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Natural Deduction and Rule I

Dr. Liam O'Connor
University of Edinburgh LFCS
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Formalisation

To talk about lang

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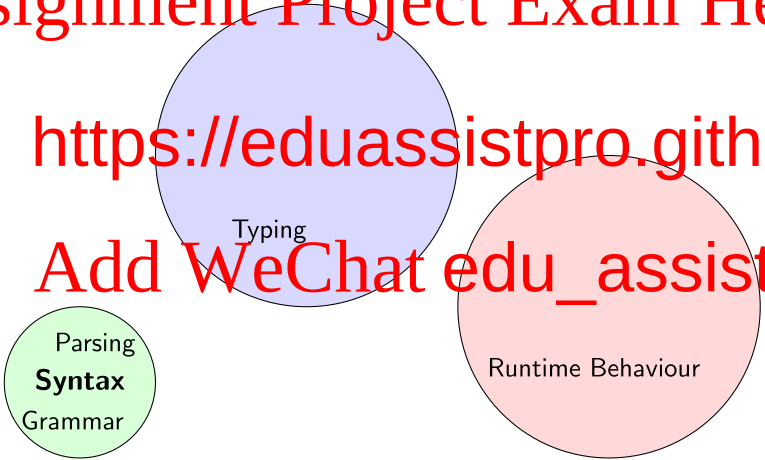
Typically, we describe the language in another language, call *language*. For implementations, it may be a programming language such as formalisations it is usually a minimal logic called a

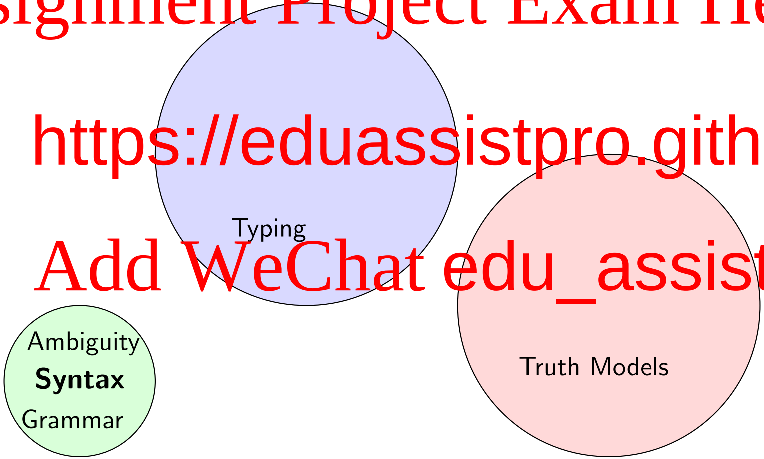
Learning from History

What sort of meta logic should we use? There are a number of things to formalise:



Learning from History

Logicians in the early 20th century had much the same desire to formalise *logics*.



Learning from History

In this course, we will use a meta-logic based on *Natural Deduction* and inductive inference rules, or
1930s.

Judgements

A *judgement* is a statement asserting a certain property for an object.

Example (Informal Judgements)

- $3 + 4 \times 5$ is
- The string
- The string

$\Longrightarrow$ Judgements do not have to hold.

Judgements

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Unary Judgements

Formally, we denote the judgement that a property A holds for a s by writing s A.

Typically, s is a string when describing syntax, and s is a term when describing semantics.

Proving Judgements

We define how a judgement may be proven by providing a set of *inference rules*.

Inference Rule

An inference rule is

$$\frac{\quad}{J}$$

This states that in order to prove judgement J ([illegible] rove all judgements J_1 through to J_n (the *premises*).

Rules with no premises are called *axioms*. Their conclusions always hold.

Examples

Example (Natural Numbers)

$$\frac{\quad}{(\mathsf{S}\ n)\ \mathsf{Nat}}$$

0 is a natural number

if n i

then t

is a n

What terms are in the set $\{n \mid n\ \mathsf{Nat}\}$?

Examples

Example (Natural Numbers)

$$\frac{}{\text{0 Nat}} \qquad \frac{n\ \text{Nat}}{(\text{S}\ n)\ \text{Nat}}$$

0 is a natural number

if n i... then t... is a n...

What terms are in the set $\{n \mid n\ \text{Nat}\}$?

$$\{0, (\text{S}\ 0), (\text{S}\ (\text{S}\ 0)), (\text{S}\ (\text{S}\ (\text{S}\ 0))), \ldots\}$$

Examples

Example (Even and Odd Numbers)

The Proof Video Game

To show that a judgement s A holds:

1. Find a rule whose conclusion matches s A.
2. The preconditions of the applied rules become new proof obligations.
3. Rince and repeat until all obligations are proven up to axioms.

Examples

Example (Even and Odd Numbers)

$$\frac{}{\mathtt{(S\ (S\ (S\ (S\ (S\ 0)))))\ Odd}}$$

Examples

Example (Even and Odd Numbers)

$$\dfrac{\dfrac{}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))\ \mathsf{Even}}}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))))\ \mathsf{Odd}}\,O_1$$

Examples

Example (Even and Odd Numbers)

O_2

$$\dfrac{\dfrac{\overline{(\mathsf{S}\ (\mathsf{S}\ 0))\ \mathsf{Even}}}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))\ \mathsf{Even}}\,E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))))\ \mathsf{Odd}}\,O_1$$

Examples

Example (Even and Odd Numbers)

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{0\ \mathsf{Even}}}{(\mathsf{S}\ (\mathsf{S}\ 0))\ \mathsf{Even}}E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))\ \mathsf{Even}}E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))))\ \mathsf{Odd}}O_1
$$

Examples

Example (Even and Odd Numbers)

O

$$\dfrac{\dfrac{\dfrac{\dfrac{}{0\ \mathsf{Even}}E_1}{(\mathsf{S}\ (\mathsf{S}\ 0))\ \mathsf{Even}}E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))\ \mathsf{Even}}E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))))\ \mathsf{Odd}}O_1$$

Defining Languages

Example (Bracket Matching Language)

Examples of st

Three rules:

Axiom	The empty string is in M
Nesting	Any string in M can be surrounded by parentheses, giving a new string in M
Juxtaposition	Any two strings in M can be concatenated to give a new string in M

With Rules

The Language M

$$\frac{}{(\,)\,(\,(\,)\,)\;\mathsf{M}}$$

With Rules

The Language M

$$\dfrac{\dfrac{}{\texttt{()}\ \mathsf{M}} \qquad \dfrac{}{\texttt{(())}\ \mathsf{M}}}{\texttt{()(())}\ \mathsf{M}}\ M_J$$

With Rules

The Language M

$$\dfrac{\dfrac{\dfrac{}{\varepsilon\ \mathsf{M}}M_E}{(\,)\ \mathsf{M}}M_N \qquad \dfrac{}{(\,(\,)\,)\ \mathsf{M}}}{(\,)\,(\,(\,)\,)\ \mathsf{M}}M_J$$

With Rules

The Language M

$$\dfrac{\dfrac{\dfrac{}{\varepsilon\ \mathsf{M}}M_E}{()\ \mathsf{M}}M_N \qquad \dfrac{\dfrac{\varepsilon\ \mathsf{M}}{()\ \mathsf{M}}M_N}{(())\ \mathsf{M}}M_N}{()(())\ \mathsf{M}}M_J$$

Getting Stuck

If we had started with rule M_N instead, we would have gotten stuck:

$$\dfrac{\overline{}}{()\,(())\ \mathrm{M}}$$

Takeaway

Getting stuck does not mean what you're trying to prove is false!

Derivability

Consider the following rule:

$$\frac{s\ M}{}$$

Does adding this ru

Derivability

Consider the following rule:

$$\frac{s\ M}{}$$

Does adding this ru

No, because we c

ositions of existing rules are called *derivable*:

$$\dfrac{\dfrac{s\ M}{(s)\ M}\ M_N}{((s))\ M}\ M_N$$

We can prove rules as well as judgements, by deriving the conclusion of the rule while taking the premises as local axioms.

Derivability

Is this rule derivable?

$$\overline{\qquad\qquad}$$

Derivability

Is this rule derivable?

$$\frac{}{\qquad\qquad}$$

We can derive it like s

$$\dfrac{\dfrac{\dfrac{}{s\ \mathsf{M}}}{(s)\ \mathsf{M}}M_N \qquad \dfrac{}{s}}{(s)\,s\ \mathsf{M}}$$

Derivability

Is this rule derivable?

$$\dfrac{(s)\ \mathrm{M}}{\qquad}\,Q$$

Derivability

Is this rule derivable?

$$\frac{(s)\ \mathrm{M}}{}\,Q$$

It is not admissible, i

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}\,M_E}{()\ \mathrm{M}}\,M_N \qquad \dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}}{()}}{()\,()\ \mathrm{M}}}{)\,(\ \mathrm{M}}\,Q$$

Derivability

Is this rule admissible? If so, is it derivable?

Derivability

Is this rule admissible? If so, is it derivable?

- It is admissible, as it doesn't let us prove any new judgement
- It is not derivable, as it is not made up of the composition of exis
- We will see how to prove these sorts of rules are admissible later on.

Hypothetical Derivations

We can write a rule in a horizontal format as well:

$$\frac{A}{\quad}$$

This allows us to nea *pothetical derivations*:

Example

$$\frac{A \vdash B}{C}$$

Read as: *If assuming A we can derive B, then we can derive C.*

Specifying Logic

With hypotheticals we can specify logic, which was the original purpose of natural deduction. Let A True be the judgement that the proposition A is true.

Example (And a

$$\frac{A\ \text{True} \qquad B\ \text{True}}{A \wedge B\ \text{True}}\wedge_I \qquad \frac{A \wedge B\ \text{True}}{A\ \text{True}} \qquad \frac{A \qquad B\ \text{True}}{}$$

$$\frac{A\ \text{True} \vdash B\ \text{True}}{A \Rightarrow B\ \text{True}}\Rightarrow_I \qquad \frac{A \Rightarrow B}{B\ \text{True}}$$

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$$\frac{A\ \text{True} \vdash B\ \text{True}}{A \Rightarrow B\ \text{True}}\Rightarrow_I \qquad \frac{A \Rightarrow B}{B\ \text{True}}$$

Specifying Logic, Continued

Example (Or, True, False and Not)

$$\frac{\quad\cdots\quad}{C \ \text{True}} \ {}_E$$

$$\frac{}{\top \ \text{True}} \ {}_I \qquad \frac{\bot}{A}$$

$$\frac{A \ \text{True} \vdash \bot \ \text{True}}{\neg A \ \text{True}} \ \neg I \qquad \frac{\neg A \ \text{True} \qquad A \ \text{True}}{B \ \text{True}} \ \neg E$$

Specifying Logic, Continued

Example (Or, True, False and Not)

$$\frac{\qquad}{C\ \text{True}}\ E$$

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$$\frac{A\ \text{True} \vdash \bot\ \text{True}}{\neg A\ \text{True}}\ \neg I \qquad \frac{\neg A\ \text{True} \qquad A\ \text{True}}{B\ \text{True}}\ \neg E$$

Minimal Definitions

The above rules are the smallest set of rules to define every string in

Therefore

If we know that a string s M, it must have been through one of these rules.

This is called an *inductive definition* of M.

Rule Induction

Suppose we want to show that a property $P(s)$ of strings s holds for any string s M.
We will use *rule induction*.

$$\frac{s\ \mathsf{M}}{(s)\ \mathsf{M}}\, M_N \qquad P(s) \text{ implies}$$

$$\frac{s_1\ \mathsf{M} \quad s_2\ \mathsf{M}}{s_1 s_2\ \mathsf{M}}\, M_J \qquad P(s_1) \text{ and}$$

Then we have shown $P(s)$ for all s M.

These assumptions are called *inductive hypotheses*.

Rule Induction

Example (Cou

Let $op(s)$ deno enote the number of closing p

$$s\ \mathsf{M} \implies op(s) =$$

by doing rule induction on s M.

Rule Induction

Example (Counting Parens)

$$\frac{}{\varepsilon}\, M_E$$

Base Case: $op(\varepsilon) = 0 = cl(\varepsilon)$

Rule Induction

Example (Counting Parens)

$$\frac{}{\varepsilon} M_E$$

Base Case: $op(\varepsilon) = 0 = cl(\varepsilon)$

$$\frac{s}{(s}$$

$op((s)) = op(s)$ $))$

Rule Induction

Example (Counting Parens)

$$\frac{}{\varepsilon}\, M_E$$

Base Case: $op(\varepsilon) = 0 = cl(\varepsilon)$

$$\frac{s}{(s}$$

$op(\,(s)\,) = op(s)$ ... $))$

$$\frac{s_1\ \mathrm{M} \quad s_2\ \mathrm{M}}{s_1 s_2\ \mathrm{M}}\, M_J$$

Inductive Case:

$op(s_1) = cl(s_1)$ and $op(s_2) = cl(s_2)$

$op(s_1 s_2) = op(s_1) + op(s_2) = cl(s_1 s_2)$

Rule Induction in General

Rule Induction Method

Given a set of rules ... ements that can be inferred wit

$$\frac{\qquad\qquad}{J}$$

that if P holds for each of $J_1 \ldots J_n$, then P hold

Therefore, axioms are the base cases of the induction, all other rules form inductive cases, and the premises of each rule give rise to inductive hypotheses.

Structural Induction

Conventional *structural induction* such as that on natural numbers, which we have encountered before, is a special case of rule induction.

Natural Numb

To show a propert

$$\frac{}{0\ \mathsf{Nat}}$$ Show that $P(0$

$$\frac{n\ \mathsf{Nat}}{(\mathsf{S}\ n)\ \mathsf{Nat}}$$ Assuming $P(n)$, show $P(n+1)$.

Another Example

Recall our definition of even numbers:

$$\boxed{n \text{ Even}}$$

We could define od

$$\boxed{n \text{ Odd}'}$$

$$\frac{}{(\mathsf{S}\ 0)\ \text{Odd}'}\ O'_1 \qquad \frac{}{(\mathsf{S}\ (}$$

Let's prove the original Odd rule, but for Odd′ (to whiteboard):

$$\frac{n \text{ Even}}{(\mathsf{S}\ n)\ \text{Odd}'}$$

Arithmetic

Example (Arithmetic Expression)

A $\in \mathbb{Z}$)

Arithmetic

Example (Arithmetic Expression)

A ... $\in \mathbb{Z}$)

$$\frac{i \in \mathbb{Z}}{i \text{ Arith}}L \qquad \frac{a \text{ Arith} \qquad b \text{ Arith}}{a \times b \text{ Arith}}P \qquad \frac{a \text{ Ar}\ldots}{\ldots} \qquad \frac{}{\ldots\text{h}}$$

Infer $1 + 2 \times 3$ Arith (both ways) to whiteboard

Ambiguity

Arith is *ambiguous*, which means that there are multiple ways to derive the same judgement.

For syntax, this is a bi

semantic inconsi

$$\dfrac{\dfrac{1 \in \mathbb{Z}}{1 \text{ Arith}} \quad \dfrac{\dfrac{2 \in \mathbb{Z}}{2 \text{ Arith}} \quad \dfrac{3 \in \mathbb{Z}}{3 \text{ Arith}}}{2 \times 3 \text{ Arith}}}{1 + 2 \times 3 \text{ Arith}}$$

$$\dfrac{\dfrac{\dfrac{1}{1 \text{ Ar}} \quad \dfrac{}{}}{}}{1 + 2 \times 3 \text{ Arith}}$$

Second Attempt

We want to specify Arith in such a way that enforces order of operations.
Here we will use multiple judgements:

Example (Arithmetic Expression)

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We want to specify Arith in such a way that enforces order of operations.
Here we will use multiple judgements:

Example (Arithmetic Expression)

$$\frac{i \in \mathbb{Z}}{i\ \mathsf{Atom}} \qquad \frac{a\ \mathsf{SExp}}{(a)\ \mathsf{Atom}} \qquad \frac{e\ \text{[illegible]}}{e\ \text{[illegible]}} \qquad \frac{\text{[illegible]}}{\text{[illegible]}}$$

$$\frac{a\ \mathsf{PExp} \qquad b\ \mathsf{PExp}}{a \times b\ \mathsf{PExp}} \qquad \frac{a\ \mathsf{SExp} \qquad b\ \mathsf{SExp}}{a + b\ \mathsf{SExp}}$$

Consider: Is there still any ambiguity here?

More ambiguity

$$
\dfrac{\dfrac{\dfrac{1}{1\ \mathsf{At}}}{1\ \mathsf{PExp}} \quad \dfrac{\dfrac{\dfrac{\dfrac{2 \quad \mathbb{Z}}{}}{}}{} \quad \dfrac{\dfrac{\dfrac{3 \quad \mathbb{Z}}{}}{}}{}}{2 \times 3\ \mathsf{PExp}}}{1 \times 2 \times 3\ \mathsf{PExp}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{1 \quad \mathbb{Z}}{}}{}}{} \quad \dfrac{\dfrac{\dfrac{2 \quad \mathbb{Z}}{}}{}}{}}{1 \quad 2\ \mathsf{PExp}} \quad \dfrac{\dfrac{3 \in \mathbb{Z}}{\mathsf{Atom}}}{3\ \mathsf{PExp}}}{}
$$

This ambiguity seems harmless, but it would not be harmless fo operations. Which ones?

More ambiguity

$$\dfrac{\dfrac{\dfrac{1}{1\ \mathsf{At}}}{1\ \mathsf{PExp}} \quad \dfrac{\dfrac{\dfrac{2 \quad \mathbb{Z}}{}}{} \quad \dfrac{\dfrac{3 \quad \mathbb{Z}}{}}{}}{2 \times 3\ \mathsf{PExp}}}{1 \times 2 \times 3\ \mathsf{PExp}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{1 \quad \mathbb{Z}}{}}{} \quad \dfrac{\dfrac{2 \quad \mathbb{Z}}{}}{}}{1 \quad 2\ \mathsf{PExp}} \quad \dfrac{\dfrac{3 \in \mathbb{Z}}{\mathsf{Atom}}}{3\ \mathsf{PExp}}}{}$$

This ambiguity seems harmless, but it would not be harmless fo operations. Which ones? Operators that are not

We have to specify the *associativity* of operators. How?

Associativities

Operators have various *associativity* constraints:

Associative

Left-Associative $A \odot B \odot C = (A \odot B) \odot C$

Right-Associative $A \odot B \odot C = A \odot (B \odot C)$

Try to think of some examples!

Enforcing associativity

We force the grammar to accept a smaller set of expressions on one side of the operator only. Show how this works on the whiteboard.

Example (Arithmetic Expression)

Enforcing associativity

We force the grammar to accept a smaller set of expressions on one side of the operator only. Show how this works on the whiteboard.

Example (Arithmetic Expression)

$$\frac{i \in \mathbb{Z}}{i \ \text{Atom}} \qquad \frac{a \ \text{SExp}}{(a) \ \text{Atom}} \qquad \frac{e}{e}$$

$$\frac{a \ \text{Atom} \qquad b \ \text{PExp}}{a \times b \ \text{PExp}} \qquad \frac{a \ \text{PExp} \qquad b \ \text{SExp}}{a + b \ \text{SExp}}$$

Here we made multiplication and addition right associative. How would we do left?

Bring Back Parentheses

The Parenthetical Language

$$\frac{}{\varepsilon\ \mathsf{M}}\,M_E \qquad \frac{s\ \mathsf{M}}{(s)\ \mathsf{M}}\,M_N \qquad \frac{s\ \mathsf{M} \qquad s\ \mathsf{M}}{}$$

Is this language ambiguous? to whiteboard

Ambiguity in Parentheses

Not only is it ambiguous, it is infinitely so. Strings like ( ) ( ) ( ) could be split at two different locations by rule M_J, but if we use ε, then even the string ( ) is ambiguous:

$$\dfrac{\dfrac{}{\varepsilon\ \mathsf{M}}M_E \qquad \dfrac{\dfrac{}{\varepsilon\ \mathsf{M}}M_E \qquad \dfrac{\dfrac{}{\varepsilon}}{(\,)\ \mathsf{M}}N}{(\,)\ \mathsf{M}}M_J}{(\,)\ \mathsf{M}}M_J$$

We will eliminate the ambiguity by once again splitting M into two judgements, N and L.

The crucial observation is that terms in M are a list (L) of terms nested within parentheses (N)

Example (Una

$\boxed{s \; \mathsf{L}}$ $\boxed{s \quad}$

$$\frac{}{\varepsilon \; \mathsf{L}}\, L_E \qquad \frac{s \; \mathsf{L}}{(s) \; \mathsf{N}}\, N_N \qquad \frac{}{s_1 s_2 \; \mathsf{L}}\, J$$

Proving Equivalence

Now we shall prove

The first case requires proving a *lemma*. The secon [illegible] *us induction*. These proofs will be carried out on the "board" [illegible] he proof will also be uploaded.