
COMPGV19 Assignment 5: Exercise 3

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Marta Betcke and Kiko Rullan

```
clear all, close all;
```

Function, gradient and Hessian

```
% % Diffe
% % For c
% F.f = @
% F.df =
%
%           200*(x(2) - x(1)^2)
% F.d2f = @(x) [-400*(x(2) - 3*x(1)
%           200];
%
% % For visualisation proposes define as a function of 2 variables
(x,y)
% FV.f = @(x,y) 100.*(y - x.^2).^2 + (1 - x).^2;
% FV.dfx = @(x,y) -400.*(y - x.^2).*x - 2.*(1 - x);
% FV.dfy = @(x,y) 200.*(y - x.^2);
% FV.d2fxx = @(x,y) -400.*(y - 3*x.^2) + 2;
% FV.d2fxy = @(x,y) -400.*x;
% FV.d2fyx = @(x,y) -400.*x;
% FV.d2fyy = @(x,y) 200;
%
% % For computation define as function of 1 vector variable
% F.f = @(x) x(1)^2 + 5*x(1)^4 + 10*x(2)^2;
% F.df = @(x) [2*x(1) + 20*x(1)^3; 20*x(2)];
% F.d2f = @(x) [2 + 60*x(1)^2, 0; 0, 20];
%
% % For visualisation proposes define as a function of 2 variables
(x,y)
% FV.f = @(x,y) x.^2 + 5*x.^4 + 10.*y.^2;
% FV.dfx = @(x,y) 2*x + 20*x.^3;
% FV.dfy = @(x,y) 20*y;
% FV.d2fxx = @(x,y) 2 + 60*x.^2;
% FV.d2fxy = @(x,y) 0;
```

```
% FV.d2fyx = @(x,y) 0;
% FV.d2fyy = @(x,y) 20;

% For computation define as function of 1 vector variable
F.f = @(x) (x(1) - 3*x(2)).^2 + x(1).^4;
F.df = @(x) [2*(x(1) - 3*x(2)) + 4*x(1).^3; -6*(x(1) - 3*x(2))];
F.d2f = @(x) [2 + 12*x(1).^2, -6; -6, 18];

% For visualisation proposes define as a function of 2 variables (x,y)
FV.f = @(x,y) (x-3*y).^2 + x.^4;
FV.dfx = @(x,y) 2*(x - 3*y) + 4*x.^3;
FV.dfy = @(x,y) -6*(x - 3*y);
FV.d2fxx = @(x,y) 2+12*x.^2;
FV.d2fxy = @(x,y) -6;
FV.d2fyx = @(x,y) -6;
FV.d2fyy = @(x,y) 18;

% Starting point
x0 = [10; 10];

% Parameters
maxIter = 200;
tol = 1e-4; % stopping tolerance or relative step length between
iterations
debug = 0
visualis
ep by step
options

% Visuali

% Define grid for visualisation
n = 1000;
x = linspace(-5, 10,n+1);
y = x;
[X,Y] = meshgrid(x,y);
% Compute log(f) to better highlight the function behaviour
Z = log(max(FV.f(X,Y), 1e-3)); % take max for better distribution of
the contours (the log as a very steep deep at (0,0))
% Compute f for true contours
% Z = FV.f(X,Y);
```

Minimise f using BFGS

```
%=====
% Line search parameters
alpha0 = 1;
c1 = 1e-4;
% Backtracking (only for comparison purposes)
lsOpts.rho = 0.9;
lsOpts.c1 = c1;
lsFunB = @(x_k, p_k, alpha0) backtracking(F, x_k, p_k, alpha0,
lsOpts);
% Strong Wolfe LS
lsOpts_LS.c1 = c1;
```

```
lsOpts_LS.c2 = 0.5; % 0.1 Good for Newton, 0.9 - good for steepest
descent, 0.5 compromise.
lsFunS = @(x_k, p_k, alpha0) lineSearch(F, x_k, p_k, alpha0,
lsOpts_LS);
lsFun = lsFunS;

% Minimisation with Newton, Steepest descent and BFGS line search
methods
[xLS_BFGS, fLS_BFGS, nIterLS_BFGS, infoLS_BFGS] =
descentLineSearch(F, 'steepest', lsFun, alpha0, x0, tol, maxIter)
[xLS_BFGS, fLS_BFGS, nIterLS_BFGS, infoLS_BFGS] =
descentLineSearch(F, 'bfgs', lsFun, alpha0, x0, tol, maxIter)

Rescaling H0 with 0.0032602

xLS_BFGS =

    0.0237
    0.0079

fLS_BFGS =
3.1629e-07

nIterLS_BFGS =
21

infoLS_BFGS =

    xs: [2x22 double]
    alphas: [1x22 double]
    H: {1x21 cell}
```

Trajectory: line search

```
visualizeConvergence(infoLS_BFGS,X,Y,Z,'final');
box on;
title(['BFGS: ' num2str(size(infoLS_BFGS.xs,2))]);
saveas(gcf, '../figs/01_BFGS_trajectory', 'png');
```

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```
H_k * F.df2(
difNorm = [];
for n = 1:niterLS_BFGS
    B = F.d2f(infoLS_BFGS.xs(:, n))
    H_BFGS = infoLS_BFGS.H{n}(eye(1
    difNorm = [difNorm norm(eye(length(x0)) - B*H_BFGS)];
end
figure;
semilogy(difNorm, 'LineWidth', 2);
title('BFGS inverse Hessian approximation') ;
ylabel(' || H_k * F.df2(x_k) - I || ');
xlabel('k');
grid on;
saveas(gcf, '../figs/02_BFGS_hessian', 'png');
```

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Minimise f

ds

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```
%=====
% Trust region parameters
maxIter = 200;
eta = 0.2; % Step acceptance relative progress threshold
Delta = 5; % Trust region radius

% Minimisation with 2d subspace and dogleg trust region methods
Fsrl = rmfield(F, 'd2f');
[xTR_SR1, fTR_SR1, nIterTR_SR1, infoTR_SR1] = trustRegion(Fsrl, x0,
    @solverCM2dSubspaceExt, Delta, eta, tol, maxIter, debug)

xTR_SR1 =

    0.0268
    0.0089

fTR_SR1 =

    5.1313e-07

nIterTR_SR1 =
```

21

```
infoTR_SR1 =  
  
    xs: [2x22 double]  
    xind: [1 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20  
21]  
    B: {1x21 cell}  
    rhos: [1x21 double]  
    Deltas: [1x21 double]  
    stopCond: 1
```

Trajectory: trust region

```
visualizeConvergence(infoTR_SR1,X,Y,Z,'final');  
box on;  
title(['SR1: using 2d subspace solver: '  
    num2str(size(infoTR_SR1.xs,2))])  
saveas(gcf, '../figs/03_TR_trajectory','png');
```

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```
B_k - F.df2(x_k)  
  
difNorm = [];  
for n = 1:min(length(infoTR_SR1.B), length(infoTR_SR1.xs))
```

```
B = F.d2f(infoTR_SR1.xs(:, n));
B_SR1 = infoTR_SR1.B{n}(eye(length(x0)));
difNorm = [difNorm norm(B-B_SR1)];
end
figure;
semilogy(difNorm, 'LineWidth', 2);
title('SR-1 Hessian approximation');
ylabel(' || B_k - F.df2(x_k) || ');
xlabel('k');
grid on;
saveas(gcf, '../figs/04_TR_hessian', 'png');
```

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===== Subfunctions =====

descentLineSearch.m

Descent line search

```
function [xMin, fMin, nIter, info] = descentLineSearch(F, descent, ls,
    alpha0, x0, tol, maxIter)
% DESCENTLINESEARCH Wrapper function executing descent with line
% search
% [xMin, fMin, nIter, info] = descentLineSearch(F, descent, ls,
% alpha0, x0, tol, maxIter)
%
```

```
% INPUTS
% F: structure with fields
%   - f: function handler
%   - df: gradient handler
%   - d2f: Hessian handler
% descent: specifies descent direction {'steepest', 'newton', 'bfgs'}
% alpha0: initial step length
% ls: line search algorithm
% x0: initial iterate
% tol: stopping condition on minimal allowed step
%       norm(x_k - x_k_1)/norm(x_k) < tol;
% maxIter: maximum number of iterations
%
% OUTPUTS
% xMin, fMin: minimum and value of f at the minimum
% nIter: number of iterations
% info: structure with information about the iteration
%   - xs: iterate history
%   - alphas: step lengths history
%
% Copyright (C) 2017  Marta M. Betcke, Kiko Rullan
% Parameters
% Stopping condition {'step', 'grad'}
stopType

% Extract
extractH

% Initialization
nIter = 0;
x_k = x0;
info.xs = x0;
info.alphas = alpha0;
stopCond = false;

switch lower(descent)
case 'bfgs'
    H_k = @(y) y;
    % Store H matrix in columns
    info.H = [];
end

% Loop until convergence or maximum number of iterations
while (~stopCond && nIter <= maxIter)

    % Increment iterations
    nIter = nIter + 1;

    % Compute descent direction
    switch lower(descent)
    case 'steepest'
        p_k = -F.df(x_k); % steepest descent direction
    case 'newton'
```

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```
p_k = -F.d2f(x_k)\F.df(x_k); % Newton direction
if p_k'*F.df(x_k) > 0 % force to be descent direction (only
active if F.d2f(x_k) not pos.def.)
    p_k = -p_k;
end
case 'bfgs'
    p_k = -H_k(F.df(x_k));
end

% Call line search given by handle ls for computing step length
alpha_k = ls(x_k, p_k, alpha0);

% Update x_k and f_k
x_k_1 = x_k;
x_k = x_k + alpha_k*p_k;

switch lower(descent)
case 'bfgs'
    % Compute s_k and y_k
    s_k = x_k - x_k_1;
    y_k = F.df(x_k) - F.df(x_k_1);
    % Verify  $s_k^T y_k > 0$ 
    if s_k'*y_k <= 0
        error('Positivity condition: <s_k,y_k> > 0 not satisfied.
Check if
e
i
    % Update initial guess H_1 p_0 = -
F.df(x_0) and x_1 = x_0 + alpha * p
else 'Rescale H_k with
H_k = @(x) (s_k'*y_k)/(y_
end

% Efficient update H
rho_k = 1/(s_k'*y_k);
projL = @(x) (x - rho_k*s_k*(y_k'*x));
projR = @(x) (x - rho_k*y_k*(s_k'*x));
H_k = @(x) projL( H_k( projR(x) ) ) + rho_k*s_k*(s_k'*x);

if extractH
    % Extraction of H_k as handler
    info.H{length(info.H)+1} = H_k;
end
end

% Store iteration info
info.xs = [info.xs x_k];
info.alphas = [info.alphas alpha_k];

switch stopType
case 'step'
    % Compute relative step length
```

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```
normStep = norm(x_k - x_k_1)/norm(x_k_1);
stopCond = (normStep < tol);
case 'grad'
    stopCond = (norm(F.df(x_k), 'inf') < tol*(1 + abs(F.f(x_k))));
end

end

% Assign output values
xMin = x_k;
fMin = F.f(x_k);
```

trustRegion.m

Trust region

```
function [x_k, f_k, k, info] = trustRegion(F, x0, solverCM, Delta,
eta, tol, maxIter, debug, F2)
% TRUSTSQTRON Trust region iteration
% [x_k, f_k, k, info] = trustRegion(F, x0, solverCM, Delta, eta, tol,
maxIter,
% INPUTS
% F: struct
% - f:
% - df: gradient handler
% optional, if missing quasi Newt
% - d2f: Hessian handler
% x_k: current iterate
% solverCM: handle to solver to quadratic constraint trust region
problem
% Delta: upper limit on trust region radius
% eta: step acceptance relative progress threshold
% tol: parameter in stopping condition (specified with stopType
parameter inside the function)
% on minimal allowed step
% norm(x_k - x_k_1)/norm(x_k) < tol
% or norm of the gradient
% norm(F.df(x_k), 'inf') < tol*(1 + abs(F.f(x_k)))
% maxIter: maximum number of iterations
% debug: debugging parameter switches on visualization of quadratic
model
% and various step options. Only works for functions in R^2
% F2: needed if debug == 1. F2 is equivalent of F but formulated as
function of (x,y)
% to enable meshgrid evaluation
% OUTPUT
% x_k: minimum
% f_k: objective function value at minimum
% k: number of iterations
% info: structure containing iteration history
```

```
% - xs: taken steps
% - xind: iterations at which steps were taken
% - stopCond: shows if stopping criterium was satisfied, otherwise
k = maxIter
%
% Reference: Algorithms 4.1, 6.2 in Nocedal Wright
%
% Copyright (C) 2017 Marta M. Betcke, Kiko Rullan

% Parameters
% Choose stopping condition {'step', 'grad'}
stopType = 'grad';

% Extract Hessian approximation handler
extractB = 1;

% Initialisation
Delta_k = 0.5*Delta;
r = 1e-8; % skipping tolerance for SR-1 update

stopCond = false;
k = 0;
x_k = x0;
nTaken =

info.xs =
info.xs(:)
info.xind = zeros(1,maxIter);
info.xind(1) = 1;

% Determine if Hessian is available or method SR-1
is used.
sr1 = ~isfield(F, 'd2f');
if sr1
    % Initialise with B_0
    F.d2f = @(x) eye(length(x0)); % B_0 = Identity matrix.
    info.B = [];
end

while ~stopCond && (k < maxIter)
    k = k+1;

    % Construct and solve quadratic model
    Mk.m = @(p) F.f(x_k) + F.df(x_k)'*p + 0.5*p'*F.d2f(x_k)*p;
    Mk.dm = @(p) F.df(x_k) + F.d2f(x_k)*p;
    Mk.d2m = @(p) F.d2f(x_k);

    p = solverCM(F, x_k, Delta_k); % for SR-1 s_k = p_k
    if sr1
        % Compute y_k. Note, that B update happens even if the step is not
        taken.
        y_k = F.df(x_k + p) - F.df(x_k);
    end
```

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```
if debug
    % Visualise quadratic model and various steps
    figure(1); clf;
    plotTaylor(F2, x_k, [x_k - 4*Delta_k, x_k + 4*Delta_k], Delta_k,
p);
    hold on,
    g = -F.df(x_k);
    gu = -F.d2f(x_k)\g;
    plot(x_k(1) + g(1)*Delta_k/norm(g), x_k(2) + g(2)*Delta_k/
norm(g), 'rs')
    plot(x_k(1) + gu(1)*Delta_k/norm(gu), x_k(2) + gu(2)*Delta_k/
norm(gu), 'bo')
    pause
end

% Evaluate actual to predicted reduction ratio
rho_k = (F.f(x_k) - F.f(x_k + p)) / (Mk.m(0*p) - Mk.m(p)) ;
if (Mk.m(0*p) < Mk.m(p))
    disp(strcat('Ascent - iter ', num2str(k)))
end
% Record iteration information
info.slog(k) = slog;
info.Deltas(k) = Delta_k;

if rho_
    % Shi
    Delta
else
    if rho_k > 0.75 && abs(p'*p - D
        % Expand trust region
        Delta_k = min(2*Delta_k, Delt
    end
end

% Accept step if rho_k > eta
x_k_1 = x_k; % if step is not accepted x_k_1 = x_k
if rho_k > eta
    % x_k_1 = x_k;
    x_k = x_k + p;

% Record all taken steps including iteration index
nTaken = nTaken + 1;
info.xs(:,nTaken+1) = x_k;
info.xind(nTaken+1) = k;

% Evaluate stopping condition:
switch stopType
case 'step'
    % relative step length
    stopCond = (norm(x_k - x_k_1)/norm(x_k_1) < tol);
case 'grad'
    % gradient norm
    %stopCond = (norm(F.df(x_k)) < tol);
```

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```
stopCond = (norm(F.df(x_k), 'inf') < tol*(1 + abs(F.f(x_k))));
end
elseif Delta_k < 1e-6*Delta
    % Stop iteration if Delta_k shrank below 1e-6*Delta. Otherwise, if
    the model
    % does not improve inspite of shrinking, the algorithm would shrink
    Delta_k indefinitely.
    warning('Region of interest is too small. Terminating iteration.')
    break;
end

% SR1 quasi Newton: Hessian update
% Note: this implementation constructs Hessian matrices explicitly
hence is not suitable for large scale.
% Efficient large scale implementation either needs to use iterative
solvers to invert the Hessian inside solverCM
% or needs to update the Hessian and its inverse at the same time.
% Then they both can be implemented as their action on a vector
H_k(x) = H_k*x, B_k(x) = B_k*x as for H_k in BFGS.
if srl
    % Residual vector of the secant equation.
    % B_k_1 = F.d2f(x_k_1). This is legacy notation as F.d2f =
    F.d2f(s_k_1) is actually a matrix and hence does not depend on x.
    rSec = y_k - F.d2f(x_k_1)*p; % recall notation s_k = p_k,

    % Upd lds, otherwise skip
update.
    if (a
        F.d2f = @(x) F.d2f(x_k_1) + (
        update is only valid at x_k. F.d2f
        end matrix.

    if extractB
        % Extraction of B_k function handler
        info.B{length(info.B)+1} = F.d2f;
    end
end
end

f_k = F.f(x_k);
info.stopCond = stopCond;
info.xs(:,nTaken+2:end) = [];
info.xind(nTaken+2:end) = [];
info.rhos(k+1:end) = [];
info.Deltas(k+1:end) = [];
```

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