3D Modelling Transformations

Intended Learning Outcomes

- Understand the use of homogeneous coordinates
- Learn different types of 3D transforms and the concept of composite transform
- Able to use co itch between one coordinate fra
- Able to use OpenGL to imple inate transform

Homogeneous coordinates

- Represent a n-dimensional entity as a (n+1)-dimensional entity
- Allow all lin expressed as matrix multi matrix addition/subtraction

Linear Transform

- $\mathbf{P}_2 = \mathbf{M}_1\mathbf{P}_1 + \mathbf{M}_2$

  $\mathbf{P}_1$ n-dimensional points (n x 1 column vector)

  $\mathbf{P}_2$ Transformed n-dimensional points

  (n x 1 col

  $\mathbf{M}_1$ n x n squ

  $\mathbf{M}_2$ n x 1 column transfor

- Homogeneous coordinates allow us to express the multiplicative term $\mathbf{M}_1$ and the addition term $\mathbf{M}_2$ in a common 4 x 4 matrix. This is achieved by adding one dimension w.

3D Point

A 3D point (n = 3) can be expressed as

- (X, Y, Z) Euclidean coordinates
- (X_W, Y_W, Z_W, coordinates

$$X = \frac{X_W}{W} \quad Y = \frac{Y_W}{W} \quad Z = \frac{{}_W}{W}$$

- W can be any non-zero value.

3D Translation

- Euclidean

$\mathbf{P}_2 = \mathbf{P}_1 + \mathbf{T}(t_X, t_Y, t_Z)$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} + \begin{pmatrix} t_X \\ t_Y \\ t_Z \end{pmatrix}$$

- Homogeneous

$\mathbf{P}_2 = \mathbf{T}(t_X, t_Y, t_Z)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_X \\ 0 & 1 & 0 & t_Y \\ 0 & 0 & 1 & t_Z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

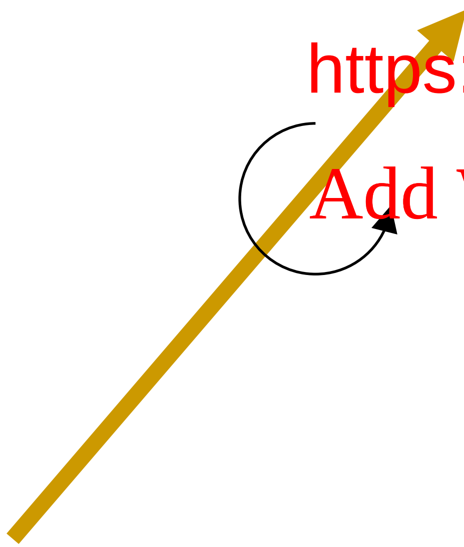
Note : $W_2 = W_1 = 1$

3D Rotations

- Rotation about an axis

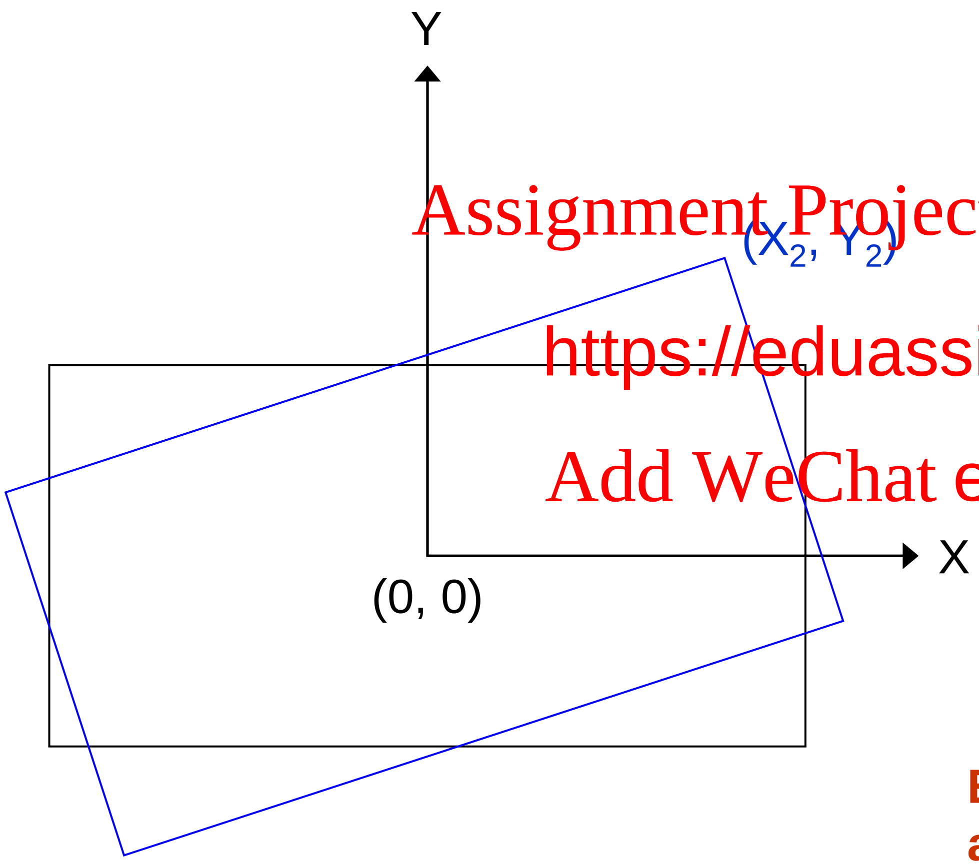
CCW $\Rightarrow$ POSITIVE rotation

d Rule



2D Rotations about the origin

- About a common coordinate system X-Y



$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$$

Equivalent to rotation about Z axis, which is pointing out of paper

Rotation about Z

- Euclidean

  $\mathbf{P}_2 = \mathbf{R}_Z(\theta)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

- Homogeneous

  $\mathbf{P}_2 = \mathbf{R}_Z(\theta)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

Rotation about X

- Euclidean

  $\mathbf{P}_2 = \mathbf{R}_X(\theta)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

- Homogeneous

  $\mathbf{P}_2 = \mathbf{R}_X(\theta)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

Rotation about Y

- Euclidean

  $\mathbf{P}_2 = \mathbf{R}_Y(\theta)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

- Homogeneous

  $\mathbf{P}_2 = \mathbf{R}_Y(\theta)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} & & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

Scaling about the origin

- Euclidean

  $\mathbf{P}_2 = \mathbf{S}(s_X, s_Y, s_Z)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} s_X & 0 & 0 \\ 0 & s_Y & 0 \\ 0 & 0 & s_Z \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

- Homogeneous

  $\mathbf{P}_2 = \mathbf{S}(s_X, s_Y, s_Z)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} s_X & 0 & 0 & 0 \\ 0 & s_Y & 0 & 0 \\ 0 & 0 & s_Z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

Reflection about the X-Y plane

- Euclidean

  $\mathbf{P}_2 = \mathbf{RF}_Z \mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

- Homogeneous

  $\mathbf{P}_2 = \mathbf{RF}_Z \mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

Shearing about the Z axis

- Euclidean

  $\mathbf{P}_2 = \mathbf{Sh}_z(a,b)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix}$$

- Homogeneous

  $\mathbf{P}_2 = \mathbf{Sh}_Z(a,b)\mathbf{P}_1$

$$\begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ W_1 \end{pmatrix}$$

Affine Transform

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- a_{ij} and b_i are c
- a linear transformation

- // lines are transformed to // li
- Translation, rotation, scaling, reflection, shearing are special cases
- Any affine transform can be expressed as composition of the above 5 transforms

Composite Transformation

- A number of (relative) transformations applied in sequence
- Models the complex movement of an object in the world coordinate sys
- The transform here possible.
- In practice, ONLY the final 4 ite transformation needs to be st

E.g. 1 Rotation about an axis // to X axis.

- Let (X_f, Y_f, Z_f) be a point on the axis. The composite rotation is

$\mathbf{P}_2 = \mathbf{T}^{-1}\mathbf{R}_x(\theta$ 1

$\mathbf{T} = \mathbf{T}(-X_f, -Y_f, -Z_f)$

- For the composite transformation

$$\begin{pmatrix} 1 & 0 & 0 & X_f \\ 0 & 1 & 0 & Y_f \\ 0 & 0 & 1 & Z_f \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -X_f \\ 0 & 1 & 0 & -Y_f \\ 0 & 0 & 1 & -Z_f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Only the produ

$$\begin{pmatrix} 1 & 0 & 0 & \\ 0 & \cos\theta & -\sin\theta & -Y_f\cos\theta + \quad {}_f \quad\quad {}_f \\ 0 & \sin\theta & \cos\theta & -Y_f\sin\theta - Z_f\cos\theta + Z_f \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is stored

E.g. 2 Scaling about (X_f, Y_f, Z_f)

- $\mathbf{P}_2 = \mathbf{T}^{-1}\mathbf{S}(s_X, s_Y, s_Z)\mathbf{T}\ \mathbf{P}_1$

$\mathbf{T} = \mathbf{T}(-X_f, -Y$

Similarly, only the final 4 x 4 c ransformation is stored

Concept

- A composite transformation may have two physical meaning:

- Either

It represent

- Or

It represents a change of coordinate system

3 Kinds of Coordinate System in CG

- Each object defined in their own natural coordinate system – Modelling coordinate system (MC)

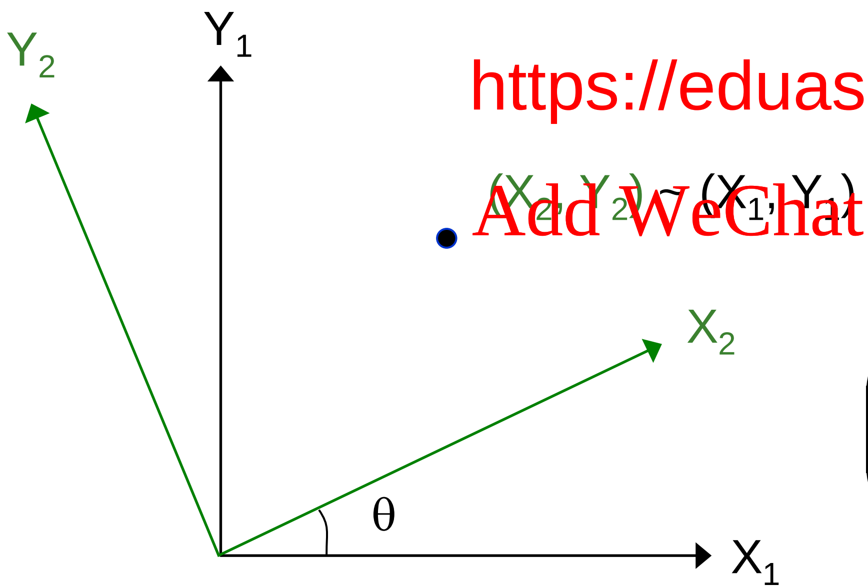
- All objects being placed in a common world coordinate

- For correct a, objects need to be expressed i on viewer or camera coordinate system (VC, CC)

$$\text{MC} \rightarrow \text{WC} \rightarrow \text{VC/CC}$$

A point in two different coordinate sy.

- The SAME point has DIFFERENT coordinates in DIFFERENT coordinate systems



$$\mathbf{R}(-\theta)\mathbf{P}^{(1)}$$

$$\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$$

- $\mathbf{P}^{(i)}$ A point in coordinate system i
- $\mathbf{M}_{j \leftarrow i}$ 4 x 4 transformation that transforms

a p ystem i to
co

- $\mathbf{P}^{(j)} = \mathbf{M}_{j \leftarrow i}\, \mathbf{P}^{(i)}$

- Rule 1 for computing $\mathbf{M}_{j\leftarrow i}$:

$\mathbf{M}_{j\leftarrow i}$ is the inverse of the transformation that takes the $\mathbf{i}^{\mathbf{th}}$ coordinate system frame

o the $\mathbf{j}^{\mathbf{th}}$ coordi position, all the time using the $\mathbf{i}^{\mathbf{th}}$ c system as the reference coordinate system

As $\mathbf{M}_{j\leftarrow i} = \mathbf{M}_{i\leftarrow j}^{-1}$, we have the alternative rule:

- Alternative rule (rule 2) for computing $\mathbf{M}_{j\leftarrow i}$:

$\mathbf{M}_{j\leftarrow i}$ is the transformation that takes the $\mathbf{j}^{\mathbf{th}}$ coordinate system frame <u>as if it is an object</u> to the $\mathbf{i}^{\mathbf{th}}$ coordinate system frame positio the $\mathbf{j}^{\mathbf{th}}$ coordi reference coordinate system

- which rule to use depends on which coordinate system is easier to get on hand

$M_{j\leftarrow i}$ is the INVERSE of the transformation that takes the ith coordinate system frame as if it is an object to the jth coordinate system frame

Proof

Suppose we have two coordinate systems x_i-y_i-z_i and x_j-y_j-z_j. Treat x_i-y_i-z_i and x_j-y_j-z_j as two objects that consist of two sets of points, both defined in the x_i-y_i-z_i coordinate system. Let

$x_i = (1, 0, 0)^T \rightarrow x_j = (a_{11}, a_{21}, a_{31})^T + (t_x, t_y, t_z)^T$
$y_i = (0, 1, 0)^T \rightarrow y_j = (a_{12}, a_{22}, a_{32})^T + (t_x, t_y, t_z)^T$
$z_i = (0, 0, 1)^T \rightarrow z_j = (a_{13}, a_{23}, a_{33})^T + (t_x, t_y, t_z)^T$

where all the coordinates are defined in the x_i-y_i-z_i coordinate system. $\rightarrow$ means "corresponds to".

The transformation T that transforms the three points x_i, y_i, z_i to x_j, y_j, z_j in the x_i-y_i-z_i coordinate system is thus

$$\mathbf{T} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{31} & a_{32} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

However, it can also be interpreted as changing f

$\mathbf{P}^{(j)} = (1, 0, 0)^T \rightarrow \mathbf{P}^{(i)} = (a_{11}, a_{21}, a_{31})^T + (t_x, t_y, t_z)^T$
$\mathbf{P}^{(j)} = (0, 1, 0)^T \rightarrow \mathbf{P}^{(i)} = (a_{12}, a_{22}, a_{32})^T + (t_x, t_y, t_z)^T$
$\mathbf{P}^{(j)} = (0, 0, 1)^T \rightarrow \mathbf{P}^{(i)} = (a_{13}, a_{23}, a_{33})^T + (t_x, t_y, t_z)^T$

Since any arbitrary $\mathbf{P}^{(j)}$ can be written as $\lambda_1(1,0,0)^T + \lambda_2(0,1,0)^T + \lambda_3(0,0,1)^T$, where $\lambda_1, \lambda_2, \lambda_3$ are constants, it follows that

$$\mathbf{M}_{i\leftarrow j} = \mathbf{T}$$

Since $\mathbf{M}_{j\leftarrow i} = \mathbf{M}_{i\leftarrow j}^{-1}$,

$$\mathbf{M}_{j\leftarrow i} = \mathbf{T}^{-1}$$

This gives the rule
$\mathbf{M}_{j\leftarrow i}$ is the INVERSE of the transformation that takes the ith coordinate system frame as if it is an object to the jth coordinate system frame

OpenGL Geometric Transformations

- 4 x 4 translation matrix

  glTranslatef (tx, ty, tz);
- 4 x 4 rotation matrix

  glRotatef (theta, vx, vy, vz);

- 4 x 4 scaling m

  glScalef (s

- 4 x 4 reflection matrix

  glScalef (1, 1, -1); // re ut Z axis
- 4 x 4 shearing matrix

  glMultMatrixf (matrix); // matrix is a 16 element

  // matrix in column-major order

OpenGL Matrix Operations

- Calls the current matrix, responsible for geometrical transformation

glMatrixMode (GL_MODELVIEW);

(do not confuse with *glMatrix* *PROJECTION)*, which is responsible for projecti ation)

- Assign identity matrix to current matrix

glLoadIdentity ( );

- Current matrix is modified by (relative) transformations
  - E.g. glTranslatef, glScalef, glRotatef ...
  - The meaning o s may either be physical action ns

- Current matrix are *postmultip peration specified is first operation performed, like a LIFO stack*

Let **C** be the composite matrix

- Example 1

glMatrixMode (

glLoadIdentity (matr

glTranslatef (-25, 50, 25); // **C** = **T**(-25,50,25)

glRotatef (45, 0, 0, 1); // **C** = **T** (-25,50,25)$\mathbf{R}_Z(45^o)$

glScalef (1, 2, 1); // **C** = **T** (-25,50,25)$\mathbf{R}_Z(45^o)$ **S**(1,2,1)

- Example 2

glMatrixmode (GL_MODELVIEW)

glLoadIdentity ( );

glScalef (1, 2, 1);

glRotatef (45, 0, 0, 1);

glTranslatef (-25, 50, 25); // $\mathbf{C} = \mathbf{S}(1,2,1)\mathbf{R}_Z(45^o)\mathbf{T}(-25,50,25)$

Note: the order of the transformation is important

OpenGL Matrix Stacks

- OpenGL has a stack for storing the relative transformations
- Stack is a LIFO data structure
- Stores interme

- Push the current matrix into t
  glPushMatrix ( );
- Pop the current matrix from the stack
  glPopMatrix ( );

Note: Very useful for modelling hierarchical structures

- Example

glMatrixMode (GL_MODELVIEW)
glLoadIdentity ( ); // **MV** = identity matrix

glTranslatef (-25, 50, 25); // **MV** = **T**(-25,50,25)
glRotatef (45, ,50,25)$\mathbf{R}_Z$(45°)
glPushMatrix ( ); d to the stack
glScalef (1, 2, 1); // **MV** = 5)$\mathbf{R}_Z$(45°) **S**(1,2,1)
glTranslatef (0, 0, 10); // **MV** = $_Z$ (1,2,1) **T**(0, 0, 10)
glPopMatrix ( );
// **MV** = **T** (-25,50,25)$\mathbf{R}_Z$(45°)

References

- Text: Sec 7.2 -7.3, 9.1 – 9.7 (except quaternion method), 9.8. The text uses a different exposition of the coordinate transformation method.

- Our discussion ation follows: Foley et. al., C d., 222-226

- The two methods of coordina ation are conceptually the same.