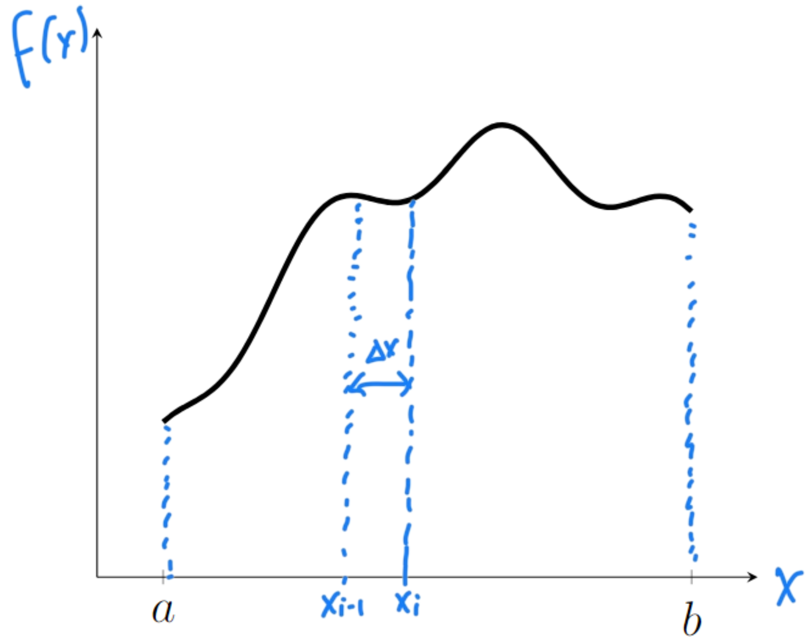
6.5 – Average Value of a Function

Suppose we have a function $f(x)$ and its graph. Let's get a formula to find the average of n values of f.



$$\Delta x = \frac{b-a}{n}$$

$$\Rightarrow \frac{\Delta x}{b-a} = \frac{1}{n}$$

$$x_i = a + i\Delta x$$

Avg value of $f(x)$ $\approx \dfrac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$

$$\approx \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

$$\approx \frac{\Delta x}{b-a}\sum_{i=1}^{n} f(x_i) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

Exact value of $f(x)$: $\displaystyle\lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x\right]$

Definition

If f is integrable on $[a, b]$, then its average value is on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex) Find the average value of the function $f(x) = \frac{1}{x}$ on the interval $[1, 7]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{7-1}\int_1^7 \frac{1}{x}\,dx$$

$$= \frac{1}{6}\Big[\ln(x)\Big]_1^7$$

$$= \frac{1}{6}\left[\ln 7 - \ln 1\right]$$

$$= \frac{1}{6}\ln 7$$

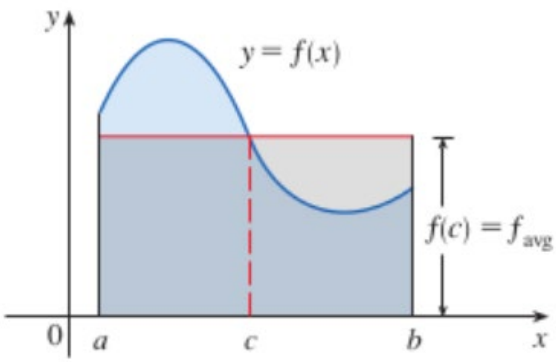
Mean Value Theorem for Integrals

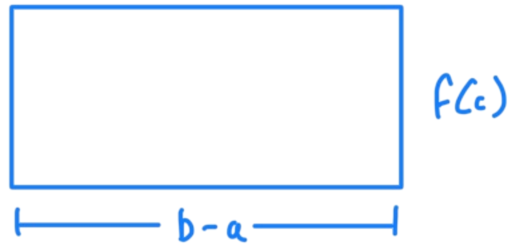
If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

or $\int_a^b f(x)\,dx = f(c)\cdot[b-a]$

* Note: The geometric interpretation of the Mean Value Theorem for Integrals is that, for positive functions f, there is a number c such that the rectangle with base $[a, b]$ and height $f(c)$ has the same area as the region under the graph of f from a to b.





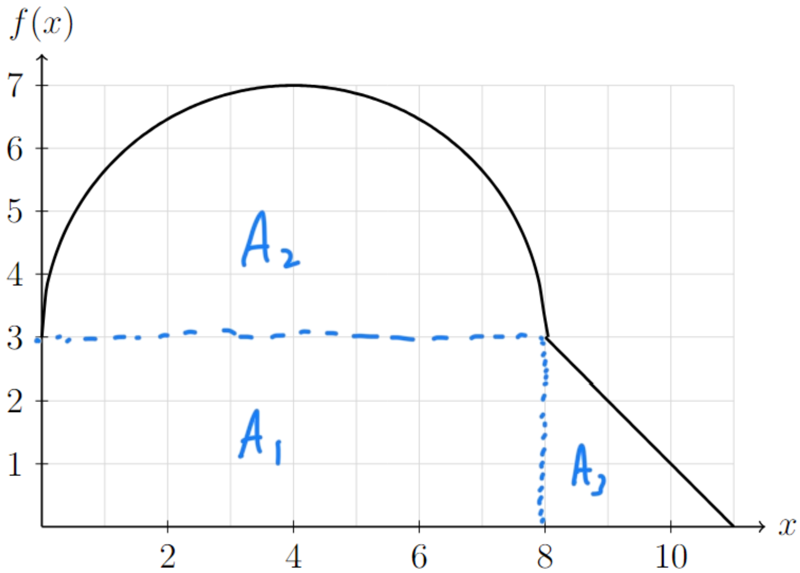
Ex) Consider again $f(x) = \frac{1}{x}$ on the interval $[1, 7]$. Find c such that $f(c) = f_{\text{avg}}$.

$$f(c) = \frac{1}{c}$$

$$f_{avg} = \frac{1}{6} \ln 7 = \frac{1}{c}$$

$$\Rightarrow c = \frac{6}{\ln 7}$$

Ex) Consider the graph of $f(x)$ below and find the average value of f on $[0, 11]$.



$$\text{Area}: \underbrace{(3\times 8)}_{A_1} + \underbrace{\left(\frac{\pi 4^2}{2}\right)}_{A_2} + \underbrace{\left(\frac{1}{2}\times 3\times 3\right)}_{A_3}$$

$$\Rightarrow 24 + 8\pi + \frac{9}{2}$$

$$\Rightarrow \frac{57}{2} + 8\pi$$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{11-0}\int_0^{11} f(x)\,dx$$

$$= \frac{1}{11}\left[\frac{57}{2} + 8\pi\right]$$