Lecture17-ShortestPaths1

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Applications:

Paths in graph

Paths in graphs

Minimize the path length among all possible length

Source	Destination
All	Single
All	All

- Complexity wise, there i e first three
- The output size of the fo

Shortest Path

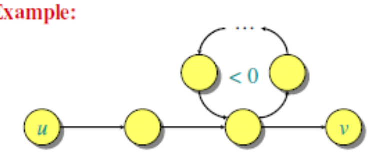
A shortest path from u to v is a path of minimum weight from u to v. The shortest-path weight from u to v is defined as δ(u, v) = min{w(p) : p is a path from u to v}.
Note: δ(u, v) = ∞ if **no path** from u to v exists.

Well-refinedness of shortest paths

Well-definedness of shortest paths

If a graph G contains a negative-weight cycle, then some shortest paths do not exist.

Example:



- Keep taking that negative cycle, then the path get shorter and shorter.

➢ We assume negative weight doesn't exist

Optimal substructure

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.
Proof: cut and phase

? If the optimal substructure exist in longest single path problem (allow visit each vertex only once)

Triangle inequality

Triangle inequality

Shortest path satisfies.

Theorem. For all u, v, x ∈ V, we have δ(u, v) <= δ(u, x) + δ(x, v).

Single -source shortest pahts

Single-source shortest paths

(nonnegative edge weights)

Problem. Assume that w(u, v) >= 0 for all (u, v) ∈ E. (Hence, all shortest-path weights must exist.) From a given source ver -path weights δ(s, v) for all v ∈ V.

Idea: Greedy.

1. Maintain a set S of vertices whose shortestpath distances from s are known.
2. At each step, add to S the vertex v ∈ V – S whose distance estimate from s is minimum.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra

- We can also maintain an arrow based on who changed the value.

Lemma 1

Correctness - part I

Lemma. Initializing d[s] ← 0 and d[v] ← ∞ for all v ∈ V – {s} establishes d[v] >= δ(s, v) for all v ∈ V, and this invariant is maintained over any sequence of relaxation steps.

- d[v] >= δ(s, v) the inequality is maintain through the algorithm, meaning the estimate that we are making always an upper bound of the actural shortest path. And they can never less than the sortest path.

Proof. Suppose not. Let v be the first vertex for which $d[v] < \delta(s, v)$, and let u be the vertex that caused $d[v]$ to change: $d[v] = d[u] + w(u, v)$. Then,

$$\begin{aligned} d[v] &< \delta(s, v) && \text{supposition} \\ &\le \delta(s, u) + \delta(u, v) && \text{triangle inequality} \\ &\le \delta(s,u) + w(u, v) && \text{sh. path} \le \text{specific path} \\ &\le d[u] + w(u, v) && v \text{ is first violation} \end{aligned}$$

Contradiction. $\square$

- Focus on the first violation

Lemma 2

Correctness - part II

from s to v. Then, if d[u] =

, v) after the relaxation.

Theorem: correctness

Correctness - part III

Theorem. Dijkstra's algorithm terminates with d[v] = δ(s, v) for all v ∈ V.

Since u is the first vertex violating the claimed invariant, we have $d[x] = \delta(s, x)$. When x was added to S, the edge (x, y) was relaxed, which implies that $d[y] = \delta(s, y) \le \delta(s, u) < d[u]$. But, $d[u] \le d[y]$ by our choice of u. Contradiction. $\square$

Running time analysis

Running time analysis:

```
|V| times {                while Q ≠ ∅
                              do u ← EXTRACT-MIN(Q)
                                 S ← S ∪ {u}
           degree(u) times {     for each v ∈ Adj[u]
                                    do if d[v] > d[u] + w(u, v)
                                          then d[v] ← d[u] + w(u, v)
```

Handshaking Lemma $\Rightarrow \Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$

Note: Same formula as in the analysis of Prim's minimum spanning tree algorithm.

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- Same formula with Prim

What if the edge weight are all 1?
Instead of using priority queue, we can use a queue (simplify the problem, the cost of PQ is complex here.).