# Semantic

Dr. Liam O'Connor
University of Edinburgh LFCS
UNSW, Term 3 2020

## Semantics

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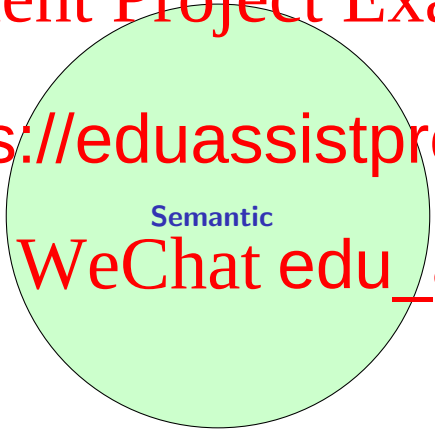

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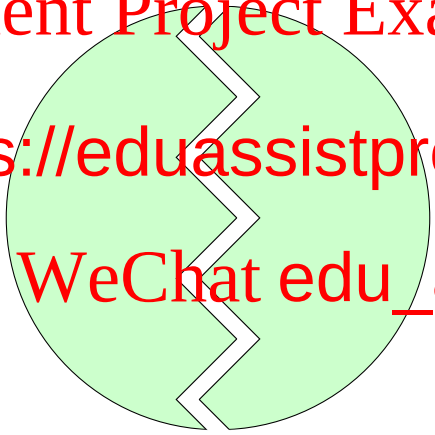
$\sigma\eta\mu\alpha\nu\tau\iota\chi\omega\varsigma$





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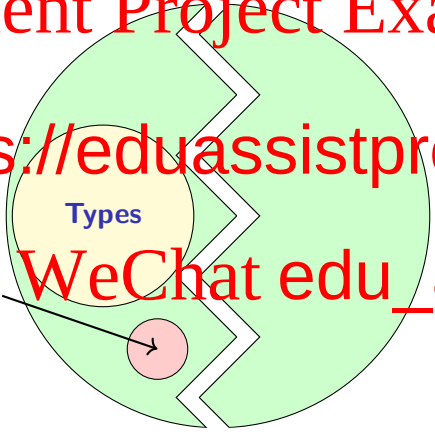

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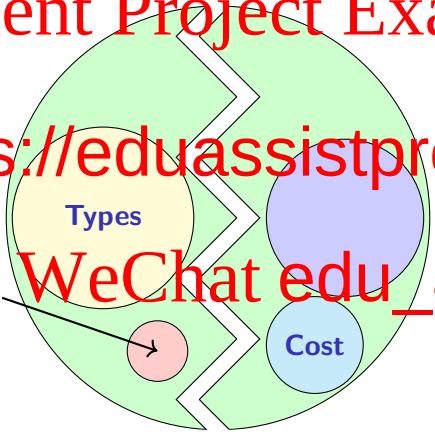

## Static Semantics



**Definition**

The *static sema* *P* that can be determine program.



Recall our arithmetic expression language. What propertie about those terms?

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The *static sema* *P* that can be determine program.



Recall our arithmetic expression language. What propertie about those terms?
The only thing we can check is that the program is well-scoped (as

## Scope-Checking



$$\frac{}{x \ \mathbf{ok}} \qquad \frac{e_1 \ \mathbf{ok} \qquad e_2 \ \mathbf{ok}}{\lambda e_1 \, e_2 \ \mathbf{ok}} \qquad \frac{e_1 \ \mathbf{ok} \qquad e_2 \ \mathbf{ok}}{e_1 \, e_2 \ \mathbf{ok}}$$



**Key Idea**



We keep a *context* $\Gamma$, a set of assumptions, on the left hand of $e$ **ok**, indicating what is required in order for $e$ to be *well-scoped*.

This could be read as hypothetical derivations for the judgement $e$ **ok** or as a binary judgement $\Gamma \vdash e$ **ok**; whichever you prefer.

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**In this course**

We focus mostly on operational semantics. We will use axiomatic semantics (Hoare Logic) on Thursday in the imperative programming topic. Denotational semantics are mostly an extension topic, except for the very next slide.

## Denotational Semantics



$$
\begin{aligned}
[\![\texttt{Var}\ x]\!] &= \\
[\![\texttt{Plus}\ e_1\ e_2]\!] & \\
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$$[\![\cdot]\!] : \textbf{AST} \to (\textbf{Var} \rightharpoonup \mathbb{Z}) \to \mathbb{Z}$$

Our denotation fo ents (mapping from variables to v



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\end{aligned}$$



Where $E[x := n]$ is a new environment just like $E$ except the variable $x$ now maps to $n$.

## Operational Semantics

There are two main kinds of operational semantics.

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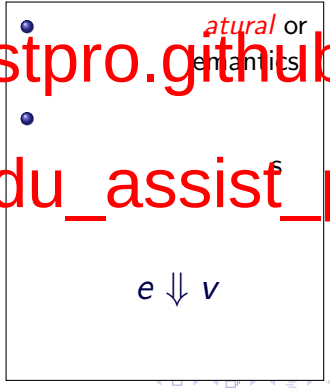
Small Step

Big Step



# Operational Semantics

There are two main kinds of operational semantics.



## Small Step

- Also cal
  *operat*
  *seman*
- A judgement that specifies transitions between *states*:

$$e \mapsto e'$$

## Big Step

- *atural* or
  emantics
- s

$$e \Downarrow v$$

## Big-Step Semantics

We need:

- A set of evaluable expressions $E$
- A set of values $V$
- A relation



**Example (Arith**

$E$ is the set of all clos $\{ \mid \}$ $\mathbb{Z}$.

$$\frac{}{(\texttt{Num}\ n) \Downarrow n}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{(\texttt{Plus}\ e_1\ e_2) \Downarrow (v_1 + v_2)} \qquad \frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{(\texttt{Times}\ e_1\ e_2) \Downarrow (v_1 \times v_2)}$$

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## Evaluation Strategies

$$\frac{e_1 \Downarrow v_1 \qquad e_2[x := (\texttt{Num}\ v_1)] \Downarrow v_2}{(\texttt{Let}\ e_1\ (x)\ e_2) \Downarrow v_2}$$



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Haskell uses *call-by-need* or lazy evaluation, which optimises cases like this.

## Small Step Semantics



For small step semantics, we need:

- A set of states
- A set of initial s
- A set of final st
- A relation $\mapsto \subseteq \Sigma \times \Sigma$, which specifies only "one st





An *execution* or trace $\sigma_1 \mapsto \sigma_2 \mapsto \sigma_3 \mapsto \cdots$ is $\sigma$ e exists no $\sigma_{n+1}$ such that $\sigma_n \mapsto \sigma_{n+1}$; and is called *comple* $_n \in F$.

# Example

**Example (Arithmetic Expressions)**

$\Sigma$ and $V$ are the set of all closed expressions $\{e \mid e \text{ ok}\}$. $F$ is the set of evaluated expressions $\{(\texttt{Num}\ n) \mid n \in \mathbb{Z}\}$.

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$$\frac{}{(\texttt{Let}\ (\texttt{Num}\ n)\ (x.\ e_2)) \mapsto e_2[x := \texttt{Num}\ n]}$$

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# Equivalence



**Comparing small step and big step**

Small step seman
step semantics gi



Having specified t
show they are equivalent, that is:



*If there exists a trace* $e \mapsto \cdots \mapsto (\mathtt{Num}$

We will need to define some notation to remove those blasted magic dots.

## Notation



Let $\overset{\star}{\mapsto}$ be the *reflexive, transitive closure* of $\mapsto$.



We can now state our property formally as:



$$e \overset{\star}{\mapsto} (\mathtt{Num}\ n) \iff$$

# Doing the Proof

The proof will be done on the iPad with typeset versions being uploaded as usual. The big-step to small-step direction can be proven by reasonably straightforward rule induction:





$$\frac{}{v \Downarrow v}$$

The other direction requires the lemma:



$$\frac{e \mapsto e' \quad e' \Downarrow n}{e \Downarrow n}$$

The abridged proof is presented in this lecture, with all cases left for the course website.