COMP9319 Web Data Compression and Search

Search on Suffix Array,
FM Index,
Backward Search,
Compressed BWT

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Suffix Array

- We loose some of the functionality but we save space.

Let s = abab

Sort the suffixes lexicographically:
ab, abab, b, bab

The suffix array gives the indices of the suffixes in sorted order

3	1	4	2

Suffix Array

- We loose some of the functionality but we save space.

Let s = abab

Sort the suffixes lexic
ab, abab, b, bab

The suffix array gives the indices of the suffixes in sorted order

2	0	3	1

Note: If 0-based index: some papers assume 1-based, some are 0-based.

Example

Let S = mississippi

L →	11	i
	8	ippi
	5	issippi
	2	ississippi
	1	mississippi
	10	pi
	9	ppi
	7	sippi
	4	sisippi
	6	ssippi
R →	3	ssissippi

Example

Let S = mississippi

Let P = issi

i.e., To find every suffix that begins with issi.
How???

L →	11	i
	8	ippi
	5	issippi
	2	ississippi
	1	mississippi
M →	10	pi
	9	ppi
	7	sippi
	4	sisippi
	6	ssippi
R →	3	ssissippi

Example

Let S = mississippi

Let P = issi

Two binary searches !!
So total O(m log n)

L →	11	i
	8	ippi
	5	issippi
	2	ississippi
	1	mississippi
M →	10	pi
	9	ppi
	7	sippi
	4	sisippi
	6	ssippi
R →	3	ssissippi

EXAMPLE

Let S = mississippi

Let P = issi

M = (L+R) / 2
if P > S[M]:
  L = M+1
else:
  R = M

L →	11	i
	8	ippi
	5	issippi
	2	ississippi
	1	mississippi
M →	10	pi
	9	ppi
	7	sippi
	4	sisippi
	6	ssippi
R →	3	ssissippi

EXAMPLE

Let S = mississippi

Let P = issi

M = (L+R) / 2
if P < S[M]:
  R = M
else:
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	8	ippi
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	2	ississippi
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M →	10	pi
	9	ppi
	7	sippi
	4	sisippi
	6	ssippi
R →	3	ssissippi

BACKWARD SEARCH

FM-I

(FULL-TEXT INDE

Paper by
Ferragina & Manzini

Modified from slides by Yuval Rikover

MOTIVATION

- Combine: Text Compression + Indexing

length n by looking at only
mpressed text.

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HOW DOES IT WORK?

- Exploit the relationship between the *Burrows-Wheeler Transform* and the Suffix Array data structure.
- Compressed suffix array that encapsulates both the ***compressed text*** and the ***full-text indexing information.***
- Supports two basic operations:
  - **Count** – return number of occurrences of P in T.
  - **Locate** – find all positions of P in T.

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BURROWS-WHEELER TRANSFORM

- Every column is a permutation of T.
- Given row i, char L[i] precedes F[i] in original T.
- Consecutive char's in L are adjacent to similar strings in T.
- Therefore – L usually contains long runs of identical char's.

F	L
# mississipp	i
i #mississip	p
i ppi#missis	s
i ssippi#mis	s
i ssissippi#	m
m ississippi	#
p i#mississi	p
p pi#mississ	i
s ippi#missi	s
s issippi#mi	s
s sippi#miss	i
s sissippi#m	i

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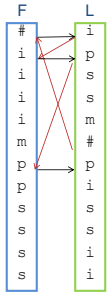
BURROWS-WHEELER TRANSFORM

Reminder: Recovering T from L

1. Find F by sorting L
2. Last char of T? #
3. Find # in L
4. L[i] precedes F[i] in T. Therefore we get i#
5. How do we choose the correct i in F?
   - The i's are in the same order in L and F
   - As are the rest of the char's
6. p precedes i: pi#
7. And so on….



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NEXT: COUNT P IN T

- **Backward-search algorithm**
- Uses only L (output of BWT)
- Relies on 2 structures:
  - C[1,…,|Σ|] : C[c] contains the total number of text chars in T which are alphabetically smaller then c (including repetitions of chars)
  - Occ(c,q): number of occurrences of char c in prefix L[1,q]

Example

- C[ ] for T = mississippi#

1	5	6	8
i	m	p	s

- occ(s, 5) = 2
- occ(s,12) = 4

Occ ≡ Rank

F	L	
#	mississipp	i 1
i	#mississip	p 2
i	ppi#missis	s 3
i	ssippi#mis	s 4
i	ssissippi#	m 5
m	ississippi	# 6
p	i#mississi	p 7
p	pi#mississ	i 8
s	ippi#missi	s 9
s	issippi#mi	s 10
s	sippi#miss	i 11
s	sissippi#m	i 12

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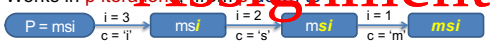
BACKWARD-SEARCH

- Works in p iterations, from p down to



- Remember that the BWT matrix r
  - All suffixes prefixed by pattern P, o
  - This set of rows has starting positio
  - and ending position ***Last***
  - So, (Last – First +1) gives total patte
- At the end of the i-th phase, ***First*** points to the first row prefixed by P[i,p], and ***Last*** points to the last row prefiex by P[i,p].

Algorithm backward_search($P[1,p]$)
(1) $i \leftarrow p$, $c \leftarrow P[p]$, First $\leftarrow C[c]+1$, Last $\leftarrow C[c+1]$;
(2) **while** ((First ≤ Last) **and** ($i \geq 2$)) **do**
(3) $c \leftarrow P[i-1]$;
(4) First $\leftarrow C[c] + \text{Occ}(c, \text{First}-1) + 1$;
(5) Last $\leftarrow C[c] + \text{Occ}(c, \text{Last})$;
(6) $i \leftarrow i - 1$;
(7) **if** (Last < First) **then return** "no rows prefixed by $P[1,p]$" **else return** (First, Last).

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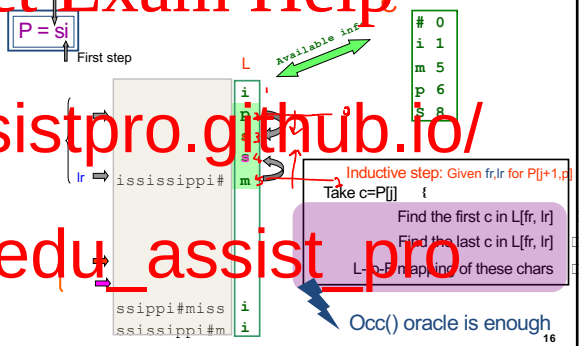
SUBSTRING SEARCH IN T (COUNT THE PATTERN OCCURRENCES)

P = si

First step



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BACKWARD-SEARCH EXAMPLE

- **P = pssi**
  - i = 3
  - c = 's'
  - First = C['s'] + Occ('s',1) +1 = 8+0+1 = 9
  - Last = C['s'] + Occ('s',5) = 8+2 = 10
  - (Last – First + 1) = 2

First

Last

F	L	
#	mississipp	i 1
i	#mississip	p 2
i	ppi#missis	s 3
i	ssippi#mis	s 4
i	ssissippi#	m 5
m	ississippi	# 6
p	i#mississi	p 7
p	pi#mississ	i 8
s	ippi#missi	s 9
s	issippi#mi	s 10
s	sippi#miss	i 11
s	sissippi#m	i 12

C[ ] = | 1 | 5 | 6 | 8 |
i m p s

Algorithm backward_search($P[1,p]$)
(1) $i \leftarrow p$, $c \leftarrow P[p]$, First $\leftarrow C[c]+1$, Last $\leftarrow C[c+1]$;
(2) **while** ((First ≤ Last) **and** ($i \geq 2$)) **do**
(3) $c \leftarrow P[i-1]$;
(4) First $\leftarrow C[c] + \text{Occ}(c, \text{First}-1) + 1$;
(5) Last $\leftarrow C[c] + \text{Occ}(c, \text{Last})$;
(6) $i \leftarrow i - 1$;
(7) **if** (Last < First) **then return** "no rows prefixed by $P[1,p]$" **else return** (First, Last).

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BACKWARD-SEARCH EXAMPLE

- **P = pssi**
  - i = 2
  - c = 's'
  - First = C['s'] + Occ('s',8) +1 = 8+2+1 = 11
  - Last = C['s'] + Occ('s',10) = 8+4 = 12
  - (Last – First + 1) = 2

First

Last

F	L	
#	mississipp	i 1
i	#mississip	p 2
i	ppi#missis	s 3
i	ssippi#mis	s 4
i	ssissippi#	m 5
m	ississippi	# 6
p	i#mississi	p 7
p	pi#mississ	i 8
s	ippi#missi	s 9
s	issippi#mi	s 10
s	sippi#miss	i 11
s	sissippi#m	i 12

C[ ] = | 1 | 5 | 6 | 8 |
i m p s

Algorithm backward_search($P[1,p]$)
(1) $i \leftarrow p$, $c \leftarrow P[p]$, First $\leftarrow C[c]+1$, Last $\leftarrow C[c+1]$;
(2) **while** ((First ≤ Last) **and** ($i \geq 2$)) **do**
(3) $c \leftarrow P[i-1]$;
(4) First $\leftarrow C[c] + \text{Occ}(c, \text{First}-1) + 1$;
(5) Last $\leftarrow C[c] + \text{Occ}(c, \text{Last})$;
(6) $i \leftarrow i - 1$;
(7) **if** (Last < First) **then return** "no rows prefixed by $P[1,p]$" **else return** (First, Last).

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BACKWARD-SEARCH EXAMPLE

- P = pssi
  - i = 1
  - c = 'p'
  - First = C['p'] + Occ('p',10) +1 = 6+2+1 = 9
  - Last = C['p'] + Occ('p',12) = 6+2 = 8
  - (Last – First + 1) = 0

	F	L
1	#	i
2	i	p
3	i	s
4	i	s
5	i	m
6	m	#
7	p	p
8	p	i
9	s	i
10	s	s
11	s	i
12	s	i

First
Last

C[ ] =

1	5	6	8
i	m	p	s

Algorithm backward_search($P[1,p]$)
(1) $i \leftarrow p$, $c \leftarrow P[p]$, First $\leftarrow C[c]+1$, Last $\leftarrow C[c+1]$;
(2) **while** ((First $\leq$ Last) **and** ($i \geq 2$)) **do**
(3) $c \leftarrow P[i-1]$;
(4) First $\leftarrow C[c] + \text{Occ}(c, \text{First}-1) + 1$;
(5) Last $\leftarrow C[c] + \text{Occ}(c, \text{Last})$;
(6) $i \leftarrow i - 1$;
(7) **if** (Last < First) **then return** "no rows prefixed by $P[1,p]$" **else return** (First, Last)

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COMPRESSED SUFFIX ARRAY / BWT

SIMPLE FM-INDEX

- Construct the *Burrows-Wheeler-transformed* text

ble to construct the suffix
r time.
the whole sa(T), one can
s besides bwt(T) to

Slides modified from the original Makinen & Navarro's

BURROWS-WHEELER TRANSFORMATION

- Construct a matrix M that contains as rows all rotations of T.
- Sort the rows in the lexicographic order.
- Let L be the last column and F be the first column.
- bwt(T)=L associated with the row number of T in the sorted M.

EXAMPLE

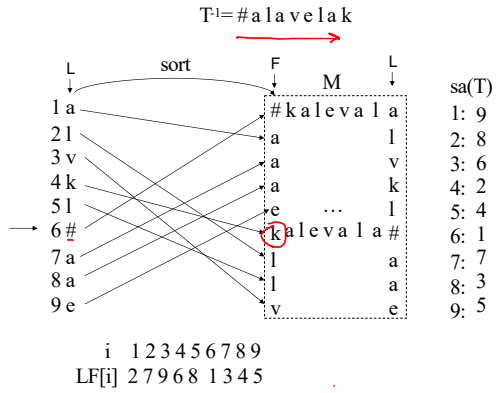
```
pos  123456789
T  = kalevala#
```

```
sa  F  M  L
1:9 #kalevala
2:8 a#kalevala
3:6 ala#kalev
4:2 alevala#k
5:4 evala#kal
6:1 kalevala#
7:7 la#kaleva
8:3 levala#ka
9:5 vala#kale
```

⟹

L = alvkl#aae, row 6

Exercise: Given L and the row number, we know how to compute T. What about sa(T)?



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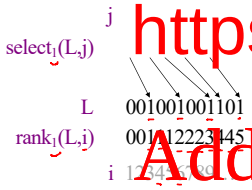
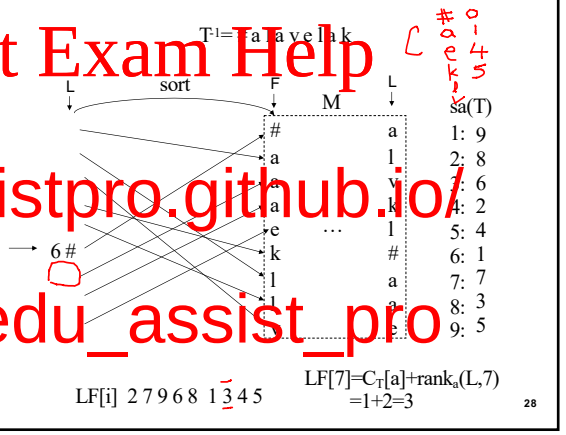
IMPLICIT LF[I]

- Ferragina and Manzini (2000) noticed the following connection:
- $LF[i]=C_T[L[i]]+rank_{L[i]}(L,i)$
- Here
  $C_T[c]$: amount of letters 0,1,...,c-1 in L=bwt(T)
  $rank_c(L,i)$: amount of letters c in the prefix L[1,i]

RANK/SELECT





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RECALL: BACKWARD SEARCH ON BWT(T)

- **Observation**: If [i,j] is the range of rows of M that start with string X, then the range [i',j'] containing cX can be computed as

i' := $C_T[c]+rank_c(L,i-1)+1$,
j' := $C_T[c]+rank_c(L,j)$.

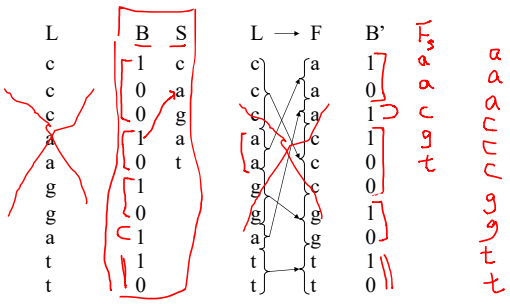
BACKWARD SEARCH ON BWT(T)...

- Array $C_T[1,\sigma]$ takes $O(\sigma \log |T|)$ bits.
- L=Bwt(T) takes $O(|T| \log \sigma)$ bits.
- Assuming $rank_c(L,i)$ can be computed in constant time for each (c,i), the algorithm takes O(|P|) time to count the occurrences of P in T.

RUN-LENGTH FM-INDEX

- We make the following changes to the previous FM-index variant:
  - L=Bwt(T) is replaced by a sequence S[1,n'] and two bit-vectors B[1,|T|] and B'[1,|T|],
  - Cumulative array C_T[1,c] is replaced by C_S[1,c],
  - some formulas are changed.

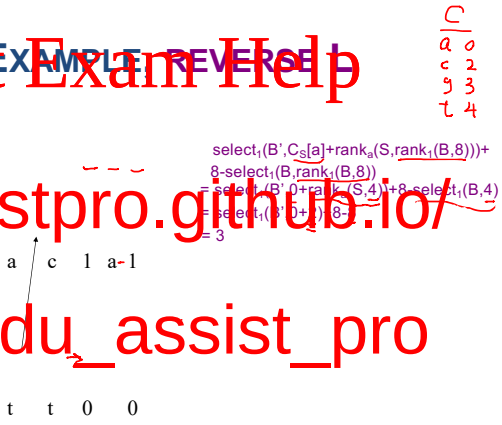
RUN-LENGTH FM-INDEX...



CHANGES TO FORMULAS

- Recall that we need to co
  $C_T[c]+rank_c(L,i)$ in the b
- **Theorem:** $C[c]+rank_c(L,i)$ = $select_1(B',C_S[c]+1+rank_c($ when $L[i] \neq c$ (e.g., when otherwise (e.g., when reverse, sometimes backward search too) to $select_1(B',C_S[c]+rank_c(S,rank_1(B,i))+$ $i-select_1(B,rank_1(B,i))$.

EXAMPLE: REVERSE L



- For more details, read the paper

EXERCISE

- ipsm$pisi
- 111011111010

What is B'

i	B	S
1	1	i
2	1	p
3	1	s
4	0	m
5	1	$
6	1	p
7	1	i
8	1	s
9	1	i
10	0	
11	1	
12	0	

Usually B' is given to save computations

i	B	S	B'
1	1	i	1
2	1	p	1
3	1	s	1
4	0	m	1
5	1	$	0
6	1	p	1
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

Reverse BWT from row 6

i	B	S	B'
1	1	i	
2	1	p	
3	1	s	
4	0	m	
5	1	$	
6	1	p	
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

Reverse BWT

i	B	S	B'
		i	1
		p	1
		s	1
		m	1
		$	0
6	1	p	1
			1
			1
			0
			1
12	0		0

$S[rank_1(B, 6)] = \$$

Reverse BWT

i	B	S	B'
1	1	i	1
2	1	p	1
3	1	s	1
4	0	m	1
5	1	$	0
6	1	p	1
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

$S[rank_1(B, 6)] = \$$

LF[6]

$= select_1(B', C_S[\$] + rank_S(S, rank_1(B, 6))) + 6 - select_1(B, rank_1(B, 6)))$

$= select_1(B', 0 + rank_S(S, 5)) + 6 - select_1(B\ 5)$

$= 1 + 6 - 6 = 1$

Reverse BWT

i	B	S	B'
1	1	i	1
2	1	p	1
3	1	s	1
4	0	m	1
5	1	$	0
6	1	p	1
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

$S[rank_1(B, 1)] = i$

LF[1]

$= select_1(B', C_S[i] + rank_i(S, rank_1(B, 1))) + 1 - select_1(B, rank_1(B, 1)))$

$= select_1(B', 1 + rank_i(S, 1)) + 1 - select_1(B, 1)$

$= 2 + 1 - 1 = 2$

REVERSE BWT

i	B	S	B'
1	1	i	1
2	1	p	1
3	1	s	1
4	0	m	1
5	1	$	0
6	1	p	1
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

$S[rank_1(B, 1)]= i$

LF[1]

$= select_1(B', C_S[i] + rank_i(S, rank_1(B, 1))) + 1 – select_1(B, rank_1(B, 1)))$

$= select_1(B', 1 + rank_i(S, 1)) + 1 – select_1(B\ 1)$

$= 2 + 1 – 1 = 2$

You can also construct the SA in this way:

12, 11, ….

12,11,8,5,2,1,10,9,7,4,6,3

BACKWARD SEARCH

i	B	S	B'
1	1	i	1
2	1	p	1
3	1	s	1
4	0	m	1
5	1	$	0
6	1	p	1
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

Suppose search for si:

c = i, First = 2, Last = 5

c = s

First = C[c] + Occ(c, First – 1) + 1

Last = C[c] + Occ(c, Last)

BACKWARD SEARCH

i	B	S	B'
1	1	i	1
2	1	p	1
3	1	s	1
4	0	m	1
5	1	$	0
6	1	p	1
7	1	i	1
8	1	s	1
9	1	i	1
10	0		0
11	1		1
12	0		0

c = i, First = 2, Last = 5

c = s

First = $select_1$ 1))) -1 + 1

$=select_1(B',7$ $_s$

$=select_1(B', 8) = 9$

Last = $select_1(B, C_S[c]+1+rank_s(S, rank_1(B,5))$ -1

$=select_1(B',7+1+rank_s(S,4)) – 1$

$=select_1(B', 9)$ -1 = 11 – 1 = 10