5.5 – Substitution Method

The Substitution Rule (Indefinite Integrals)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

$u = g(x)$

$du = g'(x)\, dx$

Evaluate.

Ex) $\int 3y\sqrt{7-3y^2}\, dy$

Let $u = 7-3y^2$

$\Rightarrow du = -6y\, dy$

$= -\frac{1}{2}\int -6y\sqrt{7-3y^2}\, dy$

$= -\frac{1}{2}\int \sqrt{u}\, du$

$= -\frac{1}{\cancel{2}}\frac{u^{3/2}}{3/\cancel{2}} + C$

$= \frac{-1}{3u\sqrt{u}} + C$

$= \frac{-1}{3(7-3y^2)\sqrt{7-3y^2}} + C$

(Hint: use log rules)

Ex) $\int \frac{\sec^2(\ln\sqrt{t})}{t}\,dt$

$\ln\sqrt{t}$

$= \ln t^{1/2}$

$= \frac{1}{2}\ln t$

Let $u = \frac{1}{2}\ln t$

$\Rightarrow du = \frac{1}{2}\frac{1}{t}\,dt$

$= 2\int \frac{\sec^2(u)}{2t}\,dt$

$= 2\int \sec^2(u)\,du$

$= 2\tan(u) + C$

$= 2\tan(\ln\sqrt{t}) + C$

Ex) $\int 3x^5\sqrt{x^3+1}\,dx$

Let $x^3+1 = u$

$\Rightarrow du = 3x^2\,dx$

$u-1 = x^3$

$= \int \sqrt{x^3+1}\cdot 3x^2\cdot x^3\,dx$

$= \int \sqrt{x^3+1}\cdot(x^3+1-1)\cdot 3x^2\,dx$

$= \int \sqrt{u}\cdot(u-1)\,du$

$= \int \left(u^{3/2} - u^{1/2}\right)du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$

Ex) $\int \frac{w^2\cos(w^3+1)}{\sin^2(w^3+1)}dw$

Let $u = \sin(w^3+1)$

$\Rightarrow du = \cos(w^3+1) \cdot 3w^2\, dw$

$= \int [\sin(w^3+1)]^{-2}\,[w^2\cos(w^3+1)]\,dw$

$= \frac{1}{3}\int [\sin(w^3+1)]^{-2}\,\{3w^2\cos(w^3+1)\,dw\}$

$= \frac{1}{3}\int u^{-2}\,du$

$\Rightarrow \frac{1}{3}\cdot\frac{u^{-3}}{-3} + C$

$\Rightarrow -\frac{1}{9u^3} \Rightarrow \frac{1}{9\{\sin(w^3+1)\}} + C$

Ex) $\int \frac{\tan^{-1}(2x)}{1+4x^2}\,dx$

Let $u = \tan^{-1}(2x)$

$\Rightarrow du = \frac{1}{1+(2x)^2}\cdot 2\cdot dx$

$\Rightarrow du = \frac{2}{1+4x^2}dx$

$\Rightarrow \frac{1}{2}\int \tan^{-1}(2x)\cdot\frac{2}{1+4x^2}dx$

$\Rightarrow \frac{1}{2}\int u\,du$

$= \frac{1}{2}\cdot\frac{u^2}{2} + C$

$= \frac{u^2}{4} + C$

$\Rightarrow \frac{(\tan^{-1}(2x))^2}{4} + C$

The Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)du$$

This rule says that when using a substitution in a definite integral, we must put everything in terms of the new variable u , not only x and dx but also the limits of integration. The new limits of integration are the values of u that correspond to $x = a$ and $x = b$.

Evaluate using the Substitution Rule for Definite Integrals.

Ex) $\int_0^1 \frac{y^2+4y-4}{\sqrt{y^3+6y^2-12y+9}}dy$

Let $u = y^3 + 6y^2 - 12y + 9$

$\Rightarrow du = (3y^2 + 12y - 12)\, dx$

$\Rightarrow du = 3(y^2 + 4y - 4)\, dx$

When,

$y = 0$

$u = 0^3 + 6(0)^2 - 12(0) + 9 = 9$

When,

$y = 1$

$u = 1 + 6 - 12 + 9$

$= 4$

$= \frac{1}{3}\int_4^9 \frac{1}{\sqrt{u}}\, du$

$= \frac{1}{3}\int_4^9 u^{-1/2}\, du$

$= \frac{1}{3} \cdot \frac{u^{1/2}}{1/2}\Big|_4^9$

$= \frac{1}{3} \cdot \frac{2}{\sqrt{u}}\Big|_4^9$

$= \frac{1}{3}\left[\frac{2}{3} - 1\right]$

$= \frac{1}{3} \cdot \frac{-1}{3}$

$= -\frac{1}{9}$

Evaluate using the Substitution Rule for Definite Integrals.

Ex) $\int_0^1 x^4 e^{9x^5}\, dx$

$$= \frac{1}{45} \int_0^9 e^u \, du$$

Let $u = 9x^5$

$\Rightarrow du = 45x^4 \, dx$

when, $x = 0$, $u = 0$

When, $x = 1$, $u = 9$

$$= \frac{1}{45} e^u \Big|_0^9$$

$$= \frac{1}{45} \left[e^9 - 1 \right]$$

Ex) $\int_0^{\frac{\pi}{4}} \tan x \, dx$

$$= \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$$

Let $u = \cos x$

$\Rightarrow du = -\sin x \, dx$

When,

$x = 0$, $u = 1$

$x = \frac{\pi}{4}$, $u = \frac{1}{\sqrt{2}}$

$$= -\int_0^{1/\sqrt{2}} \frac{1}{u} \, du$$

$$= -\left[\ln |u| \right]_0^{1/\sqrt{2}}$$