Week 14

December 22, 2023

Exercise 1. Show that for two independent random variables $V(X + Y) = V(X) + V(Y)$.

Exercise 2. Let X be a random variable that outputs a uniformly chosen number in $\{-10, -9, \ldots, 9, 10\}$, let Y be a random variable that outputs a uniformly chosen number in $\{10, 11, \ldots, 29, 30\}$, and let Z be the random variable that outputs $X + 20$.

Hint: $\mathbb{E}[X^2] = \sum_{x \in X(s)} x^2 p(X = x)$

1. Compute $\mathbb{E}[X], \mathbb{E}[Y]$ and $\mathbb{E}[Z]$.

2. Compute $V(X), V(Y)$ and $V(Z)$.

Exercise 3. You are given five dice, each in the shape of a different Platonic solid. The five Platonic solids are tetrahedron, cube, octahedron, dodecahedron, and icosahedron with 4,6,8,12,20 sides respectively. Every die has numbers written on its faces starting from 1 up to the number of faces. You throw the dice. What is the expected sum of thrown values? What is the expected product of thrown values?

Exercise 4. Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.
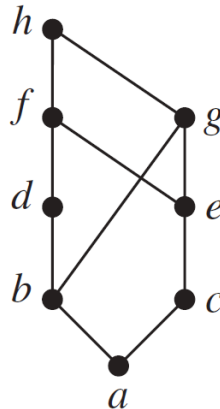
1. What is the expected value of X_n?

2. What is the variance of X_n?

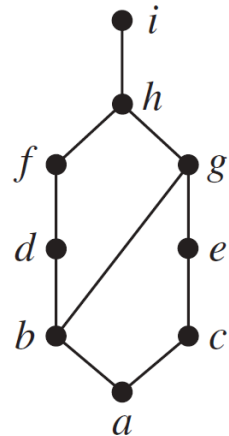**Exercise 5.** Determine whether the posets with these Hasse diagrams are lattices.
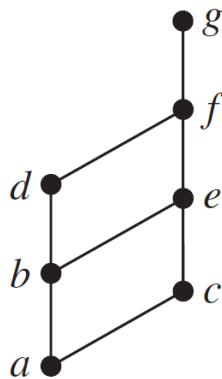
Exercise 6. Find the least number of cables required to connect 15 computers to 10 printers to guarantee that any group of 10 computers can always directly access 10 different printers simultaneously:

○ 60

○ 150

○ 10

○ None of the above

Exercise 7. Let set B be a set of strings made by left and right brackets [and]. The set is recursively defined as follows:

$$\text{Basis step: } \lambda \in B.$$

$$\text{Recursion step:} \forall x, y \in B \rightarrow ([x] \in B) \wedge (xy \in B)$$

We define for $\forall x \in B$, $R(x)$ and $L(x)$ as follows:

$$R(x) = \text{number of [s in x}$$

$$L(x) = \text{number of] s in x}$$

Prove by structural induction that $\forall x \in B, L(x) = R(x)$.

Exercise 8. You absolutely love cats. It has been your biggest dream since you were a little kid to adopt one of your own, take care of it, and enjoy all of the cuddles and the chaotic shenanigans of your furry baby. You have finally reached a point in your life where you are financially stable and can actually adopt one or two cats. But you are young and still travel a lot, which may not be ideal. Every week, you go to the animal shelter in the "haut de Lausanne". Every time you go, there is approximately a 0.7 chance that you go back with a cat. What is the probability that after 5 weeks, you adopted exactly 2 cats, and what principle of probability did you use?

○ ≈0.13 and conditional probability

○ ≈0.13 and Bernoulli trial

○ ≈0.013 and conditional probability

○ ≈0.013 and Bernoulli trial

Exercise 9. Consider the following relation defined on the set of integers **Z**:

$$R = \{(x, y) \mid x^2 \leq y^2\}$$

1. Is R a partial order on the set of integers?

2. If it is a partial order, find the least upper bound (lub) and the greatest lower bound (glb) of the integers 16 and -16 under this relation.

Exercise 10. Given the relations on **Z**$^+$:

$$R1 = \{(a, b) \mid a \text{ divides } b\}$$

$$R2 = \{(a, b) \mid a \text{ is a multiple of } b\}$$

Find $R1 \oplus R2$.

Exercise 11. Use the cashier's algorithm to make changes for the following amounts using quarters (25 cents), dimes (10 cents), and pennies (1 cent), but no nickels (5 cents). For which amount does the algorithm **fail to** use the minimum number of coins?

○ 51 cents

○ 69 cents

○ 76 cents

○ 60 cents

Exercise 12. Find the coefficient of x^6 in the expansion of $\left(x^2 - \frac{1}{x}\right)^9$.

○ 126

○ 84

○ -84

○ -126

Exercise 13. Find the number of solutions (x_1, x_2, x_3) where $x_1, x_2, x_3 \in \mathbb{Z}^+$ such that $x_1 + x_2 + x_3 = 37$ and $x_1 > 7, x_2 > 2, x_3 > 3$?

○ 741

○ 276

○ 351

○ 260,091

○ 204,516