**, **A[**
    A[5,7] equal to 1.

for $k \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to n **do**
    if $A[i,k]$ **then**
      for $j \leftarrow 1$ to n **do**
        if $A[k,j]$ **then**
          $A[i,j] \leftarrow 1$

Warshall's algorithm

- Let's look at the final steps:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

k=3

Warshall's algorithm

- Let's look at the final steps:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

k=3

k=4

Warshall’s algorithm

- Let’s look at the final steps:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

k=3 k=4 k=6

- At k=5 and k=7, there is no changes to the matrix.

Warshall’s algorithm

- Warshall's algorithm’s complexity is $\Theta(n^3)$. There is **no difference** between the best, average, and worst cases.

- The algorithm has an i op, making it **ideal for dense graphs**.

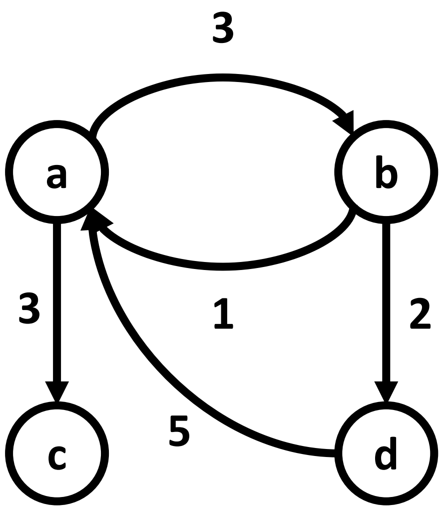
- However, it is **not the best** transitive-closure algorithm to use for **sparse graphs**.
  - For sparse graphs, you may be better off just doing DFS from each node v in turn, keeping track of which nodes are reached from v.

Floyd's Algorithm

- Floyd's algorithm solves the **all-pairs shortest-path** problem for weighted graphs with **positive weights**.
  - It works for **directed** as well as **undirected** graphs.

- We assume we are given a **weight** at holds all the edges' weights

  - If there is no edge from node *i* to node *j*, we set $W[i,j] = \infty$.
- We will construct the **distance matrix** *D*, step by step.



$$\begin{bmatrix} \infty & 3 & 3 & \infty \\ 1 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \\ 5 & \infty & \infty & \infty \end{bmatrix}$$

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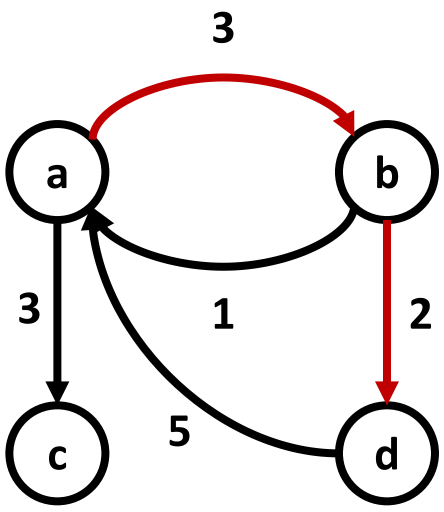

$$\begin{bmatrix} \infty & 3 & 3 & \boxed{5} \\ 1 & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \\ 5 & \infty & \infty & \infty \end{bmatrix}$$

Floyd's Algorithm

- As we did in the Warshall's algorithm, assume nodes are numbered 1 to n.

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- As we did in the Warshall's algorithm, assume nodes are numbered 1 to *n*.

- **What is the shortest p** *j* using nodes [1 … *k*] as "stepping stones"?

- Such path will exist if and only if we can:
  - step from *i* to *j* using only nodes [1 … *k*-1], or
  - step from *i* to *k* using only nodes [1 … *k*-1], and then step from *k* to *j* using only nodes [1 … *k*-1].

Floyd's Algorithm

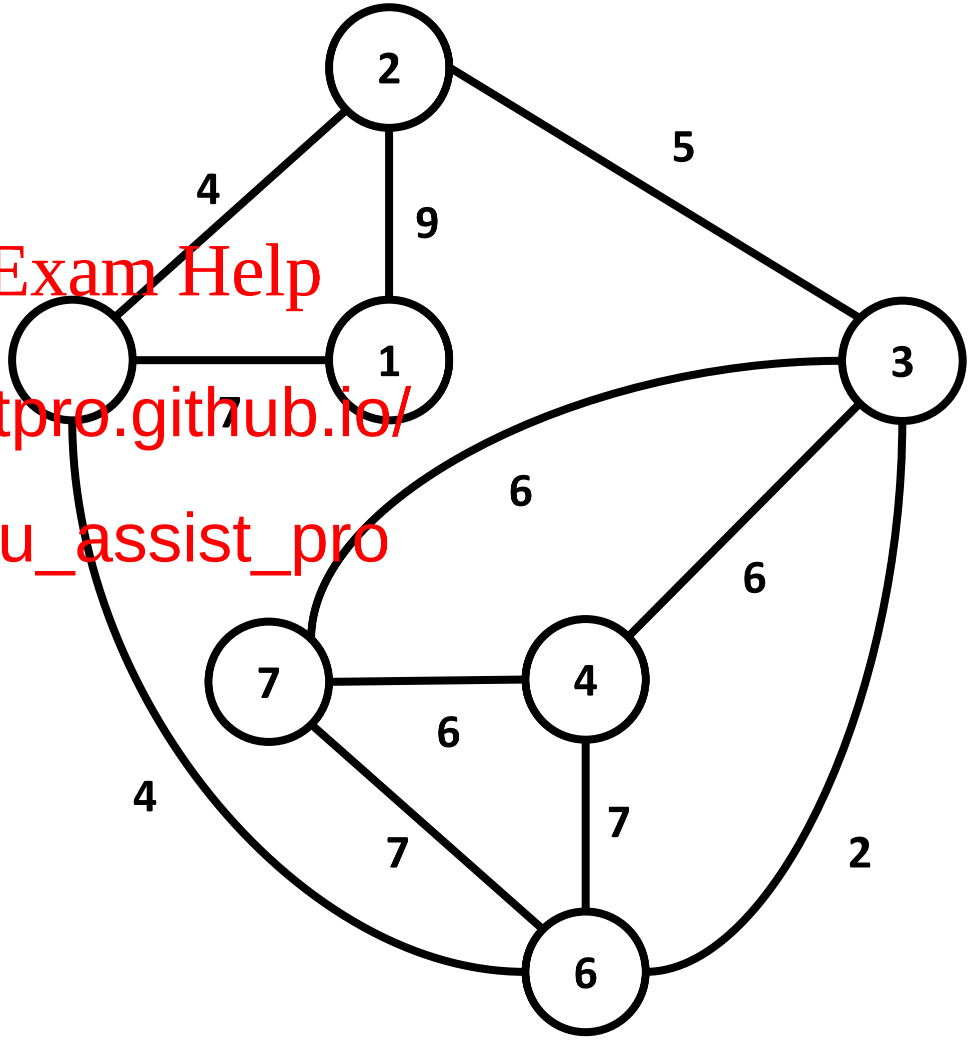
- If G's weight matrix is W then we can express the recurrence relation as:

- A simpler version updating D:

$$
\begin{array}{l}
\textbf{function } \textsc{Floyd}(W[\cdot,\cdot], n) \\
\quad D \leftarrow W \\
\quad \textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do} \\
\quad\quad\quad\quad D[i,j] \leftarrow \min\left(D[i,j], D[i,k] + D[k,j]\right) \\
\quad \textbf{return } D
\end{array}
$$

Floyd's Algorithm

- Let’s examine this algorithm. Let our graph be.



- Then, the weight matr

$$\begin{bmatrix} 0 & 9 & \infty & \infty & 7 & \infty & \infty \\ 9 & 0 & 5 & \infty & 4 & \infty & \infty \\ \infty & 5 & 0 & 6 & \infty & 2 & 6 \\ \infty & \infty & 6 & 0 & \infty & 7 & 6 \\ 7 & 4 & \infty & \infty & 0 & 4 & \infty \\ \infty & \infty & 2 & 7 & 4 & 0 & 7 \\ \infty & \infty & 6 & 6 & \infty & 7 & 0 \end{bmatrix}$$

Floyd's Algorithm

- For *k*=1 there are no changes.

function $\textsc{Floyd}(W[\cdot,\cdot], n)$
    $D \leftarrow W$
    for $k \leftarrow 1$ to n **do**
        for $i \leftarrow 1$ to n **do**
            for $j \leftarrow 1$ to n **do**
                $D[i,j] \leftarrow \min\left(D[i,j], D[i,k] + D[k,j]\right)$
    return D

Floyd's Algorithm

- For k=2, **D[1,2] = 9** and **D[2,3]=5**; and **D[4,2] = 4** and **D[2,3]=5**.
  - Hence, we can make **D[1,3]=14** and **D[4,3]=9**
  - Note that the graph is u which makes the matrix

$$
\begin{aligned}
&\textbf{function } \textsc{Floyd}(W[\cdot,\cdot], n)\\
&\quad D \leftarrow W\\
&\quad \textbf{for } k \leftarrow 1 \text{ to } n \textbf{ do}\\
&\quad\quad \textbf{for } i \leftarrow 1 \text{ to } n \textbf{ do}\\
&\quad\quad\quad \textbf{for } j \leftarrow 1 \text{ to } n \textbf{ do}\\
&\quad\quad\quad\quad D[i,j] \leftarrow \min\left(D[i,j], D[i,k] + D[k,j]\right)\\
&\quad \textbf{return } D
\end{aligned}
$$

Floyd's Algorithm

- For k=2, **D[1,2] = 9** and **D[2,3]=5**; and **D[4,2] = 4** and **D[2,3]=5**.
  - Hence, we can make **D[1,3]=14** and **D[4,3]=9**
  - Note that the graph is u
    which makes the matrix

function FLOYD($W[\cdot,\cdot], n$)
  $D \leftarrow W$
  for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
      for $j \leftarrow 1$ to n **do**
        $D[i,j] \leftarrow \min\left(D[i,j], D[i,k] + D[k,j]\right)$
  return D

Floyd's Algorithm

- For k=3, we can reach all other nodes in the graph.
  - However, these may not be the shortest paths.

function $\textsc{Floyd}(W[\cdot,\cdot], n)$
    $D \leftarrow W$
    for $k \leftarrow 1$ to n **do**
        for $i \leftarrow 1$ to n **do**
            for $j \leftarrow 1$ to n **do**
                $D[i,j] \leftarrow \min\left(D[i,j], D[i,k] + D[k,j]\right)$
    return D

Floyd's Algorithm

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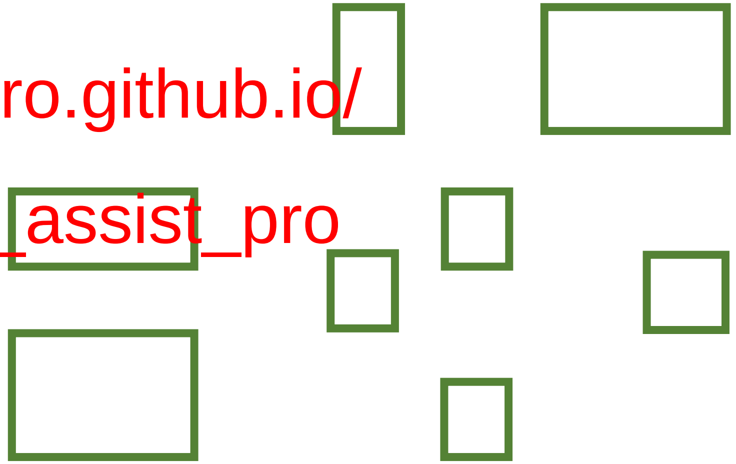


function $\textsc{Floyd}(W[\cdot,\cdot], n)$
    $D \leftarrow W$
    for $k \leftarrow 1$ to n **do**
        for $i \leftarrow 1$ to n **do**
            for $j \leftarrow 1$ to n **do**
                $D[i,j] \leftarrow \min(D[i,j], D[i,k] + D[k,j])$
    return D

Floyd's Algorithm

- Let’s look at the final steps:

$$
\begin{bmatrix}
0 & 9 & 14 & 20 & 7 & 16 & 20 \\
9 & 0 & 5 & 11 & 4 & 7 & 11 \\
14 & 5 & 0 & 6 & 9 & 2 & 6 \\
20 & 11 & 6 & 0 & 15 & 7 & 6 \\
7 & 4 & 9 & 15 & 0 & 4 & 15 \\
16 & 7 & 2 & 7 & 4 & 0 & 7 \\
20 & 11 & 6 & 6 & 15 & 7 & 0
\end{bmatrix}
$$

k=4

Floyd's Algorithm

- Let's look at the final steps:

$$\begin{bmatrix} 0 & 9 & 14 & 20 & 7 & 16 & 20 \\ 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ 14 & 5 & 0 & 6 & 9 & 2 & 6 \\ 20 & 11 & 6 & 0 & 15 & 7 & 6 \\ 7 & 4 & 9 & 15 & 0 & 4 & 15 \\ 16 & 7 & 2 & 7 & 4 & 0 & 7 \\ 20 & 11 & 6 & 6 & 15 & 7 & 0 \end{bmatrix}$$

k=4

k=5

Floyd's Algorithm

- Let's look at the final steps:

$$\begin{bmatrix} 0 & 9 & 14 & 20 & 7 & 16 & 20 \\ 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ 14 & 5 & 0 & 6 & 9 & 2 & 6 \\ 20 & 11 & 6 & 0 & 15 & 7 & 6 \\ 7 & 4 & 9 & 15 & 0 & 4 & 15 \\ 16 & 7 & 2 & 7 & 4 & 0 & 7 \\ 20 & 11 & 6 & 6 & 15 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 & \boxed{13} & \boxed{18} & 7 & 11 & \boxed{18} \\ 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ \boxed{13} & 5 & 0 & 6 & \boxed{6} & 2 & 6 \\ \boxed{18} & 11 & 6 & 0 & \boxed{11} & 7 & 6 \\ 7 & 4 & \boxed{6} & \boxed{11} & 0 & 4 & \boxed{11} \\ 11 & 7 & 2 & 7 & 4 & 0 & 7 \\ \boxed{18} & 11 & 6 & 6 & \boxed{11} & 7 & 0 \end{bmatrix}$$

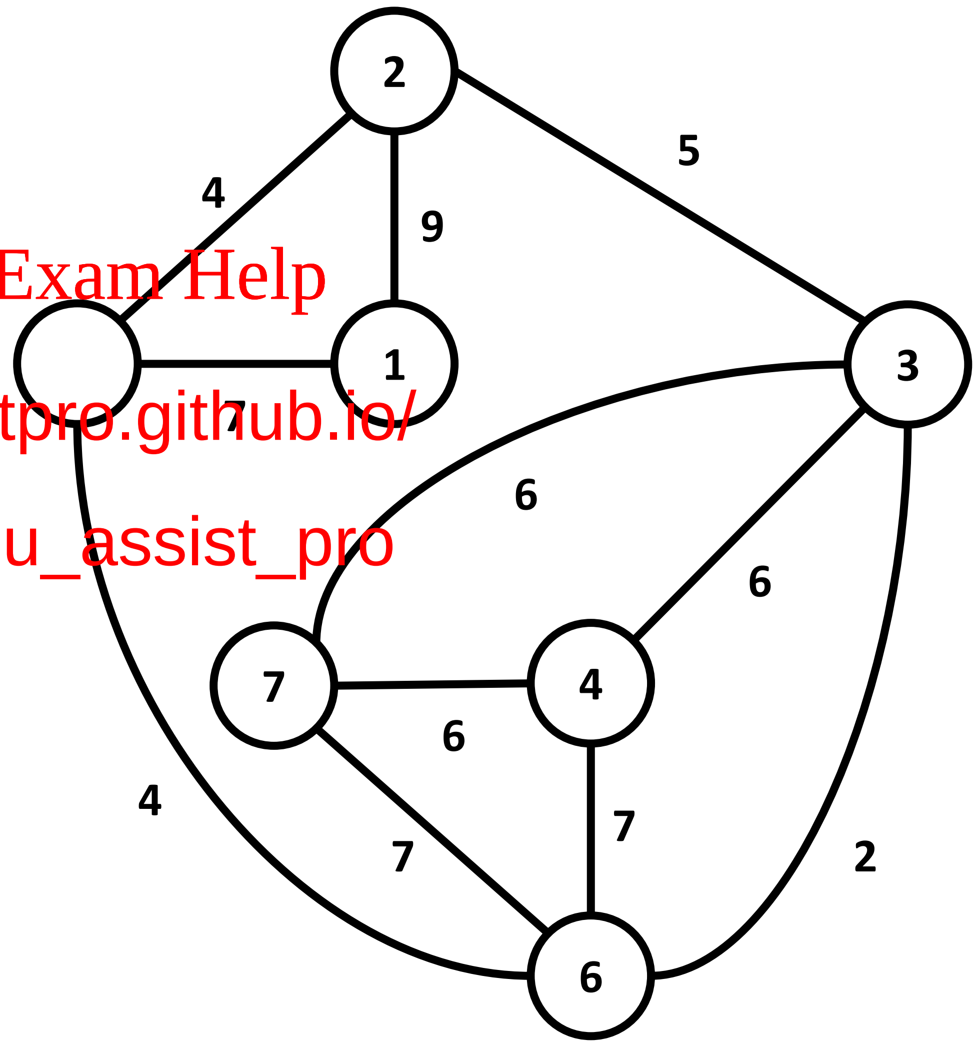
k=4 k=5 k=6

- For k=7, it is unchanged. So we have found the best paths.

A Sub-Structure Property

- For a DP approach to be applicable, the problem must have a "**sub-structure**" that allows for a compositional solution.
  - Shortest-path problems ha property. For example, if { x_n} is a shortest path from $\{x_1, x_2, \ldots, x_i\}$ is a shortest path from x_1 to x_n.

- Longest-path problems don't have that property.
  - In our sample graph, {1,2,5,6,7,4,3} is a longest path from 1 to 3, but {1,2} is not a longest path from 1 to 2 (since {1,5,6,7,4,3,2} is longer).



Next lecture

- Greedy algorithms