

COMP3121/9101

ALGORITHM DESIGN

PROBLEM SET 5 – DYNAMIC PROGRAMMING

[K] – key questions [H] – harder questions [E] – extended questions [X] – beyond the scope of this course

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§ SECTION ONE: OPTIMISATION

[K] **Exercise 1.** You are given n intervals on an axis. The i th interval $[l_i, r_i)$ has integer endpoints $l_i < r_i$ and has a score of s_i . Your task is to select a set of disjoint intervals with maximum total score. Note that if intervals i and j satisfy $r_i = l_j$ then they are still disjoint. Design an algorithm which solves this problem and runs in $O(n^2)$ time.

[K] **Exercise 2.** Due to the recent droughts, n proposals have been made to dam the Murray river. The i th proposal asks to place a dam x_i meters from the head of the river (i.e., from the source of the river) and requires that there is not another dam within r_i metres (upstream or downstream). Design an algorithm that returns the maximal number of dams that can be built, you may assume that $x_i < x_{i+1}$ for all $i = 1, \dots, n - 1$.

[K] **Exercise 3.** You are given a set of n types of rectangular boxes, where the i^{th} box has height h_i , width w_i and depth d_i . You want to create a stack of boxes which is as tall as possible, but you can only stack a box on top of another box if the dimensions of the 2-D base of the lower box are each strictly larger than those of the 2-D base of the higher box. Of course, you can rotate a box so that any side functions as its base. It is also allowable to use multiple instances of the same type of box. Design an $O(n^2)$ algorithm that returns the maximum height of the stack of boxes.

Hint: How can we reduce this to a problem without rotations?

[K] **Exercise 4.** You have n_1 items of size s_1 and n_2 items of size s_2 . You would like to pack all of these items into bins, each of capacity $C > s_1, s_2$, using as few bins as possible. Design an algorithm that returns the minimal number of bins required.

[K] **Exercise 5.** Consider a row of n coins, indexed 1 to n , with values v_1, v_2, \dots, v_n . Two players take turns selecting a coin from the row, either the first or last coin, until all coins are taken. Each player's goal is to maximize their total value. Determine the maximum possible value for the first player, assuming both players play optimally.

[K] **Exercise 6.** Given a sequence of n positive or negative integers A_1, A_2, \dots, A_n , determine a contiguous subsequence A_i to A_j for which the sum of elements in the subsequence is maximised.

[H] **Exercise 7.** Skiers go fastest with skis whose length is close to their height. Consider n skiers with heights h_1, h_2, \dots, h_n , gets a delivery of $m \geq n$ pairs of skis, with lengths l_1, l_2, \dots, l_m . Design an algorithm to assign to each skier one pair of skis to minimise the sum of the absolute differences between the height h_i of the skier and the length of the corresponding ski they got, i.e., to minimise

$$S = \sum_{i=1}^n |h_i - l_{s(i)}|$$

where $s(i)$ is the index of the skis assigned to the skier of height h_i .

[H] **Exercise 8.** You have been handed responsibility for a business in Texas for the next n days. Initially, you have K illegal workers. At the beginning of each day, you may hire an illegal worker, keep the number of illegal workers the same or fire an illegal worker. At the end of each day, there will be an inspection. The inspector on the i^{th} day will check that you have between l_i and r_i illegal workers (inclusive). If you do not, you will fail the inspection. Design an

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algorithm that determines the fewest number of inspections you will fail if you hire and fire illegal employees optimally.

[H] **Exercise 9.** A company is organising a party for its employees. The organisers of the party want it to be a fun party, and so have assigned a fun rating to every employee. The employees are organised into a strict hierarchy, i.e. a tree rooted at the president. There is one restriction, though, on the guest list to the party: an employee and their immediate supervisor (parent in the tree) cannot both attend the party (because that would be no fun at all). Give an algorithm that makes a guest list for the party that maximises the sum of the fun ratings of the guests.

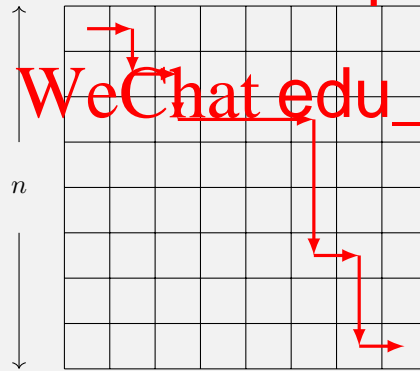
[H] Exercise 10. You have to cut a wood stick into several pieces. The most affordable company, Analog Cutting Machinery (ACM), charges money according to the length of the stick being cut. Their cutting saw allows them to make only one cut at a time. It is easy to see that different cutting orders can lead to different prices. For example, consider a stick of length 10 m that has to be cut at 2, 4, and 7 m from one end. There are several choices. One can cut first at 2, then at 4, then at 7. This leads to a price of $10 + 8 + 6 = 24$ because the first stick was of 10 m, the resulting stick of 8 m, and the last one of 6 m. Another choice could cut at 4, then at 2, then at 7. This would lead to a price of $10 + 4 + 6 = 20$, which is better for us. Your boss demands that you design an $O(n^2)$ algorithm to find the minimum possible cutting cost for any given stick.

[H] **Exercise 11.** You are given an $n \times n$ chessboard with an integer in each of its n^2 squares. You start from the top left corner of the board; at each move you can go either to the square immediately below or to the square immediately to the right of the square you are at the moment; you can never move diagonally. The goal is to reach the right bottom corner so that the sum of integers at all squares visited is maximized. Write an algorithm to find the maximum possible sum.

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- Describe a greedy algorithm which attempts to find such a minimal sum path and show by an example that such a greedy algorithm might fail to find such a minimal sum path.
- Describe an algorithm which always correctly finds a minimal sum path and runs in time n^2 .
- Describe an algorithm which computes the number of such minimal paths.
- Assume now that such a chessboard is stored in a read only memory. Describe an algorithm which always correctly finds a minimal sum path and runs in linear space (i.e., amount of read/write memory used is $O(n)$) and in time $O(n^2)$.

Hint: For (d), combine divide and conquer with dynamic programming.

§ SECTION TWO: ENUMERATION

[K] **Exercise 12.** Some people think that the bigger an elephant is, the smarter it is. To disprove this, you want to analyse a collection of elephants and place as large a subset of elephants as possible into a sequence whose weights are increasing but their IQ's are decreasing. Design an $O(n \log n)$ algorithm which, given the weights and IQ's of n elephants, will find a longest sequence of elephants such that their weights are increasing but IQ's are decreasing.

[K] **Exercise 13.** There are n levels to complete in a video game. Completing a level takes you to the next level, however each level has a secret exit that lets you skip to another level later in the game. Determine if there is a path through the game that plays exactly K levels.

[K] **Exercise 14.** A partition of a number n is a sequence $\langle p_1, p_2, \dots, p_t \rangle$ (we call the p_k *parts*) such that $1 \leq p_1 \leq p_2 \leq \dots \leq p_t \leq n$ and such that $p_1 + \dots + p_t = n$. Define $\text{numpart}(k, n)$ to be the number of partitions of n such that every part is at most k .

- Explain why $\text{numpart}(1, n) = 1$ for each n .
- Devise a recursion to determine the number of partitions of n in which every part is smaller or equal than k , where n, k are two given numbers such that $2 \leq k \leq n$.
- Hence, find the total number of partitions of n .

[K] **Exercise 15.** Given an array of n integers, find the maximum number of contiguous blocks of sum at most k .

[H] **Exercise 16.** You are given a boolean expression consisting of a string of the symbols true (T) and false (F) and with exactly one operation of $\text{and (}\wedge\text{)}$, $\text{or (}\vee\text{)}$ or $\text{xor (}\oplus\text{)}$ between two truth values). Count the number of ways to place brackets in the expression such that it will evaluate to true .

[H] **Exercise 17.** We are given a checkerboard which has 4 rows and n columns, and has an integer written in each square. We are also given a set of $2n$ pebbles, and we want to place some or all of these on the checkerboard (each pebble can be placed on exactly one square) so as to maximise the sum of the integers in the squares that are covered by pebbles. There is one constraint: for a placement of pebbles to be legal, no two of them can be on horizontally or vertically adjacent squares (diagonal adjacency is fine).

- Determine the number of legal *patterns* that can occur in any column (in isolation, ignoring the pebbles in adjacent columns) and describe these patterns.

Call two patterns *compatible* if they can be placed on adjacent columns to form a legal placement. Let us consider sub-problems consisting of the first k columns $1 \leq k \leq n$. Each sub-problem can be assigned a type, which is the pattern occurring in the last column.

- Using the notions of compatibility and type, give an $O(n)$ -time algorithm for computing an optimal placement.

§ SECTION THREE: KNAPSACK AND RELATED PROBLEMS

[K] **Exercise 18.** You have an amount of money M and you are in a candy store. There are n kinds of candies and for each candy, you know how much pleasure you get by eating it, which is a number between 1 and K , as well as the price of each candy. Your task is to choose which candies you are going to buy to maximise the total pleasure you will get by gobbling them all.

[K] **Exercise 19.** Your shipping company has just received N shipping requests (jobs). For each request i , you know it will require t_i trucks to complete, paying you d_i dollars. You have T trucks in total. Out of these N jobs you can take as many as you would like, as long as no more than T trucks are used total. Devise an efficient algorithm to select jobs that will bring you the largest possible amount of money.

[K] **Exercise 20.** Again your shipping company has just received N shipping requests (jobs). This time, for each request i , you know it will require e_i employees and t_i trucks to complete, paying you d_i dollars. You have E employees and T trucks in total. Out of these N jobs you can take as many of them as you would like, as long as no more than E employees and T trucks are used in total. Devise an efficient algorithm to select jobs which will bring you the largest possible amount of money.

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§ SECTION FOUR: STRINGS

[K] **Exercise 21.** We say that a string A occurs as a subsequence of another string B if we can obtain A by deleting some of the letters of B . Design an algorithm that for two strings A and B gives the number of different occurrences of A in B , i.e., the number of ways one can delete some of the symbols of B to get A .

[K] **Exercise 22.** For bit strings $X = x_1 \dots x_n$, $Y = y_1 \dots y_m$ and $Z = z_1 \dots z_{n+m}$, we say that Z is an interleaving of X and Y if it can be obtained by interleaving the bits in X and Y in a way that maintains the left-to-right order of the bits in X and Y .

[K] **Exercise 23.** A palindrome is a sequence of at least two letters which reads equally from left to right and from right to left. For example, the string $S = ababa$ is a palindrome of length 5.

Given a sequence of letters (a string), find efficiently its longest subsequence (not necessarily contiguous) that is a palindrome. In other words, we are looking for the longest palindrome which can be obtained by crossing out some of the letters of the initial sequence without permuting the remaining letters.

[X] **Exercise 24.** A *context-free grammar* is a tuple $\langle V, \Sigma, R, S \rangle$, where:

- V is a set of variables;
- Σ is a finite set of symbols used to define characters of a word;
- R is a set of production rules th
- S is the start variable.

The rules of a grammar come in the form:

- $A \rightarrow \alpha$ where A is a variable and α is a symbol;
- $A \rightarrow BC$ where A, B, C are variables and B, C are not the start variable;
- $S \rightarrow \epsilon$ where ϵ is the empty character.

To generate a string, we use the production rules given above and keep performing substitutions until we get to a point where there are no variables. That is, we repeatedly replace variables with the corresponding symbol or variable given by the production rules.

(a) Consider the context-free grammar, $G = \langle V, \Sigma, R, S \rangle$, where

- $V = \{A, B, C\}$,
- $\Sigma = \{0, 1\}$,
- R is given by the production rules:
 - $A \rightarrow \epsilon$ or $A \rightarrow BC$,
 - $B \rightarrow 0$ or $B \rightarrow BB$,
 - $C \rightarrow 1$ or $C \rightarrow CC$,
- $S = A$.

We can generate the string 01 because, starting with $A \rightarrow BC$, we can replace B with 0 and C with 1. Thus, we generate $A \rightarrow 01$ and so, G generates 01. Using these production rules, find another string that can be generated by G .

We now attempt to solve the membership problem; that is, given an arbitrary context-free grammar G and a string w , does G generate w ?

- (b) Suppose that w is a symbol. Devise an $O(1)$ algorithm to determine if G generates w .
- (c) Now suppose that w is a string of length 2. Devise a recursion to determine if G generates w .
- (d) Now suppose that w is a string of length n . Devise an $O(n^3)$ algorithm to determine if G generates w .

Note. This formally depends on the size of the grammar but the size of the grammar is independent of the length of the string, so we treat the size of the grammar as a fixed constant.

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§ SECTION FIVE: SHORTEST PATHS

[K] **Exercise 25.** You are travelling by canoe down a river and there are n trading posts along the way. Before starting your journey, you are given for each $1 \leq i < j \leq n$ the fee $F(i, j)$ for renting a canoe from post i to post j . These fees are arbitrary, for example it is possible that $F(1, 3) = 10$ and $F(1, 4) = 5$. You begin at trading post 1 and must end at trading post n (using rented canoes). Your goal is to design an efficient algorithm that produces the sequence of trading posts where you change your canoe which minimizes the total rental cost.

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