

Assignment Project Exam Help

Maths Preliminaries

<https://eduassistpro.github.io>

Add WeChat edu_assist_pr

Introduction

Assignment Project Exam Help

- This review serves two purposes.
 - Recap relevant maths contents that you may have learned a

<https://eduassistpro.github.io>

Machine Learning.

- Contents

Add WeChat edu_assist_pr

Note

Assignment Project Exam Help

- You've probably learned Linear Algebra from matrix/system of linear equations, etc. We will review key concepts in LA

[https://eduassistpro.github.io](https://eduassistpro.github.io/cs-and-maths-prelims/)

- Here we emphasize more on intuitions; We deliberately skip many concepts and present some content
- It is a great exercise for you to view related mathematical models/operations in this perspective

A Common Trick in Maths I

Question

Calculate 2^{10} , 2^{-1} , $2^{\ln 5}$ and 2^{4-3i} ?

Assignment Project Exam Help

<https://eduassistpro.github.io>

- Properties:

- $f_a(n) = f_a(n-1) \cdot a$, for $n \geq 1$;

- $f(u) * f(v) = f(u + v)$.

- $f(x) = y \Leftrightarrow \ln(y) = x \ln(a) \Leftrightarrow f(x) = \exp\{x \ln a\}$.

- $e^{ix} = \cos(x) + i \cdot \sin(x)$.

- The trick:

- Same in Linear algebra

Objects and Their Representations

Goal

- We need to study the objects
- On one side.
 - A good representation helps (a lot)!

<https://eduassistpro.github.io>

Add WeChat edu_assist_pr

Basic Concepts I

Algebra

Assignment Project Exam Help

- a set of objects



<https://eduassistpro.github.io>

- constraints:

Add WeChat edu_assist_pro

- Closed for both operations
- Some nice properties of these operations
 - Commutative: $a + b = b + a$.
 - Associative: $(a + b) + c = a + (b + c)$.
 - Distributive: $\lambda(a + b) = \lambda a + \lambda b$.

Basic Concepts II

Think: What about subtraction and division?

Assignment Project Exam Help

Tips

Alwa

ra on

Poly

<https://eduassistpro.github.io>

Why these constraints are natural and useful?

Add WeChat edu_assist_pr

Basic Concepts III

Representation matters?

Consider even geometric vectors. $c = a + b$

What if we represent vectors by a column of their coordinates?

Wha

<https://eduassistpro.github.io>

Notes

- Informally, the objects we are concerned with are (column) vectors.
- The set of all n -dimensional real vectors is called \mathbb{R}^n .

(Column) Vector

Assignment Project Exam Help

Vector



- <https://eduassistpro.github.io>

Operations

- Addition: $v_1 + v_2 =$

- (Scalar) Multiplication: $\lambda v_1 \in$

Add WeChat edu_assist_pr

Linearity I

Linear Combination: Generalization of Univariate Linear Functions

- Let $\lambda_i \in \mathbb{R}$, given a set of k vectors v_i ($i \in [k]$), a linear combination of them is

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_k v_k = \sum_{i \in [k]} \lambda_i v_i$$

<https://eduassistpro.github.io>

$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix}$$

Add WeChat edu_assist_pro

- Span: All linear combination of a set of vectors of them.
- Basis: The minimal set of vectors whose span is exactly the whole \mathbb{R}^n .

Linearity II

Assignment Project Exam Help

- Benefit: every vector has a unique decomposition into basis

Think: Why uniqueness is desirable?

Exa

- <https://eduassistpro.github.io>

- Span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is \mathbb{R}^2 . But one o

Think: Who

- Decompose $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$

Add WeChat edu_assist_pr

Linearity III

Exercises

- What are the (natural) basis of all (univariate) Polynomials of degrees up to d ?
- Decompose $3x^2 + 4x - 7$ into the linear combination of 2,

<https://eduassistpro.github.io>

- The “same” polynomial is mapped to two di
under two different bases.

Think

Add WeChat edu_assist_pr

Matrix I

Linear Transformation

• is a "nice" linear function that maps a vector in \mathbb{R}^n to another vector in \mathbb{R}^m .

<https://eduassistpro.github.io>

3

- The general form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{f} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \implies \begin{aligned} y_2 &= M_{21}x_1 + M_{22}x_2 \\ y_3 &= M_{31}x_1 + M_{32}x_2 \end{aligned}$$

Matrix II

Nonexample

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \begin{matrix} y_1 = \alpha x_1^2 + \beta x_2 \\ y_2 = \gamma x_1^2 + \theta x_1 + \tau x_2 \end{matrix}$$

<https://eduassistpro.github.io>

Why Only Linear Transformation?

- Simple and nice properties:
 - $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
 - $(\lambda f)(x) = \lambda \cdot f(x)$
 - What about $f(g(x))$?
- Useful

Matrix I

Definition

- A $m \times n$ matrix corresponds to a linear transformation from \mathbb{R}^n to \mathbb{R}^m

<https://eduassistpro.github.io>

- *Transformation* or *Mapping* emphasizes more on the mapping between two sets, rather than the detailed mapping; the latter is more or less the understanding of a *function*. The *morphism* in category theory.

Semantic Interpretation

Matrix II

- Linear combination of columns of M:

$$y = x_1 M_{\bullet 1} + \dots + x_m M_{\bullet m}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = M \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

- Example:

$$\begin{bmatrix} 1 & 2 \\ -4 & 9 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -4 \\ 25 \end{bmatrix} + 10 \begin{bmatrix} 2 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 86 \\ 35 \end{bmatrix}$$

Matrix III

Assignment $\begin{bmatrix} 1 & 2 \\ -4 & 9 \end{bmatrix}$ Project $\begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$ Exam Help $\begin{bmatrix} 2 \\ 9 \\ 86 \end{bmatrix}$

<https://eduassistpro.github.io>

Add WeChat $\begin{bmatrix} 1 & 2 \\ -4 & 9 \\ 25 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 10 \\ 20 \end{bmatrix}$ edu_assist_pro

System of Linear Equations I

Assignment Project Exam Help

$$y_1 = M_{11}x_1 + M_{12}x_2$$

<https://eduassistpro.github.io>

- Add WeChat edu_assist_pro
- Interpretation. Find a vector $x \in \mathbb{R}^2$ such that Mx is exactly the given vector $y \in \mathbb{R}^2$
 - How to solve it?

System of Linear Equations II

Assignment Project Exam Help

<https://eduassistpro.github.io>

Add WeChat ^{Domain} ^M edu_assist_pro

The above transformation is *injec*

A Matrix Also Specifies a (Generalized) Coordinate System

Yet another interpretation

- $y = Mx \Rightarrow y = Mx$
- The vector y wrt standard coordinate system, I , is the same as s of M .

<https://eduassistpro.github.io>

Exa

for I $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow M:$

for x^2 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow M:$

for $2x^2+5x-4$ $\begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$

Let $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \Rightarrow Mx = I \begin{bmatrix} -7 \\ 13 \\ 6 \end{bmatrix}$

Exercise 1

Assignment Project Exam Help

- What if y is given in the above example?



<https://eduassistpro.github.io>

0 0 2 0 0 1

- Think about representing polynomials $(x-1)^2$, x^2-1 , x^2+1 .

Inner Product

THE binary operator – some kind of “similarity”

- Type signature: vector \times vector \rightarrow scalar: $\langle x, y \rangle$.

- In \mathbb{R}^n , usually called *dot product*: $x \cdot y \stackrel{\text{def}}{=} x^T y = \sum_{i=1}^n x_i y_i$.

- For certain functions, $f, g = \int_a^b f(t)g(t) dt$. leads to the

- <https://eduassistpro.github.io>

- linearity in the first argument: $\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$

- positive definiteness: $\langle x, x \rangle \geq 0$

- Generalizes many geometric concepts to V (orthogonal), projection, norm

- $\langle \sin nt, \sin mt \rangle = 0$ within $[-\pi, \pi]$ ($m \neq n$) \Rightarrow they are orthogonal to each other.

- $C = A^T B$: $C_{ij} = \langle A_i, B_j \rangle$

- Special case: $A^T A$.

Eigenvalues/vectors and Eigen Decomposition

“Eigen” means “characteristic of” (German)

- A (right) eigenvector of a square matrix A is u such that $Au = \lambda u$.

-

- <https://eduassistpro.github.io>

- We can use all eigenvectors of A to construct a matrix U (columns). Then $AU = U\Lambda$, or equiv is the Eigen Decomposition.

- We can interpret U as a transformation between coordinate systems. **Note** that vectors in U are not necessarily orthogonal.
- Λ as the scaling on each of the directions in the “new” coordinate system.

Similar Matrices

Assignment Project Exam Help

- Let A and B be two $n \times n$ matrix. A is similar to B (denoted $A \sim B$)

- <https://eduassistpro.github.io>

- Think of P as a *change of basis*
 - Relationship with the Eigen decompos
- Similar matrices have the same value wrt n (e.g., rank, trace, eigenvalues, determin

SVD

Singular Vector Decomposition

- Let M be $n \times d$ ($n \geq d$).
- Reduced SVD: $M = \hat{U} \hat{\Sigma} V^T$ exists for any M , such that

<https://eduassistpro.github.io>

- \hat{U} consists of a set of basis vectors v in \mathbb{R}^n ($d \times d$ reduced space)
- Full SVD: $M = U \Sigma V^T$.
 - Add the remaining $(n - d)$ basis vectors to \hat{U} (thus becomes $n \times n$).
 - Add the $n - d$ rows of 0 to $\hat{\Sigma}$ (thus becomes $n \times d$).

Geometric Illustration of SVD

Assignment Project Exam Help

Geometric Meaning

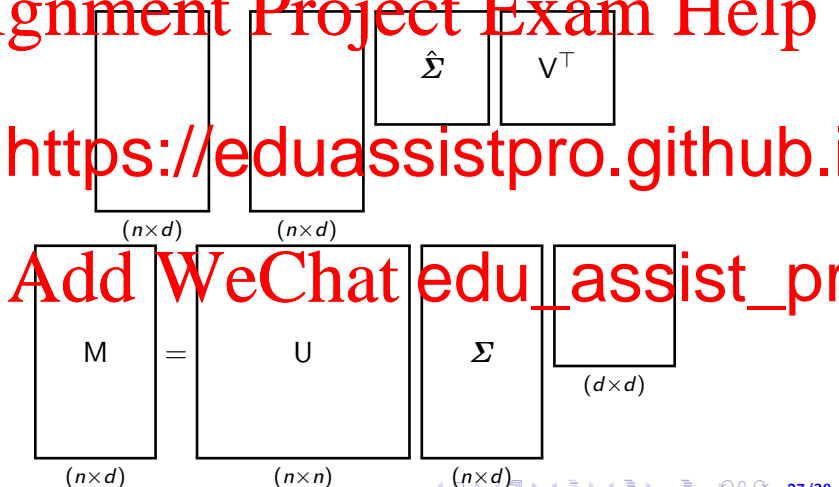
- $Mv_j = \sigma_j u_j$

<https://eduassistpro.github.io>

Add WeChat edu_assist_pr

Graphical Illustration of SVD I

Figure: Reduced SVD vs Full SVD



Graphical Illustration of SVD II

Assignment Project Exam Help

Mea

- <https://eduassistpro.github.io>
- Rows of V are the basis of \mathbb{R}

Add WeChat edu_assist_pr

SVD Applications I

Relationship between Singular Values and Eigenvalues

- What are the eigenvalues of $M^T M$?
- Hint: $M = U \Sigma V^T$ and U and V are unitary (i.e., rotations)

Assignment Project Exam Help

<https://eduassistpro.github.io>

Add WeChat edu_assist_pr

- Related to *PCA (Principle Component Analysis)*

References and Further Reading I

Assignment Project Exam Help

- Gaussian Quadrature:

<https://www.youtube.com/watch?v=k-yUdqRXijo>

- <https://eduassistpro.github.io/pdf>

- Scipy LA tutorial. <https://docs.scipy.org/doc/scipy/reference/tutorial/linalg.htm>

- We Recommend a Singular Value Decomposition
<http://www.ams.org/samplings/> [vd](#)