6.4 – Work

Reminder: Force = mass x acceleration

$F = ma = m\frac{d^2s}{dt^2}$ and Newtons (which are units of Force) = $N = (kg) \cdot \frac{meters}{second^2}$

	Force F	distance d	Work W
units (US/Imperial System)	Pounds (lbs)	Feet (ft)	Foot - pounds (ft - lbs)
units (SI/International System)	Newton (N) $N = kg \cdot \frac{m}{s^2}$	meter (m)	Joules (J) $J = Nm$

Work done by a constant force: $W = Fd$

Ex) Determine the work done by a weightlifter in raising a 60 kg barbell from the floor to a height of 2 m.

$F = \text{mass} \cdot \text{accel}$

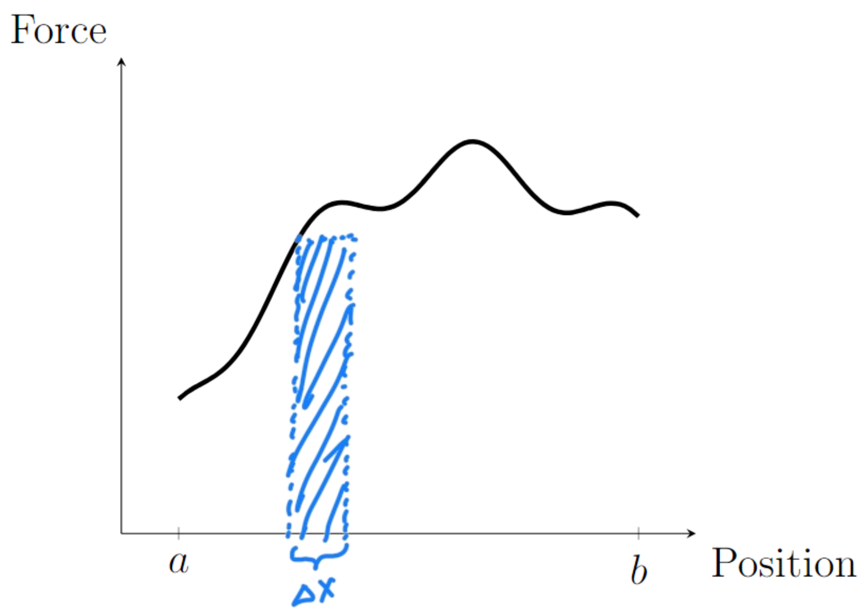
$= (60)(9.8) = 588\ N$

$W = F \cdot \text{dist}$

$= (588)(2)$

$= 1176\ J$

Work involving variable force that depends on position:



Work done over i-th subinterval: $W_i = F(x_i^*)\Delta x$

Approximation of total work done:

$$W \approx \sum_{i=1}^{n} F(x_i^*)\Delta x$$

Exact amount of total work done:

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)\,dx$$

Definition:

The work done by a variable force $F(x)$ in moving an object from $x = a$ to $x = b$ is:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)dx$$

Ropes/Cables

Ex) A uniform cable 30 ft. long and weighing 60 lbs. hangs over the edge of a building. A steel beam weighs 500 lbs and is attached to the end of the cable.

a) Set up an integral to find the work required to pull the beam to the top.

Note: Cable weight: $\frac{60\text{lbs}}{30\text{ft}} = 2 \text{ lbs/ft}$

Total work: $\underbrace{\text{work to pull up beam}}_{\text{const.}} + \underbrace{\text{work to pull up cable}}_{\text{variable}}$

500 lbs

$W_{beam}: 500 \cdot 30$

$= 15000 \text{ lbs/ft}$

cable segment length

$F_i = (2 \text{ lbs/ft})(\Delta x \text{ ft})$

$W_i = F_i \cdot d_i$ } dist cable moves for each i

$= (2\Delta x)(x_i^*)$

$$W_{ttl} = \sum_{i=1}^{n} 2x_i^*\Delta x = \int_0^3 2x\,dx$$

b) Find the work required to pulll the beam to the top.

$$W_{tot} = W_{beam} + W_{tot\ cable} = 15000 + \int_0^{30} 2x\,dx = 15000 + x^2\Big]_0^{30} = 15{,}900 \text{ ft.lbs}$$

OR $$\int_0^{30} (2x + 500)\,dx = x^2 + 500x\Big]_0^{30} = 15{,}900 \text{ ft.lbs}$$

c) Set up an integral to find the work required to pull the beam up 10 feet.

$$W = \underbrace{\int_0^{10} 2x\,dx}_{\text{variable force of cable}} + \underbrace{\int_0^{10} 500\,dx}_{\text{constant force of beam}} + \underbrace{\int_0^{10} 2(20)\,dx}_{\text{const force of remaining 20ft of cable}}$$

d) Find the work required to pulll the beam up 10 feet.

$$W = \int_0^{10} [2x + 500 + 40]\,dx = \int_0^{10} [2x + 540]\,dx = x^2 + 540x\Big]_0^{10}$$

$$= 100 + 5400 - 0 = 5500 \text{ ft.lbs}$$

Springs

Hooke's Law for Springs: The force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x.

$$F = kx$$

where "k" is the spring constant measured in force units per unit length.

Ex) A spring has a natural length of 50 cm. A force of 50 N stretches the spring to a length of 60 cm.

a) What force is needed to stretch the spring x meters?

$F = Kx$

$\Rightarrow 50 = K(0.1\text{m})$

$\Rightarrow K = 500$

$\therefore F = 500x$

b) How long is the spring when stretched by a force of 200 N?

$F = kx$

$\Rightarrow 200 = 500x$

$\Rightarrow x = \frac{200}{500} = \frac{2}{5} = \underbrace{0.4}_{\text{amt. of stretch}}$

$\therefore \text{New length} = \underbrace{0.5}_{\text{natural length}} + 0.4$

$= 0.9 \text{ m}$

c) Find the work done in stretching the spring 0.2 m.

$W = \int_0^{0.2} 500x \, dx = 250x^2 \Big]_0^{0.2}$

$= 250(0.04)$

$= 10 \text{ J}$

0.5m beyond natural length

0.6 m beyond natural length

d) Find the work done in stretching the spring from a length of 1 meter to 1.1 meters.

$$W = \int_{0.5}^{0.6} 500x\,dx$$

$$= 250x^2 \Big]_{0.5}^{0.6}$$

$$= 27.5\text{J}$$

e) 20 J of work is done in stretching the spring a certain distance. How far was the spring stretched?

$$20\text{ J} = \int_{0}^{d} 500x\,dx$$

$$\Rightarrow 20 = 250x^2 \Big]_{0}^{d}$$

$$\Rightarrow 20 = 250(d^2) - 0$$

$$\Rightarrow d^2 = \frac{20}{250}$$

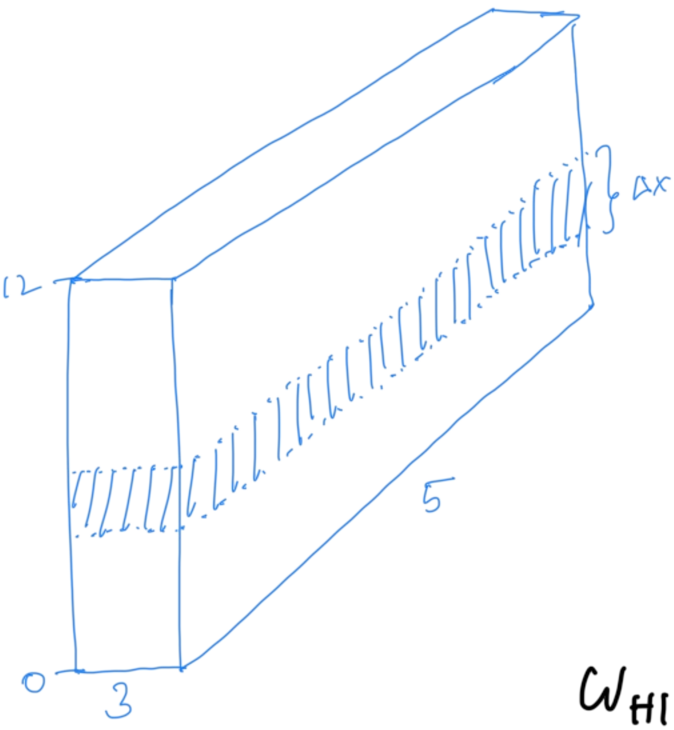
$$\Rightarrow d = \pm\frac{\sqrt{2}}{5}$$

$\therefore d = \frac{\sqrt{2}}{5}$ m of stretch

Pump liquids

Ex) A rectangular storage tank full of rain water has height 12 ft, width 3 ft, and length 5 ft.

a) Set up the integral that would find the work done in pumping all of the water out just over the top of the tank given that the density (weight per cubic ft) of the water is 62.4 lbs./ft^3



$$V_i = 3(5)\Delta x$$

$$F_i = V_i \cdot (\text{density})$$

$$= (15\Delta x)(62.4)$$

$$W_i = F_i \cdot d_i$$

$$= (15\Delta x)(62.4)\cdot(12 - x_i^*)$$

$$W_{HI} = \lim_{n\to\infty} \sum_{i=1}^{n} (15)(62.4)(12 - x_i^*)\Delta x$$

$$= \int_0^{12} \overbrace{\overbrace{\underbrace{(15)}_{\text{len x width}}}^{\text{Volume}}\underbrace{(62.4)}_{\text{density}}}^{\text{Force}}\underbrace{(12 - x)}_{\text{change in height}}\,dx$$

b) Set up the integral that would find the work done if only half of the tank is pumped out just over the top.

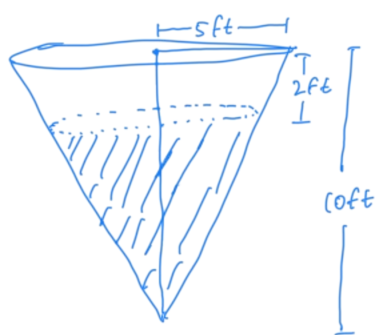
$$W = \int_6^{12} (15)(62.4)(12 - x)\,dx$$

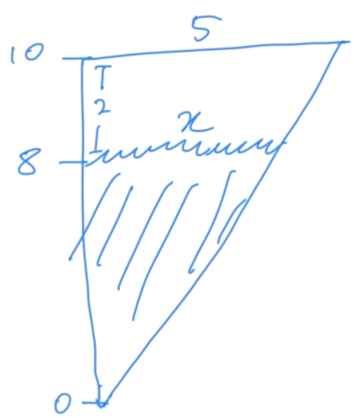
c) Set up the integral that would find the work done if the tank is only half full and the water is pumped out just over the top.

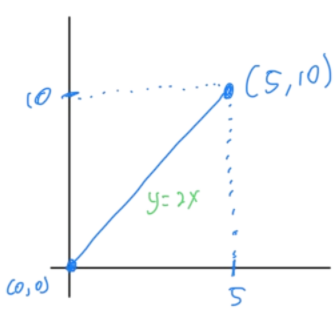
$$W = \int_0^6 (15)(62.4)(12-x)\, dx$$

Ex) Suppose we have a conical tank (point down) with radius 5 ft and height 10 ft, and suppose the tank is filled to 2 ft from the top with milk of density 64.5 lbs./ft^3.

a) How much work will it take to pump the contents to the rim? (Set up the integral)







* radius of each circulor slab varies by $y = 2x$ or $x = \frac{y}{2}$

V_i = Area of circle $\cdot \Delta y$

$= \pi \left(\frac{y}{2}\right)^2 \cdot \Delta y$

$F_i = V_i \cdot (\text{density})$

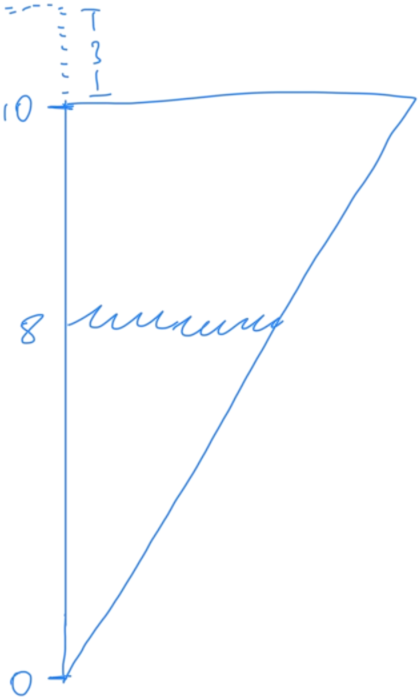
$= \pi \left(\frac{y}{2}\right)^2 \Delta y \cdot (64.5)$

$W_i = F_i \cdot d_i$

$= \pi \left(\frac{y}{2}\right)^2 \Delta y \cdot (64.5) \cdot (10 - y_i^*)$

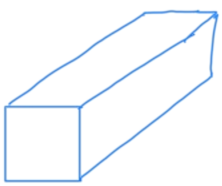
$$\therefore W = \int_0^8 64.5\pi \left(\frac{y^2}{4}\right)(10-y)\, dy$$

b) Set up the integral that would find the work done if the milk were pumped to a level 3 feet above the rim.

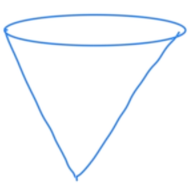


$$W = \int_0^8 64.5\pi\left(\frac{y^2}{4}\right)(13-y)\,dy$$

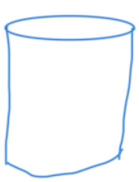
*tanks of other shapes… consider what the shapes of the slabs are



* All slabs have same volume



* Slabs have varying volume



* All slabs have same volume

* Slubs have varying volume