

NUMERICAL OPTIMISATION ASSIGNMENT 8

MARTA BETCKE
KIKO RUL-LAN

EXERCISE 1

Consider a problem to minimise the function

$$\min_x f(x) = \frac{1}{2}x^T Gx + c^T x$$

subject to the constraint

$$Ax \leq b,$$

where $G \in \mathbb{R}^{n \times n}$ symmetric positive semidefinite, $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$.

- (a) State the KKT conditions for this problem.

We can rewrite $Ax \leq b$ as $Ax - b \leq 0$. The Lagrangian for this problem is

$$\mathcal{L}(x, \lambda, \nu) = \frac{1}{2}x^T Gx + c^T x + \lambda^T (Ax - b),$$

therefore, the KKT

<https://eduassistpro.github.io/>

Add WeChat [edu_assist_pro](#)

- (b) Rewrite the constraint using a vector of slack variables $y \in \mathbb{R}^m$, $y \geq 0$ and give the corresponding KKT conditions.

We set $Ax - b + y = 0$ and $y \geq 0$. The Lagrangian for this problem is

$$\mathcal{L}(x, \lambda, \nu) = \frac{1}{2}x^T Gx + c^T x - \lambda^T y + \nu^T (Ax - b + y)$$

and the KKT conditions can be expressed as

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda, \nu) &= Gx + c + A^T \nu = 0, \\ \nabla_y \mathcal{L}(x, \lambda, \nu) &= \lambda - \nu = 0 \Rightarrow \lambda = \nu, \\ Ax - b + y &= 0, \\ y &\geq 0, \\ \lambda &\geq 0, \\ \lambda_i \cdot y_i &= 0 \quad i = 1 \dots m. \end{aligned}$$

- (c) Formulate the dual to the problems in (b) and discuss its properties.

Since the gradient of the Lagrangian for (b) is

$$\nabla_x \mathcal{L}(x, \lambda, \nu) = Gx + c + A^T \nu,$$

and $\lambda = \nu$ (according to the KKT conditions) we find the dual for problem (a) with Lagrangian

$$\nabla_x \mathcal{L}(x, \lambda, \nu) = Gx + c + A^T \lambda.$$

This function has a unique zero that corresponds to the minimum of the quadratic form

$$x^* = -G^{-1}(A^T \lambda + c).$$

We substitute it into the Lagrangian to obtain the dual problem:

$$\begin{aligned} \mathcal{L}(x^*, \lambda, \nu) &= \frac{1}{2}[-G^{-1}(A^T \lambda + c)]^T G[-G^{-1}(A^T \lambda + c)] + c^T(-G^{-1}(A^T \lambda + c)) + \lambda^T[A(-G^{-1}(A^T \lambda + c)) - b] = \\ &= \frac{1}{2}(A^T \lambda + c)^T G^{-1}(A^T \lambda + c) - (A^T \lambda + c)^T G^{-1}(A^T \lambda + c) - \lambda^T b = \\ &= -\frac{1}{2}(A^T \lambda + c)^T G^{-1}(A^T \lambda + c) - \lambda^T b \end{aligned}$$

Therefore the dual problem is

$$\begin{aligned} \max_{\lambda} & -\frac{1}{2}(A^T \lambda + c)^T G^{-1}(A^T \lambda + c) - \lambda^T b \\ \text{subject to } & \lambda \geq 0 \end{aligned}$$

The Lagrangian for (b) is the same as for (a).

EXERCISE 2 Assignment Project Exam Help

Solve the following constr

<https://eduassistpro.github.io/>

- (a) Formulate the KKT system.

This problem can be reformulated as

$$\begin{aligned} \min_{(x,y)} f(x, y) &= \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 4 \\ \text{subject to } & (1 \quad -1) \begin{pmatrix} x \\ y \end{pmatrix} - 4 = 0 \end{aligned}$$

Therefore, the Lagrangian is

$$\mathcal{L}(x, \lambda, \nu) = \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 4 + \nu \left((1 \quad -1) \begin{pmatrix} x \\ y \end{pmatrix} - 4 \right),$$

and the KKT conditions for this problem are

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda, \nu) &= \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \\ (1 \quad -1) & \begin{pmatrix} x \\ y \end{pmatrix} - 4 = 0 \end{aligned}$$

- (b) Solve the KKT system with a method of your choice. Explain briefly your results. We have a problem of the form

$$\min_x \frac{1}{2} x^T P x + q^T x + r$$

subject to $Ax = b$. Its unique solution satisfies

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

In our problem

$$P = \begin{pmatrix} 4 & -4 \\ -4 & 8 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad b = 4$$

Then we have

$$\begin{pmatrix} 4 & -4 & 1 \\ -4 & 8 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x^* \\ \nu^* \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix}.$$

Solving for x^*, ν^* we obtain

$$x^* = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \quad \nu^* = -12.$$

Assignment Project Exam Help

<https://eduassistpro.github.io/>

Add WeChat edu_assist_pro