Advanced Probabilities

Advanced Information, Computation, Communication - 1

December 13, 2023

Overview

Problems in Computer Science

Figure out a solution | Figure out how well the solution scales

Logic	Structures	Algorithms	Counting	**Probabilities**
To understand proofs	To understand the structures underlying our solutions	To do useful stuff	To figure out algorithm speed	**To know what to expect**

Today's Topic

Week	Area	Topic
9	Algorithms	Induction, Recursion
10	Counting	Numbers, Counting
11		Advanced Counting
12		Counting / Probabilities
13	Probability Theory	**Advanced Probabilities**
14		Probabilities, Q&A

Short quiz about the past…

End

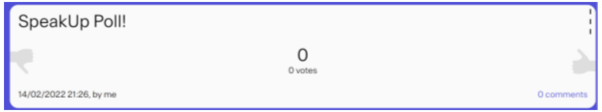
Assume that a coin is biased, where rolling a 6 is twice as likely as rolling any other number (with all other numbers being equally likely). What is the probability of having exactly 4 times a 6 when rolling the biased die seven times?

a) $\binom{7}{4} / 6^7$

b) $\binom{7}{4} \cdot \left(\frac{2}{7}\right)^4 \cdot \left(\frac{5}{7}\right)^3$

c) $\binom{7}{4} \cdot \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^3$

d) $\binom{7}{4} \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^3$



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Agenda

- Random Variables
- Expected Value
- Variance

Learning Objectives

You are able to:

- calculate probabilities for random variables
- calculate the expectation and variance of random variables by applying the according laws

Random Variables vs. Events

Let the random variable Y again represent the number of heads on three coin flips.

Event	Outcomes	Meaning	Probability Statement
$Y = 1$	(TTH),(HTT),(THT)	Y takes on the value 1	$\mathrm{p}(Y = 1)$
$Y < 2$	(TTH),(HTT),(THT),(TTT)	Y takes on the value 0 or 1	$\mathrm{p}(Y < 2)$
$Y \geq 2$	(HHT),(HHT),(THH),(HHH)	Y takes on the value 2 or 3	$\mathrm{p}(Y \geq 2)$
$Y = y$	-	Y takes on the value y	$\mathrm{p}(Y = y)$

Distribution of a Random Variable

The **distribution** of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$ where $p(X = r)$ is the probability that X takes the value r

$$p(X = r) = \sum_{s \in S : X(s) = r} p(s)$$

- Assigns a probability to each possible value of the random variable

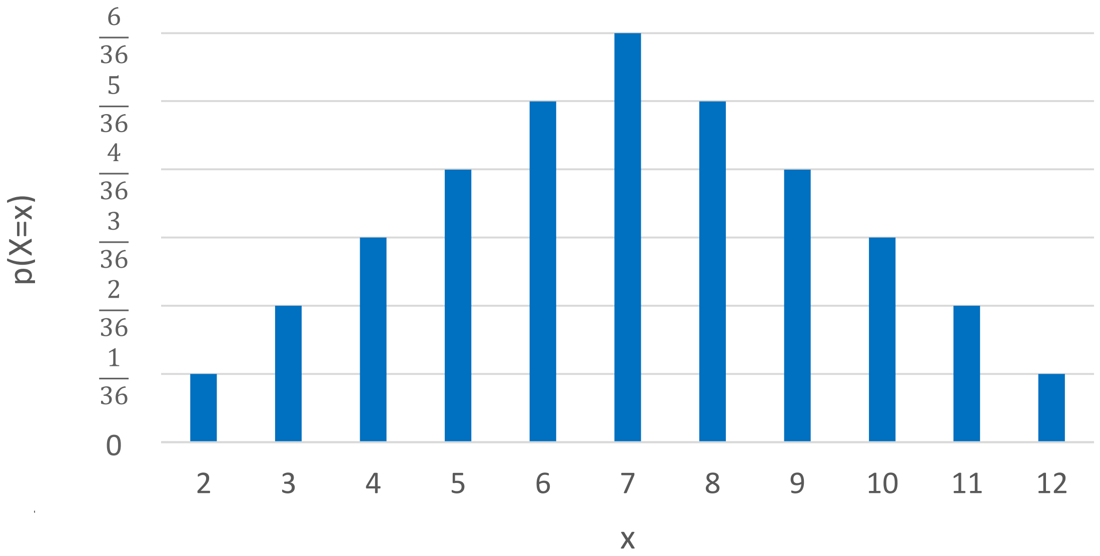
Probability Mass Function

- If the range of the function X is countable, then $p(X = r)$ can be interpreted as a function $p: X(S) \rightarrow R$.
- This function is called **probability mass function** and it is a probability distribution over the sample space $X(S)$**.**

➢ In our context, we will only consider this case

Probability Mass Function

- Suppose that two dice are rolled.
- Let $X(\mathrm{t})$ be the random variable that equals the sum of the two dices of an outcome t.



Your Turn - Coins

End

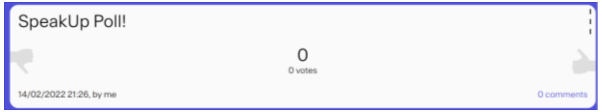
Suppose that a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. What is the probability $p(X = 1)$?

a) $\frac{1}{2}$

b) $\frac{1}{8}$

c) $\frac{3}{8}$

d) $\frac{3}{4}$



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Example: Bernoulli Trials

- The number of successes in n Bernoulli trials is
$$b(k{:}\, n, p) = C(n, k)p^k q^{n-k}$$
- We can interpret $b(k{:}\, n, p)$ as probability distribution
$$p(X = k) = b(k{:}\, n, p)$$
- **Example**: probability of 1 head in 3 coin flips
$$p(X = 1) = C(3,1)\left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2$$

Expected Value

The expected value (or expectation or mean) of the random variable X on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s)$$

Example - Quiz

Let X be the number of points obtained in a multiple-choice question with 1 correct answer and 4 answer options. Choosing the correct option results in 1 point, while choosing one of the incorrect options will result in $-{}^{1}/_{3}$ points. What is the expected number of points when guessing?

Expected Value

Theorem: if X is a random variable and $p(X = r)$ is the probability distribution with $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then $E(x) = \sum_{r \in X(s)} p(X = r) \cdot r$.

Expected Value

Proof:

$$
\begin{aligned}
E(X) &= \sum_{s \in S} p(s) X(s) \\
&= \sum_{r \in X(S)} \sum_{s \in S : X(s) = r} p(s) X(s) \\
&= \sum_{r \in X(S)} \sum_{s \in S : X(s) = r} p(s) \cdot r \\
&= \sum_{r \in X(S)} r \sum_{s \in S : X(s) = r} p(s) \\
&= \sum_{r \in X(S)} r \cdot p(X = r)
\end{aligned}
$$

End

Your Turn

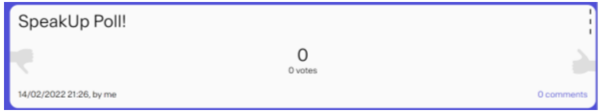
A fair coin is flipped two times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

a) 1

b) 0,5

c) 1,5

d) 2



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Linearity of Expectation

Theorem: if $X_i, i = 1, \ldots, n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

i. $E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$

ii. $E(a \cdot x + b) = a \cdot E(x) + b$

- ➢ The theorem states that expected values are linear.

Linearity of Expectation

(Partial) Proof (i):

Linearity of Expectation

Proof (ii):

Example – Rolling Dice

Let X be the random variable equal to the sum of the numbers that appear when a pair of fair dice is rolled. What is the expected value of X when the pair of dice is rolled once?

Expected value of Bernoulli Trials

Theorem: Let X denote the number of successes, when n mutually independent Bernoulli trials are performed, where p is the probability of each trial. Then the expected value of X is np.

Example – Points in quiz

Assume that a quiz has 10 multiple choice questions. Each question has 4 answer options, one of them is correct. If you answer correctly, you get 1 point, if you answer wrong, you get 0 points. If X denotes the total number of points, what is the expected value of X when guessing?

Example – Coin Flips

Suppose that the probability that a coin comes up tails is p. This coin is flipped repeatedly until it comes up tails. What is the expected number of flips until this coin comes up tails?

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End

Your Turn

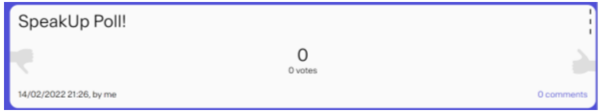
Suppose that we roll a pair of fair dice until the sum of the numbers on the dice is seven. What is the expected number of times we roll the dice?

a) 7

b) 6

c) 12

d) 10



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Independent Random Variables

The random variables X and Y on a sample space S are independent if

$$p(X = r_1 \wedge Y = r_2) = p(X = r_1) \cdot p(Y = r_2)$$

Theorem: if X and Y are independent random variable on sample space S, then:

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

Variance

- Let X be a random variable on the sample space S. The **variance** of X, denoted by $V(X)$ is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 \cdot p(s)$$

Variance

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- The **standard deviation** of X, denoted by $\sigma(X)$ is defined as $\sqrt{V(X)}$

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- Variance and standard deviation are used to quantify how widely a random variable is distributed

Example

We again look into rolling a die.

- Let X and Y be random variables on $S = \{1, 2, 3, 4, 5, 6\}$
- Let $X(s) = 0$ all $s \in S$
- Let $Y(s) = -1$ for s $\in \{1,2,3\}$ and $Y(s) = 1$ for s $\in \{4,5,6\}$

Characterization of Variance

Theorem: if X is a random variable on a sample space S, then $V(X) = E(X^2) - E(X)^2$

Corollary: if X is a random variable on a sample space S and $E(X) = \mu$, then $V(X) = E\left((X - \mu)^2\right)$

Characterization of Variance

Proof of theorem: if X is a random variable on a sample space S, then $V(X) = E(X^2) - E(X)^2$

Example - Die

What is the variance of a random variable X, where X is the number that comes up when a fair die is rolled?

Law of the Unconscious Statistician

In the die example, we assumed: $\mathrm{E}\left(X^2\right) = \sum_{x \in X(S)} x^2 p(X = x)$

The law of the unconscious statistician tells us that this relation holds for any function:

$$E\left(g(X)\right) = \sum_{x \in X(S)} g(x) p(X = x)$$

Your Turn - Variance

End

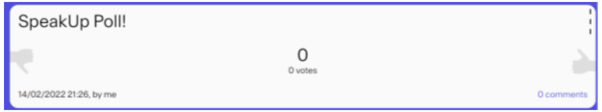
What is the variance of the number of heads that come up when a fair coin is flipped 2 times?

a) 1/4

b) 1

c) 1/2

d) 3/2



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Variance for Independent Random Variables

Bienaymé‘s Formula: If X and Y are two independent random variables on a sample space S, then $V(X + Y) = V(X) + V(Y)$.

Furthermore, if X_i, $i = 1,2, \ldots, n$, with n a positive integer, are pairwise independent random variables on S, then
$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n)$.

Example – Bernoulli Trials

Find the variance of the number of successes when n independent Bernoulli trials are performed, where on each trial, p is the probability of success and q is the probability of failure.

Summary

- Expected Value
- Variance
  - Independent Variables