COMP90038

Algorit plexity

Lecture 22: NP Problems and n Algorithms

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Recap

- We continued discussing **greedy algorithms**:
  - A problem solving strategy that takes the **locally best** choice among all feasible ones. Such choice is **irrevocable**.
  - Usually, **locally best** ch **est** results.
  - In some exceptions a g **and** .
  - Also, a greedy algorithm can provide g **nations**.

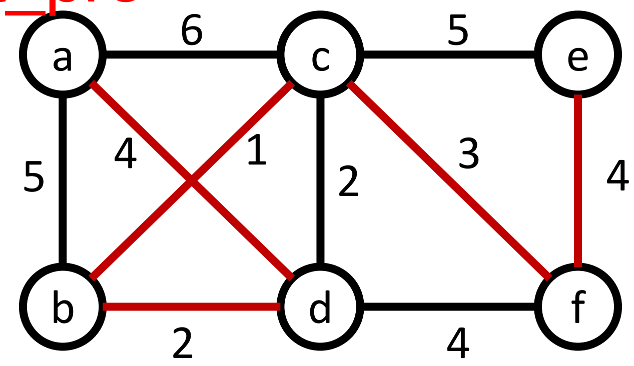
- We applied this idea to graphs and data compression:
  - Prim’s and Djikstra Algorithms
  - **Huffman Algorithms and Trees** for variable length encoding.

Prim's Algorithm

- Starting from different nodes produces a different sequence.
  - However, the tree will have the same edges.
  - Unless there are edges weights, as tie breaking would influen which one to take.

START	SEQUENCE	EDGES
a	a-d-b-c-f-e	(a,d)(b,d)(b,c)(c,f)(e,f)
	b-c-d-f-a-e	(b,c)(b,d)(c,f)(a,d)(e,f)
	c-d-b-f-a-e	(c,d)(b,d)(c,f)(a,d)(e,f)

- The following example has only one tree. Tie breaking was done alphabetically.



Variable-Length Encoding

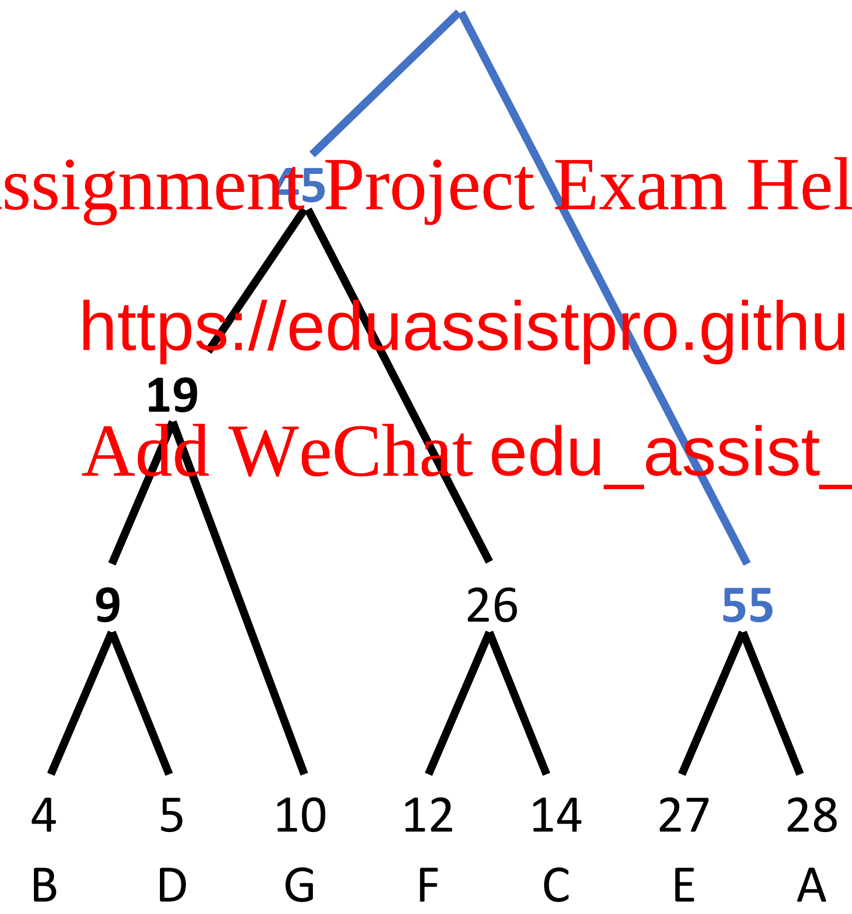
- Variable-Length encoding assigns shorter codes to common characters.
  - In English, the most common character is **E**, hence, we could assign **0** to it.
  - However, no other characte

- That is, no character's code should be a prefix other character's code (unless we somehow p separators between characters, which would space).

- The table shows the occurrences and some sensible codes for the alphabet {A,B,C,D,E,F,G}
  - This table was generated using **Huffman's algorithm** – another example of a **greedy method**.

SYMBOL	OCCURRENCE	CODE
A	28	11
B	4	0000
	14	011
	5	0001
E	27	10
F	12	010
G	10	001

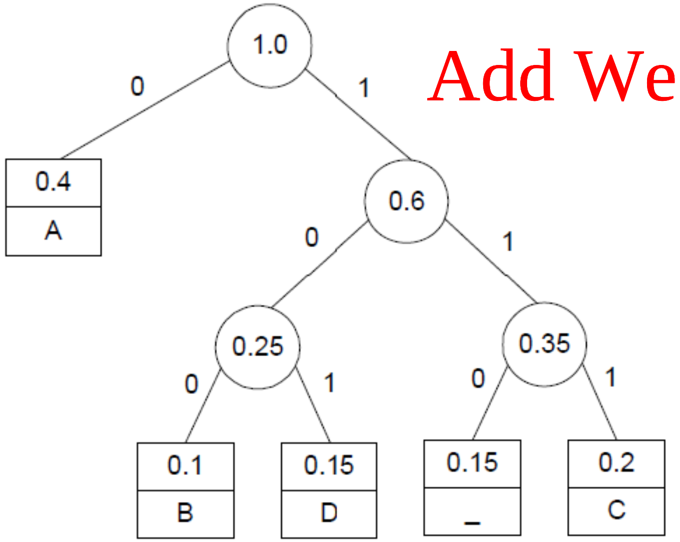
Huffman Trees (example)



An exercise

- Construct the Huffman code for data in the table, placing in the tree from left to right [A,B,D,C,_]
- Then, encode **ABACABAD** and decode **100010111001010**
- 0100011101000101 / BAD_A

SYMBOL	FEQUENCY	CODE
A	0.40	0
B	0.10	100
	0.20	111
	0.15	101
_	0.15	110



Concrete Complexity

- So far our concern has been the analysis of algorithms from the running time point of orst cases)

- Our approach has been to determin **ptotic** behavior of the running time **as a function of the input size**.
  - For example, the quicksort algorithm is $O(n^2)$ in the worst case, whereas mergesort is $O(n \log n)$.

Abstract Complexity

- The field of **complexity theory** focuses on the question:

“What is the inhe ?”

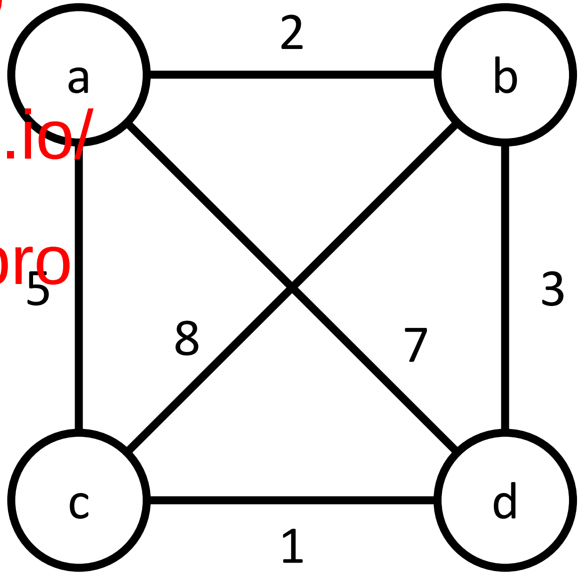
- How do we know that an algorithm is **optimal** (in the asymptotic sense)?

Difficult problems

- Which problems are difficult to solve?

- The Travelling Salesm
solved through brute f
instances.
    - One solution is: a-b-d-c-a

- However, it becomes very difficult as the number of nodes and connections increase.
    - However, you can check the solution and determine if it is a good solution or not?



Does P=NP?

- The **“P versus NP”** problem comes from **computational complexity theory**

- P means with polynomial time complexity
  - That is, algorithms that h
  - Sorting is a type of polyn

- NP means non-deterministic polynomi
  - You can check the answer in polynomial time, but cannot find the answer in polynomial time for large n
  - The TSP problem is an NP problem

- This is the most important question in Computer Science

Algorithmic problems

- When we talk about a **problem**, we almost always mean a family of **instances** of a general problem

- An **algorithm** for the p r all possible instances

- Examples:
  - The **sorting** problem – an instance is a sequence of items.
  - The **graph k-colouring** problem – an instance is a graph.
  - **Equation solving** problems – an instance is a set of, say, linear equations.

Easy and hard problems

- A path in a graph *G* is **simple** if it visits each node of *G* at most once.

- Consider these two pr graphs *G*:

  - **SPATH**: Given *G* and tw ere a simple path from *a* to *b* of length **at most** *k*?

  - **LPATH**: Given *G* and two nodes *a* and *b* e a simple path from *a* to *b* of length **at least** *k*?

- If you had a large graph *G*, which of the two problems would you rather have to solve?

Easy and hard problems

- There are fast algorithms to solve SPATH.
  - For example, we can do a BFS over raph.

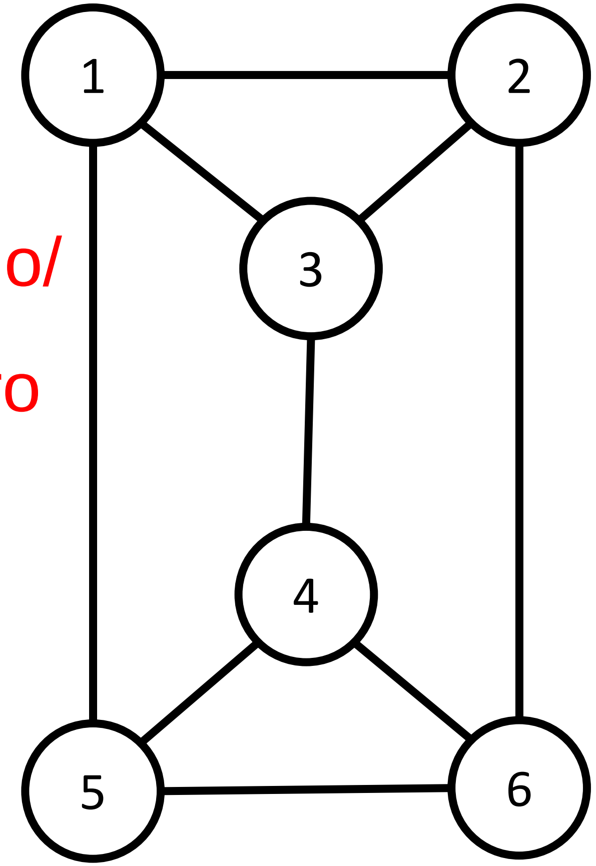
- nows of a fast algorithm for LPATH.

- It is likely that the LPATH problem cannot be solved in polynomial time.

Easy and hard problems

- Other two related problems:
  - The Eulerian tour problem: In a given graph, is there a path which visits each **edge** of the graph once, returning to the origin?
  - The Hamiltonian tour pr ere a path which visits each **n** ning to the origin?

- Is the Eulerian tour problem P?
  - We just need to know whether the edge distribution is even.
- Is the Hamiltonian tour P?
  - No. As the nodes increase, runtime becomes exponential.



Easy and hard problems

- Some more examples:
  - **SAT**: Given a propositi able?
  - **SUBSET-SUM**: Given a and a positive integer t, is there a subset of S that adds up to t?
  - **3COL**: Given a graph G, is it possible to odes of G using only three colours, so that no edge connects two nodes of the same colour?
- Although these problems are very different they share an interesting property

Polynomial time verifiability

- While most instances of these problems cannot be solved in polynomial time, we can test a solution in polynomial time

- In other words, while they **seem hard t** y allow for **efficient verification**.

- This is called **polynomial-time verifiable**

- To understand this concept we need to talk about **Turing Machines**

Turing Machines

- Turing Machines are an **abstract model of a computer**.

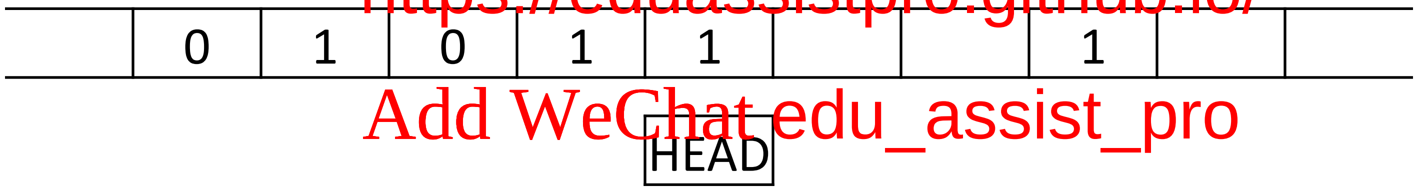
- Despite of their simpli ve the same **computational power** ting device

  - That is, any function that can be imple , Java, etc. can be implemented in a Turing Machine

- Moreover, a Turing Machine is able to **simulate** any other Turing Machine.
  - This is known as the **universality** property

Turing Machines

- A Turing machine is represented as an **infinity sized memory space**, and a **read/write head**

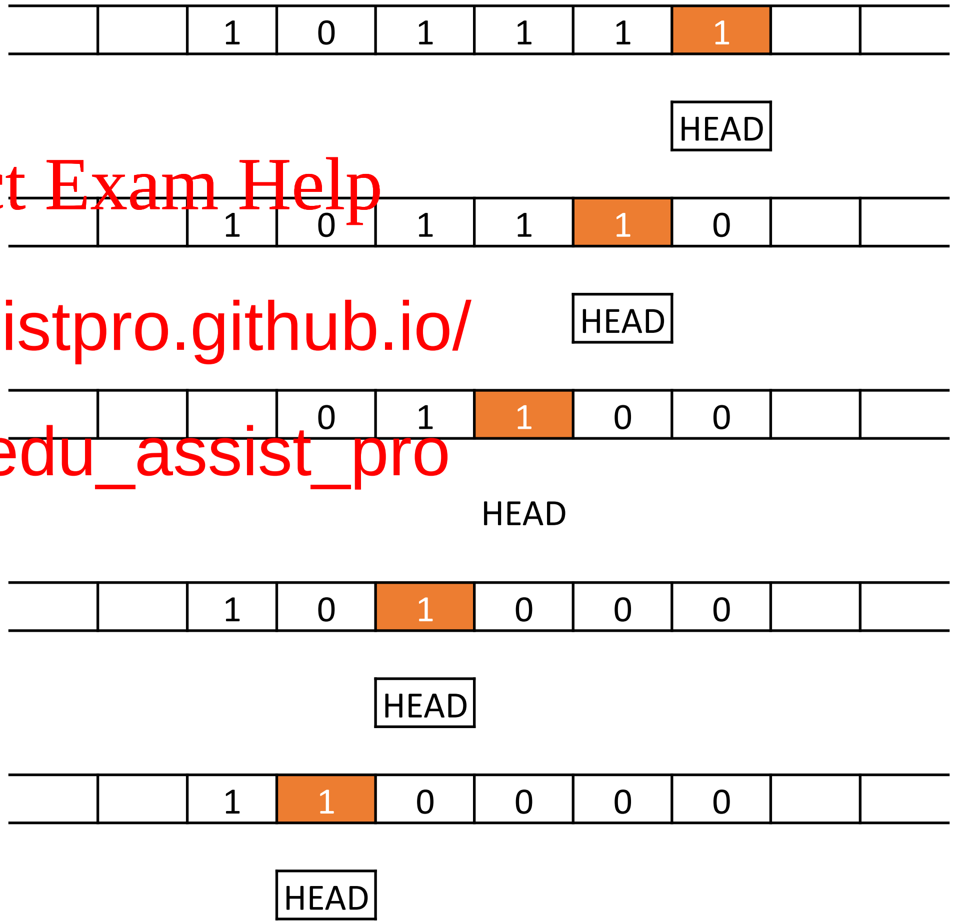


- Whether the head reads, writes or moves to left or right depends of a **control sequence**

An example

- Let the control sequence be:
  - If read **1**, write **0**, go **LEFT**
  - If read **0**, write **1**, **HALT**
  - If read _, write **1**, **HAL**

- The input will be $47_{10} = 101111_2$
- The output is $48_{10} = 11000_2$
  - In other words, this rules add one to a number



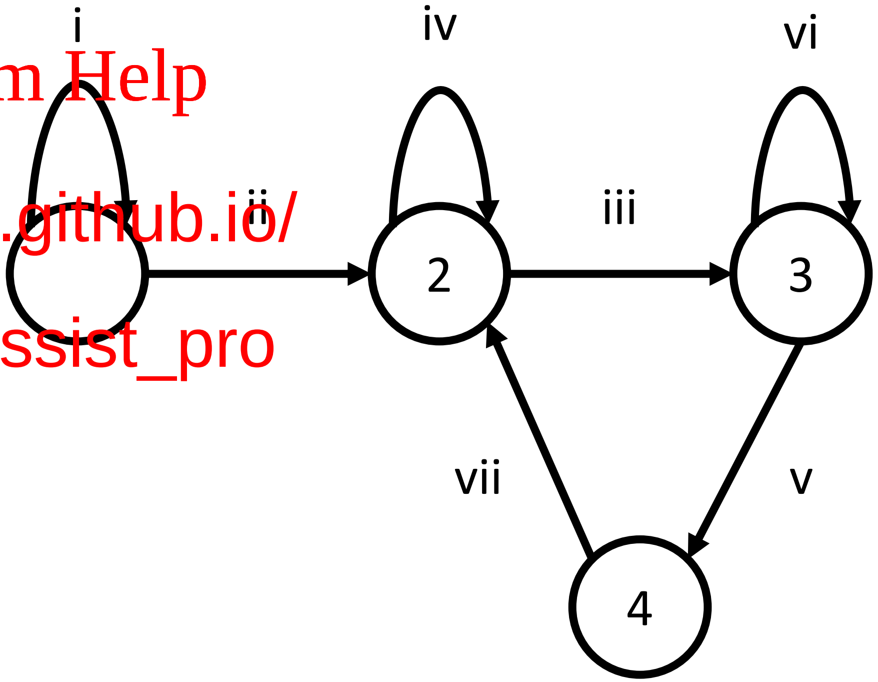
A more complex control sequence

- We will develop an state automaton:
  1. If $\mathbf{S_1}$ and **a**, go **RIGHT** stay in $\mathbf{S_1}$
  2. If $\mathbf{S_1}$ and **b**, go **RIGHT** go to $\mathbf{S_2}$

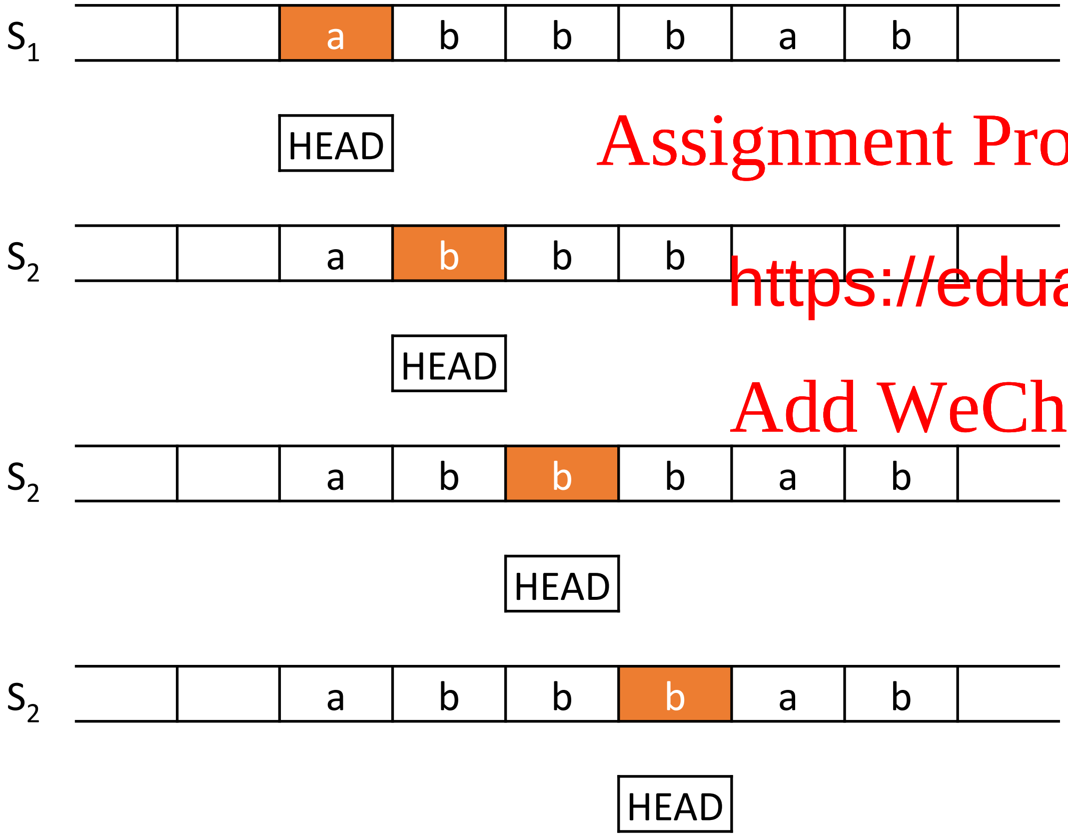
  3. If $\mathbf{S_2}$ and **a**, write **b** g
  4. If $\mathbf{S_2}$ and **b**, go **RIGHT** stay in $\mathbf{S_2}$

  5. If $\mathbf{S_3}$ and **a** or _, go **RIGHT** go to $\mathbf{S_4}$
  6. If $\mathbf{S_3}$ and **b**, go **LEFT** stay in $\mathbf{S_3}$

  7. If $\mathbf{S_4}$ and **b**, write **a** go **RIGHT** go to $\mathbf{S_2}$

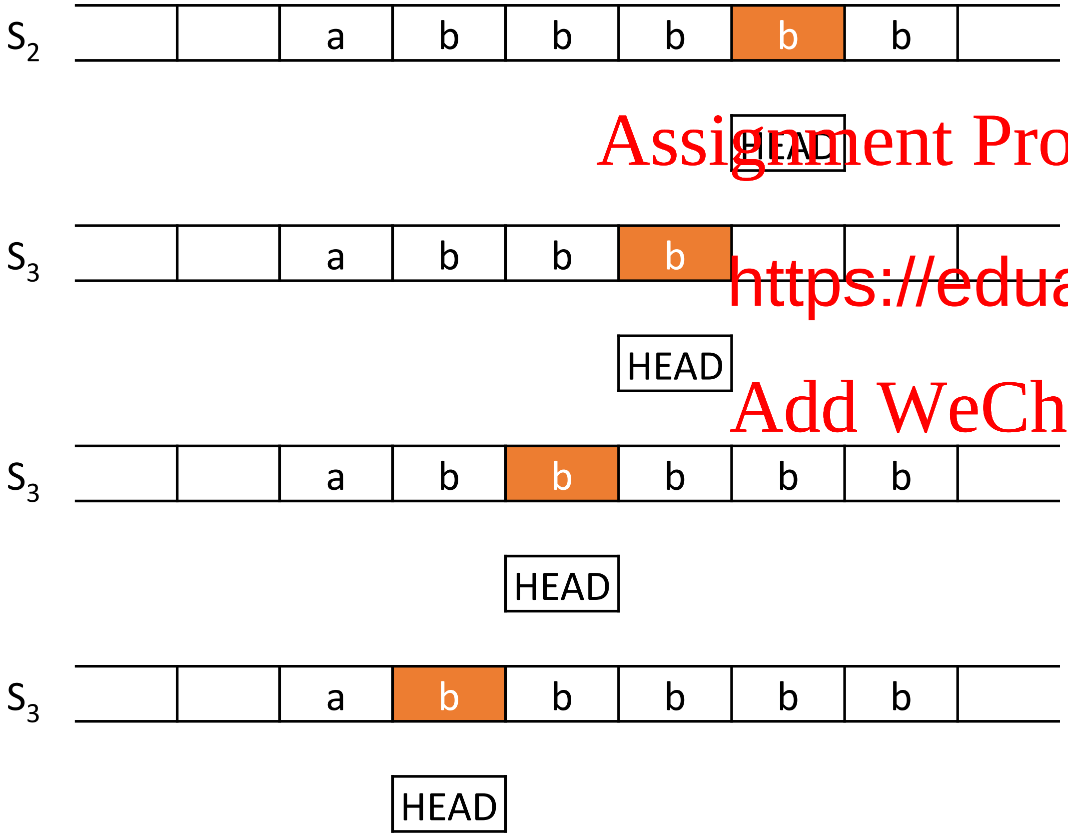


Example



- What would this machine do for the input **abbbab**?
  i. If **S_1** and **a**, go **RIGHT** stay in **S_1**
  ii. If **S_1** and **b**, go **RIGHT** go to **S_2**
  $_2$ and **a**, write **b** go **LEFT** go to **S_3**
  nd **b**, go **RIGHT** stay in **S_2**
  $_3$ nd **a** or **_**, go **RIGHT** go to **S_4**
  vi. If **S_3** and **b**, go **LEFT** stay in **S_3**
  vii. If **S_4** and **b**, write **a** go **RIGHT** go to **S_2**

Example



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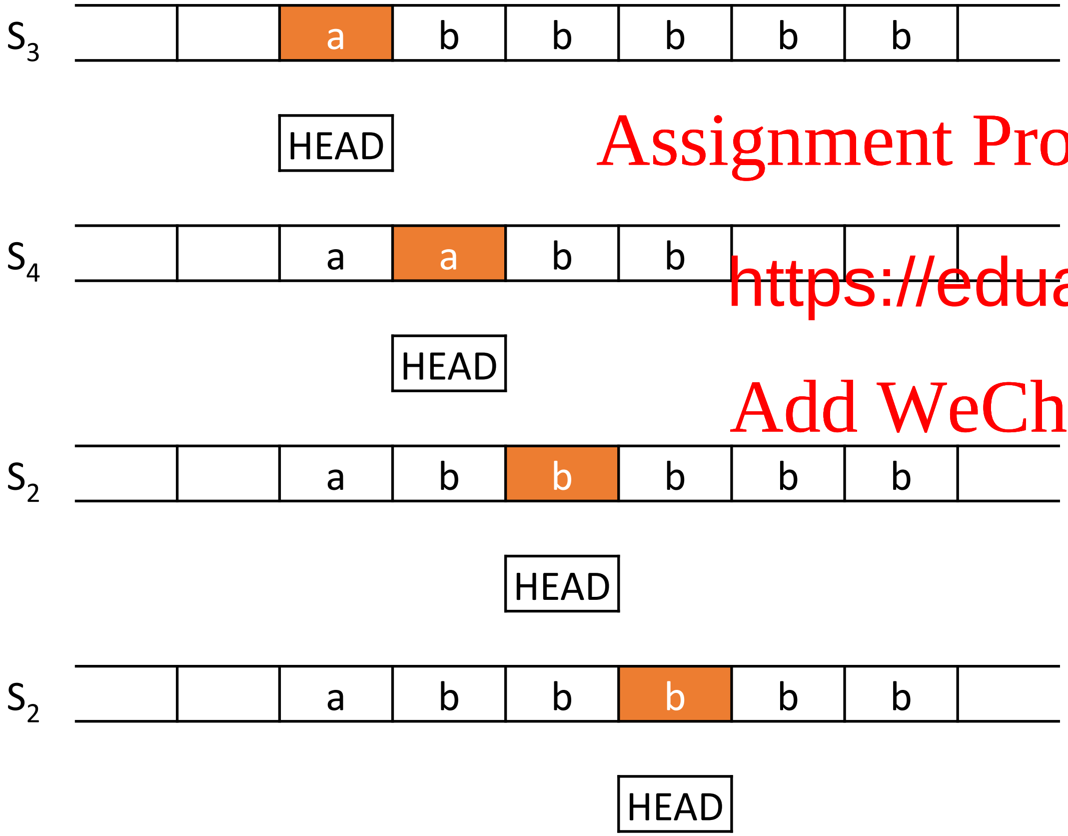

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- The machine **sorts** the letters upon completion

Non-deterministic Turing Machines

- From now onwards we will assume that a Turing Machine will be used to implement **decision procedures**
  - That is an algorithm with **YES/NO** answers

- Now, lets assume that has a powerful **guessing** capability:

  - If different moves are available, the m avour one that leads to a **YES** answer

- Adding this **non-deterministic** capability does not change **what** the machine can compute, but affects its **efficiency**

Non-deterministic Turing Machines

- What a non-deterministic Turing machine can compute in polynomial time corresponds exactly to the class of polynomial-time verifiable problems.

- In other words:
  - **P** is the class of problems solvable in polynomial time by a **deterministic** Turing Machine
  - **NP** is the class of problems solvable in polynomial time by a **non-deterministic** Turing Machine

- Clearly P $\subseteq$ NP. Is P = NP?

Problem reduction

- The main tool used to determine the class of a problem is **reducibility**

- Consider two problems

- Suppose that we can transform, **witho** **h effort**, any instance *p* of *P* into an instance *q* of *Q*

- Such transformation should be **faithful**. That is we can extract a solution to *p* from a solution of *q*

A very simple example

- **Multiplication** and **squaring**:
  - Suppose all we know to do is how to add, subtract, take squares and divide by two.

  - Then, we can use this f[illegible]roduct of any two numbers:

$$a \times b = \frac{\left((a+b)^2 - a \quad - b \quad\right)}{2}$$

  - We can also go the other direction, that is, if we can multiply two numbers, we can calculate the square.

Another example

- The Hamiltonian cycle (HAM) and the Travelling Salesman (TSP) problems have similari

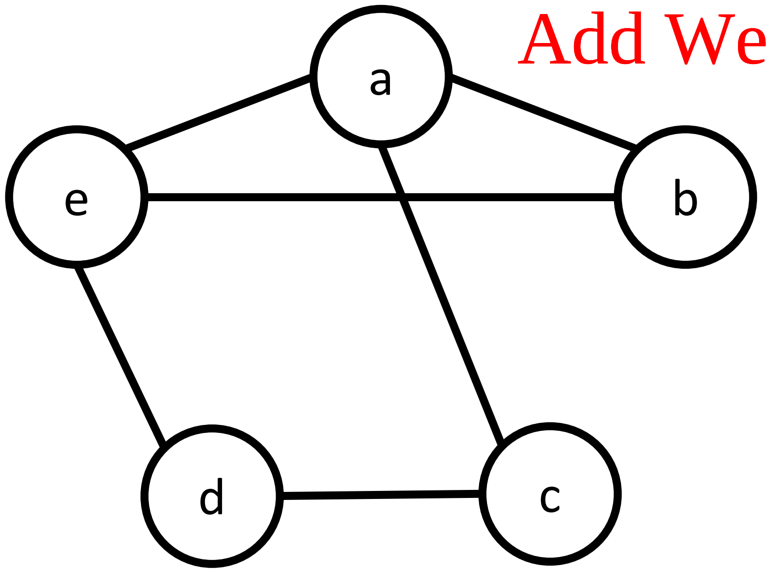
  - Both operate on graph

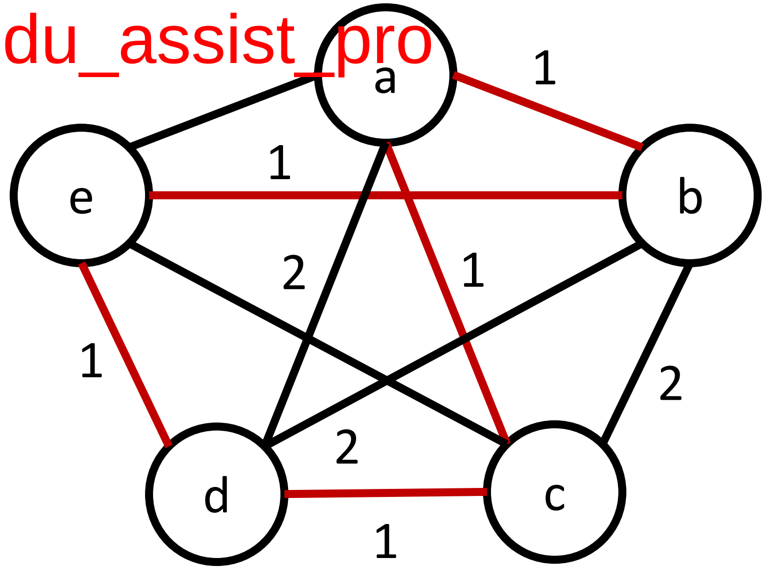
  - Both try to find a tour that visits the v nce

- The only difference is that the HAM works in unweighted graphs and TSP does in weighted graphs

Reducing HAM to TSP

- We can transform a HAM problem into a TSP problem:
  - By assigning **1** to all the edges in the unweighted graph
  - By creating paths between unconnected edges with weight of **2**
  - If there is a TSP tour of amiltonian cycle.





Problem reduction

- Problem reduction allows us to make a few conclusions:

  - If a reduction from P t **east as hard** as Q

  - If Q is known to be hard, then we may decide **not to waste more time** trying to find an efficient algorithm for P

Dealing with difficult problems

- **Pseudo-polynomial problems** (SUBSET-SUM and KNAPSACK are in this class): Unless you have really large instance, there is no need to panic. For small enough instances the bad behavior is not yet present.

- **Clever engineering** to p [illegible]: SAT solvers.

- **Approximation algorithms**: Settle for less than perfection.

- **Live happily** with intractability: Sometimes the bad instances never turn up in practice.

Approximation Algorithms

- For intractable optimi kes sense to look for **approximation algorit** still find solutions that are reasonably close to the optimal.

Example: Bin packing

- **Bin packing** is closely related to the knapsack problem.

- Given a finite set U = { nd a rational size $s(u) \in [0,1]$ for each ite

to disjoint subsets U_1, U_2, …, U_k such that

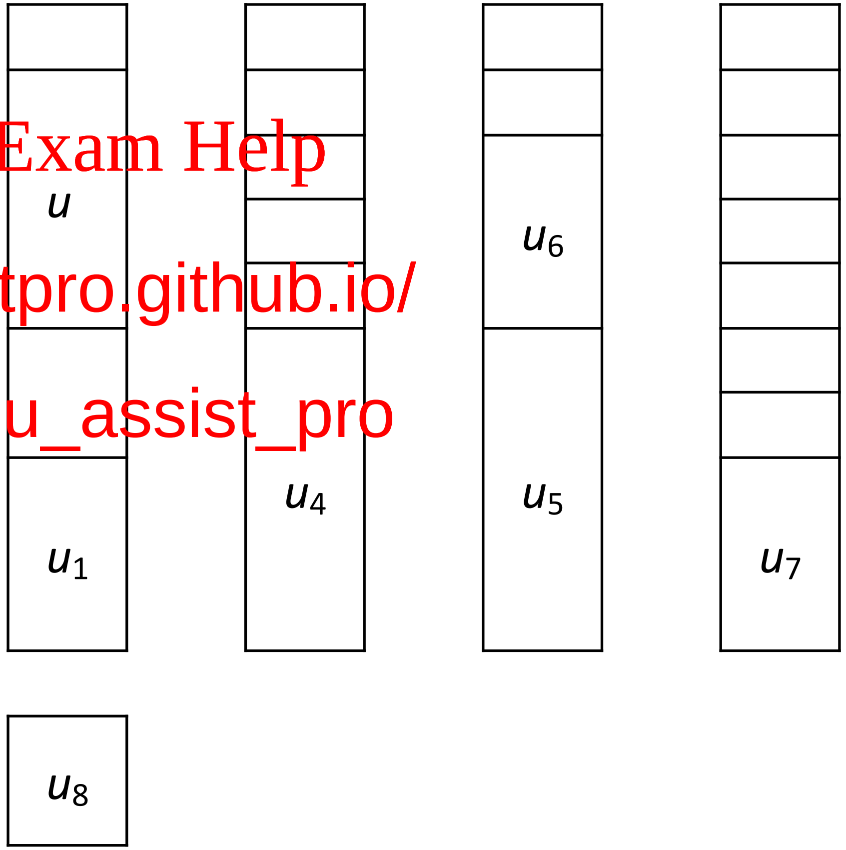
  - the sum of the sizes of items in U_i is at
  - k is as small as possible.

- The bin-packing problem is NP-hard.

Bin packing

- In plain English, Each subset U_i gives the set of items to be placed in a unit-sized "bin", with the objective of using bins as possible.

- There some **heuristics** that can be used.
  - First Fit: Use the first bin that has the necessary capacity



Bin packing

- For First Bin, the number of bins used Fit is never more than **twice** the minimal number required.
  - First Fit behaves worst when we are left with many large items towards the end.

- The variant in which the r of decreasing size performs better.

- The added cost (for sorting the items) is not large.

- This variation guarantees that the number of bins used cannot exceed $\frac{11n}{9} + 4$ where *n* is the optimal solution.

Next week

- We will review the co