# 1. Ejercicio 2.2

$\omega \in Alt^2\mathbb{R}^4$

$$\omega \wedge \omega \neq 0$$

$\{e_1, \ldots, e_4\}$

$\mathbb{R}^4$

$\{\varepsilon_1, \ldots, \varepsilon_4\}$

$Alt^1(\mathbb{R}^4)$

$Alt^2(\mathbb{R}^4)$

$$\{\varepsilon_i \wedge \varepsilon_j\}_{i,j,i<j}$$

$$\omega \in Alt^2(\mathbb{R}^4), \omega \wedge \omega \in Alt^4(\mathbb{R}^4)$$

$$\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4$$

$$\omega = \lambda_1 \varepsilon_1 \wedge \varepsilon_2 + \cdots + \lambda_6 \varepsilon_3 \wedge \varepsilon_4$$

$$\omega = \lambda_1 \varepsilon_1 \wedge \varepsilon_2 + \cdots + \lambda_6 \varepsilon_3 \wedge \varepsilon_4 \tag{1}$$

$$\text{si } \lambda_1 = \lambda_6 = 1, \quad \omega \wedge \omega = 2\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4$$

## 2. Ejercicio 2.3

$$\mathbb{R}^3 \xrightarrow{i} Alt^1(\mathbb{R}^3), \quad \mathbb{R}^3 \xrightarrow{j} Alt^2\mathbb{R}^3$$

$$i(v)(w) = \langle v, w \rangle, \quad j(v)(w_1, w_2) = det(v, w_1, w_2),$$

<, >

$v_1, v_2 \in \mathbb{R}^3$

$$i(v_1) \wedge i(v_2) = j(v_1 \times v_2).$$

*i*

*i*

$$i(\lambda v_1 + v_2) = \lambda i(v_1) + i(v_2)$$

$$i(\lambda v_1 + v_2) = \langle \lambda v_1 + v_2, w \rangle = \lambda \langle v_1, w \rangle + \langle v_2 + w \rangle = \lambda i(v_1(w) + i(v_2)(w)$$

*i*

$$i(e_1)(w) =< e, w >= \lambda_1, i(e_j)(w) = \lambda_j, i(e_j) = \varepsilon_j$$

$$w = \sum_i \lambda_i e_i$$

$j$

$$v \in \mathbb{R}^3 \Rightarrow j(v) \in Alt^2(\mathbb{R}^3)$$

$$j(v)$$

$$det(v, \lambda w_1 + w_2, w_3) = j(v)(\lambda w_1 + w_2, w_3) = \lambda j(w_1, w_3) + j(w_2, w_3) = \lambda det(v, w_1, w_3) + det(v, w_2, w_3)$$

$$j(v)$$

$$j(v)(w,w) = det(v,w,w) = 0$$

$j$

$$j(\lambda v_1 + v_2)(w_1, w_2) = det(\lambda v_1 + v_2, w_1, w_2) = \lambda det(v_1, w_1, w_2) + det(v_2, w_1, w_2) = (\lambda j(v_1) + j(v_2))(w_1, w_2)$$

$j$

$$\{\varepsilon_1 \wedge \varepsilon_2, \varepsilon_1 \wedge \varepsilon_3, \varepsilon_2 \wedge \varepsilon_3\}$$

$$Alt^2(\mathbb{R}^3)$$

$$j(e_1) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

$$(w_1, w_2) \mapsto det \begin{pmatrix} 1 & 0 & 0 \\ w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \end{pmatrix} = w_{12}w_{23} - w_{13}w_{22} =^{?} \varepsilon_2 \wedge \varepsilon_3(w_1, w_2).$$

$$\varepsilon_2 \wedge \varepsilon_3(w_1, w_2) = \sum_{\sigma \in S(1,1)} sign(\sigma)\varepsilon_2(w_{\sigma(1)})\varepsilon_3(w_{\sigma(2)}) = \varepsilon_2(w_1)\varepsilon_3(w_2) - \varepsilon(w_2)\varepsilon_3(w_1) = w_{12}w_{23} - w_{22}w_{13}.$$

$$j(e_2) = \pm\varepsilon_1 \wedge \varepsilon_3(w_1, w_2)$$

$$j(e_3) = \mp \varepsilon_1 \wedge \varepsilon_2$$

$$i(v_1) \wedge i(v_2) \stackrel{?}{=} j(v_1 \times v_2)$$

$$i(v_1) \wedge i(v_2) = \sum_{\sigma \in S(2,1)} i(v_1) \wedge i(v_2)(e_{\sigma(1)}, e_{\sigma(2)}) \varepsilon_{\sigma(1)} \wedge \varepsilon_{\sigma(2)} = A_{03} \varepsilon_1 \wedge \varepsilon_2 + A_{02} \varepsilon_1 \wedge \varepsilon_3 + A_{01} \varepsilon_2 \wedge \varepsilon_3$$

$$i(v_1) \wedge i(v_2)(e_1, e_2) = i(v_1)(e_2)i(v_1)(e_2) - i(v_1)(e_2)i(v_2)(e_1) = v_{11}v_{22} - v_{12}v_{21} = A_{03}$$

## 3. Ejercicio 2.5

$Alt^p(V)$

$$\langle \omega_1 \wedge \cdots \wedge \omega_p, \tau_1 \wedge \cdots \wedge \ tau_p \rangle,$$

$\omega_i, \tau_j \in Alt^1(V)$

$$\langle \omega, \tau \rangle = \langle i^{-1}(\omega), i^{-1}(\tau) \rangle.$$

$$\{b_1, \ldots, b_n\}$$

$V$

$$\beta_j = i(b_j)$$

$$\{\beta_{\sigma(1)} \wedge \cdots \wedge \beta_{\sigma(p)} | \sigma \in S(p, n-p)\}$$

$Alt^p(V)$

$$< w_i, \tau_j > = < i^{-1}(\omega_i), i^{-1}(\tau_j) >$$

$$Alt^1(V)$$

$$< w_i, \tau_j >=< i^{-1}(\omega_i), i^{-1}(\tau_j) >= i(i^{-1}(\omega_i))(i^{-1}(\tau_j)) = \omega_i(i^{-1}(\tau_j))$$

$$< \tau_j, \omega_i > = < i^{-1}(\tau_j), i^{-1}(\omega_i) > = i(i^{-1}(\tau_j))(i^{-1}(\omega_i)) = \tau_j(i^{-1}(\omega_i))$$

$$
\begin{aligned}
det(<\omega_i, \tau_j>) &= det(\omega_i(i^{-1}(\tau_j)))_{i,j\in\{1,\ldots,p\}} = \\
&= det\begin{pmatrix} \omega_1(i^{-1}(\tau_1)) & \cdots & \omega_1(i^{-1}(\tau_p)) \\ \vdots & & \vdots \\ \omega_p(i^{-1}(\tau_1)) & \cdots & \omega_p(i^{-1}(\tau_p)) \end{pmatrix} \\
&= \omega_1 \wedge \cdots \wedge \omega_p(i^{-1}(\tau_1), \ldots, i^{-1}(\tau_p)) \\
&= (\tau_1 \wedge \cdots \wedge \tau_p)(i^{-1}(\omega_1, \ldots, i^{-1}(\omega_p)).
\end{aligned} \tag{2}
$$

$$\langle \omega_1 \wedge \cdots \wedge \omega_p, \tau_1 \wedge \cdots \wedge \tau_p \rangle = \omega_1 \wedge \cdots \wedge \omega_p(i^{-1}(\tau_1), \ldots, i^{-1}(\tau_p)).$$

$$\langle \cdot, \cdot \rangle$$

$$
\begin{aligned}
\langle \omega_1 \wedge \cdots \wedge \omega_p, \tau_1 \wedge \cdots \wedge \tau_p \rangle &= \\
&= \omega_1 \wedge \cdots \wedge \omega_p(i^{-1}(\tau_1), \ldots, i^{-1}(\tau_p)) \\
&= \langle \tau_1 \wedge \cdots \wedge \tau_p, \omega_1 \wedge \cdots \wedge \omega_p \rangle \\
&= (\tau_1 \wedge \cdots \wedge \tau_p)(i^{-1}(\omega_1, \ldots, i^{-1}(\omega_p)).
\end{aligned} \tag{3}
$$

$\omega, \tau \in Alt^p(V)$

$$
\begin{aligned}
\omega &= \sum_{\sigma \in S(p,n-p)} \omega(e_{\sigma(1)}, \ldots, e_{\sigma(p)}) \varepsilon_\sigma \\
\tau &= \sum_{\bar\sigma \in S(p,n-p)} \omega(e_{\bar\sigma(1)}, \ldots, e_{\bar\sigma(p)}) \varepsilon_{\bar\sigma} \\
< \omega, \tau > &= \sum_\sigma \sum_{\bar\sigma} \omega_\sigma \tau_{\bar\sigma} < \varepsilon_\sigma, \varepsilon_{\bar\sigma} > . \\
< \omega, \omega > &= \sum_\sigma (\omega_\sigma)^2 \geq 0 \\
< \varepsilon_\sigma, \varepsilon_{\bar\sigma} > &= \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} (e_{\bar\sigma(1)}, \ldots, e_{\bar\sigma(p)}) = I_\sigma(\bar\sigma) \\
< \omega, \omega > &= 0 \Leftrightarrow \omega_\sigma = 0 \forall \sigma \in S(p, n-p) \Rightarrow \omega = 0 \in Alt^p(V).
\end{aligned} \tag{4}
$$

$$Alt^p(V)$$

$$
\begin{aligned}
\langle \omega, i(\xi_1) \wedge \cdots \wedge i(\xi_p) \rangle &= \\
&= \langle \sum_\sigma \omega_\sigma \varepsilon_\sigma, i(\xi_1 \wedge \cdots \wedge i(\xi_p) \rangle \\
&= \sum_\sigma \omega_\sigma \langle \varepsilon_\sigma, i(\xi_1) \wedge \cdots \wedge i(\xi_p) \rangle \\
&= \sum_\sigma \omega_\sigma \varepsilon_\sigma (\xi_1, \ldots, \xi_p) \\
&= \omega(\xi_1, \ldots, \xi_p).
\end{aligned} \tag{5}
$$

$$\langle \omega, \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} \rangle = \langle \omega, i(e_{\sigma(1)}) \wedge \cdots \wedge i(e_{\sigma(p)} \rangle = \omega(e_{\sigma(1)}, \ldots, e_{\sigma(p)}).$$

$$\omega = \sum_\sigma \omega(e_{\sigma(1)}, \ldots, e_{\sigma(p)}) \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} = \sum_\sigma \langle \omega, \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} \rangle \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)}.$$

$$\{b_1, \ldots, b_n\}$$

$V$

$$\langle b_i, b_j \rangle = \begin{cases} 0 & \text{si } i \neq j \\ 1 & \text{si } i = j \end{cases}$$

$$\{\beta_1, \ldots, \beta_n\}$$

$$Alt^1(V)$$

$$\{\beta_{\sigma(1)} \wedge \cdots \wedge \beta_{\sigma(p)}\}_{\sigma \in S(p,n-p)}$$

$$Alt^p(V)$$

$$\langle \beta_{\sigma(1)} \wedge \cdots \wedge \beta_{\sigma(p)}, \beta_{\bar{\sigma}(1)} \wedge \cdots \wedge \beta_{\bar{\sigma}(p)} \rangle = \beta_{\sigma(1)} \wedge \cdots \wedge \beta_{\sigma(p)}(\beta_{\bar{\sigma}(1)}, \ldots, \beta_{\bar{\sigma}(p)}) = \begin{cases} 0 & \text{si } \sigma \neq \bar{\sigma} \\ 1 & \text{si } \sigma = \bar{\sigma} \end{cases}$$

# 4. Ejercicio 2.6

$\omega \in Alt^p(V)$

$v_1, \ldots, v_p$

$\omega \in Alt^p(V)$

$v_1, \ldots, v_p \in V$

$$A = (a_{ij}) \in M_{p \times p}$$

$$\omega_i = \sum_{j=1}^{p} a_{ij} v_j, i = 1, \ldots, p$$

$$\omega(\omega_1, \ldots, \omega_p) = det(A) \cdot \omega(v_1, \ldots, v_p)$$

$p = 2$

$$A = (a_{i,j})_{i,j \in \{1,2\}}$$

$$\omega_1 = a_{11}v_1 + a_{12}v_2$$

$$\omega_2 = a_{21}v_2 + a_{22}v_2$$

$$\begin{aligned}\omega(\omega_1,\omega_2) &= \omega(a_{11}v_1 + a_{12}v_2, a_{21}v_2 + a_{22}v_2) = \\ &= a_{11}a_{22}\omega(v_1,v_2) + a_{12}a_{21}\omega(v_2,v_1) = \\ &= a_{11}a_{22}\omega(v_1,v_2) - a_{12}a_{21}\omega(v_1,v_2) = \\ &= (a_{11}a_{22} - a_{12}a_{21})\omega(v_1,v_2) = det(A)\cdot\omega(v_1,v_2).\end{aligned} \tag{6}$$

$p$

$$
\begin{aligned}
\omega(\omega_1, \ldots, \omega_p) &= \omega(\sum_{j=1}^{p} a_{ij} v_j, \ldots, \sum_{j=1}^{p} a_{pj} v_j) = \\
&= \sum_{\tau \in S(p)} \prod_{k=1}^{p} a_{k\tau(k)} \omega(v_{\tau(1)}, \ldots, v_{\tau(p)}) = \\
&= \underbrace{\sum_{\tau \in S(p)} sgn(\tau) \prod_{k=1}^{p} a_{k\tau(k)}}_{det(A)} \cdot \omega(v_1, \ldots, v_p).
\end{aligned}
\tag{7}
$$

# 5. Ejercicio 2.7

$$
\begin{aligned}
Alt^{p+q}(f)(\omega_1 \wedge \omega_2) = & \\
& = \omega_1 \wedge \omega_2(f(\xi_1), \ldots, f(\xi_{p+q})) \\
& = \sum_{\sigma \in S(p,n-p)} sgn(\sigma)\omega_1(f(\xi_{\sigma(1)}), \ldots, f(\xi_{\sigma(p)}))\omega_2(f(\xi_{\sigma(p+1)}), \ldots, f(\xi_{\sigma(p+q)})) \\
& = \sum_{\sigma \in S(p,n-p)} sgn(\sigma) Alt^p(f)\omega_1(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(p)}) Alt^q(f)\omega_2(\xi_{\sigma(p+1)}, \ldots, \xi_{\sigma(p+q)}) \\
& = Alt^p(f)(\omega_1) \wedge Alt^q(f)(\omega_2)(\xi_1, \ldots, xi_{p+q}).
\end{aligned}
\tag{8}
$$