COMP3161/COMP9164

Preliminaries Exercises

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1. **Strange Loops:** The following system, based on a system called $\textsc{Miu}$, is perhaps famously mentioned in Douglas Hofstadter's book, *Gödel, Escher, Bach.*

$$\frac{}{\texttt{MI}\ \textsc{Mi}}$$

(a) [⋆] Is $\texttt{MUII}\ \textsc{Miu}$ deriva

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{MI}\ \textsc{Miu}}\,1}{\texttt{MII}\ \textsc{Miu}}\,3}{\texttt{MIIII}\ \textsc{Miu}}\,3}{\texttt{MIIIIIIII}\ \textsc{Miu}}\,3}{\texttt{MIIIIIIIIU}\ \textsc{Miu}}\,2}{\dfrac{\texttt{}}{\texttt{}}}\,4$$

(b) [⋆⋆] Is $\dfrac{x\texttt{IU}\ \textsc{Miu}}{x\texttt{I}\ \textsc{Miu}}$ admissible? Is it derivable? Justify your ans

Solution: It is not derivable, but it is admissible. It is not deriva construct a tree that looks like this:

$$\dfrac{x\texttt{IU}\ \textsc{Miu}}{\dfrac{\vdots}{x\texttt{I}\ \textsc{Miu}}}$$

I believe it is, however, *admissible* because it does not change the language $\textsc{Miu}$. I think there is no string x that could be judged $x\ \textsc{Miu}$ with this rule that could not be so judged without it, although I welcome any counterexample or proof.

(c) [⋆⋆⋆⋆] Perhaps famously, $\texttt{MU}\ \textsc{Miu}$ is not admissible. Prove this using rule induction. *Hint*: Try proving something related to the number of `I`s in the string.

Solution: We will prove that the number of `I`s in any string in $\textsc{Miu}$ is not divisible by three. Seeing as `MU` has zero `I`s (a multiple of three), if we prove the above, we prove that `MU` is not admissible.

Base Case (From rule 1). We see that the string `MI` has only one `I`, which is not a multiple of three, hence we have shown our goal.

Inductive case (From rule 2). Given that the number of `I`s in $x\texttt{I}$ is not divisible by three (our inductive hypothesis), we can easily see that the number of `I`s in $x\texttt{IU}$ is identical and therefore is similarly not divisible by three.

Inductive case (From rule 3). Let n be the number of `I`s in $\texttt{M}x$. Our inductive hypothesis is that $3 \nmid n$. The number of `I`s in $\texttt{M}xx$, clearly $2n$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid 2n$.

Inductive case (From rule 4). Let n be the number of Is in $x\texttt{III}y$. Our inductive hypothesis is that $3 \nmid n$. The number of Is in $x\texttt{U}y$, clearly $n-3$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid (n-3)$

Inductive case (From rule 5). Given that the number of Is in $x\texttt{UU}y$ is the same as the number of Is in xy, our inductive hypothesis trivially proves our goal.

Thus, by induction, no string in Miu has a number of Is divisible by three. Therefore, MU Miu is not admissible. □

(d) Here is another language, which we'll call Mi:

$$\frac{}{\texttt{MI}\ \textsc{Mi}}A \qquad \frac{\texttt{M}x\ \textsc{Mi}}{\texttt{M}xx\ \textsc{Mi}}B \qquad \frac{x\texttt{IIIIII}y\ \textsc{Mi}}{xy\ \textsc{Mi}}C$$

i. [⋆⋆⋆] Prove using rule induction that all strings in Mi could be expressed as follows, for some k and some i, where $2^k - 6i > 0$ (where $\texttt{C}^n$ is the character C repeated n times):

Solution:

Base case (From rule A). $\texttt{MI} = \texttt{MI}^{2^k-6i}$ when $2^k - 6i = 1$, i.e when $k = 0$ and $i = 0$.

Inductive case (From rule B) Given that $\texttt{M}x = \texttt{MI}^{2^a-6b}$ (our inductive hypothesis), we must show that $\texttt{M}xx = \texttt{MI}^{2^k-6i}$ for some k and some i. As $x = \texttt{I}^{2^a-6b}$ (from I.H), it is easy to see that $xx = \texttt{I}^{2(2^a-6b)} = \texttt{I}^{2^{a+1}-6(2b)} = \texttt{I}^{2^k-6i}$ for $k = a+1$ and $i = 2b$.

Inductive c nductive hypothesis). We must show t his rule simply subtracts si $= a$ and $i = b + 1$.

Thus, all strings in Mi can be expressed as M □

ii. We will now prove the opposite claim that, for all 6i > 0:

$$\texttt{M}\,\texttt{I}^{2^k-6i}\ \textsc{Mi}$$

To prove this we will need a few lemmas which we will prove separately.

α) [⋆⋆] Prove, using induction on the natural number k (i.e when $k = 0$ and when $k = k'+1$), that $\texttt{M}\,\texttt{I}^{2^k}$ Mi

Solution:

Base case (when $k = 0$). We have to show $\texttt{MI}$ Mi, which is true by rule A.

Inductive case (when $k = k'+1$) We have to show $\texttt{MI}^{2^{k'+1}}$ Mi, with the inductive hypothesis that $\texttt{MI}^{2^{k'}}$ Mi. Equivalently, we have to show $\texttt{MI}^{2^{k'}}\,\texttt{I}^{2^{k'}}$ Mi, as follows:

$$\cfrac{\cfrac{}{\texttt{MI}^{2^{k'}}\ \textsc{Mi}}I.H}{\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}\ \textsc{Mi}}B$$

Therefore, by induction on the natural number k, we have shown $\forall k.\texttt{M}\,\texttt{I}^{2^k}$ Mi. □

β) [⋆⋆] Prove, using induction on the natural number i, that $\texttt{M}\,\texttt{I}^k$ Mi implies $\texttt{M}\,\texttt{I}^{k-6i}$ Mi, assuming $k - 6i > 0$.

Solution:

Base case (when $i = 0$). We must show that $\texttt{M}\,\texttt{I}^k$ Mi implies $\texttt{M}\,\texttt{I}^{k-0}$ Mi, which is obviously a tautology.

Inductive case (when $i = i' + 1$) We must show that $\texttt{M}\,\texttt{I}^k\ \textsc{Mi}$ implies $\texttt{M}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Mi}$, given the inductive hypothesis that $\texttt{M}\,\texttt{I}^{k-6i'}\ \textsc{Mi}$. Note that our I.H can be restated as $\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Mi}$ due to our assumption that $k - 6(i' + 1) > 0$. With this, we can prove our goal as shown:

$$\dfrac{\dfrac{}{\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Mi}}\,I.H}{\texttt{M}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Mi}}\,C$$

□

Therefore, our goal is shown by induction.

Hence, as we know $\texttt{M}\,\texttt{I}^{2^k}\ \textsc{Mi}$ for all k from lemma α, we can conclude from lemma β that $\texttt{M}\,\texttt{I}^{2^k-6i}\ \textsc{Mi}$ for all k and all i where $2^k - 6i > 0$ by modus ponens.

These two parts prove that the language $\textsc{Mi}$ is exactly characterised by the formulation $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$. A very useful res

iii. [⋆] Hence prove or disp

$$\dfrac{}{\texttt{M}x\ \textsc{Mi}}\,\textsc{Lem}_1$$

Solution: We know from part i that $\texttt{M}x\ \textsc{Mi} \implies x = \texttt{I}^{2^k-6i}$ for some k and some i where $2^k - 6i > 0$.

This rule is *not admissible* as it adds strings to the language. As $2^4 - 6 = 10$, we know $\texttt{MI}^{10}$ is in the langu 5 ch it is not as there is no k an

iv. [⋆] Why is the foll

$$\dfrac{xy}{x\texttt{IIIII}}$$

Solution: The rule is not admissible as it adds strings to the language. This allows us to *add* six $\texttt{I}$ characters to any string in $\textsc{Mi}$ and judge it in $\textsc{Mi}$, which results in additional strings. For example, applying the rule to $\texttt{MI}$ (which is in $\textsc{Mi}$), gives us $\texttt{M}\,\texttt{I}^7$, when our existing formulation of $\textsc{Mi}$ ($\texttt{M}\,\texttt{I}^{2^k-6i}$) clearly only allows for even amounts of $\texttt{I}$s.

v. [⋆⋆⋆] Prove that, for all s, $s\ \textsc{Mi} \implies s\ \textsc{Miu}$. Note that using straightforward rule induction appears to necessitate $\textsc{Lem}_2$ above, which we know is not admissible. Try proving using the characterisation we have already developed.

Solution: We shall show that all strings in $\textsc{Mi}$, characterised by $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$, are also in $\textsc{Miu}$. That is, we shall show that $\texttt{M}\,\texttt{I}^{2^k-6i}\ \textsc{Miu}$.
To start, we shall prove inductively on k that $\texttt{M}\,\texttt{I}^{2^k}\ \textsc{Miu}$ for all k.

Base case (Where $k = 0$). We must show $\texttt{MI}\ \textsc{Miu}$, which we know trivially from rule 1.

Inductive case (where $k = k' + 1$). We must show $\texttt{M}\,\texttt{I}^{2^{k'+1}}\ \textsc{Miu}$, given the inductive hypothesis that $\texttt{M}\,\texttt{I}^{2^{k'}}\ \textsc{Miu}$. Note we can restate our proof goal as $\texttt{M}\,\texttt{I}^{2^{k'}}\,\texttt{I}^{2^{k'}}\ \textsc{Miu}$

$$\dfrac{\dfrac{}{\texttt{M}\,\texttt{I}^{2^{k'}}\ \textsc{Miu}}\,I.H}{\texttt{M}\,\texttt{I}^{2^{k'}}\,\texttt{I}^{2^{k'}}\ \textsc{Miu}}\,B$$

Thus we have shown by induction that $\texttt{M}\,\texttt{I}^{2^k}\ \textsc{Miu}$ for all k.

Next we must prove that $\texttt{M}\,\texttt{I}^k\ \textsc{Miu}$ implies $\texttt{M}\,\texttt{I}^{k-6i}\ \textsc{Miu}$ for all i, assuming $k - 6i > 0$.

Base case (where $i = 0$), we must show that $\texttt{M}\,\texttt{I}^k$ $\textsc{Miu}$ implies $\texttt{M}\,\texttt{I}^{k-0}$ $\textsc{Miu}$, which is trivially a tautology.
Inductive case (where $i = i' + 1$) we must show that $\texttt{M}\,\texttt{I}^{k-6(i'+1)}$ $\textsc{Miu}$ given the inductive hypothesis $\texttt{M}\,\texttt{I}^{k-6i'}$ $\textsc{Miu}$. As we know $k - 6(i'+1) > 0$, we can restate our inductive hypothesis as $\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}$ $\textsc{Miu}$, and easily prove our goal:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,I.H}{\texttt{MUIII}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,4}{\texttt{MUU}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,4}{\texttt{M}\,\texttt{I}^{k-6(i'+1)}\ \textsc{Miu}}\,5$$

Thus, by modus ponens we can see that $\texttt{M}\,\texttt{I}^{2^k-6i}$ $\textsc{Miu}$ for all k and i where $2^k - 6i > 0$. As this is the exact characterisation of $\textsc{Mi}$, we have proven that s $\textsc{Mi} \implies s$ $\textsc{Miu}$ for all s. $\square$

2. **Counting Sticks:** The follo
the original invention is not his) is calle iscussed above, this
language is not comprised of a single ju $\Phi\ y\ \Psi\ z$, where x, y
and z are strings of hyphens (i.e s.

$$\dfrac{}{\epsilon\ \Phi\ x\ \Psi\ x}\,B \qquad \dfrac{x\ \Phi\ y\ \Psi\ z}{\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z}\,I$$

(a) [$\star$] Prove that `--` Φ `---` Ψ `-----`.

Solution:

$$\dfrac{\dfrac{\dfrac{}{\epsilon\ \Phi\ \texttt{---}\ \Psi\ \texttt{---}}\,B}{\qquad}\,I}{\qquad}$$

(b) [$\star$] Is the following rul

$$\dfrac{\texttt{-}x\ \Phi\ y}{x\ \Phi\ y}$$

Solution: It is not derivable (as it cannot be shown with a proof tree), but it is admissible. We know this because the language definition for $\Phi\Psi$ is unambiguous, so the only way for $\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z$ to hold is if this was established by rule I. Therefore, we can deduce that $x\ \Phi\ y\ \Psi\ z$, as this is the premise of rule I. We can often "flip" or invert rules in this way, but *only if the language definition is unambiguous.*

(c) [$\star\star$] Show that $x\ \Phi\ \epsilon\ \Psi\ x$, for all hyphen strings x, by doing induction on the length of the hyphen string (where $x = \epsilon$ and $x = \texttt{-}x'$).

Solution:

Base case (where $x = \epsilon$). We must show that $\epsilon\ \Phi\ \epsilon\ \Psi\ \epsilon$, which is trivially true by rule B.
Inductive case (where $x = \texttt{-}x'$) We have the inductive hypothesis $x'\ \Phi\ \epsilon\ \Psi\ x'$, and must show that $\texttt{-}x'\ \Phi\ \epsilon\ \Psi\ \texttt{-}x'$. Our goal trivially reduces to our induction hypothesis by rule I.
Therefore we have shown $x\ \Phi\ \epsilon\ \Psi\ x$ for all x by induction on x. $\square$

(d) [$\star\star\star$] Show that if $\texttt{-}x\ \Phi\ y\ \Psi\ z$ then $x\ \Phi\ \texttt{-}y\ \Psi\ z$, for all hyphen strings x, y and z, by doing induction on the size of x.

Solution: We shall do induction on x where we keep y and z as arbitrary.

Base case. (where $x = \epsilon$). We must show that if $\texttt{-}\ \Phi\ y\ \Psi\ z$ then $\epsilon\ \Phi\ \texttt{-}y\ \Psi\ z$. Observe that the only way for $\texttt{-}\ \Phi\ y\ \Psi\ z$ to hold is if $z = \texttt{-}y$. Therefore we must show that $\epsilon\ \Phi\ \texttt{-}y\ \Psi\ \texttt{-}y$ which is true by rule B.

Inductive case. (From rule I, where $x = -x'$). We have the inductive hypothesis that $-x' \;\Phi\; y' \;\Psi\; z'$ implies $x' \;\Phi\; -y' \;\Psi\; z'$ for any y', z'.
We must show that $--x' \;\Phi\; y \;\Psi\; z$ implies $-x' \;\Phi\; -y \;\Psi\; z$. Observe that the only way that $--x' \;\Phi\; y \;\Psi\; z$ could hold is if $z = -k$ for some k, then by the rule I' which we have already shown to be admissible, we know that $-x' \;\Phi\; y \;\Psi\; k$. Using our induction hypothesis where $y' = y$ and $z' = k$, we can establish that $x' \;\Phi\; -y \;\Psi\; k$, and therefore by rule I we can finally conclude that $-x' \;\Phi\; -y \;\Psi\; -k$ as required. □

(e) [★★] Show that $x \;\Phi\; y \;\Psi\; z$ implies $y \;\Phi\; x \;\Psi\; z$.

Solution: We show this by rule induction on the premise with the rules of $\Phi\Psi$.

Base case. (From rule B, where $\epsilon \;\Phi\; y \;\Psi\; y$). We must show that $y \;\Phi\; \epsilon \;\Psi\; y$. We proved this, most fortunately, above in part (c).

Inductive case. (From ru is that

We must show that $y \;\Phi\; -x' \;\Psi\; -z'$.

$$\dfrac{\dfrac{\dfrac{}{y \;\Phi\; x' \;\Psi\; z'}\,I.H.}{-y \;\Phi\; x' \;\Psi\; -z'}\,I}{y \;\Phi\; -x' \;\Psi\; -z'}\,(d)$$

Thus we have shown by induction that $x \;\Phi\; y \;\Psi\; z$ implies $y \;\Phi\; x \;\Psi\; z$. □

(f) [★★] Have you figure $-^x \;\Phi\; -^y \;\Psi\; -^z$, then $z = -^{x+y}$ (where $-^x$ is a hyp

Solution: We proceed by rule induction on the premise.

Base case. (From rule B, where $-^0 \;\Phi\; -^y \;\Psi\; -^y$), we which holds trivially.
Inductive case (From rule I, where $-^{x'+1} \;\Phi\; -^y$ pothesis that $z' = x' + y$. We must show that $z' + 1 = (x' + 1) + y$, or, equivalently, that $z' = x' + y$, which is just our I.H.
Thus we have shown by rule induction that the $\Phi\Psi$ system is in fact unary addition. □

3. **Ambiguity and Simultaneity**: Here is a simple grammar for a functional programming language [1]:

$$\dfrac{x \in \mathbb{N}}{x \;\mathit{Expr}}\,\textsc{Var.} \qquad \dfrac{e_1 \;\mathit{Expr} \quad e_2 \;\mathit{Expr}}{e_1 e_2 \;\mathit{Expr}}\,\textsc{Appl.} \qquad \dfrac{e \;\mathit{Expr}}{\lambda e \;\mathit{Expr}}\,\textsc{Abst.} \qquad \dfrac{e \;\mathit{Expr}}{(e) \;\mathit{Expr}}\,\textsc{Paren.}$$

(a) [★] Is this grammar ambiguous? If not, explain why not. If so, give an example of an expression that has multiple parse trees.

Solution: Yes, the expression 1 2 3 could be parsed two different ways, i.e:

$$\dfrac{\dfrac{\dfrac{}{1 \;\mathit{Expr}}\,\textsc{Var.} \quad \dfrac{}{2 \;\mathit{Expr}}\,\textsc{Var.}}{1\;2 \;\mathit{Expr}}\,\textsc{Appl.} \quad \dfrac{}{3 \;\mathit{Expr}}\,\textsc{Var.}}{1\;2\;3 \;\mathit{Expr}}\,\textsc{Appl.}$$

Or:

$$\dfrac{\dfrac{}{1 \;\mathit{Expr}}\,\textsc{Var.} \quad \dfrac{\dfrac{}{2 \;\mathit{Expr}}\,\textsc{Var.} \quad \dfrac{}{3 \;\mathit{Expr}}\,\textsc{Var.}}{2\;3 \;\mathit{Expr}}\,\textsc{Appl.}}{1\;2\;3 \;\mathit{Expr}}\,\textsc{Appl.}$$

[1] if you're interested, it's called *lambda calculus*, with *de Bruijn indices* syntax, not that it's relevant to the question!

(b) [$\star\star$] Develop a new (unambiguous) grammar that encodes the left associativity of application, that is 1 2 3 4 should be parsed as ((1 2) 3) 4 (modulo parentheses). Furthermore, lambda expressions should extend as far as possible, i.e $\lambda 1\ 2$ is equivalent to $\lambda(1\ 2)$ not $(\lambda 1)2$.

Solution:

$$\frac{x \in \mathbb{N}}{x\ \mathit{AExpr}}\textsc{AVar}. \qquad \frac{e_1\ \mathit{PExpr} \quad e_2\ \mathit{AExpr}}{e_1 e_2\ \mathit{PExpr}}\textsc{AAppl}. \qquad \frac{e\ \mathit{LExpr}}{\lambda e\ \mathit{LExpr}}\textsc{AAbs}.$$

$$\frac{e\ \mathit{LExpr}}{(e)\ \mathit{AExpr}}\textsc{AParen}.y \qquad \frac{e\ \mathit{PExpr}}{e\ \mathit{LExpr}}\textsc{Shunt}_1 \qquad \frac{e\ \mathit{AExpr}}{e\ \mathit{PExpr}}\textsc{Shunt}_2$$

(c) [$\star\star\star$] Prove that all expressions in your grammar are representable in *Expr*, that is, that your grammar describes only strings that are in *Expr*.

Solution: We shall prove the following simultaneously:

- $x\ \mathit{LExpr} \Rightarrow x\ E$
- $x\ \mathit{PExpr} \Rightarrow x\ E$
- $x\ \mathit{AExpr} \Rightarrow x\ \mathit{Expr}$

Proof. Base case (From rule AVar., where $x\ \mathit{AExpr}$ for some $x \in \mathbb{N}$). We must show $x\ \mathit{Expr}$, trivial by rule Var.

Inductive case. (From rule AAppl., where $e_1 e_2\ \mathit{PExpr}$). We know $e_1\ \mathit{PExpr}$, and $e_2\ \mathit{AExpr}$ which give rise to inductive hypotheses $e_1\ \mathit{Expr}$ (I.H$_1$) and $e_2\ \mathit{Expr}$ (I.H$_2$). We must show that $e_1 e_2\ \mathit{Expr}$.

Inductive case. (From rule AAbs., where $\lambda x\ LE$ xpr, giving inductive hypothesis $x\ \mathit{Expr}$. The rule Abs. then proves our go

Inductive case. (From rule AParen., where $(x)\ A$ Expr, giving inductive hypothesis $x\ \mathit{Expr}$. Then by rule Paren. we show our goal $(x)\ \mathit{Expr}$.

The inductive case for the rules Shunt$_1$ and Shunt$_2$ are trivial as they do not alter the expression.

Thus, by induction, $s\ \mathit{LExpr} \lor s\ \mathit{PExpr} \lor s\ \mathit{AExpr} \implies s\ \mathit{Expr}$. We can state this more succinctly thanks to the Shunt rules as $s\ \mathit{LExpr} \implies s\ \mathit{Expr}$.

$\square$

4. **Regular Expressions:** Consider this language used to describe regular expressions consisting of:

- single characters, written `c`
- Sequential composition, written $R; R$
- Nondeterministic choice, written $R \mid R$.
- Kleene star, written $R\star$.
- Grouping parentheses.

$$\frac{\texttt{c}\ \mathbf{Char}}{\texttt{c}\ \mathbf{R}} \qquad \frac{a\ \mathbf{R} \quad b\ \mathbf{R}}{a; b\ \mathbf{R}} \qquad \frac{a\ \mathbf{R} \quad b\ \mathbf{R}}{a \mid b\ \mathbf{R}} \qquad \frac{a\ \mathbf{R}}{a\star\ \mathbf{R}} \qquad \frac{a\ \mathbf{R}}{(a)\ \mathbf{R}}$$

(a) [$\star$] In what way is this grammar *ambiguous*? Identify an expression with multiple parse trees.

> **Solution:** Precedence between choice and sequential composition is not enforced, and the associativity of the two binary operators is not clarified either.
> `a; b | c | d` encounters both the precedence and the associativity issue.

(b) [$\star$] Devise an alternative grammar that is unambiguous, order of operations should be such that

$$\texttt{a; b; c}\star \mid \texttt{a; d} \mid \texttt{e}$$

is parsed with the grouping indicated by the parentheses in:

$$(\texttt{a}; (\texttt{b}; (\texttt{c}\star))) \mid ((\texttt{a}; \texttt{d}) \mid \texttt{e})$$

> **Solution:**
>
> $$\frac{\texttt{c}\ \textbf{Char}}{\texttt{c}\ \textbf{RAtom}} \qquad \frac{a\ \textbf{RA}}{\quad} \qquad \frac{\quad}{\quad} \qquad \frac{\quad}{\quad\textbf{m}} \qquad \frac{a\ \textbf{RChoice}}{(a)\ \textbf{RAtom}}$$
>
> $$\frac{\quad}{a\ \textbf{RSeq}} \qquad \frac{\quad}{a\ \textbf{RChoice}}$$

5. **Key Combinations:** Consider the language used to document key combinations

$$\frac{x \in \{\text{a}, \text{b}, \ldots, \text{Shift}\}}{\boxed{}}\,\textit{Key} \qquad \frac{c_1\ \textbf{K} \quad c_2\ \textbf{K}}{\quad}\,\textit{Hold} \qquad \frac{c_1\ \textbf{K} \quad c_2\ \textbf{K}}{\quad}\,\textit{Then} \qquad \frac{c\ \textbf{K}}{(c)\ \textbf{K}}\,\textit{Paren}$$

For example $\boxed{\text{Ctrl}}$ ✚ $\boxed{}$

(a) [$\star$] Find an example of a

> **Solution:** The very next question offers an example:
>
> $$\boxed{q}\ \ \boxed{w}\text{✚}\boxed{}\ \ \boxed{}\ \ \boxed{}$$
>
> This can be parsed using *Hold* first or using *Then* first. It also shows an associativity issue with the *Then* rule.

(b) [$\star$] Eliminate ambiguity such that

$$\boxed{q}\ \ \boxed{w}\text{✚}\boxed{e}\ \ \boxed{r}\ \ \boxed{t}$$

is parsed with this grouping:

$$(\boxed{q}\ ((\boxed{w}\text{✚}\boxed{e})(\boxed{r}\ \ \boxed{t})))$$

and such that

$$\boxed{\text{Ctrl}}\text{✚}\boxed{\text{Shift}\Uparrow}\text{✚}\boxed{\text{Q}}$$

is parsed with the following grouping:

$$(\boxed{\text{Ctrl}}\text{✚}\boxed{\text{Shift}\Uparrow})\text{✚}\boxed{\text{Q}}$$

> **Solution:**
>
> $$\frac{x \in \{\text{a}, \text{b}, \ldots, \text{Shift}\}}{\boxed{x}\ \ \textbf{KA}}\,\textit{Key} \qquad \frac{c_1\ \textbf{KH} \quad c_2\ \textbf{KA}}{c_1\text{✚}c_2\ \textbf{KH}}\,\textit{Hold} \qquad \frac{c_1\ \textbf{KH} \quad c_2\ \textbf{KT}}{c_1 c_2\ \textbf{KT}}\,\textit{Then} \qquad \frac{c\ \textbf{KT}}{(c)\ \textbf{KA}}\,\textit{Paren}$$
>
> $$\frac{a\ \textbf{KA}}{a\ \textbf{KH}} \qquad \frac{a\ \textbf{KH}}{a\ \textbf{KT}}$$