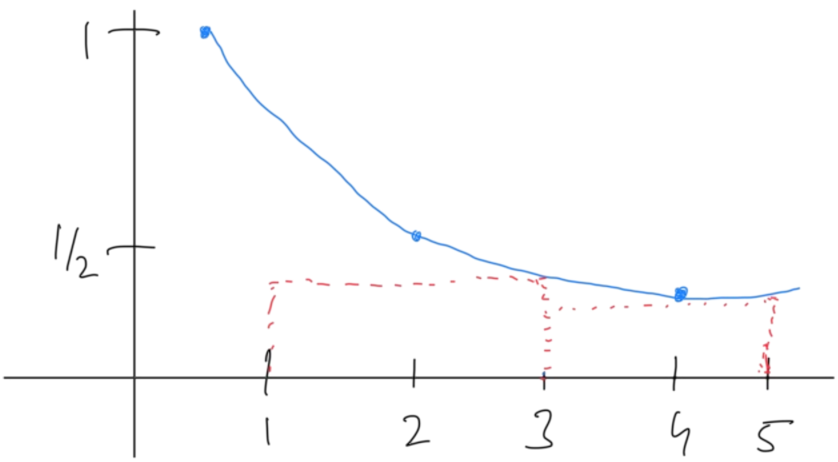
5.2 – The Definite Integral (and review of Riemann sums)

Consider the function on the interval [1, 5]

Ex) $f(x) = \frac{1}{x}$

a) Sketch the graph on the interval.



b) Estimate the area under the graph of the function using the **right endpoints** with two rectangles of equal width. We'll call this R_2. Is this an overestimation or an underestimation?

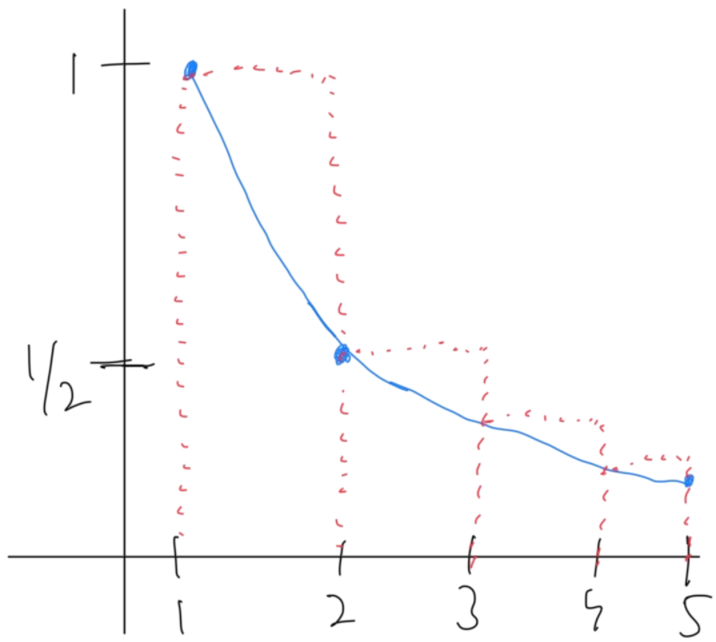
rectangle width : $\frac{b-a}{n} = \frac{5-1}{2} = 2 = \Delta x$

rectangle height : function value on the right side of rectangle

$$A = \underset{w}{2} \cdot \underset{h}{f(3)} + \underset{w}{2} \cdot \underset{h}{f(5)}$$

$$= 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{5} = \frac{16}{15}$$

c) Estimate the area under the graph of the function using the **left endpoints** with four rectangles of equal width. We'll call this L_4. Is this an overestimation or an underestimation?



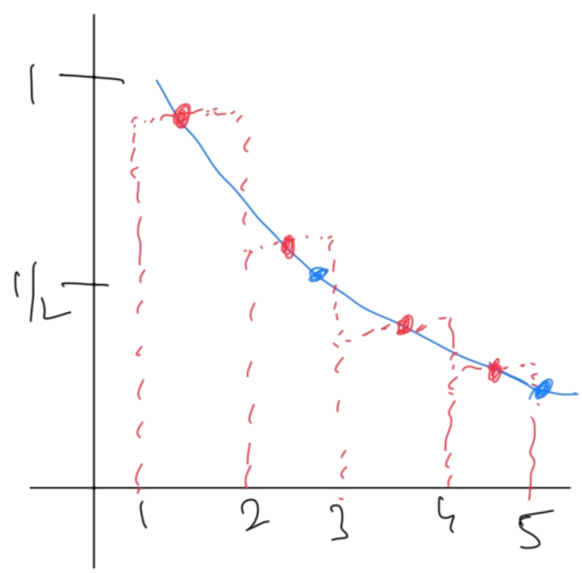
Width $= \frac{5-1}{4} = 1 = \Delta x$

Height = function value on left side of the function

$$A = \underbrace{1}_{w} \cdot \left[\underbrace{\frac{1}{1}}_{h} + \underbrace{\frac{1}{2}}_{h} + \underbrace{\frac{1}{3}}_{h} + \underbrace{\frac{1}{4}}_{h} \right]$$

$$= \frac{25}{12}$$

d) Estimate the area under the graph of the function using the **midpoints** with four rectangles of equal width. We'll call this M_4.



Width $= \frac{5-1}{4} = 1 = \Delta x$

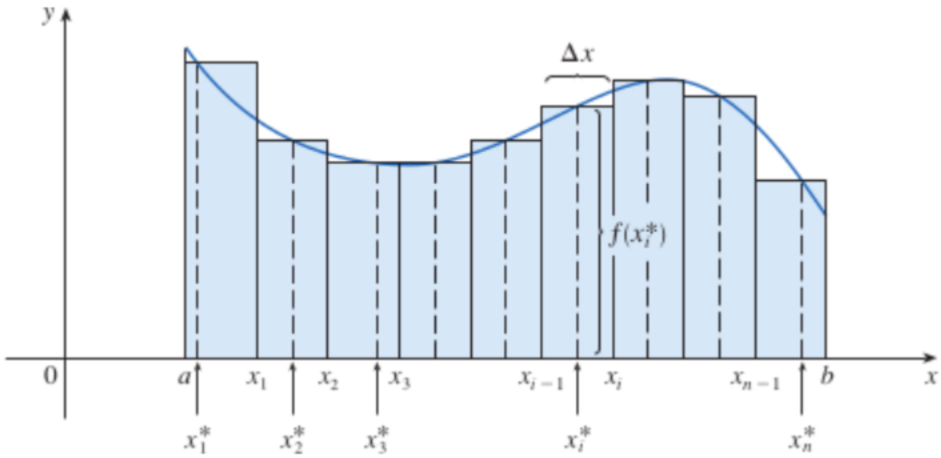
Height = function value at mid points of each rectangle

$$A = \underbrace{1}_{w} \cdot \left[\underbrace{f(3/2)}_{h} + \underbrace{f(5/2)}_{h} + \underbrace{f(7/2)}_{h} + \underbrace{f(9/2)}_{h} \right]$$

$$= \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}$$

* Instead of using right or left endpoints for the height of our rectangles, we could use the height of the i-th rectangle to be the value of f and any number x_i^* in the i-th subinterval $[x_{i-1}, x_i]$. We call the x_i^* 's **sample points**. A more general expression (using the sample points) for the area is:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x]$$



Definition of a Definite Integral

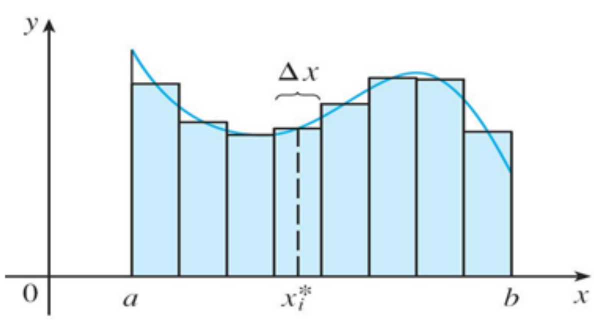
If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \cdots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

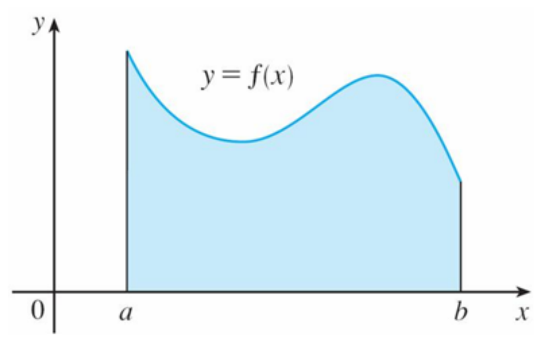
provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Ex) For x in $[0, \frac{\pi}{4}]$, write $\lim_{n\to\infty} \sum_{i=1}^{n} \left(e^{(x_i^*)^2 + 2x_i^*} \tan x_i^* \right) \Delta x$ as a definite integral.

$$\int_0^{\pi/4} \left(e^{x^2+2x} + \tan x \right) dx$$



If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.



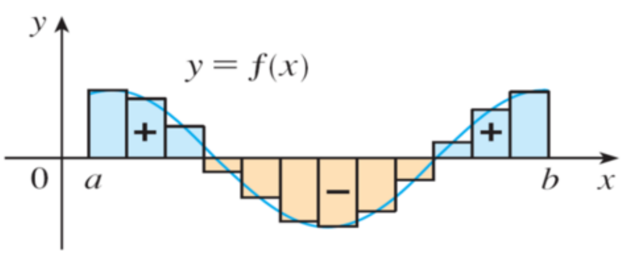
If $f(x) \geq 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Note: If f takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis.

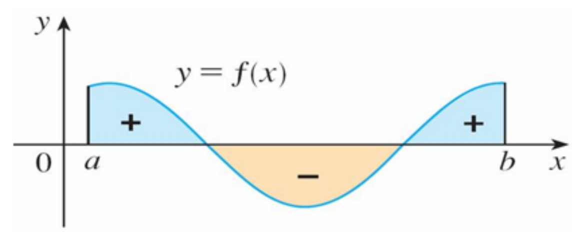
* A definite integral can be interpreted as a **net area** (a difference of areas):

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.



$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area.



$\int_a^b f(x)\,dx$ is the net area.