Numerical Optimisation: Line search methods

f.rullan@cs.uc

Department of Computer Sc
Centre for Medical Image Com
Centre for Inverse Problems
University College London

Lecture 2 & 3

Descent direction is a vector $p \in \mathbb{R}^n$ for which the function decreases.

From Taylor's theorem

$$f(x_k \quad - \quad p)p, \; t \in (0, \alpha)$$

$$<0$$

Thus for $\alpha > 0$ small enough, $f(x_k +$

$$p^{\mathrm{T}} \nabla f(x_k) = \|p\| \|\nabla f(x_k)\| \cos\theta < 0 \Leftrightarrow |\theta| > \pi/2,$$

where θ is the angle between p and $\nabla f(x_k)$.

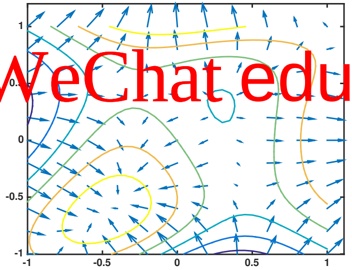
Steepest descent direction p

$$\min_p p^{\mathrm{T}} \nabla f(x_k), \quad \text{subject to } \|p\| = 1.$$

$$\min_p p^{\mathrm{T}} \nabla f(x_k) = \min_p \|p\| \|\nabla f(x_k)\| \cos(\theta) = -\|\nabla f(x_k)\|,$$

attai re θ is

the a



Newton direction

Consider the second order Taylor polynomial

$$f(x_k + p) \approx f(x_k) + p^{\mathrm{T}} \nabla f(x_k) + \frac{1}{2} p^{\mathrm{T}} \nabla^2 f(x_k) p =: m_2(p)$$

and assume $\nabla^2 f(x_k)$ is positive definite.

New

m_2. Se

$2 \quad k \quad k$

yields

$$p = -(\nabla^2 f(x_k))^{-}$$

The Newton direction is reliable when $m_2(p)$ is a close approximation to $f(x_k + p)$ i.e. $\nabla^2 f(x_k + tp)$, $t \in (0, 1)$ and $\nabla^2 f(x_k)$ are close. This is the case if $\nabla^2 f$ is sufficiently smooth and the difference is of order $\mathcal{O}(\|p\|^3)$.

$$p^{\mathrm{T}} \nabla f(x_k) = -p^{\mathrm{T}} \nabla^2 f(x_k) p \le -\sigma \|p\|^2$$

for some $\sigma > 0$. Thus unless $\nabla f(x_k) = 0$ (and hence $p = 0$), $p^{\mathrm{T}} \nabla f(x_k) < 0$ and p is a descend direction.

The step length 1 is optimal for $f(x \quad + p) = m_2(p)$, thus 1 is used unles

If ∇

be defined: if $\nabla^2 f(x_k)$ is singular, $\nabla^2 f$ 1

Otherwise, p may not be a descent direction wh

remedied.

Fast local convergence (quadratic) close to the solution.

Computing the Hessian is expensive.

Use symmetric positive definite (s.p.d.) approximation B_k to the Hessian $\nabla^2 f(x_k)$ in the Newton step

$$p = -B_k^{-1}\nabla f(x_k), \quad \nabla f(x_k)^{\mathrm{T}} B_k^{-1} \nabla f(x_k) > 0,$$

such that superlinear convergence is retained.

B_k is
infor

the fact that changes in gradient provide information about the second derivative of f along the search dire

Secant equation

$$\nabla f(x_k + p) = \nabla f(x_k) + B_k p$$

This equation is underdetermined, different methods quasi-Newton methods differ in the way they solve it.

Line search

Given the search direction p the optimal reduction of the f amounts to minimising the function of one variable

This is i

inex

Choice of step size is important. To small steps mea convergence, to large steps may not lead to reducti objective function f.

Conditions for decrease

Simple condition: require $f(x_k + \alpha p) < f(x_k)$.

Consider a sequence $f(x_k) = 5/k, k = 1, 2, \ldots$. This sequence is decreasing but its limiting value is 0, while the minimum of a convex function can be smaller than 0.

Figure: Nocedal Wright Fig 3.2

The decrease is insufficient to converge to the minimum of a convex function. Hence we need conditions for sufficient decrease.

Sufficient decrease condition

Armijo condition

$$f(x_k + \alpha p) \leq f(x_k) + c_1 \alpha p^{\mathrm{T}} \nabla f(x_k) =: \ell(\alpha),$$

for some $c_1 \in (0, 1)$. [Typically small, $c_1 = 10^{-4}$]

$\ell(\alpha)$ is a linear function with negative slope $c_1 p^{\mathrm{T}} \quad f(x_k) < 0$,

$\ell(\alpha)$ $+\alpha\phi'(0)$.

Fro $0, \alpha)$,
thus f

Figure: Nocedal Wright Fig 3.3

Armijo condition is satisfied for all sufficiently small α, so we need another condition to avoid very small steps.

Curvature condition

$$p^{\mathrm{T}} \quad f(x_k + \alpha p) \quad c_2\, p^{\mathrm{T}} \quad f(x_k), \quad c_2 \quad (c_1, 1).$$

- ection p.
- If $\phi'(\alpha)$ is strongly negative, there is a good p significant decrease along p.

- If $\phi'(\alpha)$ is slightly negative (or even positiv prospect of little decrease and hence we terminate the line search.
- Typically $c_2 = 0.9$ for a Newton or quasi Newton direction, $c_2 = 0.1$ for nonlinear conjugate gradient.

Curvature condition

$$\underbrace{p^{\mathrm{T}}\nabla f(x_k+\alpha p)}_{\phi'(\alpha)} \geq c_2 \underbrace{p^{\mathrm{T}}\nabla f(x_k)}_{\phi'(0)}, \quad c_2 \in (c_1, 1).$$

Figure: Nocedal Wright Fig 3.4

The sufficient decrease (Armijo rule) and curvature conditions together are called **Wolfe conditions**

$$
\begin{aligned}
f(x_k + \alpha p) &\le f(x_k) + c_1 \alpha p^{\mathrm{T}} \nabla f(x_k), \\
p^{\mathrm{T}} \nabla f(x_k + \alpha p) &\ge c_2 p^{\mathrm{T}} \nabla f(x_k),
\end{aligned}
$$

for 0

Poss

strong Wolfe conditions to disallow “to

$\phi'(\alpha)$

$$
\begin{aligned}
f(x_k + \alpha p) &\le f(x_k) + c_1 \alpha p^{\mathrm{T}} \nabla f(x_k), \\
|p^{\mathrm{T}} \nabla f(x_k + \alpha p)| &\le c_2 |p^{\mathrm{T}} \nabla f(x_k)|,
\end{aligned}
$$

for $0 < c_1 < c_2 < 1$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable and p be a descent direction at x_k. If f is bounded below along the ray $\{x_k + \alpha p \mid \alpha > 0\}$, then there exists an interval of step lengths satisfying both the Wolfe conditions and strong Wolfe conditions.

Pro

For

$\ell(\alpha)$

$c_1 p^{\mathrm{T}} \nabla f(x_k) < 0$ and for small α, $\ell(\alpha)$

$\ell(\alpha)$ has to intersect $\phi(\alpha)$ at least once. Let

value for which

$$\phi(\alpha') = f(x_k + \alpha' p) = f(x_k) + \alpha' c_1 p^{\mathrm{T}} \nabla f(x_k) = \ell(\alpha').$$

Then the sufficient decrease condition holds for all $\alpha \leq \alpha'$.

Furthermore, by the mean value theorem

$$\exists \alpha'' \in (0, \alpha') : \; f(x_k + \alpha' p) - f(x_k) = \alpha' p^{\mathrm{T}} \nabla f(x_k + \alpha'' p)$$

and we obtain

$$p^{\mathrm{T}} \nabla f(x_k + \alpha'' p) = c_1 p^{\mathrm{T}} \nabla f(x_k) > c_2 p^{\mathrm{T}} \nabla f(x_k)$$

since s Wolfe
cond holds
stric

conditions) also holds in an interval containing
as all terms in the last equation are negative strong W
conditions hold for the same interval.

Wolfe conditions are scale-invariant in the sense that are unaffected by scaling the function or affine change of variables. They can be used in most line search methods and are particularly important for quasi-Newton methods.

$$f(x_k) + (1-c)\alpha p^{\mathrm{T}} \nabla f(x_k) \leq f(x_k + \alpha p) \leq f(x_k) + c\alpha p^{\mathrm{T}} \nabla f(x_k)$$

with $0 < c < 1/2$.

The second inequality is the sufficient decrease condition. The first inequality controls the step length from below. Disadvantage w.r.t ϕ.

Figure: Nocedal Wright Fig 3.6

Backtracking line search

1: Choose $\bar{\alpha} > 0, \rho \in (0, 1), c \in (0, 1)$
2: Set $\alpha = \bar{\alpha}$
3: **repeat**
4: $\alpha = \rho\alpha$
5: T

-
- Prevents too short step lengths: the accept factor ρ of the previous value, $\alpha/$ violating the sufficient decrease conditio

- ρ can vary in $[\rho_{\min}, \rho_{\max}] \subset (0, 1)$ b
- In Newton and quasi-Newton methods $\bar{\alpha} = 1$, but different values can be appropriate for other algorithms.
- Well suited for Newton methods, less appropriate for quasi-Newton and conjugate gradient methods.

Consider an iteration

$$x_{k+1} = x_k + \alpha_k p_k, \;\; k = 0, 1, \ldots,$$

where p_k is a descent direction and α_k satisfied the Wolfe cond

Let f an

open s

If ∇f

$$\exists L > 0: \;\; \|\nabla f(x) - \nabla f(\bar{x})\| \leq$$

then

$$\sum_{k \geq 0} \cos^2 \theta_k \|\nabla f(x_k)\|^2 < \infty,$$

where $\theta_k = \angle(p_k, -\nabla f(x_k))$.

Subtracting $p_k^{\mathrm{T}}\nabla f(x_k)$ from both sides of curvature condition

$$p_k^{\mathrm{T}}\nabla f(\underbrace{x_k+\alpha p_k}_{=x_{k+1}}) \geq c_2 p_k^{\mathrm{T}}\nabla f(x_k)$$

we obtain

On the other hand the Lipschitz condition implies

$$p_k^{\mathrm{T}}(\nabla f(x_{k+1}) - \nabla f(x_k)) \leq \|\nabla f(x_{k+1}) \ldots {}_k\|^2.$$

Combining the two inequalities we obtain a lower bound on the step size

$$\alpha_k \geq \frac{c_2-1}{L}\frac{p_k^{\mathrm{T}}\nabla f(x_k)}{\|p_k\|^2}.$$

Substituting this inequality into the sufficient decrease condition

$$f(x_{k+1}) \leq f(x_k) + c_1 \frac{c_2 - 1}{L} \frac{(p_k^{\mathrm{T}} \nabla f(x_k))^2}{\|p_k\|^2}$$

with $\cos\theta_k = -\frac{p_k^{\mathrm{T}} \nabla f(x_k)}{\|\nabla f(x_k)\| \|p_k\|}$ yields

$$f(x_{k+1}) \leq f(x_k) - c \cos^2\theta_k \|\nabla f(x_k)\|^2,$$

wher

Sum

$$f(x_{k+1}) \leq f(x_0) - c \sum_{j=0} \cos^2\theta \quad f(x) \quad ^2$$

and since f is bounded from below

$$\sum_{j=0}^{k} \cos^2\theta_j \|\nabla f(x_j)\|^2 \leq (f(x_0) - f(x_{k+1}))/c < C$$

where $C > 0$ is some positive constant. Taking limits
$\sum_{k=0}^{\infty} \cos^2\theta_k \|\nabla f(x_k)\|^2 < \infty$.

Goldstein or strong Wolfe conditions also imply the Zoutendijk condition

$$\sum_{k\geq 0} \cos^2 \theta_k \|\nabla f(x_k)\|^2 < \infty.$$

The Zoutendijk condition implies

which

search algorithms.

If the method ensures that $\cos\theta_k \geq \delta >$

away from $\pi/2$, it follows that

$$\lim_{k\to\infty} \|\nabla f(x_k)\| = 0.$$

This is the strongest global convergence result that can be obtained for such iteration (convergence to a stationary point) without additional assumptions.

In particular, the steepest descent ($p_k = -\nabla f(x_k)$) produces a gradient sequence which converges to 0 if it uses a line search satisfying Wolfe or Goldstein conditions.

For so

only a

k

$k \to \infty$

i.e. only subsequence of gradient norms

rather than the whole sequence.

Those limits can be proved by contradiction:
Suppose that $\|\nabla f(x_k)\| \geq \gamma$ for some $\gamma > 0$ for all k sufficiently large. Then from $\cos^2 \theta_k \|\nabla f(x_k)\|^2 \to 0$ we conclude that $\cos \theta_k \to 0$ i.e. the entire sequence $\{\cos \theta_k\}$ converges to 0.

Thus to show the weak convergence result it is enough to show that a subsequence $\cos \theta_{k_j}$ is bounded away from 0.

Cons

(i) dec
(ii) ev
satisfying the Wolfe or Goldstein conditions.
Since $\cos \theta_k = 1$ for steepest descent steps, this p

subsequence bounded away from 0. The algorit
something better in remaining $m - 1$ iterates, while the occasional steepest descent step will guarantee the overall (weak) global convergence.

Unfortunately, rapid convergence sometimes conflicts with global convergence.

Exa
step si
whil

solution, the Newton step may not even be a descent direction far away from the solution.

The challenge: design algorithms with good glob
properties and rapid convergence rate.

Steepest descent with exact line search for strictly convex quadratic function

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Qx - b^{\mathrm{T}}x$$

wher

Figure: Nocedal Wright Fig 3.7

Characteristic zig-zag due to elongated shape of the ellipse. If the level sets were circles instead, the steepest descent would need one step only.

Convergence rate of steepest descent with exact line search:

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the ei
$\|x\|_Q^2 = x^{\mathrm{T}} Qx$. Note: for quadratic strictly con
obtain objective function and (for free) iterate co

The objective function convergence rate is essentially the same for steepest descent with exact line search when applied to a twice continuously differentiable nonlinear function satisfying sufficient conditions at $x^\star$.

Local convergence rate: Newton methods

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable with Lipschitz continuous Hessian in a neighbourhood of the solution $x^\star$ satisfying the sufficient conditions. Note that the Hessian $\nabla^2 f$ is positive definite also in the vicinity of the solution $x^\star$.

The iterates x computed by the Newton method (note step length 1)

converge locally quadratically i.e. for starting point x_0 sufficiently close to $x^\star$.

The sequence of gradient norms $\|\nabla f($ quadratically to 0.

Local convergence: note that away from the solution ∇f_k may not be positive definite and hence p_k may not be a descent direction. Global convergence with Hessian modification is discussed later.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable.
Let $\{x_k\}$ be a sequence generated by a descent method

$$x_{k+1} = x_k + \alpha p_k$$

for step sizes satisfying Wolfe conditions with $c_1 \leq 1/2$.
If the se
suffi

$$\lim_{k\to\infty} \frac{\quad}{\|p_k\|}$$

then for all $k > k_0$, the step length α_k
that choice of $\alpha_k = 1$, $k > k_0$, the sequen x^*
superlinearly.

Note: once close enough to the solution so that $\nabla^2 f(x_k)$ became s.p.d., the limit is trivially satisfied and for $\alpha_k = 1$ we recover local quadratic convergence.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable.
Let $\{x_k\}$ be a sequence generated by a quasi-Newton method (note step length 1, B_k s.p.d.)

$$x_{k+1} = x_k - B_k^{-1}\nabla f(x_k).$$

Assu ... ng the
suffi ... y
if

$$\lim_{k\to\infty} \frac{\|(B_k - \nabla^2 f(x^\star) \ldots}{\|p_k\|}$$

Note: the superlinear convergence rate can be attained even if the sequence $\{B_k\}$ does not converge to $\nabla^2 f(x^\star)$. It suffices that B_k becomes increasingly accurate approximation to $\nabla^2 f(x^\star)$ along the search direction p_k. Quasi-Newton methods use it to construct B_k.

Away from the solution, the Hessian may not be positive definite, and the Newton direction may not be a descent direction. The general solution is to consider positive definite approximations.

B_k is suffi

Global convergence results can be established for Newton method with Hessian modification and step satisfying W Armijo backtracking conditions provided tha

$\kappa(B_k) = \|B_k\|\|B_k^{-1}\| \leq C$ for some $C > 0$ and all k whenever the sequence of the Hessians $\{\nabla^2 f(x_k)\}$ is bounded.

Eigenvalue decomposition

$$\nabla^2 f(x_k) = Q\Lambda Q^{\mathrm{T}} = \sum_{i=1}^{n} \lambda_i q_i q_i^{\mathrm{T}}.$$

Exa

$$Q = I, \quad \Lambda = \mathrm{diag}($$

The Newton step: $p_k = (-0.1, 1, 2)^{\mathrm{T}}$

As $p_k^{\mathrm{T}} \nabla f(x_k) > 0$, it is not a descent direction.

Eigenvalue modifications (not practical):
Replace all negative eigenvalues with $\delta = \sqrt{\mathbf{u}} = 10^{-8}$, where $\mathbf{u} = 10^{-16}$ is the machine precision.

$$B_k = \sum_{i=1}^{2} \lambda_i q_i q_i^{\mathrm{T}} + \delta q_3 q_3^{\mathrm{T}} = \mathrm{diag}(10, 3, 10^{-8})$$

B_k is e
direc

$$p_k = -B_k^{-1} \nabla f_k = -\sum_{i=1}^{2} \frac{1}{\lambda_i} q_i q_i^{\mathrm{T}} \nabla f_k - \quad)q_3.$$

p_k is a descent direction but the length is very large, not in line with local validity of the Newton approximation. Thus p_k may be ineffective.
Adapt choice of δ to avoid excessive lengths. Even $\delta = 0$ which eliminates direction q_3.

Let A is symmetric $A = Q\Lambda Q^{\mathrm{T}}$.
The correction matrix ΔA of minimum Frobenius norm that ensures $\lambda_{\text{min}}(A + \Delta A) \geq \delta$ is given by

$$\Delta A = Q \,\mathrm{diag}(\tau_i) Q^{\mathrm{T}}, \quad \text{with} \quad \tau_i = \begin{cases} 0, & \lambda_i \geq \delta \\ \delta - \lambda_i, & \lambda_i < \delta. \end{cases}$$

and th

Frobenius norm is defined $\|A\|_F^2 = \sum_{i,}^{n}$ n

The correction matrix ΔA of minimum F
satisfies $\lambda_{\text{min}}(A + \Delta A) \geq \delta$ is given by

$$\Delta A = \tau I, \quad \text{with} \quad \tau = \max(0, \delta - \lambda_{\text{min}}(A)).$$

and the modified matrix has the form $A + \tau I$.

Simple idea

If min

(e.g. 1

Atte

If not successful, increase $\tau_{k+1} = \max($

Drawback: possibly multiple failed attempts to f

Figure: Nocedal Wright Ex. 3.1

For A indefinite:

- The factorisation $A = LDL^T$ may not exist.
- Even if it does exist, the algorithm can be unstable
-

  A.
- Instead, modify A during the factoris

  elements of D are sufficiently positiv L

  and D are not to large.

Modified Cholesky decomposition

Choose $\delta, \beta > 0$. While computing jth column of L, D ensure

$$d_j \geq \delta, \quad |m_{ij}| \leq \beta, \quad i = j+1, j+2, \dots, n,$$

where $m_{ij} = l_{ij}\sqrt{d_j}$.

To satisfy these bounds we only need to change how d_j is com

$\leq$

where $c_{ij} = l_{ij} d_j$. Note: θ_j can be comput
computing c_{ij}, $j < i \leq n$ only needs previo

Verification:
$d_j \geq \delta$ due to taking maximum

$$|m_{ij}| = |l_{ij}\sqrt{d_j}| = \frac{|c_{ij}|}{\sqrt{d_j}} \leq \frac{|c_{ij}|\beta}{\theta_j} \leq \beta, \quad \forall i > j.$$

Properties:

- Modifies the Hessian during factorization where necessary.
- The modified Cholesky factors exist and are bounded relative to the norm of the actual Hessian.
- It does not modify Hessian if it is sufficiently positive definite.

This is
intro
size of t

$$PAP^{\mathrm{T}} + E = LDL^{\mathrm{T}}$$

where E is a nonnegative diagonal matrix that i
sufficiently positive definite.

It has been shown, that the matrices obtained by this modified Cholesky algorithm to the exact Hessian $\nabla^2 f(x_k)$ have bounded condition numbers, hence some global convergence results can be obtained.

How to find a step length satisfying one of the termination conditions e.g. Wolfe etc. for

$$\phi(\alpha) = f(x_k + \alpha p_k),$$

wher

If f i … as a global minimiser along the ray $x_k + \alpha p$ which can be calculated analytically

$$\alpha_k = \frac{p_k^{\mathrm{T}} \nabla f}{p_k^{\mathrm{T}} Q}$$

For general nonlinear functions iterative approach is necessary.

Line search algorithms can be classified according to the information they use:

Methods using only function evaluations can be very inefficient as they need to continue iterating until a very small interval has been foun

Met
curre
which require gradients to evaluate.

Typically, they consist of two phases: ds
an interval containing acceptable step lengths a
phase which locates the final step in the interval.

Aim: find a step length α that satisfies sufficient decrease condition without being to small. [similarity to backtracking]

Sufficient decrease condition

.

We w

Initial guess α_0: check sufficient decrease con

$$\phi(\alpha_0) \leq \phi(0) + c_1 \quad 0$$

If satisfied terminate the search. Otherwise, $[0, \alpha_0]$ contains acceptable step lengths.

Quadratic approximation $\phi_q(\alpha)$ to ϕ by interpolating the available information: $\phi_q(0) = \phi(0)$, $\phi_q'(0) = \phi'(0)$ and $\phi_q(\alpha_0) = \phi(\alpha_0)$ yields

$$\phi_q(\alpha) = \frac{\phi(\alpha_0) - \phi(0) - \alpha_0\phi'(0)}{\alpha_0^2}\alpha^2 + \phi'(0)\alpha + \phi(0).$$

The new trial value α_1 is defined as the minimiser of ϕ_q i.e.

If sufficient decrease condition is satisfied, termi

Otherwise, construct cubic interpolating the fo

$\phi_c(0) = \phi(0)$, $\phi_c'(0) = \phi'(0)$, $\phi_c(\alpha_0) =$ $\alpha_1)$

yields

$$\phi_c(\alpha) = a\alpha^3 + b\alpha^2 + \alpha\phi'(0) + \phi(0),$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\alpha_0^2\alpha_1^2(\alpha_1 - \alpha_0)} \begin{bmatrix} \alpha_0^2 & -\alpha_1^2 \\ -\alpha_0^3 & \alpha_1^3 \end{bmatrix} \begin{bmatrix} \phi(\alpha_1) - \phi(0) - \phi'(0)\alpha_1 \\ \phi(\alpha_0) - \phi(0) - \phi'(0)\alpha_0 \end{bmatrix}$$

By differentiating ϕ_c we find the minimiser $\alpha_2 \in [0, \alpha_1]$

$$\alpha_2 = \frac{-b + \sqrt{b^2 - 3a\phi'(0)}}{3a}.$$

If necessary repeat the cubic interpolation with $\phi_c(0) = \phi(0)$,
$\phi_c'(0$
$\phi_c(\alpha$ ... fies the
suffi

Safeguard: If any α_i is either too close to
than α_{i-1}, we reset $\alpha_i = \alpha_{i-1}/2$.

If derivatives can be computed along the function v
additional cost, we can also devise variant interpolating ϕ, ϕ' at
two most recent values.

Initial step length

For Newton and quasi Newton $\alpha_0 = 1$. This ensures that unit step lengt
cond

For m
steepest descent or conjugate gradient it is impor
available information to make the initial guess e.

- First order change in function at iterate x_k will be the same as that obtained at previous step
  i.e. $\alpha_0 p_k^{\mathrm{T}} \nabla f(x_k) = \alpha_{k-1} p_{k-1}^{\mathrm{T}} \nabla f(x_{k-1})$

$$\alpha_0 = \alpha_{k-1} \frac{p_{k-1}^{\mathrm{T}} \nabla f(x_{k-1})}{p_k^{\mathrm{T}} \nabla f(x_k)}.$$

-

$$\alpha_0 = \frac{2(f(x_k) - }{\phi'}$$

It can be shown that if $x_k \to x^\star$ s converges to 1. If we adjust by setting $\alpha_0 = \min(1, 1.01\alpha_0)$ we find that the unit step length will eventually always be tried and accepted and the superlinear convergence will be observed.